Handling First Order Logic

HE Mingxin, Max
CS104: program07 @ yeah.net CS108:
mxhe1 @ yeah.net

I2ML(H) Spring 2023 (CS104|CS108)

Exercises 11 : Reading and More

Record your time spent (in 0.1 hours) with brief tasks and durations in your learning log by hand writing!

1) Read textB-ch02-2.1+2.2-basics-fol.pdf (cont.)
2) Read textI-ch02-2.1+2.2-basics-fol.pdf (cont.)
3) Read textI-ch03-3.1-FormalProofs.pdf (in 2 weeks)
4) Read textB-ch02-2.3-Proofs-fol.pdf (in 2 weeks)

Topic 11.1

Supporting Definitions

Clubbing Similar Quantifiers

If we have a chain of same quantifier then we write the quantifier once followed by the list of variables.

Example 11.1

- $\forall z, x. \exists y. G(x, y, z) = \forall z. (\forall x. (\exists y. G(x, y, z)))$
- $\exists z, x, y. G(x, y, z) = \exists z. (\exists x. (\exists y. G(x, y, z)))$

Subterm and Subformulas

Definition 11.1

A term t is subterm of term t', if t is a substring of t'.

Thinking Exercise 11.1

- *Is $f(x)$ a subterm of $g(f(x), y)$?*
- *Is c a subterm of c?*
- *x is a subterm of $f(x)$*

Definition 11.2

A formula F is subformula of formula F', if F is a substring of F'.

Example 11.2

- *$G(x, y, z)$ is a subformula of $\forall z, x.\exists y.G(x, y, z)$*
- *$P(c)$ is a subformula of $P(c)$*
- *$\exists y.G(x, y, z)$ is a subformula of $\forall z, x.\exists y.G(x, y, z)$*

Closed Terms and Quantifier Free

Definition 11.3

A closed term is a term without any variable. Let $\hat{T}_{\mathbf{S}}$ be the set of closed $\mathbf{S}$-terms.

Sometimes closed terms are also referred as ground terms.

Example 11.3

Let $\mathbf{F} = \{f/1, c/0\}$. $f(c)$ is a closed term, and $f(x)$ is not, where x is a variable.

Thinking Exercise 11.2

Which of the following terms are closed with respect to $\mathbf{F} = \{f/1, g/2, c/0\}$?

- $g(c, y)$
- x
- c
- $f(g(c, c))$

Quantifier-Free

Definition 11.4

A formula F is quantifier-free if there are no quantifiers in F.

Example 11.4

$H(c)$ is a quantifier-free formula and $\forall x.H(x)$ is not a quantifier-free formula.

Thinking Exercise 11.3

For signature $(\{f/1, c/0\}, \{H/1\})$, which of the following are quantifier-free?

- $\forall x.H(y)$
- $H(y) \vee \perp$
- $f(c)$
- $H(f(c))$

Free Variables

Definition 11.5

A variable $x \in \text{Vars}$ is free *in formula F if*

- $F \in A_{\mathbf{S}}$: *x occurs in F,*
- $F = \neg G$: *x is free in G,*
- $F = G \circ H$: *x is free in G or H, for some binary operator $\circ$, and*
- $F = \exists y.G$ *or* $F = \forall y.G$: *x is free in G and $x \neq y$.*

Let $FV(F)$ denote the set of free variables in F.

Thinking Exercise 11.4

Is x free?

- $H(x)$
- $H(y)$
- $\forall x.H(x)$
- $x = y \Rightarrow \exists x.G(x)$

Sentence

Definition 11.6

In $\forall x.(G)$, we say the quantifier $\forall x$ has scope G and bounds x.
In $\exists x.(G)$, we say the quantifier $\exists x$ has scope G and bounds x.

Definition 11.7

A formula F is a sentence if it has no free variable.

Thinking Exercise 11.5

Which of the following formulas are sentence(s)?

- $H(x)$
- $\forall x.H(x)$
- $x = y \Rightarrow \exists x.G(x)$
- $\forall x.\exists y.\ x = y \Rightarrow \exists x.G(x)$

Topic 11.2

Understanding FOL Semantics

No Free Variables

Definition 11.8
Let t be a closed term. $m(t) \triangleq m^{\nu}(t)$ *for any* ν.

If F is a sentence, ν has no influence in the satisfaction relation. (why?)

