First Order Logic (Predicate Logic)

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Exercises 10 : Reading and More

Record your time spent (in 0.1 hours) with brief tasks and durations in your learning log by hand writing!

1) Read textB-ch02-2.1+2.2-basics-fol.pdf (in 2 weeks)
2) Read textI-ch02-2.1+2.2-basics-fol.pdf (in one week)

Formal Language

Examples:

1. (Digital sequence understood by computer)

   0010101010000010111101000

2. (Programme Language, eg. Java or C)

   ```
   s = 1; i = n; while (i > 0) { s *= a; i--; }
   ```

3. (Propositional Logic)

   $$(\neg((p \vee q) \rightarrow p))$$

4. (First-Order Logic)

   $$\forall \epsilon \exists \delta \forall x (|x - a| < \delta \rightarrow |f(x) - c| < \epsilon)$$

5. (Modal Logic)

   $$\neg(\Diamond p) \leftrightarrow \Box(\neg p)$$

Topic 10.1

First-Order Logic (FOL) Syntax

First-Order Logic (FOL)

First-order logic(FOL)
$=$
propositional logic $+$ quantifiers over individuals $+$ functions/predicates

"First" comes from this property

Example 10.1

Consider argument: Humans are mortal. Socrates is a human. Therefore, Socrates is mortal.

In symbolic form,
$\forall x.(H(x) \Rightarrow M(x)) \wedge H(s) \Rightarrow M(s)$

- $H(x) = x$ *is a human*
- $M(x) = x$ *is mortal*
- $s =$ *Socrates*

A note on FOL syntax

The FOL syntax may appear non-intuitive and cumbersome.

FOL requires getting used to it like many other concepts such as complex numbers.

Connectives and Variables

An FOL consists of three disjoint kinds of symbols

- variables
- logical connectives
- non-logical symbols : function and predicate symbols

Variables

We assume that there is a set Vars of variables, which is countably infinite in size.

- Since Vars is countable, we assume that variables are indexed.

$$\text{Vars} = \{x_1, x_2, \ldots, \}$$

- The variables are just names/symbols without any inherent meaning
- We may also sometimes use x, y, z to denote the variables

Now forget all the definitions of the propositional logic. We will redefine everything and the new definitions will subsume the PL definitions.

Logical Connectives

The following are a finite set of symbols that are called logical connectives.

formal name	symbol	read as	
true	$\top$	top	0-ary
false	$\bot$	bot	
negation	$\neg$	not	unary
conjunction	$\wedge$	and	binary
disjunction	$\vee$	or	
implication	$\Rightarrow$	implies	
exclusive or	$\oplus$	xor	
equivalence	$\Leftrightarrow$	iff	
equality	$=$	equals	binary predicate
existential quantifier	$\exists$	there is	quantifiers
universal quantifier	$\forall$	for each	
open parenthesis	(		punctuation
close parenthesis	)		
comma	,		

Non-Logical Symbols

FOL is a parameterized logic

The parameter is a signature $\mathbf{S} = (\mathbf{F}, \mathbf{R})$, where

- $\mathbf{F}$ is a set of function symbols and
- $\mathbf{R}$ is a set of predicate symbols (aka relational symbols).

Each symbol is associated with an arity ≥ 0.

We write $f/n \in \mathbf{F}$ and $P/k \in \mathbf{R}$ to explicitly state the arity

Example 10.2

We may have $\mathbf{F} = \{c/0, f/1, g/2\}$ *and* $\mathbf{R} = \{P/0, H/2, M/1\}$*.*

Example 10.3

We may have $\mathbf{F} = \{+/2, -/2\}$ *and* $\mathbf{R} = \{< /2\}$*.*

Commentary: We are familiar with predicates, which are the things that are either true or false. However, the functions are the truly novel concept.

Non-Logical Symbols (contd.)

F and **R** may either be finite or infinite.

