Propositional Language L_0

Formal languages usually

1. translate a restricted class of natural language.
2. have a fix set of atomic symbols and formation rules.
3. are precise and unambiguous.

Propositional logic formalizes certain type of assertions in natural language.

Definition 1

An **assertion** is a sentence that is either true or false.

Symbols

The following 3 types of elements are extracted from our natural language:

1. parenthesis（括号）：(,).
2. propositional connectives（命题连接词）:

$$\begin{array}{ll} \neg & \text{not} \\ \to & \text{if } \cdots, \text{ then } \cdots \end{array}$$

3. proposition symbols（命题符号）:

$$A_1, A_2, \cdots, A_n, \cdots$$

$\mathcal{L}_0$-Formulas

Definition 4

The **propositional language** $\mathcal{L}_0$ is the smallest set L such that L is a set of finite sequences of symbols in

$$S_0 = \{(,), \neg, \to\} \cup \{A_n \mid n \in \mathbb{N}\}.$$

and such that

1. $\langle A_n \rangle \in L$, for each $n \in \mathbb{N}$. [1]
2. If $s \in L$, then $(\neg s) \in L$.
3. If $s, t \in L$, then $(s \to t) \in L$.

[1] $\langle A \rangle$ denote the length-1 sequence that consists of only one symbol A.

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The existence of the "smallest" such L needs some explanation. To see that $\mathcal{L}_0$ is well defined, we give an equivalent definition of $\mathcal{L}_0$.

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$\mathcal{L}_0$ is well defined

Let $(*)$ denote the three conditions in the previous definition.

Theorem 5

Let $\mathcal{L}_0^* = \bigcap \{L \mid L$ *satisfies* $(*)\}$*. Then* $\mathcal{L}_0 = \mathcal{L}_0^*$*.*

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Let $(*)$ denote the three conditions in the previous definition.

Theorem 5

Let $\mathcal{L}_0^* = \bigcap \{L \mid L \text{ satisfies } (*)\}$. Then $\mathcal{L}_0 = \mathcal{L}_0^*$.

Proof.

- Let $\Lambda = \{L \subseteq (S_0)^{<\omega} \mid L \text{ satisfies } (*)\}$. Then $\Lambda \neq \varnothing$, as $(S_0)^{<\omega} =_{\text{def}} \bigcup_{n \in \mathbb{Z}^+} (S_0)^n$, the set of all finite sequences of symbols in S_0, belongs to Λ. Thus $\mathcal{L}_0^*$ is well defined.
- Check that $\mathcal{L}_0^*$ satisfies $(*)$.
- Clearly $\mathcal{L}_0^* \subseteq L$, for all $L \in \Lambda$, i.e. $\mathcal{L}_0^*$ is the $\subseteq$-smallest among the L's in Γ, therefore $\mathcal{L}_0^* = \mathcal{L}_0$.

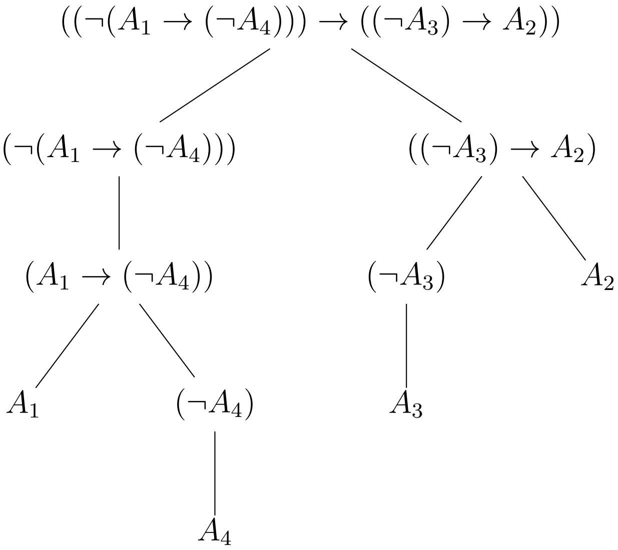
$\square$

Well-formed formula

Definition 6

A finite sequence of elements in S_0 is called **well-formed formulas** (or simply **formula** or **wff**) if it can be built-up from $\{A_n \mid n \in \mathbb{N}\}$ by applying the following *formula-building operations* finitely many times:

