Assignment#1 CS201 Fall 2023

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PROBLEM 1. Propositions: p: You get an A on the final, q: You do all the assignments, r: You get an A in this course.

SOLUTION. The formulae are as follows

(a) $p \vee q$

(b) $q \to r$

(c) $(q \wedge \neg p) \to r$

(d) $\neg r \to (\neg p \vee \neg q)$

(e) $(p \wedge q) \to r$

PROBLEM 2. Construct a truth table for each of the formulae.

SOLUTION.

(a)

p	$\neg p$	$p \oplus \neg p$
0	1	1
1	0	1

(b)

p	q	$\neg p$	$p \to q$	$\neg p \leftrightarrow q$	$(p \to q) \wedge (\neg p \leftrightarrow q)$
0	0	1	1	0	0
0	1	1	1	1	1
1	0	0	0	1	0
1	1	0	1	0	0

(c)

p	q	$\neg q$	$p \oplus q$	$p \vee \neg q$	$(p \oplus q) \to (p \vee \neg q)$
0	0	1	0	1	1
0	1	0	1	0	0
1	0	1	1	1	1
1	1	0	0	1	1

(d)

p	q	r	$\neg q$	$p \to \neg q$	$p \vee \neg q$	$r \to (p \vee \neg q)$	$(p \to \neg q) \leftrightarrow (r \to (p \vee \neg q))$
0	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1
0	1	0	0	1	0	1	1
0	1	1	0	1	0	0	0
1	0	0	1	1	1	1	1
1	0	1	1	1	1	1	1
1	1	0	0	0	1	1	0
1	1	1	0	0	1	1	0

Problem 3. Use logical equivalences to prove the following statements.

Solution.

(a) $\neg(p \to q) \to p$ is a tautology.

Proof:

$$
\begin{aligned}
\neg(p \to q) \to p &\equiv \neg\neg(p \to q) \lor p && \text{Implies Definition} \\
&\equiv (p \to q) \lor p && \text{Double Negation} \\
&\equiv (\neg p \lor q) \lor p && \text{Implies Definition} \\
&\equiv (\neg p \lor p) \lor q && \text{OR Distributive} \\
&\equiv \top \lor q && \text{Negation Law} \\
&\equiv \top && \text{Domination Law}
\end{aligned}
$$

(b) $(p \land \neg q) \to r$ and $p \to (q \lor r)$ are equivalent.

Proof:

$$
\begin{aligned}
p \to (q \lor r) &\equiv \neg p \lor (q \lor r) && \text{Implies Definition} \\
&\equiv (q \lor \neg p) \lor r && \text{OR Distributive} \\
&\equiv \neg\neg(q \lor \neg p) \lor r && \text{Double Negation} \\
&\equiv \neg(\neg q \land \neg\neg p) \lor r && \text{De Morgan's Law} \\
&\equiv \neg(\neg q \land p) \lor r && \text{Double Negation} \\
&\equiv (\neg q \land p) \to r && \text{Implies Definition}
\end{aligned}
$$

(c) $(p \to q) \to ((r \to p) \to (r \to q))$ is a tautology.

Proof:

