Assignment#2 CS201 Fall 2023

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PROBLEM 1. Suppose that A, B and C are finite sets, determine whether the following statements are true of false and explain.

SOLUTION. **a)** True. Because

$$\begin{aligned} A \cap B \neq \emptyset &\equiv \exists x,\ x \in A \cap B && \text{(Premise)} \\ A - B &\equiv A \cap \bar{B} && \text{(Difference Definition)} \\ A \cap \bar{B} \subseteq A &\equiv A - B \subseteq A && \text{(Equivalence)} \\ A \cap B \neq \emptyset &\equiv \exists x,\ (x \in A) \wedge (x \notin A \cap \bar{B}) && \text{(Derived from Premise)} \\ \text{Thus, } A \cap B \neq \emptyset &\rightarrow (A - B \subset A) && \text{(Proper Subset Definition)} \end{aligned}$$

b) False. Counterexample: since $A \subseteq B$, if $A = B$ then $|A \cup B| = |B| = |A| \ngeq 2|A|$, which is contrary to $(A \subseteq B) \rightarrow (|A \cup B| \geq |A|)$.

c) False. Since

$$\begin{aligned} \overline{(A - B)} \cap \overline{(B - A)} &= \overline{(A \cap \bar{B})} \cap \overline{(B \cap \bar{A})} \\ &= (\bar{A} \cup B) \cap (\bar{B} \cup A) \\ &= \big((\bar{A} \cup B) \cap \bar{B}\big) \cup \big((\bar{A} \cup B) \cap A\big) \\ &= \big((\bar{A} \cap \bar{B}) \cup (\bar{B} \cap B)\big) \cup \big((\bar{A} \cap A) \cup (A \cap B)\big) \\ &= (\bar{A} \cap \bar{B}) \cup (A \cap B) \\ &= \overline{(A \cup B)} \cup (A \cap B) \neq \overline{(A \cup B)} \end{aligned}$$

Problem 2. Let A, B and C be sets, and prove the following set identities.

Solution. **a)** Proof

$$\begin{aligned}
\overline{A \cap (B \cup C)} &= \bar{A} \cup \overline{(B \cup C)} && \text{De Morgan} \\
&= \bar{A} \cup (\bar{B} \cap \bar{C}) && \text{De Morgan} \\
&= \bar{A} \cup (\bar{C} \cap \bar{B}) && \text{Commutative} \\
&= (\bar{C} \cap \bar{B}) \cup \bar{A} && \text{Commutative}
\end{aligned}$$

b) Proof

$$\begin{aligned}
(A - B) \cap (B - A) &= (A \cap \bar{B}) \cap (B \cap \bar{A}) && \text{Difference} \\
&= A \cap \bar{B} \cap (B \cap \bar{A}) && \text{Associative} \\
&= A \cap \bar{B} \cap B \cap \bar{A} && \text{Associative} \\
&= (\bar{B} \cap B) \cap (A \cap \bar{A}) && \text{Commutative} \\
&= \emptyset \cap \emptyset && \text{Complement} \\
&= \emptyset && \text{Identity}
\end{aligned}$$

Problem 3. Proof the following statements.

Solution. **a)** Suppose A_n is a set of subsets of A with n elements. And let $B = \{(k_1, k_2, \cdots, k_n) \big| k_i \in \mathbb{N}\}$, which means B is the set includes finite cartesian products of the $\mathbb{N}$, so B is countable. Thus, there exists a bijective function $f : A_n \to B$, and since B is countable set, A_n is also countable set and $F = \bigcup_{n=0}^{\infty} A_n$ is countable set. F is the set of all subsets of A.
b) Since there's a surjective function from A to B,let A_s be the subset of A that satisfy there exists a bijective function $f : B \to A_s$. From (a), since A is countable, A_i is also countable. Thus, B is also countable.

Problem 4. The *symmetric difference* of A and B, denoted as $A \Delta B$, is the set containing those elements in A or B (not in both A and B). Answer the following questions.

