CSE5014 CRYPTOGRAPHY AND NETWORK SECURITY

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Perfect security

- **Definition 1.6** *Perfect secrecy*. An encryption scheme (Gen, Enc, Dec) with message space $\mathcal{M}$ and ciphertext space $\mathcal{C}$ is *perfectly secure* if and only if for every two distinct plaintexts $\{x_0, x_1\} \in \mathcal{M}$, and for every strategy used by Eve, if we choose at random $b \in \{0, 1\}$ and a random key $k \in \{0, 1\}^n$, then the probability that Eve guesses x_b after seeing the ciphertext $c = Enc_k(x_b)$ is at most $1/2$.

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  Theorem 1.10 (Limitations of perfect secrecy) There is no *perfectly secure* encryption schemes (Gen, Enc, Dec) with n-bit plaintexts and $(n-1)$-bit keys.

  - The key is as long as the message
  - Only secure if each key is used to encrypt a single message
  - Trivially broken by a known-plaintext attack

ϵ-Statistical Security

- **Definition 2.1** Let X and Y be two distributions over $\{0,1\}^n$. The *statistical distance* of X and Y, denoted by $\Delta(X, Y)$ is defined to be
$$\max_{T \subseteq \{0,1\}^n} |\Pr[X \in T] - \Pr[Y \in T]|.$$
If $\Delta(X, Y) \le \epsilon$, we say that $X \equiv_\epsilon Y$.

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  Definition 2.2 *ϵ-Statistical Security*. An encryption scheme (Gen, Enc, Dec) is *ϵ-statistically secure* if for every pair of plaintexts m, m', we have $Enc_{U_n}(m)\equiv_\epsilon Enc_{U_n}(m')$.

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- **Lemma 2.3**

$$\Delta(X, Y) = \frac{1}{2} \sum_{w \in Supp(X) \cup Supp(Y)} |Pr[X = w] - Pr[Y = w]|$$

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Observations:

$0 \leq \Delta(X,Y) \leq 1$

$\Delta(X,Y) = 0$ if $X = Y$

$0 \leq \Delta(X,Y) \leq \Delta(X,Z) + \Delta(Z,Y)$

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Δ is a *metric*.

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- **Lemma 2.4** Eve has at most $1/2 + \epsilon$ success probability if and only if for every pair of m_1, m_2,
  $$\Delta(Enc_{U_n}(m_1), Enc_{U_n}(m_2)) \leq 2\epsilon.$$

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- **Lemma 2.4** Eve has at most $1/2 + \epsilon$ success probability if and only if for every pair of m_1, m_2,
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  Theorem 2.5 Let (Gen, Enc, Dec) be a valid encryption with $Enc : \{0,1\}^n \times \{0,1\}^{n+1} \rightarrow \{0,1\}^*$. Then there exist plaintexts m_1, m_2 with $\Delta(Enc_{U_n}(m_1), Enc_{U_n}(m_2)) > 1/2$.

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- **Idea**: Would be OK if a scheme leaked information with *tiny probability* to eavesdroppers with *bounded computational resources*
  - Allowing security to "fail" with tiny probability
  - Restricting attention to "efficient" attackers

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- Say security fails with probability 2^{-60}
  - Should we be concerned about this?
  - With probability $> 2^{-60}$, the sender and receiver will both be struck by lightning in the next year …
  - Something that occurs with probability $2^{-60}/sec$ is expected to occur once every 100 billion years

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Modern key space: 2^{128} keys or more ...

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"Breaking E is very hard"?
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$\mathcal{Q}$: How do we model the resources of Eve (the adversary)?

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  Define a randomized experiment $PrivK_{A,\Pi}$:

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  Claim: Π is *perfectly indistinguishable* $\Leftrightarrow$ Π is *perfectly secure*

Computational security?

