Existance and Uniqueness of Solutions.

Wronskian

Theorem-1: Existance and Uniqueness Theorem for initial value problem:

If $p(x)$ and $q(x)$ are continous functions on some open interval I and x_0 is in I, then the initial value problem consisting of

$$y'' + p(x)y' + q(x)y = 0 \quad -(1)$$

and

$$y(x_0) = K_0 \quad , \quad y'(x_0) = K_1 \quad -(2)$$

has a unique solution $y(x)$ on the interval I.

Theorem-2: Linear Dependence and Independence of Solutions:-

Let the ODE (1) have continous coefficients $p(x)$ and $q(x)$ on an open interval I. Then two solutions y_1 and y_2 of (1) on I are linearly dependent on I if and only if their "Wronskian"

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1' = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

is 0 at some x_0 in I. Furthermore, if $W = 0$ at an $x = x_0$ in I, then $W = 0$ on I. hence, if there is an x_1 in I at which W is not 0, then y_1, y_2 are linearly independent on I.

Theorem-3 Existence of a General solution.

If $p(x)$ and $q(x)$ are continuous on an open interval I, then (1) has a general solution on I.

Theorem-4: A General Solution includes all solutions:-

If the ODE (1) has continuous coefficients $p(x)$ and $q(x)$ on some open interval I, then every solution $y = Y(x)$ of (1) on I is of the form

$$Y(x) = c_1 y_1(x) + c_2 y_2(x).$$

where y_1, y_2 is any basis of solutions of (1) on I and c_1, c_2 are suitable constants.

Hence, (1) does not have singular solutions (that is, solutions not obtained from a general solution).

Example:- Find the Wronskian. State the linear independence or dependence of functions.

① $e^{4x}, e^{-1.5x}$

Solution:- Let $y_1 = e^{4x}$ & $y_2 = e^{-1.5x}$

$y_1' = 4e^{4x}$ $\quad$ $y_2' = -1.5e^{-1.5x}$.

$$\therefore W(y_1, y_2) = \begin{vmatrix} e^{4x} & e^{-1.5x} \\ 4e^{4x} & -1.5e^{-1.5x} \end{vmatrix} = -1.5e^{2.5x} - 4e^{2.5x}$$

$$= -5.5e^{2.5x} \neq 0.$$

$\therefore e^{4x}$ and $e^{-1.5x}$ are L.I.

(2) $x, \frac{1}{x}$

Ans Let $y_1 = x$ & $y_2 = \frac{1}{x}$

$\Rightarrow y_1' = 1$ $\quad y_2' = -\frac{1}{x^2}$

$\therefore W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix}$

$= -\frac{1}{x} - \frac{1}{x} = -\frac{2}{x} \neq 0.$

$\therefore x, \frac{1}{x}$ are L.I.

(3) $e^{5x}, 6e^{5x}$

$y_1 = e^{5x}$ $\quad y_2 = 6e^{5x}$

$\Rightarrow y_1' = 5e^{5x}$ $\quad y_2' = 30e^{5x}.$

$\therefore W(y_1, y_2) = \begin{vmatrix} e^{5x} & 6e^{5x} \\ 5e^{5x} & 30e^{5x} \end{vmatrix}$

$= 30e^{10x} - 30e^{10x} = 0$

$\therefore e^{5x}, 6e^{5x}$ are linearly dependent.

Non-Homogeneous ODEs:

The second order linear ode of non-homogeneous is of the form

$$y'' + p(x)\, y' + q(x)\, y = r(x) \qquad -(1).$$

where $r(x) \neq 0$.

The homogeneous linear equation corresponding to (1) is

$$y'' + p(x)\, y' + q(x)\, y = 0 \qquad -(2)$$

General solution:-

A general solution of the non homogeneous ODE (1) on an open interval I is a solution of the form

$$y(x) = y_h(x) + y_p(x). \qquad -(3).$$

here $y_h = c_1 y_1 + c_2 y_2$ is the general solution of (2) on I and y_p is any solution of (1) on I containing no arbitrary constants.

particular Solution:

A particular solution of (1) on I is a solution obtained from (3) by assigning specific values to the arbitrary constants c_1 and c_2 in y_h.

Theorem-1: Relations of Solutions of (1) to those of (2).

(a) The sum of a solution y of (1) on some open interval I and a solution $\tilde{y}$ of (2) on I is a solution of (1) on I. In particular, (3) is a solution of (1) on I.

(b) The difference of two solutions of (1) on I is a solution of (2) on I.

Theorem-2: A General solution of Nonhomogeneous ODE includes all solutions.

If the coefficients $p(x)$, $q(x)$ and the function $r(x)$ in (1) are continuous on some open interval I, then every solution of (1) on I is obtained by assigning suitable values to the arbitrary constants C_1 and C_2 in a general solution (3) of (1) on I.

Working procedure to obtain Solution

To solve ~~or~~ find a general solution of Non homogeneous differential equation ①.

① first, we need to find the general solution y_h to ②.

② Second, we need to find a y_p solution, without ~~to eqn ①~~ arbitrary constants to eqn ①.

