Chapter 5
Arithmetic: Gödel's Incompleteness Theorems

H.C.M. (Harrie) de Swart

Abstract We *formalize* elementary number theory, i.e., we introduce a formal language $\mathscr{L}$ for expressing properties of addition and multiplication of natural numbers, and a set $\mathscr{P}$ of non-logical axioms (of Peano) in order to be able to formally deduce those properties from $\mathscr{P}$.
Gödel's *first incompleteness theorem* says that not every formula in $\mathscr{L}$, which is true in the intended interpretation, can be deduced from $\mathscr{P}$; even worse, extending $\mathscr{P}$ consistently with further axioms does not remedy this incompleteness. Gödel's *second incompleteness theorem* follows from his first one and says that the consistency of $\mathscr{P}$ cannot be formally deduced from $\mathscr{P}$; similar results hold for consistent extensions of $\mathscr{P}$. A sketch of Gödel's incompleteness proofs is given.
It turns out that there are two non-isomorphic models of $\mathscr{P}$ (or of any consistent extension Γ of $\mathscr{P}$). However, if we also allow in our language quantifiers of the type $\forall X$, where X is a variable over properties of natural numbers (or subsets of $\mathbb{N}$), as is done in second-order logic, then there is one single formula $\mathscr{PA}$ such that any model of $\mathscr{PA}$ is isomorphic to the standard (or intended) interpretation.

5.1 Formalization of Elementary Number Theory

In *elementary number theory* or *arithmetic* one studies the properties of natural numbers with respect to addition and multiplication. In doing arithmetic one needs only a very restricted sub-language of English containing the following expressions:
1. The binary predicate or relation 'is equal to'.
2. The natural numbers: zero, one, two, three, and so on.
3. The functions of addition (plus) and multiplication (times).
4. Variables n, m for natural numbers. For instance, in: (n plus m) times (n plus m) equals (n times n) plus two times (n times m) plus (m times m).
5. The connectives 'if ..., then ...', 'and', 'or', 'not' and 'if and only if'. For instance, in: if n equals m, then (n times n) equals (m times m).
6. The quantifiers 'for all n, ...' and 'there is at least one n such that ...'. For in-

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stance, in: for all natural numbers n, n plus zero equals n. And in: there is a natural number n such that (n times n) equals n.

Below we present a formal language $\mathscr{L}$ (Language for Arithmetic), rich enough to express properties of addition and multiplication of natural numbers. This language should contain non-logical symbols for:
1. the equality relation,
2. the individual natural numbers, and
3. the addition and multiplication functions.

Instead of introducing an individual constant c_n for each individual natural number n, we can take only one individual constant c_0 together with a unary function symbol s, to be interpreted as the successor function. Then $s(c_0)$ can play the role of c_1, $s(s(c_0))$ can play the role of c_2, and so on.

Definition 5.1 (Formal Language $\mathscr{L}$ for Arithmetic).
Alphabet of $\mathscr{L}$:

non-logical symbols:	$\equiv$	binary predicate symbol
	c_0	individual constant
	s	unary function symbol
	$\oplus, \otimes$	binary function symbols
logical symbols:	$a_1, a_2, a_3, \ldots$	free individual variables
	$x_1, x_2, x_3, \ldots$	bound individual variables
	$\rightleftarrows$, $\rightarrow$, $\wedge$, $\vee$, $\neg$	connectives
	$\forall$, $\exists$	quantifiers
	$(,), [,]$	parentheses

Definition 5.2 (Standard model of arithmetic).
$\mathscr{N} = \langle \mathbb{N};\ =;\ 0;\ ',\ +,\ \cdot\ \rangle$ is the *intended interpretation* of $\mathscr{L}$, i.e., $\mathscr{N}$ interprets the individual variables as natural numbers (i.e., as elements of $\mathbb{N}$), the symbol $\equiv$ as the equality relation $=$ between natural numbers, the symbol c_0 as the natural number 0, the symbol s as the successor function $' : \mathbb{N} \to \mathbb{N}$, defined by $n' = n+1$, the symbol $\oplus$ as addition $+$ of natural numbers and the symbol $\otimes$ as multiplication $\cdot$ of natural numbers. The intended interpretation $\mathscr{N}$ of (the symbols in the formal language) $\mathscr{L}$ is also called the *standard model* for the formal language $\mathscr{L}$ or the *standard model of arithmetic*.

Warning: $\equiv$, c_0, s, $\oplus$ and $\otimes$ are just non-logical symbols in (the alphabet of) our object-language, which under different interpretations may get many different non-intended meanings: $\equiv$ might be interpreted as $<$ (less than), c_0 might be interpreted as 5, s might be interpreted as taking the square, $\oplus$ might be interpreted as exponentiation and so on. One should clearly distinguish between the symbols in our formal language, which under different interpretations may get many different meanings, and the intended interpretation of these symbols. c_0 is a symbol in the object-language and not (the name of) a natural number; 0 (zero), on the other hand, is the name of a natural number. Similarly, $\oplus$ is a function symbol, not a function; $+$ is (the name of) a function from $\mathbb{N}^2$ to $\mathbb{N}$, it is not a function symbol in the object-language. However, for reasons of easy notation, the following convention is adopted, mostly implicitly.

Convention: One uses $=$ instead of $\equiv$; 0 instead of c_0; $'$ instead of s; and $+$ and $\cdot$ instead of $\oplus$ and $\otimes$, respectively. So, the symbols =, 0, ', + and $\cdot$ are used in two ways: 'par abus de language' as symbols in the formal language $\mathscr{L}$ for arithmetic with many possible interpretations and as the intended interpretation of the corresponding symbols in the language $\mathscr{L}$.

Under this convention the *alphabet of* $\mathscr{L}$ contains the following symbols.

Symbols	**Name**	**Intended interpretation**
$=$	binary predicate symbol	equality
0	individual constant	zero
$'$	unary function symbol	successor function
$+\,;\,\cdot$	binary function symbols	addition; multiplication
$a_1, a_2, \ldots$	free individual variables	natural numbers
$x_1, x_2, \ldots$	bound individual variables	natural numbers
$\rightleftarrows$, $\rightarrow$, $\wedge$, $\vee$, $\neg$	connectives	
$\forall$, $\exists$	quantifiers	
$(,), [,]$	parentheses	

Definition 5.3 (Terms of $\mathscr{L}$).
The *terms* of the language $\mathscr{L}$ for formal arithmetic are defined as follows:
1. Each free individual variable a is a term.
2. 0 is a term.
3. If r and s are terms, then $(r)'$, $(r+s)$ and $(r \cdot s)$ are also terms.

