Logic & Bayesian Network Questions

Propositional Logic

- The best combination of the six important players in the volleyball team (1, 3, 4, 6, 9, 12) should follow the following rules:
  - Players 4 and 6 need to play together.
  - Player 3 does not play the game if and only if player 1 does not play.
  - Either player 3 or player 6 should appear on the court, and they can not appear at the same time.
  - If players 9 and 12 are on the court, then player 4 must be on the court.

- **If players 1 and 12 need to play at the same time in one game, who of the other 4 players should play?**

(1) **Please use propositional logic to represent the above statement.**

(2) **Convert them into a CNF.**

(3) **Apply the DPLL algorithm to derive which players should be on the court.**

Answer:

(1) Propositional logical expression:

$$((\mathrm{P}_4 \leftrightarrow P_6))\wedge(\sim\mathrm{P}_3 \leftrightarrow \sim P_1)\wedge((\sim P_3 \wedge P_6)\vee(P_3 \wedge \sim P_6))\wedge((P_9 \wedge P_{12}) \rightarrow P_4)\wedge(P_1 \wedge P_{12})$$

(2) Convert to CNF:

$$(\sim\mathrm{P}_4 \vee P_6)\wedge((\mathrm{P}_4 \vee \sim P_6)\wedge(\mathrm{P}_3 \vee \sim P_1)\wedge(\sim\mathrm{P}_3 \vee P_1)\wedge(\sim P_3 \vee \sim P_6)\wedge(P_3 \vee P_6)\wedge(\sim P_9 \vee \sim P_{12} \vee P_4)\wedge P_1 \wedge P_{12}$$

(3)Apply DPLL:

1. Split to Clause :

$$(\sim P_4 V P_6)\ ((P_4 V \sim P_6)\ (P_3 V \sim P_1)\ (\sim P_3 V P_1)\ (\sim P_3 V \sim P_6)\ (P_3 V P_6)\ (\sim P_9 V \sim P_{12} V P_4)\ \ P_1\ \ P_{12}$$

2. Find Unit Clause: P_1 , P_{12}

Model：{P_1:True; P_{12}:True;}

3.Unknown Clause : $(\sim P_4 V P_6)\ ((P_4 V \sim P_6)\ (P_3 V \sim P_1)\ \cancel{(\sim P_3 V P_1)}\ (\sim P_3 V \sim P_6)\ (P_3 V P_6)\ (\sim P_9 V \sim P_{12} V P_4)$

Unit Clause: P_3

Model：{P_1:True; P_{12}:True; P_3: True}

4. Unknown Clause:

$$(\sim P_4 V P_6)\ ((P_4 V \sim P_6)\ \cancel{(P_3 V \sim P_1)}\ \cancel{(\sim P_3 V P_1)}\ (\sim P_3 V \sim P_6)\ \cancel{(P_3 V P_6)}\ (\sim P_9 V \sim P_{12} V P_4)$$

Unit Clause: $\sim P_6$

Model: {P_1:True; P_{12}:True; P_3: True; P_6: False}

5. Unknown Clause:

$$(\sim P_4 V P_6)\ \cancel{(P_4 V \sim P_6)}\ \cancel{(P_3 V \sim P_1)}\ \cancel{(\sim P_3 V P_1)}\ \cancel{(\sim P_3 V \sim P_6)}\ \cancel{(P_3 V P_6)}\ (\sim P_9 V \sim P_{12} V P_4)$$

Unit Clause: $\sim P_4$

Model：{P_1:True; P_{12}:True; P_3: True; P_6: False; P_4: False}

6. Unknown Clause:

$$\cancel{(\sim P_4 V P_6)}\ \cancel{(P_4 V \sim P_6)}\ \cancel{(P_3 V \sim P_1)}\ \cancel{(\sim P_3 V P_1)}\ \cancel{(\sim P_3 V \sim P_6)}\ \cancel{(P_3 V P_6)}\ (\sim P_9 V \sim P_{12} V P_4)$$

Unit Clause: $\sim P_9$

Model：{P_1:True; P_{12}:True; P_3: True; P_6: False; P_4: False; P_9: False}

7. Unknown Clause:

$$\cancel{(\sim P_4 V P_6)}\ \cancel{(P_4 V \sim P_6)}\ \cancel{(P_3 V \sim P_1)}\ \cancel{(\sim P_3 V P_1)}\ \cancel{(\sim P_3 V \sim P_6)}\ \cancel{(P_3 V P_6)}\ \cancel{(\sim P_9 V \sim P_{12} V P_4)}$$

No unknown clause and every clause can be true.