$$\mathcal{E}_\neg(s) = (\neg s),$$
$$\mathcal{E}_\to(s, t) = (s \to t).$$



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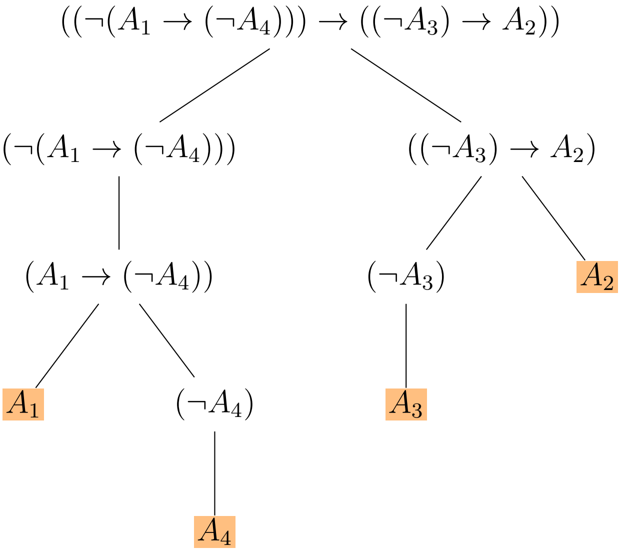
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Proposition 8

Suppose $\varphi \in$ wff. Then one of the following applies.

1. *There is an n such that $\varphi = \langle A_n \rangle$.*
2. *There is a wff ψ such that $\varphi = (\neg\psi)$.*
3. *There are wffs ψ_1 and ψ_2 such that $\varphi = (\psi_1 \to \psi_2)$.* $\square$

Readability

Corollary 9 (Readability)

Suppose $\varphi \in \mathcal{L}_0$. Then exactly one of the following applies.

1. *There is an n such that $\varphi = \langle A_n \rangle$.*
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For the "exact"-ness, it suffices to verify that the three cases are mutually exclusive.

- Case 1 has only one symbol
- Case 2 starts with "$(\neg$"
- Case 3 starts with "$(($" or "$(A_n$" for some A_n. $\square$

Subformulas

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Definition 10 (Subformula, an inductive definition)

The set $S(\varphi)$ of all subformulas of a given $\varphi \in \mathcal{L}_0$ is defined inductively as follows:

$$\begin{aligned} S(\langle A_n \rangle) &= \{\langle A_n \rangle\}, \quad \text{for } n \in \mathbb{N} \\ S((\neg\alpha)) &= S(\alpha) \cup \{(\neg\alpha)\} \\ S((\alpha \to \beta)) &= S(\alpha) \cup S(\beta) \cup \{(\alpha \to \beta)\} \end{aligned}$$

For the proof of Unique Readability, we use a bit-by-bit definition of subformulas.

Definition 11

Suppose s, t are finite sequences, φ, ψ are formulas.

1. t is a block-subsequence of s $\quad \cdots [\cdots] \cdots$
2. t is a (proper) initial segment of s $\quad [\cdots] \cdots\cdots$
3. an occurrence of s in φ $\quad \cdots [\text{-}s\text{-}] \cdots$
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Question

Let s be a finite sequence of length n. How many block-subsequence of s are there?

Unique Readability

Theorem 12 (Unique Readability)

Suppose $\varphi \in \mathcal{L}_0$. Then exactly one of the following applies.

1. *There is an n such that $\varphi = A_n$.*
2. *There is a $\psi \in \mathcal{L}_0$ such that $\varphi = (\neg\psi)$.*
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Further, in cases (2) and (3), the subformulas ψ, ψ_1 and ψ_2 are unique, respectively.

