CSE5014 CRYPTOGRAPHY AND NETWORK SECURITY

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Digital signatures

- Provide *integrity* in the public-key setting
- Analogous to *message authentication codes*, but some key differences

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	Private Key	Public Key
Secrecy	private key encryption	public key encryption
Integrity	MAC	??

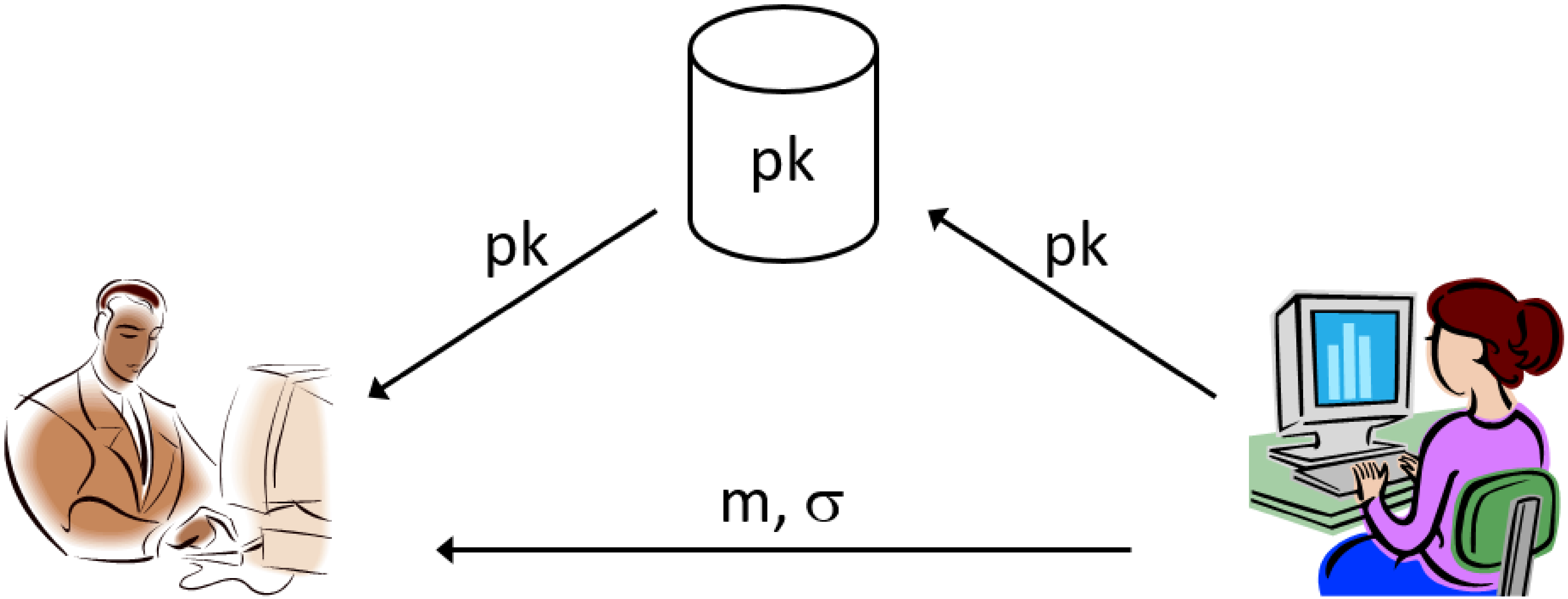
Digital signatures

- A *signature scheme* is defined by three PPT algorithms (*Gen*, *Sign*, *Vrfy*):
  - *Gen*: takes as input 1^n; outputs pk, sk
  - *Sign*: takes as input a private key sk and a message $m \in \{0,1\}^*$; outputs *signature* σ: $\sigma \leftarrow Sign_{sk}(m)$
  - *Vrfy*: takes public key pk, message m, and signature σ as input; outputs 1 or 0

For all m and all pk, sk output by *Gen*,

$$Vrfy_{pk}(m, Sign_{sk}(m)) = 1$$

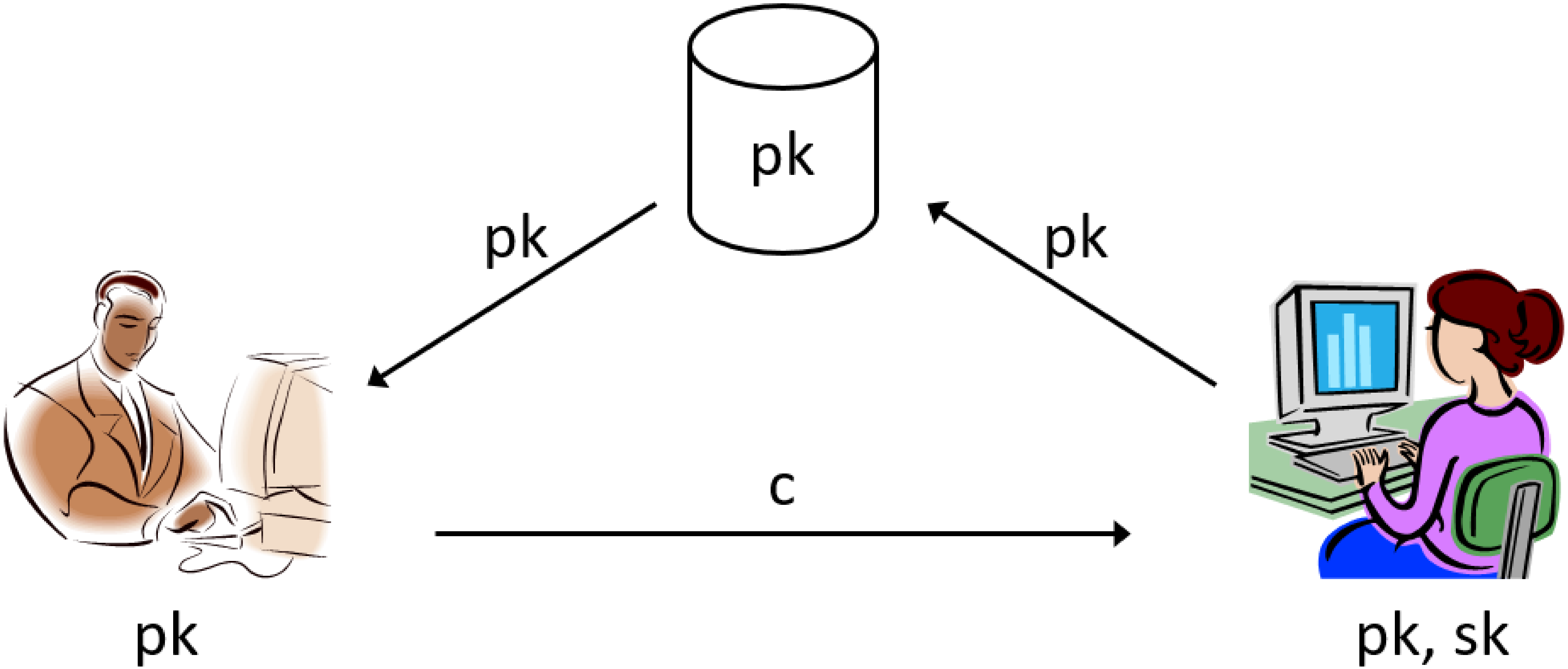
Digital signatures



pk $\qquad\qquad\qquad\qquad$ pk, sk

$1 \stackrel{?}{=} \mathrm{Vrfy}_{pk}(m, \sigma)$ $\qquad\qquad$ $\sigma = \mathrm{Sign}_{sk}(m)$