Suppose that $\neg((p \to q) \to ((r \to p) \to (r \to q)))$ is true

$$\neg\left((p \to q) \to ((r \to p) \to (r \to q))\right)$$

$$\begin{aligned}
&\equiv \neg\left(\neg(p \to q) \lor ((r \to p) \to (r \to q))\right) && \text{Implies Definition}\\
&\equiv \neg\left(\neg(p \to q) \lor (\neg(r \to p) \lor (r \to q))\right) && \text{Implies Definition}\\
&\equiv \neg\left(\neg(\neg p \lor q) \lor (\neg(r \to p) \lor (r \to q))\right) && \text{Implies Definition}\\
&\equiv \neg\left(\neg(\neg p \lor q) \lor (\neg(\neg r \lor p) \lor (r \to q))\right) && \text{Implies Definition}\\
&\equiv \neg\left(\neg(\neg p \lor q) \lor (\neg(\neg r \lor p) \lor (\neg r \lor q))\right) && \text{Implies Definition}\\
&\equiv \neg\left((\neg\neg p \land \neg q) \lor (\neg(\neg r \lor p) \lor (\neg r \lor q))\right) && \text{De Morgan's Law}\\
&\equiv \neg\left((\neg\neg p \land \neg q) \lor ((\neg\neg r \land \neg p) \lor (\neg r \lor q))\right) && \text{De Morgan's Law}\\
&\equiv \neg\left((p \land \neg q) \lor (r \land \neg p) \lor (\neg r \lor q)\right) && \text{(Double Negation)}\\
&\equiv \neg(p \land \neg q) \land \neg(r \land \neg p) \land \neg(\neg r \lor q) && \text{(De Morgan's Law)}\\
&\equiv (\neg p \lor \neg\neg q) \land (\neg r \lor \neg\neg p) \land (\neg\neg r \land \neg q) && \text{De Morgan's Law}\\
&\equiv (\neg p \lor q) \land (\neg r \lor p) \land (r \land \neg q) && \text{Double Negation}\\
&\equiv (\neg p \lor q) \land (\neg r \lor p) \land r \land \neg q && \text{AND Distributive}\\
&\equiv ((\neg p \lor q) \land \neg q) \land ((\neg r \lor p) \land r) && \text{AND Distributive}\\
&\equiv \neg p \land p && \text{Modus Ponens}\\
&\equiv \bot && \text{Contradiction}
\end{aligned}$$

Thus, the assumption is false and therefore, by the proof of contradiction, the formula $(p \to q) \to ((r \to p) \to (r \to q))$ always holds.

PROBLEM 4. Determine whether the following pairs of statements are logically equivalent and explain why.

SOLUTION.

(a) $p \oplus q$ and $\neg p \vee \neg q$ are **not** logical equivalent.

$$\begin{aligned} \neg p \vee \neg q &\equiv \neg\neg\left(\neg p \vee \neg q\right) && \text{Double Negation} \\ &\equiv \neg\left(\neg\neg p \wedge \neg\neg q\right) && \text{De Morgan} \\ &\equiv \neg\left(p \wedge q\right) && \text{Double Negation} \end{aligned}$$

So, $\neg p \vee \neg q$ is zero if and only if p and q are zero. Otherwise it's one and therefore

$$\begin{aligned} p \oplus q &\equiv (\neg p \wedge q) \vee (p \wedge \neg q) \\ &\not\equiv \neg p \vee \neg q \end{aligned}$$

(b) $\neg q \wedge (p \leftrightarrow q)$ and $\neg p$ are **not** logical equivalent.

p	q	$\neg p$	$\neg q$	$p \leftrightarrow q$	$\neg q \wedge (p \leftrightarrow q)$
0	0	1	1	1	1
0	1	1	0	0	0
1	0	0	1	0	0
1	1	0	0	1	1

(c) $(p \to q) \vee (p \to r)$ and $p \to (q \vee r)$ are logically equivalent.

$$\begin{aligned} (p \to q) \vee (p \to r) &\equiv (\neg p \vee q) \vee (p \to r) && \text{Implies Definition} \\ &\equiv (\neg p \vee q) \vee (\neg p \vee r) && \text{Implies Definition} \\ &\equiv (\neg p \vee \neg p) \vee (q \vee r) && \text{OR Distributive} \\ &\equiv \neg p \vee (q \vee r) && \text{Idempotene Law} \\ &\equiv p \to (q \vee r) && \text{Implies Definition} \end{aligned}$$

(d) $(p \to q) \to r$ and $p \to (q \to r)$ are **not** logically equivalent.

$$\begin{aligned}
(p \to q) \to r &\equiv \neg(p \to q) \vee r && \text{Implies Definition} \\
&\equiv \neg(\neg p \vee q) \vee r && \text{Implies Definition} \\
&\equiv (\neg\neg p \wedge \neg q) \vee r && \text{De Morgan's Law} \\
&\equiv (p \wedge \neg q) \vee r && \text{Double Negation}
\end{aligned}$$

And we got

$$\begin{aligned}
p \to (q \to r) &\equiv \neg p \vee (q \to r) && \text{Implies Definition} \\
&\equiv \neg p \vee (\neg q \vee r) && \text{Implies Definition} \\
&\equiv (\neg p \vee \neg q) \vee r && \text{OR Distributive}
\end{aligned}$$

It's obvious that $p \wedge \neg q$ and $\neg p \vee \neg q$ are not equivalent.