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  Two approaches

  – *Concrete* security

  – *Asympototic* security

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- (t, ϵ)-*indistinguishability* (concrete)
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Does not lead to a clean theory ...
  - Sensitive to exact computational model
  - Π can be (t, ϵ)-secure for many choices of t, ϵ

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  - For now, can view n as the key length
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Computational indistinguishability:

- Security may fail with probability *negligible* in n
- Restrict attention to attackers running in time (at most) polynomial in n

Definitions

- A function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ is (at most) *polynomial* if there exists c s.t. $f(n) < n^c$ for large enough n.

  A function $f : \mathbb{Z}^+ \to [0, 1]$ is *negligible* if every polynomial p it holds that $f(n) < 1/p(n)$ for large enough n.

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  – Typical example: $f(n) = poly(n) \cdot 2^{-cn}$
- "Efficient" $=$ "(probabilistic) polynomial-time (**PPT**)" borrowed from *complexity theory*
- Convenient closure properties
  – $poly * poly = poly$
    – Poly-many calls to PPT subroutine (with poly-size input) is still PPT
  – $poly * negl = negl$
    – Poly-many calls to subroutine that fails with negligible probability fails with negligible probability overall

(Re)defining encryption

- A *private-key encryption scheme* is defined by three PPT algorithms (Gen, Enc, Dec):
  - Gen: takes as input 1^n; outputs k
  - Enc: takes as input a key k and message $m \in \{0,1\}^*$; outputs ciphertext c: $\quad c \leftarrow Enc_k(m)$
  - Dec: takes key k and ciphertext c as input; outputs a message m or "error" ($\perp$)

Computational indistinguishability (asymptotic)

- Fix Π, A
  Define a randomized experiment $PrivK_{A,\Pi}(n)$:
  1. $A(1^n)$ outputs $m_0, m_1 \in \{0,1\}^*$ of equal length
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  Definition 3.1 Π is *computationally indistinguishable* (aka *EAV-secure*) if for all PPT attackers (algorithms) A, there is a *negligible* function ϵ such that

$$\Pr[PrivK_{A,\Pi}(n) = 1] \leq 1/2 + \epsilon(n)$$

Example

- Consider a scheme where the best attack is *brute-force search* over the key space, and $Gen(1^n)$ generates a uniform n-bit key
  - So if A runs in time $t(n)$, then
    $\Pr[PrivK_{A,\Pi}(n) = 1] < 1/2 + O(t(n)/2^n)$

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  - So if A runs in time $t(n)$, then
    $\Pr[PrivK_{A,\Pi}(n) = 1] < 1/2 + O(t(n)/2^n)$
  - The scheme is EAV-secure: for any polynomial t, the function $t(n)/2^n$ is negligible.

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  - This does not contradict asymptotic security
  - What about real-world security (against this attacker)?
    - $n = 40$: A breaks with prob. 1 in 6 weeks
    - $n = 50$: A breaks with prob. $1/1000$ in 3 months
    - $n = 500$: A breaks with prob. 2^{-500} in 200 years

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  What if computers get $4\times$ faster?

  – Users double n; maintain same running time

  – Attacker's work is (roughly) squared!

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- In practice, we want encryption schemes that can encrypt arbitrary-length messages.

- In general, encryption does not hide the plaintext length
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- But leaking plaintext length can often lead to problems in the real world!
  - Databases searches
  - Encrypting compressed data

Micali & Goldwasser

Silvio Micali

Shafi Goldwasser

1984: semantic security, indistinguishability
(Turing Award 2012)

Micali & Blum

Silvio Micali

Manuel Blum

1984: defined notion of pseudo-random generator
(Turing Award 1995)

Pseudorandomness

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- If we generate a uniform 16-bit string, each of the above occurs with probability 2^{-16}

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- Informal: Cannot be distinguished from uniform ("random")
- Which of the following is pseudorandom?
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- *Pseudorandomness* is a property of a *distribution*, not a string.