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The uniqueness of case 1 and 2 are self-clear.
To prove the uniqueness of ψ_1 and ψ_2 in case 3, we need

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- $|\varphi| = 1$. Then $s = \varnothing$. Vacuously true: $\varnothing \notin \mathcal{L}_0$.

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  - $\varphi \equiv (\neg\psi)$, if $s \in \mathcal{L}_0$, then it must be $s \equiv (\neg\theta)$, some $\theta \in \mathcal{L}_0$. But then $\theta \subsetneq_{\text{init}} \psi$ and $|\psi| < |\varphi|$. Contradiction!

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  - $\varphi \equiv (\psi_1 \to \psi_2)$, if $s \in \mathcal{L}_0$, it must be that $s \equiv (\theta_1 \to \theta_2)$, for some $\theta_1, \theta_2 \in \mathcal{L}_0$.

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    - $\psi_1 \neq \theta_1$, one of $\{\psi_1, \theta_1\}$ is a proper initial segment of the other. Contradiction!
    - $\psi_1 = \theta_1$, one of $\{\psi_2, \theta_2\}$ is a proper initial segment of the other. Contradiction! $\square$

Exercises

Question

Fix a 1-1 enumeration of S_0. Give an algorithm to enumerate
(1) $(S_0)^{<\omega}$ = the set of finite sequences of members in S_0;
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§1.1.2: (2), (3)

Hints:

(2) Show that there are no wffs of length 2, 3 or 6, but that any other positive length is possible.

(3) The sequence $\langle \varphi_1, \ldots, \varphi_n \rangle$ is called a construction sequence for φ_n, which is obtained from the construction tree for φ_n.

Polish Notation

Though parentheses are helpful for human eyes, it is possible to drop parentheses without loss of clarity. Let $S_0^* = S_0 - \{(,)\}$.

Definition 14

Let $\mathcal{P}_0$ be the smallest set $P \subseteq (S_0^*)^{<\omega}$ such that

1. For each n, $A_n \in P$.
2. If ψ_1 and ψ_2 belong to P, then so do $\neg\psi_1$ and $\rightarrow \psi_1\psi_2$.

Theorem 15

For any $s \in (S_0^*)^{<\omega}$, $s \in \mathcal{P}_0 \Leftrightarrow s \in \mathcal{P}_0$*-wff.*

$\mathcal{P}$-wff

Definition 16

A finite sequence of elements in S_0 is called $\mathcal{P}_0$-**wff** if it can be built-up from $\{A_n \mid n \in \mathbb{N}\}$ by applying the following formula-building operations finitely many times:

$$\mathcal{D}_{\neg}(s) = \neg s,$$
$$\mathcal{D}_{\to}(s, t) = {\to} s\, t.$$

Example 17

$$\to \neg \to A_1 \, \neg A_4 \to \neg A_3 \, A_2.$$

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Assignment

Find out more about Polish and reverse Polish notations, as well as SVO, SOV, VSO, etc.

Priority of operators

To establish a more compact notation,

1. The outermost parentheses are omitted.
2. The priority of operators are ordered as: $\neg$ is higher than $\rightarrow$. [2] e.g.
$$B \rightarrow \neg A \quad \text{is} \quad (B \rightarrow (\neg A))$$
3. When connectives of the same priority are repeated, grouping is to the right:
$$A \rightarrow B \rightarrow C \quad \text{is} \quad (A \rightarrow (B \rightarrow C))$$

---

[2] When $\vee$, $\wedge$ and $\leftrightarrow$ are considered:
$$\neg \qquad (\vee, \wedge) \qquad (\rightarrow, \leftrightarrow).$$

Other connectives

Other connectives $\vee$, $\wedge$, $\leftrightarrow$ are treated as abbreviations of formulas (involving $\{\neg, \rightarrow\}$ only) as follows:

$$\begin{array}{rcl} p \vee q & \text{iff} & \neg p \rightarrow q \\ p \wedge q & \text{iff} & \neg(p \rightarrow \neg q) \\ p \leftrightarrow q & \text{iff} & (p \rightarrow q) \wedge (q \rightarrow p) \end{array}$$

This treatment will be justified later.