(3) The general solution of (1) is

$$y = y_h + y_p.$$

~~Note~~

Method of undetermined Coefficients :-

It is ~~a~~ used to find ~~the~~ a solution to Non homogeneous linear differential equation (1). Specifically, this method is suitable for Non-homogeneous linear differential equation with Constant coefficients. That is.

$$y'' + a y' + by = r(x) \qquad -(4)$$

where $r(x)$ is an exponential function, a power of x (algebraic expression), a Cosine or Sine or Sums or products of such functions.

We choose a form for y_p similar to $r(x)$, but with unknown Coefficients to be determined by substituting that y_p and its derivatives into ODE (4). The following table shows the ~~these~~ choice of y_p for given $r(x)$. and the associated rules for correct choice of y_p.

Terms in $r(x)$	Choice for $y_p(x)$
$k e^{\gamma x}$	$C e^{\gamma x}$
$k x^n \quad (n = 0, 1, 2, \ldots)$	$k_n x^n + k_{n-1} x^{n-1} + \cdots + k_1 x + k_0$
$k \cos \omega x$, $k \sin \omega x$	$k \cos \omega x + M \sin \omega x$
$k e^{\alpha x} \cos \omega x$ $k e^{\alpha x} \sin \omega x$	$e^{\alpha x} (K \cos \omega x + M \sin \omega x)$

Choice Rules for the method of undetermined Coefficients :

(a) Basic Rule :-
If $r(x)$ in (4) is one of the functions in the first Column in the above table, choose y_p in the same line and determine its undetermined Coefficients by Substituting y_p and its derivatives into (4).

(b) Modification Rule :-
If a term in your choice for y_p happens to be a solution of the homogeneous ODE corresponding to (4), multiply this term by x (or by x^2 if this solution corresponds to a double root of the characteristic equation of the homogeneous ODE).

(c). Sum rule;

If $r(x)$ is a sum of functions in the first column of above table. Choose for y_p the sum of the functions in the corresponding lines of the second column.

Example 1.

Solve $y'' + y = 0.001x^2$. $y(0) = 0$. $y'(0) = 1.5$.

Solution; Given non homogeneous differential equation is

$$y'' + y = 0.001x^2 \qquad -(1)$$

Its corresponding homogeneous differential equation is

$$y'' + y = 0 \qquad -(2)$$

The auxiliary equation for (2) is

$$\lambda^2 + 1 = 0.$$

$$\Rightarrow \lambda = \pm i$$

Thus, the general solution of (2) is

$$\boxed{y_h = c_1 \cos x + c_2 \sin x.} \qquad -(3)$$

c_1, c_2 are arbitrary constants.

In equation (1), $r(x) = 0.001x^2$. So, we then choose y_p for (1) as $y_p = Ax^2 + Bx + C$.

Now, $y_p' = 2Ax + B$ and $y_p'' = 2A$.

Replacing y, y' & y'' by y_p, y_p' and y_p'' in (1),

$$2A + Ax^2 + Bx + C = 0.001x^2$$

$$\Rightarrow Ax^2 + Bx + (2A + C) = 0.001x^2 \qquad -(4)$$

Equating both the sides, we have.

$A = 0.001 \qquad \Rightarrow A = 0.001$

$B = 0. \qquad \Rightarrow B = 0$

$2A + C = 0. \qquad \Rightarrow C = -2A = -0.002.$

$$\therefore y_p = 0.001x^2 - 0.002. \qquad -(5)$$

Hence, the general solution of (1) is

$$y = y_h + y_p.$$

$$\boxed{y = c_1 \cos x + c_2 \sin x + 0.001x^2 - 0.002.} \qquad -(6)$$

Again,

$$y' = -c_1 \sin x + c_2 \cos x + 0.002x \qquad -(7)$$

Using initial condition $y(0) = 0$ for eqn (6), we have.

$$0 = y(0) = c_1 \cdot \cos 0 + c_2 \sin 0 + 0.001 \times 0^2 - 0.002$$

$$\Rightarrow 0 = c_1 - 0.002 \Rightarrow \boxed{c_1 = 0.002}$$

Again, using $y'(0) = 1.5$ for (7),

$1.5 = y'(0) = -C_1 \cdot \sin 0 + C_2 \cos 0 + 0.002 \times 0$

$\Rightarrow 1.5 = -C_1 \times 0 + C_2 \cdot 1 + 0$

$\Rightarrow \boxed{C_2 = 1.5}$

$\therefore$ The solution (6) becomes.

$y = 0.002 \cos x + 1.5 \sin x + 0.001 x^2 - 0.002$

This is the required solution for given IVP.

Example-2 : Solve $y'' + 3y' + 2.25y = -10e^{-1.5x}$

Solution : Given differential equation is

$y'' + 3y' + 2.25y = -10e^{-1.5x}$ — (1)

The corresponding homogeneous differential equation is

$y'' + 3y' + 2.25y = 0$ — (2).