Pseudorandomness

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- Historically, D was considered *pseudorandom* if it "*passed a bunch of statistical tests*"
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  - $\Pr_{x \leftarrow D}[\text{parity of } x \text{ is } 1] \approx 1/2$
  - $\Pr_{x \leftarrow D}[A_i(x) = 1] \approx \Pr_{x \leftarrow U_n}[A_i(x) = 1]$
    for $i = 1, \ldots, 20$

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  This is not sufficient, since it is not possible to know what statistical test an attacker will use.

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(Concrete) Let D be a distribution on p-bit strings. D is (t, ϵ)-*pseudorandom* if for all A running in time at most t,

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(Asymptotic) *Security parameter n*, polynomial p

Definiton 3.2 Let D_n be a distribution over $p(n)$-bit strings. $\{D_n\}$ is *pseudorandom* if for all probabilistic, polynomial-time (PPT) distinguishers A, there is a negligible function ϵ such that

$$|\Pr_{x \leftarrow D_n}[A(x) = 1] - \Pr_{x \leftarrow U_{p(n)}}[A(x) = 1]| \leq \epsilon(n)$$

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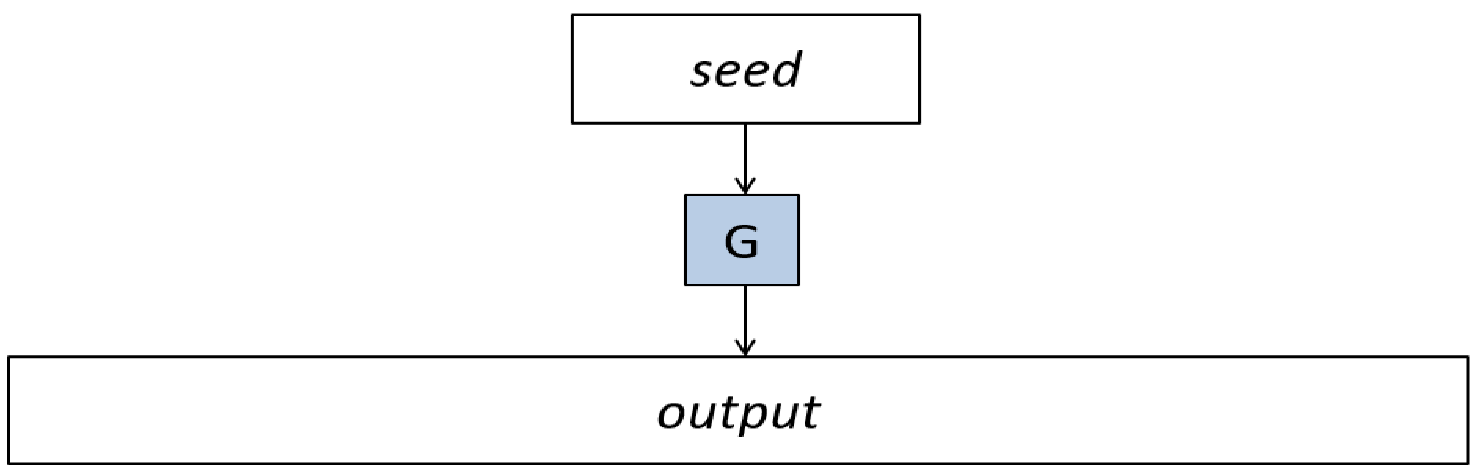
Let G be a deterministic, poly-time algorithm that is *expanding*, i.e., $|G(x)| = p(|x|) > |x|$.



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G defines a sequence of distributions.

- D_n = the distribution on $p(n)$-bit strings defined by choosing $x \leftarrow U_n$ and outputting $G(x)$
- $\Pr_{D_n}[y] = \Pr_{U_n}[G(x) = y] = \sum_{x:\ G(x)=y} \Pr_{U_n}[x]$

$$= \sum_{x:\ G(x)=y} 2^{-n}$$

$$= |\{x :\ G(x) = y\}|/2^n$$

PRGs

- For all efficient distinguishers A, there is a negligible function ϵ such that
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- PRGs are limited
  - They have fixed-length output
  - They produce the entire output in "one shot"
  - In practice, PRGs are based on *stream ciphers*
  - Can be viewed as producing an "unbounded" stream of pseudorandom bits, on demand
  - Will be revisited later

Do PRGs/stream ciphers exist?

