07 Counting

CS201 Discrete Mathematics

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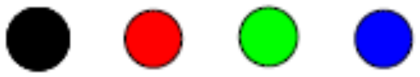
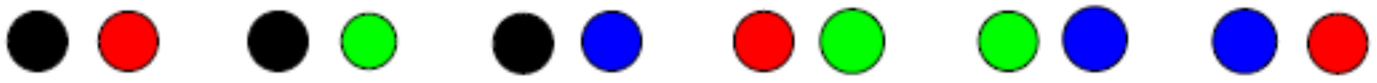
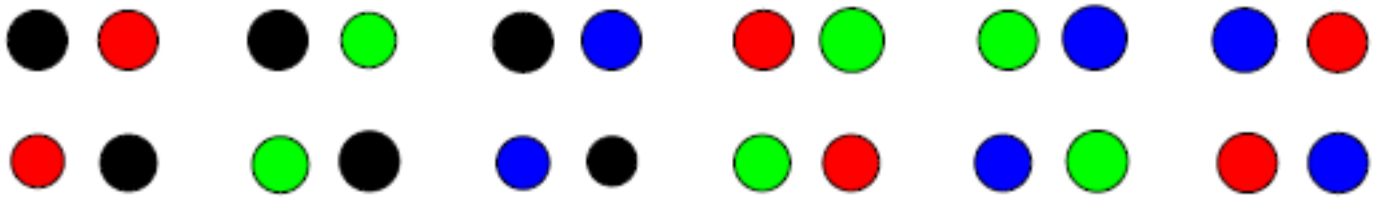
Counting

- Assume we have a set of objects with certain properties. Counting is used to determine the number of these objects.

- Example: 
  - How many different ways to choose 2 balls out of 4 colored balls?



  - What about when order matters?



Counting

- Assume we have a set of objects with certain properties. Counting is used to determine the number of these objects.

- More examples:
  - the number of steps in a computer program
  - the number of passwords between *6 ~ 10* characters
  - the number of telephone numbers with *8* digits

- Counting may be very hard and not trivial.
  - usually can be simplified by decomposing the problem

The Counting Basics

The Sum Rule

- A count decomposes into a set of independent counts:
  - elements of different counts are alternatives
- Example: You need to travel from city A to B. You may either fly, take a train, or a bus. There are *12* different flights, *5* different trains and *10* buses.
  - How many options do you have to travel from A to B?

    $12 + 5 + 10 = 27$
- **The sum rule:** If a count of elements can be broken down into a set of independent counts where the first count yields n_1 elements, the second n_2 elements, and k-th n_k elements, then the total number of elements is $n = n_1 + n_2 + \cdots + n_k$.

The Product Rule

- A count decomposes into a sequence of dependent counts:
  - each element in one count is associated with all elements of the next count

- Example: In an auditorium, the seats are labeled by a letter and numbers in between *1* to *50* (e.g., *A23*).
  - What is the total number of seats?

    $26 \cdot 50 = 1300$

- **The product rule:** If a count of elements can be broken down into a sequence of dependent counts where the first count yields n_1 elements, the second n_2 elements, and k-th n_k elements, then the total number of elements is $n = n_1 \cdot n_2 \cdot \cdots \cdot n_k$.

Other Rules

- **The subtraction rule:** If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.
  - E.g., $|A \cup B| = |A| + |B| - |A \cap B|$

- **The division rule:** If a task can be done using a procedure that can be carried out in n “fine-grained” ways, and for every “giant” way w, exactly d of the n “fine-grained” ways correspond to way w, then there are n/d “giant” ways to do it.
  - E.g., how many kilobytes in one megabyte? $10^6 / 10^3 = 10^3$

More Complex Counting

- Typically, a counting problem requires a combination of more than one rule.

- Example: Each password is 6 to 8 characters long, where each character is a lowercase letter or a digit. Each password must contain at least one digit.
  - How many possible passwords are there?

$P = P_6 + P_7 + P_8$

$P_6 = 36^6 - 26^6$

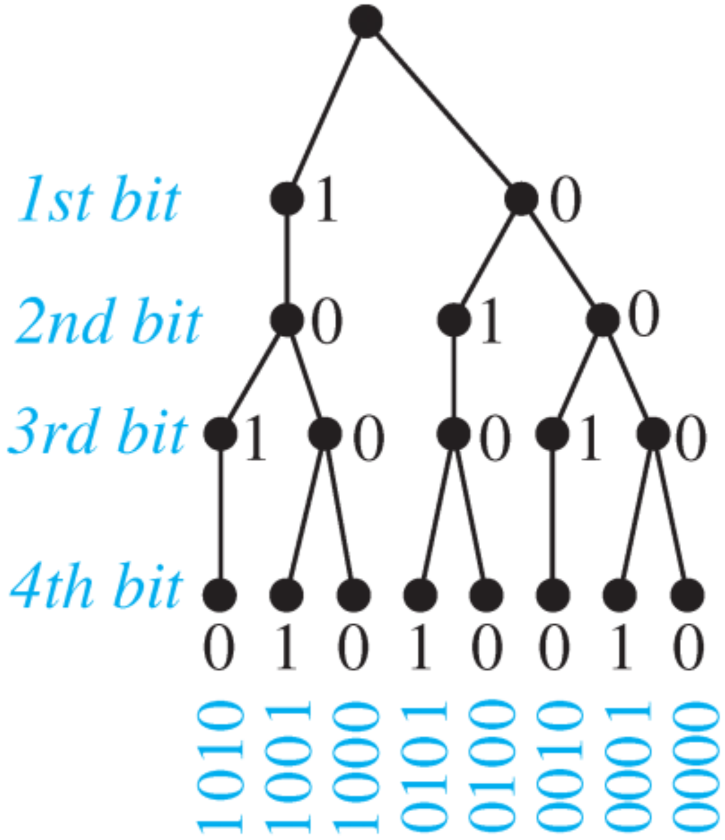
$P_7 = 36^7 - 26^7$

$P_8 = 36^8 - 26^8$

Tree Diagrams

- A tree is a structure that consists of a root, branches and leaves.
  - can represent a counting problem and record the choices we made for alternatives, with the count appears on the leaves

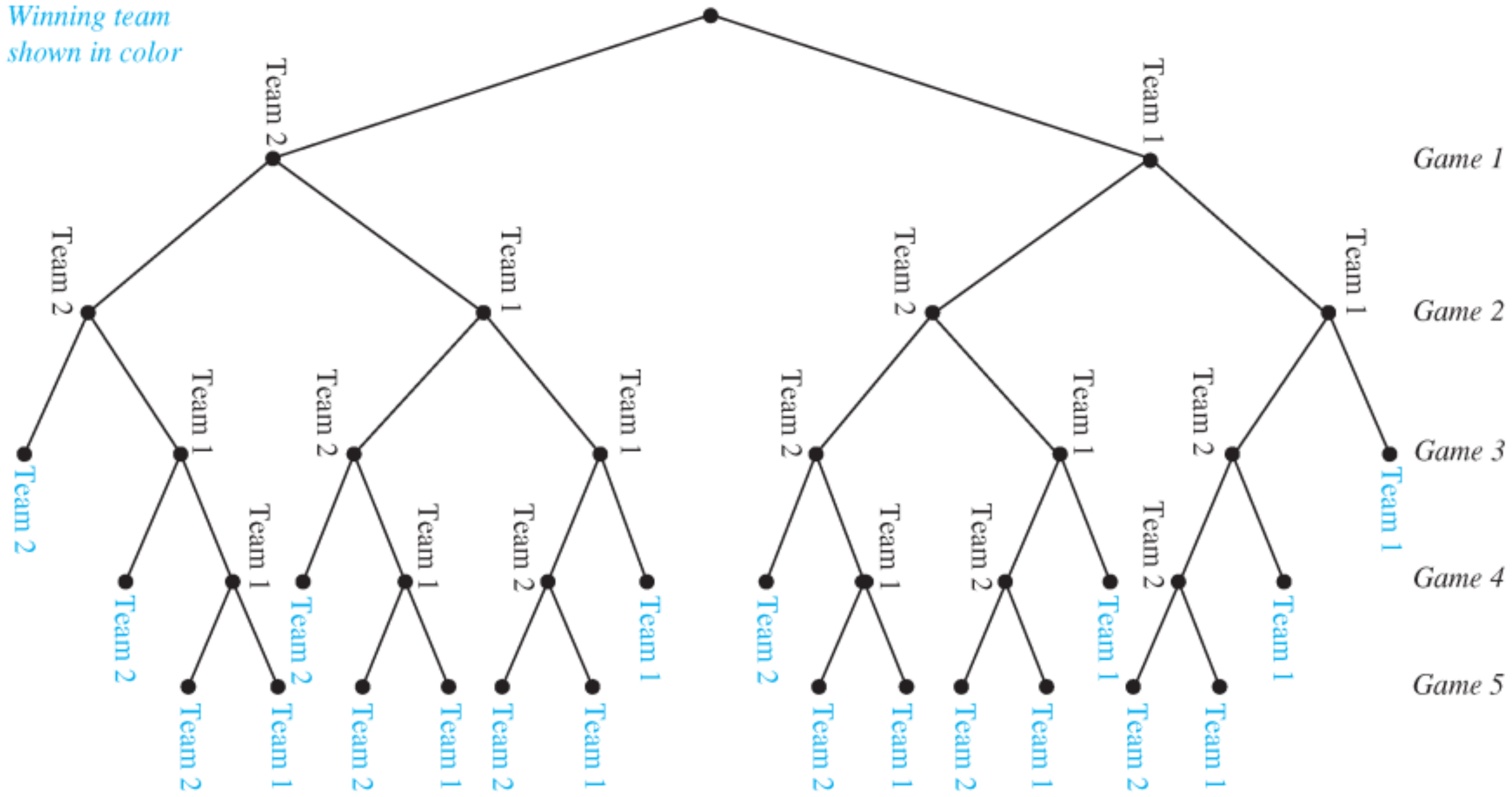
- Example: What is the number of bit strings of length *4* that do not have two consecutive *1*s?



