Reference Answers for Assign 0304 of I2ML-s23

Q1 Show that the following sequents are not valid by finding a valuation (model) in which the truth values of the formulas to the left of *are 1s (Ts) and the truth value of* the formula to the right of *is 0 (F).* *(6 * 10pts = 60 pts)*

(a) $\neg p \lor (q \to p) \vdash \neg p \land q$

(b) $\neg r \to (p \lor q),\ r \land \neg q \vdash r \to q$

(c) $\neg p,\ p \lor q \vdash \neg q$

(d) $p \to (\neg q \lor r),\ \neg r \vdash \neg q \to \neg p$

(e) $p \to q \vdash p \lor q$

(f) $p \to (q \lor r) \vdash (p \to q) \land (p \to r)$

Reference Answers for Assign 0304 of I2ML-s23

Q1 ***Ref. Ans.*** (The model may not be unique for each sequent, the ref. ans. given may not include all models.)

(a) $\neg p \lor (q \rightarrow p) \vdash \neg p \land q$

m = { p --> 1, q --> 0 } or { p --> 0, q --> 0 } or { q --> 0 }

(b) $\neg r \rightarrow (p \lor q),\ r \land \neg q \vdash r \rightarrow q$

m = { p --> 1, q --> 0, r --> 1 } or { p --> 0, q --> 0, r --> 1 } or { q --> 0, r --> 1 }

(c) $\neg p,\ p \lor q \vdash \neg q$

m = { p --> 0, q --> 1 }

(d) $p \rightarrow (\neg q \lor r),\ \neg r \vdash \neg q \rightarrow \neg p$

m = { p --> 1, q --> 0, r --> 0 }

(e) $p \rightarrow q \vdash p \lor q$

m = { p --> 0, q --> 0 }

(f) $p \rightarrow (q \lor r) \vdash (p \rightarrow q) \land (p \rightarrow r)$

m = { p --> 1, q --> 1 , r --> 0 } or { p --> 1, q --> 0 , r --> 1 }

Q2 Prove the validity of the following sequents (study all of them first, and then choose 4 of the hardest sequents for you to prove and do not duplicate with formulas or sequents you have proved in the assignment 1 or 2) by formal proof rules and format learnt in Lecture Notes 05 & 06. (Pay attention that F1 $\vdash$ F2 is a shorthand of { F1 } $\vdash$ F2 in formal proof.)
(*4 * 10pts = 40pts*)

(a) $\phi_1 \wedge \neg\phi_2 \vdash \neg(\phi_1 \rightarrow \phi_2)$
(b) $\neg\phi_1 \wedge \neg\phi_2 \vdash \phi_1 \rightarrow \phi_2$
(c) $\neg\phi_1 \wedge \phi_2 \vdash \phi_1 \rightarrow \phi_2$
(d) $\phi_1 \wedge \phi_2 \vdash \phi_1 \rightarrow \phi_2$
(e) $\neg\phi_1 \wedge \phi_2 \vdash \neg(\phi_1 \wedge \phi_2)$
(f) $\neg\phi_1 \wedge \neg\phi_2 \vdash \neg(\phi_1 \wedge \phi_2)$
(g) $\phi_1 \wedge \neg\phi_2 \vdash \neg(\phi_1 \wedge \phi_2)$
(h) $\neg\phi_1 \wedge \neg\phi_2 \vdash \neg(\phi_1 \vee \phi_2)$
(i) $\phi_1 \wedge \phi_2 \vdash \phi_1 \vee \phi_2$
(j) $\neg\phi_1 \wedge \phi_2 \vdash \phi_1 \vee \phi_2$
(k) $\phi_1 \wedge \neg\phi_2 \vdash \phi_1 \vee \phi_2$.

(No Ref. Ans. for Q2 !)

Reference Answers for Assign 0304 of I2ML-s23

Q3 Use mathematical induction on *n to prove the following equivalence: (20 pts)*

$$((\phi_1 \land (\phi_2 \land (\cdots \land \phi_n) \dots) \to \psi) \equiv (\phi_1 \to (\phi_2 \to (\dots (\phi_n \to \psi) \dots)))).$$

Ref. Proof

The basic step, n = 1, the *lhf* (left-hand formula) and *rhf* are the same:
$(\phi_1 \to \psi)$.
In induction step, assume the equivalence is true when n is no more than n:

$$((\phi_1 \land (\phi_2 \land (\cdots \land \phi_n) \dots) \to \psi) \equiv (\phi_1 \to (\phi_2 \to (\dots (\phi_n \to \psi) \dots)))).$$

Now we try to prove the *equivalence when n is n+1:*

$$((\phi_1 \land (\phi_2 \land (\cdots \land \phi_{n+1}) \dots) \to \psi) \equiv (\phi_1 \to (\phi_2 \to (\dots (\phi_{n+1} \to \psi) \dots))))$$

Let $F_n = (\phi_1 \land (\phi_2 \land (\cdots \land \phi_n)))$, we have

$F_{lhf} = ((F_n \land \phi_{n+1}) \to \psi)$ *; by associative law of* $\land$

$= (F_n \to (\phi_{n+1} \to \psi))$ *; by induction assumption with n is 2*

$= F_{rhf}$ *; by induction assumption with n is n*

; in which $(\phi_{n+1} \to \psi)$ *views as one formula*

QED.