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  - Would imply $P \neq NP$
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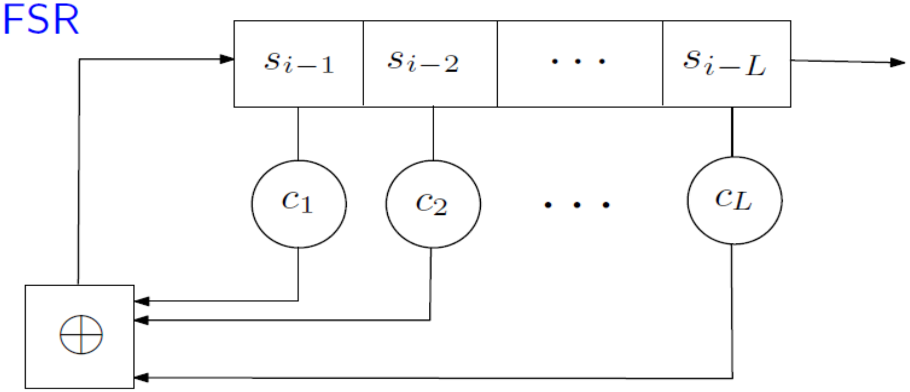
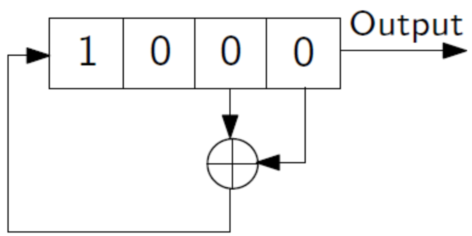
LFSR



Linear Feedback Shift Register (LFSR)

Example

$\mathbf{s} = (000100110101111)^{15}$



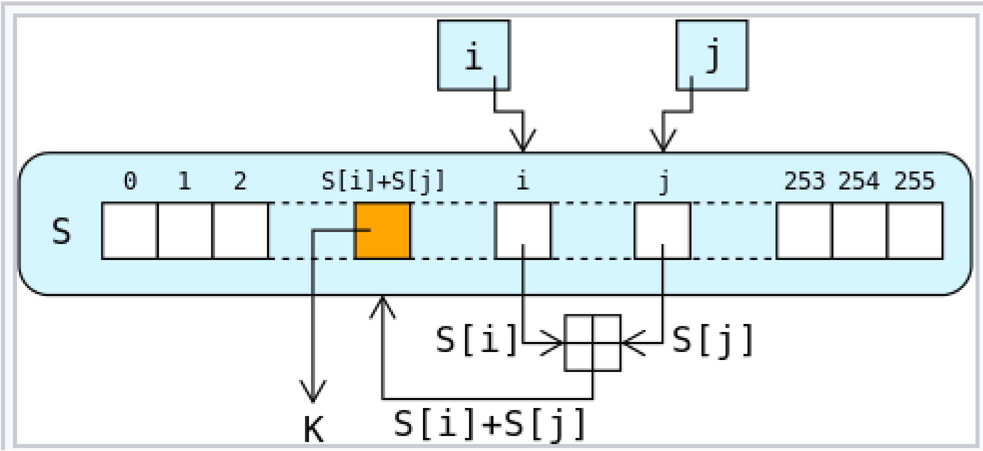
$\mathrm{LC}(\mathbf{s}) = 4$

$P_{\mathbf{s}}(x) = x^4 + x + 1$

Practical "PRGs"

