The proposition can then be proved by induction on n in a similar manner to Proposition 3.14. We leave the details as Exercise 3.15. $\square$

It follows from the previous propositions that any formula is provably equivalent to a formula in which the quantifiers preceed all other fixed symbols. Informally, the quantifiers can be "pulled out in front" of any formula. We make this idea precise and prove it in the following section.

3.2 Normal forms

One of our goals in this chapter is to develop resolution for first-order logic. Recall that, in propositional logic, we needed to have the formulas in CNF before we could proceed with resolution. Likewise, in first-order logic the formulas will need to be in a nice form. In this section, we define what we mean by "nice."

3.2.1 Conjunctive prenex normal form.

Definition 3.16 A formula φ is in *prenex normal form* (PNF) if it has the form $Q_1x_1 \cdots Q_nx_n\psi$ where each Q_i is a quantifier (either $\exists$ or $\forall$) and ψ is a quantifier-free first-order formula. Moreover, if ψ is a conjunction of disjunctions of literals (atomic or negated atomic formulas), then φ is in *conjunctive prenex normal form.*

So a formula is in prenex normal form if all of its quantifiers are in front.

Example 3.17 $\forall y\exists x(f(x) = y)$ is in PNF, and $\neg\forall x\exists yP(x,y,z)$ and $\exists x\forall y \neg P(x,y,z) \wedge \forall x\exists yQ(x,y,z)$ are not.

Theorem 3.18 For any formula of first-order logic, there exists an equivalent formula in conjunctive prenex normal form.

Proof Let φ be an arbitrary formula. First we show that there exists an equivalent formula φ' in prenex normal form. We prove this by induction on the complexity of φ.

If φ is atomic, then φ is already in PNF, so we can just let φ' be φ.

Suppose ψ and θ are formulas and there exist ψ' and θ' in PNF such that $\psi \equiv \psi'$ and $\theta \equiv \theta'$. Clearly, if $\varphi \equiv \psi$ then we can let φ' be ψ'. To complete the induction step, we must consider three cases corresponding to $\neg$, $\wedge$, and $\exists$.

First, suppose φ is the formula $\neg\psi$. Then $\varphi \equiv \neg\psi'$. Since ψ' is in PNF, ψ' has the form $Q_1x_1 \cdots Q_mx_m\psi_0$ for some quantifier-free formula ψ_0 and quantifiers $Q_1, \ldots, Q_m$. So $\varphi \equiv \neg Q_1x_1 \cdots Q_mx_m\psi_0$. By Proposition 3.15, this is equivalent to $\overline{Q}_1x_1 \cdots \overline{Q}_mx_m\neg\psi_0$ where $\{Q_i, \overline{Q}_i\} = \{\exists, \forall\}$. This formula is in PNF, and so it may serve as φ'.

Next, suppose φ is the formula $\psi \wedge \theta$. Then $\varphi \equiv \psi' \wedge \theta'$. Since ψ' and θ' are in PNF,

$$\psi' \text{ is } Q_1x_1 \cdots Q_mx_m\psi_0(x_1, \ldots, x_m), \text{ and}$$
$$\theta' \text{ is } q_1x_1 \cdots q_nx_n\theta_0(x_1, \ldots, x_n)$$

for some quantifiers Q_i and q_i and some quantifier-free formulas ψ_0 and θ_0. Let $y_1, \ldots, y_m$ and $z_1, \ldots, z_n$ be new variables (that is, variables not occurring in ψ' or θ'). Then by Corollaries 3.8 and 3.9,

$$\psi' \equiv Q_1y_1 \cdots Q_my_m\psi_0(y_1, \ldots, y_m),$$
$$\theta' \equiv q_1z_1 \cdots q_nz_n\theta_0(z_1, \ldots, z_n), \text{ and so}$$
$$\varphi \equiv Q_1y_1 \cdots Q_my_m\psi_0(y_1, \ldots, y_m) \wedge q_1z_1 \cdots q_nz_n\theta_0(z_1, \ldots, z_n).$$

Applying Proposition 3.14 twice,

$$\varphi \equiv Q_1y_1 \cdots Q_my_mq_1z_1 \cdots q_nz_n(\psi_0(y_1, \ldots, y_m) \wedge \theta_0(z_1, \ldots, z_n))$$

which is in PNF. Let φ' be this formula.