Tree Diagrams

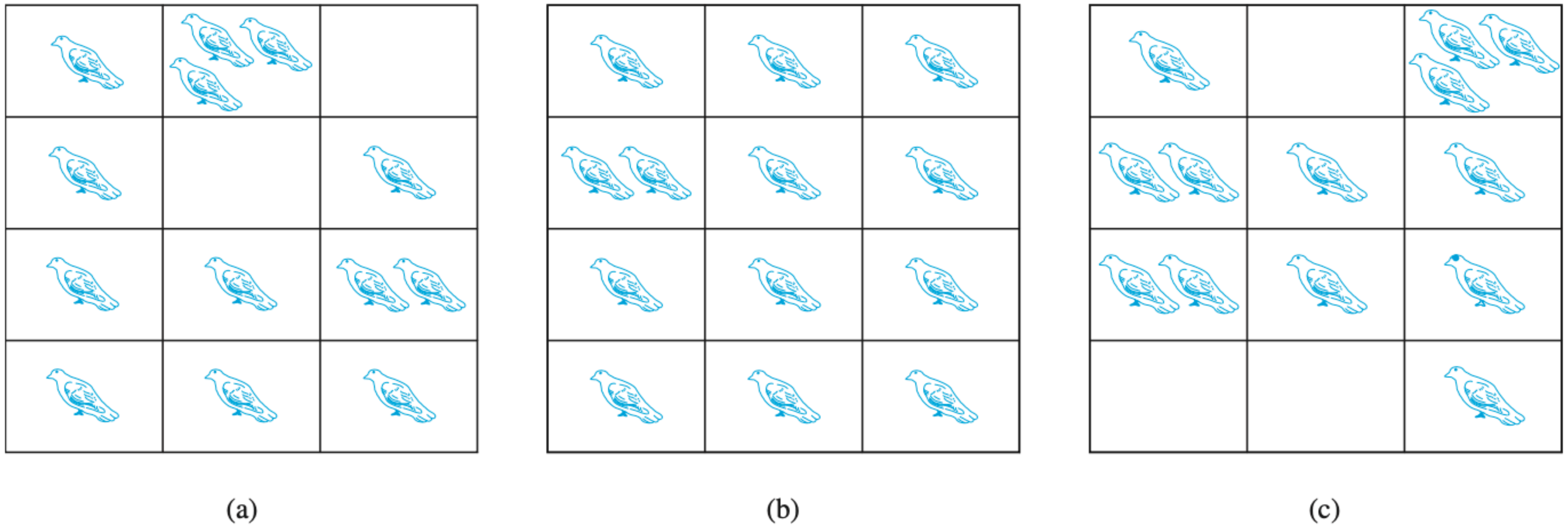
- A tree is a structure that consists of a root, branches and leaves.
  - can represent a counting problem and record the choices we made for alternatives, with the count appears on the leaves

- Example: The first team that wins 3 out of 5 games wins the playoff. In how many different ways can the playoff occur?



The Pigeonhole Principle

- If a flock of *13* pigeons flies into a set of *12* pigeonholes to roost, then at least one pigeonhole must have at least two pigeons in it.



- **The pigeonhole principle:** If *k* is a positive integer and *k* + 1 or more objects are placed into *k* boxes, then there is at least one box containing two or more of the objects.
  - proof by contradiction

The Generalized Pigeonhole Principle

- **The generalized pigeonhole principle:** If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.
  - proof by contradiction

- Example:
  - We have *144* students registered CS201. At least how many of you were born in the same month?

    $\lceil 144/12 \rceil = 12$

  - Now we have *137* students left. What about now?

    $\lceil 137/12 \rceil = 12$

Permutations and Combinations

- Many counting problems can be solved by finding the number of ways to arrange or select some distinct elements from a set.

- A permutation of a set of distinct objects is an ordered arrangement of these objects.
  - E.g., in how many ways can we select three students from a group of five students to stand in line for a picture?

- A combination of a set of distinct objects is an unordered selection of these objects.
  - E.g., how many different committees of three students can be formed from a group of five students?

r-Permutations

- An ordered arrangement of *r* distinct elements from a set is called a *r*-permutation.
  - an *n*-permutation of a set of size *n* is simply called a permutation

- Example: what are the 3-permutations of *{1, 2, 3, 4}*?
  - *L = {123, 124, 132, 134, 142, 143, 213, 214, 231, 234, 241, 243 312, 314, 321, 324, 341, 342, 412, 413, 421, 423, 431, 432}*
  - This type of "dictionary" ordering (where we treat numbers as letters) is called a lexicographic ordering and is used quite often.

r-Permutations

- **Theorem:** Let n, r be integers and $0 \le r \le n$, then there are

  $$P(n, r) = n(n-1)(n-2) \cdots (n-r+1) = n! / (n-r)!$$

  r-permutations of a set with n distinct elements.

  * *note that $P(n, 0) = 1$, i.e., there is one way to order 0 element*

- Proof:
  - There are n choices for the first number.
    For each way of choosing the first number, there are $n - 1$ choices for the second number.
    For each way of choosing the first two numbers, there are $n - 2$ choices for the third number.
    …
    For each way of choosing the first $r - 1$ numbers, there are $n - r + 1$ choices for the *r-th* number.
  - Therefore, by the product rule, there are $n(n-1)(n-2) \cdots (n-r+1)$ r-permutations, which is equal to $n! / (n-r)!$.

r-Combinations

- An unordered selection of r distinct elements from a set is called a r-combination.

- Example: what are the 3-combinations of $\{1, 2, 3, 4\}$?
  - $L = \{123, 124, 134, 234\}$

- **Theorem:** Let n, r be integers and $0 \leq r \leq n$, then there are

$$C(n, r) = P(n, r) / P(r, r) = n! / r!\,(n - r)!$$

  r-combinations of a set with n distinct elements.

  * *note that $C(n, 0) = 1$, i.e., there is one way to choose 0 element*

  - Proof: Since the order of elements in a combination does not matter and there are $P(r, r)$ different ways to order the r elements in an r-combination, each of the $C(n, r)$ r-combinations corresponds to exactly $P(r, r) = r!$ r-permutations. Therefore, by the **division rule**, we have $C(n, r) = P(n, r) / P(r, r) = n! / r!\,(n - r)!$.

Exercise *(5 mins)*

- Answer the following questions:
  - How many different bit strings of length *7* are there?
  - How many different functions from a set with *m* elements to a set with *n* elements?
  - How many injective functions from a set with *m* elements to a set with *n* elements $(m \leq n)$?
  - How many onto functions from a set with *m* elements to a set with *n* elements $(m \geq n)$?

