Lecture 2
Algorithm Analysis

Bo Tang @ SUSTech, Fall 2023

Several pages are based on the notes by Dr. Ken Yiu (PolyU) and Dr. Yufei Tao(CUHK)

Our Roadmap

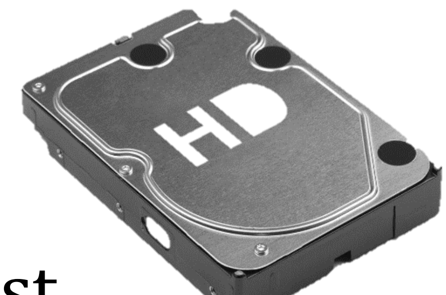
- RAM Computation Model
  - Memory, CPU, Algorithm
  - Algorithm, Pseudocode
- Worst Case Analysis
  - Binary Search Problem
  - Big O notation

RAM Computation Model

CPU

RAM

Hard Disk



CPU cost

I/O cost

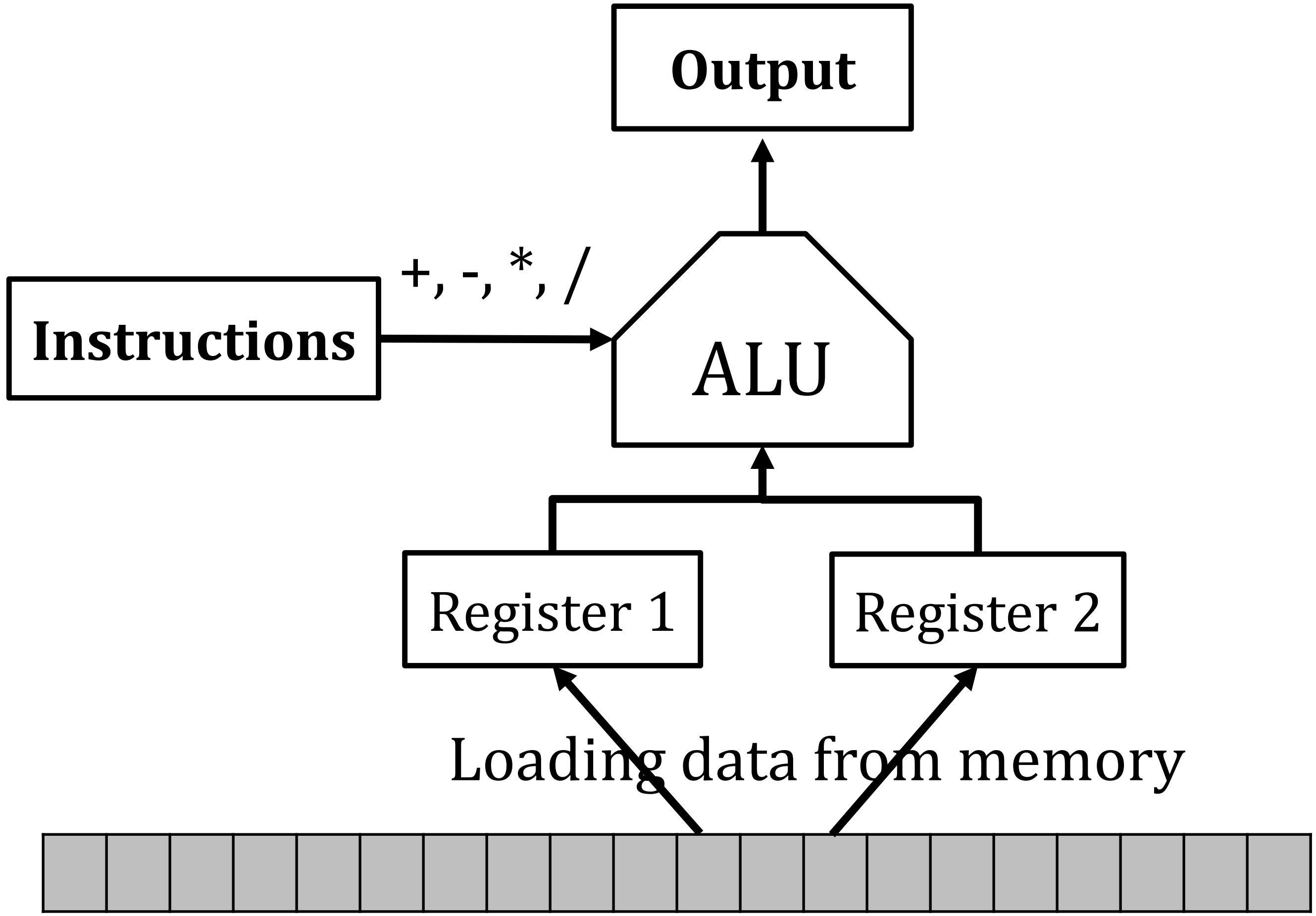
ALU

GB level

TB Level

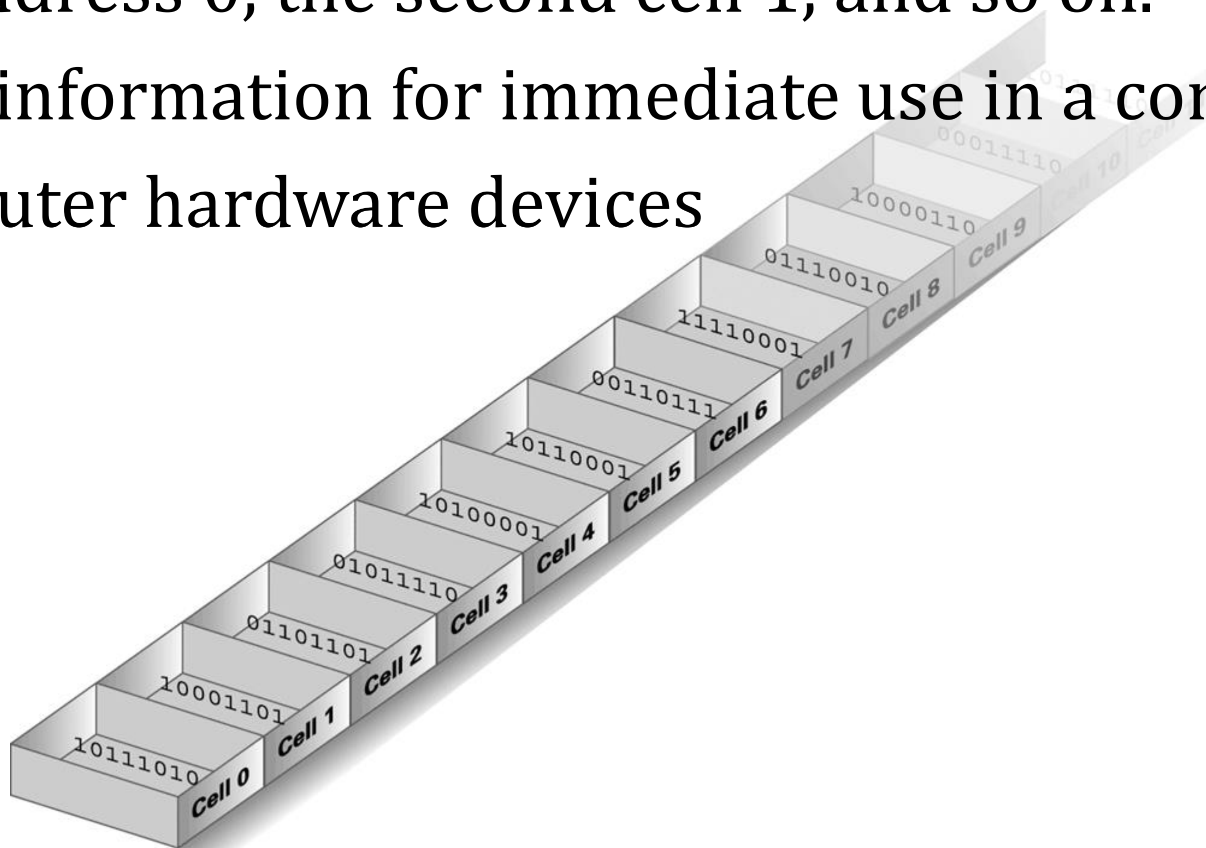
We focus on this part in CS203

RAM Computation Model



Memory

- A finite sequence of cells, each cell has the same number of bits.
- Every cell has an address: the first cell of memory has address 0, the second cell 1, and so on.
- Store information for immediate use in a computer
- Computer hardware devices

