Department of
Computer Science and Engineering
计算机科学与工程系

CSE5014 CRYPTOGRAPHY AND NETWORK SECURITY

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(Asymptotic) *Security parameter* n, polynomial p

Definiton 3.2 Let D_n be a distribution over $p(n)$-bit strings. $\{D_n\}$ is *pseudorandom* if for all probabilistic, polynomial-time (PPT) distinguishers A, there is a negligible function ϵ such that

$$|\Pr_{x \leftarrow D_n}[A(x) = 1] - \Pr_{x \leftarrow U_{p(n)}}[A(x) = 1]| \leq \epsilon(n)$$

PRGs

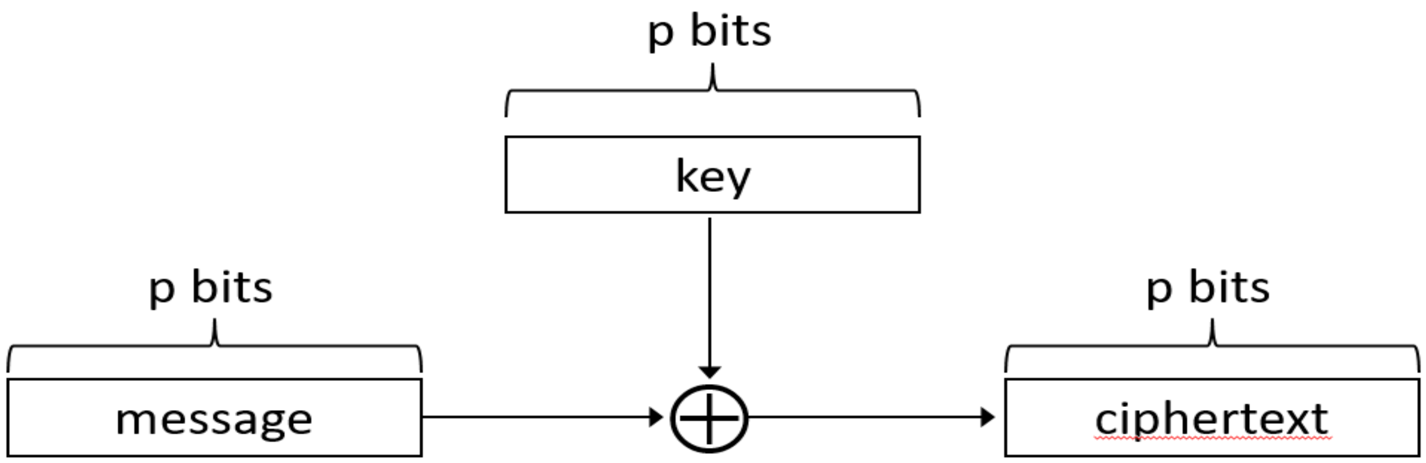
- A *PRG* is an efficient, deterministic algorithm that expands a *short*, *uniform seed* into a *longer*, *pseudorandom* output

  Let G be a deterministic, poly-time algorithm that is *expanding*, i.e., $|G(x)| = p(|x|) > |x|$.

- For all efficient distinguishers A, there is a negligible function ϵ such that
  $$|\Pr_{x \leftarrow U_n}[A(G(x)) = 1] - \Pr_{y \leftarrow U_{p(n)}}[A(y) = 1]| \leq \epsilon(n)$$

  No efficient A can distinguish whether it is given $G(x)$ (for uniform x) or a uniform string y!

