Natural Deduction
PL: Syntax & Parsing

Dr. 何明昕，He Mingxin, Max

CS104: program07 @ yeah.net

CS108: mxhe1 @ yeah.net

Email Subject：CS104/108 + *ID* + *Name*: *TOPIC*

Sakai: CS104/CS108-数理逻辑导论/(H)

Introduction to Mathematical Logic (I2ML)

Content

- Exercise 03 Assignment
- Natural Deduction for PL
- Syntax of PL
- Formula Parsing

Next Lecture:

- Semantics of PL and Truth Tables

Exercise 03: Reading (Assignment 1 to be launched …)

Record your time spent (in 0.1 hours) with brief tasks and durations in your learning log by hand writing!

Reading for lecture 3:

- textD-ch03-NaturalDeduction4PL.pdf (in 1 week)

  (textD: "Logic and Proof" is an open book online: https://avigad.github.io/logic_and_proof/)
- textB-ch01-1.3-PLasFormalLanguage.pdf (in 1 week)
- textB中文-ch01-命题逻辑.pdf (reference, in 3rd wk ~ 6th wk)

Assignment 1 is to be launched on sakai soon. ☺

Formal Language

Examples:

1. (Digital sequence understood by computer)

   0010101010000010111101000

2. (Programme Language, eg. Java or C)

   ```
   s = 1; i = n; while (i > 0) { s *= a; i--; }
   ```

3. (Propositional Logic)

   $$(\neg((p \vee q) \to p))$$

4. (First-Order Logic)

   $$\forall\epsilon\exists\delta\forall x(|x-a| < \delta \to |f(x)-c| < \epsilon)$$

5. (Modal Logic)

   $$\neg(\Diamond p) \leftrightarrow \Box(\neg p)$$

Formality versus Precision

A description is precise if it expresses as briefly as possible what is intended.

It is formal if it uses a language for which formal, meaning-preserving manipulation rules have been defined.

Formal descriptions can be very imprecise.

Precise descriptions can be very informal.

In practice, the attempt to be precise has priority over the attempt to be formal.

BUT in Math and Logic ...

After explaining to a student through various lessons and examples that:

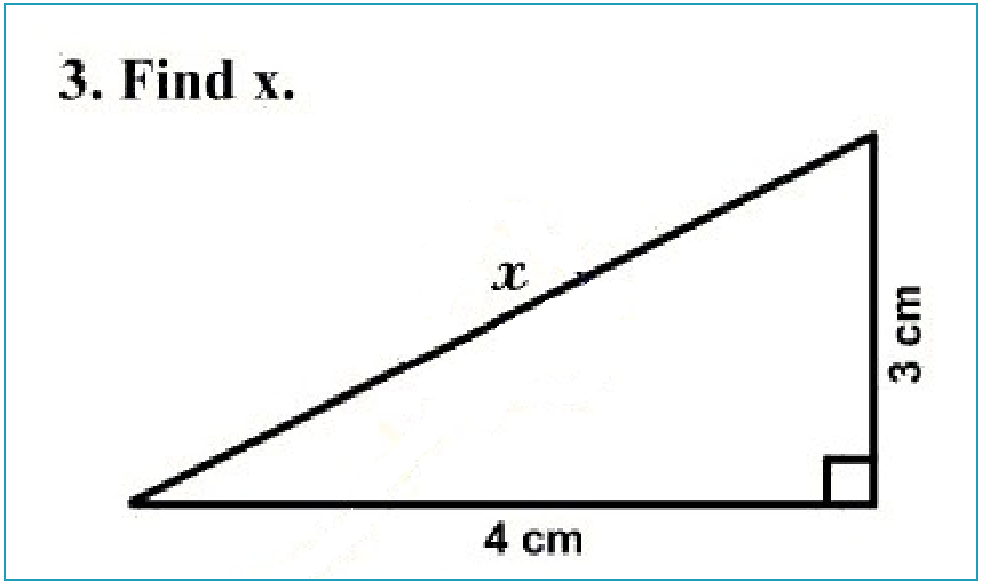
$$\lim_{x \to 8} \frac{1}{x-8} = \infty$$