Public-key encryption



$c \leftarrow \mathrm{Enc}_{pk}(m)$

$m = \mathrm{Dec}_{sk}(c)$

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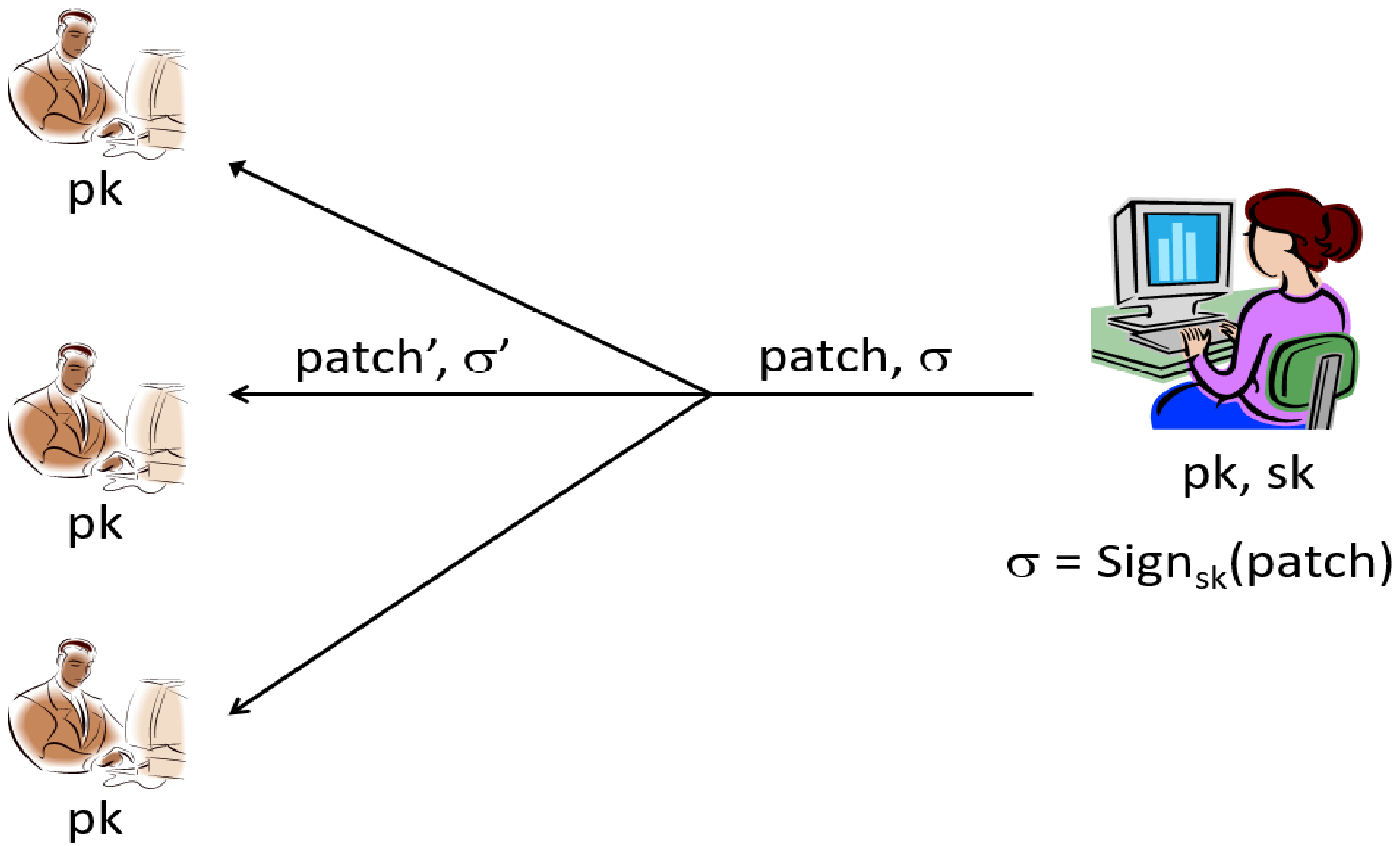
- Prototypical application



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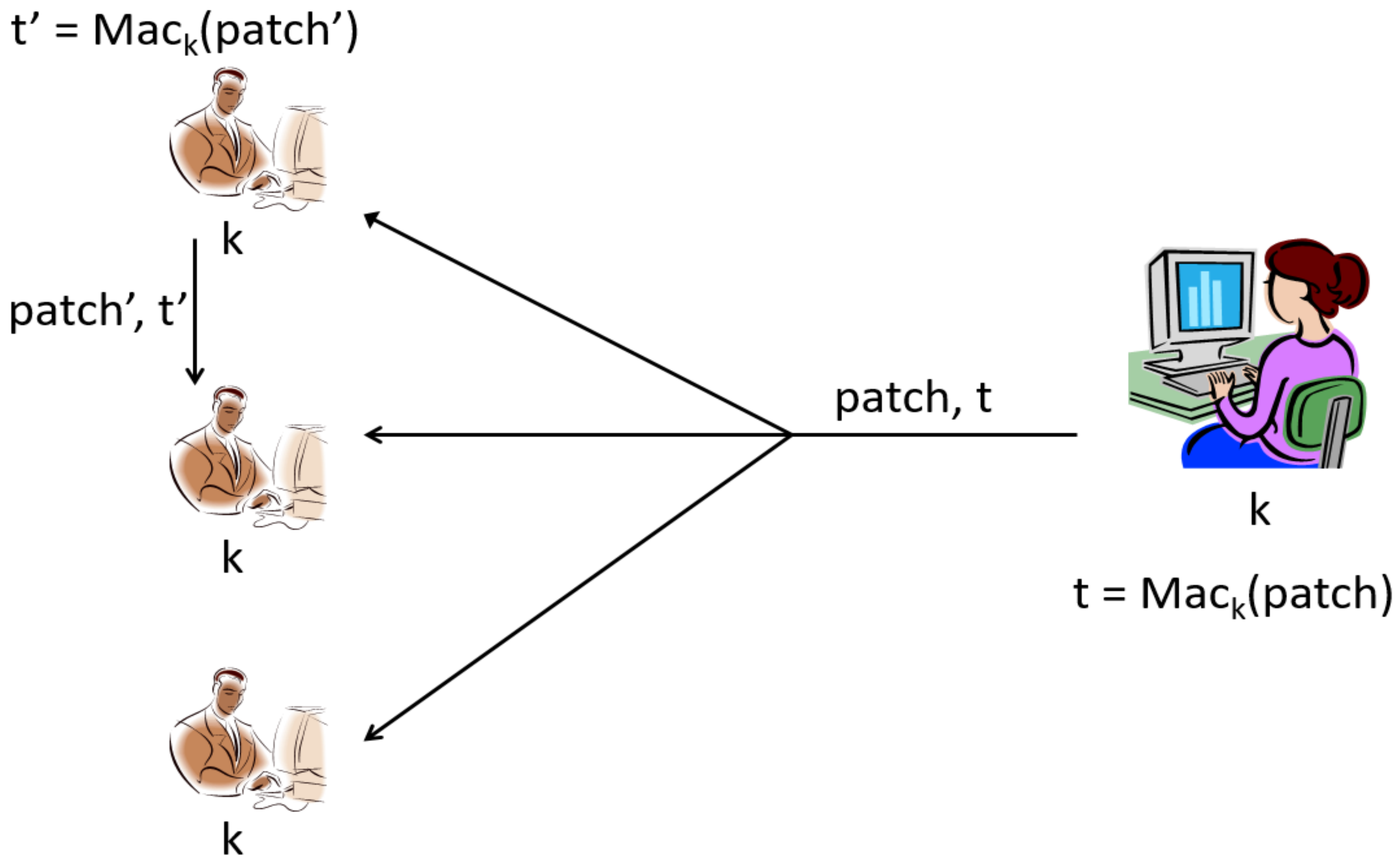

