Assignment#5 CS201 Fall 2023

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January 5, 2024

Problem 1. Let S be the set of all strings of English letters. Determine whether the following relations are *reflexive*, *irreflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution. **a)** irreflexive, symmetric

b) reflexive, symmetric, transitive

c) transitive, irreflexive, antisymmetric

d) transitive, irreflexive, symmetric

e) transitive, antisymmetric, reflexive

Problem 2. Consider relations on a set A. Prove or disprove the following statements.

Solution. **a)** True. For reflexive, we have

$$\forall x \in A, (x, x) \in R$$

and for symmetric, we have

$$\forall a, b \in A, (a, b) \Rightarrow (b, a)$$

so we have

$$\forall a, b \in A, (a, b) \in R, (b, a) \in R \Rightarrow (a, a) \in R, (b, b) \in R$$

always holds. So if R is reflexive and symmetric, then R is also transitive.

b) True. Consider that R_1 and R_2 are subsets of $R_1 \cup R_2$, so if R_1 and R_2 is reflexive, we have

$$\forall a \in A,\ ((a, a) \in R_1) \wedge ((a, a) \in R_2)$$

$$\Rightarrow \forall a \in A,\ (a, a) \in R_1 \cup R_2$$

which gives that $R_1 \cup R_2$ is also reflexive.

c) False. Counterexample: if we have $(a, b) \in R_1$ and $(b, a) \notin R_1$ and $(b, a) \in R_2$ and $(a, b) \notin R_2$ then we have $(a, b) \in R_1 \cup R_2$ and $(b, a) \in R_1 \cup R_2$ which gives that $R_1 \cup R_2$ is symmetric.

PROBLEM 3. Prove the statements about n-ary relations.

SOLUTION. **a)** The selection operator combined is the set

$$S_{C_1 \wedge C_2}(R) = \{a \in R | C_1(a) \wedge C_2(a)\}$$

and we have if

$$x \in S_{C_1 \wedge C_2}(R)$$

then $x \in R$ satisfy $C_1 \wedge C_2$, which is $x \in R$ satisfy C_1 and C_2, so we have

$$x \in S_{C_2}(R) \text{ and } x \text{ satisfy } C_1$$

which gives

$$S_{C_1}(S_{C_2}(R))$$

b) Both sides project into the $\{i_k\} = \{i_1, i_2, \cdots, i_m\}$th elements in tuples of R and S. So for the left side, we have

$$a = (a_1, a_2, \cdots, a_n)$$

$$P_{\{i_k\}}(R \cup S) = \{(a_{i_1}, a_{i_2}, \cdots, a_{i_m}) | a \in R \cup S\}$$

and for the right side, we have

$$P_{\{i_k\}}(R) \cup P_{\{i_k\}}(S) = \{(a_{i_1}, a_{i_2}, \cdots, a_{i_m}) | a \in R\} \cup \{(a_{i_1}, a_{i_2}, \cdots, a_{i_m}) | a \in S\}$$

$$\Rightarrow \{(a_{i_1}, a_{i_2}, \cdots, a_{i_m}) | a \in R \cup S\}$$

Problem 4. Suppose that a relation R on a set A is symmetric.

Solution. **a)** We can use induction to prove this.
For $k = 1$, R^1 is symmetric since R is symmetric.
For $k = n - 1$, assume that R^{n-1} is symmetric, then we have

$$R^n = R^{n-1} \circ R = R \circ R^{n-1}$$

so for all $(x, y) \in R^n$ we always have

$$(x, z) \in R \wedge (z, y) \in R^{n-1}$$

since R and R^{n-1} are symmetric, we have

$$(z, x) \in R \wedge (y, z) \in R^{n-1}$$

and the composition of them gives

$$(y, x) \in R \circ R^{n-1} = R^n$$

which means R^n is symmetric. So by induction, we have R^n is symmetric for all $n \in \mathbb{N}^+$.

b) The $R^* = \bigcup R^k$ and from previous proof we have R^n is symmetric for all $n \in \mathbb{N}^+$, so we need to prove that the union of symmetric relations is symmetric. Suppose that R_1 and R_2 are symmetric relations on A, then

$$\Big((a,b) \in R_1 \to (b,a) \in R_1\Big) \vee \Big((c,d) \in R_2 \to (d,c) \in R_2\Big)$$

$$\Rightarrow (a,b),(c,d) \in R_1 \cup R_2 \to (b,a),(d,c) \in R_1 \cup R_2$$

which gives that $R_1 \cup R_2$ is symmetric. So we have R^* is symmetric since

$$R^* = (R^1 \vee R^2) \vee \cdots \vee R^n$$

the union of symmetric relations is symmetric.

Problem 5. Prove that the transitive closure of the symmetric closure of a relation must contain the symmetric closure of the transitive closure of the relation.