PROBLEM 5. Determine for which values of p, q, r the statement $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ is true and false. Try to explain without the truth table.

SOLUTION. If the statement is true, it's known that $p \vee q \vee r$ and $\neg p \vee \neg q \vee \neg r$ are true at the same place. Only when p, q, r have different values can both of the terms be true and the statement will be true, say,

$$\begin{array}{rcccccc}
p = & 0 & 0 & 1 & 0 & 1 & 1 \\
q = & 0 & 1 & 0 & 1 & 0 & 1 \\
r = & 1 & 0 & 0 & 1 & 1 & 0
\end{array}$$

If the statement is false, it's known that $p \vee q \vee r$ and $\neg p \vee \neg q \vee \neg r$ are false at the same place. In that case, p, q, r have the same value, which means they are zeros or ones. And such that one of the two terms will be false, making the statement false.

PROBLEM 6. Prove that if $p \to q, \neg p \to \neg r, s \vee r$,then $q \vee s$.

SOLUTION. Premises are $p \to q, \neg p \to \neg r, s \vee r$

Proof

1. $\neg p \to \neg r$	Premise
2. $\neg r \vee \neg\neg p$	Implies Definition using (1)
3. $\neg r \vee p$	Double Negation using (2)
4. $r \to p$	Implies Definition using (3)
5. $p \to q$	Premise
6. $r \to q$	Hypothetical syllogism using (4) (5)
7. $\neg r \vee q$	Implies Definition using (7)
8. $s \vee r$	Premise
9. $q \vee s$	Resolution using (7) (8)

PROBLEM 7. Prove that if $p \wedge q, q \to \neg (p \wedge r), s \to r$ then $\neg s$.

SOLUTION. Premises are $p \wedge q, q \to \neg (p \wedge r), s \to r$.

Proof

1. $p \wedge q$	Premise
2. q	Simplification using (1)
3. $q \rightarrow \neg(p \wedge r)$	Premise
4. $\neg(p \wedge r)$	Modus Ponens using (2) (3)
5. $\neg p \vee \neg r$	De Morgan using (4)
6. p	Simplification using (1)
7. $\neg r$	Disjunctive syllogism using (5) (6)
8. $s \rightarrow r$	Premise
9. $\neg s$	Modus Tollens using (7) (8)

PROBLEM 8. Let $P(x)$ be the statement "x can speak Russian" and $Q(x)$ be the statement "x knows the C Plus Plus". The domain consists of all students at SUSTech. Translate the statements into predicates logic.

SOLUTION.
(a) $\exists x\ P(x) \vee Q(x)$
(b) $\exists x\ P(x) \wedge \neg Q(x)$
(c) $\forall x\ P(x) \wedge Q(x)$
(d) $\neg\exists x\ P(x) \vee Q(x)$
(e) $\exists x\ P(x) \rightarrow Q(x)$

PROBLEM 9. Let $L(x, y)$ be the statement "x loves y", where the domain consists of all people in the world.

SOLUTION.
(a) $\forall x \exists y\ L(x, y) \wedge (x \neq y)$
(b) $\exists x\ L(x, x)$
(c) $\exists x \exists y\ L(\text{Lynn}, x) \wedge L(\text{Lynn}, y) \wedge (x \neq y)$

PROBLEM 10. Express the negations of the following statements with negation ahead of predicates.

SOLUTION.
(a) $\forall z \exists y \exists x\ \neg T(x, y, z)$
(b) $\forall x \forall y\ \neg P(x, y) \vee \exists x \exists y\ \neg Q(x, y)$
(c) $\exists x \forall y\ \neg Q(x, y) \rightarrow \neg P(x, y)$

PROBLEM 11. Consider the argument "All movies produced by John Sayles are wonderful. John Sayles produced a movie about coal miners. Therefore, there is a wonderful movie about coal miners." and answer the questions.

