

FOL -- Conjunctive Normal Form

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Exercises 12 : Reading and More

Record your time spent (in 0.1 hours) with brief tasks and durations in your learning log by hand writing!

- 1) Read [textI-ch03-3.1-FormalProofs.pdf](#) (cont.)
- 2) Read [textB-ch02-2.3-Proofs-fol.pdf](#) (cont.)
- 3) Read [textI-ch03-3.2-NormalForms.pdf](#) (in 2 weeks)

CNF normalization steps

We can convert any FOL sentence into a first-order logic conjunctive normal form (FOL CNF).

We will define FOL CNF by following the process of transformation.

The following transformations result in the CNF.

1. **Rename apart** : rename variables for each quantifier
2. **Negation normal form** : push negation inside
3. **Prenex form** : pull quantifiers to front
4. **Skolemization**: remove existential quantifiers (only satisfiability preserving)
5. **CNF transformation**: turn the quantifier-free part of the sentence into CNF
6. **Syntactical removal of universal quantifiers**: a CNF with free variables.

Topic 12.1

Step 1: Rename Apart

Name does not matter

Theorem 12.1

If $x, y \notin FV(F(z))$, then $\forall x.F(x)$ and $\forall y.F(y)$ are provably equivalent .

Proof.

1. $\{\forall x.F(x)\} \vdash \forall x.F(x)$ Assumption
2. $\{\forall x.F(x)\} \vdash F(y)$ \forall -Instantiation applied to 1
3. $\{\forall x.F(x)\} \vdash \forall y.F(y)$ \forall -Intro applied to 2, since $y \notin FV(\forall x.F(x))$

We can run the proof in both the directions. □

Thinking Exercise 12.1

- a. Prove: if $x, y \notin FV(F(z))$, then $\exists x.F(x)$ and $\exists y.F(y)$ are provably equivalent.
- b. Give proof for renaming a quantified variable to a fresh name that is not on the top.

Step 1: rename apart

Definition 12.1

A formula F is *renamed apart* if no quantifier in F use a variable that is used by another quantifier or occurs as free variable in F .

Due to the previous theorem, we can assume that every quantifier has different variable. If not, we can *rename quantified variables apart*.

Example 12.1

Consider formula $\neg(\exists x.\forall y.R(x,y) \Rightarrow \forall y.\exists x.(R(x,y) \wedge P(x)))$. After renaming apart we obtain the following

$$\neg(\exists x.\forall y.R(x,y) \Rightarrow \forall z.\exists w.(R(w,z) \wedge P(w)))$$

Topic 12.2

Step 2: Negation normal form

Relating \forall and \exists

Theorem 12.2 *If we have $\Sigma \vdash \neg \exists x. \neg F(x)$, we can prove $\Sigma \vdash \forall x. F(x)$.*

Proof.

1. $\Sigma \vdash \neg \exists x. \neg F(x)$ Premise
2. $\Sigma \cup \{\neg F(y)\} \vdash \neg F(y)$ Assumption (choose fresh y (why?))
3. $\Sigma \cup \{\neg F(y)\} \vdash \exists x. \neg F(x)$ \exists -Intro
4. $\Sigma \vdash F(y)$ propositional rules applied to 1 and 3
5. $\Sigma \vdash \forall x. F(x)$ \forall -Intro on 4

□

Thinking Exercise 12.2

- a. *Prove: if we have $\Sigma \vdash \neg \forall x. F(x)$, we can prove $\Sigma \vdash \exists x. \neg F(x)$.*
- b. *Prove: if we have $\Sigma \vdash \neg \exists x. F(x)$, we can prove $\Sigma \vdash \forall x. \neg F(x)$. (Hint: replace $\neg F(\cdot)$ by $F(\cdot)$ in the above proof)*

Step 2: Negation normal form(NNF)

Definition 12.2

*A formula F is in **negation normal form** if all the negation symbols in the formula occur in form of atomic formulas.*

Due to the previous theorems and the properties of propositional connectives, we can translate any formula in negation normal form.