Finally, suppose φ is the formula $\exists x\psi$. Then $\varphi \equiv \exists x_0\psi'$ for some variable x_0. Since ψ' is in PNF, $\exists x_0\psi'$ is in PNF. So in this case, we can let φ' be $\exists x_0\psi'$.

Given an arbitrary formula φ we have shown that there exists an equivalent formula φ' in prenex normal form. Let $Q_1x_1 \cdots Q_nx_n\varphi_0$ be the formula φ'. Each Q_i denotes a quantifier and φ_0 is a quantifier-free formula. We want to show that φ is equivalent to a formula in conjunctive prenex normal form. It remains to be shown that φ_0 is equivalent to a formula that is a conjunction of disjunctions. This can be done by induction on the complexity of φ_0. Since it is quantifier-free, we do not have to consider the part of the induction step corresponding to $\exists$. Therefore, the proof is identical to the proof of Theorem 1.57 where it was shown that every formula of propositional logic is equivalent to a formula in CNF. □

Example 3.19 Let φ be the formula $\neg(\forall x\exists yP(x, y, z) \vee \exists x\forall y\neg Q(x, y, z))$ having free variable z. By the previous theorem, there exists a formula φ' in PNF that is equivalent to φ. Moreover, the proof of the theorem indicates a method for finding such φ'. First, noting that φ has the form $\neg\psi$, we distribute the negation to obtain

$$\varphi \equiv \exists x\forall y\neg P(x, y, z) \wedge \forall x\exists yQ(x, y, z).$$

So φ is equivalent to a formula of the form $\psi \wedge \theta$. By renaming variables, we get

$$\varphi \equiv \exists x\forall y\neg P(x, y, z) \wedge \forall u\exists vQ(u, v, z).$$

By applying Proposition 3.14 twice,

$$\varphi \equiv \exists x\forall y\forall u\exists v(\neg P(x, y, z) \wedge Q(u, v, z))$$

which is in PNF. Moreover, this formula is in conjunctive PNF.

Our goal is to find a method for determining whether a given formula is satisfiable or not. By Theorem 3.18, it suffices to have a method that works for formulas in conjunctive prenex normal form (although, as we shall see in later chapters, no method "works" entirely). Next we show that we can simplify our formulas further. We show that we need only consider formulas that are *universal*: formulas in PNF in which the existential quantifier $\exists$ does not occur.

3.2.2 Skolem normal form.

Definition 3.20 A formula is in Skolem normal form (SNF), if it is universal and in conjunctive prenex normal form.

Given any formula φ of first-order logic we define a formula φ^S that is in SNF. We prove in Theorem 3.22 that φ is satisfiable if and only if φ^S is satisfiable. The formula φ^S is called a *Skolemization* of φ. The following is a step-by-step procedure for finding φ^S.

- First we find a formula φ' in conjunctive prenex normal form such that $\varphi' \equiv \varphi$. So

$$\varphi' \text{ is } Q_1 x_1 \cdots Q_m x_m \varphi_0(x_1, \ldots, x_m)$$

  for some quantifier-free formula φ_0 and quantifiers $Q_1, Q_2, \ldots, Q_m$.
- If each Q_i is $\forall$, then φ' is a universal formula. In this case let φ^S be φ'.
- Otherwise, φ' has existential quantifiers. In this case we define a formula $s(\varphi')$ that has fewer existential quantifiers than φ'. (So if φ' has just one existential quantifier, then $s(\varphi')$ is universal.) Let i be least such that Q_i is $\exists$.

  If $i = 1$, then φ' is $\exists x_1 Q_2 x_2 \cdots Q_m x_m \varphi_0(x_1, \ldots, x_m)$.

  Let $s(\varphi')$ be $Q_2 x_2 \cdots Q_m x_m \varphi_0(c, x_2, \ldots, x_m)$ where c is a constant symbol that does not occur in φ'.