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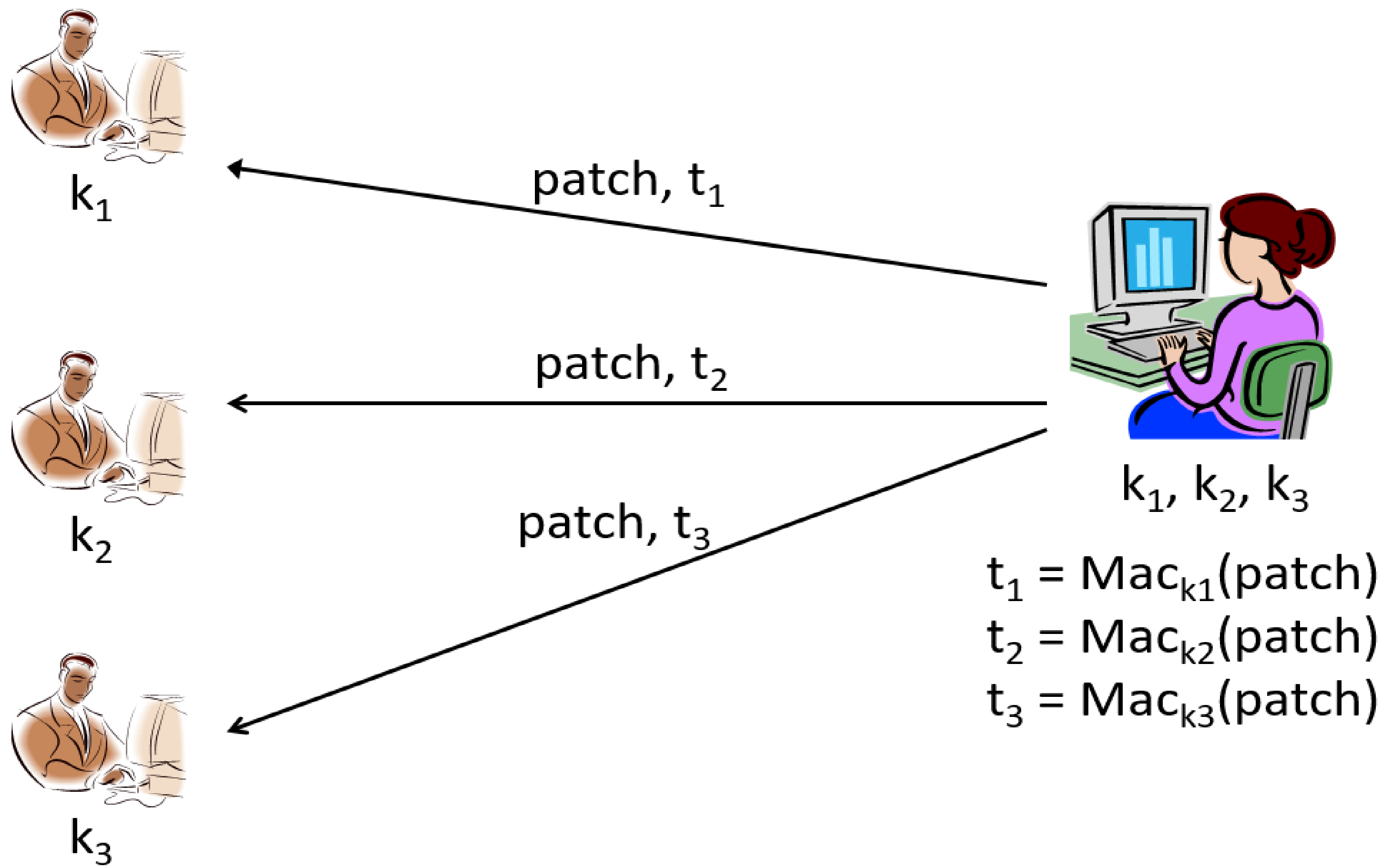

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  - Even if receiver leaks key to judge, how can the judge verify that the key is correct?
    - Even if key is correct, receiver could have generated the tag also!

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- Attacker gets the public key

- **Definition 14.1** Fix A, Π. Define randomized experiment $Forge_{A,\Pi}(n)$:
  1. $pk, sk \leftarrow Gen(1^n)$
  2. A is given pk, and interacts with *oracle* $Sign_{sk}(n)$; let M be the set of messages sent to this oracle
  3. A outputs (m, σ)
  4. A *succeeds*, and the experiment evaluates to 1, if $Vrfy_{pk}(m, \sigma) = 1$ and $m \notin M$

  Π is *secure* if for all PPT attackers A, there is a negligible function ϵ such that

$$\Pr[Forge_{A,\Pi}(n) = 1] \leq \epsilon(n)$$

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- Given
  - A signature scheme $\Pi = (Gen, Sign, Vrfy)$ for "short" messages of length n
  - Hash function $H : \{0,1\}^* \to \{0,1\}^n$
- Construct a signature scheme $\Pi' = (Gen, Sign', Vrfy')$ for arbitrary-length messages:
  - $Sign'_{sk}(m) = Sign_{sk}(H(m))$
  - $Vrfy'_{pk}(m, \sigma) = Vrfy_{pk}(H(m), \sigma)$

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  Two cases:
  – $H(m) = h_i$ for some i
    – Collision in H!
  – $H(m) \neq h_i$ for all i
    – Forgery in the underlying signature scheme!

"Plain" RSA signatures

$\xleftarrow{\quad N,\ e \quad}$

$\xleftarrow{\quad m,\ \sigma \quad}$

$m \stackrel{?}{=} [\sigma^e \bmod N]$

$(N, e, d) \leftarrow \mathrm{RSAGen}(1^n)$

$pk = (N, e)$

$sk = d$

$\sigma = [m^d \bmod N]$

Key generation: choose two random p, q and compute $N = p \cdot q$. Run $GenRSA(1^n)$. The secret key is (N, e). The public key is (N, d).

Signing: To sign a message m, output $\sigma = m^d \pmod{n}$.

Verification: To verify that σ is a valid signature for m, check whether $\sigma^e = m \pmod{n}$.