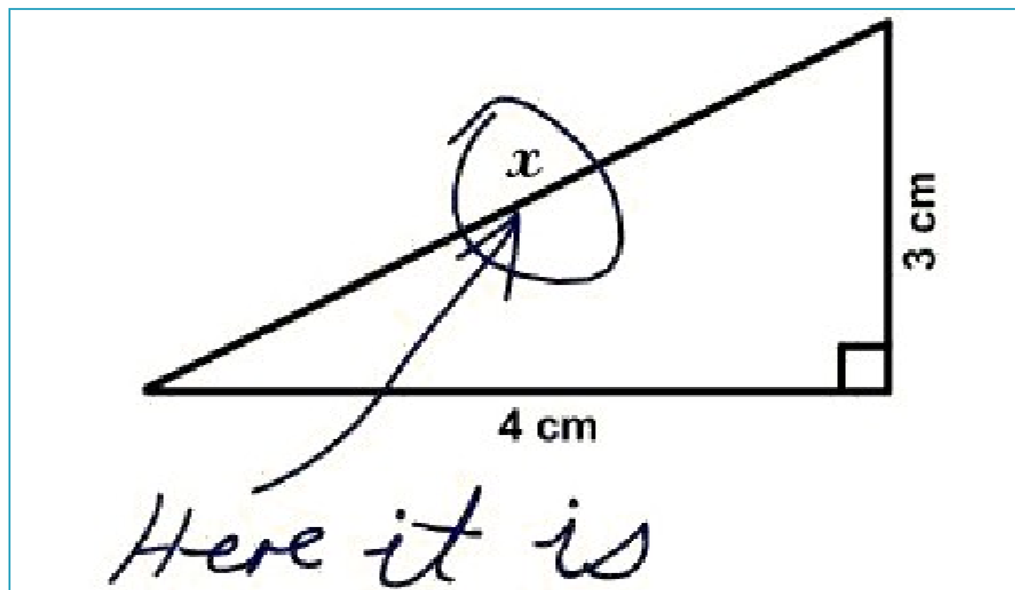
I gave a different example.

$$\lim_{x \to 5} \frac{1}{x-5} = \ ?$$

This was the result:

$$\lim_{x \to 5} \frac{1}{x-5} = 5$$


3. Find x.
x
3 cm
4 cm

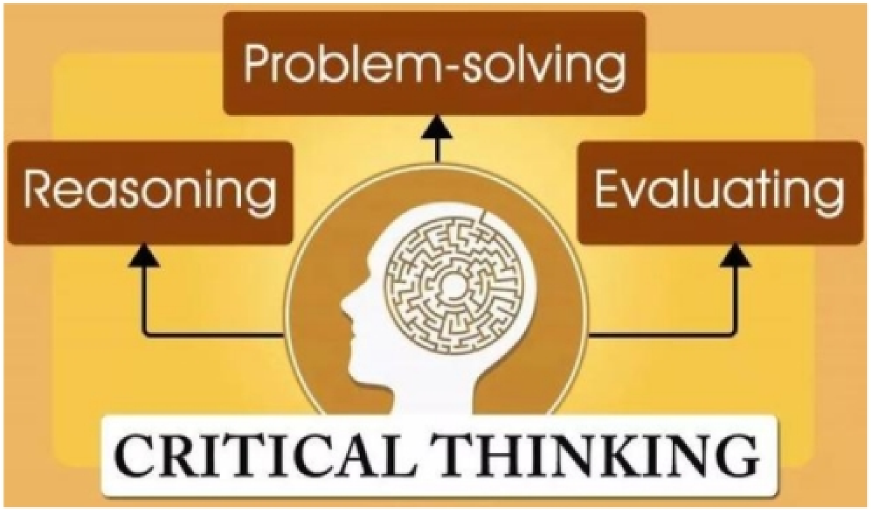

x
3 cm
4 cm
Here it is

$$\frac{1}{n}\sin x = ?$$

$$\frac{1}{\cancel{n}}\mathrm{si}\cancel{n}\, x =$$

$$six = 6$$

What does it mean?
What really matters?