Solution. Suppose that R is a relation on a set A, the symmetric closure is

$$S = R \cup \{(b,a)|(a,b) \in R\}$$

and the transitive closure of it is

$$T = S \cup \{(a,c)|(a,b),(b,c) \in S\}$$

and by symmetric closure, we have

$$(c,b),(b,a) \in S \Rightarrow (c,a) \in T$$

so the transitive closure of the symmetric closure of R is symmetric and hence it must contain the symmetric closure of the transitive closure of R.

PROBLEM 6. Use the Floyd-Warshall algorithm to find the transitive closures of the relation

$$R = \{(a,b),(a,c),(a,e),(b,a),(b,c),(c,a),(c,b),(d,a),(e,d)\}$$

on the set $A = \{a,b,c,d,e\}$.

SOLUTION. The initial matrix is

$$M = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

for $k = 1$, we have

$$M_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

for $k = 2$, we have

$$M_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

for $k = 3$, we have

$$M_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

for $k = 4$, we have

$$M_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

so the transitive closure of R is

$$M^* = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Problem 7. Consider the relation $R = \{(x, y) | x - y \in \mathbb{Z}\}$.

Solution. **a)** R is symmetric since for any $x, y \in \mathbb{R}$, we have

$$x - y \in \mathbb{Z} \Rightarrow y - x = -(x - y) \in \mathbb{Z}$$

R is reflexive since for any $x \in \mathbb{R}$, we have

$$x - x = 0 \in \mathbb{Z}$$

R is transitive since for any $x, y, z \in \mathbb{R}$, we have

$$x - y \in \mathbb{Z} \wedge y - z \in \mathbb{Z} \Rightarrow x - z = (x - y) + (y - z) \in \mathbb{Z}$$

so R is an equivalence relation.

b) The equivalence class of 1 is

$$[1] = \{x \in \mathbb{R} | 1 - x \in \mathbb{Z}\} = \mathbb{Z}$$

The equivalence class of 1/2 is

$$\left[\frac{1}{2}\right] = \{x \in \mathbb{R} | 1/2 - x \in \mathbb{Z}\} = \{1/2 + n | n \in \mathbb{Z}\}$$

The equivalence class of π is

$$[\pi] = \{x \in \mathbb{R} | \pi - x \in \mathbb{Z}\} = \{\pi + n | n \in \mathbb{Z}\}$$

PROBLEM 8. For any functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. We say f is *dominated* by g, denoted by $f \preceq g$, if and only if $\forall x \in \mathbb{R}, f(x) \leq g(x)$ holds. Prove or disprove the following statements.

SOLUTION. **a)** The relation is antisymmetric since if $f \preceq g$ then we have

$$\forall x \in \mathbb{R}, f(x) \leq g(x)$$

and the opposite is false since if $g \preceq f$ then we have

$$\forall x \in \mathbb{R}, g(x) \leq f(x)$$

which is contradictory to the previous statement. So the relation is antisymmetric.

The relation is reflexive since for any $f : \mathbb{R} \to \mathbb{R}$, we have

$$\forall x \in \mathbb{R}, f(x) \leq f(x)$$

which is $f \preceq f$.

The relation is transitive since for any $f, g, h : \mathbb{R} \to \mathbb{R}$, we have

$$\forall x \in \mathbb{R}, f(x) \leq g(x) \wedge g(x) \leq h(x) \Rightarrow f(x) \leq h(x)$$

which is $f \preceq g \wedge g \preceq h \Rightarrow f \preceq h$. Thereby, the relation is a partial order.

b) The statement is false since functions in the poset are not comparable. For example, if we have

$$f(x) = x, \quad g(x) = x^2$$

then $f(x) \leq g(x)$ only holds for $x \leq 0$ or $x \geq 1$, so $f \preceq g$ and $g \preceq f$ are false.

Problem 9. Answer the questions about the partial order represented by the Hasse diagram.

Solution. **a)** l and m

b) a, b and c

c) No.

d) No.

e) l, k and e

f) k

g) a, d and b

h) d

Problem 10. Topological sorting. Find all compatible total orderings for the poset $(\{2, 3, 4, 6, 12\}, |)$.

Solution. The Hasse diagram of the poset is

$$\begin{tikzcd}[arrows=-]
& 12 \arrow[dl] \arrow[dr] & \\
6 \arrow[d] & & 4 \arrow[d] \\
3 & & 2
\end{tikzcd}$$

so the compatible total orderings are

$$2 \prec 3 \prec 4 \prec 6 \prec 12$$

$$3 \prec 2 \prec 4 \prec 6 \prec 12$$

$$2 \prec 4 \prec 3 \prec 6 \prec 12$$

$$3 \prec 4 \prec 2 \prec 6 \prec 12$$