Each **S** defines an FOL. We say, consider an FOL with signature $\mathbf{S} = (\mathbf{F}, \mathbf{R})$ ...

We may not mention **S** if from the context the signature is clear.

Example 10.4

In the propositional logic, $\mathbf{F} = \emptyset$ *and*

$$\mathbf{R} = \{p_1/0, p_2/0, ...\}.$$

Constants and Propositional Variable

There are special cases when the arity is zero.

$f/0 \in \mathbf{F}$ is called a constant.

$P/0 \in \mathbf{R}$ is called a propositional variable.

Building FOL Formulas

Let us use the ingredients to build the FOL formulas.

It will take a few steps to get there.

- terms
- atoms
- formulas

Syntax : Terms

Commentary: Terms are defined using grammar notation. If unfamiliar with the notation Please look
`https://en.wikipedia.org/wiki/Formal_grammar`

Definition 10.1

For signature $\mathbf{S} = (\mathbf{F}, \mathbf{R})$, **S**-*terms* $T_{\mathbf{S}}$ *are given by the following grammar:*

$$t ::= x \mid f(\underbrace{t, \dots, t}_{n}),$$

where $x \in \mathrm{Vars}$ *and* $f/n \in \mathbf{F}$.

Example 10.5

Consider $\mathbf{F} = \{c/0, f/1, g/2\}$. *Let* x_i*s be variables. The following are terms.*

- $f(x_1)$
- $g(f(c), g(x_2, x_1))$
- c
- x_1

You may be noticing some similarities between variables and constants

Infix Notation

We may write some functions and predicates in infix notation.

Example 10.6

we may write $+(a, b)$ *as* $a + b$ *and similarly* $< (a, b)$ *as* $a < b$*.*

Syntax: Atoms

Definition 10.2

S-*atoms* $A_{\mathbf{S}}$ *are given by the following grammar:*

$$a ::= P(\underbrace{t, \ldots, t}_{n}) \mid t = t \mid \bot \mid \top,$$

where $P/n \in \mathbf{R}$.

Thinking Exercise 10.1

Consider $\mathbf{F} = \{s/0\}$ *and* $\mathbf{R} = \{H/1, M/1\}$. *Which of the following are atom?*

- $H(x)$
- s
- $M(s)$
- $H(M(s))$

Commentary: Remember: you can nest terms but not atoms.

Equality within logic vs. equality outside logic

We have an equality $=$ within logic and the other when we use to talk about logic.

Both are distinct objects.

Some notations use same symbols for both and the others do not to avoid confusion.

Whatever is the case, we must be very clear about this.

Syntax: Formulas

Definition 10.3

S-*formulas* $\mathsf{P}_{\mathbf{S}}$ *are given by the following grammar:*

$$F ::= a \mid \neg F \mid (F \wedge F) \mid (F \vee F) \mid (F \Rightarrow F) \mid (F \Leftrightarrow F) \mid (F \oplus F) \mid \forall x.(F) \mid \exists x.(F)$$

where $x \in \mathsf{Vars}$.

Example 10.7

Consider $\mathbf{F} = \{s/0\}$ *and* $\mathbf{R} = \{H/1, M/1\}$

The following is a $(\mathbf{F}, \mathbf{R})$*-formula:*

$$\forall x.(H(x) \Rightarrow M(x)) \wedge H(s) \Rightarrow M(s)$$

Commentary: Notice we have dropped some parentheses. We will not discuss the minimal parentheses issue at length.

Unique Parsing

For FOL we will ignore the issue of unique parsing,

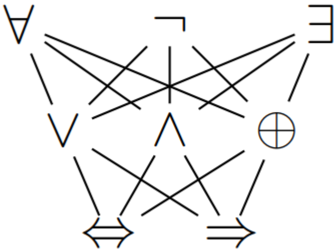
and assume

all the necessary precedence and associativity orders are defined

for ensuring human readability and unique parsing.