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- Attack2: Can sign "*random*" messages
  - Choose arbitrary σ; set $m = \sigma^e \pmod N$
- Attack3: Can combine two signatures to obtain a third
  - Say σ_1, σ_2 are valid signatures on m_1, m_2 w.r.t. public key N, e
  - Then $\sigma' = \sigma_1 \cdot \sigma_2 \bmod N$ is a valid signature on the message $m' = m_1 \cdot m_2 \bmod N$

RSA-FDH Signature Scheme

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- **Construction 14.3**: Construct a signature scheme as follows:
  - *Gen*: on input 1^n, run $GenRSA(1^n)$ to compute (N, e, d). The public key is (N, e), and the private key is d. As part of key generation, a function $H : \{0,1\}^* \rightarrow \mathbb{Z}_N^*$ is specified.
  - $Sign_{sk}(m)$: on input a private key (N, d) and a message $m \in \{0,1\}^*$, compute $Sign_{sk}(m) = \sigma = H(m)^d \bmod N$
  - $Vrfy_{pk}(m, \sigma)$: On input a public key (N, e), a message m, and a signature σ, output 1 if and only if $\sigma^e = H(m) \bmod N$

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- **Theorem 14.4** If the *RSA assumption* holds, and H is modeled as a *random oracle* (mapping onto $\mathbb{Z}_N^*$), then RSA-FDH is secure.

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RSA-FDH in practice

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- The RSA PKCS $\#1$ v2.1 standard includes a signature scheme inspired by RSA-FDH
  - Essentially a randomized variant of RSA-FDH
- DSS: NIST standard for digital signatures
  - DSA, based on *discrete-logarithm problem* in subgroup of $\mathbb{Z}_p^*$
  - ECDSA, based on elliptic-curve groups

"Plain" Rabin signatures

- **Key generation**: choose two random p, q with $p, q \equiv 3 \pmod 4$, as secret keys. The public key is $n = p \cdot q$.

  Signing: To sign a message m, output $\sigma = \sqrt{m} \pmod n$ (fix some choice for one of the four possible roots).

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- **Note**: Assuming the *factoring problem* is hard, if m is chosen at random, then it should be hard to forge a signature for m.

- However, this scheme is *insecure* against *chosen-message attack*.
  - Choose an $x \in \mathbb{Z}_n^*$ at random, and let $m = x^2 \pmod n$
  - Given $\sigma = \sqrt{m} \pmod n$ there is probablity $1/2$ that $\sigma \neq \pm x \pmod n$ in which case $\gcd(\sigma - x, n)$ will yield a nontrivial factor of n

Zero knowledge proofs

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- In a *Zero Knowledge Proof*, Alice will prove to Bob that a statement P is true. Bob will be completely convinced that P is true, but will not learn anything as a result of this process. That is, Bob will gain zero knowledge

Zero knowledge proofs

- S. Goldwasser, S. Micali, C. Rackoff, STOC'85

The Knowledge Complexity of Interactive Proof-Systems

(Extended Abstract)

Shafi Goldwasser, MIT

Silvio Micali, MIT

Charles Rackoff, University of Toronto

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The Knowledge Complexity of Interactive Proof-Systems

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Shafi, with Micali (and later Rackoff) [6], had been thinking for a while about expanding the traditional notion of "proof" to an interactive process in which a "prover" can convince a probabilistic "verifier" of the correctness of a mathematical proposition with overwhelming probability if and only if the proposition is correct. They called this interactive process an "interactive proof" (a name suggested by Mike Sipser). They wondered if one could prove some non-trivial statement (for example, membership of a string in a hard language) without giving away any knowledge whatsoever about why it was true. They defined that the verifier receives no knowledge from the prover if the verifier could simulate on his own the probability distribution that he obtains in interacting with the prover.The idea that "no knowledge" means simulatability was a very important contribution. They also gave the first example of these "zero knowledge interactive proofs" using quadratic residuosity. This paper won the first ACM SIGACT Gödel Prize. This zero-knowledge work led to a huge research program in the community that continues to this day, including results showing that (subject to an assumption such as the existence of one-way functions) a group of distrusting parties can compute a function of all their inputs without learning any knowledge about other people's inputs beyond that which follows from the value of the function.

`https://amturing.acm.org/award_winners/goldwasser_8627889.cfm`

Applications of ZKPs

- *Protocol design*. A *protocol* is an algorithm for interactive parties to achieve a certain goal. However, in crypto, we often want to design protocols that should achieve security even when one of the parties is "cheating". Alice can prove in *zero knowledge* that she followed the instructions.

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Proofs that Yield Nothing But their Validity
and a Methodology of Cryptographic Protocol Design

(Extended Abstract)

Oded Goldreich	Silvio Micali	Avi Wigderson
Dept. of Computer Sc.	Lab. for Computer Sc.	Inst. of Math. and CS
Technion	MIT	Hebrew University
Haifa, Israel	Cambridge, MA 02139	Jerusalem, Israel

Applications of ZKPs

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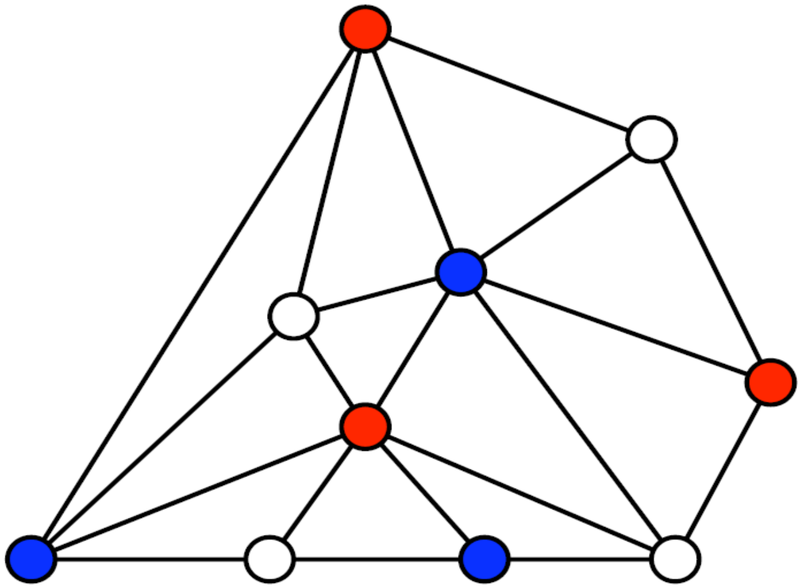
- **Ideas using ZKPs**:
  - Let the box contain an *instance* of a hard problem.
  - Give the authorized people the *solution* to the instance.
  - The authorized people will *prove* to the box that they know the solution in zero knowledge.

An example



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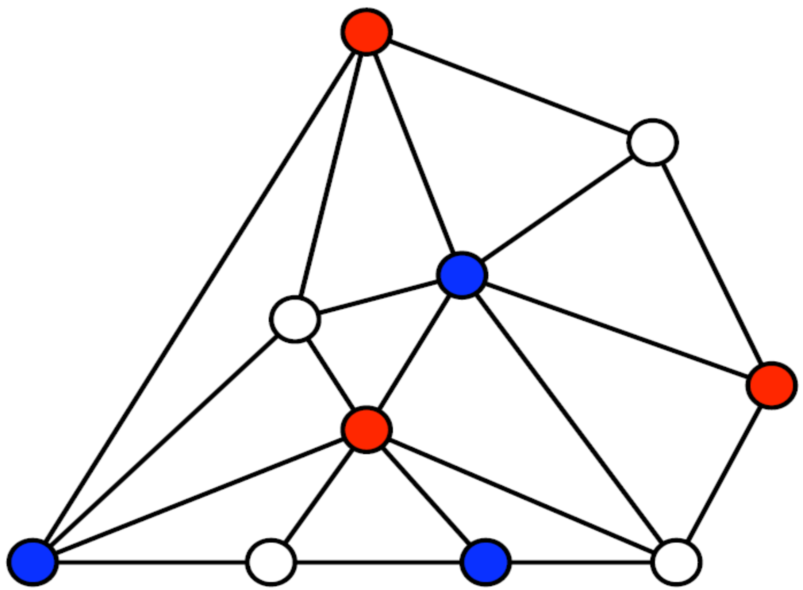