Exercise (5 mins)

- Answer the following questions:
  - How many different bit strings of length 7 are there?

    2^7

  - How many different functions from a set with m elements to a set with n elements?

    n^m

  - How many injective functions from a set with m elements to a set with n elements $(m \leq n)$?

    $n(n - 1) \cdots (n - m + 1)$

  - How many onto functions from a set with m elements to a set with n elements $(m \geq n)$?

    $n^m - C(n, 1)(n - 1)^m + C(n, 2)(n - 2)^m - \ldots + (-1)^{n-1}C(n, n - 1)1^m$

    ** to be proved soon in later sections*

The Birthday Problem

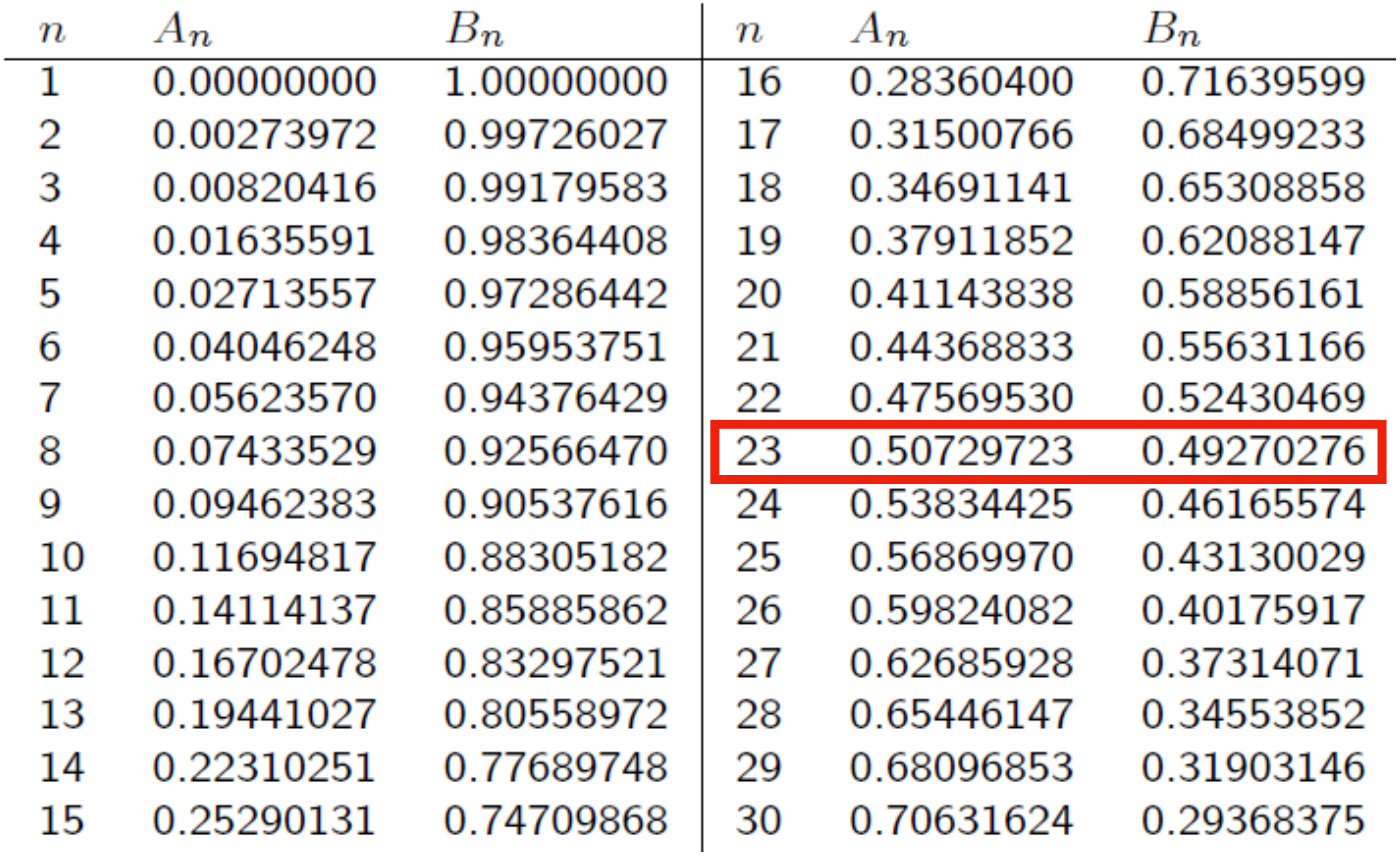
- **The birthday paradox:** Suppose that *23* students are in a room. What is the probability that at least two of them share a birthday?
  - It's greater than a half!

- Assume a year has 365 days and there are no twins in the room.
  - sample space: $|S| = 365^n$ * *all cases occur equally likely*

- A_n : "for n students in a room ≥ 2 of them share a birthday"
  B_n : "for n students in a room none of them share a birthday"
  $\#E$: number of cases favorable to event E
  - $\#A_n + \#B_n = |S| = 365^n$
  - $\#B_n = C(365, n) = 365 \times 364 \times \cdots \times (365 - (n - 1))$
  - $\Pr[A_n] = \#A_n / |S| = 1 - \#B_n / |S|$ * *classical probability*

The Birthday Problem

- Probabilities of A_n and B_n:



n	A_n	B_n
1	0.00000000	1.00000000
2	0.00273972	0.99726027
3	0.00820416	0.99179583
4	0.01635591	0.98364408
5	0.02713557	0.97286442
6	0.04046248	0.95953751
7	0.05623570	0.94376429
8	0.07433529	0.92566470
9	0.09462383	0.90537616
10	0.11694817	0.88305182
11	0.14114137	0.85885862
12	0.16702478	0.83297521
13	0.19441027	0.80558972
14	0.22310251	0.77689748
15	0.25290131	0.74709868
16	0.28360400	0.71639599
17	0.31500766	0.68499233
18	0.34691141	0.65308858
19	0.37911852	0.62088147
20	0.41143838	0.58856161
21	0.44368833	0.55631166
22	0.47569530	0.52430469
23	0.50729723	0.49270276
24	0.53834425	0.46165574
25	0.56869970	0.43130029
26	0.59824082	0.40175917
27	0.62685928	0.37314071
28	0.65446147	0.34553852
29	0.68096853	0.31903146
30	0.70631624	0.29368375

The Birthday Attack

- In cryptography, the birthday attack is an attack that uses the probabilistic model shown in the birthday problem to reduce the complexity of finding a collision for a hash function.
  - assume a hash function has independent random outputs
  - each hash output can be viewed as a student's birthday
- Recall the birthday problem:
  - A_n : "for n students in a room ≥ 2 of them share a birthday"
  - B_n : "for n students in a room none of them share a birthday"

$$\Pr[B_n] = \left(1 - \frac{1}{365}\right) \cdot \left(1 - \frac{2}{365}\right) \cdot \cdots \cdot \left(1 - \frac{n-1}{365}\right) = \prod_{i=1}^{n-1} \left(1 - \frac{i}{365}\right)$$

$$\Pr[A_n] = 1 - \Pr[B_n] = 1 - \prod_{i=1}^{n-1} \left(1 - \tfrac{i}{365}\right) \qquad p(n; H) := 1 - \prod_{i=1}^{n-1} (1 - \tfrac{i}{H})$$

The Birthday Attack

- In cryptography, the birthday attack is an attack that uses the probabilistic model shown in the birthday problem to reduce the complexity of finding a collision for a hash function.