Precedence order

We will use the following precedence order in writing the FOL formulas



Example 10.8

The following are the interpretation of the formulas after dropping parenthesis

- $\forall x.H(x) \Rightarrow M(x) = \forall x.(H(x)) \Rightarrow M(x)$
- $\exists z \forall x.\exists y.G(x,y,z) = \exists z.(\forall x.(\exists y.G(x,y,z)))$

Topic 10.2

FOL - Semantics

Semantics : Models

Definition 10.4

For signature $\mathbf{S} = (\mathbf{F}, \mathbf{R})$*, a* **S***-model* m *is a*

$$(D_m; \{f_m : D_m^n \to D_m | f/n \in \mathbf{F}\}, \{P_m \subseteq D_m^n | P/n \in \mathbf{R}\}),$$

where D_m *is a nonempty set. Let* **S***-Mods denotes the set of all* **S***-models.*

Some terminology

- D_m is called domain of m.
- f_m assigns meaning to f under model m.
- Similarly, P_m assigns meaning to P under model m.

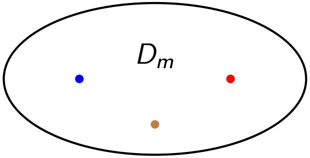
Commentary: Models are also known as interpretations/structures.

Commentary: Ideally, we should write $H_m = \{({\color{blue}\bullet}), ({\color{brown}\bullet})\}$ but we conventionally drop (..) for singleton tuples.

Example: Model

Example 10.9

Consider $\mathbf{S} = (\{c/0, f/1, g/2\}, \{H/1, M/2\})$.
Let us suppose our model m has domain $D_m = \{{\color{red}\bullet}, {\color{blue}\bullet}, {\color{brown}\bullet}\}$.



We need to assign value to each function.

- $c_m = {\color{blue}\bullet}$
- $f_m = \{{\color{blue}\bullet} \mapsto {\color{brown}\bullet}, {\color{brown}\bullet} \mapsto {\color{red}\bullet}, {\color{red}\bullet} \mapsto {\color{red}\bullet}\}$
- $g_m = \{({\color{blue}\bullet}, {\color{blue}\bullet}) \mapsto {\color{brown}\bullet}, ({\color{blue}\bullet}, {\color{brown}\bullet}) \mapsto {\color{blue}\bullet}, ({\color{blue}\bullet}, {\color{red}\bullet}) \mapsto {\color{red}\bullet},$
  $({\color{red}\bullet}, {\color{blue}\bullet}) \mapsto {\color{brown}\bullet}, ({\color{red}\bullet}, {\color{brown}\bullet}) \mapsto {\color{brown}\bullet}, ({\color{red}\bullet}, {\color{red}\bullet}) \mapsto {\color{brown}\bullet},$
  $({\color{brown}\bullet}, {\color{blue}\bullet}) \mapsto {\color{red}\bullet}, ({\color{brown}\bullet}, {\color{brown}\bullet}) \mapsto {\color{brown}\bullet}, ({\color{brown}\bullet}, {\color{red}\bullet}) \mapsto {\color{blue}\bullet}\}$

We also need to assign values to each predicate.

- $H_m = \{{\color{blue}\bullet}, {\color{brown}\bullet}\}$ $\qquad M_m = \{({\color{blue}\bullet}, {\color{blue}\bullet}), ({\color{brown}\bullet}, {\color{red}\bullet})\}$

Thinking Exercise 10.2

a. How many models are there for the signature with the above domain?
b. Suppose $P/0 \in \mathbf{R}$, *give a value to* P_m.

Semantics: Assignments

Recall, We also have variables. Who will assign to the variables?

Definition 10.5

An assignment *is a map* $\nu : \text{Vars} \rightarrow D_m$

Thinking Example 10.10

In our running example the domain is $\mathbb{N}$*. We may have the following assignment.*

$$\nu = \{x \mapsto 2, y \mapsto 3, ...\}$$

Commentary: ν is a function. It needs to map each variable. However, we only care about the mappings for variables that are relevant to our context. Therefore, in our slides we write mapping for only those variables that are important. For others, we assume there is some mapping.