- Alice knows how to 3-color a graph: no two adjacent vertices have the same color; this is an NPC problem.
  - can impress your friends
  - useful for identification

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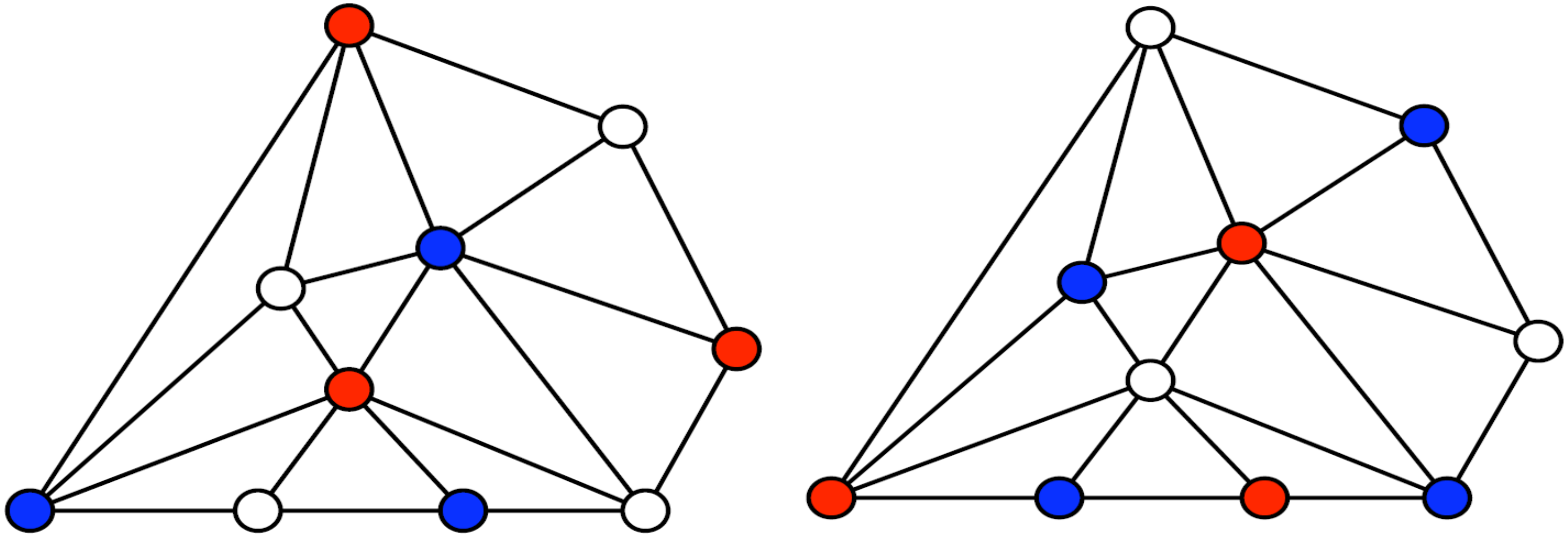
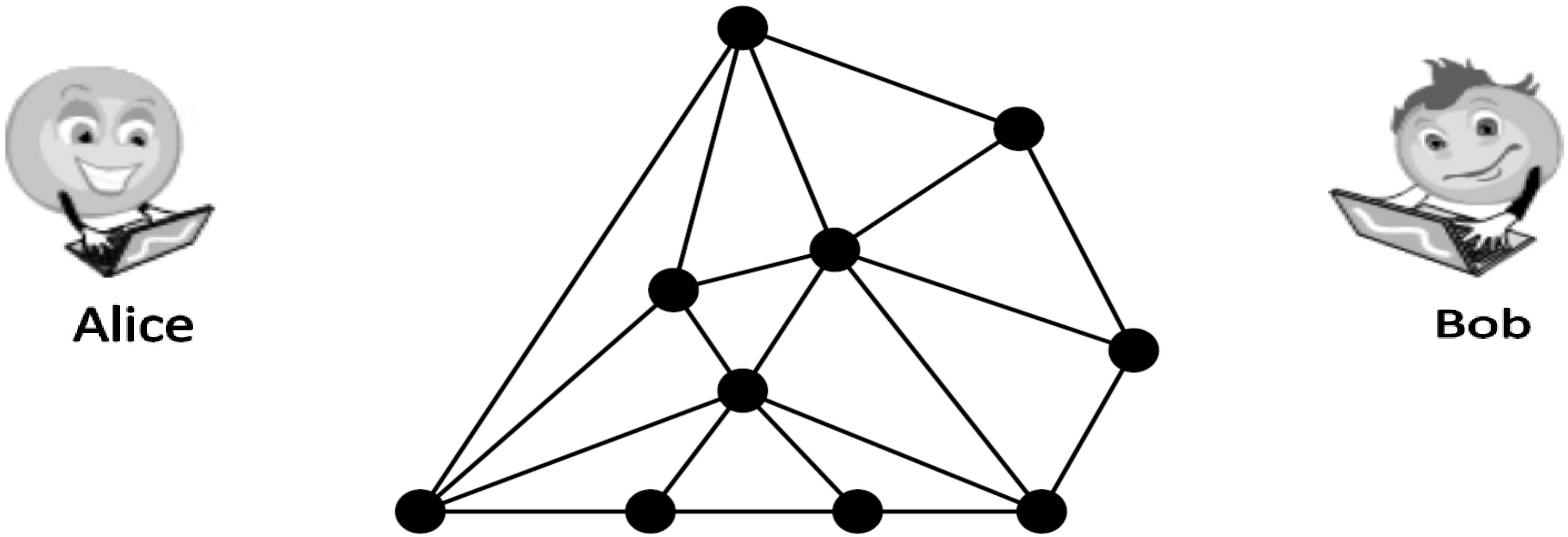
Alice may permute the vertex colors.



An example

- Alice then encrypts all vertex colors (one key per vertex), and sends the graph to Bob.