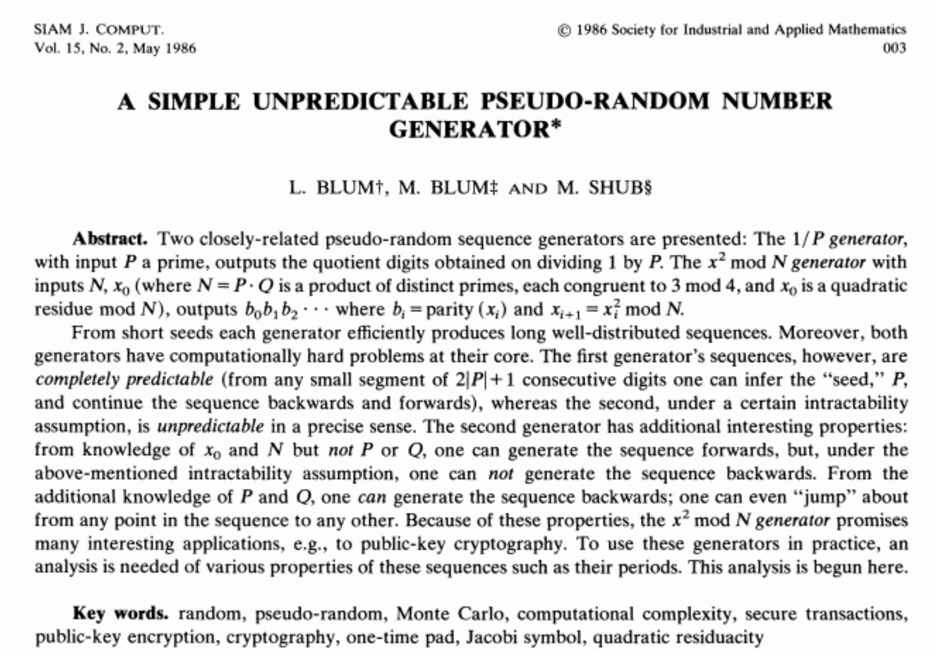
- RC4

```
i := 0
j := 0
while GeneratingOutput:
    i := (i + 1) mod 256
    j := (j + S[i]) mod 256
    swap values of S[i] and S[j]
    K := S[(S[i] + S[j]) mod 256]
    output K
endwhile
```



Blum-Blum-Shub

```
num_outputted = 0;
while num_outputted < m:
    X := X*X mod N
    num_outputted := num_outputted + 1
    output least-significant-bit(X)
```



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A SIMPLE UNPREDICTABLE PSEUDO-RANDOM NUMBER GENERATOR*

L. BLUM†, M. BLUM‡ AND M. SHUB§

Abstract. Two closely-related pseudo-random sequence generators are presented: The $1/P$ *generator*, with input P a prime, outputs the quotient digits obtained on dividing 1 by P. The $x^2 \bmod N$ *generator* with inputs N, x_0 (where $N = P \cdot Q$ is a product of distinct primes, each congruent to 3 mod 4, and x_0 is a quadratic residue mod N), outputs $b_0 b_1 b_2 \cdots$ where $b_i = \text{parity}(x_i)$ and $x_{i+1} = x_i^2 \bmod N$.

From short seeds each generator efficiently produces long well-distributed sequences. Moreover, both generators have computationally hard problems at their core. The first generator's sequences, however, are *completely predictable* (from any small segment of $2|P|+1$ consecutive digits one can infer the "seed," P, and continue the sequence backwards and forwards), whereas the second, under a certain intractability assumption, is *unpredictable* in a precise sense. The second generator has additional interesting properties: from knowledge of x_0 and N but *not* P or Q, one can generate the sequence forwards, but, under the above-mentioned intractability assumption, one can *not* generate the sequence backwards. From the additional knowledge of P and Q, one *can* generate the sequence backwards; one can even "jump" about from any point in the sequence to any other. Because of these properties, the $x^2 \bmod N$ *generator* promises many interesting applications, e.g., to public-key cryptography. To use these generators in practice, an analysis is needed of various properties of these sequences such as their periods. This analysis is begun here.