Recall: one-time pad



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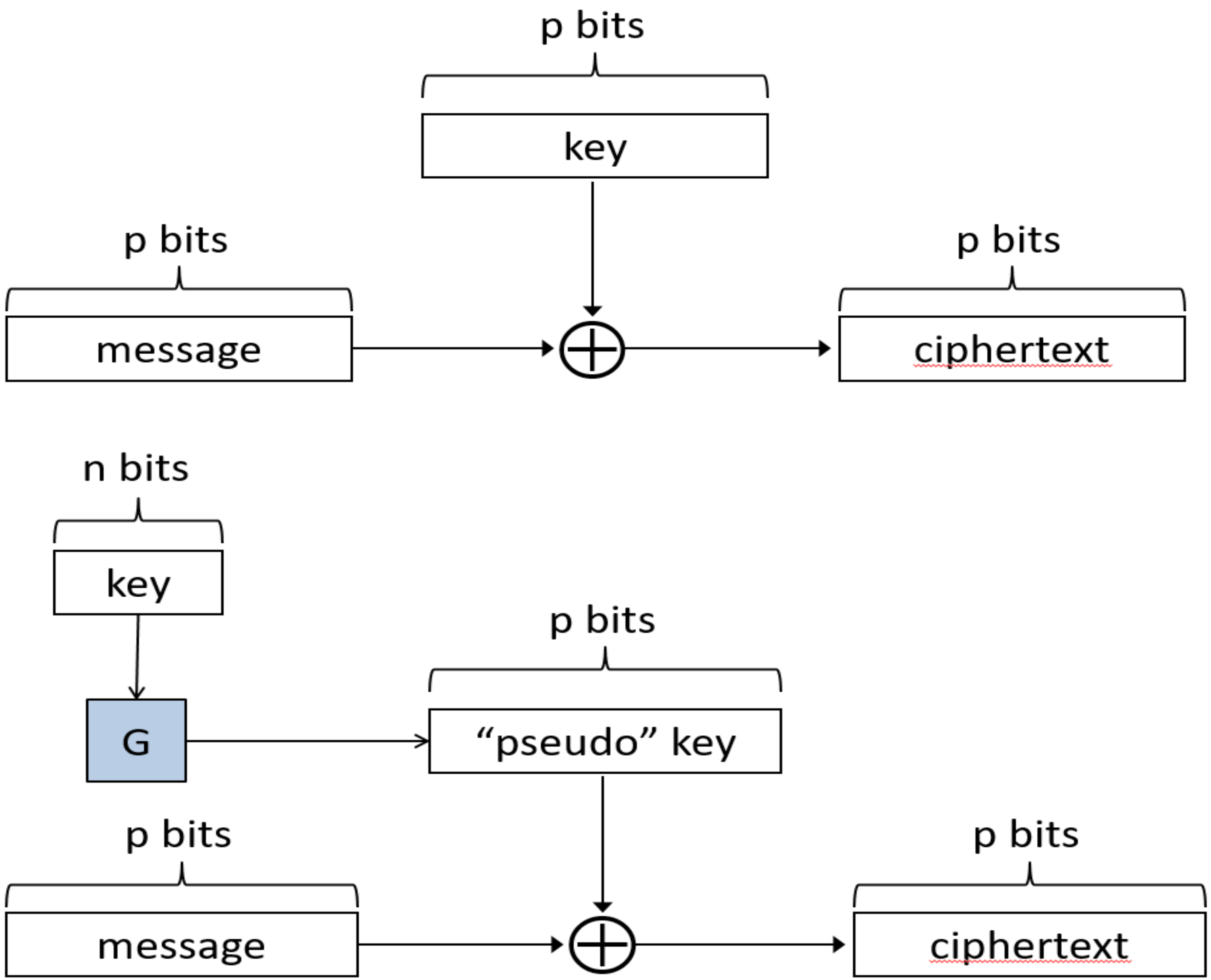