- 批判性思维
  关键性思考
  建设性思考

- Critical Observing:
  建设性观察

- 系统化认知问题 与 构建解决方案的能力

Critical thinking is a mental process involving the evaluation of arguments or statements to determine whether they are valid, accurate, or logical.

It involves the ability to think analytically, identify and assess arguments, draw conclusions, and make decisions.

Critical thinking is essential in many areas, such as problem-solving, decision-making, creativity, and research.

C.J. Smith, Synonyms Discriminated. G. Bell and Sons, Ltd., 1926.

Precise denotes the quality of exact limitation, as distinguished from the vague, loose, doubtful, inaccurate; ... The idea of precision is that of casting aside the useless and superfluous.

Exactness is that kind of truth which consists in conformity to an external standard or measure, or has an internal correspondence with external requirement ... an exact mount is that which is required."

Accuracy, by contrast, refers to the attention spent upon a thing, and the exactness which may be expected from it. Accuracy is designed whereas exactness may be coincidental.

Correctness, finally, applies to what is conformable to a moral standard as well as to truth generally, as correct behavior.

C.J. Smith, Synonyms Discriminated. G. Bell and Sons, Ltd., 1926.

"It is most desirable that men should be

- exact in duties and obligations,
- accurate in statements and representations,
- correct in conduct,
- and precise in the use of words."

Omit needless words

W. Strunk Jr. & E.B. White, The Elements of Style. Fourth edition. Allyn and Bacon, 2000.

Rule 17

Vigorous writing is concise. A sentence should contain no unnecessary words, a paragraph no unnecessary sentences, for the same reason that a drawing should have no unnecessary lines and a machine no unnecessary parts.

This requires not that the writer make all sentences short, or avoid all detail and treat subjects only in outline, but that every word tell.

Derivations in Natural Deduction

Suppose we have a set of formulas $\varphi_1, \varphi_2, \varphi_3, \ldots, \varphi_n$, *which we will call premises, and another formula* ψ, *which we* will call a *conclusion.*

By applying proof rules to the premises, we hope to get some more formulas, and by applying more proof rules to those, to eventually obtain the conclusion. This intention we denote by

$$\boldsymbol{\varphi_1, \varphi_2, \varphi_3, \ldots, \varphi_n \vdash \psi}$$

This expression is called a *sequent* (矢列) *; it is valid if a proof for it can be found.*

Derivations in Natural Deduction

In natural deduction, every proof is a proof from hypotheses. In other words, in any proof, there is a finite set of hypotheses {B,C, . . .} and a conclusion A, and what the proof shows is that A follows from B,C, . . .

Like formulas, proofs are built by putting together smaller proofs, according to the rules. For instance, the way to read the and-introduction rule

$$\frac{A \qquad B}{A \wedge B}$$

is as follows: if you have a proof P1 of A from some hypotheses, and you have a proof P2 of B from some hypotheses, then you can put them together using this rule to obtain a proof of A ∧ B, which uses all the hypotheses in P1 together with all the hypotheses in P2.

Derivations in Natural Deduction

For example, this is a proof of (A ∧ B) ∧ (A ∧ C) from three hypotheses, A, B, and C:

$$\dfrac{\dfrac{A \qquad B}{A \wedge B} \qquad \dfrac{A \qquad C}{A \wedge C}}{(A \wedge B) \wedge (A \wedge C)}$$

In some presentations of natural deduction, a proof is written as a sequence of lines in which each line can refer to any previous lines for justification. But here we will adopt a rigid two-dimensional diagrammatic format in which the premises of each inference appear immediately above the conclusion. This makes it easy to look over a proof and check that it is correct: each inference should be the result of instantiating the letters in one of the rules with particular formulas.