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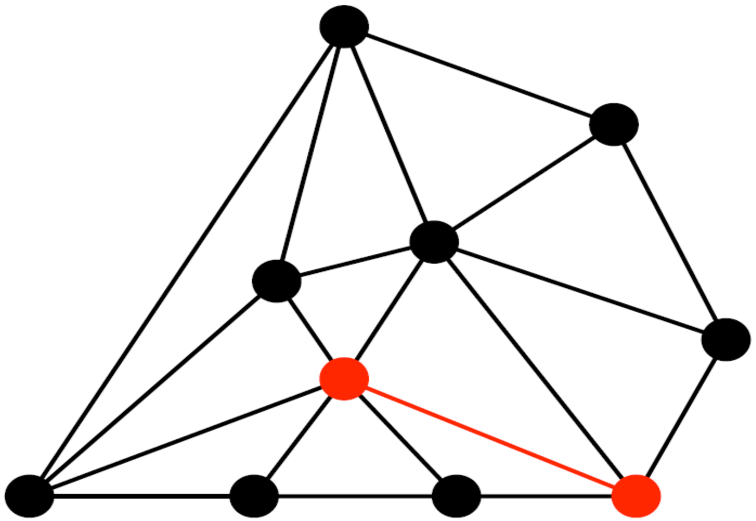
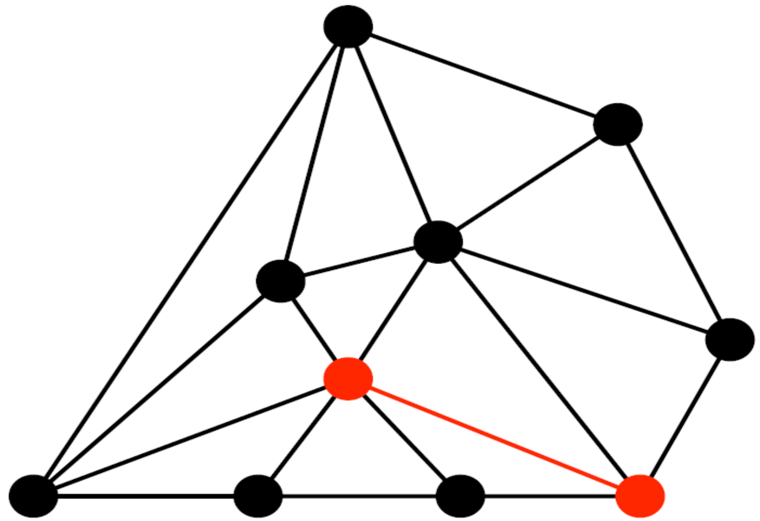
- Bob picks an edge at random.

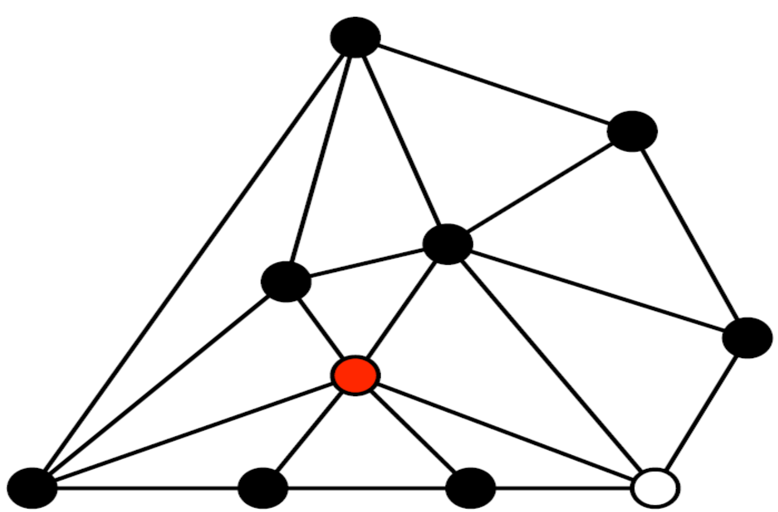


An example

- Bob picks an edge at random.



Alice reveals colors of those two keys.



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- Repeat as much as needed:
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After k repetitions, the probability she fools Bob is $(1 - \frac{1}{|E|})^k$.

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Claim. Every NP-statement can be proven in zero-knowledge.

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- **Example.** Alice can distinguish between Coke and Pepsi:
  Alice turns her back, Bob flips a coin and puts either Coke and Pepsi into a paper cup according the result, Alice tastes and announces whether she thinks it is Coke or Pepsi.

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- *Interactive probabilistic proofs*: the verifier does not convince with absolute certainty that P is true, but with high certainty.

  No matter what the prover does, and how she tries to to cheat, if the statement P is false, she will fail with this probability.

- **Example.** Alice can distinguish between Coke and Pepsi:
  Alice turns her back, Bob flips a coin and puts either Coke and Pepsi into a paper cup according the result, Alice tastes and announces whether she thinks it is Coke or Pepsi.

- If they repeat this k times, and Alice always answers correctly, then Bob can conclude with $1 - 2^{-k}$ probability that she really can tell the difference.

Protocol QR

- **Recall** If n is an integer, then $x \in \mathbb{Z}_n^*$ is a *quadratic residue* modulo n if there is some s such that $x = s^2 \pmod{n}$.

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  Some useful **facts**:

  ◇ if n is prime, then $\mathbb{Z}_n^*$ has a generator g and x is a QR iff $x = g^i$ for an even i.

  ◇ All the QRs form a *group*. If x is a QR, and y is a random QR, then xy is a random QR. For every $z \in QR_n$,

$$\Pr[xy = z] = 1/|QR_n|.$$

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  Public input: x, n; Prover – Alice; Verifier – Bob
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  We will analyze this protocol in *completeness, soundness, zero knowledge*.

Protocol QR – completeness

- *Completeness*: Whenever x is really a QR, Alice is given s such that $x = s^2 \pmod{n}$, and Alice and Bob follow the protocol, then Bob will *accept* the proof with probability 1.

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  Alice may not follow the instructions in this protocol, and may possibly cheat. We model her strategy as a function P^*. We think of P^* as follows: on input the empty word, it gives a string y, and on input b, it gives a string z.

Protocol QR – soundness

- **Lemma 15.1** For every (possibly not efficiently computable) P^*, and (x, n) such that x is not a QR modulo n, we have

$$\Pr_{b \leftarrow \{0,1\}}[out_V \langle P^*, V_{x,b} \rangle = accept] \leq 1/2.$$

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If two interactive algorithms A and B are running a protocol, we denote this execution by $\langle A, B \rangle$.

$out_A \langle A, B \rangle$ – the output of A after this interaction is finished.
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Case 2: $y \notin QR_n$. With probability $1/2$, Bob sends $b = 0$. However, if $b = 0$, Alice has to come up with some z such that $z^2 = y$, impossible! Bob will also reject with probability $\geq 1/2$.

Protocol QR – zero knowledge

- We think of a possibly cheating verifier V^*. He can only sends either $b = 0$ or $b = 1$. Our goal is to show that: in both cases, he gets a random element in $\mathbb{Z}_n^*$, leaking no info about the QR of x.

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An Example

- What does Bob see?
  - randomly-generated keys
  - randomly-generated colors

  Because Bob could have generated those keys and colors by himself, he learns nothing from the graph coloring.

Protocol QR – zero knowledge

- **Definition 15.2** A prove strategy P is (T, ϵ)-zero knowledge if for every T-time cheating strategy V^* there exists a *poly*(T)-time non-interactive algorithm S (called the *simulator* for V^*) such that for every valid public input x and private input w, the following two random variables are (T, ϵ)-computationally indistinguishable:
  - $\mathit{view}_{V^*}\langle P_{U_m,x,w}, V^*\rangle$, where m is the number of random coins P uses.
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The *simulator* S only gets the public input and has no interaction with P, but still manages to output something indistinguishable from whatever V^* learned in the interaction.

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  1. **Input**: x, n such that $x \in QR_n$.
  2. Choose $b' \leftarrow_R \{0, 1\}$.
  3. Choose $z \leftarrow_R QR_n$.
  4. If $b' = 0$, compute $y = z^2$. Otherwise ($b' = 1$), compute $y = z^2 x^{-1}$.
  5. Invoke V^* on the message y to obtain a bit b.
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  We do not even know whether this algorithm loops forever or not.