Solution. **a)** Yes. Explanation

$$
\begin{aligned}
A \,\Delta\, B &= (A \cap \bar{B}) \cup (B \cap \bar{A}) \\
A \,\Delta\, (B \,\Delta\, C) &= \big(A \,\Delta\, (B \cap \bar{C}) \cup (C \cap \bar{B})\big) \\
&= \Big(A \cap \overline{(B \cap \bar{C}) \cup (C \cap \bar{B})}\Big) \cup \big(((B \cap \bar{C}) \cup (C \cap \bar{B})) \cap \bar{A}\big) \\
&= \Big(A \cap \overline{(B \cap \bar{C})} \cap \overline{(C \cap \bar{B})}\Big) \cup \big(((B \cap \bar{C}) \cup (C \cap \bar{B})) \cap \bar{A}\big) \\
&= \big(A \cap (\bar{B} \cup C) \cap (\bar{C} \cup B)\big) \cup \big(((B \cap \bar{C}) \cup (C \cap \bar{B})) \cap \bar{A}\big) \\
&= \big(A \cap (\bar{B} \cup C) \cap (\bar{C} \cup B)\big) \cup \big((B \cap \bar{C} \cap \bar{A}) \cup (C \cap \bar{B} \cap \bar{A})\big) \\
&= \big(A \cap \big((C \cap (\bar{C} \cup B)) \cup \big(\bar{B} \cap (\bar{C} \cup B)\big)\big)\big) \cup \big((B \cap \bar{C} \cap \bar{A}) \cup (C \cap \bar{B} \cap \bar{A})\big) \\
&= (A \cap B \cap C) \cup (A \cap \bar{B} \cap \bar{C}) \cup (\bar{A} \cap B \cap \bar{C}) \cup (\bar{A} \cap \bar{B} \cap C) \\
(A \,\Delta\, B) \,\Delta\, C &= \big((A \cap \bar{B}) \cup (B \cap \bar{A})\big) \;\Delta\, C \\
&= \Big(C \cap \overline{(A \cap \bar{B}) \cup (B \cap \bar{A})}\Big) \cup \big(((A \cap \bar{B}) \cup (B \cap \bar{A})) \cap \bar{C}\big) \\
&= \Big(C \cap \overline{(A \cap \bar{B})} \cap \overline{(B \cap \bar{A})}\Big) \cup \big(((A \cap \bar{B}) \cup (B \cap \bar{A})) \cap \bar{C}\big) \\
&= \big(C \cap (A \cup \bar{B}) \cap (B \cup \bar{A})\big) \cup \big(((A \cap \bar{B}) \cup (B \cap \bar{A})) \cap \bar{C}\big) \\
&= \big(C \cap (A \cup \bar{B}) \cap (B \cup \bar{A})\big) \cup \big((A \cap \bar{B} \cap \bar{C}) \cup (B \cap \bar{A} \cap \bar{C})\big) \\
&= \big(C \cap \big((A \cap (B \cup \bar{A})) \cup \big(\bar{B} \cap (\bar{A} \cup B)\big)\big)\big) \cup \big((A \cap \bar{B} \cap \bar{C}) \cup (B \cap \bar{A} \cap \bar{C})\big) \\
&= (A \cap B \cap C) \cup (A \cap \bar{B} \cap \bar{C}) \cup (\bar{A} \cap B \cap \bar{C}) \cup (\bar{A} \cap \bar{B} \cap C)
\end{aligned}
$$

Thus, $A \,\Delta\, (B \,\Delta\, C) = (A \,\Delta\, B) \,\Delta\, C$.

b) True. Proof

$$
\begin{gathered}
A \,\Delta\, C = B \,\Delta\, C \\
\Rightarrow (A \,\Delta\, C) \,\Delta\, C = (B \,\Delta\, C) \,\Delta\, C \\
\Rightarrow A \,\Delta\, (C \,\Delta\, C) = B \,\Delta\, (C \,\Delta\, C)
\end{gathered}
$$

$$\Rightarrow A \,\Delta\, \emptyset = B \,\Delta\, \emptyset$$
$$\Rightarrow A = B$$

c) Example: $A = \mathbb{R}$, $B = \overline{\mathbb{Q}}$, meaning A is the set of real number and B is the set of irrational number. From class, we've proved that $\mathbb{R}$ and $\overline{\mathbb{Q}}$ are uncountable. We got $A \,\Delta\, B = \mathbb{Q}$ and we've proved that $\mathbb{Q}$ is countable set.