Derivations in Natural Deduction

A proof of (A ∧ B) ∧ (A ∧ C) from three hypotheses, A, B, and C

The two-dimensional diagrammatic format

$$\dfrac{\dfrac{A \qquad B}{A \wedge B} \qquad \dfrac{A \qquad C}{A \wedge C}}{(A \wedge B) \wedge (A \wedge C)}$$

is equivalent to
the sequence :

1.	A	Premise
2.	B	Premise
3.	A ∧ B	∧I, 1,2
4.	A	Premise
5.	C	Premise
6.	A ∧ C	∧I, 4,5
7.	(A ∧B) ∧(A ∧ C)	∧I, 3,6

Derivations in Natural Deduction

The proof sequence:

1.	A	Premise
2.	B	Premise
3.	A ∧ B	∧I, 1,2
4.	A	Premise
5.	C	Premise
6.	A ∧ C	∧I, 4,5
7.	(A ∧B) ∧(A ∧ C)	∧I, 3,6

$$\dfrac{\dfrac{A \qquad B}{A \wedge B} \qquad \dfrac{A \qquad C}{A \wedge C}}{(A \wedge B) \wedge (A \wedge C)}$$

is also equivalent to the sequence :

1.	A	Premise
2.	B	Premise
3.	C	Premise
4.	A ∧ B	∧I, 1,2
5.	A ∧ C	∧I, 1,3
6.	(A ∧B) ∧(A ∧ C)	∧I, 4,5

Derivations in Natural Deduction

One thing that makes natural deduction confusing is that when you put together proofs in this way, hypotheses can be eliminated, or, as we will say, canceled. For example, we can apply the implies-introduction rule to the last proof, and obtain the following proof of B → (A ∧ B) ∧ (A ∧ C) from only two hypotheses, A and C:

$$\dfrac{\dfrac{\dfrac{A \qquad \overline{B}^{\,1}}{A \wedge B} \qquad \dfrac{A \qquad C}{A \wedge C}}{(A \wedge B) \wedge (A \wedge C)}}{B \to (A \wedge B) \wedge (A \wedge C)}\,1$$

Here, we have used the label 1 to indicate the place where the hypothesis B was canceled. Any label will do, though we will tend to use numbers for that purpose.

Derivations in Natural Deduction

We can continue to cancel the hypothesis A:

$$
\cfrac{\cfrac{\cfrac{\cfrac{\overline{A}^{\,2} \qquad \overline{B}^{\,1}}{A \wedge B} \qquad \cfrac{\overline{A}^{\,2} \qquad C}{A \wedge C}}{(A \wedge B) \wedge (A \wedge C)}\;1}{B \rightarrow (A \wedge B) \wedge (A \wedge C)}\;2}{A \rightarrow (B \rightarrow (A \wedge B) \wedge (A \wedge C))}
$$

The result is a proof using only the hypothesis C.

Derivations in Natural Deduction

We can continue to cancel that hypothesis as well:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A}\,{}^{2} \qquad \dfrac{}{B}\,{}^{1}}{A \wedge B} \qquad \dfrac{\dfrac{}{A}\,{}^{2} \qquad \dfrac{}{C}\,{}^{3}}{A \wedge C}}{(A \wedge B) \wedge (A \wedge C)}}{B \to (A \wedge B) \wedge (A \wedge C)}\,{}^{1}}{A \to (B \to (A \wedge B) \wedge (A \wedge C))}\,{}^{2}}{C \to (A \to (B \to (A \wedge B) \wedge (A \wedge C)))}\,{}^{3}
$$

The resulting proof uses no hypothesis at all. In other words, it establishes the conclusion outright.

Notice that in the second step, we canceled two "copies" of the hypothesis A.

Derivations in Natural Deduction

In natural deduction, we can choose which hypotheses to cancel; we could have canceled either one, and left the other hypothesis open. In fact, we can also carry out the implication-introduction rule and cancel zero hypotheses. For example, the following is a short proof of A → B from the hypothesis B:

$$\frac{B}{A \to B}$$

In this proof, zero copies of A are canceled.