Key words. random, pseudo-random, Monte Carlo, computational complexity, secure transactions, public-key encryption, cryptography, one-time pad, Jacobi symbol, quadratic residuacity

Where things stand

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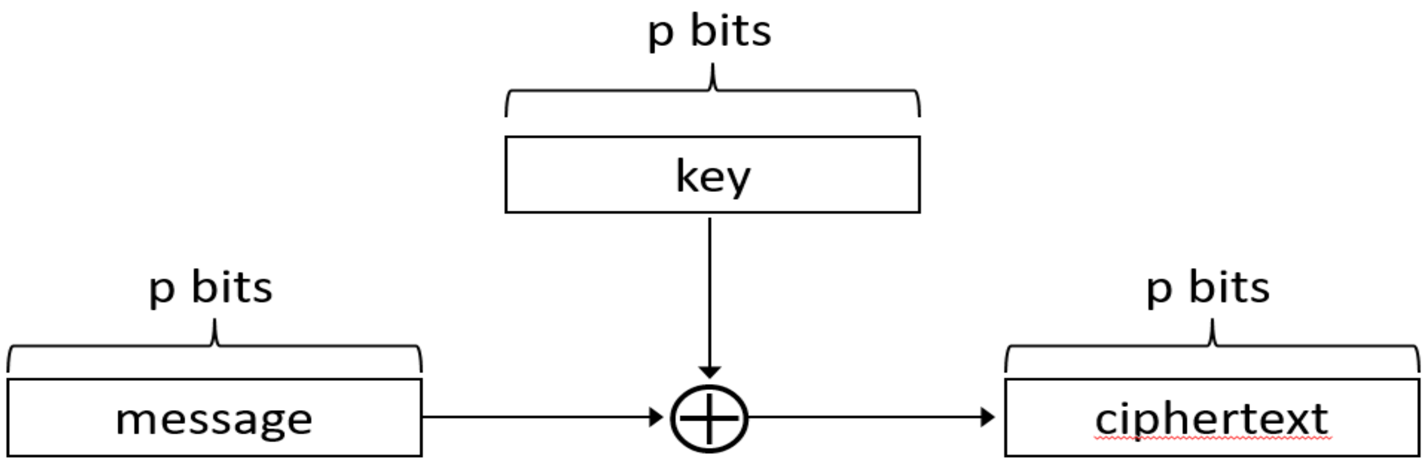
  - In particular, key must be as long as the message