Example: Negation normal form

Exercise 12.3

We convert $\neg(\exists x.\forall y.R(x,y) \Rightarrow \forall z.\exists w.(R(w,z) \wedge P(w)))$ into NNF as follows

$$\begin{aligned}\neg(\exists x.\forall y.R(x,y) \Rightarrow \forall z.\exists w.(R(w,z) \wedge P(w))) &\equiv (\exists x.\forall y.R(x,y) \wedge \neg\forall z.\exists w.(R(w,z) \wedge P(w))) \\ &\equiv (\exists x.\forall y.R(x,y) \wedge \exists z.\neg\exists w.(R(w,z) \wedge P(w))) \\ &\equiv (\exists x.\forall y.R(x,y) \wedge \exists z.\forall w.\neg(R(w,z) \wedge P(w))) \\ &\equiv (\exists x.\forall y.R(x,y) \wedge \exists z.\forall w.(\neg R(w,z) \vee \neg P(w)))\end{aligned}$$

Topic 12.3

Step 3: Prenex form

No occurrence; no issues

Theorem 12.3

Let x be a variable such that $x \notin FV(F)$. Then F , $\exists x.F$, and $\forall x.F$ are provably equivalent.

Proof.

We have already seen $\forall x.F$ to $\exists x.F$.

Proving from F to $\forall x.F$

1. $\Sigma \vdash F$ Premise
2. $\Sigma \vdash \forall x.F$ \forall -Intro applied to 1

Since x is not in F , we choose $y, z \notin FV(\Sigma \cup \{F\})$ and say $F(z)\{z \mapsto y\} = F$. \forall -Intro conditions are met.(why?)

Proving from $\exists x.F$ to F

1. $\Sigma \vdash \exists x.F$ Premise
2. $\Sigma \cup \{F\} \vdash F$ Assumption
3. $\Sigma \vdash F \Rightarrow F$ propositional rules applied to 2
4. $\Sigma \vdash \exists x.F \Rightarrow F$ \exists -Elim applied to 3
5. $\Sigma \vdash F$ propositional rules applied to 4 and 1

□

No occurrence; we can pull quantifiers to top

Theorem 12.4

If $x \notin FV(G)$, then $\exists x.F(x) \wedge \exists x.G$ and $\exists x.(F(x) \wedge G)$ are provably equivalent.

Proof.

Reverse direction is trivial. Consider the forward direction.

Commentary: If x occurs in G , which step of the following proof does not work?

1. $\Sigma \vdash \exists x.F(x) \wedge \exists x.G$

2. $\Sigma \vdash \exists x.G$

3. $\Sigma \vdash G$

4. $\Sigma \cup \{F(x)\} \vdash F(x) \wedge G$

5. $\Sigma \cup \{F(x)\} \vdash \exists x.(F(x) \wedge G)$

6. $\Sigma \vdash F(x) \Rightarrow \exists x.(F(x) \wedge G)$

7. $\Sigma \vdash \exists x.F(x) \Rightarrow \exists x.(F(x) \wedge G)$

8. $\Sigma \vdash \exists x.(F(x) \wedge G)$

Premise

propositional rules applied to 1

previous theorem applied to 2

propositional rules applied to 3

\exists -Intro applied to 4

\Rightarrow -Intro applied to 5

\exists -Elim applied to 6

propositional rules applied to 7 and 1 \square

Thinking Exercise 12.4

If $x \notin FV(G)$, then $\forall x.F(x) \vee \forall x.G$ and $\forall x.(F(x) \vee G)$ are provably equivalent.

Step 3: Prenex form

Definition 12.3

A formula F is in *prenex form* if all the quantifiers of the formula occur as prefix of F . The quantifier-free suffix of F is called *matrix of F* .

Due to the previous theorems, we move quantifiers to the front.

