Proof System for $\mathcal{L}_0$

Recall

Two aspects of a formal language.

- Syntax
  - formulas
  - connectives
- Semantics
  - truth value/truth assignment
  - truth table/truth function

A proof system for $\mathcal{L}_0$

Suppose that φ_1, φ_2 and φ_3 are $\mathcal{L}_0$-formulas. Then each of the following $\mathcal{L}_0$-formulas is a logical axiom:

(Group I axioms)

- $(\varphi_1 \to (\varphi_2 \to \varphi_3)) \to ((\varphi_1 \to \varphi_2) \to (\varphi_1 \to \varphi_3))$
- $\varphi_1 \to \varphi_1$
- $\varphi_1 \to (\varphi_2 \to \varphi_1)$

(Group II axioms)

- $\varphi_1 \to (\neg\varphi_1 \to \varphi_2)$

(Group III axioms)

- $(\neg\varphi_1 \to \varphi_1) \to \varphi_1$

(Group IV axioms)

- $\neg\varphi_1 \to (\varphi_1 \to \varphi_2)$
- $\varphi_1 \to (\neg\varphi_2 \to \neg(\varphi_1 \to \varphi_2))$

Δ_0 denote the set of these four groups of logical axioms.

(Group III axioms)

- $(\neg\varphi_1 \to \varphi_1) \to \varphi_1$

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- $\neg\varphi_1 \to (\varphi_1 \to \varphi_2)$
- $\varphi_1 \to (\neg\varphi_2 \to \neg(\varphi_1 \to \varphi_2))$

Δ_0 denote the set of these four groups of logical axioms.

Proposition 1

Every logical axioms above is a tautology.

Γ-proof

Suppose that $\Gamma \subseteq \mathcal{L}_0$.

Definition 2

Suppose that $s = \langle \varphi_i : i \leq n \rangle$ is a finite sequence of propositional formulas. s is a Γ-proof if for each $i \leq n$ at least one of the following happens:

- $\varphi_i \in \Gamma$;
- φ_i is a logical axiom;
- there exists $j_1, j_2 < i$ such that $\varphi_{j_2} = \varphi_{j_1} \to \varphi_i$. This rule is called Modus Ponens.

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Definition 3

$\Gamma \vdash \varphi$ (Γ proves φ) iff there exists a finite sequence $s = \langle \varphi_i : i \leq n \rangle$ such that s is a Γ-proof and such that $\varphi_n = \varphi$.

Such sequence s is called a proof from Γ to φ, and φ is called a consequence of Γ.

When $\Gamma = \varnothing$, write $\vdash \varphi$.

Some properties of Γ-proofs

1. If s is a Γ-proof, and t is an initial segment of s, then t is also a Γ-proof.

2. If $s = \langle \varphi_i : i \leq n \rangle$ and $t = \langle \psi_i : i \leq m \rangle$ are two Γ-proofs, then so is

$$s + t = \langle \varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_m \rangle.$$

Definition 4

Suppose that $\Gamma \subseteq \mathcal{L}_0$.

1. Γ is inconsistent if for some formula φ, $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$.
2. Γ is consistent if Γ is not inconsistent.
3. Γ is maximally consistent if and only if for each formula ψ if $\Gamma \cup \{\psi\}$ is consistent then $\psi \in \Gamma$.

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Compare with Γ being **satisfiable**. Our ultimate goal is to show that

$$\Gamma \text{ is consistent if and only if } \Gamma \text{ is satisfiable.}$$

We first show the 'if" direction.

Soundness

Theorem 5 (Soundness, version I)

If $\Gamma \subseteq \mathcal{L}_0$ *is satisfiable. Then* Γ *is consistent.*

Definition 6 (Logical implication)

Suppose $\Gamma \subseteq \mathcal{L}_0$ and $\varphi \in \mathcal{L}_0$. Then Γ logically implies φ, write $\Gamma \models \varphi$, if and only if for every truth assignment ν, $\nu \models \Gamma$ implies $\nu \models \varphi$.