If no confusion is possible, parentheses are omitted as much as possible.

Example 5.1. Examples of terms of $\mathscr{L}$: 0, a_1, $0+a_1$, $(0+a_1)\cdot a_1$, $a_1 \cdot a_2$, $0+a_1 \cdot a_1$, $0'' \cdot a_1 + a_2 \cdot a_3$.

Since there is only one predicate symbol in the alphabet, the atomic formulas in the language $\mathscr{L}$ for formal number theory are of the form $=(r,s)$, where r and s are terms. Instead of $=(r,s)$ one usually writes $r = s$.

Definition 5.4 (Atomic formulas of $\mathscr{L}$).
If r and s are terms, then $r = s$ is an *atomic formula* of the language $\mathscr{L}$ for formal number theory.

From these atomic formulas complex formulas can be built in the usual way by means of connectives and quantifiers:

Definition 5.5 (Formulas of $\mathscr{L}$).
1. Every atomic formula of $\mathscr{L}$ is a formula of $\mathscr{L}$.
2. If A and B are formulas of $\mathscr{L}$, then also $(A \rightleftarrows B)$, $(A \rightarrow B)$, $(A \wedge B)$, $(A \vee B)$ and $(\neg A)$ are formulas of $\mathscr{L}$.
3. If $A(a)$ is a formula of $\mathscr{L}$ and x is a bound individual variable, then also $\forall x[A(x)]$ and $\exists x[A(x)]$ are formulas of $\mathscr{L}$, where $A(x)$ results from $A(a)$ by replacing one or more occurrences of a in $A(a)$ by x.

English sentences about addition and multiplication of natural numbers can be translated into formulas of the language $\mathscr{L}$ for formal number theory. Here are some examples:
(i) For all natural numbers n, m, (n plus m) times (n plus m) equals (n times n) plus two times (n times m) plus (m times m): $\forall x \forall y[(x+y)\cdot(x+y) = x\cdot x + 0'' \cdot x \cdot y + y \cdot y]$.
(ii) For all natural numbers n, m, if n equals m, then n square equals m square: $\forall x \forall y[x = y \rightarrow x \cdot x = y \cdot y]$.
(iii) For all natural numbers n, n plus zero equals n: $\forall x[x+0 = x]$.
(iv) There is at least one natural number n such that n square equals n: $\exists x[x \cdot x = x]$.

Now consider the formula $\forall x[x+0 = x]$, or rather $\forall x[x \oplus c_0 \equiv x]$. This formula is true under the intended interpretation $\mathscr{N}$, in other words $\mathscr{N} \models \forall x[x+0 = x]$, but this formula is not under every interpretation true. For instance, let M be the structure $\langle \mathbb{Q};\ >;\ 5,\ ',\ -,\ \cdot \rangle$, i.e., M has the set of rational numbers as domain, interprets $\equiv$ as 'is greater than ($>$)', c_0 as 5, and $\oplus$ as subtraction ($-$). Under this interpretation $\forall x[x \oplus c_0 \equiv x]$ reads as follows: for all rational numbers x, $x - 5 > x$; and this happens to be false. Therefore, $M \not\models \forall x[x \oplus c_0 \equiv x]$. So, although $\forall x[x \oplus c_0 \equiv x]$ is true under the intended interpretation, it is not always true,, i.e., not under every interpretation true, in other words $\not\models \forall x[x \oplus c_0 \equiv x]$.

Of course, $\models \forall x[x \oplus c_0 \equiv x \vee \neg(x \oplus c_0 \equiv x)]$, i.e., $\forall x[x \oplus c_0 \equiv x \vee x \oplus c_0 \not\equiv x]$ is true in every interpretation. The validity of this formula rests upon the fixed meaning of the connectives and quantifiers, which for that reason are called *logical symbols*. The symbols $\equiv$, c_0, s, $\oplus$ and $\otimes$ are called *non-logical symbols*, because they do not belong to logic but come from mathematics; their meaning can vary depending on the context, in other words, they allow many different interpretations.

Since valid patterns of reasoning should be applicable universally, i.e., in any domain, mathematics, physics, economics or whatever, in logic we are interested in valid formulas, i.e., in formulas which are always true, in other words, which yield a true proposition in every interpretation of the non-logical symbols occurring in them. *But in elementary number theory (arithmetic) we are* of course *only interested in the intended interpretation*, and not in all possible interpretations.

Notice that $\mathscr{N} \models (a+1)\cdot(a+1) = a\cdot a + 2\cdot a + 1$, in other words, $\mathscr{N} \models \forall x[(x+1)\cdot(x+1) = x\cdot x + 2\cdot x + 1]$, because for every $n \in \mathbb{N}$, $\mathscr{N} \models (a+1)\cdot(a+1) = a\cdot a + 2\cdot a + 1\ [a/n]$. But $\mathscr{N} \not\models (a+1)\cdot(a+1) = 4$, because, for instance, $\mathscr{N} \not\models (a+1)\cdot(a+1) = 4\ [a/2]$, although $\mathscr{N} \models (a+1)\cdot(a+1) = 4\ [a/1]$.

So far we have introduced a (first-order) formal language $\mathscr{L}$ for elementary number theory, in which propositions about addition and multiplication of natural numbers can be formulated. The next step is to select a number of arithmetic (non-logical) axioms, formulated in this language, in order to be able to deduce formally properties of natural numbers. To that purpose Guiseppe Peano formulated in 1891 the following set $\mathscr{P}$ of arithmetic axioms, named after him.

The Peano axioms are formulas in the formal language $\mathscr{L}$ for elementary number theory and these axioms are true in the intended interpretation. The induction axiom *schema* yields an induction axiom for any formula A in the language $\mathscr{L}$.

Definition 5.6 (Axioms of Peano).