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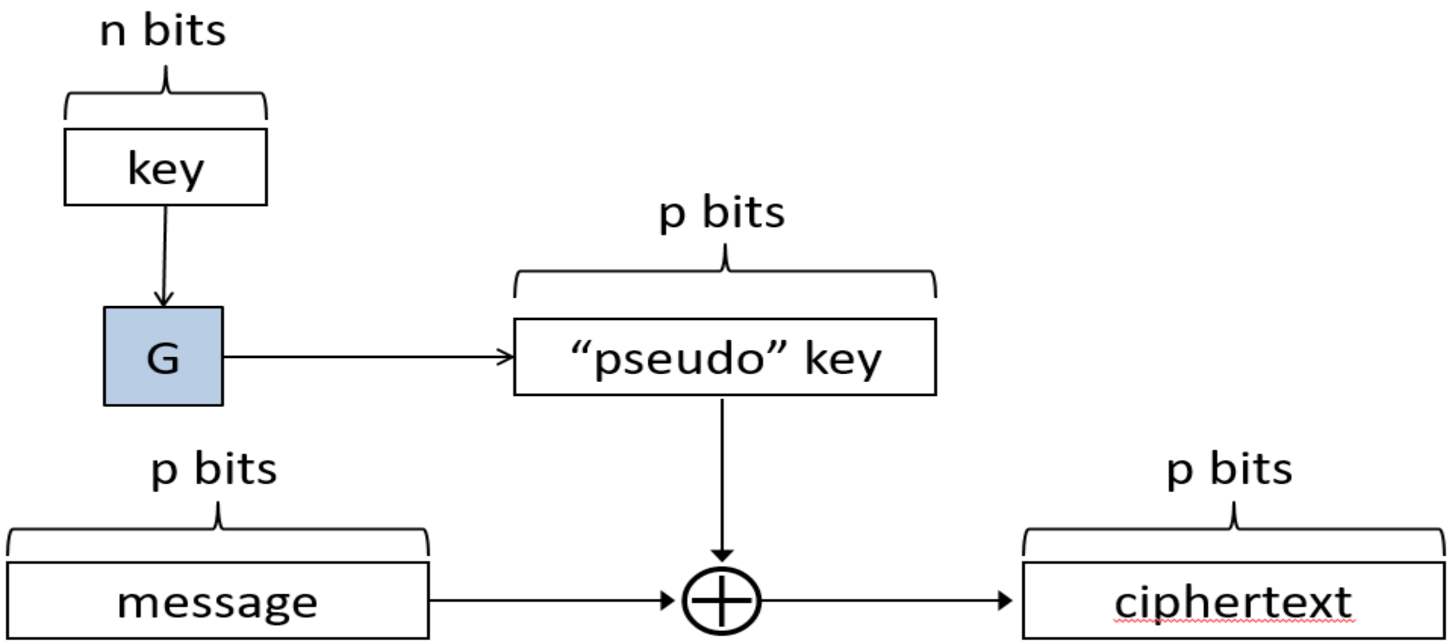

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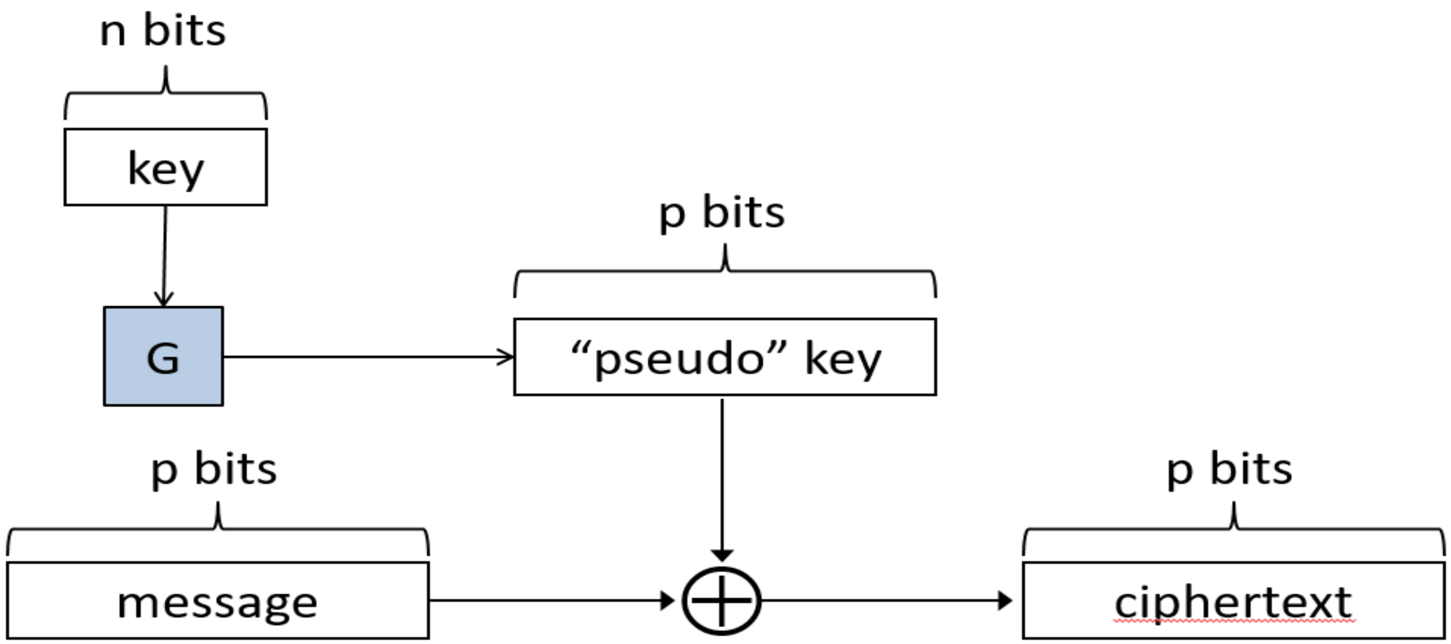