For sentence F, we say

- F is *true* in m if $m \models F$
- Otherwise, F is *false* in m.

Why Nonempty Domain?

We are required to have nonempty domain in the model. Why?

Example 11.5

Consider formula $\forall x.(H(x) \land \neg H(x))$.

Should any model satisfy the formula? *Noooooooo..*

But, if we allow $m = \{\emptyset; H_m = \emptyset\}$ *then*

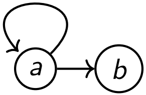
$$m \models \forall x.(H(x) \land \neg H(x)).$$

Due to this counter-intuitive behavior, the empty domain is disallowed.

Example: Graph Models

Example 11.6

Consider $\mathbf{S} = (\{\}, \{E/2\})$ *and* $m = (\{a, b\}; \{(a, a), (a, b)\})$.
m may be viewed as the following graph.



$$m, \{x \to a\} \models E(x, x) \land \exists y.(E(x, y) \land \neg E(y, y))$$

Thinking Exercise 11.6

Give another model and assignment that satisfies the above formula

Example : Counting

Example 11.7

Consider $\mathbf{S} = (\{\}, \{E/2\})$

The following sentence is false in all the models with one element domain

$$\forall x. \neg E(x, x) \land \exists x \exists y. E(x, y)$$

Exercise: Counting

Thinking Exercise 11.7

Give a sentence that is true only in the models with more than two elements

Thinking Exercise 11.8

a. Give a sentence that is true only in infinite models

b. Do only finite models satisfy the negation of the sentence in (a)? If not, give an example of infinite model.

Thinking Exercise 11.9

a. Give a sentence that is true only in models with less than or equal to two element domains.
b. Can you answer (a) without using $=$?

A Limit: Impossibility of Expressing Finite

Theorem 11.1

No FOL sentence can express that all satisfying models are finite.

Commentary: Proof of the above is not part of this course.

Topic 11.3

Substitution

Substitution

In first-order logic, we have terms and formulas. We need a more elaborate notion of substitution for terms.

Definition 11.9

A substitution σ is a map from Vars $\rightarrow T_{\mathbf{S}}$. *We will write $t\sigma$ to denote $\sigma(t)$.*

Definition 11.10

We say σ has finite support if only finite variables do not map to themselves. σ with finite support is denoted by $[t_1/x_1, .., t_n/x_n]$ or $\{x_1 \mapsto t_1, .., x_n \mapsto t_n\}$.

We may write a formula as $F(x_1, \dots, x_k)$, where variables $x_1, \dots, x_k$ play a special role in F.

Let $F(t_1, \dots, t_n)$ be $F[t_1/x_1, \dots, t_n/x_n]$.

Commentary: We have seen substitution in propositional logic. Now in FOL, substitution is not a simple matter. We were replacing a formula by another. Now we need another kind of substitution that replaces terms.

Substitution on Terms

Definition 11.11

For $t \in T_{\mathbf{S}}$, let the following naturally define $t\sigma$ as extension of σ.

- $c\sigma \triangleq c$
- $(f(t_1, \ldots, t_n))\sigma \triangleq f(t_1\sigma, \ldots, t_n\sigma)$

Example 11.8

Consider $\sigma = \{x \mapsto f(x, y), y \mapsto f(y, x)\}$

- $x\sigma = f(x, y)$
- $f(x, y)\sigma = f(f(x, y), f(y, x))$
- $(f(x, y)\sigma)\sigma = ?$
- $f(x, g(y))\{x \mapsto g(z), z \mapsto y\} = f(g(z), g(y))$

Substitution on Atoms

We further extend the substitution σ to atoms.

Definition 11.12

For $F \in A_{\mathbf{S}}$, $F\sigma$ is defined as follows.

- $\top\sigma \triangleq \top$
- $\bot\sigma \triangleq \bot$
- $P(t_1, \ldots, t_n)\sigma \triangleq P(t_1\sigma, \ldots, t_n\sigma)$
- $(t_1 = t_2)\sigma \triangleq t_1\sigma = t_2\sigma$

Substitution Projection

Sometimes, we may need to remove variable x from the support of σ.

Definition 11.13

Let $\sigma_x = \sigma[x \mapsto x]$*.*

Example 11.9

Consider $\sigma = \{x \mapsto f(x, y), y \mapsto f(y, x)\}$*.* $\sigma_x = \{y \mapsto f(y, x)\}$

Commentary: The need of the definition will be clear soon.