Theorem 7 (Soundness, version II)

Suppose that $\Gamma \subseteq \mathcal{L}_0$, $\varphi \in \mathcal{L}_0$ *and that* $\Gamma \vdash \varphi$. *Then* $\Gamma \models \varphi$.

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Proof.

By induction on the length of Γ-proofs. $\square$

Some lemmas of this proof system

Lemma 8 (Inference)

Suppose that $\Gamma \subseteq \mathcal{L}_0$, $\varphi, \psi \in \mathcal{L}_0$. *Suppose that* $\Gamma \vdash \psi$ *and* $\Gamma \vdash (\psi \to \varphi)$. *Then* $\Gamma \vdash \varphi$. [1]

Lemma 9 (Deduction)

Suppose that $\Gamma \subseteq \mathcal{L}_0$, $\varphi, \psi \in \mathcal{L}_0$ *and* $\Gamma \cup \{\varphi\} \vdash \psi$. *Then* $\Gamma \vdash (\varphi \to \psi)$.[2]

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[1] No logical axioms required.

[2] Group I axioms are needed.

Lemma 10

Suppose that $\Gamma \subseteq \mathcal{L}_0$ *and that* Γ *is inconsistent. Suppose that* $\psi \in \mathcal{L}_0$. *Then* $\Gamma \vdash \psi$.[3]

Lemma 11

Suppose that $\Gamma \subseteq \mathcal{L}_0$ *and that* Γ *is consistent. Suppose that* $\varphi \in \mathcal{L}_0$. *Then at least one of* $\Gamma \cup \{\varphi\}$ *or* $\Gamma \cup \{\neg\varphi\}$ *is consistent, possibly both.*[4]

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[3] Uses the Deduction lemma and Group II axioms.

[4] Needs Group III axioms.

Corollary 12

Suppose that $\Gamma \subseteq \mathcal{L}_0$ *and that* Γ *is maximally consistent. Suppose that* $\varphi \in \mathcal{L}_0$. *Then*

1. *Either* $\varphi \in \Gamma$ *or* $(\neg\varphi) \in \Gamma$.
2. *If* $\Gamma \vdash \varphi$ *then* $\varphi \in \Gamma$.

Lemma 13

Suppose that $\Gamma \subseteq \mathcal{L}_0$ *and that* Γ *is maximally consistent. Suppose that* $\varphi_1, \varphi_2 \in \mathcal{L}_0$. *Then* $(\varphi_1 \to \varphi_2) \in \Gamma$ *iff either* $\varphi_1 \notin \Gamma$ *or* $\varphi_2 \in \Gamma$.[5]

[5] Uses Group IV axioms.

Lemma 14

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Lemma 15

Suppose that $\Gamma \subseteq \mathcal{L}_0$ *and that* Γ *is consistent. Then there exists a set* $\Gamma^* \subset \mathcal{L}_0$ *such that* $\Gamma \subseteq \Gamma^*$ *and such that* Γ^* *is maximally consistent.*

Completeness Theorem

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Theorem 17 (Completeness, version II)

Suppose that $\Gamma \subseteq \mathcal{L}_0$, $\varphi \in \mathcal{L}_0$ *and that* $\Gamma \models \varphi$. *Then* $\Gamma \vdash \varphi$.

Exercise 1 (Slaman & Woodin)

Exercise 1.4.1 (1)–(3)

Exercise (思考题)

Show that

1. $\vdash \neg\neg\alpha \to \alpha$
2. $\vdash (\alpha \to \neg\beta) \to (\beta \to \neg\alpha)$
3. $\vdash \alpha \to \neg\neg\alpha$
4. $\vdash (\alpha \to \beta) \leftrightarrow (\neg\beta \to \neg\alpha)$
5. If $\Gamma \cup \{\alpha\} \vdash \beta$ and $\Gamma \cup \{\neg\alpha\} \vdash \beta$, then $\Gamma \vdash \beta$.

Proof of Inference.