Thinking Exercise 12.5

Show that the following equivalences hold.

$$\blacktriangleright \forall x.F \Rightarrow G \equiv \exists x.(F \Rightarrow G)$$

$$\blacktriangleright \exists x.F \Rightarrow G \equiv \forall x.(F \Rightarrow G)$$

$$\blacktriangleright F \Rightarrow \forall x.G \equiv \forall x.(F \Rightarrow G)$$

$$\blacktriangleright F \Rightarrow \exists x.G \equiv \exists x.(F \Rightarrow G)$$

Example: Prenex form

Thinking Exercise 12.6

We convert $(\exists x. \forall y. R(x, y) \wedge \exists z. \forall w. (\neg R(w, z) \vee \neg P(w)))$ into prenex form as follows

- ▶ $(\exists x. \forall y. R(x, y) \wedge \exists z. \forall w. (\neg R(w, z) \vee \neg P(w)))$
- ▶ $\exists z. (\exists x. \forall y. R(x, y) \wedge \forall w. (\neg R(w, z) \vee \neg P(w)))$
- ▶ $\exists z. \forall w. (\exists x. \forall y. R(x, y) \wedge (\neg R(w, z) \vee \neg P(w)))$
- ▶ $\exists z. \forall w. \exists x. (\forall y. R(x, y) \wedge (\neg R(w, z) \vee \neg P(w)))$
- ▶ $\underbrace{\exists z. \forall w.}_{\text{Quantifiers}} \underbrace{\exists x. \forall y. (R(x, y) \wedge (\neg R(w, z) \vee \neg P(w)))}_{\text{body/matrix of the formula}}$

We move quantifier forward step by step.

In the standard definition of prenex, the body need not be in NNF. Our body is in NNF due to the order of steps we have followed.

Topic 12.4

Step 4: Skolemization

Step 4: Skolemization

Skolemization removes \exists quantifiers from prenex sentences and only \forall quantifiers are left.

Example 12.2

Let us suppose. We know "for every man there is a woman".

$$\forall m. \exists w. \text{Relationship}(m, w)$$

To satisfy the sentence, we need to find a woman for each man.

In other words, there is a function $f: \text{Men} \rightarrow \text{Women}$.

In terms of FOL, we may write

$$\forall m. \text{Relationship}(m, f(m))$$

The replacement of \exists by a function is called **skolemization** and f is called **skolem function**.

Introduction of skolem function with free variables

Theorem 12.5

Let F be a **S**-formula, $FV(F) = \{x, y_1, \dots, y_n\}$ and $f/n \in \mathbf{F}$ does not occur in F . For each model m' , there is a model m such that

Commentary: m is not with any assignment, which means for any assignment.

$$m \models \exists x.F \Rightarrow F\{x \mapsto f(y_1, \dots, y_n)\}.$$

and m and m' only differ on the interpretation of f .

Proof.

Consider a model m' . We will construct m . Before, let us construct an interpretation $f' : D_{m'}^n \rightarrow D_{m'}$ of f as follows.

$$f'(d_1, \dots, d_n) \triangleq \begin{cases} d & \text{if } m', \{y_1 \mapsto d_1, \dots, y_n \mapsto d_n\} \models \exists x.F, \\ & \text{Choose } d \in D_{m'} \text{ such that } m', \{y_1 \mapsto d_1, \dots, y_n \mapsto d_n, x \mapsto d\} \models F \\ d & \text{otherwise choose any } d \in D_{m'} \end{cases}$$

Why d exists?

Introduction of skolem function with free variables(contd.)

Proof(contd.)

Let us define $m \triangleq m'[f \mapsto f']$.

Since f does not occur in F , if $m, \nu \models \exists x.F$ then $m', \nu \models \exists x.F$.