- Let G be a deterministic, with $|G(k)| = p(|k|)$
  $Gen(1^n)$: output uniform n-bit key k
  – Security parameter $n \Rightarrow$ message space $\{0,1\}^{p(n)}$
  $Enc_k(m)$: output $G(k) \oplus m$
  $Dec_k(m)$: output $G(k) \oplus c$

Proof by reduction

- 1. Assume that G is a *PRG*

  2. Assume toward a contradiction that there is an efficient attacker A who "breaks" the pseudo-OTP scheme

  3. Use A as a subroutine to build an efficient D that "breaks" *pseudorandomness* of G

  – By assumption, no such D exists!

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Theorem 3.3 If G is a pseudorandom generator (PRG), then the pseudo one-time pad (pseudo-OTP) Π is *EAV-secure* (i.e., *computationally secure*)

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- **Proof.**

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  Fix Π, A
  Define a randomized experiment $PrivK_{A,\Pi}(n)$:

  1. $A(1^n)$ outputs $m_0, m_1 \in \{0,1\}^*$ of equal length
  2. $k \leftarrow Gen(1^n)$, $b \leftarrow \{0,1\}$, $c \leftarrow Enc_k(m_b)$
  3. $b' \leftarrow A(c)$

  Adversary A *succeeds* if $b = b'$, and we say the experiment evaluates to 1 in this case.

  Definition 3.1 Π is *computationally indistinguishable* (aka *EAV-secure*) if for all PPT attackers (algorithms) A, there is a *negligible* function ϵ such that

$$\Pr[PrivK_{A,\Pi}(n) = 1] \leq 1/2 + \epsilon(n)$$

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  Since $U_{p(n)} \oplus m \equiv U_{p(n)}$, this contradicts that G is a PRG.

Proof by reduction (alternatively)

1. Assume that G is a *PRG*
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   $\Rightarrow$ Bound the success probability of A

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  $\Rightarrow \Pr[PrivK_{A,\Pi}(n) = 1] \leq 1/2 + negl(n)$

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How can we circumvent the second limitation?

But first...

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  - Security goal
  - Threat model