The auxiliary equation for (2) is

$\lambda^2 + 3\lambda + 2.25 = 0.$ — (3)

$\therefore \lambda = \dfrac{-3 \pm \sqrt{9 - 9}}{2} = -1.5$

$\Rightarrow \lambda = -1.5$ a double root

Now, the general solution for ② is

$y_h = (c_1 + c_2 x) e^{-1.5x}$ ——③.

The $r(x)$ in ① is, $r(x) = e^{-1.5x}$.

Here, $r(x)$ is a solution of ② corresponding to double root (see ③). Thus, we can choose y_p as $y_p = c x^2 e^{-1.5x}$ (instead of $y_p = ce^{-1.5x}$).

Then, $y_p' = 2cx e^{-1.5x} - 1.5 c x^2 e^{-1.5x}$.

$y_p'' = 2c e^{-1.5x} - 3cx e^{-1.5x} - 3cx e^{-1.5x} + 2.25 c x^2 e^{-1.5x}$.

$\Rightarrow y_p'' = 2c e^{-1.5x} - 6cx e^{-1.5x} + 2.25 c x^2 e^{-1.5x}$.

Now, replace y by y_p, y' by y_p' and y'' by y_p'' in ①,

$2c e^{-1.5x} - 6cx e^{-1.5x} + 2.25 c x^2 e^{-1.5x}$

$+ 3(2cx e^{-1.5x} - 1.5 c x^2 e^{-1.5x})$

$+ 2.25 c x^2 e^{-1.5x} = -10 e^{-1.5x}$

$\Rightarrow 2c e^{-1.5x} - \cancel{6cx} e^{-1.5x} + 2.25 \cancel{cx^2} e^{-1.5x}$

$+ \cancel{6cx e^{-1.5x}} - \cancel{4.5cx^2} e^{-1.5x} + 2.25 \cancel{c} x^2 e^{-1.5x}$

$= -10 e^{-1.5x}$.

$\Rightarrow 2c \; e^{-1.5x} = -10\, e^{-1.5x}$.

Equating both the sides, we have,

$2c = -10 \Rightarrow c = -5$.

$\therefore \boxed{y_p = -5x^2 e^{-1.5x}}$

Hence, the general solution of (1) is

$y = y_h + y_p$.

That is $\boxed{y = (c_1 + c_2 x)\, e^{-1.5x} - 5x^2 e^{-1.5x}}$.

Example :-

Solve $y'' + 2y' + 0.75y = 2\cos x - 0.25\sin x + 0.09x$.

Solution :-

Given Non-homogeneous differential equation is

$$y'' + 2y' + 0.75y = 2\cos x - 0.25\sin x + 0.09x \quad -(1).$$

It's corresponding homogeneous equation is

$$y'' + 2y' + 0.75y = 0 \quad -(2)$$

The auxiliary equation for (2) is

$$\lambda^2 + 2\lambda + 0.75 = 0 \quad -(3).$$

The roots of eqn (3) are

$$\lambda = \frac{-2 \pm \sqrt{4-3}}{2} = \frac{-2 \pm 1}{2}$$

$$= -3/2, -1/2$$

Thus, the general solution of eqn (2) is

$$y = c_1 e^{-x/2} + c_2 e^{-3x/2} \quad -(4)$$

In eqn (1), $r(x) = 2\cos x - 0.25\sin x + 0.09x.$

Then, we can choose y_p as follow:

$$y_p = A\cos x + B\sin x + Cx + D. \quad (5).$$

$$\Rightarrow \quad y_p' = -A\sin x + B\cos x + C$$

$$y_p'' = -A\cos x - B\sin x$$

Substituting all these in eqn (1),

$$-A\cos x - B\sin x + 2(-A\sin x + B\cos x + C)$$
$$+ 0.75(A\cos x + B\sin x + Cx + D)$$
$$= 2\cos x - 0.25\sin x + 0.09x.$$

$$\Rightarrow (-A + 2B + 0.75A)\cos x$$
$$+ (-B - 2A + 0.75B)\sin x + 0.75Cx$$
$$+ 2C + 0.75D = 2\cos x - 0.25\sin x$$
$$+ 0.09x.$$

Equating the coefficients of same terms in both sides we have.

$-0.25A + 2B = 2$ —(i)

$-2A - 0.25B = -0.25$ —(ii)

$0.75C = \cancel{0.009}\ 0.09$ —(iii)

$2C + 0.75D = 0$ —(iv)

(i) $\quad -0.25A + 2B = 2.$

(ii)×8 $\quad -16A - 2B = -2$

(+) $\quad -16.25A = 0.$

$\boxed{A = 0}$

$\Rightarrow \boxed{B = 1.}$

$\Rightarrow C = \dfrac{0.09}{0.75} = 0.12 \Rightarrow \boxed{C = 0.12}$

$\Rightarrow D = -\dfrac{2C}{0.75} = \dfrac{0.24}{0.75} = -0.32$

$\therefore \quad y_p = \sin x + 0.12x - 0.32$

Hence the general solution of (i) is

$y = y_h + y_p.$

that is, $\quad y = C_1 e^{-x/2} + C_2 e^{-3x/2} + \sin x + 0.12x - 0.32$

Ans///