SOLUTION.
(a) Let predicate $W(x)$ stand for "The movie x is wonderful" and $P(x, y)$ be "The movie x is produced by y". Let C be "A movie about coal miners", and S be "John Sayles". Thus, the sentences are

All movies produced by John Sayles are wonderful: $\forall x\ W(x) \wedge P(x, S)$

John Sayles produced a movie about coal miners: $P(C, S)$

There is a wonderful movie about coal miners: $\exists x\ W(C) \wedge P(C, x)$

(b) Premises are $\forall x\ W(x) \wedge P(x, S)$ and $P(C, S)$

Proof

1. $\forall x\ W(x) \wedge P(x, S)$	Premise
2. $P(C, S)$	Premise
3. $W(C) \wedge P(C, S)$	UI using (1)(2)
4. $\exists x\ W(C) \wedge P(C, x)$	EG using (3)

PROBLEM 12. Prove that $\sqrt[3]{2}$ is irrational

SOLUTION.

Suppose $\sqrt[3]{2}$ is rational, thus we got $\sqrt[3]{2} = \frac{p}{q}(p, q \in \mathbb{N},\ gcd(p, q) = 1)$

Then $p^3 = 2q^3 \implies p^3 = 0(mod\ 2)$ so let $p = 2k(k \in \mathbb{N})$,

Then $p^3 = 8k^3 = 2q^3 \implies q^3 = 4k^3 \implies q^3 = 0(mod\ 2)$,

Thus p and q has a cofactor 2, which is contrary to the assumption that $gcd(p, q) = 1$.

Therefore, $\sqrt[3]{2}$ is irrational.

PROBLEM 13. Prove that there is an irrational number between every two distinct rational numbers.

SOLUTION.

Suppose $\frac{p}{q}$ and $\frac{m}{n}$ are arbitrary distinct rational number and $\frac{p}{q} < \frac{m}{n}$

Then $\frac{m}{n} - \frac{p}{q} = \frac{mq-np}{nq}$ is also a rational number.

Let Q be a positive rational number and $\sqrt{2} < Q \times \frac{mq-np}{nq}$.

Thus, $0 < \frac{\sqrt{2}}{Q} < \frac{mq-np}{nq}$.
And we got, $\frac{p}{q} < \frac{\sqrt{2}}{Q} + \frac{p}{q} < \frac{mq-np}{nq} + \frac{p}{q}$, that is, $\frac{p}{q} < \frac{\sqrt{2}}{Q} + \frac{p}{q} < \frac{m}{n}$.
Thus, $\frac{p}{q}$ and $\frac{m}{n}$ are rational number and $\frac{\sqrt{2}}{Q} + \frac{p}{q}$ is irrational number between $\frac{p}{q}$ and $\frac{m}{n}$.
Therefore, there must be an irrational number between every distinct rational number.

PROBLEM 14. Prove that all integral solutions to the equation that satisfies $m, n \geq 3$ and $e > 0$ are in the table.

$$\frac{1}{m} + \frac{1}{n} = \frac{1}{e} + \frac{1}{2}$$

SOLUTION. The proof can be divided into 4 parts. Since m and n are symmetric, we only need to discuss one of them. Firstly, ensure

$$e = \frac{2mn}{2(m+n) - mn} > 0 \implies 2m + 2n > mn$$

Case 1: $n = 3$

If $n = 3$, then we have $m < 6$ so m can barely be $3, 4, 5$, and all of the values statisfy the condition that e is an integral.

Case 2: $n = 4$

If $n = 4$, then $m < 4$. So m can only be 3 and e is an integral.

Case 3: $n = 5$

$n = 5$ makes $m < \frac{10}{3}$, so m is 3 and e is an integral.

Case 4: $n = 6$

In that case, m will be less than 3, which is contradiction. And by solving the inequality we got $n < 6$.

From the cases above, we know that $m, n \leq 5$ and there are only 5 combinations of m and n, which fits the cases in the table. Therefore, all integral solutions to the equation are in the table.