PROBLEM 5. For finite sets A, B and C, explain why the following inclusion-exclusion formula is true. $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

SOLUTION. Proof

$$\begin{aligned}
|A \cup B \cup C| &= |A \cup (B \cup C)| \\
&= |A| + |(B \cup C)| - |A \cap ((B \cup C))| \\
&= |A| + |B| + |C| - |B \cap C| - |A \cap ((B \cup C))| \\
&= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)| \\
&= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|
\end{aligned}$$

PROBLEM 6. Show that if A, B, C and D are (probably infinite) sets with $|A| = |B|$ and $|C| = |D|$, then $|A \times C| = |B \times D|$.

SOLUTION. Since $|A| = |B|$ and $|C| = |D|$, we are given bijections $f : A \to B$ and $g : C \to D$. Then there must exist a bijection that $F : (a, c) \to (f(a), g(c))$ where $a \in A$ and $c \in C$, which is also $F : A \times C \to B \times D$. Thus, we have $|A \times C| = |B \times D|$.

PROBLEM 7. Consider two functions $g : A \to B$ and $f : B \to C$ and its composition function $f \circ g$. Answer the following questions.

Solution. **a)** No. Counterexample: let $A = C = \{1, 2\}$, $B = \{a, b, c\}$ and define the functions $f : a \mapsto 1, b \mapsto 2, c \mapsto 2$ and $g : 1 \mapsto a, 2 \mapsto b$. Then it's obvious that $f \circ g$ and g is one-to-one, but f is not one-to-one.
b) Yes. Similar to (c) and the condition f is one-to-one is unnecessary.
c) Yes. Suppose $x, y \in A$ and $g(x) = g(y)$, then $f\big(g(x)\big) = f\big(g(y)\big)$. Since $f \circ g$ is one-to-one and $(f \circ g)(x) = (f \circ g)(y)$, we have $x = y$, which means g is one-to-one.
d) Yes. Since $f \circ g$ is onto, for any $c \in C$ there exists $a \in A$ such that $c = (f \circ g)(a) = f\big(g(a)\big)$. Let $b = g(a) \in B$, then $c = f(b)$. Because c is arbitary elements in C, f is onto.
e) No. Counterexample: let $A = C = \{1, 2\}$, $B = \{a, b, c\}$ and define the functions $f : a \mapsto 1, b \mapsto 2, c \mapsto 2$ and $g : 1 \mapsto a, 2 \mapsto b$. It can be verified that $f \circ g$ and f is onto, but g is not onto.

Problem 8. Let x be a real number, prove that

$$\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$$

Solution. By definition, $x = \lfloor x \rfloor + y$ where $0 \le y < 1$
Case 1: If $0 \le y < \frac{1}{3}$ then $0 \le 3y < 1, \quad 0 \le y + \frac{1}{3} < \frac{2}{3}, \quad 0 \le y + \frac{2}{3} < 1$

$$\begin{aligned}
\lfloor 3x \rfloor &= \lfloor 3\lfloor x \rfloor + 3y \rfloor = 3\lfloor x \rfloor + 3\lfloor y \rfloor = 3\lfloor x \rfloor \\
\lfloor x + \frac{1}{3} \rfloor &= \lfloor \lfloor x \rfloor + y + \frac{1}{3} \rfloor = \lfloor x \rfloor + \lfloor y + \frac{1}{3} \rfloor = \lfloor x \rfloor \\
\lfloor x + \frac{2}{3} \rfloor &= \lfloor \lfloor x \rfloor + y + \frac{2}{3} \rfloor = \lfloor x \rfloor + \lfloor y + \frac{2}{3} \rfloor = \lfloor x \rfloor
\end{aligned}$$