$\forall x \forall y \forall z[x = y \rightarrow (x = z \rightarrow y = z)]$	axiom for $=$
$\forall x \forall y[x' = y' \rightarrow x = y]$	
$\forall x \forall y[x = y \rightarrow x' = y']$	axioms for $'$
$\forall x[\neg(x' = 0)]$	
$\forall x[x + 0 = x]$	axioms for $+$
$\forall x \forall y[x + y' = (x + y)']$	
$\forall x[x \cdot 0 = 0]$	axioms for $\cdot$
$\forall x \forall y[x \cdot y' = x \cdot y + x]$	
$A(0) \wedge \forall x[A(x) \rightarrow A(x')] \rightarrow \forall x[A(x)]$	induction axiom *schema*

Now let $\mathscr{P}$ be the set of the axioms of Peano. One can verify that, for instance, $\mathscr{P} \vdash \forall x[x = x]$ and $\mathscr{P} \vdash \forall x \forall y[x + y = y + x]$ (see Exercise 5.1). And from experience we know that any formula which is true in the intended interpretation and *which one encounters in practice* can be formally deduced from $\mathscr{P}$. In fact, in [4], Sections 38-40, S.C. Kleene formally deduces a great number of such formulas from $\mathscr{P}$.

By the completeness theorem (for the predicate logic with equality) we know that for any formula A in $\mathscr{L}$, $\mathscr{P} \vdash A$ iff $\mathscr{P} \models A$, i.e., $\mathscr{P} \vdash A$ iff every interpretation that makes $\mathscr{P}$ true also makes A true, in other words, $\mathscr{P} \vdash A$ iff every model of $\mathscr{P}$ is also a model of A. In particular: if $\mathscr{P} \vdash A$, then A is true in the standard interpretation, in other words, if $\mathscr{P} \vdash A$, then $\mathscr{N} \models A$. But the question arises if the following holds:

$$\mathscr{P} \vdash A \text{ iff } A \text{ is true in the intended interpretation } \mathscr{N}\text{, i.e.,}$$
$$\mathscr{P} \vdash A \text{ iff } \mathscr{N} \models A.$$

In Section 5.2 it will be made clear that this is not the case. Even worse, there is no consistent and axiomatizable extension Γ of $\mathscr{P}$ such that any formula A in $\mathscr{L}$ which is true in the intended interpretation can be formally deduced from Γ. This is Gödel's first *incompleteness theorem* (for formal number theory; 1931).

Summarizing: In this Section we have given a *formalization* of elementary number theory (arithmetic). That is:
1. We have introduced a formal language $\mathscr{L}$ for elementary number theory in which we can express properties of natural numbers with respect to addition and multiplication.
2. We have introduced an axiom system $\mathscr{P}$ for (formal) number theory in order to be able to deduce formally formulas from $\mathscr{P}$ which are true in the intended interpretation.

The result is called *formal number theory*, consisting of two components: the formal language $\mathscr{L}$ and the axioms $\mathscr{P}$ of Peano. For any formula A,

if $\mathscr{P} \vdash A$, then A is true in the intended interpretation, i.e., $\mathscr{N} \models A$.

But according to Gödel's incompleteness theorem (1931), the converse,

if A is true in the intended interpretation, then $\mathscr{P} \vdash A$
is not for all formulas A in $\mathscr{L}$ true.

Therefore, Gödel's incompleteness theorem says that the proof power of $\mathscr{P}$ is restricted; more generally, that the proof power of any consistent and axiomatizable extension Γ of $\mathscr{P}$ is restricted.

Exercise 5.1. Prove: a) $\mathscr{P} \vdash \forall x[x = x]$; b) $\mathscr{P} \vdash \forall y \forall x[x' + y = (x+y)']$;
c) $\mathscr{P} \vdash \forall x \forall y[x + y = y + x]$.

5.2 Gödel's first Incompleteness Theorem

Definition 5.7 (Consistency). Let Γ be a set of formulas (in $\mathscr{L}$ or in any other language). Γ is *consistent* := there is no formula A such that $\Gamma \vdash A$ and $\Gamma \vdash \neg A$. (1)

Theorem 5.1. *Γ is consistent iff*
there is some formula A such that not $\Gamma \vdash A$ iff (2)
Γ is satisfiable. (3)

Proof. (1) implies (2), by the completeness theorem (2) implies (3), and (3) implies (1). □

Definition 5.8 (Axiomatizable). Let Γ be a set of formulas (in $\mathscr{L}$ or in any other language). Γ is *axiomatizable* := there is a subset Γ' of Γ such that:
1. Γ' is decidable, i.e., there is a decision method which decides for any formula A in the language whether A is in Γ' or A is not in Γ', and
2. for any formula A in the language, $\Gamma' \vdash A$ iff $\Gamma \vdash A$.
The elements of Γ' are called *axioms* for Γ.

The hope that any formula in $\mathscr{L}$ which is true under the intended interpretation, can be formally deduced from Peano's axioms, was dashed in 1931 by the incompleteness theorem of Kurt Gödel.

Theorem 5.2 (First Incompleteness Theorem for Arithmetic).
Let Γ be a consistent and axiomatizable extension of $\mathscr{P}$. Then there is a closed formula A_Γ (depending on Γ) in $\mathscr{L}$ such that
1. *A_Γ is true in the intended interpretation, i.e., $\mathscr{N} \models A_\Gamma$, but*
2. *not $\Gamma \vdash A_\Gamma$, and*
3. *not $\Gamma \vdash \neg A_\Gamma$.*

2 and 3 together say that A_Γ is undecidable on the basis of Γ; i.e., the proof power of any consistent and axiomatizable extension Γ of $\mathscr{P}$ is restricted.

Of course, Gödel's incompleteness theorem does not hold if we take for Γ the set of all formulas in $\mathscr{L}$ which are true in the intended interpretation. But this set cannot be seen as an axiom system, more precisely, it is not axiomatizable.

Gödel's incompleteness theorem says that, given any consistent and axiomatizable extension Γ of $\mathscr{P}$, not every formula which is true in the intended interpretation can be formally deduced from Γ. Given any such Γ, the truth of A_Γ (in the standard model) can be seen semantically, but A_Γ cannot be formally deduced from Γ.

Since the set $\mathscr{P}$ of Peano's axioms satisfies the conditions in Theorem 5.2, Gödel's incompleteness theorem says in particular that there is a formula A_1 which is true in the intended interpretation, but which cannot be deduced from $\mathscr{P}$ (not $\mathscr{P} \vdash A_1$). Because A_1 is true in the intended interpretation, we might extend $\mathscr{P}$ with the formula A_1 to the set $\mathscr{P} \cup \{A_1\}$. But then, taking $\Gamma = \mathscr{P} \cup \{A_1\}$, Gödel's incompleteness theorem says that there is a formula A_2, depending on $\mathscr{P} \cup \{A_1\}$, such that A_2 is true in the intended interpretation and not $\mathscr{P}, A_1 \vdash A_2$. In a similar way we can find a formula A_3 such that A_3 is true in the intended interpretation and such that not $\mathscr{P}, A_1, A_2 \vdash A_3$, and so on.