Semantics: term value

Definition 10.6

For a model m and assignment ν, we define $m^\nu : T_{\mathbf{S}} \to D_m$ as follows.

$$\begin{aligned} m^\nu(x) &\triangleq \nu(x) && x \in \text{Vars} \\ m^\nu(f(t_1,\ldots,t_n)) &\triangleq f_m(m^\nu(t_1),\ldots,m^\nu(t_n)) \end{aligned}$$

Example 10.11

Consider $\mathbf{S} = (\{s/1, +/2\}, \{\})$ *and term* $s(x) + y$

Consider model $m = (\mathbb{N}; succ, +^{\mathbb{N}})$ *and assignment* $\nu = \{x \mapsto 3, y \mapsto 2\}$

$$m^\nu(s(x) + y) = m^\nu(s(x)) +^{\mathbb{N}} m^\nu(y) = succ(m^\nu(x)) +^{\mathbb{N}} 2 = succ(3) +^{\mathbb{N}} 2 = 6$$

Semantics: satisfaction relation

Definition 10.7

We define the satisfaction relation $\models$ *among models, assignments, and formulas as follows*

- $m, \nu \models \top$
- $m, \nu \models P(t_1, \ldots, t_n)$ *if* $(m^{\nu}(t_1), \ldots, m^{\nu}(t_n)) \in P_m$
- $m, \nu \models t_1 = t_2$ *if* $m^{\nu}(t_1) = m^{\nu}(t_2)$
- $m, \nu \models \neg F$ *if* $m, \nu \not\models F$
- $m, \nu \models F_1 \vee F_2$ *if* $m, \nu \models F_1$ *or* $m, \nu \models F_2$

  skipping other propositional connectives
- $m, \nu \models \exists x.(F)$ *if there is* $u \in D_m : m, \nu[x \mapsto u] \models F$
- $m, \nu \models \forall x.(F)$ *if for each* $u \in D_m : m, \nu[x \mapsto u] \models F$

Example: Satisfiability

Example 10.12

Consider $\mathbf{S} = (\{s/1, +/2\}, \{\})$ *and formula* $\exists z.s(x) + y = s(z)$

Consider model $m = (\mathbb{N}; succ, +^{\mathbb{N}})$ *and assignment* $\nu = \{x \mapsto 3, y \mapsto 2\}$

We have seen $m^{\nu}(s(x) + y) = 6$.

$m^{\nu[z \mapsto 5]}(s(x) + y) = m^{\nu}(s(x) + y) = 6.$ *//Since z does not occur in the term*

$m^{\nu[z \mapsto 5]}(s(z)) = 6$

Therefore, $m, \nu[z \mapsto 5] \models s(x) + y = s(z)$.

$m, \nu \models \exists z.s(x) + y = s(z)$.

Satisfiable, True, Valid, and Unsatisfiable

We say

- F is *satisfiable* if there are m and ν such that $m, \nu \models F$
- Otherwise, F is called unsatisfiable (written $\not\models F$)
- F is *true* in m ($m \models F$) if for all ν we have $m, \nu \models F$
- F is *valid* ($\models F$) if for all ν and m we have $m, \nu \models F$

Exercise: Model

Consider $\mathbf{S} = (\{c/0, f/1\}, \{H/1, M/2\})$. Let us suppose model m has $D_m = \{\textcolor{red}{\bullet}, \textcolor{blue}{\bullet}, \textcolor{brown}{\bullet}\}$ and the values of the symbols in m are

- $c_m = \textcolor{blue}{\bullet}$
- $f_m = \{\textcolor{blue}{\bullet} \mapsto \textcolor{brown}{\bullet}, \textcolor{brown}{\bullet} \mapsto \textcolor{red}{\bullet}, \textcolor{red}{\bullet} \mapsto \textcolor{red}{\bullet}\}$
- $H_m = \{\textcolor{blue}{\bullet}, \textcolor{brown}{\bullet}\}$
- $M_m = \{(\textcolor{blue}{\bullet}, \textcolor{blue}{\bullet}), (\textcolor{brown}{\bullet}, \textcolor{red}{\bullet})\}$

Thinking Exercise 10.3

Which of the following hold?