Substitution in Formulas (Incorrect)

Now we extend the substitution σ to all the formulas.

Definition 11.14

For $F \in \mathrm{P}_{\mathbf{S}}$, $F\sigma$ is defined as follows.

- $(\neg G)\sigma \triangleq \neg(G\sigma)$
- $(G \circ H)\sigma \triangleq (G\sigma) \circ (H\sigma)$ *for some binary operator* $\circ$
- $(\forall x.G)\sigma \triangleq \forall x.(G\sigma_x)$
- $(\exists x.G)\sigma \triangleq \exists x.(G\sigma_x)$

Example 11.10

- $(P(x) \Rightarrow \forall x.Q(x))\{x \mapsto y\} = (P(y) \Rightarrow \forall x.Q(x))$
- $(\exists y.\ x \neq y)\{x \mapsto z\} = (\exists y.\ z \neq y)$
- $(\exists y.\ x \neq y)\{x \mapsto y\} = (\exists y.\ y \neq y)$ ☹ *Undesirable!!!*

Some substitutions should be disallowed.

Commentary: The above näive definition of the substitution in formulas appears to be incorrect. In the next slide, we present the accepted definition. Please note that that the substitution is a syntactic operation. It does not provide any semantic guarantee. One could consider the above definition correct, but it would be a useless definition. The definition in the next slide is correct because it is useful.

Substitution in Formulas(Correct)

Definition 11.15

σ is suitable *with respect to formula G and variable x if for all $y \neq x$, if $y \in FV(G)$ then x does not occur in $y\sigma$.*

Now we correctly extend the substitution σ to all formulas.

Definition 11.16

For $F \in \mathrm{P}_{\mathbf{S}}$, $F\sigma$ is defined as follows.

- $(\neg G)\sigma \triangleq \neg(G\sigma)$
- $(G \circ H)\sigma \triangleq (G\sigma) \circ (H\sigma)$ *for some binary operator* $\circ$
- $(\forall x.G)\sigma \triangleq \forall x.(G\sigma_x)$, *where σ is suitable with respect to G and x*
- $(\exists x.G)\sigma \triangleq \exists x.(G\sigma_x)$, *where σ is suitable with respect to G and x*

It is not a true restriction.
We will see later.

Composition

Definition 11.17

Let σ_1 and σ_2 be substitutions. The composition $\sigma_1\sigma_2$ of the substitutions is defined as follows.

$$\text{For each } x \in \mathrm{Vars},\ x(\sigma_1\sigma_2) \triangleq (x\sigma_1)\sigma_2.$$

Example 11.11

- $\sigma_1 = \{x \mapsto f(x,y)\}$ *and* $\sigma_2 = \{y \mapsto c\}$. $\sigma_1\sigma_2 = \{x \mapsto f(x,c), y \mapsto c\}$.
- $\sigma_1 = \{x \mapsto y\}$ *and* $\sigma_2 = \{y \mapsto x\}$. $\sigma_1\sigma_2 = \{x \mapsto x, y \mapsto x\} = \{y \mapsto x\}$.

Thinking Exercise 11.10

Show $\sigma_1(\sigma_2\sigma_3) = (\sigma_1\sigma_2)\sigma_3$, i.e., substitution is associative.

Commentary: Type check composition definition. Convince yourself that composition is well-defined.
Solution for exercise: Consider variable x. $(x\sigma_1)(\sigma_2\sigma_3) = ((x\sigma_1)\sigma_2)\sigma_3 = (x(\sigma_1\sigma_2))\sigma_3$

Composition works on terms and atoms

Theorem 11.2

For each $t \in T_{\mathbf{S}}$, $t(\sigma_1\sigma_2) = (t\sigma_1)\sigma_2$

Proof.

Proved by trivial structural induction. $\square$

Theorem 11.3

For each $F \in A_{\mathbf{S}}$, $F(\sigma_1\sigma_2) = (F\sigma_1)\sigma_2$

Proof.

Proved by trivial structural induction. $\square$

Commentary: Why do we need this theorem? In the definition, the composition is defined only for variables but not for arbitrary terms. We need to show that the definition extends for any term.