  If $i > 1$, then φ' is $\forall x_1 \cdots \forall x_{i-1} \exists x_i Q_{i+1} x_{i+1} \cdots Q_m x_m \varphi_0(x_1, \ldots, x_m)$. Let $s(\varphi')$ be the formula

$$\forall x_1 \cdots \forall x_{i-1} Q_{i+1} x_{i+1} \cdots Q_m x_m$$
$$\varphi_0(x_1, \ldots, x_{i-1}, f(x_1, \ldots, x_{i-1}), x_{i+1}, \ldots, x_m),$$

  where f is an $(i-1)$-ary function symbol that does not occur in φ'.

  So if the first quantifier in φ' is $\exists$, we replace x_1 with a new constant. And if the ith quantifier in φ' is $\exists$ and all previous quantifiers are $\forall$, replace x_i with $f(x_1, \ldots, x_{i-1})$ where f is a new function symbol.

- Since $s(\varphi')$ has fewer existential quantifiers than φ', by repeating this process, we will eventually obtain the required universal formula φ^S. That is, φ^S is $s^n(\varphi') = s(s(s \cdots s(\varphi')))$ for some n.

Example 3.21 Suppose φ is the formula $\neg(\forall x \exists y P(x, y, z) \vee \exists x \forall y \neg Q(x, y, z))$. First, we find a formula φ' in conjunctive prenex normal form that is equivalent to φ. In Example 3.19 it was shown that φ is equivalent to

$$\exists x \forall y \forall u \exists v(\neg P(x, y, z) \wedge Q(u, v, z)).$$

Let φ' be this formula.

Next we find $s(\varphi')$ as defined above. Then we find $s(s(\varphi'))$ and $s(s(s(\varphi')))$, and so forth, until we get a formula in SNF. In this example, since φ' has only two existential quantifiers, we will stop at $s(s(\varphi'))$.

We have $s(\varphi')$ is $\forall y \forall u \exists v(\neg P(c, y, z) \wedge Q(u, v, z))$, and
$s(s(\varphi'))$ is $\forall y \forall u(\neg P(c, y, z) \wedge Q(u, f(y, u), z))$

which is in SNF. So we have successfully *Skolemized* the given formula φ and obtained the formula $\forall y \forall u(\neg P(c, y, z) \wedge Q(u, f(y, u), z))$. This is the formula denoted by φ^S.

Theorem 3.22 Let φ be a formula of first-order logic and let φ^S be the Skolemization of φ. Then φ is satisfiable if and only if φ^S is satisfiable.

Proof By Theorem 3.18, we may assume that φ is in conjunctive prenex normal form. By induction, it suffices to show that φ is satisfiable if and only if $s(\varphi)$ is satisfiable. There are two possibilities for $s(\varphi)$.

Case 1: If φ' has the form $\exists x_1 Q_2 x_2 \cdots Q_m x_m \varphi_0(x_1, \ldots, x_m)$, then $s(\varphi')$ is $Q_2 x_2 \cdots Q_m x_m \varphi_0(c, x_2, \ldots, x_m)$ for some constant c. Let $\psi(x_1)$ be the formula

$$Q_2 x_2 \cdots Q_m x_m \varphi_0(x_1, \ldots, x_m)$$

so that φ' is $\exists x_1 \psi(x_1)$ and $s(\varphi')$ is $\psi(c)$. By the semantics for $\exists$,

$$M \models \exists x_1 \psi(x_1) \text{ if and only if } M_C \models \psi(c),$$

where M_C is an expansion of M by constants one of which is c . It follows that $\exists x_1 \psi(x_1)$ is satisfiable if and only if $\psi(c)$ is satisfiable.