- What is the smallest number of values that we have to choose, such that the probability of finding a hash collision is $\geq p$?
  - Let H be the number of possible hash outputs.
  - The collision probability when choosing n hash values is

$$p(n;H) := 1 - \prod_{i=1}^{n-1}(1 - \tfrac{i}{H})$$

Since $e^x = 1 + x + \frac{x^2}{2!} + \cdots$, for $|x| \ll 1$, $e^x \approx 1 + x$

Thus, we have $e^{-i/H} \approx 1 - \frac{i}{H}$. $p(n;H) \approx 1 - e^{-n(n-1)/2H} \approx 1 - e^{-n^2/2H}$

  - By inverting the above expression, we have the smallest number n:

$$n(p;H) \approx \sqrt{2H \ln \frac{1}{1-p}}.$$

Binomial Coefficients and Identities

Binomial Coefficients

- **Theorem:** For integers *n* and *k* with *0 ≤ k ≤ n*, the number of *k*-element subsets of an *n*-element set is

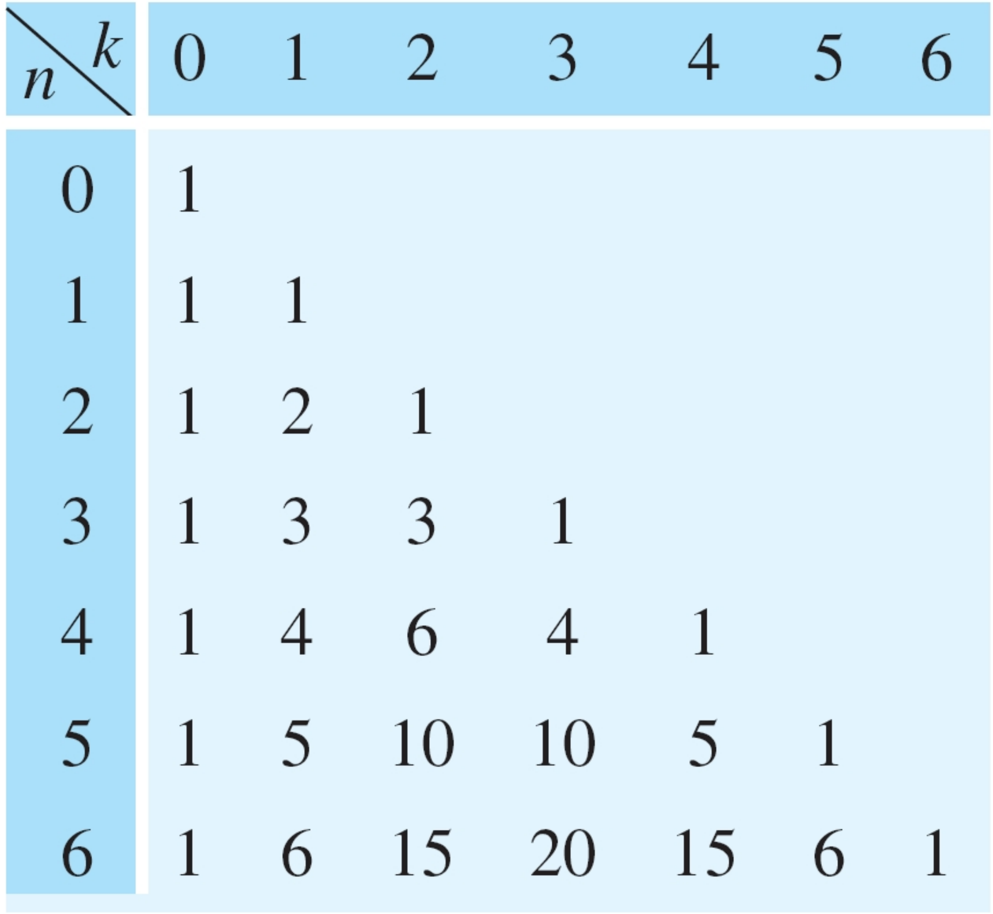
$$\binom{n}{k} = C(n,k) = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!}.$$

  This is the number of *k*-combinations of a set with *n* elements.

- Properties:
  - $\binom{n}{0} = \binom{n}{n} = 1$ * *only one subset of size 0 and one of size n*
  - $\binom{n}{k} = \binom{n}{n-k}$ * *obvious by definition*
  - $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ * *the number of subsets of an n-element set*

Binomial Coefficients

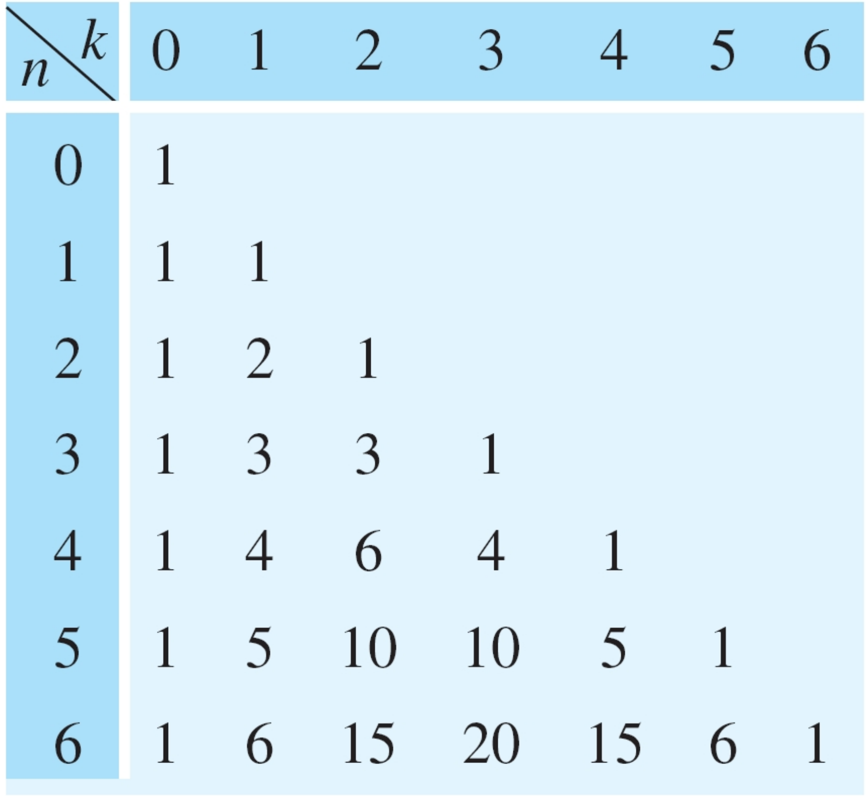
- Each row starts with a *1*
  - $\binom{n}{0} = 1$
- Each row ends with a *1*
  - $\binom{n}{n} = 1$
- Second half of each row is the reverse of the first half.
  - $\binom{n}{k} = \binom{n}{n-k}$
- Sum on the *n*-th row is 2^n
  - $\sum_{k=0}^{n} \binom{n}{k} = 2^n$



$n \backslash k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

Pascal’s Triangle

- Take the table and shift each row slightly such that middle element is in the center