Substitution composition on formulas

Theorem 11.4

if $F\sigma_1$ and $(F\sigma_1)\sigma_2$ are defined then $(F\sigma_1)\sigma_2 = F(\sigma_1\sigma_2)$

Proof.

We prove it by induction. Non-quantifier cases are simple structural induction.

Assume $F = \forall x.G$

Since $F\sigma_1$ is defined, $G\sigma_{1x}$ is defined. Since $(F\sigma_1)\sigma_2$ is defined, $(G\sigma_{1x})\sigma_{2x}$ is defined (why?).

By induction hypothesis, $(G\sigma_{1x})\sigma_{2x} = G(\sigma_{1x}\sigma_{2x})$

claim: $G(\sigma_{1x}\sigma_{2x}) = G(\sigma_1\sigma_2)_x$

Choose $y \in FV(G)$ and $y \neq x$

$$y(\sigma_{1x}\sigma_{2x}) \underbrace{= ((y\sigma_{1x})\sigma_{2x})}_{\text{Def. substitution}} \underbrace{= ((y\sigma_1)\sigma_{2x})}_{y \neq x} \underbrace{= ((y\sigma_1)\sigma_2)}_{x \notin FV(y\sigma_1)\text{(why?)}} = y(\sigma_1\sigma_2) = y(\sigma_1\sigma_2)_x$$

$$(\forall x.G\sigma_1)\sigma_2 = (\forall x.G(\sigma_{1x}\sigma_{2x})) = (\forall x.G(\sigma_1\sigma_2)_x) = F(\sigma_1\sigma_2)$$

□

Commentary: The substitution notation may be new to you. Please follow the argument for each step.

Topic 11.4

Problems

Properties of FOL

Thinking Exercise 11.11

If $x, y \notin Vars(F(z))$, then $\forall x.F(x) \Leftrightarrow \forall y.F(y)$

Thinking Exercise 11.12

Let us suppose x does not occur in formula G. Show that the following formulas are valid.

- $\exists x.G \Leftrightarrow G$
- $\forall x.G \Leftrightarrow G$
- $(\forall x.F(x) \lor G) \Leftrightarrow \forall x.(F(x) \lor G)$
- $(\forall x.F(x) \land G) \Leftrightarrow \forall x.(F(x) \land G)$
- $(\exists x.F(x) \lor G) \Leftrightarrow \exists x.(F(x) \lor G)$
- $(\exists x.F(x) \land G) \Leftrightarrow \exists x.(F(x) \land G)$

Encode mod k

Thinking Exercise 11.13

Give an FOL sentence that encodes that there are n elements in any satisfying model, such that $n \bmod k = 0$ for a given k.

Unique Quantifier

Thinking Exercise 11.14

We could consider enriching the language by the addition of a new quantifier. The formula $\exists ! x.F$ *(read "there exists a unique* x *such that* F*") is to be satisfied in model* m *and assignment* ν *iff there is one and only one* $d \in D_m$ *such that* $m, \nu[x \to d] \models F$*.*

Show that this apparent enrichment does not increase expressive power of FOL.

Exercise: equality propagation

Thinking Exercise 11.15

Which of the following equivalences are correct?

- $\exists x, x'.\ (x' = x \land F(x, x')) \equiv \exists x.\ F(x, x)$
- $\exists x, x'.\ (x' = x \Rightarrow F(x, x')) \equiv \exists x.\ F(x, x)$
- $\forall x, x'.\ (x' = x \land F(x, x')) \equiv \forall x.\ F(x, x)$
- $\forall x, x'.\ (x' = x \Rightarrow F(x, x')) \equiv \forall x.\ F(x, x)$

Topic 11.5

Extra Slides: Not-so-useful Definitions

Bounded Variables

Definition 11.18

A variable $x \in \mathrm{Vars}$ is bounded in formula F if

- $F = \neg G$: *x is bounded in* G,
- $F = G \circ H$: *x is bounded in* G *or* H, *for some binary operator* $\circ$, *and*
- $F = \exists y.G$ *or* $F = \forall y.G$: *x is bounded in* G *or* x *is equal to* y.

Let $bnd(F)$ denote the set of bounded variables in F.

Thinking Exercise 11.16

Is x bounded?

- $H(x)$
- $H(y)$
- $\forall x.H(x)$
- $x = y \Rightarrow \exists x.G(x)$

End of Lecture 11A