First-order Logic

- Anyone who passes the AI exam and wins a prize is happy.
- Anyone willing to learn or lucky can pass all the exams.
- Zhansan is not willing to learn but he is lucky.
- Any lucky person can win a prize.

- Show $ZhangSan$ is Happy using resolution.

Steps of resolution proof:

1. Logic Expression of KB
2. KB to CNF
3. $\sim\alpha$ to CNF:
4. add $\sim\alpha$ to KB
5. resolve to empty

Answer:

1. Logic Expression of KB

R1: Anyone who passes the AI exam and wins a prize is happy.

$$(\forall x)\Big(\big(Pass(x, AIExam)\big) \wedge Win(x, prize)\Big) \rightarrow Happy(x)$$

R2: Anyone willing to learn or lucky can pass all the exams.

$$(\forall x)(\forall y)(Study(x) \vee Lucky(x) \rightarrow Pass(x, y))$$

R3: Zhansan is not willing to learn but he is lucky.

$$\sim Study(ZhangSan) \wedge Lucky(ZhangSan)$$

R4: Any lucky person can win a prize.

$$(\forall x)(Lucky(x) \rightarrow Win(x, prize))$$

2.KB to CNF:

R1: $\sim Pass(x, AIExam) \vee \sim Win(x, prize) \vee Happy(x)$

R2: $\sim Study(y) \vee Pass(y, z), \sim Lucky(u) \vee Pass(u, v)$

R3: $\sim Study(ZhangSan), Lucky(ZhangSan)$

R4: $\sim Lucky(w) \vee Win(w, prize)$

3. $\sim\boldsymbol{\alpha}$ **to CNF:**

$\sim Happy(ZhangSan)$

4. And 5. given as follows:

$\sim Lucky(u) \vee Pass(u, v)$

$Lucky(ZhangSan)$

$Pass(ZhangSan, v)\{u/ZhangSan\}$

$\sim Pass(x, AIExam) \vee \sim Win(x, prize) \vee Happy(x)$

$\sim Win(ZhangSan, prize) \vee Happy(ZhangSan)\{x/ZhangSan, v/AIExam\}$

$\sim Lucky(w) \vee Win(w, prize)$

$\sim Lucky(ZhangSan) \vee Happy(ZhangSan)\{w/ZhangSan\}$

$\sim Happy(ZhangSan)$

$\sim Lucky(ZhangSan)$

$Lucky(ZhangSan)$

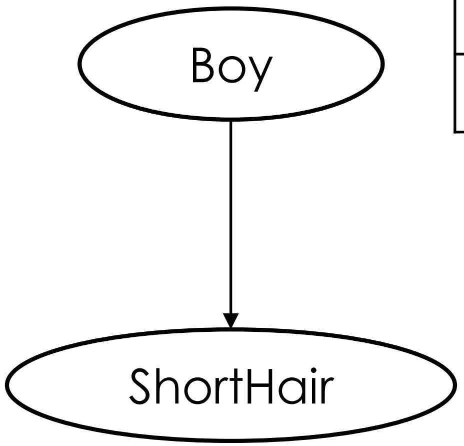
$\square$

Bayesian Network

- Suppose the ratio of boys and girls in SUSTech is 7:3, the ratio of long hair and short hair among boys is 1:9, and the ratio of long hair and short hair among girls is 8:2, please do:
  - Draw the Bayesian network that represents the above relationship.
  - If you meet a student with short hair on the campus, what is the probability that it is a girl?

Answer:

(1) Bayesian Network



P(B)
0.7

B	P(S|B)
T	0.9
F	0.2

(2) The probability:

$$P(\neg B|S) = P(\neg BS) / P(S) = \alpha P(\neg BS) = \frac{P(\neg BS)}{P(\neg BS)+P(BS)} = \frac{0.2*0.3}{0.2*0.3+0.7*0.9} = 0.087$$

Approximate Inference with Bayesian Networks

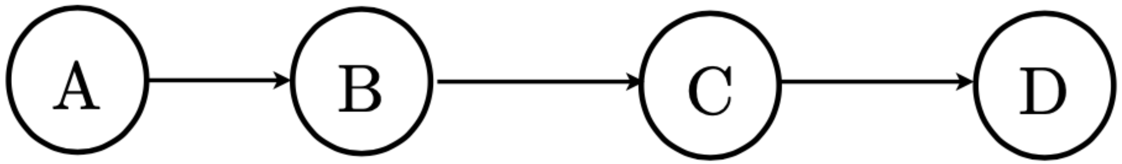
- Assume the right Bayes' net, and the corresponding distributions over the variables in the Bayes' net:
  - You are given the following samples:

  (A, B, ¬C, ¬D), (A, ¬B, C, ¬D), (¬A, B, C, ¬D), (¬A, ¬B, C, ¬D)

  (A, ¬B, ¬C, D), (A, B, C, ¬D), (¬A, B, ¬C, D), (¬A, ¬B, C, ¬D)

  (i) If these samples came from doing Prior Sampling, calculate our sample estimate of P (C).