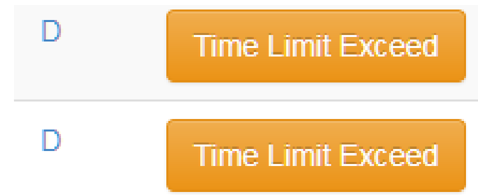


Center Process Unit (CPU)

- Contains a fixed number of registers
- Basic (atomic) operations
  - **Initialization**
    - Set a register to a fixed values (e.g., 100, 1000, etc.)
  - **Arithmetic** (ALU)
    - Take integers a, b stored in two registers, calculate one of {+, -, *, /} and store the result in a register
  - **Comparison / Branching**
    - Take integers a, b stored in two registers, compare them, and learn which of {a<b, a=b, a>b} is true.
  - **Memory Access**
    - Take a memory address A currently stored in a register, Do the READ (i.e., load data from memory) or WRITE (i.e., flush data to memory)operator

Algorithm Analysis

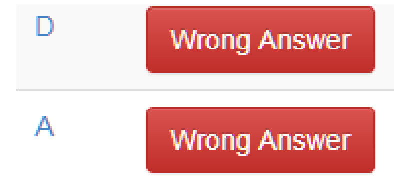
- **Algorithm**
  - A sequence of basic operations



- **Algorithm Analysis**
  - **Cost analysis**
    - Algorithm cost (running time) is the length of the sequences, i.e., the number of basic operations
    - My algorithm is correct, why my submission is **TLE**?
    - Is your algorithm fast?
      - Focus on the order of growth (how the running time grows for large n)

  - Unless otherwise stated, we refer algorithm analysis as cost analysis in CS203

Algorithm Correctness Analysis



- **Correctness** analysis
  - I have passed all test cases, why is still **WA**?
  - It is not enough even if you have tested your algorithm on many instances
    - Will your algorithm fail on some other instances?
  - Proof your algorithm is correct
  - Guarantee your implementation is correct

- Software testing is an individual course in many universities
  - We will not introduce software testing techniques in this course.

Example 1: Summation

- **Problem**: given integer n, calculate 1+2+3+…+n

- **Algorithm**:
  - Initialize variable a to 1, b to n, c to 0
  - Repeat the following until a > b:
    - Calculate c plus a, and store the result to c.
    - Calculate a plus 1, and store the result to a.

- **Cost** of the algorithm:
  - $3 + n + n + n = 3n + 3$
- Which atomic operations are performed?
- Algorithm is described by English words

Example I: Summation

- **Algorithm**:

```
load n from memory to register b
register a ← 1, c ← 0
repeat
        c ← c + a
        a ← a + 1
until a > b
return c
```

- The above is **pseudocode**, it serves the purpose of express (without **ambiguity**) how our algorithm runs.
- **Pseudocode** does not reply on any particular programming language

Example II: Summation

- **Problem**: given integer n, calculate 1+2+3+…+n
- **Cost** of the above algorithm: 3n + 3
- Can we make it faster?
- In our middle school math course:

$$1+2+3+\ldots+n = (1+n)*n / 2$$

Example II: Summation

- **Algorithm**:

```
1. load n from memory to register b
2. register a ← 1
3. a ← a + b
4. a ← a * b
5. a ← a / 2
6. return a
```

- **Cost of the algorithm = 5**
  - This is significantly faster than the previous algorithm
  - The time of the previous algorithm increases linearly with ***n***
  - The time of this algorithm remains constant with ***n***

Our Roadmap

- RAM Computation Model
  - Memory, CPU, Algorithm
  - Algorithm, Pseudocode
- Worst Case Analysis
  - Binary Search Problem
  - Big O notation

Search Problem

- An array A of n integers have been sorted in ascending order. Design an algorithm to determine whether given value t exists in A.

