Assignment 07 & 08 of I2ML-s23

Q1 *(20 pts)*

For each $n \in \mathbb{N}$, let Φ_n be the sentence

$$\exists x_1 \cdots \exists x_n \left(\bigwedge_{i \neq j} x_i \neq x_j \right)$$

asserting that there exist at least n elements.

Let φ_1 be the sentence

$$\forall x \forall y ((f(x) = f(y)) \to (x = y))$$

saying that f is one-to-one and let φ_2 be the sentence

$$\forall y \exists x (f(x) = y)$$

saying that f is onto.

Show that $\{\varphi_1, \varphi_2, \Phi_n\} \vdash \Phi_{n+1}$ by giving a sketch of a formal proof.

Q2 *(3 * 12 = 36 pts)* For each of the following formulas, find an equivalent formula in Conjunctive Prenex Normal Form. Note that each of these formulas have *x and y* as free variables (view them as global universal quantifiers).

(a) $\neg\exists z Q(x,y,z) \vee \forall z \exists w P(w,x,y,z)$

(b) $\forall z(R(x,z) \wedge R(x,y) \to \exists w(R(x,w) \wedge R(y,w) \wedge R(z,w)))$

(c) $\exists z(S(y,z) \wedge \exists y(S(z,y) \wedge \exists z(S(x,z) \wedge (S(z,y))))).$

Q3 *(3 * 10 = 30 pts)* Find the Skolemization of each of the formulas in the above Q2.

Q4 *(14 pts)* Is the following set of literals unifiable? If so, give the most general unifier and another unifier that is not most general. If not, give an interpretation for why not.

$$\{R(f(x), g(z)), R(y, g(x)), R(v, w), R(w, g(x))\}.$$

Q5 *(2 * 20 = 40 pts)* Use resolution to prove that the following are tautologies:

(a) $(\exists x \forall y Q(x, y) \wedge \forall x(Q(x, x) \rightarrow \exists y R(y, x))) \rightarrow \exists y \exists x R(x, y)$

(b) $(\exists x \forall y R(x, y)) \leftrightarrow (\neg \forall x \exists y \neg R(x, y))$

Q6 *(15 pts)*

A familiar command missing from our language is the for-statement. It may be used to sum the elements in an array, for example, by programming as follows:

```
s = 0;
for (i = 0; i <= max; i = i+1) {
   s = s + a[i];
}
```

After performing the initial assignment `s = 0`, this executes `i = 0` first, then executes the body `s = s + a[i]` and the incrementation `i = i + 1` continually until `i <= max` becomes false. Explain how `for` $(C_1; B; C_2)$ $\{C_3\}$ can be defined as a derived program in our core language.

Q7 *(3 * 10 = 30pts)*

Use the proof rule for assignment and logical implication as appropriate to show the validity of

(a) $\vdash_{\mathsf{par}} (\!| x > 0 |\!)$ `y = x + 1` $(\!| y > 1 |\!)$

(b) $\vdash_{\mathsf{par}} (\!| \top |\!)$ `y = x; y = x + x + y` $(\!| y = 3 \cdot x |\!)$

(c) $\vdash_{\mathsf{par}} (\!| x > 1 |\!)$ `a = 1; y = x; y = y - a` $(\!| y > 0 \wedge x > y |\!)$.

Q8 *(20 pts)*

Show that $\vdash_{\text{par}} (\!| y \geq 0 |\!)$ `Multi1` $(\!| z = x \cdot y |\!)$ is valid, where `Multi1` is:

```
a = 0;
z = 0;
while (a != y) {
  z = z + x;
  a = a + 1;
}
```

Q9 *(15 pts)* Prove the validity of the following total-correctness sequent based on the above Q8.

$$\vdash_{\text{tot}} (\!| y \geq 0 |\!)\ \texttt{Multi1}\ (\!| z = x \cdot y |\!)$$