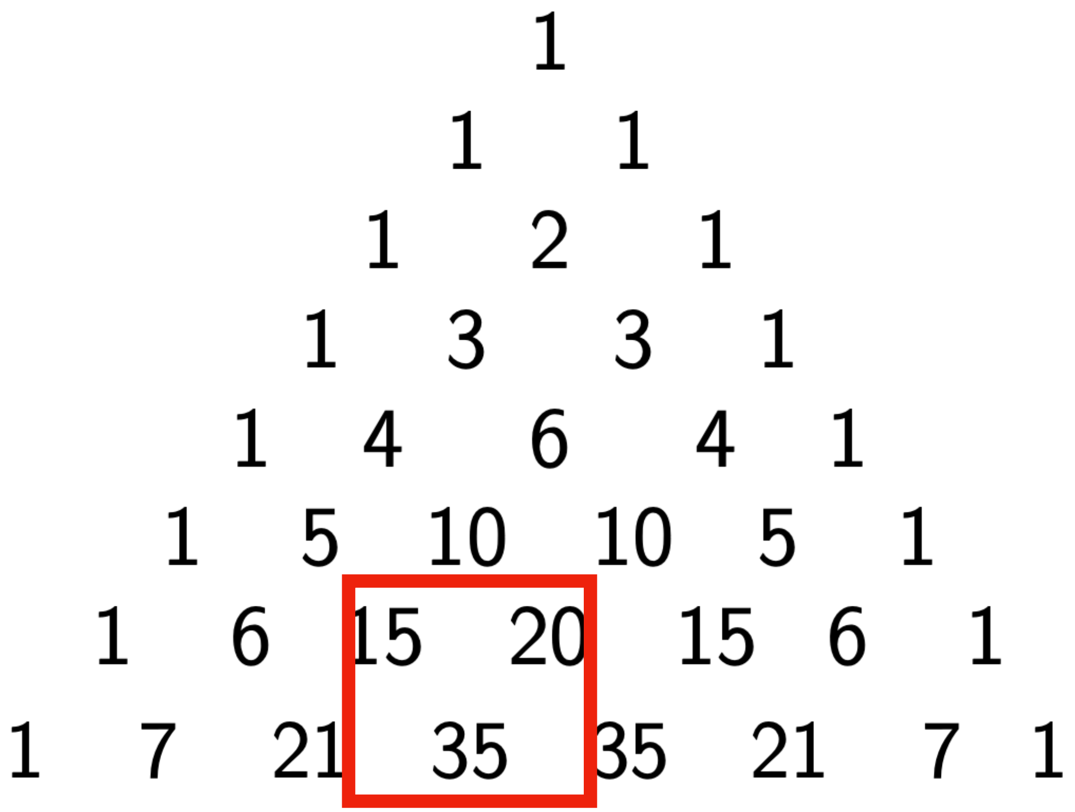
$n \setminus k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5  10  10   5   1
1   6  15  20  15   6   1
```

Pascal's Identity

- **Q:** What is the next row?
  - Each (non-*1*) entry in Pascal's triangle is the sum of the two entries directly above it.

- **Pascal's identity:**
  - $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$
  - A purely algebraic proof (i.e., by manipulating formulas) is possible but complicated.
  - We will apply a so-called combinatorial proof (or combinatorial argument).



1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

A Combinatorial Proof

- **Pascal's identity:** $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

- A combinatorial proof:
  - Let S_1 be the set of all k-element subsets and x_n be the n-th element. Partition S_1 into S_2 and S_3 and apply the **sum rule**:

    S_2: the set of k-element subsets that contain x_n.

    S_3: the set of k-element subsets that don't contain x_n.
  - $\binom{n}{k}$ is the number of k-subsets of an n-element set
  - $\binom{n-1}{k-1}$ is the number of $(k-1)$-subsets of an $(n-1)$-element set
  - $\binom{n-1}{k}$ is the number of k-subsets of an $(n-1)$-element set

Blaise Pascal

- French mathematician (*1623~1662*)
  - a founder of probability theory
  - inventor of one of the first mechanical calculating machines
  - Pascal programming language named in honor of him

The Binomial Theorem

- **The binomial theorem:** Let *x* and *y* be variables, and let *n* be a nonnegative integer. Then $\sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^{k} = (x+y)^{n}$.

  ** the proof is easy: how many ways one can derive the term $x^{n-k}y^{k}$?*

  - Let $x = y = 1$. We have $\sum_{k=0}^{n} \binom{n}{k} = 2^{n}$.
  - Let $x = 1$, $y = -1$. We have $\sum_{k=0}^{n} (-1)^{k} \binom{n}{k} = 0$.
  - Let $y = 1$. We have $\sum_{k=0}^{n} \binom{n}{k} x^{k} = (1+x)^{n}$.

- Why the name **"binomial coefficients"**?
  - because those numbers occur as coefficients in the expansion of powers of binomial expressions such as $(x + y)^n$.

Trinomial Coefficients

- **Q:** What is the coefficient of $x^{k_1} y^{k_2} z^{k_3}$ in $(x + y + z)^n$?

- **A:** If we have k_1 red labels, k_2 blue labels, and $k_3 = n - k_1 - k_2$ purple labels, then in how many different ways can we apply these labels to n objects?
  - How many ways to choose the k_1 red items? How many ways to choose the k_2 blue items from the remaining $n - k_1$ items? Finally, the remaining $k_3 = n - k_1 - k_2$ items get labelled purple.

$$\binom{n}{k_1}\binom{n-k_1}{k_2} = \frac{n!}{k_1!(n-k_1)!}\frac{(n-k_1)!}{(k_2)!(n-k_1-k_2)!} = \frac{n!}{k_1!k_2!k_3!}$$

- If $k_1 + k_2 + k_3 = n$, we call $n!/(k_1!\, k_2!\, k_3!)$ a trinomial coefficient and denote it as $\binom{n}{k_1\ k_2\ k_3}$.

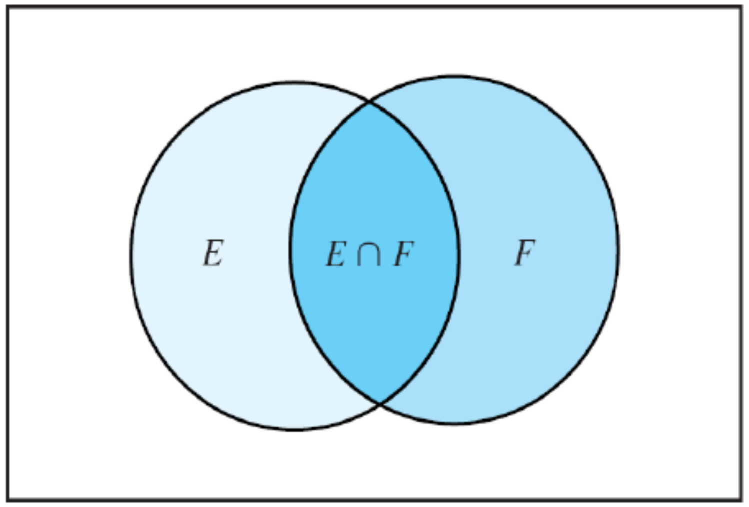
Inclusion-Exclusion

The Inclusion-Exclusion Principle

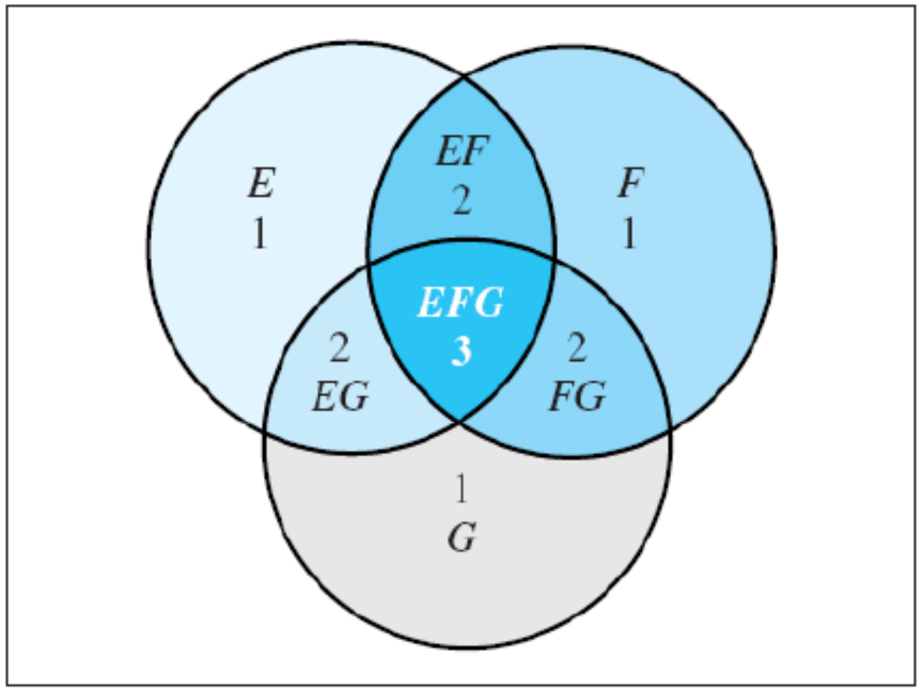
- The principle is used in counts where the decomposition yields two dependent counting tasks with overlapping elements
  - If we use the **sum rule**, some elements would be counted twice.
- The principle uses the **subtraction rule** to correct for the overlapping elements after the sum.
  - two-set case: $|A \cup B| = |A| + |B| - |A \cap B|$
- Example: How many bit strings of length *8* that start with a '*1*' bit or end with the two bits '*00*'?
  - It is easy to count bit strings starting with '*1*': 2^7
  - It is easy to count bit strings ending with '*00*': 2^6
  - Deduct the overcounted number of strings starting with '*1*' and ending with '*00*': 2^5

The Inclusion-Exclusion Principle

- Two sets:
  - $|E \cup F| = |E| + |F| - |E \cap F|$
  - $E \cap F$ got counted twice and deducted once



- Three sets:
  - $|E \cup F \cup G| = |E| + |F| + |G| - |E \cap F| - |E \cap G| - |F \cap G| + |E \cap F \cap G|$
  - $E \cap F$, $E \cap G$, $F \cap G$ got counted twice and then deducted once
  - $E \cap F \cap G$ got counted three times then deducted three times and finally got counted once



The Inclusion-Exclusion Principle

- **The principle of inclusion-exclusion:** Let E_1, E_2, …, E_n be finite sets, then

$$|\cup_{i=1}^{n} E_i| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} |E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|$$

- Proof by induction:
  - **Basis step:** obviously true for $n = 1, 2$
  - **Inductive step:** By the two-set case of this principle, i.e., thinking of $\cup_{i=1}^{n-1} E_i$ and E_n as two sets, we have

$$|\cup_{i=1}^{n} E_i| = \left|\cup_{i=1}^{n-1} E_i\right| + |E_n| - \left|\left(\cup_{i=1}^{n-1} E_i\right) \cap E_n\right|$$

Let $G_i = E_i \cap E_n$, by **distributive law**, we have

$$\left|\left(\cup_{i=1}^{n-1} E_i\right) \cap E_n\right| = \left|\cup_{i=1}^{n-1} (E_i \cap E_n)\right| = \left|\cup_{i=1}^{n-1} G_i\right|$$

Therefore, $|\cup_{i=1}^{n} E_i| = \left|\cup_{i=1}^{n-1} E_i\right| + |E_n| - \left|\cup_{i=1}^{n-1} G_i\right|$

The Inclusion-Exclusion Principle

- **The principle of inclusion-exclusion:** Let E_1, E_2, ..., E_n be finite sets, then

$$|\cup_{i=1}^{n} E_i| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} |E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|$$

- Proof by induction:
  - **Inductive step:** By the two-set case and $G_i = E_i \cap E_n$, we have:

$$|\cup_{i=1}^{n} E_i| = \left|\cup_{i=1}^{n-1} E_i\right| + |E_n| - \left|\cup_{i=1}^{n-1} G_i\right|$$

By inductive hypothesis, we have:

$$|\cup_{i=1}^{n-1} E_i| = \sum_{k=1}^{n-1} \sum_{0 \le i_1 < i_2 < \cdots < i_k \le n-1} (-1)^{k+1} |E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|$$

$$-|\cup_{i=1}^{n-1} G_i| = \sum_{k=1}^{n-1} \sum_{0 \le i_1 < i_2 < \cdots < i_k \le n-1} -(-1)^{k+1} |G_{i_1} \cap G_{i_2} \cap \cdots \cap G_{i_k}|$$

By definition, $|G_{i_1} \cap G_{i_2} \cap \cdots \cap G_{i_k}| = |E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k} \cap E_n|$.

The Inclusion-Exclusion Principle

- **The principle of inclusion-exclusion:** Let E_1, E_2, ..., E_n be finite sets, then

$$|\cup_{i=1}^{n} E_i| = \sum_{k=1}^{n}(-1)^{k+1} \sum_{1\le i_1<i_2<\cdots<i_k\le n} |E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|$$

- Proof by induction:
  - **Inductive step:** By the two-set case and $G_i = E_i \cap E_n$, we have:

$$|\cup_{i=1}^{n} E_i| = \left|\cup_{i=1}^{n-1} E_i\right| + |E_n| - \left|\cup_{i=1}^{n-1} G_i\right|$$

By inductive hypothesis and noting that $G_i = E_i \cap E_n$, we have:

$$|\cup_{i=1}^{n-1} E_i| = \sum_{k=1}^{n-1} \sum_{0\le i_1<i_2<\cdots<i_k\le n-1} (-1)^{k+1} |E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|$$

$$-|\cup_{i=1}^{n-1} G_i| = \sum_{k=1}^{n-1} \sum_{0\le i_1<i_2<\cdots<i_k\le n-1} (-1)^{(k+1)+1} |E_{i_1} \cap \cdots \cap E_{i_k} \cap E_n|$$

The 1st big sum captures all $|E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|$ where $i_k < n$.

The 2nd big sum captures all $|E_{i_1} \cap \cdots \cap E_{i_k} \cap E_n|$ where $i_k < n$.

The Inclusion-Exclusion Principle

- **The principle of inclusion-exclusion:** Let $E_1, E_2, \ldots, E_n$ be finite sets, then

$$|\cup_{i=1}^{n} E_i| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} |E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|$$

- Proof by induction:
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By inductive hypothesis and noting that $G_i = E_i \cap E_n$, we have:

$$|\cup_{i=1}^{n-1} E_i| = \sum_{k=1}^{n-1} \sum_{0 \le i_1 < i_2 < \cdots < i_k \le n-1} (-1)^{k+1} |E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|$$

$$-|\cup_{i=1}^{n-1} G_i| = \sum_{k=1}^{n-1} \sum_{0 \le i_1 < i_2 < \cdots < i_k \le n-1} (-1)^{(k+1)+1} |E_{i_1} \cap \cdots \cap E_{i_k} \cap E_n|$$

Therefore, the above two big sums together captures all possible combinations of $|E_{i_1} \cap \cdots \cap E_{i_k}|$ for $1 \le k \le n$ except $|E_n|$. * *why?*

The Number of Onto Functions

- **The principle of inclusion-exclusion:** Let E_1, E_2, ..., E_n be finite sets, then

$$|\cup_{i=1}^{n} E_i| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} |E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|$$

- We can use this principle to find the number of onto functions:
  - Let A, B be two sets with $|A| = m$ and $|B| = n$.

    $\#(a)$: the number of onto functions from A to B

    $\#(b)$: the number of non-onto functions from A to B, i.e., the functions with at least one element of B having no preimage
  - Since there are n^m functions from A to B, we have $\#(a) + \#(b) = n^m$. So, in order to find $\#(a)$, we only need to calculate $\#(b)$.
  - E_i : set of functions such that the i-th element of B has no preimage

$$\#(b) = | \cup_{i=1}^{n} E_i|$$

The Number of Onto Functions

- **The principle of inclusion-exclusion:** Let $E_1, E_2, \ldots, E_n$ be finite sets, then

$$|\cup_{i=1}^{n} E_i| = \sum_{k=1}^{n}(-1)^{k+1} \sum_{1\le i_1<i_2<\cdots<i_k\le n} |E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|$$

- We can use this principle to find the number of onto functions:
  - E_i : set of functions such that the i-th element of B has no preimage
  - By the **principle of inclusion-exclusion**,

$$\#(b) = |\cup_{i=1}^{n} E_i| = \sum_{k=1}^{n}(-1)^{k+1} \sum_{1\le i_1<i_2<\cdots<i_k\le n} |E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|$$

  - Given any index list $i_1, i_2, \ldots, i_k$ *(how many such lists?)*, functions in $E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}$ do not map to those k indexed elements of B, so there are $(n-k)^m$ such functions in total. Therefore,

$$\#(b) = |\cup_{i=1}^{n} E_i| = \sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}(n-k)^m$$

Solving Linear Recurrence Relations

Linear Recurrence Relations

- **Definition:** A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

  - linear: it is a linear combination of previous terms
  - homogeneous: all terms are multiples of a_j s
  - degree k : a_n is expressed by the previous k terms
  - constant coefficients: coefficients are constants
  - By induction, such a recurrence relation is uniquely determined by this recurrence relation, and k initial conditions $a_0, a_1, \ldots, a_{k-1}$.

Linear Recurrence Relations

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where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

- Examples: are these linear homogeneous recurrence relations?
  - $P_n = \pi P_{n-1}$
  - $f_n = f_{n-1} + f_{n-2}$
  - $a_n = a_{n-1} + a_{n-2} \cdot a_{n-2}$
  - $H_n = 2H_{n-1} + 1$
  - $B_n = nB_{n-1}$

Linear Recurrence Relations

- **Definition:** A linear homogeneous recurrence relation of degree *k* with constant coefficients is a recurrence relation of the form

  $$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

  where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

- Examples: are these linear homogeneous recurrence relations?

• $P_n = \pi P_{n-1}$	*Yes, of degree 1*
• $f_n = f_{n-1} + f_{n-2}$	*Yes, of degree 2*
• $a_n = a_{n-1} + a_{n\text{-}2} \cdot a_{n\text{-}2}$	*No, not linear*
• $H_n = 2H_{n-1} + 1$	*No, not homogeneous*
• $B_n = nB_{n-1}$	*No, coefficients are not constants*

Solving Recurrences of Degree 2

- Consider an arbitrary linear homogeneous relation of degree 2 with constant coefficients:

  $$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

  The characteristic equation (CE) is:

  $$r^2 - c_1 r - c_2 = 0$$

- **Theorem:** If this CE has two distinct roots r_1, r_2, then sequence $\{a_n\}$ is a solution of the recurrence relation if and only if

  $$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n \text{ for } n \geq 0,$$

  where α_1, α_2 are constants.

  - *see the textbook for the proof ("if" part is easy but "only if" is tricky)*

Example

- Solve the recurrence: $a_n = 7a_{n-1} - 10a_{n-2}$, with $a_0 = 2, a_1 = 1$.

- Solution:
  - The characteristic equation (CE) is

$$r^2 - 7r + 10 = 0$$

  - Two roots are 2 and 5. So, assume that

$$a_n = \alpha_1 2^n + \alpha_2 5^n$$

  - By the two initial conditions, we have

$$a_0 = \alpha_1 + \alpha_2 = 2$$

$$a_1 = 2\alpha_1 + 5\alpha_2 = 1$$

  - We get $\alpha_1 = 3$ and $\alpha_2 = -1$. Therefore

$$a_n = 3 \cdot 2^n - 5^n$$

Exercise (5 mins)

- What is the closed-form expression of Fibonacci sequence F_n?
  - $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$

- Solve the recurrence: $a_n = 7a_{n-1} - 10a_{n-2}$, with $a_0 = 2, a_1 = 1$.

- Solution:
  - The characteristic equation (CE) is

$$r^2 - 7r + 10 = 0$$

  - Two roots are 2 and 5. So, assume that

$$a_n = \alpha_1 2^n + \alpha_2 5^n$$

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$$a_0 = \alpha_1 + \alpha_2 = 2$$