- $m, \{x \mapsto \textcolor{red}{\bullet}\} \models M(f(x), x)$
- $m, \{\} \models \exists x. H(x)$
- $m, \{\} \models \exists x. H(f(x))$
- $m, \{x \mapsto \textcolor{red}{\bullet}\} \models H(x)$
- $m, \{\} \models \forall x. H(x)$
- $m, \{\} \models H(c)$

Extended satisfiability (repeat from propositional logic)

We extend the usage of $\models$. Let Σ be a (possibly infinite) set of formulas.

Definition 10.8

$m, \nu \models \Sigma$ if $m, \nu \models F$ for each $F \in \Sigma$.

Definition 10.9

$\Sigma \models F$ if for each model m and assignment ν if $m, \nu \models \Sigma$ then $m, \nu \models F$.
$\Sigma \models F$ is read Σ implies F. If $\{G\} \models F$ then we may write $G \models F$.

Definition 10.10

Let $F \equiv G$ if $G \models F$ and $F \models G$.

Definition 10.11

Formulas F and G are equisatisfiable if

$$F \text{ is sat} \quad \text{iff} \quad G \text{ is sat.}$$

Commentary: These definitions are identical to the propositional case.

Topic 10.3

Problems

FOL to PL

Thinking Exercise 10.4

Give the restrictions on FOL such that it becomes the propositional logic. Give an example of FOL model of a non-trivial propositional formula.

Valid Formulas

Thinking Exercise 10.5

Prove/Disprove the following formulas are valid.

- $\forall x.P(x) \Rightarrow P(c)$
- $\forall x.(P(x) \Rightarrow P(c))$
- $\exists x.(P(x) \Rightarrow \forall x.P(x))$
- $\exists y \forall x.R(x,y) \Rightarrow \forall x \exists y.R(x,y)$
- $\forall x \exists y.R(x,y) \Rightarrow \exists y \forall x.R(x,y)$

Properties of FOL

Thinking Exercise 10.6

Show the validity of the following formulas.

1. $\neg\forall x.\ P(x) \Leftrightarrow \exists x.\ \neg P(x)$
2. $\neg\exists x.\ P(x) \Leftrightarrow \forall x.\ \neg P(x)$
3. $(\forall x.\ (P(x) \wedge Q(x))) \Leftrightarrow \forall x.\ P(x) \wedge \forall x.\ Q(x)$
4. $(\exists x.\ (P(x) \vee Q(x))) \Leftrightarrow \exists x.\ P(x) \vee \exists x.\ Q(x)$

Thinking Exercise 10.7

Show $\forall$ does not distribute over $\vee$.

Show $\exists$ does not distribute over $\wedge$.

Example: Non-Standard Models

Consider $\mathbf{S} = (\{0/0, s/1, +/2\}, \{\})$ and formula $\exists z.s(x) + y = s(z)$

Unexpected model: Let $m = (\{a, b\}^*; \epsilon, append\ a, concat)$.

- The domain of m is the set of all strings over alphabet $\{a, b\}$.
- $append\ a$: appends a in the input and
- $concat$: joins two strings.

Let $\nu = \{x \mapsto ab, y \mapsto ba\}$.
Since $m, \nu[z \mapsto abab] \models s(x) + y = s(z)$, we have $m, \nu \models \exists z.s(x) + y = s(z)$.