Sketch of proof of Gödel's first incompleteness theorem

A detailed proof of Gödel's incompleteness theorem requires many pages. See, for instance, Kleene [4], Boolos, Burgess and Jeffrey [2], Smith [8], Nagel [5]. However, the heart of the proof can be explained in a few lines, if we postulate in addition that the formulas in Γ are true in $\mathscr{N}$, which only slightly strengthens the condition that Γ is consistent. The formula A_Γ in the language $\mathscr{L}$ for formal number theory, which is constructed given a set Γ satisfying the conditions of Theorem 5.2, means that A_Γ is not formally deducible from Γ; more precisely:

$$A_\Gamma \text{ is true (in the intended interpretation) if and only if not } \Gamma \vdash A_\Gamma. \qquad (*)$$

Hence, A_Γ is a sentence in $\mathscr{L}$ that says of itself that it is not deducible from Γ.

Now suppose A_Γ were false (in the intended interpretation). Then it follows from $(*)$ that $\Gamma \vdash A_\Gamma$. Because of the Soundness Theorem it follows that $\Gamma \models A_\Gamma$ and because Γ is supposed to be true (in the intended interpretation), it follows that A_Γ is true (in the intended interpretation). Contradiction. Therefore, A_Γ is not false, and hence true, in the intended interpretation. And hence it follows from $(*)$ that not $\Gamma \vdash A_\Gamma$.

Because A_Γ is true, $\neg A_\Gamma$ is false (in the intended interpretation). Now suppose $\Gamma \vdash \neg A_\Gamma$. Then by soundness, $\Gamma \models \neg A_\Gamma$. So, assuming that $\mathscr{N} \models \Gamma$, it would follow that $\mathscr{N} \models \neg A_\Gamma$, i.e., $\neg A_\Gamma$ is true in the intended interpretation. Contradiction. Therefore, not $\Gamma \vdash \neg A_\Gamma$. □

Corollary 5.1. *There exists a model of Peano's arithmetic $\mathscr{P}$ that is not the standard model $\mathscr{N}$.*

Proof. Since $\mathscr{P} \nvdash A_{\mathscr{P}}$, we know by the completeness theorem for predicate logic that $\mathscr{P} \not\models A_{\mathscr{P}}$, i.e., there is a model M of $\mathscr{P}$ such that $M \not\models A_{\mathscr{P}}$. However, the standard model $\mathscr{N}$ is a model of $A_{\mathscr{P}}$, in other words, $A_{\mathscr{P}}$ is true in the intended interpretation. Therefore, M cannot be the standard model $\mathscr{N}$. □

5.2.1 Gödel-numbering

However, it still costs a lot of energy, given any Γ satisfying the conditions of Theorem 5.2, to construct a (closed) formula A_Γ in $\mathscr{L}$ satisfying the property $(*)$. The key idea is the *Gödel-numbering* of the symbols (letters) in the alphabet of $\mathscr{L}$, of the terms and of the formulas in the language $\mathscr{L}$ for formal number theory. Each symbol in the alphabet for formal number theory can be identified with a natural number, called the *Gödel-number* of that symbol. Different symbols are identified with different Gödel-numbers. For example, if we replace the free individual variables $a_1, a_2, \ldots$ by $a, (|, a), (|, (|, a)), \ldots$ respectively, then we can take the following correlation (identification) of natural numbers with the symbols in $\mathscr{L}$:

$\rightarrow$	$\wedge$	$\vee$	$\neg$	$\forall$	$\exists$	$=$	$+$	$\cdot$	$'$	0	a	$\vert$
3	5	7	9	11	13	15	17	19	21	23	25	27

Many other correlations are possible. There is nothing special about our particular Gödel-numbering.

A *Gödel-numbering* assigns to symbols, terms, formulas and deductions a natural number, called the *Gödel-number* of the expression, such that:

(i) it assigns different Gödel-numbers to different expressions;

(ii) the Gödel-number of any expression is effectively calculable;

(iii) one can effectively decide whether a natural number is the Gödel-number of some expression, and, if so, of what expression.

If A is an expression with Gödel-number n, we define $\ulcorner A \urcorner$ to be the expression $\bar{n}$, the *numeral* for n; $\bar{0} := 0,\ \bar{1} := 0',\ \bar{2} := 0'', \ldots$; so, $\bar{n}$ is the term in $\mathscr{L}$ that corresponds to the natural number n.

Terms and formulas of $\mathscr{L}$ are finite sequences of symbols of (the alphabet of) $\mathscr{L}$ formed according to certain rules and hence they can be identified with finite sequences of natural numbers. And in its turn each finite sequence $k_1, \ldots, k_n$ of natural numbers can be identified with another natural number, for instance, with $p_1^{k_1} \cdots p_n^{k_n}$, where $p_1, \ldots, p_n$ are the first n prime numbers. Then the individual variable a_1, that is $(|, a)$, is identified with $2^{27} \cdot 3^{25}$ and the atomic formula $a_1 = 0$, that is $= (a_1, 0)$, is then identified with the natural number $2^{15} \cdot 3^{2^{27} \cdot 3^{25}} \cdot 5^{23}$.

Given a specific Gödel-numbering, if n is the Gödel-number of some formula, let $A_n(a)$ be the formula with Gödel-number n, so $\ulcorner A_n(a) \urcorner = \bar{n}$.

Now let Γ be a set of arithmetic axioms formulated in $\mathscr{L}$. Then a formal deduction of A from Γ is a finite sequence of formulas in $\mathscr{L}$, constructed according to certain rules, and hence can be identified with a finite sequence of natural numbers, and therefore with a natural number.

By correlating to different formal objects different natural numbers and by talking about the correlated natural numbers instead of the formal objects themselves, the meta-mathematical predicate '$A(a)$ is a formula, k is a natural number and b is a formal deduction of $A(\bar{k})$ from Γ' can be rendered by an arithmetical predicate $Ded_\Gamma(n, k, m)$ saying:

n is the Gödel-number of a formula, namely $A_n(a)$, and m is the Gödel-number of a formal deduction of $A_n(\bar{k})$ from Γ.

So, using the Gödel-numbering, meta-mathematics becomes part of arithmetic. $\Gamma \vdash A_n(\bar{k})$ if and only if there is a natural number m such that $Ded_\Gamma(n,k,m)$.