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$$a_n = 3 \cdot 2^n - 5^n$$

Exercise (5 mins)

- What is the closed-form expression of Fibonacci sequence F_n?
  - $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$

- Solution:
  - Consider the characteristic equation (CE): $x^2 - x + 1 = 0$. There are two different roots

$$\phi = \frac{1+\sqrt{5}}{2}, \quad \psi = \frac{1-\sqrt{5}}{2}$$

  - Then F_n is of the form $\alpha_1 \phi^n + \alpha_2 \psi^n$.
  - By $F_0 = 0$ and $F_1 = 1$, we have $\alpha_1 + \alpha_2 = 0$ and $\alpha_1 \phi + \alpha_2 \psi = 1$, leading to $\alpha_1 = 1/\sqrt{5}$, $\alpha_2 = -1/\sqrt{5}$. Therefore,

$$F_n = \frac{\phi^n - \psi^n}{\sqrt{5}}$$

Solving Recurrences of Degree *k*

- Consider an arbitrary linear homogeneous relation of degree k with constant coefficients:

  $$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$$

  The characteristic equation (CE) is:

  $$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$$

  - *e.g.*, $a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3}$ with *CE:* $r^3 - 2r^2 - 5r + 6 = 0$

- **Theorem:** If this CE has k distinct roots $r_1, r_2, \cdots, r_k$, then the solutions to the recurrence $\{a_n\}$ is of the form

  $$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n \text{ for } n \geq 0,$$

  where $\alpha_1, \alpha_2, \dots, \alpha_k$ are constants.

  - *the proof is left as an exercise*

The Case of Degenerate Roots

- **Theorem:** If the CE $r^2 - c_1 r - c_2 = 0$ has only one root r_0, then

  $$a_n = a_1 r_0^n + a_2 n r_0^n \text{ for } n \geq 0,$$

  where a_1, a_2 are constants.

  - *the proof is left as an exercise*

- **Theorem:** Suppose the CE $r^k - c_1 r^{k-1} - \cdots - c_k = 0$ has t distinct roots $r_1, \ldots, r_t$ with multiplicities $m_1, \ldots, m_t$, respectively, where $m_i \geq 1$ and $m_1 + \cdots + m_t = k$. Then

  $$a_n = \sum_{i=1}^{t} \left(\sum_{j=0}^{m_i - 1} \alpha_{i,j} n^j \right) r_i^n \quad \text{for } n \geq 0,$$

  where $a_{i,j}$ are constants.

  - *the proof is left as an exercise*

Exercise (3 mins)

- Solve the recurrence: $a_n = 4a_{n-1} - 4a_{n-2}$ with $a_0 = 1, a_1 = 0$.

- **Theorem:** If the CE $r^2 - c_1 r - c_2 = 0$ has only one root r_0, then

$$a_n = a_1 r_0^n + a_2 n r_0^n \text{ for } n \geq 0,$$

where a_1, a_2 are constants.

Exercise (3 mins)

- Solve the recurrence: $a_n = 4a_{n-1} - 4a_{n-2}$ with $a_0 = 1, a_1 = 0$.

- Solution:
  - The characteristic equation (CE) is

    $$r^2 - 4r + 4 = 0$$

  - The only root is 2. So, assume that

    $$a_n = \alpha_1 2^n + \alpha_2 n 2^n$$

  - By the two initial conditions, we have

    $$a_0 = \alpha_1 = 1$$

    $$a_1 = 2\alpha_1 + 2\alpha_2 = 0$$

  - We get $\alpha_1 = 1$ and $\alpha_2 = -1$. Therefore

    $$a_n = 2^n - n2^n$$

Solving Nonhomogeneous Recurrences

- **Theorem:** If $a_n = p(n)$ is any particular solution to the linear nonhomogeneous relation with constant coefficients,

  $$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$$

  Then all its solutions are of the form

  $$a_n = p(n) + h(n)$$

  where $a_n = h(n)$ is any solution to the associated homogeneous recurrence relation

  $$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

  - *the proof is left as an exercise*

Example

- Solve the recurrence: $a_n = 3a_{n-1} + 2n$ with $a_1 = 3$.
- Solution:
  - The characteristic equation (CE) of the associated linear homogeneous recurrence relation is $r - 3 = 0$.
  - The root is 3. So, assume $a_n = p(n)$ is a particular solution to the original recurrence relation, then all of its solutions are of the form
$$a_n = \alpha_1 3^n + p(n)$$
  - It is natural to try a degree-1 polynomial, i.e., $p(n) = cn + d$. Then, $cn + d = 3(c(n - 1) + d) + 2n$, i.e., $(2c+ 2)n + 2d - 3c = 0$.
  - We get $c = -1$ and $d = -3/2$. Therefore, $a_n = \alpha_1 3^n - n - 3/2$.
  - By the initial condition $a_1 = 2\alpha_1 - 1 - 3/2 = 3$, we get $\alpha_1 = 11/4$.

Generating Functions

Generating Functions

- **Definition:** The generating function for the sequence *{a_k}* of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

- We use generating functions to characterize sequences:
  - $\sum_{k=0}^{\infty} 3x^k$: generating function for the sequence $\{a_k\}$ with $a_k = 3$
  - $\sum_{k=0}^{\infty} 2^k x^k$: generating function for the sequence $\{a_k\}$ with $a_k = 2^k$

Generating Functions

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$$G(x) = a_0 + a_1 x + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

- We use generating functions to characterize sequences:
  - $\sum_{k=0}^{\infty} 3x^k = \lim_{n\to\infty} \sum_{k=0}^{n} 3x^k = 3 \lim_{n\to\infty} \frac{1 - x^{n+1}}{1 - x} = \frac{3}{1-x}$ for $|x| < 1$
  - $\sum_{k=0}^{\infty} 2^k x^k = \lim_{n\to\infty} \sum_{k=0}^{n} (2x)^k = \lim_{n\to\infty} \frac{1 - (2x)^{n+1}}{1 - 2x} = \frac{1}{1-2x}$ for $|2x| < 1$

Operations of Generating Functions

- **Theorem:** Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$, then

$$f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k)x^k \text{ and } f(x)g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) x^k$$

** the proof can be found in a calculus course*

- Example: $f(x) = g(x) = \sum_{k=0}^{\infty} a^k x^k$
  - We know $G(x) = \sum_{k=0}^{\infty} a^k x^k = 1/(1-ax)$ for $|ax| < 1$
  - Therefore,

$$f(x)g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a^j a^{k-j} \right) x^k = \sum_{k=0}^{\infty} (k+1)a^k x^k = \frac{1}{(1-ax)^2}$$

The Case of Finite Sequences

- **Definition:** The generating function for the finite sequence $a_0, a_1, \ldots, a_n$ of real numbers is a polynomial of degree n

$$G(x) = a_0 + a_1 x + \cdots + a_k x^n$$

  - A finite sequence $a_0, a_1, \ldots, a_n$ can be easily extended to infinity by setting $a_{n+1} = a_{n+2} = \cdots = 0$.

- Example: What is the generating function for the finite sequence $a_0, a_1, \ldots, a_n$, with $a_k = C(n, k)$?

$$G(x) = C(n,0) + C(n,1)x + \cdots + C(n,n)x^n = (1+x)^n$$

Useful Generating Functions

$$(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k$$

$$(1+ax)^n = \sum_{k=0}^{n} C(n,k)a^k x^k$$

$$(1+x^r)^n = \sum_{k=0}^{n} C(n,k)x^{rk}$$

$$\frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$$

$$\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + \cdots$$