- Example

A	5	8	10	13	16	19	27	46	51	86

  - $t = 16$, the result is "TRUE"

  - $t = 17$, the result is "FALSE"

Search Problem

- The First Algorithm
  - Simply read the value of $\mathbf{A}[i]$ for each $i \in [1, n]$
  - If any of those cell equals to $\boldsymbol{t}$, return “TRUE”, otherwise return “FALSE”

- Pseudocode:

```
1. variable i ← 1
2. Repeat
3.     if A[i] = t then
4.           return “TRUE”
5.     i ← i + 1
6. until i > n
7. return “FALSE”
```

Running Time of the First Algorithm

A	5	8	10	13	16	19	27	46	51	86

- How much time does the algorithm require?
  - If $\boldsymbol{t}$ is 5, the algorithm has running time = 3
  - If $\boldsymbol{t}$ is 6, the algorithm has running time = $4\boldsymbol{n} + 1 = 41$

- In computer science, it is an art to design algorithms with performance guarantees.

- What is the largest running time on the worst input with $\boldsymbol{n}$ integers?

Worst-Case Running Time

The worst-case running time (or worst case cost) of an algorithm under a problem size n, is defined to be the largest running time of the algorithm on all the inputs of the same size n.

Worst-Case Time of Search Problem

- Our algorithm has worst-case time

$$f(n) = 4n + 1$$

- In other words, the algorithm will terminates with a cost at most $4n+1$.

- This is a performance guarantee on every n

- Can we make it faster?
  - **Binary search algorithm**

Binary Search Algorithm

- We utilize the fact that array ***A*** has been sorted in ascending order.

- Let us compare t to the element ***x*** in the middle of ***A*** (i.e., **A[n/2]**)
  - If ***t = A[n/2],*** we have found ***t***, return "TRUE", terminate
  - If ***t < A[n/2],*** we can ignore ***A[n/2+1] to A[n]***
  - If ***t > A[n/2],*** we can ignore ***A[1] to A[n/2]***

- In the 2^{nd} and 3^{rd} cases, we have at most ***n/2*** elements. Then repeat the above on these left elements.

Binary Search Algorithm

A	5	8	10	13	16	19	27	46	51	86	***t=27***
A	5	8	10	13	**16**	19	27	46	51	86	**< t**
A						19	27	**46**	51	86	**> t**
A						19	**27**	46			**= t**

Binary Search Algorithm

- Binary Search in Pseudocode

```
left ← 1, right ← n
repeat
 mid ← (left+right)/2
 if (t = A[mid]) then
      return TRUE
 else if (t < A[mid]) then
      right ← mid -1
 else
      left ← mid + 1
 until left > right
 return FLASE
```

Worst-Case Time of Binary Search

- We call the elements from left to right as surviving elements

- Line 1: initialization: 2 basic operations
- Line 2 – 10: iteration, each iteration performs at most 9 basic operations
- Line 11: termination

- How many iterations in the algorithm?

Worst-Case Time of Binary Search

- How many iterations in the algorithm?
  - After the ***1^st^*** iteration, the number of surviving elements is at most **n/2**
  - After the ***2^nd^*** iteration, the number of surviving elements is at most **n/4**
  - In general, after ***i-th*** iteration, the number of surviving elements is at mots $n / 2^i$
  - Suppose that there are ***h*** iterations in total, it holds that ***h*** is the smallest integer satisfying (why?):

$$n / 2^h < 1$$

  - Then, $h > \log_2 n \Rightarrow h = 1 + \log_2 n$
- Thus, the worst case time of binary search is at most:

$$g(n) = 2 + 9h = 2 + 9(1 + \log_2 n)$$

- This is a performance guarantee that holds on all values of ***n***.

Search Problem

- Running time of two algorithms, with input size *n*
  - Algorithm 1: $f(n) = 4n + 1$ (operations)
  - Algorithm 2: $g(n) = 9\log_2 n + 11$ (operations)

- Which algorithm is better?
  - Algorithm 2. Why?
  - We care about the running time at *large input size*
  - Constant factors do not affect *the order of growth*

Asymptotic Analysis

- Running time of two algorithms, with input size *n*
  - Algorithm 1: $f(n) = 4n + 1$ (operations)
  - Algorithm 2: $g(n) = 9\log_2 n + 11$ (operations)
- In computer science, we rarely calculate the time to such a level.
- We ignore all the constants, but only worry about the dominating term.
  - Why not constant? 10n VS. 5n? Which one is faster?
  - "it depends", 10n comparison, 5n multiplication
  - Why dominating term: 3n VS. $\log_2 n$? Which one is faster
  - "$\log_2 n$ " is better than 3n in theoretical computer science

Big-O notation

- Let $f(n)$ and $g(n)$ be two functions of n.
- We say that $f(n)$ grows asymptotically no faster than $g(n)$ if there is a constant $c_1 > 0$ such that:

$$f(n) \leq c_1 \cdot g(n)$$

holds for all $n \geq c_2$.