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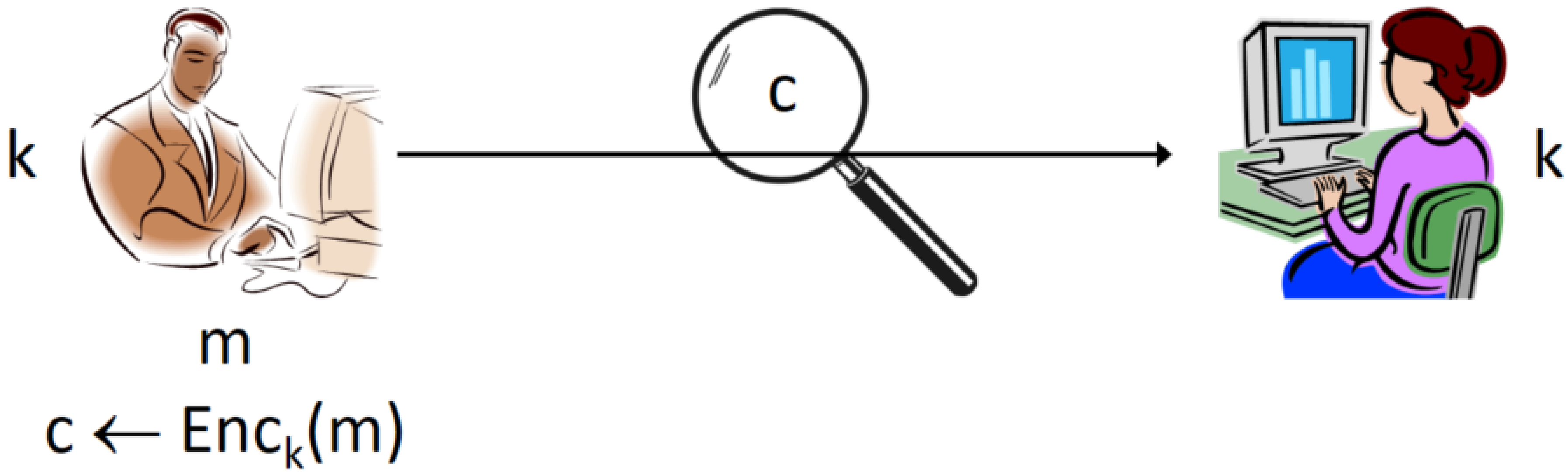

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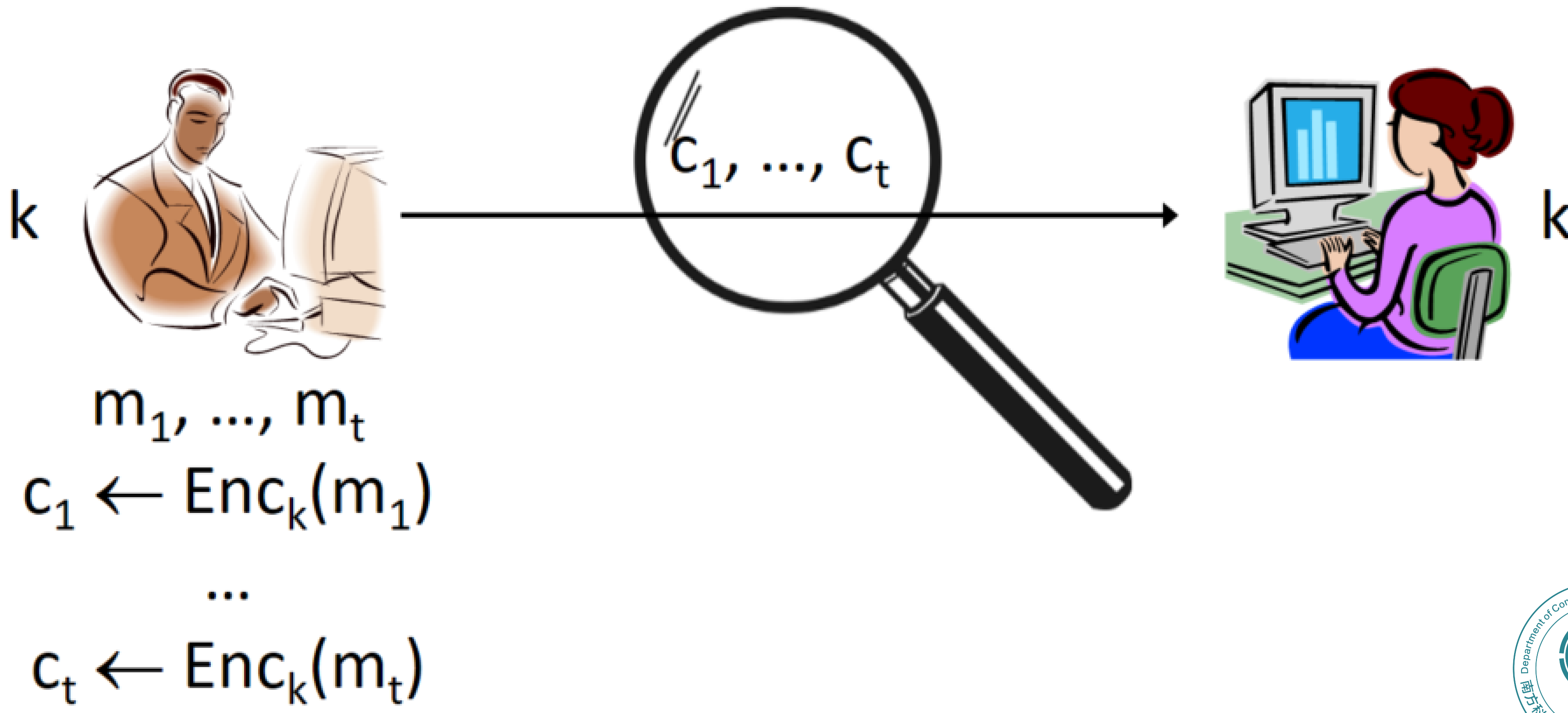