Due to the construction of m ,

$$m, \nu \models F\{x \mapsto f(y_1, \dots, y_n)\}_{(\text{why?})}.$$



Thinking Exercise 12.7

Show there is m such that $m \models F\{x \mapsto f(y_1, \dots, y_n)\} \Rightarrow \forall x.F$

Introduction of skolem functions under quantifiers

Theorem 12.6

Let $F(x)$ be a (\mathbf{F}, \mathbf{R}) -formula with $FV(F) = \{x, y_1, \dots, y_n\}$ and $f/n \in \mathbf{F}$ such that f does not occur in $F(x)$.

$$\forall y_1, \dots, y_n. \exists x. F(x) \text{ is sat} \quad \text{iff} \quad \forall y_1, \dots, y_n. F(f(y_1, \dots, y_n)) \text{ is sat}$$

Proof.

Forward direction:

Assume $m' \models \forall y_1, \dots, y_n. \exists x. F(x)$. Therefore, $m' \models \exists x. F(x)$ (why?).

Due to the last theorem, there is m such that $m \models \exists x. F(x) \Rightarrow F(f(y_1, \dots, y_n))$.

Since m and m' only differ on f , $m \models \exists x. F(x)$.

Therefore, $m \models F(f(y_1, \dots, y_n))$. Therefore, $m \models \forall y_1, \dots, y_n. F(f(y_1, \dots, y_n))$.

...

Introduction of skolem functions under quantifiers(contd.)

Proof.

Reverse direction

1. $\{\forall y_1, \dots, y_n. F(f(y_1, \dots, y_n))\} \vdash \forall y_1, \dots, y_n. F(f(y_1, \dots, y_n))$ Assumption
2. $\{\forall y_1, \dots, y_n. F(f(y_1, \dots, y_n))\} \vdash F(f(y_1, \dots, y_n))$ \forall -Elim
3. $\{\forall y_1, \dots, y_n. F(f(y_1, \dots, y_n))\} \vdash \exists x. F(x)$ \exists -Intro
4. $\{\forall y_1, \dots, y_n. F(f(y_1, \dots, y_n))\} \vdash \forall y_1, \dots, y_n. \exists x. F(x)$ \forall -Intro



Skolemization of prenex sentence

Since the quantifiers are in prenex form, all \exists s can be removed using skolem functions.

Skolemization should be applied from out to inside, i.e.,

remove outermost \exists first.

Example 12.3

Let us skolemize the following sentence

- ▶ $\exists z. \forall w. \exists x. \forall y. (R(x, y) \wedge (\neg R(w, z) \vee \neg P(w)))$
- ▶ *Since there are no universals before $\exists z$, we introduce a function $c/0$.*
 $\forall w. \exists x. \forall y. (R(x, y) \wedge (\neg R(w, c) \vee \neg P(w)))$
- ▶ *Since there is a universal $\forall w$ before $\exists x$, we introduce a function $f/1$.*
 $\forall w. \forall y. (R(f(w), y) \wedge (\neg R(w, c) \vee \neg P(w)))$

Topic 12.5

Step 5-6: FOL CNF

Step 5: Convert body of the sentence to CNF

Consider skolemized prenex sentence $\forall x_1, \dots, x_n. F$.

Since F is quantifier-free, we can use propositional logic methods to convert F into CNF

$$\forall x_1, \dots, x_n. C_1 \wedge \dots \wedge C_k.$$

Example 12.4

In our running example, the body of the sentence was already in CNF

$$\forall w. \forall y. (R(f(w), y) \wedge (\neg R(w, c) \vee \neg P(w))).$$

Thinking Exercise 12.8

We may use Tseitin encoding to obtain CNF, which introduces fresh propositional predicates. Is there a quantifier over the propositional predicates? (Hint: there are no propositional variables in FOL and we cannot quantify over predicates.)

Step 6: Drop of explicit mention of quantifiers

Consider skolemized prenex clauses $\forall x_1, \dots, x_n. C_1 \wedge \dots \wedge C_k$.