Rules of Inference in Summary

Implication:

$$\dfrac{\dfrac{\overline{A}^{\,1}}{\vdots}\\ B}{A \to B}\,1\ \to\mathrm{I} \qquad \dfrac{A \to B \qquad A}{B}\,\to\mathrm{E}$$

Conjunction:

$$\dfrac{A \qquad B}{A \wedge B}\,\wedge\mathrm{I} \qquad \dfrac{A \wedge B}{A}\,\wedge\mathrm{E_l} \qquad \dfrac{A \wedge B}{B}\,\wedge\mathrm{E_r}$$

Negation:

$$\dfrac{\dfrac{\overline{A}^{\,1}}{\vdots}\\ \bot}{\neg A}\,1\ \neg\mathrm{I} \qquad \dfrac{\neg A \qquad A}{\bot}\,\neg\mathrm{E}$$

Rules of Inference in Summary

Disjunction:

$$\frac{A}{A \vee B}\ \vee\mathrm{I_l} \qquad \frac{B}{A \vee B}\ \vee\mathrm{I_r} \qquad \frac{A \vee B \quad \begin{matrix}\overline{A}^{\,1} \\ \vdots \\ C\end{matrix} \quad \begin{matrix}\overline{B}^{\,1} \\ \vdots \\ C\end{matrix}}{C}\ 1\ \vee\mathrm{E}$$

Truth and falsity:

$$\frac{\bot}{A}\ \bot\mathrm{E} \qquad \frac{}{\top}\ \top\mathrm{I}$$

Bi-implication:

$$\frac{\begin{matrix}\overline{A}^{\,1} \\ \vdots \\ B\end{matrix} \quad \begin{matrix}\overline{B}^{\,1} \\ \vdots \\ A\end{matrix}}{A \leftrightarrow B}\ 1\ \leftrightarrow\mathrm{I} \qquad \frac{A \leftrightarrow B \quad A}{B}\ \leftrightarrow\mathrm{E}_l \qquad \frac{A \leftrightarrow B \quad B}{A}\ \leftrightarrow\mathrm{E}_r$$

Rules of Inference in Summary

Reductio ad absurdum (proof by contradiction):

$$\dfrac{\dfrac{\dfrac{}{\neg A}\,1}{\vdots}\quad}{}$$

$$\cfrac{\begin{matrix}\overline{\neg A}^{\,1}\\ \vdots\\ \bot\end{matrix}}{A}\;1\ \text{RAA}$$

The basic rules of natural deduction:

	introduction	*elimination*
$\wedge$	$\dfrac{\phi \quad \psi}{\phi \wedge \psi}\ \wedge\mathrm{i}$	$\dfrac{\phi \wedge \psi}{\phi}\ \wedge\mathrm{e}_1 \qquad \dfrac{\phi \wedge \psi}{\psi}\ \wedge\mathrm{e}_2$
$\vee$	$\dfrac{\phi}{\phi \vee \psi}\ \vee\mathrm{i}_1 \qquad \dfrac{\psi}{\phi \vee \psi}\ \vee\mathrm{i}_2$	$\dfrac{\phi \vee \psi \quad \boxed{\begin{array}{c}\phi\\ \vdots\\ \chi\end{array}} \quad \boxed{\begin{array}{c}\psi\\ \vdots\\ \chi\end{array}}}{\chi}\ \vee\mathrm{e}$
$\rightarrow$	$\dfrac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \psi\end{array}}}{\phi \rightarrow \psi}\ \rightarrow\mathrm{i}$	$\dfrac{\phi \quad \phi \rightarrow \psi}{\psi}\ \rightarrow\mathrm{e}$

The basic rules of natural deduction:

	introduction	*elimination*
$\neg$	$\dfrac{\boxed{\begin{matrix}\phi \\ \vdots \\ \bot\end{matrix}}}{\neg\phi}\ \neg\mathrm{i}$	$\dfrac{\phi \quad \neg\phi}{\bot}\ \neg\mathrm{e}$
$\bot$	(no introduction rule for $\bot$)	$\dfrac{\bot}{\phi}\ \bot\mathrm{e}$
$\neg\neg$		$\dfrac{\neg\neg\phi}{\phi}\ \neg\neg\mathrm{e}$

Some useful derived rules:

$$\frac{\phi \to \psi \quad \neg\psi}{\neg\phi}\ \mathrm{MT} \qquad\qquad \frac{\phi}{\neg\neg\phi}\ \neg\neg\mathrm{i}$$

$$\frac{\boxed{\begin{array}{c}\neg\phi\\ \vdots \\ \bot\end{array}}}{\phi}\ \mathrm{PBC} \qquad\qquad \frac{}{\phi \vee \neg\phi}\ \mathrm{LEM}$$

Figure 1.2. Natural deduction rules for propositional logic.