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  Lemma 15.5 The output of the simulator S is distributed identically to the view of V^* in an interaction with an honest prover.

  Proof. For both the prover and the simulator, if $b = 0$, then z is a random root of y; if $b = 1$, then z is a random root of xy.

Schnorr's identification protocol

- Statement P: Alice knows DL of h, w.r.t. g, these are in group $G = \mathbb{Z}_p$.

  Public input: g, h; Prover – Alice; Verifier – Bob
  Prover's private input: x such that $h = g^x$

  $P \rightarrow V$: Alice chooses random $r \leftarrow_R \mathbb{Z}_p$ and sends $a = g^r$ to Bob
  $P \leftarrow V$: Bob chooses $b \leftarrow_R \mathbb{Z}_p$ and sends b to Alice
  $P \rightarrow V$: Alice sends $c = r + xb \pmod{p}$ to Bob.

  Verification: Bob verifies that $ah^b = g^c$.

Schnorr's identification protocol

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  Proof of knowledge: if $b \neq b'$ then given a and $b \neq b'$ and $c \neq c'$ such that $ah^b = g^c$ and $ah^{b'} = g^{c'}$, we get $h^{b-b'} = g^{c-c'}$. Since we know b and b', we can take this to the power $(b - b')^{-1} \pmod{p}$ to get an equation $h = g^x$.

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  Honest verifier zero knowledge: The simulator S does the following: choose $b, c \leftarrow_R \mathbb{Z}_p$, choose a as $h^{-b}g^c$.

- **Definition 15.6** (*Group homomorphism*)
  Two groups G and G' are *homomorphic* if there exists a function (*homomorphism*) $f : G \to G'$ such that for all $x, y \in G$, $f(x +_G y) = f(x) +_{G'} f(y)$.

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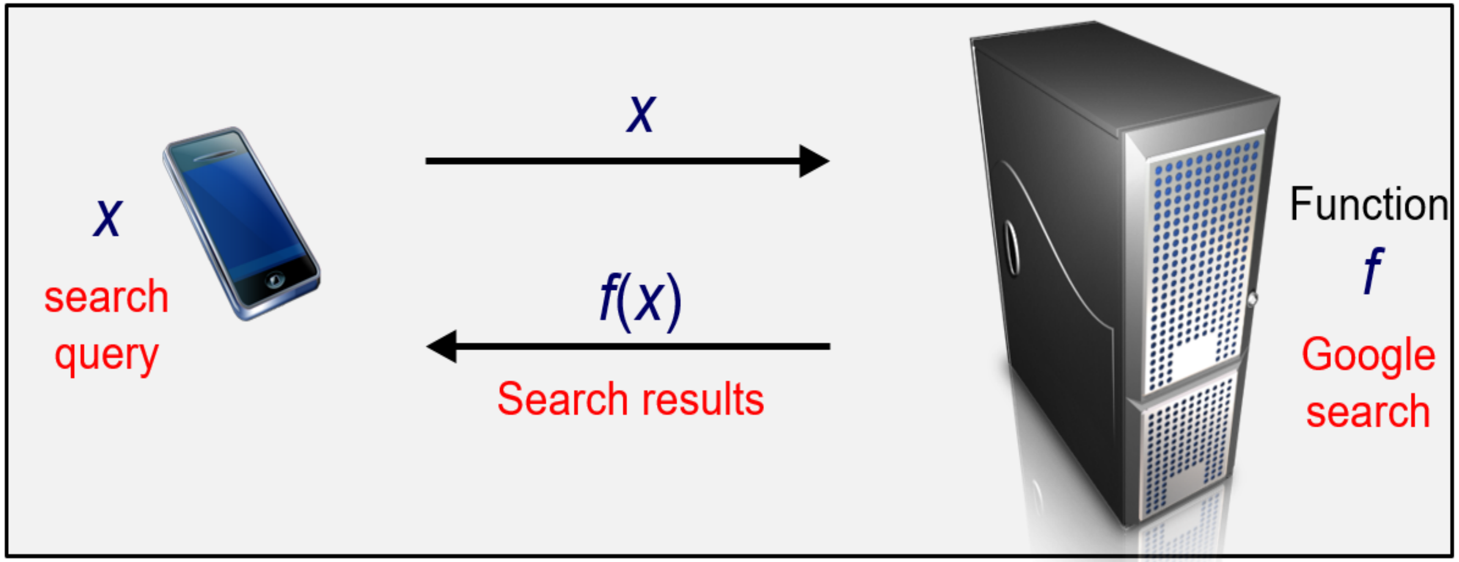

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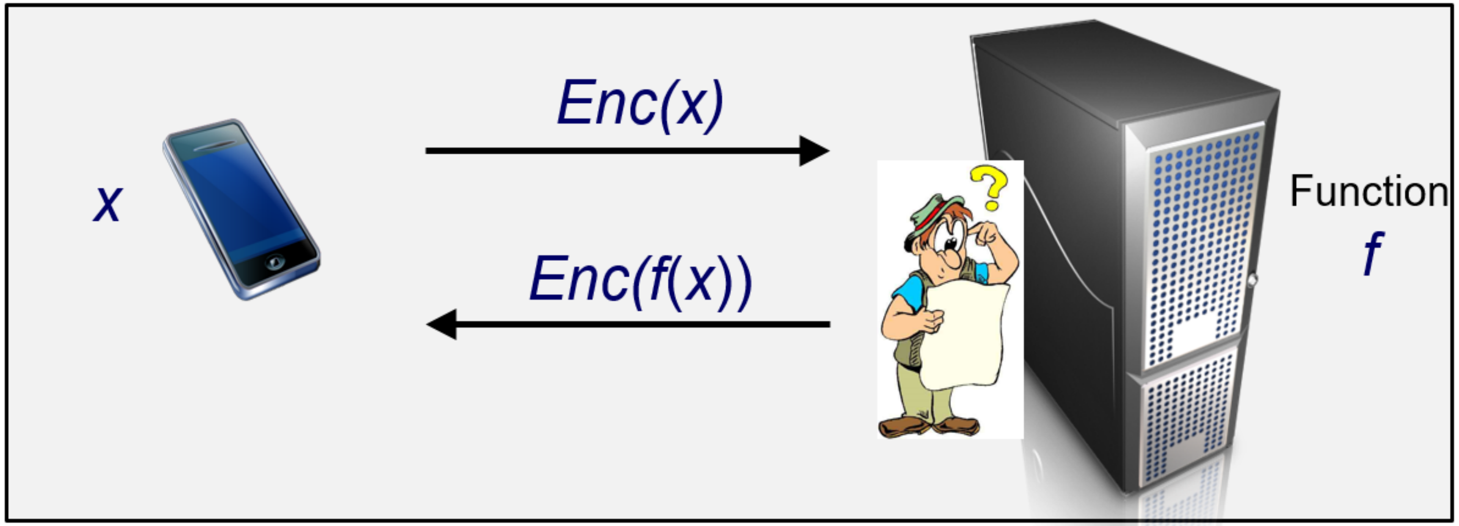

Computing on encrypted data

- Recall RSA encryption

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  What people really wanted was the ability to do arbitrary computing on encrypted data, and this requires the abibility to compute both sums and products.

Computing on encrypted data

- Why SUMs and PRODUCTs?