Now consider the arithmetic predicate $Ded_\Gamma(n,n,m)$, which expresses: m is the Gödel-number of a formal deduction of $A_n(\bar{n})$ from Γ.

In section 52 of [4] S.C. Kleene proves that there is a formula $DED_\Gamma(a,a_1)$ of $\mathscr{L}$, such that for all natural numbers n,m:
(i) if $Ded_\Gamma(n,n,m)$ is true, then $\Gamma \vdash DED_\Gamma(\bar{n},\bar{m})$, and
(ii) if $Ded_\Gamma(n,n,m)$ is false, then $\Gamma \vdash \neg DED_\Gamma(\bar{n},\bar{m})$.

In order to prove (i) and (ii), one uses the supposition that Γ contains the axioms of Peano and that Γ is axiomatizable.

Next consider the formula $\neg\exists y[DED_\Gamma(a,y)]$, having a as the only free variable. This formula has a Gödel-number, say p, and hence equals $A_p(a)$ according to the notation introduced before.

Finally, consider the formula

$$A_\Gamma := A_p(\bar{p}) : \ \neg\exists y[DED_\Gamma(\bar{p},y)].$$

Then it holds that A_Γ is true in the intended interpretation if and only if there is no formal deduction of the formula $A_p(\bar{p})$ from Γ. But this latter formula $A_p(\bar{p})$ is A_Γ itself! Therefore:

$$A_\Gamma \text{ is true (in the intended interpretation) if and only if not } \Gamma \vdash A_\Gamma \qquad (*)$$

So, using the Gödel-numbering, it is possible to construct a formula A_Γ of $\mathscr{L}$, which says of itself that it cannot be deduced from Γ.

Now it is easy to see that if Γ satisfies the conditions in Theorem 5.2, then not $\Gamma \vdash A_\Gamma$ and hence, by $(*)$, A_Γ is true (in the intended interpretation). For suppose Γ is consistent and $\Gamma \vdash A_\Gamma$. Let k be the Gödel-number of a formal deduction of A_Γ from Γ. Then $Ded_\Gamma(p,p,k)$ is true. So it follows from (i) that $\Gamma \vdash DED_\Gamma(\bar{p},\bar{k})$. Therefore $\Gamma \vdash \exists y[DED_\Gamma(\bar{p},y)]$. But we supposed that $\Gamma \vdash A_\Gamma$, i.e., $\Gamma \vdash \neg\exists y[DED_\Gamma(\bar{p},y)]$. Contradiction with the consistency of Γ. Therefore, if Γ is consistent, then not $\Gamma \vdash A_\Gamma$. And then according to $(*)$, A_Γ is true (in the intended interpretation).

This finishes our sketch of the proof of Gödel's first incompleteness theorem. For further details the reader is referred to section 42 and Chapter X of Kleene [4]. For a popular exposition of Gödel's work see Nagel and Newman [5], Hofstadter [3], and Smullyan [9].

Remark 5.1. The Liar's paradox results from considering a sentence A which says of itself that it is not true. By replacing 'A is not true' by 'A is not deducible from Γ', Gödel escapes a paradox and finds a deep philosophical insight instead.

Remark 5.2. In his proof of the incompleteness theorem, K. Gödel constructs – given any Γ satisfying the hypotheses of the theorem – a formula A_Γ, which in the intended interpretation says of itself that it is not deducible from Γ. By thinking *about* Γ and A_Γ, we then see that not $\Gamma \vdash A_\Gamma$ and hence that A_Γ is true (in the intended interpretation). The proof of Gödel's incompleteness theorem is – although very long and technically very smart – in essence very elementary. One can raise no objections against it which would not be at the same time objections against parts of traditional mathematics, which are generally considered to be unproblematic.

Remark 5.3. The formula A_Γ refers to itself, because it says about itself that it is not deducible from Γ. Such sentences are not of particular interest for mathematicians. However, Paris and Harrington [6] gave a strictly mathematical example of an incompleteness in first-order Peano arithmetic, which is mathematically simple and interesting and which does not require a numerical coding of logical notions.

Remark 5.4. From the definition of $\mathscr{P} \models A$ it follows immediately that for any formula A, if $\mathscr{P} \models A$, then A is true in the intended interpretation. (α)
By Gödel's completeness theorem for the predicate logic, $\mathscr{P} \models A$ iff $\mathscr{P} \vdash A$. Therefore, by Gödel's incompleteness theorem for formal number theory, the converse of (α) does not hold, i.e., *not* for every formula A, if A is true in the intended interpretation, then $\mathscr{P} \models A$.

5.2.2 Provability predicate for $\mathscr{P}$

If A is a formula of the formal language $\mathscr{L}$ for arithmetic (see Section 5.1) with Gödel-number n, we define $\ulcorner A \urcorner$ to be the expression $\bar{n}$, the numeral for n; $\bar{1} = 0'$, $\bar{2} = 0''$, etc.

We shall assume, but not prove, the following FACT: By 'straightforwardly transcribing' in $\mathscr{L}$ the definition of *being deducible from* $\mathscr{P}$, where $\mathscr{P}$ is the set of Peano's axioms for arithmetic, making reference to Gödel-numbers instead of expressions, one can construct a formula $Prov(a)$ of $\mathscr{L}$, with the following properties:
(a) $Prov(a)$ expresses that a is the Gödel-number of a formula which is deducible from $\mathscr{P}$, and
(b) $Prov(a)$ is a *provability predicate* for $\mathscr{P}$, i.e.,

(i) if $\mathscr{P} \vdash A$, then $\mathscr{P} \vdash Prov(\ulcorner A \urcorner)$;
(ii) $\mathscr{P} \vdash Prov(\ulcorner B \to C \urcorner) \to (Prov(\ulcorner B \urcorner) \to Prov(\ulcorner C \urcorner))$;
(iii) $\mathscr{P} \vdash Prov(\ulcorner A \urcorner) \to Prov(\ulcorner Prov(\ulcorner A \urcorner) \urcorner)$.

(c) In addition,

(iv) if $\mathscr{P} \vdash Prov(\ulcorner A \urcorner)$, then $\mathscr{P} \vdash A$.