Difference in the two sets of rules

From TextD:

Truth and falsity:

$$\frac{}{\top}\ \top\mathrm{I}$$

From TextB:

Some useful derived rules:

$$\frac{}{\phi \vee \neg\phi}\ \mathrm{LEM}$$

From the view of Constructive Mathematics, double-negative is a non-constructive rule.

$$\frac{\neg\neg\phi}{\phi}\ \neg\neg\mathrm{e}$$

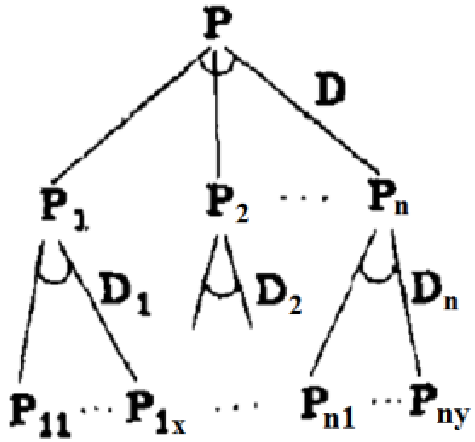
Forward and Backward Reasoning

Natural deduction is supposed to clarify the form and structure of our logical arguments, describe the appropriate means of justifying a conclusion, and explain the sense in which the rules we use are valid.

In the “official” description, natural deduction proofs are constructed by putting smaller proofs together to obtain bigger ones (Bottom-Up solving).

However, when we read natural deduction proofs, we often read them backward (Top-Down solving).

Bottom-Up or Top-Down
Problem-solving strategies.



Some Logical Identities

Two propositional formulas, A and B, are said to be logically equivalent if $A \leftrightarrow B$ is provable.

1. Commutativity of $\wedge$: $A \wedge B \leftrightarrow B \wedge A$
2. Commutativity of $\vee$: $A \vee B \leftrightarrow B \vee A$
3. Associativity of $\wedge$: $(A \wedge B) \wedge C \leftrightarrow A \wedge (B \wedge C)$
4. Associativity of $\vee$ $(A \vee B) \vee C \leftrightarrow A \vee (B \vee C)$
5. Distributivity of $\wedge$ over $\vee$: $A \wedge (B \vee C) \leftrightarrow (A \wedge B) \vee (A \wedge C)$
6. Distributivity of $\vee$ over $\wedge$: $A \vee (B \wedge C) \leftrightarrow (A \vee B) \wedge (A \vee C)$
7. $(A \to (B \to C)) \leftrightarrow (A \wedge B \to C)$.
8. $(A \to B) \to ((B \to C) \to (A \to C))$
9. $((A \vee B) \to C) \leftrightarrow (A \to C) \wedge (B \to C)$
10. $\neg(A \vee B) \leftrightarrow \neg A \wedge \neg B$
11. $\neg(A \wedge B) \leftrightarrow \neg A \vee \neg B$
12. $\neg(A \wedge \neg A)$
13. $\neg(A \to B) \leftrightarrow A \wedge \neg B$
14. $\neg A \to (A \to B)$
15. $(\neg A \vee B) \leftrightarrow (A \to B)$
16. $A \vee \bot \leftrightarrow A$
17. $A \wedge \bot \leftrightarrow \bot$
18. $A \vee \neg A$
19. $\neg(A \leftrightarrow \neg A)$
20. $(A \to B) \leftrightarrow (\neg B \to \neg A)$
21. $(A \to C \vee D) \to ((A \to C) \vee (A \to D))$
22. $(((A \to B) \to A) \to A)$

Content

- Exercise 03 Assignment
- Natural Deduction for PL
- Syntax of PL
- Formula Parsing

Next Lecture:

- Semantics of PL and Truth Tables