A formal definition

- Fix Π, A

  Define a randomized experiment $PrivK^{mult}_{A,\Pi}(n)$:

  1. $A(1^n)$ outputs two vectors $(m_{0,1}, \ldots, m_{0,t})$ and $(m_{1,1}, \ldots, m_{1,t})$

     Required that $|m_{0,i}| = |m_{1,i}|$ for all i
  2. $k \leftarrow Gen(1^n)$, $b \leftarrow \{0,1\}$, for all i, $c_i \leftarrow Enc_k(m_{b,i})$
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  Definition 3.4 Π is *multiple-message indistinguishable* if for all PPT attackers A, there is a *negligible* function ϵ such that

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  $\mathcal{Q}$: Show that the pseudo OTP is not multiple-message indistinguishable

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  Instead, define someting *stronger*: security against chosen-plaintext attacks (*CPA-security*)

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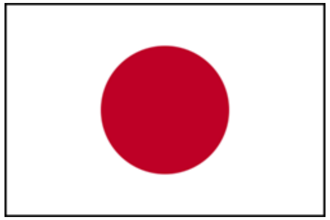
  In practice, there are many ways an attacker can *influence* what gets encrypted

  – Not clear how best to model

  – Chosen-plaintext attacks encompasses any such influence

“Midway” story



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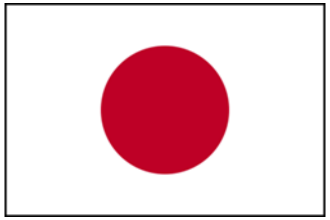

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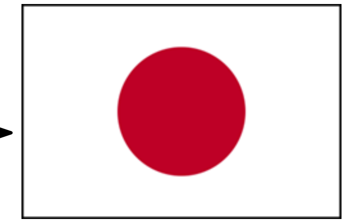

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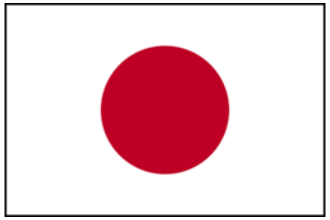

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  3. $b \leftarrow \{0, 1\}$, $c \leftarrow Enc_k(m_b)$, give c to A
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Definition 4.1 Π is *secure against chosen-plaintext attacks (CPA-secure)* if for all PPT attackers A, there is a *negligible* function ϵ such that

$$\Pr[PrivKCPA_{A,\Pi}(n) = 1] \leq 1/2 + \epsilon(n)$$

Impossible?

- Consider the following attacker A;
  - Using a chosen-plaintext attack, get $c_0 = Enc_k(m_0)$ and $c_1 = Enc_k(m_1)$
  - Output m_0, m_1; get challenge ciphertext c
  - If $c = c_0$ output '0'; if $c = c_1$ output '1'
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- This attack only works if encryption is deterministic!
  - randomized encryption must be used!
  - It really is a problem if an attacker can tell when the same message is encrypted twice

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  $\mathcal{Q}$: how many functions are there mapping from $\{0,1\}^n$ to $\{0,1\}^m$?