That $Prov(a)$ satisfies (i) may be seen as follows: Suppose $\mathscr{P} \vdash A$. Then there is a formal proof of A from $\mathscr{P}$. Let $\ulcorner A \urcorner$ be the Gödel number of A. Then the formula $Prov(\ulcorner A \urcorner)$ expresses that $\ulcorner A \urcorner$ is the Gödel number of a formula which is deducible from $\mathscr{P}$. Then $\mathscr{P} \vdash Prov(\ulcorner A \urcorner)$. (ii) A deduction of C can be obtained from deductions of B and of $B \to C$ by one more application of Modus Ponens. This argument can be formalized in $\mathscr{P}$. Showing that $Prov(a)$ satisfies (iii) is much harder: it involves showing that the argument that $Prov(a)$ satisfies (i) can be formalized in $\mathscr{P}$. To show (iv), suppose that $\mathscr{P} \vdash Prov(\ulcorner A \urcorner)$. Then $Prov(\ulcorner A \urcorner)$ is true in $\mathscr{N}$. Hence A is deducible from $\mathscr{P}$.

However, $Prov(a)$ does NOT meet the stronger condition $\mathscr{P} \vdash Prov(\ulcorner A \urcorner) \to A$. Löb's theorem says that if $\mathscr{P} \vdash Prov(\ulcorner A \urcorner) \to A$, then $\mathscr{P} \vdash A$.

For more details the reader is referred to Boolos and Jeffrey [1], Chapter 16, or to Boolos, Burgess and Jeffrey [2], Chapter 18.

There is an interesting connection between the provability predicate for arithmetic and the necessitation operator $\Box$ of a particular modal logic *GL* (Gödel's modal Logic); see Section 6.12.

5.3 Gödel's second Incompleteness Theorem

Theorem 5.3 (Second Incompleteness Theorem for Arithmetic). *Let Γ be a consistent and axiomatizable extension of $\mathscr{P}$. Let $Cons_\Gamma$ be a formula in $\mathscr{L}$, expressing the consistency of Γ. Then not $\Gamma \vdash Cons_\Gamma$.*

Gödel's second incompleteness theorem says that the consistency of Γ – provided that Γ satisfies the conditions mentioned above – cannot be proved by means which are available in Γ itself.

Since the standard model $\mathscr{N}$ is a model of the axioms $\mathscr{P}$ of Peano, we know that $\mathscr{P}$ is consistent. By Gödel's second theorem, the consistency proof for $\mathscr{P}$ just given cannot be formalized in $\mathscr{P}$ itself.

First we have to construct a formula $Cons_\Gamma$ in $\mathscr{L}$ expressing the consistency of Γ. Because $\mathscr{P} \subseteq \Gamma$, $\Gamma \vdash \neg(0 = 1)$. Consequently, Γ is consistent if and only if not $\Gamma \vdash 0 = 1$. Now let k be the Gödel-number of the formula $0 = 1$; therefore, $A_k(a)$ is the formula $0 = 1$ and $A_k(\bar{k})$ is the same formula, since a does not occur in $0 = 1$. The consistency of Γ can be expressed in $\mathscr{L}$ by the formula $\neg\exists y[DED_\Gamma(\bar{k}, y)]$: there is no y such that y is the Gödel-number of a formal deduction of $A_k(\bar{k})$, i.e., $0 = 1$, from Γ. Let $Cons_\Gamma := \neg\exists y[DED_\Gamma(\bar{k}, y)]$. Then $Cons_\Gamma$ is a formula in $\mathscr{L}$ expressing the consistency of Γ.

Proof (of Gödel's second theorem). Let Γ be an axiomatizable extension of $\mathscr{P}$. In Gödel's first incompleteness theorem we have shown informally:

(I) if Γ is consistent, then not $\Gamma \vdash A_\Gamma$, where A_Γ is the formula $A_p(\bar{p})$.

The statement that A_Γ is not deducible from Γ is expressed via the Gödel-numbering by $\neg\exists y[DED_\Gamma(\bar{p}, y)]$, this is A_Γ itself. The statement that Γ is consistent is expressed by the formula $Cons_\Gamma$. Because the informal proof of (I) is so elementary, it can be completely formalized in $\mathscr{P}$ via the Gödel-numbering, and hence in Γ. Therefore,

$$\text{(II) } \Gamma \vdash Cons_\Gamma \to A_\Gamma.$$

Now suppose that $\Gamma \vdash Cons_\Gamma$. Then it follows from (II) that $\Gamma \vdash A_\Gamma$. Supposing that Γ is also consistent, this is in contradiction to Gödel's first incompleteness theorem. Therefore, if Γ is a consistent and axiomatizable extension of $\mathscr{P}$, then not $\Gamma \vdash Cons_\Gamma$. $\Box$

5.3.1 Implications of Gödel's Incompleteness Theorems

In Chapter X, *Minds and Machines*, of his book *From Mathematics to Philosophy*, Hao Wang [10] discusses the implications of Gödel's incompleteness results with respect to the superiority of man over machine. In section 7 of this chapter Hao Wang presents as Gödel's opinion that the two most interesting rigorously proved results about minds and machines are:

> 1 The human mind is incapable of formulating (or mechanizing) all its mathematical intuitions. That is, if it has succeeded in formulating some of them, this very fact yields new intuitive knowledge, e.g., the consistency of this formalism. This fact may be called the 'incompletability' of mathematics. On the other hand, on the basis of what has been proved so far, it remains possible that there may exist (and even be empirically discoverable) a theorem-proving machine which in fact *is* equivalent to mathematical intuition, but cannot be *proved* to be so, nor even be proved to yield only *correct* theorems of finitary number theory.
>
> 2 The second result is the following disjunction: Either the human mind surpasses all machines (to be more precise: it can decide more number theoretical questions than any machine) or else there exist number theoretical questions undecidable for the human mind.
>
> Gödel thinks Hilbert was right in rejecting the second alternative. If it were true, it would mean that human reason is utterly irrational by asking questions it cannot answer, while asserting emphatically that only reason can answer them. Human reason would then be very imperfect ....

Wang also explains that Gödel considered the attempted proofs for the equivalence of mind and machines as fallacious. See also Searle [7].

5.4 Non-standard Models of Peano's Arithmetic

Let $\mathscr{N}$ be the intended interpretation or standard model of $\mathscr{L}$, the language for formal number theory, i.e., $\mathscr{N} := \langle \mathbb{N}; =; 0, ', +, \cdot \rangle$. Trivially, $\mathscr{N}$ is a model of $\mathscr{P}$, Peano's axioms. But $\mathscr{N}$ is not the only model of $\mathscr{P}$. Given $\mathscr{N}$, one can construct another model of $\mathscr{P}$ that is *isomorphic* but not identical to $\mathscr{N}$ by 'replacing' some element in the domain $\mathbb{N}$ of $\mathscr{N}$ by another object that is not in $\mathbb{N}$. We leave it to the reader to verify that the same sentences are true in isomorphic interpretations.