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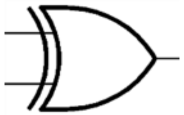
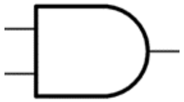
SUM



XOR

PRODUCT



AND

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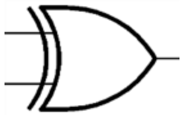
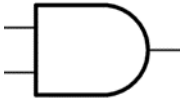


XOR

$x + y \bmod 2$

PRODUCT



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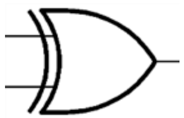

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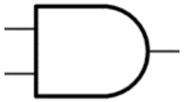

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{XOR, AND} is complete, i.e.,
any function is a combination of XOR and AND. (e.g., OR)

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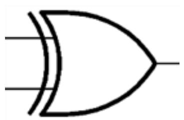

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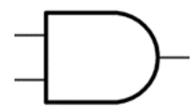

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Example
$x \ OR \ y = x + y + x \cdot y \bmod 2.$

Computing on encrypted data

- Because {XOR, AND} is *complete*, if we can compute SUMs and PRODUCTs on encrypted bits, we can compute any function on encrypted inputs.

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We can delegate arbitrary processing of data without giving away access to it.

Applications: *private cloud computing, private information retrieval, multi-party secure computation, encrypted search, ...*

Fully homomorphic encryption

Fully Homomorphic Encryption Using Ideal Lattices

Craig Gentry
Stanford University and IBM Watson
cgentry@cs.stanford.edu

ABSTRACT

We propose a fully homomorphic encryption scheme – i.e., a scheme that allows one to evaluate circuits over encrypted data without being able to decrypt. Our solution comes in three steps. First, we provide a general result – that, to construct an encryption scheme that permits evaluation of *arbitrary circuits*, it suffices to construct an encryption

duced by Rivest, Adleman and Dertouzos [54] shortly after the invention of RSA by Rivest, Adleman and Shamir [55]. Basic RSA is a multiplicatively homomorphic encryption scheme – i.e., given RSA public key pk $= (N, e)$ and ciphertexts $\{\psi_i \leftarrow \pi_i^e \bmod N\}$, one can efficiently compute $\prod_i \psi_i = (\prod_i \pi_i)^e \bmod N$, a ciphertext that encrypts the product of the original plaintexts. Rivest et al. [54] asked

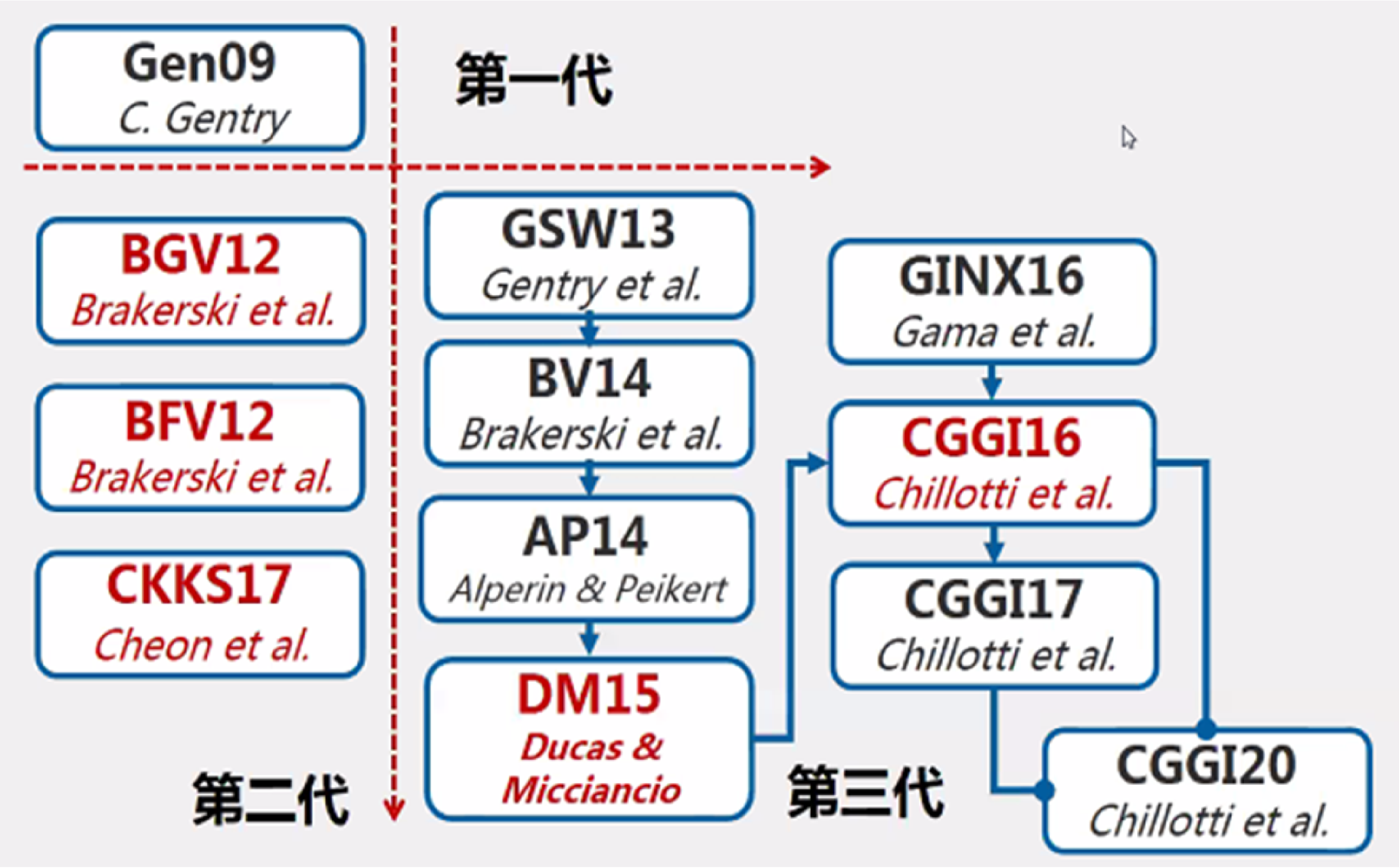
Fully Homomorphic Encryption over the Integers

Marten van Dijk[1], Craig Gentry[2], Shai Halevi[2], and Vinod Vaikuntanathan[2]

[1] MIT CSAIL
[2] IBM Research

Abstract. We construct a simple fully homomorphic encryption scheme, using only elementary modular arithmetic. We use Gentry's technique to construct a fully homomorphic scheme from a "bootstrappable" somewhat homomorphic scheme. However, instead of using ideal lattices over a

Fully homomorphic encryption



Fully homomorphic encryption

Library	Developed by	FHE Scheme
HElib	IBM	BGV/CKKS
Microsoft SEAL	Microsoft	BFV/CKKS
PALISADE	MIT, UCSD etc.	BFV/BGV etc.
HEAAN	Seoul National University	CKKS

Post-Quantum Cryptography (PQC)

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PQC Standardization Process: Third Round Candidate Announcement

July 22, 2020

PQC Standardization Process: Announcing Four Candidates to be Standardized, Plus Fourth Round Candidates

July 05, 2022

Post-Quantum Safe Algorithm Candidate Cracked in an Hour on a PC

BY MATT SWAYNE • AUGUST 5, 2022 • RESEARCH

Good Luck!