Assignment 05 & 06 of I2ML-s23

Q1 Assume predicate $S(x)$ and $W(x)$ denote "x is a student" and "x walks" respectively, constant d denotes David, translate the following sentences into proper FOL formulas. *(6 * 3pts + 3 * 4pts = 30 pts)*

(1) David walks.
(2) Every student walks.
(3) Some student walks.
(4) No student Walks.
(5) Somebody walks.
(6) Nobody walks.
(7) There are at least 2 students.
(8) There are exactly 2 students.
(9) If every student walks and David is a student, then David walks.

Q2 Let c and d be constants; f/1, g/2 and h/3 be functions; P/3 and Q/3 be predicates, which of the following strings are formulas in FOL? *(6 * 3pts = 18 pts)*

(10) $\forall x\, P(\, f(d), h(g(c, x), d, y))$

(11) $\forall x\, P(\, f(d), h(P(x, y), d, y))$

(12) $\forall x\, Q(\, g(h(x, f(d), x), g(x, x)), h(x, x, x), c)$

(13) $\exists z\, (Q(z,z,z) \rightarrow P(z))$

(14) $\forall x \forall y\, (g(x, y) \rightarrow P(x, y, x))$

(15) $Q(c, d, c)$

Q3 Let φ be $\exists x\,(\,P(y, z) \wedge \forall y\,(\,Q(y, x) \vee P(y, z)\,)\,)$, where $P/2$ and $Q/2$ are predicate symbols. (*10 * 3pts = 30pts*)

(16) Identify all bound and free variables in φ.

(17) Is there a variable in φ which has free and bound occurrences?

(18) Consider the terms w (w is a variable), f(x) and g(y, z), where f/1 and g/2 are function symbols respectively.

18.1) Compute $\varphi[w/x]$, $\varphi[w/y]$, $\varphi[f(x)/y]$ and $\varphi[g(y, z)/z]$. (*4 * 3pts = 12pts*)

18.2) Which of w, $f(x)$ and $g(y, z)$ are free for x in φ?

18.3) Which of w, f(x) and $g(y, z)$ are free for y in φ?

(19) What is the scope of $\exists x$ in φ?

(20) Suppose that we change φ to $\exists x\,(\,P(y, z) \wedge \forall x\,(\,\neg Q(x, x) \vee P(x, z)))$. What is the scope of $\exists x$ now?

Q4 . Choose 2 sequents and prove the validity of the selected sequents, using, among others, the rules =i and =e. Make sure that you indicate for each application of =e what the rule instances *φ, t1 and t2 are. (2 *10pts = 20pts)*

(21) $(y = 0) \land (y = x) \vdash 0 = x$

(22) $t1 = t2 \vdash (t + t2) = (t + t1)$

(23) $(x = 0) \lor ((x + x) > 0) \vdash (y = (x + x)) \rightarrow ((y > 0) \lor (y = (0+x)))$

Q5 Provide formal proofs for the following sequents: (*2 * 10 pts = 20 pts*)

(24) $\forall x\, (P(x) \rightarrow Q(x)) \vdash (\forall x\, \neg Q(x)) \rightarrow (\forall x\, \neg P(x))$

(25) $\forall x\, (P(x) \rightarrow \neg Q(x)) \vdash \neg(\exists x\, (P(x) \land Q(x)))$.

Q6 Choose the hardest 3 sequents you felt from the right list, prove the validity of them by formal proof in FOL, where *F*/1, *G*/1, *P*/1, Q/1 and S/0 are predicates. *3* 10pts = 30 pts)*