We now wonder whether any two models of $\mathscr{P}$ (or of some axiomatizable and consistent extension Γ of $\mathscr{P}$) are isomorphic. In that case, one would say that $\mathscr{P}$ (or Γ) characterizes its models 'up to isomorphism' and that it has 'essentially' only one model. The following theorem answers this question in the negative.

Theorem 5.4. *Let Γ be a consistent and axiomatizable extension of $\mathscr{P}$. Then there are two non-isomorphic models of Γ, both with enumerably infinite domains. (In other words, Γ is not aleph-null-categorical).*

Proof. Let Γ be a consistent and axiomatizable extension of $\mathscr{P}$. By Gödel's first incompleteness theorem, there is a sentence A_Γ such that A_Γ is true in $\mathscr{N}$, $\Gamma \not\vdash A_\Gamma$

and $\Gamma \nvdash \neg A_\Gamma$. By Gödel's completeness theorem (for predicate logic), it follows that $\Gamma \not\models A_\Gamma$ and $\Gamma \not\models \neg A_\Gamma$. Hence, there is a model M_1 of Γ such that $M_1 \models \neg A_\Gamma$ and there is a model M_2 of Γ such that $M_2 \models A_\Gamma$. By the Löwenheim-Skolem Theorem (for predicate logic), M_1 and M_2 may be assumed to have an enumerably infinite domain. Since $M_1 \models \neg A_\Gamma$ and $M_2 \models A_\Gamma$, M_1 and M_2 are non-isomorphic. □

Definition 5.9 (Non-standard model). Let M be an interpretation of the language $\mathscr{L}$ for formal number theory. M is a *non-standard model of arithmetic* := the same sentences are true in M as are true in $\mathscr{N}$, and M is not isomorphic to $\mathscr{N}$.

In Theorem 5.5 we prove the existence of non-standard models of arithmetic with enumerably infinite domains.

Theorem 5.5. *Let Δ be the set of all sentences of $\mathscr{L}$ that are true in $\mathscr{N}$. Then there is an interpretation M of $\mathscr{L}$ such that:*
1. M is a model of Δ,
2. M is not isomorphic to $\mathscr{N}$, and
3. M has an enumerably infinite domain.
1 and 2 say that M is a non-standard model of arithmetic. It follows that Δ is not aleph-null-categorical, i.e., it is not the case that any two models of Δ, which both have an enumerably infinite domain, are isomorphic.

Proof. Let Δ be the set of all sentences of $\mathscr{L}$ that are true in $\mathscr{N}$. Let $A_0, A_1, A_2, \ldots$ be an enumeration of all sentences in Δ. Now consider $\Delta' := \{A_0,\ a_1 \neq 0,\ A_1,\ a_1 \neq 0',\ A_2,\ a_1 \neq 0'', \ldots\}$. Then each finite subset of Δ' is simultaneously satisfiable. So, by the compactness theorem (for predicate logic), Δ' is simultaneously satisfiable in an enumerable domain. Say $M \models \Delta'\ [a_1^*]$, that is $M \models \Delta$ and $M \models a_1 \neq 0\ [a_1^*]$, $M \models a_1 \neq 0'\ [a_1^*]$, $M \models a_1 \neq 0''\ [a_1^*]$, and so on.

For any natural numbers m,n, if $m \neq n$, then $\bar{m} \neq \bar{n}$ is in Δ, where $\bar{1} := 0'$, $\bar{2} := 0''$, etc. Since $M \models \Delta$, the domain of M is enumerably infinite.

The element a_1^* in the domain of M is not the denotation in M of $\bar{n}$ for any natural number n, while in any interpretation isomorphic to $\mathscr{N}$ every element in the domain is denoted by $\bar{n}$ for some natural number n. Hence, M is not isomorphic to $\mathscr{N}$. □

In Chapter 17 of [1], Boolos and Jeffrey investigate what non-standard models of arithmetic do look like.

5.4.1 *Second-order Logic (continued)*

In Subsection 4.5.3 on second-order logic we have already seen that the Löwenheim-Skolem theorem fails for second-order logic. In this subsection we will indicate other important differences between first- and second-order logic with respect to arithmetic.

First of all, in Theorem 5.5 we have seen that arithmetic (i.e., the set of sentences of $\mathscr{L}$ true in the standard model $\mathscr{N}$) has at least one model which is not isomorphic to $\mathscr{N}$. Below we will show that there is a single sentence, $\mathscr{PA}$, of second-order logic such that any model of $\mathscr{PA}$ is isomorphic to $\mathscr{N}$.

Let *Ind* be the second-order sentence

$$\forall X[\, X(0) \wedge \forall x[X(x) \rightarrow X(x')] \rightarrow \forall x[X(x)]\,].$$

When interpreted over $\mathscr{N}$, *Ind* formalizes the principle of mathematical induction. Therefore, *Ind* is true in $\mathscr{N}$, interpreting $\forall X$ as 'for all subsets of $\mathbb{N}$'. All of the enumerably many induction axioms of $\mathscr{P}$ (Peano's axioms) are logical consequences of the one second-order sentence *Ind*. Now let $\mathscr{PA}$ be the conjunction of *Ind* and the finitely many axioms of Peano which are not an induction axiom. *Ind* and hence $\mathscr{PA}$ are second-order sentences.

Theorem 5.6. *If $M \models \mathscr{PA}$, then M is isomorphic to $\mathscr{N}$ (the standard model).*

Proof. Let $M = \langle D;\, =;\, e,\, s,\, p,\, t\,\rangle$ be a model of $\mathscr{PA}$, where $e,\, s,\, p$ and t are what M assigns to 0, $'$, $+$ and $\cdot$, respectively. Since M is a model of *Ind*, it follows that for any subset V of D

(†) if both e is in V and $s(d)$ is in V whenever d is in V (for all d in D), then $V = D$.