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- Informally, a *pseudorandom function* "looks like" a random function
  - It does not make sense to talk about any fixed function being pseudorandom. We look instead at *keyed* functions

Keyed functions

- Let $F : \{0,1\}^* \times \{0,1\}^* \to \{0,1\}^*$ be an efficient, deterministic algorithm
  - Define $F_k(x) = F(k, x)$
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- Choosing a uniform $k \in \{0,1\}^n$ is equivalent to choosing the function $F_k : \{0,1\}^n \to \{0,1\}^n$
  - I.e., for fixed key length n, the algorithm F defines a *distribution* over functions in $Func_n$!
  - E.g., $F(k,x) = k$, $F(k,x) = k \oplus x$

PRFs

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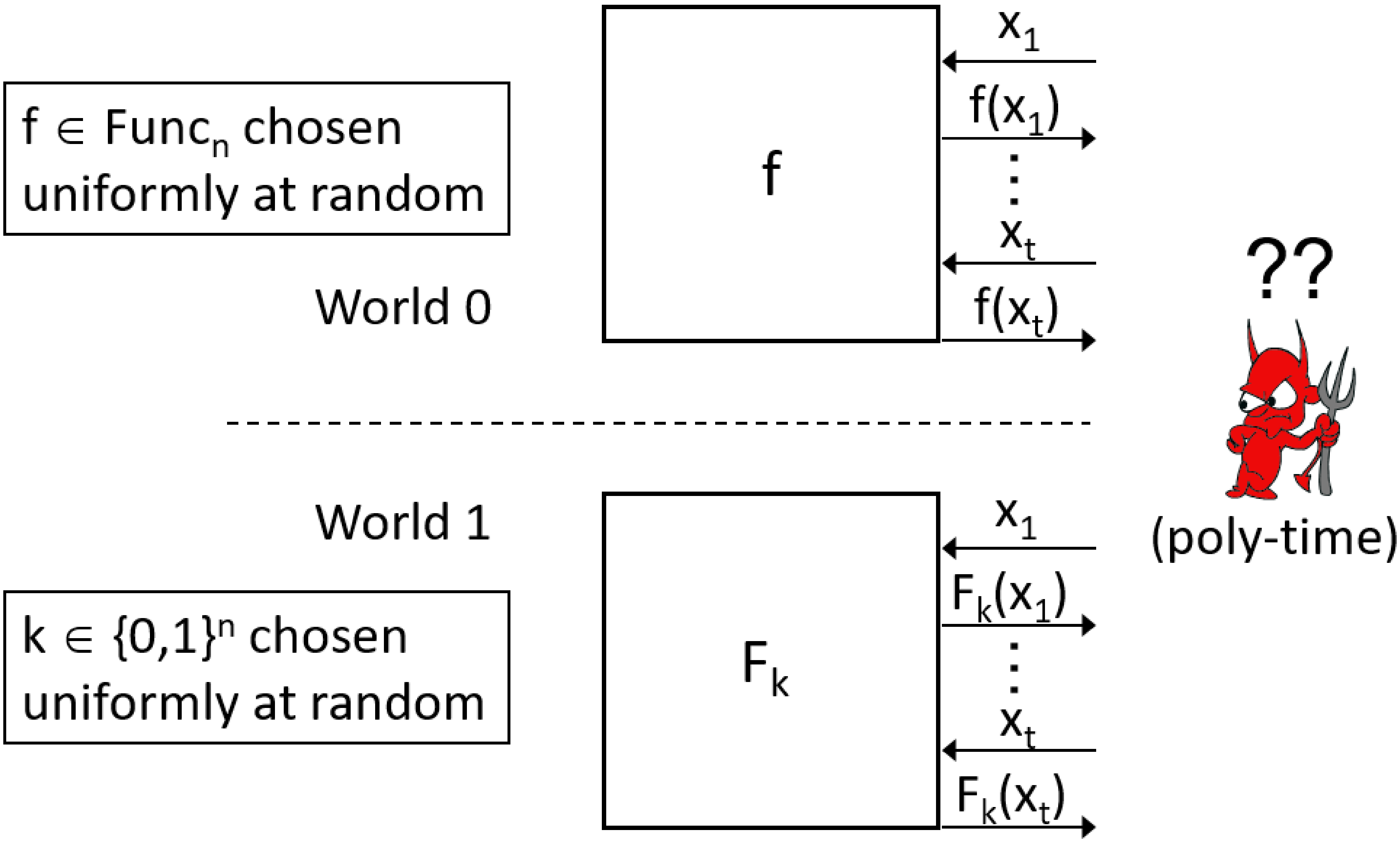
Definition 4.2 F is a *pseudorandom function* if F_k, for uniform $k \in \{0,1\}^n$ is indistinguishable from a uniform function $f \in Func_n$