Since \forall distributes over \wedge , we translate to

$$(\forall x_1, \dots, x_n. C_1) \wedge \dots \wedge (\forall x_1, \dots, x_n. C_k).$$

We may view the above sentence as conjunction of clauses

$$C_1 \wedge \dots \wedge C_k,$$

Since clauses have different quantifiers, even if two clauses share a variable name, they are referring to different variables.

without any explicit mention of quantifiers.

Since we started with sentences, we will assume that the free variables are universally quantified.

Example 12.5

We write the sentence as $R(f(w), y) \wedge (\neg R(w, c) \vee \neg P(w))$

Commentary: Observe that both the occurrences of w in $(\neg R(w, c) \vee \neg P(w))$ refer to same variable. However, the w in $R(f(w), y)$ is a different variable from the w in $(\neg R(w, c) \vee \neg P(w))$

Topic 12.6

Problems

Skolemization

Thinking Exercise 12.9

Demonstrate that skolemization does not produce equivalent formula.

Minimize skolem functions

Thinking Exercise 12.10

The order of quantifiers determines the number of parameters in the skolem functions. Give a greedy and efficient(linear) strategy for producing prenex formula such that the total number of parameters in skolem functions is minimal?

FOL CNF

Thinking Exercise 12.11

Convert the following formulas in FOL CNF

- ▶ $\exists z. (\exists x. Q(x, z) \vee \exists x. P(x)) \Rightarrow \neg(\neg\exists x. P(x) \wedge \forall x. \exists z. Q(z, x))$
- ▶ $\neg\exists n. \forall w. (Gtn(w) \Rightarrow \exists x, y, z. (f(x, y, z) = w \wedge NonEmpty(y) \wedge \neg Gtn(f(x, y)) \wedge (\forall k. L(x, y, z, k))))$

Convert into CNF

Thinking Exercise 12.12

Consider the following formulas

$$\Sigma = \{ \forall x, y, z. (z \in x \Leftrightarrow z \in y) \Rightarrow x \approx y, \\ \forall x, y. (x \subseteq y \Leftrightarrow \forall z. (z \in x \Rightarrow z \in y)), \\ \forall x, y, z. (z \in x - y \Leftrightarrow (z \in x \wedge z \notin y)) \}.$$

Convert the following formula into FOL CNF.

$$\bigwedge \Sigma \wedge \neg \forall x, y. (x \subseteq y \Rightarrow \exists z. (y - z \approx x))$$

Topic 12.7

Extra slides: Proofs for pulling negations out

Relating \forall and \exists (reverse direction)

Commentary: $c \neq c$ is proxy for false. Reflex rule allows us to have a representation of true without having true as a symbol. It may feel like cheating.

Theorem 12.7 *If we have $\Sigma \vdash \forall x.F(x)$, we can prove $\Sigma \vdash \neg \exists x. \neg F(x)$.*

Proof.

1. $\Sigma \vdash \forall x.F(x)$ Premise
2. $\Sigma \cup \{\neg F(x)\} \vdash \forall x.F(x)$ Monotonic applied to 1
3. $\Sigma \cup \{\neg F(x)\} \vdash F(x)$ \forall -Elim applied to 2
4. $\Sigma \cup \{\neg F(x)\} \vdash \neg F(x) \wedge F(x)$ propositional rules applied to 3
5. $\Sigma \vdash \neg F(x) \Rightarrow c \neq c$ Contra applied to 4
6. $\Sigma \vdash \exists x. \neg F(x) \Rightarrow c \neq c$ \exists -Elim applied to 5
7. $\Sigma \vdash c = c$ Reflex
8. $\Sigma \vdash \neg \exists x. \neg F(x)$ propositional rules applied to 6 and 7 \square

Thinking Exercise 12.14

Prove: if we have $\Sigma \vdash \forall x. \neg F(x)$, we can derive $\Sigma \vdash \neg \exists x.F(x)$. Hint : replace $F(\cdot)$ by $\neg F(\cdot)$ in the above.

End of Lecture 12