$$\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk} = 1 + x^r + x^{2r} + \cdots$$

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty}(k+1)x^k = 1 + 2x + 3x^2 + \cdots$$

Generating Functions for Counting

- **Example 1:** Find the number of solutions of $x_1 + x_2 + x_3 = 17$, such that x_1, x_2, x_3 are all nonnegative integers and $2 \leq x_1 \leq 5$, $3 \leq x_2 \leq 6$, $4 \leq x_3 \leq 7$.

- Solution: Using generating functions, the number is the coefficient of x^{17} in the expansion of

$$(x^2 + x^3 + x^4 + x^5)(x^3 + x^4 + x^5 + x^6)(x^4 + x^5 + x^6 + x^7)$$

The answer is 3.

Generating Functions for Counting

- **Example 2:** In how many ways can eight identical cookies be distributed among three distinct children if each child receives at least two and no more than four cookies?

- Solution: Using generating functions, the number of ways is the coefficient of x^8 in the expansion of

$$(x^2 + x^3 + x^4)^3$$

The answer is 6.

Generating Functions for Counting

- **Example 3:** In how many ways can *r* dollars be paid by coins that are worth *1* dollar, *2* dollar and *5* dollar each?

- Solution: Using generating functions, the number of ways is the coefficient of x^r in the expansion of the following functions:
  - If the order of coins do not matter:

    $$(1+x+x^2+x^3+\cdots)(1+x^2+x^4+x^6+\cdots)(1+x^5+x^{10}+x^{15}+\cdots)$$

  - If the order matters:

    $$1+(x+x^2+x^5)+(x+x^2+x^5)^2+(x+x^2+x^5)^3+\cdots$$

Generating Functions for Counting

- **Example 4:** Use generating functions to find the number of *k*-combinations of a set with *n* elements.

- Solution: Each of the *n* elements in the set contributes one term *(1 + x)* to the generating function $(1+x)^n = \sum_{k=0}^{n} a_k x^k$. Then, by the binomial theorem, we have

$$a_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

r-Combinations with Repetition

- An *r*-combination with repetition allowed (or a multiset of size *r*), chosen from a set of *n* elements, is an unordered selection of *r* elements from *n* elements with repetition allowed.

- Example: How many multisets of size *17* from the set *{1, 2, 3}*?
  - This is equivalent to find the number of nonnegative solutions to

    $$x_1 + x_2 + x_3 = 17$$

  - The solution is $C(3 + 17 - 1, 17) = C(19, 17) = C(19, 2)$

    How many ways one can split *17* balls into *3* groups?

    Imagine *19* boxes are aligned in a line, then one just needs to choose *17* boxes and put balls in each of them, with *2* empty boxes splitting the balls into *3* groups.

  - This is equal to the coefficient of x^{17} in the generating function:

    $$(1 + x + x^2 + \cdots)^3 = 1/(1 - x)^3 = \sum_{k=0}^{\infty} C(3 + k - 1,\ k)x^k$$

Useful Generating Functions (more)

- Hinted from *r*-combinations with repetition allowed:

$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)x^k$$
$$\frac{1}{(1+x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)(-1)^k x^k$$
$$\frac{1}{(1-ax)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)a^k x^k$$

- Taylor series:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Solving Recurrences with *G(x)*

- **Example 1:** Solve $a_k = 3a_{k-1}$ $(k \geq 1)$ with $a_0 = 2$.
- Solution:
  - Let $G(x)$ be the generating function of $\{a_k\}$, we have

$$G(x) - 3xG(x) = \sum_{k=0}^{\infty} a_k x^k - \sum_{k=1}^{\infty} 3a_{k-1}x^k$$

$$= a_0 + \sum_{k=1}^{\infty} (a_k - 3a_{k-1})x^k = a_0 = 2$$

  - Then, we know

$$G(x) = \frac{2}{1-3x} = \sum_{k=1}^{\infty} 2 \cdot 3^k x^k$$

  - Therefore, the solution is $a_k = 2 \cdot 3^k$ for $k \geq 0$.

Solving Recurrences with *G(x)*

- **Example 2:** Solve $a_n = 8a_{n-1} + 10^{n-1}$ $(n \geq 2)$ with $a_1 = 9$.
- Solution:
  - Let $a_0 = 1$, then we have $a_1 = 9 = 8a_0 + 10^0$ holds, so the recurrence relation is consistent with $n \geq 1$.
  - Then, let $G(x)$ be the generating function of $\{a_n\}$ for $n \geq 0$, we have

$$G(x) - a_0 = \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} (8a_{n-1}x^n + 10^{n-1}x^n)$$

$$= 8x \sum_{n=1}^{\infty} a_{n-1}x^{n-1} + x \sum_{n=1}^{\infty} 10^{n-1}x^{n-1}$$

$$= 8x \sum_{n=0}^{\infty} a_n x^n + x \sum_{n=0}^{\infty} 10^n x^n$$

$$= 8xG(x) + x/(1 - 10x)$$

Solving Recurrences with *G(x)*

- **Example 2:** Solve $a_n = 8a_{n-1} + 10^{n-1}$ $(n \geq 2)$ with $a_1 = 9$.
- Solution:
  - Let $a_0 = 1$, then we have $a_1 = 9 = 8a_0 + 10^0$ holds, so the recurrence relation is consistent with $n \geq 1$.
  - Then, let $G(x)$ be the generating function of $\{a_n\}$ for $n \geq 0$, we have

$$G(x) - a_0 = G(x) - 1 = 8xG(x) + x/(1 - 10x)$$

Solving for $G(x)$ shows that

$$G(x) = \frac{1 - 9x}{(1 - 8x)(1 - 10x)}$$

Solving Recurrences with *G(x)*

- **Example 2:** Solve $a_n = 8a_{n-1} + 10^{n-1}$ $(n \geq 2)$ with $a_1 = 9$.
- Solution:
  - Let $a_0 = 1$, then we have $a_1 = 9 = 8a_0 + 10^0$ holds, so the recurrence relation is consistent with $n \geq 1$.
  - Then, let $G(x)$ be the generating function of $\{a_n\}$ for $n \geq 0$, we have

$$G(x) = \frac{1-9x}{(1-8x)(1-10x)} = \frac{1}{2}\left(\frac{1}{1-8x} + \frac{1}{1-10x}\right)$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty} 8^n x^n + \sum_{n=0}^{\infty} 10^n x^n\right) = \sum_{n=0}^{\infty} \frac{1}{2}(8^n + 10^n)x^n$$

Therefore, the solution is $a_n = \frac{1}{2}(8^n + 10^n)$ for $n \geq 0$.

Solving Recurrences with *G(x)*

- Using generating functions to solve recurrence relations:
  - **Step 1:** Based on the given recurrence and its initial conditions, find its generating function *G(x)* as an explicit formula.

    E.g., $G(x) = \dfrac{1-9x}{(1-8x)(1-10x)}$

  - **Step 2:** Rewrite *G(x)* as the summation of an infinite (or finite) series.
    ** this step may be tricky*

    E.g., $G(x) = \displaystyle\sum_{n=0}^{\infty} \frac{1}{2}(8^n + 10^n)x^n$

08 Relations

To be continued…

Announcement

- Quiz 2 will take place in class on Dec 8 and it captures materials from 06 Induction and Recursion to 07 Counting.

- Again, the quiz is open-book:
  - 3~6 questions in 30 minutes
  - bring several pieces of paper to write your answers on
  - no electronic device is allowed during the quiz
  - take photos of your quiz answers and submit them as a single file via Blackboard (you will have 5 minutes after quiz to do this)
  - must attend the quiz in person