Thinking Exercise 10.8

- *Show* $m, \nu[y \mapsto bb] \not\models \exists z.s(x) + y = s(z)$
- *Give an assignment* ν *s.t.* $m, \nu \models x \neq 0 \Rightarrow \exists y.\ x = s(y)$.
  Show $m \not\models \forall x.\ (x \neq 0 \Rightarrow \exists y.\ x = s(y))$.

Find Models

Thinking Exercise 10.9

For each *of the following formula give a model that satisfies the formula. If there is no model that satisfies a formula, then report that the formula is unsatisfiable.*

1. $\forall x.\exists y R(x, y) \land \neg\exists x.\forall y R(x, y)$
2. $\neg\forall x.\exists y R(x, y) \land \exists x.\forall y R(x, y)$
3. $\neg\forall x.\exists y R(x, y) \land \neg\exists x.\forall y R(x, y)$
4. $\forall x.\exists y R(x, y) \land \exists x.\forall y R(x, y)$

Similar Quantifiers

Thinking Exercise 10.10

Show using FOL fol semantics.

- $\exists x. \exists x.\ F \equiv \exists x.\ F$
- $\exists x. \exists y.\ F \equiv \exists y. \exists x.\ F$
- $\forall x. \forall x.\ F \equiv \forall x.\ F$
- $\forall x. \forall y.\ F \equiv \forall y. \forall x.\ F$

Exercise : compact notation for terms

Since we know arity of each symbol, we need not write "," "(", and ")" to write a term unambiguously.

Example 10.13

$f(g(a,b),h(x),c)$ can be written as $fgabhxc$.

Thinking Exercise 10.11

Consider $\mathbf{F} = \{f/3, g/2, h/1, c/0\}$ and $x, y \in \text{Vars}$.
Insert parentheses at appropriate places in the following if they are valid term.

- $hc =$
- $gxc =$
- $fhxhyhc =$
- $fx =$

Thinking Exercise 10.12

Give an algorithm to insert the parentheses

Commentary: We will not use the compact notation in the course. It makes the formulas very difficult to read.

Exercise: DeBruijn index of quantified variables

DeBruijn index is a method for representing formulas without naming the quantified variables.

Definition 10.12

Each DeBruijn index is a natural number that represents an occurrence of a variable in a formula, and denotes the number of quantifiers that are in scope between that occurrence and its corresponding quantifier.

Example 10.14

We can write $\forall x.H(x)$ as $\forall.H(1)$. 1 is indicating the occurrence of a quantified variable that is bounded by the closest quantifier. More examples.

- $\exists y \forall x.M(x, y) = \exists\forall.M(1, 2)$
- $\exists y \forall x.M(y, x) = \exists\forall.M(2, 1)$
- $\forall x.(H(x) \Rightarrow \exists y.M(x, y)) = \forall.(H(1) \Rightarrow \exists.M(2, 1))$

Thinking Exercise 10.13

Give an algorithm that translates FOL formulas into DeBurjin indexed formulas.

Drinker Paradox

Thinking Exercise
10.14 *Prove*

There is someone x such that if x drinks, then everyone drinks.

Let $D(x) \triangleq x$ drinks. Formally

$$\exists x.(D(x) \Rightarrow \forall x.\ D(x))$$

https://en.wikipedia.org/wiki/Drinker_paradox

Exercise: Satisfaction Relation

Exercise 10.15

Consider $\mathbf{S} = (\{\cup/2\}, \{\in /2\})$ *and formula* $F = \exists x.\forall y.\ \neg y \in x$ *(what does it say to you!)*

Consider $\mathbf{S}$*-model* $m = (\mathbb{N}; \cup_m = max, \in_m = \{(i,j) | i < j\})$ *and* $\nu = \{x \mapsto 2, y \mapsto 3\}$.

$m, \nu \models F$?

Exercise: Implication

Thinking Exercise 10.16

Let us suppose the following formula is valid and Σ does not refer to c.

$$\Sigma \Rightarrow H(f(c)) \land \neg H(f(a))$$

Prove that Σ is unsatisfiable.