- We denote this by $f(n) = O(g(n))$
- We say that 5n is considered equally fast as on with 10n, why?
- Big-O captures this by having both of following true (can you prove that?):

$$10n = O(5n)$$

$$5n = O(10n)$$

Big-O example

- $10000\log_2 n$ is considered better than n. Big-O captures this by having both of following true:

$$10000log_2\, n = O(n)$$

$$n \neq O(10000log_2\, n)$$

- Proof of $\boldsymbol{10000log_2\, n = O(n)}$
- There are constants $c_1 = 1, c_2 = 2^{20}$ such that

$$10000log_2\, n \leq c_1 n$$

holds for all $n \geq c_2$

Big-O example

- Proof of $n \neq O(10000 log_2 n)$
- We can proof it by contradiction. Suppose that are constant c_1, c_2 such that

$$n \leq c_1 \cdot 10000 log_2 n$$

holds for all $n \geq c_2$. The above can be rewritten as:

$$\frac{n}{log_2 n} \leq c_1 \cdot 10000$$

however, $\frac{n}{log_2 n}$ tends to be ∞ as n increases.

Therefore, the inequality cannot hold for all $n \geq c_2$

Exercise

- Is $(5n^2 + 3n) = O(n^2)$?
  - Fix c=6 and n_0=3, then prove $f(n) \le c\, g(n)$
    [note: other choices also possible]
- Is $(5n^2 + 3n) = O(n^3)$?
- Is $(5n^2 + 3n) = O(n)$?

- Proof the following statements:

$$10000 = O(1)$$

$$100\sqrt{n} + 10n = O(n)$$

$$1000n^{1.5} = O(n^2)$$

$$(\log_2 n)^3 = O(\sqrt{n})$$

$$\log_a n = O(\log_b n) \quad \text{for } a>1, b>1$$

Asymptotic Analysis

- Henceforth, we will describe the running time of an algorithm only in the asymptotical (i.e., big-O) form, which is also called the algorithm's time complexity.

- Instead of saying the running time of binary search is $g(n) = 9\log_2 n + 11$, we will say $g(n)=O(\log n)$, which captures the fastest-growing term in the running time. This is also the binary search's time complexity.

Worst-Case of Algorithms

Complexity		***Algorithm***
O(1)	Constant time	E.g., Compare two numbers
O(log n)	Logarithmic	E.g., Binary search (on a sorted array)
O(n)	Linear time	E.g., Search (on a unsorted array)
O(n log n)		E.g., Merge sort
O(n^2)	Quadratic	E.g., Selection sort
O(n^3)	Cubic	E.g., Matrix multiplication
O(2^n)	Exponential	E.g., Brute-force search on boolean satisfiability
O(n!)	Factorial	E.g., Brute-force search on traveling salesman

Big-Ω notation

- Let $f(n)$ and $g(n)$ be two functions of n.
- We say that $f(n)$ grows asymptotically no slower than $g(n)$ if there is a constant $c_1 > 0$ such that:

$$f(n) \geq c_1 \cdot g(n)$$

holds for all $n \geq c_2$.

- We denote this by $f(n) = \Omega(g(n))$
- Examples:
  - $\log_2 n = \Omega(1)$
  - $0.001n = \Omega(\sqrt{n})$

Big-Θ notation

- Let $f(n)$ and $g(n)$ be two functions of n.
- If $f(n) = O(g(n))$ and $f(n) = \Omega\big(g(n)\big)$, then we define: $f(n) = \Theta\big(g(n)\big)$ to indicate $f(n)$ grows asymptotically as fast as $g(n)$

- Examples:
  - $1000 + 30\log n + 1.5\sqrt{\mathrm{n}} = \Theta(\sqrt{\mathrm{n}})$

Thank You!