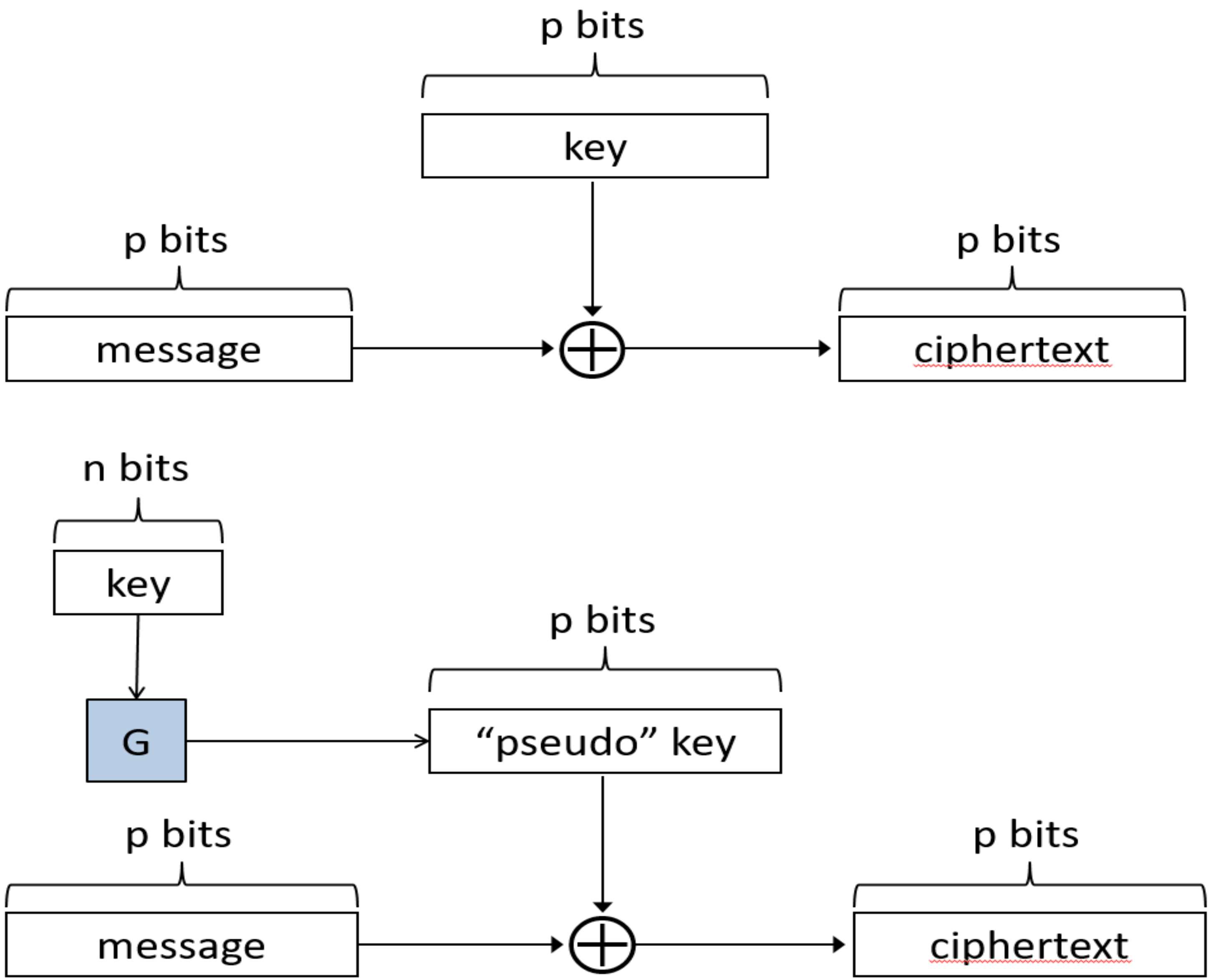
  We defined *computational security*, a relaxed notion of security

  $\mathcal{Q}$: Can we overcome prior limitations?

Recall: one-time pad



Recall: one-time pad



Pseudo one-time pad

- Let G be a deterministic, with $|G(k)| = p(|k|)$

  $Gen(1^n)$: output uniform n-bit key k

  – Security parameter $n \Rightarrow$ message space $\{0,1\}^{p(n)}$

  $Enc_k(m)$: output $G(k) \oplus m$

  $Dec_k(m)$: output $G(k) \oplus c$

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- Would like to be able to prove *computational security*
  – Based on the assumption that G is a PRG