Let s be a Γ-proof of φ, t be a Γ-proof of $(\varphi \to \psi)$, Then by MP, $s + t + \langle\psi\rangle$ is a Γ-proof of ψ. $\square$

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Proof of Deduction.

From a $(\Gamma \cup \{\varphi\})$-proof of ψ, $\langle \psi_1, \ldots, \psi_n \rangle$, produce a Γ-proof of $(\varphi \to \psi)$. Prove by induction on $i \leq n$ that $\Gamma \vdash (\varphi \to \psi_i)$. Use Inference.

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CASE 1. $\psi_i \in \Gamma \cup \Delta$.
CASE 2. $\psi_i = \varphi$.
CASE 3. ψ_i is obtained from MP.

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Case 1. $\psi_i \in \Gamma \cup \Delta$. Need $\psi_i \to (\varphi \to \psi_i)$.
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Let s be a Γ-proof of φ, t be a Γ-proof of $(\varphi \rightarrow \psi)$, Then by MP, $s + t + \langle \psi \rangle$ is a Γ-proof of ψ. $\square$

Proof of Deduction.

From a $(\Gamma \cup \{\varphi\})$-proof of ψ, $\langle \psi_1, \ldots, \psi_n \rangle$, produce a Γ-proof of $(\varphi \rightarrow \psi)$. Prove by induction on $i \leq n$ that $\Gamma \vdash (\varphi \rightarrow \psi_i)$. Use Inference.

Case 1. $\psi_i \in \Gamma \cup \Delta$. Need $\psi_i \rightarrow (\varphi \rightarrow \psi_i)$.

Case 2. $\psi_i = \varphi$. Need $\varphi \rightarrow \varphi$.

Case 3. ψ_i is obtained from MP. Need

$$(\varphi \rightarrow (\psi_{j_1} \rightarrow \psi_i)) \rightarrow [(\varphi \rightarrow \psi_{j_1}) \rightarrow (\varphi \rightarrow \psi_i)].$$

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First show $\Gamma \vdash \varphi$. Since $\Gamma \cup \{\neg\varphi\}$ is inconsistent, by Lemma **??** and Deduction, $\Gamma \vdash \neg\varphi \to \varphi$. Use $(\neg\varphi \to \varphi) \to \varphi \in \Delta$.

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As $\Gamma \cup \{\varphi\}$ is inconsistent, by Lemma **??** and Deduction, $\Gamma \vdash \varphi \to \neg\varphi$. We've shown $\Gamma \vdash \varphi$. By Inference, $\Gamma \vdash \neg\varphi$.

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Need $\varphi \to (\neg\varphi \to \psi)$. Use Inference. □

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Therefore Γ is inconsistent. Contradiction! □

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"$\Leftarrow$". Suppose $\varphi_1 \notin \Gamma$. Use Corollary **??** and a logical axiom, $(\neg\varphi_1) \to (\varphi_1 \to \varphi_2)$, and Inference to get $(\varphi_1 \to \varphi_2) \in \Gamma$.

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Suppose $\varphi_2 \in \Gamma$. Need a logical axiom $\varphi_2 \to (\varphi_1 \to \varphi_2)$.

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Suppose $\varphi_2 \in \Gamma$. Need a logical axiom $\varphi_2 \to (\varphi_1 \to \varphi_2)$.

"$\Rightarrow$". Suppose $\varphi_1 \in \Gamma$ and $\varphi_2 \notin \Gamma$. By Corollary **??**, $\neg\varphi_2 \in \Gamma$. Need $\varphi_1 \to ((\neg\varphi_2) \to \neg(\varphi_1 \to \varphi_2))$. Therefore $(\neg(\varphi_1 \to \varphi_2)) \in \Gamma$. So it can't be that $(\varphi_1 \to \varphi_2) \in \Gamma$. $\square$

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Extend Γ to either $\Gamma \cup \{\varphi\}$ or $\Gamma \cup \{\neg\varphi\}$, which ever is consistent, for every $\varphi \in \mathcal{L}_0$. $\square$