

Reference Answers for Assignment 01 of I2ML-s23

Remark: BOT for BOTTOM/falsity,
fi/fe for falsity introduction/elimination.

Use natural deduction to prove the validity of the following sequents:

$$1. \quad p \vee q \quad | - \quad q \vee p$$

Proof:

$$\begin{array}{c}
 \frac{}{p} 1 \qquad \frac{}{q} 1 \\
 \hline
 p \vee q \qquad q \vee p \qquad q \vee p \\
 \hline
 q \vee p \quad 1 \text{ Ve}
 \end{array}$$

$$2. \quad (p \wedge q) \wedge r \quad | - \quad p \wedge (q \wedge r)$$

Proof:

$$\begin{array}{c}
 \frac{}{(p \wedge q) \wedge r} \wedge e1 \qquad \frac{}{(p \wedge q) \wedge r} \wedge e1 \qquad \frac{}{(p \wedge q) \wedge r} \wedge e1 \\
 \hline
 p \wedge q \qquad p \wedge q \qquad r \\
 \hline
 p \qquad q \qquad r \\
 \hline
 p \wedge (q \wedge r) \quad \wedge i
 \end{array}$$

$$3. \quad p \rightarrow q \quad | - \quad \sim q \rightarrow \sim p$$

Proof:

$$\begin{array}{c}
 \frac{}{\sim q} 1 \qquad \frac{}{p} 2 \\
 \hline
 \sim q \qquad p \\
 \hline
 \text{BOT} \\
 \hline
 \sim p \quad 2 \text{ fe} \\
 \hline
 \sim q \rightarrow \sim p \quad 1 \rightarrow i
 \end{array}$$

4. $(p \wedge q) \vee (p \wedge r) \mid - p \wedge (q \vee r)$

Proof:

$$\begin{array}{c}
 \begin{array}{c}
 \frac{}{1} \\
 \hline
 p \wedge q \\
 \hline
 \text{---} \wedge e1 \\
 p
 \end{array}
 \quad
 \begin{array}{c}
 \frac{}{1} \\
 \hline
 p \wedge q \\
 \hline
 \text{---} \wedge e2 \\
 q
 \end{array}
 \quad
 \begin{array}{c}
 \frac{}{1} \\
 \hline
 p \wedge r \\
 \hline
 \text{---} \wedge e1 \\
 p
 \end{array}
 \quad
 \begin{array}{c}
 \frac{}{1} \\
 \hline
 p \wedge r \\
 \hline
 \text{---} \wedge e2 \\
 r
 \end{array}
 \\
 \begin{array}{c}
 \text{---} \vee i \\
 q \vee r
 \end{array}
 \quad
 \begin{array}{c}
 \text{---} \vee i \\
 q \vee r
 \end{array}
 \\
 \begin{array}{c}
 \text{---} \wedge i \\
 p \wedge (q \vee r)
 \end{array}
 \quad
 \begin{array}{c}
 \text{---} \wedge i \\
 p \wedge (q \vee r)
 \end{array}
 \\
 \hline
 (p \wedge q) \vee (p \wedge r) \quad p \wedge (q \vee r) \quad \text{---} 1 \vee e \\
 \hline
 p \wedge (q \vee r)
 \end{array}$$

5. $\mid - \sim p \vee q \rightarrow (p \rightarrow q)$

Proof:

$$\begin{array}{c}
 \begin{array}{c}
 \frac{}{2} \\
 \hline
 \sim p
 \end{array}
 \quad
 \begin{array}{c}
 \frac{}{3} \\
 \hline
 p
 \end{array}
 \\
 \text{---} fi \\
 \text{BOT} \\
 \text{---} fe \quad \frac{}{2} \\
 q \quad q
 \\
 \begin{array}{c}
 \frac{}{1} \\
 \hline
 \sim p \vee q
 \end{array}
 \quad
 \begin{array}{c}
 \text{---} 3 \rightarrow i \\
 p \rightarrow q
 \end{array}
 \quad
 \begin{array}{c}
 \text{---} \rightarrow i \\
 p \rightarrow q
 \end{array}
 \\
 \hline
 \sim p \vee q \quad p \rightarrow q \quad \text{---} 2 \vee e \\
 \hline
 p \rightarrow q
 \\
 \text{---} 1 \rightarrow i \\
 \sim p \vee q \rightarrow (p \rightarrow q)
 \end{array}$$