Case 2: If $\frac{1}{3} \le y < \frac{2}{3}$ then $1 \le 3y < 2, \quad 0 \le y + \frac{1}{3} < 1, \quad 1 \le y + \frac{2}{3} < \frac{4}{3}$

$$\begin{aligned}
\lfloor 3x \rfloor &= \lfloor 3\lfloor x \rfloor + 3y \rfloor = 3\lfloor x \rfloor + 3\lfloor y \rfloor = 3\lfloor x \rfloor + 1 \\
\lfloor x + \frac{1}{3} \rfloor &= \lfloor \lfloor x \rfloor + y + \frac{1}{3} \rfloor = \lfloor x \rfloor + \lfloor y + \frac{1}{3} \rfloor = \lfloor x \rfloor \\
\lfloor x + \frac{2}{3} \rfloor &= \lfloor \lfloor x \rfloor + y + \frac{2}{3} \rfloor = \lfloor x \rfloor + \lfloor y + \frac{2}{3} \rfloor = \lfloor x \rfloor + 1
\end{aligned}$$

Case 3: If $\frac{2}{3} \leq y < 1$ then $2 \leq 3y < 3, \quad 1 \leq y + \frac{1}{3} < \frac{4}{3}, \quad 1 \leq y + \frac{2}{3} < \frac{5}{3}$

$$\begin{aligned}
\lfloor 3x \rfloor &= \lfloor 3\lfloor x \rfloor + 3y \rfloor = 3\lfloor x \rfloor + 3\lfloor y \rfloor = 3\lfloor x \rfloor + 2 \\
\lfloor x + \frac{1}{3} \rfloor &= \lfloor \lfloor x \rfloor + y + \frac{1}{3} \rfloor = 3\lfloor x \rfloor + \lfloor y + \frac{1}{3} \rfloor = \lfloor x \rfloor + 1 \\
\lfloor x + \frac{2}{3} \rfloor &= \lfloor \lfloor x \rfloor + y + \frac{2}{3} \rfloor = 3\lfloor x \rfloor + \lfloor y + \frac{2}{3} \rfloor = \lfloor x \rfloor + 1
\end{aligned}$$

PROBLEM 9. Derive the *closed formula* for $\sum_{k=0}^{m} \lfloor \sqrt{k} \rfloor$.

SOLUTION. We have

$$\lfloor k \rfloor = p \Rightarrow p \leq \sqrt{k} < p + 1 \Rightarrow p^2 \leq k < (p+1)^2$$

so

$$\sum_{k=0}^{m} \lfloor \sqrt{k} \rfloor = \sum_{p=0}^{\lfloor \sqrt{m+1} \rfloor - 1} \sum_{k=p^2}^{(p+1)^2 - 1} \lfloor \sqrt{k} \rfloor = \sum_{p=0}^{\lfloor \sqrt{m+1} \rfloor - 1} p(2p+1)$$

Thus, the closed formula will be

$$\begin{aligned}
\sum_{k=0}^{m} \lfloor \sqrt{k} \rfloor &= \sum_{p=0}^{\lfloor \sqrt{m+1} \rfloor - 1} p(2p+1) \\
&= \sum_{p=0}^{\lfloor \sqrt{m+1} \rfloor - 1} 2p^2 + \sum_{p=0}^{\lfloor \sqrt{m+1} \rfloor - 1} p \\
&= \frac{\lfloor \sqrt{m+1} \rfloor (\lfloor \sqrt{m+1} \rfloor - 1)(2\lfloor \sqrt{m+1} \rfloor - 1)}{3} + \frac{\lfloor \sqrt{m+1} \rfloor (\lfloor \sqrt{m+1} \rfloor - 1)}{2}
\end{aligned}$$

PROBLEM 10. Apply the Schröder-Bernstein theorem to prove $(0,1)$ and $[0,2]$ have the same cardinality.

SOLUTION. Construct two one-to-one functions f and g

$$\begin{aligned}
f &: (0,1) \to [0,2] \quad f(x) = 2x \\
g &: [0,2] \to (0,1) \quad g(x) = \frac{x}{4} + \frac{1}{3}
\end{aligned}$$

Thus, there exists a bijective function between $(0,1)$ and $[0,2]$ so as $\big|(0,1)\big| = \big|[0,2]\big|$.

PROBLEM 11. Show that when Hilbert's Grand Hotel is fully occupied one can still accommodate countably infinite new guests in it.