Case 2: If φ' is $\forall x_1 \cdots \forall x_{i-1} \exists x_i Q_{i+1} x_{i+1} \cdots Q_m x_m \varphi_0(x_1, \ldots, x_m)$ then $s(\varphi')$ is the formula

$$\forall x_1 \cdots \forall x_{i-1} Q_{i+1} x_{i+1} \cdots Q_m x_m \varphi_0(x_1, \ldots, x_{i-1}, f(x_1, \ldots, x_{i-1}), x_{i+1}, \ldots, x_m),$$

where f is an $(i-1)$-ary function symbol that does not occur in φ'.

Now let $\psi(x_1, \ldots, x_i)$ be the formula $Q_{i+1} x_{i+1} \cdots Q_m x_m \varphi_0(x_1, \ldots, x_m)$. Suppose that $\forall x_1 \cdots \forall x_{i-1} \exists x_i \psi(x_1, \ldots, x_i)$ is satisfiable. Let M be a model.

Let M_f be an expansion of M_C that interprets f in such a way that for all constants $c_1, \ldots, c_{i-1}$, $M_f \models \psi(c_1, \ldots, c_{i-1}, f(c_1, \ldots, c_{i-1}))$. Then

$$M_f \models \forall x_1 \cdots \forall x_{i-1} \psi(x_1, \ldots, x_{i-1}, f(x_1, \ldots, x_{i-1})).$$

So if $\forall x_1 \cdots \forall x_{i-1} \exists x_i \psi(x_1, \ldots, x_i)$ is satisfiable, then so is

$$\forall x_1 \cdots \forall x_{i-1} \psi(x_1, \ldots, x_{i-1}, f(x_1, \ldots, x_{i-1})).$$

Conversely, if $M \models \forall x_1 \cdots \forall x_{i-1} \psi(x_1, \ldots, x_{i-1}, f(x_1, \ldots, x_{i-1}))$, then, by the meaning of $\exists$, $M \models \forall x_1 \cdots \forall x_{i-1} \exists x_i \psi(x_1, \ldots, x_i)$.

It follows that φ is satisfiable if and only if $s(\varphi)$ is satisfiable. $\square$

Note that φ and φ^S are not necessarily equivalent. Theorem 3.22 merely states that one is satisfiable if and only if the other is. For example, if φ is the formula $\exists x \psi_0(x, y)$ for atomic $\psi_0(x, y)$, then φ^S is $\psi(c, y)$ which is equivalent to $\forall x \psi(x, y)$. Of course, $\exists x \psi(x, y)$ and $\forall x \psi(x, y)$ are not equivalent formulas, but if one of these formulas is satisfiable, then so is the other. For our purposes, this is all we need. To determine whether φ is satisfiable, it suffices to determine whether φ^S is satisfiable.

3.3 Herbrand theory

In this section we "reduce" sentences of first-order logic to sets of sentences in propositional logic. More precisely, given φ in SNF we find a (possibly infinite) set $E(\varphi)$ of sentences of propositional logic such that φ is satisfiable if and only if $E(\varphi)$ is satisfiable. We know $E(\varphi)$ is unsatisfiable if and only if $\emptyset \in Res^*(E(\varphi))$. So we can use the method of resolution from propositional logic to show that a first-order sentence φ in SNF is unsatisfiable. By Theorem 3.22, we can use this method to determine whether *any* sentence of first-order logic is unsatisfiable.

The method we describe in this section will not necessarily tell us if a sentence φ is satisfiable. Since $E(\varphi)$ may be infinite, there may be no way to tell whether $\emptyset$ is *not* in $Res^*(E(\varphi))$. But if $\emptyset$ is in $Res^*(E(\varphi))$, then, by the compactness of propositional logic, we can derive it in a finite number of steps. Recall that to show that φ is satisfiable, we must exhibit a model for φ. We have done this in previous examples. But to show that φ is unsatisfiable, we must show that it does not hold in *any* structure. Previously, we had no way of doing this. Theorem 3.25 provides the key. We show that, in certain circumstances, it suffices to show that φ does not hold in a specific type of structure called a *Herbrand structure*.

3.3.1 Herbrand structures.

Definition 3.23 Let $\mathcal{V}$ be a vocabulary. The *Herbrand universe* for $\mathcal{V}$ is the set of all variable free $\mathcal{V}$-terms.