Proof by reduction

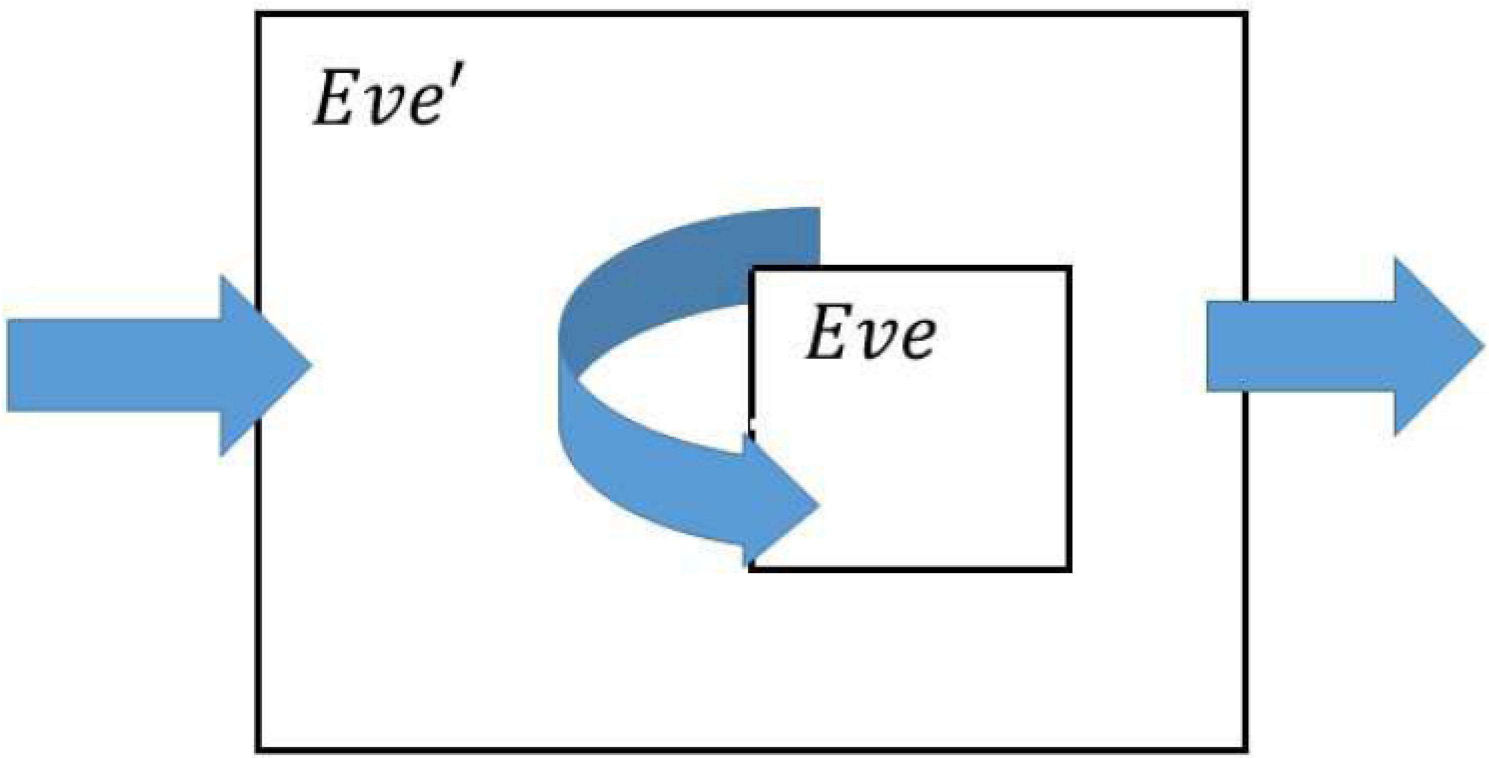

Figure 2.1: We show that the security of S' implies the security of S by transforming an adversary *Eve* breaking S into an adversary *Eve*′ breaking S'

Eve breaks $S \rightarrow$ Eve' breaks S'
S' is secure $\rightarrow S$ is secure

Proof by reduction

- 1. Assume that G is a *PRG*

  2. Assume toward a contradiction that there is an efficient attacker A who "breaks" the pseudo-OTP scheme

  3. Use A as a subroutine to build an efficient D that "breaks" *pseudorandomness* of G

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  Theorem 3.3 If G is a pseudorandom generator (PRG), then the pseudo one-time pad (pseudo-OTP) Π is *EAV-secure* (i.e., *computationally secure*)

Next Lecture

- Pseudorandom functions, block ciphers ...