Reference Answers for Assign 0304 of I2ML-s23

Q4 Find a formula of propositional logic φ which contains only the atoms *p, q* and *r* and which is true only when *p* and *q* are false, or when $\neg q \land (p \lor r)$ is true. *(10pts)*

Ref. Ans. $\varphi = ((\neg p \land \neg q) \lor (\neg q \land (p \lor r)))$

Q5 Derive the following statements by formal proof learnt in Lecture Notes 05 & 06. (*2 * 15 pts = 30 pts*)

5a) $\emptyset \vdash (p \Rightarrow (q \lor r)) \lor (r \Rightarrow \neg p)$

5b) $\{r \lor (s \land \neg t), (r \lor s) \Rightarrow (u \lor \neg t)\} \vdash t \Rightarrow u$

(No Ref. Ans. for Q5 !)

Reference Answers for Assign 0304 of I2ML-s23

Q6 Prove the following equivalences: *(2 * 15 pts = 30 pts)*

6a) $p \Rightarrow q \equiv \neg p \vee q \equiv \neg(p \wedge \neg q) \equiv \neg q \Rightarrow \neg p$

6b) $(p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r) \equiv (p \Leftrightarrow q) \wedge (q \Leftrightarrow r)$

6a) ***Ref. Proof***

All the four formulas are evaluated to 0 (false) if and only if under the model m = { p --> 1, q --> 0 }; otherwise all the four evaluations will be 1 (true).
That means, the four formulas are equivalent. QED.

6b) ***Ref. Proof***

Both formulas are evaluated to 1 (true) if and only if under the one of the 2 models { p --> 1, q --> 1, r --> 1 } or { p --> 0, q --> 0, r --> 0 } ; otherwise, both evaluations will be 0 (false).
That means, the two formulas are equivalent. QED.

Reference Answers for Assign 0304 of I2ML-s23

Q7 Simplify *(10 pts)*

$$\underbrace{p \oplus \ldots \oplus p}_{n} \oplus \underbrace{\neg p \oplus \ldots \oplus \neg p}_{k} \equiv ?$$

Ref. Ans.

The left part xor of n ps will get a 0 when n is even or a p when n is odd;
The right part xor of k ~ps will get a 0 when k is even or a ~p when k is odd.

The simplified result of the given formula may have 4 cases:
0 ; when both n and k are even
~p ; when n is even and k is odd
p ; when n is odd and k is even
1 ; when both n and k are odd. QED.

By the way, we can express the result in a single java String expression:

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n%2==0 ? (k%2==0 ? "0" : "~p") : (k%2==0 ? "p" : "1")
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Reference Answers for Assign 0304 of I2ML-s23

Q8 For the formula: $\neg(\neg((s \Rightarrow \neg(p \Leftrightarrow q))) \oplus (\neg q \vee r))$

8a) Convert the above formula into NNF;

8b) Remove $\Rightarrow$, $\Leftrightarrow$, *and* $\oplus$ before turning the above into NNF, redo 8a).

(*2 * 10pts = 20pts*)

8a) Ref. Ans.

$$\text{Formula} = (\neg\neg(s \Rightarrow \neg(p \Leftrightarrow q)) \oplus (\neg q \vee r))$$

$$= ((s \Rightarrow (p \oplus q)) \oplus (\neg q \vee r)$$

This is a NNF with $\Rightarrow$ and $\oplus$

Reference Answers for Assign 0304 of I2ML-s23

8b) Ref. Ans.

$$\text{Formula} = ((s \Rightarrow (p \oplus q)) \oplus (\sim q \vee r))$$

$$= ((\sim s \vee (\sim p \wedge q) \vee (p \wedge \sim q)) \oplus (\sim q \vee r))$$

$$= (((\sim s \vee (\sim p \wedge q) \vee (p \wedge \sim q)) \vee (\sim q \vee r)) \wedge$$
$$(\sim(\sim s \vee (\sim p \wedge q) \vee (p \wedge \sim q)) \vee (\sim(\sim q \vee r))))$$

$$= ((\sim s \vee (\sim p \wedge q) \vee (p \wedge \sim q) \vee (\sim q \vee r)) \wedge$$
$$((s \wedge (p \vee \sim q) \wedge (\sim p \wedge q)) \vee (q \wedge \sim r)))$$

This is a NNF without $\Rightarrow$, $\Leftrightarrow$ and $\oplus$.