  (ii) Now we will estimate P (C| A, ¬D). Cross out the samples that would not be used when doing Rejection Sampling for this task and write down the sample estimate of P (C| A, ¬D).



P(A)
1/4

P(B | A)	
A	1/5
¬A	3/4

P(C | B)	
B	1/2
¬B	1/3

P(D | C)	
C	1/6
¬C	7/8

Answer:

(i) P(C) = 5/8

(ii) (A, B, ¬C, ¬D), (A, ¬B, C, ¬D) ~~, (¬A, B, C, ¬D)~~ ~~, (¬A, ¬B, C, ¬D)~~

~~(A, ¬B, ¬C, D)~~, (A, B, C, ¬D) ~~, (¬A, B, ¬C, D)~~ ~~, (¬A, ¬B, C, ¬D)~~

P (C| A, ¬D) = 2/3

- Using Likelihood Weighting Sampling to estimate $P(\neg A \mid B, \neg D)$, the following samples were obtained. What is the weight of each sample?

Sample	Weight
$\neg A$ B C $\neg D$	$P(B \mid \neg A)P(\neg D \mid C)=3/4*5/6=5/8$
A B C $\neg D$	$P(B \mid A)P(\neg D \mid C)=1/5*5/6=5/30=1/6$
A B $\neg C$ $\neg D$	$P(B \mid A)P(\neg D \mid \neg C)=1/5*1/8=1/40$
$\neg A$ B $\neg C$ $\neg D$	$P(B \mid \neg A)P(\neg D \mid \neg C)=3/4*1/8=3/32$

- From the weighted samples, estimate $P(\neg A \mid B, \neg D)$.

$$\frac{\frac{5}{8}+\frac{3}{32}}{\frac{5}{8}+\frac{1}{6}+\frac{1}{40}+\frac{3}{32}} \approx 0.789$$

- Recall that during the MCMC-Ask function (Gibbs Sampling), samples are generated through an iterative process. Assume that the only evidence that is available is A ==TRUE. Which sequence(s) below could have been generated by Gibbs Sampling?

"State" of network = current assignment to all variables.

Generate next state by sampling one variable given Markov blanket
Sample each variable in turn, keeping evidence fixed

```
function MCMC-ASK(X, e, bn, N) returns an estimate of P(X|e)
    local variables: N[X], a vector of counts over X, initially zero
                     Z, the nonevidence variables in bn
                     x, the current state of the network, initially copied from e

    initialize x with random values for the variables in Y
    for j = 1 to N do
        for each Z_i in Z do
            sample the value of Z_i in x from P(Z_i|mb(Z_i))
                given the values of MB(Z_i) in x
            N[x] ← N[x] + 1 where x is the value of X in x
    return NORMALIZE(N[X])
```

Markov Blanket

Can also choose a variable to sample at random each time

Sequence 1	
1:	A ¬B ¬C D
2:	A ¬B ¬C D
3:	A ¬B C D

Sequence 2	
1:	A ¬B ¬C D
2:	A ¬B ¬C ¬D
3:	¬A ¬B ¬C D

Sequence 3	
1:	A ¬B ¬C D
2:	A ¬B ¬C ¬D
3:	A B ¬C ¬D

Sequence 4	
1:	A ¬B ¬C D
2:	A ¬B ¬C ¬D
3:	A B ¬C D

Sequence 1 and Sequence 3.
Gibbs sampling updates one variable at a time and never changes the evidence. The first and third sequences have at most one variable change per row, and hence could have been generated from Gibbs sampling. In sequence 2, the evidence variable is changed. In sequence 4, the second and third samples have both B and D changing.

Summary

- Propositional Logic
  - How to construct KB with PL
  - Inference with PL
  - DPLL
- First-order logic
  - How to construct KB with FOL
  - Inference with FOL
- Bayesian Network
  - The definition of BN
  - How to construct a CPT with BN?
  - Inference with BN