Formally, for all poly-time distinguishers D:

$$\left|\Pr_{k \leftarrow \{0,1\}^n}[D^{F_k(\cdot)}(1^n) = 1] - \Pr_{f \leftarrow Func_n}[D^{f(\cdot)}(1^n) = 1]\right| \leq \epsilon(n)$$

PRFs


$f \in Func_n$ chosen uniformly at random
World 0
f
x_1
$f(x_1)$
x_t
$f(x_t)$
??
(poly-time)
World 1
$k \in \{0,1\}^n$ chosen uniformly at random
F_k
x_1
$F_k(x_1)$
x_t
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Pseudorandom permutations (PRPs)

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- Let $Perm_n \subset Func_n$ be the set of permutations
  – What is $|Perm_n|$?

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- **Definition 4.3** F is a *pseudorandom permutation* if F_k, for uniform key $k \in \{0,1\}^n$, is indistinguishable from a uniform permutation $f \in Perm_n$
- For large enough n, a random permutation is indistinguishable from a random function.
  - In practice, PRPs are also good PRFs

PRFs vs. PRGs

- PRF F immediately implies a PRG G:
  - Define $G(k) = F_k(0\dots0)|F_k(0\dots1)$
  - I.e., $G(k) = F_k(\langle 0\rangle)|F_k(\langle 1\rangle)|F_k(\langle 2\rangle)|\dots$, where $\langle i\rangle$ denotes the n-bit encoding of i

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- PRF can be viewed as a PRG with random access to exponentially long output
  - The function F_k can be viewed as the $n2^n$-bit string $F_k(0\dots0)|\dots|F_k(1\dots1)$

Do PRFs/PRPs exist?

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  – though can be built from PRGs

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Theorem (Goldreich, Goldwasser, Micali 1984)
If the PRG Axiom is true, then there exist PRFs.

How to Construct Random Functions

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Abstract. A constructive theory of randomness for functions, based on computational complexity, is developed, and a pseudorandom function generator is presented. This generator is a deterministic polynomial-time algorithm that transforms pairs (g, r), where g is *any* one-way function and r is a random k-bit string, to polynomial-time computable functions $f_r: \{1, \ldots, 2^k\} \to \{1, \ldots, 2^k\}$. These f_r's cannot be distinguished from *random* functions by any probabilistic polynomial-time algorithm that asks and receives the value of a function at arguments of its choice. The result has applications in cryptography, random constructions, and complexity theory.

Categories and Subject Descriptors: F.0 [**Theory of Computation**]: General; F.1.1 [**Computation by Abstract Devices**]: Models of Computation—*computability theory*; G.0 [**Mathematics of Computing**]: General; G.3 [**Mathematics of Computing**]: Probability and Statistics—*probabilistic algorithms*; *random number generation*

General Terms: Algorithms, Security, Theory

Additional Key Words and Phrases: Cryptography, one-way functions, prediction problems, randomness

I have set up on a Manchester computer a small programme using only 1000 units of storage, whereby the machine supplied with one sixteen figure number replies with another within two seconds. I would defy anyone to learn from these replies sufficient about the programme to be able to predict any replies to untried values.

A. TURING

Do PRFs/PRPs exist?

- They are a stronger primitive than PRGs
  – though can be built from PRGs

- In practice, block ciphers are used

Next Lecture

- block cipher ...