Topic 10.4

Extra Slides: Some Properties of Models

Homomorphisms of Models

Definition 10.13

Consider $\mathbf{S} = (\mathbf{F}, \mathbf{R})$. *Let* m *and* m' *be* $\mathbf{S}$*-models.*
A function $h : D_m \to D_{m'}$ *is a* *homomorphism* *of* m *into* m' *if the following holds.*

- *for each* $f/n \in \mathbf{F}$, *for each* $(d_1, .., d_n) \in D_m^n$

$$h(f_m(d_1, .., d_n)) = f_{m'}(h(d_1), .., h(d_n))$$

- *for each* $P/n \in \mathbf{R}$, *for each* $(d_1, .., d_n) \in D_m^n$

$$(d_1, .., d_n) \in P_m \quad \textit{iff} \quad (h(d_1), .., h(d_n)) \in P_{m'}$$

Definition 10.14

A homomorphism h *of* m *into* m' *is called* *isomorphism* *if* h *is one-to-one.*
m *and* m' *are called* *isomorphic* *if an* h *exists that is also onto.*

Example : Homomorphism

Example 10.15

Consider $\mathbf{S} = (\{+/2\}, \{\})$.

Consider $m = (\mathbb{N}, +^{\mathbb{N}})$ *and* $m = (\mathcal{B}, \oplus^{\mathcal{B}})$,

$h(n) = n \ mod \ 2$ *is a homomorphism of* m *into* m'.

Homomorphism theorem for terms and quantifier-free formulas without =

Theorem 10.1

Let h be a homomorphism of m into m'. Let ν be an assignment.

1. *For each term t, $h(m^{\nu}(t)) = m'^{(\nu \circ h)}(t)$*
2. *If formula F is quantifier-free and has no symbol "="*

$$m^{\nu} \models F \quad \text{iff} \quad m'^{(\nu \circ h)} \models F$$

Proof.

Simple structural induction. $\square$

Thinking Exercise 10.17

For a quantifier-free formula F that may have symbol "=", show

$$\text{if} \quad m^{\nu} \models F \quad \text{then} \quad m'^{(\nu \circ h)} \models F$$

Why the reverse direction does not work?

Homomorphism theorem with $=$

Theorem 10.2

Let h be a homomorphism of m into m'. Let ν be an assignment. If h is isomorphism then the reverse implication also holds for formulas with "$=$".

Proof.

Let us suppose $m'^{(\nu\circ h)} \models s = t$.
Therefore, $m'^{(\nu\circ h)}(s) = m'^{(\nu\circ h)}(t)$.
Therefore, $h(m^{\nu}(s)) = h(m^{\nu}(t))$.
Due to the one-to-one condition of h, $m^{\nu}(s) = m^{\nu}(t)$.
Therefore, $m^{\nu} \models s = t$. $\square$

Exercise 10.18

For a formula F (with quantifiers) without symbol "$=$", show

$$\text{if} \quad m'^{(\nu\circ h)} \models F \quad \text{then} \quad m^{\nu} \models F.$$

Why the reverse direction does not work?

Commentary: Note that that implication direction is switched from the previous exercise.

Homomorphism theorem with quantifiers

Theorem 10.3

Let h be a isomorphism of m into m' and ν be an assignment.
If h is also onto, the reverse direction also holds for the quantified formulas.

Proof.

Let us assume, $m^{\nu} \models \forall x.F$.
Choose $d' \in D_{m'}$.
Since h is onto, there is a d such that $d = h(d')$.
Therefore, $m^{\nu[x \mapsto d]} \models F$.
Therefore, $m'^{\nu[x \mapsto d']} \models F$.
Therefore, $m'^{(\nu \circ h)} \models \forall x.\ F$. $\square$

Theorem 10.4

If m and m' are isomorphic then for all sentences F, $m \models F \quad$ iff $\quad m' \models F$.

Commentary: The reverse direction of the above theorem is not true.

End of Lecture 10