6. $\mid - (p \rightarrow q) \rightarrow \sim p \vee q$

Proof:

$$\begin{array}{c}
 \begin{array}{c}
 \frac{}{1} \\
 \hline
 p \rightarrow q
 \end{array}
 \quad
 \begin{array}{c}
 \frac{}{2} \\
 \hline
 p
 \end{array}
 \\
 \text{---} \rightarrow e \\
 q
 \\
 \begin{array}{c}
 \text{---} lem \\
 p \vee \sim p
 \end{array}
 \quad
 \begin{array}{c}
 \text{---} \vee i2 \\
 \sim p \vee q
 \end{array}
 \quad
 \begin{array}{c}
 \frac{}{2} \\
 \hline
 \sim p
 \end{array}
 \quad
 \begin{array}{c}
 \text{---} \vee i1 \\
 \sim p \vee q
 \end{array}
 \\
 \hline
 \sim p \vee q \quad \text{---} 2 \vee e \\
 \hline
 \sim p \vee q
 \\
 \text{---} 1 \rightarrow i \\
 (p \rightarrow q) \rightarrow \sim p \vee q
 \end{array}$$

7. $\sim(p \rightarrow q) \mid - q \rightarrow p$

Proof:

$$\begin{array}{l}
 \frac{}{q} 1 \\
 \text{-----} \rightarrow i \\
 p \rightarrow q \quad \sim(p \rightarrow q) \\
 \text{-----} \text{fi} \\
 \text{BOT} \\
 \text{--- fe} \\
 p \\
 \text{-----} 1 \rightarrow i \\
 q \rightarrow p
 \end{array}$$

8. $W \rightarrow X, Y \rightarrow Z \mid - W \vee Y \rightarrow X \vee Z$

Proof:

$$\begin{array}{l}
 \frac{}{W \vee Y} 1 \qquad \frac{W \rightarrow X \quad \frac{}{W} 2}{\text{-----} \rightarrow e} \qquad \frac{Y \rightarrow Z \quad \frac{}{Y} 2}{\text{-----} \rightarrow e} \\
 \qquad \qquad \qquad X \qquad \qquad \qquad Z \\
 \qquad \qquad \qquad \text{----- Vi1} \qquad \qquad \text{----- Vi2} \\
 \qquad \qquad \qquad X \vee Z \qquad \qquad X \vee Z \\
 \text{-----} 2 \text{ Ve} \\
 X \vee Z \\
 \text{-----} 1 \rightarrow i \\
 W \vee Y \rightarrow X \vee Z
 \end{array}$$

9. $\mid - \sim(A \wedge B) \rightarrow (A \rightarrow \sim B)$

Proof:

$$\begin{array}{l}
 \frac{}{\sim(A \wedge B)} 1 \qquad \frac{A \quad B}{\text{-----} \wedge i} \\
 \qquad \qquad \qquad A \wedge B \\
 \text{-----} \text{fi} \\
 \text{BOT} \\
 \text{--- fe} \\
 \sim B \\
 \text{-----} 1 \text{em} \qquad \text{-----} 3 \rightarrow e \qquad \frac{}{\sim B} 2 \\
 B \vee \sim B \qquad A \rightarrow \sim B \qquad A \rightarrow \sim B \\
 \text{-----} 2 \text{ Ve} \\
 A \rightarrow \sim B \\
 \text{-----} 1 \rightarrow i \\
 \sim(A \wedge B) \rightarrow (A \rightarrow \sim B)
 \end{array}$$

[illegible]