(a) $\exists x\,(S \to Q(x)) \vdash S \to \exists x\, Q(x)$
(b) $S \to \exists x\, Q(x) \vdash \exists x\,(S \to Q(x))$
(c) $\exists x\, P(x) \to S \vdash \forall x\,(P(x) \to S)$
(d) $\forall x\, P(x) \to S \vdash \exists x\,(P(x) \to S)$
(e) $\forall x\,(P(x) \vee Q(x)) \vdash \forall x\, P(x) \vee \exists x\, Q(x)$
(f) $\forall x\, \exists y\,(P(x) \vee Q(y)) \vdash \exists y\, \forall x\,(P(x) \vee Q(y))$
(g) $\forall x\,(\neg P(x) \wedge Q(x)) \vdash \forall x\,(P(x) \to Q(x))$
(h) $\forall x\,(P(x) \wedge Q(x)) \vdash \forall x\,(P(x) \to Q(x))$
(i) $\exists x\,(\neg P(x) \wedge \neg Q(x)) \vdash \exists x\,(\neg(P(x) \wedge Q(x)))$
(j) $\exists x\,(\neg P(x) \vee Q(x)) \vdash \exists x\,(\neg(P(x) \wedge \neg Q(x)))$
(k) $\forall x\,(P(x) \wedge Q(x)) \vdash \forall x\, P(x) \wedge \forall x\, Q(x)$.
(l) $\forall x\, P(x) \vee \forall x\, Q(x) \vdash \forall x\,(P(x) \vee Q(x))$.
(m) $\exists x\,(P(x) \wedge Q(x)) \vdash \exists x\, P(x) \wedge \exists x\, Q(x)$.
(n) $\exists x\, F(x) \vee \exists x\, G(x) \vdash \exists x\,(F(x) \vee G(x))$.
(o) $\forall x\, \forall y\,(S(y) \to F(x)) \vdash \exists y S(y) \to \forall x\, F(x)$.
(p) $\neg \forall x\, \neg P(x) \vdash \exists x\, P(x)$.
(q) $\forall x\, \neg P(x) \vdash \neg \exists x\, P(x)$.
(r) $\neg \exists x\, P(x) \vdash \forall x\, \neg P(x)$.

Q7 The proofs of the sequents below combine the proof rules for equality and quantifiers. We write $\varphi \leftrightarrow \psi$ *as an abbreviation for* $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$. *Choose and find 3 formal* proofs from the following 4 sequents : *(3 * 10 pts = 30 pts)*

(a) $P(b) \vdash \forall x\,(x = b \rightarrow P(x))$

(b) $P(b), \forall x \forall y\,(P(x) \wedge P(y) \rightarrow x = y) \vdash \forall x\,(P(x) \leftrightarrow x = b)$

(c) $\exists x\,\exists y\,(H(x, y) \vee H(y, x)), \neg\exists x\,H(x, x) \vdash \exists x \exists y\,\neg(x = y)$

(d) $\forall x\,(P(x) \leftrightarrow x = b) \vdash P(b) \wedge \forall x \forall y\,(P(x) \wedge P(y) \rightarrow x = y)$.

Q8 By formal proof in FOL, show the validity of the 4 hardest sequents selected from the following list according to your intuition. *(4 * 10 pts = 40 pts)*

(a) $\forall x\, P(a,x,x),\ \forall x\,\forall y\,\forall z\,(P(x,y,z) \to P(f(x),y,f(z)))$
$\vdash P(f(a),a,f(a))$

(b) $\forall x\, P(a,x,x),\ \forall x\,\forall y\,\forall z\,(P(x,y,z) \to P(f(x),y,f(z)))$
$\vdash \exists z\, P(f(a),z,f(f(a)))$

(c) $\forall y\, Q(b,y),\ \forall x\,\forall y\,(Q(x,y) \to Q(s(x),s(y)))$
$\vdash \exists z\,(Q(b,z) \wedge Q(z,s(s(b))))$

(d) $\forall x\,\forall y\,\forall z\,(S(x,y) \wedge S(y,z) \to S(x,z)),\ \forall x\,\neg S(x,x)$
$\vdash \forall x\,\forall y\,(S(x,y) \to \neg S(y,x))$

(e) $\forall x\,(P(x) \vee Q(x)),\ \exists x\,\neg Q(x),\ \forall x\,(R(x) \to \neg P(x)) \vdash \exists x\,\neg R(x)$

(f) $\forall x\,(P(x) \to (Q(x) \vee R(x))),\ \neg\exists x\,(P(x) \wedge R(x)) \vdash \forall x\,(P(x) \to Q(x))$

(g) $\exists x\,\exists y\,(S(x,y) \vee S(y,x)) \vdash \exists x\,\exists y\, S(x,y)$

(h) $\exists x\,(P(x) \wedge Q(x)),\ \forall y\,(P(x) \to R(x)) \vdash \exists x\,(R(x) \wedge Q(x))$.