Define $h : \mathbb{N} \rightarrow D$ inductively by: $h(0) = e$, and $h(n') = s(h(n))$. In order to show that h is an isomorphism from $\mathscr{N}$ to M, we still have to prove:
a) h is a surjection from $\mathbb{N}$ to D,
b) h is an injection from $\mathbb{N}$ to D,
c) $h(m+n) = p(h(m), h(n))$, and
d) $h(m \cdot n) = t(h(m), h(n))$.

It is straightforward, but tedious, to prove b), c) and d), using the hypothesis of the theorem. We leave this as an exercise to the reader; or the reader may consult Chapter 18 of [1]. Here we restrict ourselves to the most crucial part of the proof, that is the proof of a). Note:
1) e is in the range of h,
2) if d is in the range of h, then $d = h(n)$ for some n, whence $h(n') = s(d)$, and so $s(d)$ is in the range of h.

It follows from (†) that the range of h equals D, i.e., h is a surjection. □

It is important to note that the proof above does not work for $\mathscr{P}$ instead of $\mathscr{PA}$, although the infinitely many induction axioms of $\mathscr{P}$ logically follow from *Ind*. The point is that 'd is in the range of h' cannot be expressed by any first-order formula A. There are more subsets of $\mathbb{N}$ than formulas in $\mathscr{L}$: there are only denumerably many formulas in $\mathscr{L}$, while there are uncountably many subsets of $\mathbb{N}$.

If $\mathscr{P} \models A$, then A is true in $\mathscr{N}$. But, by Gödel's first incompleteness theorem, the converse does not hold. However, any sentence A, which is true in $\mathscr{N}$, is a valid consequence of the second-order sentence $\mathscr{PA}$.

Corollary 5.2. *Suppose that A is a (first- or second-order) sentence of $\mathscr{L}$. Then $\mathscr{PA} \models A$ iff A is true in $\mathscr{N}$.*

Proof. The 'only if' part is trivial. So, suppose A is true in $\mathscr{N}$. We want to show: $\mathscr{PA} \models A$. So, let M be a model of $\mathscr{PA}$. Then, by Theorem 5.6, M is isomorphic to $\mathscr{N}$. Since A is true in $\mathscr{N}$, it follows that A is true in M. □

A further Corollary of Theorem 5.6 is that the compactness theorem fails for second-order logic: there is an enumerable, unsatisfiable set of sentences (at least one of them is second-order), every finite subset of which is satisfiable.

Corollary 5.3. *Let $\Gamma = \{\mathscr{PA},\ c \neq 0,\ c \neq 0',\ c \neq 0'', \ldots\}$, where c is an individual constant. Then every finite subset of Γ is satisfiable, but Γ itself is not satisfiable.*

Proof. One easily sees that every finite subset of Γ is satisfiable. Now suppose Γ itself were satisfiable. Let M' be a model of Γ and let M be like M', but assigning nothing to c. Then M is a model of $\mathscr{PA}$ and hence, by Theorem 5.6, M is isomorphic to $\mathscr{N}$. On the other hand, because all of $c \neq 0,\ c \neq 0',\ c \neq 0'', \ldots$ are true in M', M – having the same domain as M' – cannot be isomorphic to $\mathscr{N}$. Contradiction. Therefore Γ has no model. □

In Subsection 4.5.1 we have given an effective positive test for validity of first-order formulas. However, there is no effective positive test for validity of second-order sentences. The existence of such a test would imply that there is a decision procedure for truth in $\mathscr{N}$, which is not the case. For proofs of these results the reader is referred to Chapter 15 and 18 of [2].

5.5 Solutions

Solution 5.1. a) To show that $\mathscr{P} \vdash \forall x[x = x]$, we use the following abbreviations:
$A := \forall x \forall y \forall z[x = y \to (x = z \to y = z)]$
$B := \quad \forall y \forall z[a_1 + 0 = y \to (a_1 + 0 = z \to y = z)]$
$C := \qquad \forall z[a_1 + 0 = a_1 \to (a_1 + 0 = z \to a_1 = z)]$
$D := \qquad\quad a_1 + 0 = a_1 \to (a_1 + 0 = a_1 \to a_1 = a_1)$
Below we present a deduction of $\forall x[x = x]$ from Peano's axioms.
1. A; one of the axioms of Peano.
2. $A \to B$; one of the axioms of predicate logic.
3. B; Modus Ponens, 1, 2.
4. $B \to C$; one of the axioms of predicate logic.
5. C; Modus Ponens, 3, 4.
6. $C \to D$; one of the axioms of predicate logic.
7. D; Modus Ponens, 5, 6.
8. $\forall x[x + 0 = x]$; one of the axioms of Peano.
9. $\forall x[x + 0 = x] \to a_1 + 0 = a_1$; one of the axioms of predicate logic.
10. $a_1 + 0 = a_1$; Modus Ponens, 8, 9.
11. $a_1 + 0 = a_1 \to a_1 = a_1$; Modus Ponens, 7, 10.
12. $a_1 = a_1$; Modus Ponens, 10, 11.
13. $a_1 = a_1 \to (\text{axiom} \to a_1 = a_1)$; axiom schema 1.

14. axiom $\rightarrow a_1 = a_1$; Modus Ponens, 12, 13.
15. axiom $\rightarrow \forall x[x = x]$; $\forall$-rule, 14.
16. axiom.
17. $\forall x[x = x]$; Modus Ponens, 15, 16.

b) To show that $\mathscr{P} \vdash \forall y \forall x[x' + y = (x + y)']$. We use induction on y.
$y = 0$: $\forall x[x' + 0 = (x + 0)']$; from the definition of +: $x' + 0 = x'$ and $x + 0 = x$.
Induction hypothesis: $\forall x[x' + y = (x + y)']$. To show: $\forall x[x' + y' = (x + y')']$. Proof: $x' + y' := (x' + y)' =_{indhyp} ((x + y)')' := (x + y')'$.

c) To show that $\mathscr{P} \vdash \forall x \forall y[x + y = y + x]$. We use induction on x, using induction on y in the basis.
$x = 0$: To show $\forall y[0 + y = y + 0]$. We use induction on y:

$y = 0$: 0 + 0 = 0 + 0. Induction hypothesis: $0 + y = y + 0$.
To show: $0 + y' = y' + 0$. Proof: $0 + y' := (0 + y)' =_{indhyp} (y + 0)' := y'$.

Induction hypothesis: $\forall y[x + y = y + x]$. To show: $\forall y[x' + y = y + x']$.
Proof: $y + x' := (y + x)' =_{indhyp} (x + y)'$ and according to b) $(x + y)' = x' + y$. □

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