SOLUTION. As the hotel has countably infinite number of rooms, we can define two functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ and $g : \mathbb{Z}^+ \to \mathbb{Z}^+$ that $f(x) = 2x - 1$ and $g(x) = 4x - 2$. It's guarantee that there doesn't exist $i, j \in \mathbb{Z}^+$ that makes $f(i) = g(j)$ because the solution $i = 2j - \frac{1}{2}$ is invalid. Thus, we can move the i-th current guest to the room No.$f(i)$ and accommodate the j-th new guest to the room No.$g(j)$. Thus, we can always accommodate countably infinite new guests without evicting any current guests.

PROBLEM 12. To show that there exists uncomputable funcions, it suffices to prove the following two parts: **a)**The set of all computer programs in all existing programming languages is countable **b)**The set of all functions from $\mathbb{Z}^+$ to the set of digits $\{0, 1, \cdots, 9\}$ is uncountable.

SOLUTION. **a)** Let S be the set of *finite strings* constructed from the the *finite alphabet* consisting of the characters used in all computer programs. The order is defined by ASCII (excluding the control characters). As the theorem "the set of finite strings S over a finite alphabet A is countable", we enumerate the strings in S and do as follows,

1. feed s into the the corresponding compiler
2. if the compiler accpets the string, add s to the list; otherwise, skip
3. move on to the next string s

Thus, we got a bijection $f : \mathbb{Z}^+ \to S$
b) Assume that the set of all functions from $\mathbb{Z}^+$ to the set of digits $\{0, 1, \cdots, 9\}$

is countable. Then, denote the set as $F = \{f_1, f_2, \cdots, f_n\}$. And we have

$$
\begin{aligned}
&f_1 : \mathbb{Z}^+ \to \{d_{11}d_{12}d_{13}\cdots\} \\
&f_2 : \mathbb{Z}^+ \to \{d_{21}d_{22}d_{23}\cdots\} \\
&f_3 : \mathbb{Z}^+ \to \{d_{31}d_{32}d_{33}\cdots\} \\
&\ldots \\
&d_{ij} \in \{0, 1, 2, \cdots, 9\}
\end{aligned}
$$

And we can construct a function such that

$$
\begin{aligned}
g : \mathbb{Z}^+ &\to \{d_1 d_2 d_3 \cdots\} \\
\text{where} \quad & d_i \neq d_{ii}
\end{aligned}
$$

the function is valid but not included in F. Thus, the set is uncountable.

Problem 13. Prove that for any $a > 1$, $\Theta(\log_a n) = \Theta(\log_2 n)$.

Solution. Since

$$\log_a n = \frac{\log_2 n}{log_2 a}$$

there exists a constant $c_1 = \frac{1}{log_2 a} > 0$ and $c_2 = log_2 a > 0$ such that

$$
\begin{aligned}
\log_a n \leq c_1 \log_2 n \quad &\text{and} \quad c_2 \log_a n \geq \log_2 n \\
\log_2 n \leq c_2 \log_a n \quad &\text{and} \quad c_1 \log_2 n \geq \log_a n
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\log_a n = O(\log_2 n) \quad &\text{and} \quad \log_a n = \Omega(\log_2 n) \\
\log_2 n = O(\log_a n) \quad &\text{and} \quad \log_2 n = \Omega(\log_a n)
\end{aligned}
$$

That is,

$$\Theta(\log_a n) = \Theta(\log_2 n)$$

PROBLEM 14. Given a pseudocode of the Binary Search Algorithm with a increasing sequence of n distinct integers $a_1, a_2, \cdots, a_n$ and answer the questions.

SOLUTION. **a)** $\Theta(\log n)$. The while loop takes $\log_2 n$ comparison operations and the rest takes 1 operations.
b) Add one line of code between 4th and 5th line that

$$\textbf{if } x = a_m \textbf{ then return } m$$

c) $\Theta(n)$. The input take $\Theta(n)$ memory and algorithm takes $\Theta(1)$ memory.
d) If the computer uses *size_t* as integer, then each integer takes 8 *bytes* and the input takes $8(n+1)$ *bytes*.