Day-5, Nov-20, 2024 (Mangshir 5, 2081 B.S).

Continuity.

The intuitive idea of a continuous function f in the interval $[a,b]$ gives impression that the graph of the function f in this interval is a smooth curve without any break it.

So, in simple language curve drawn without lifting it in a sheet of paper

A discontinuous function gives the picture consisting of disconnected curves.

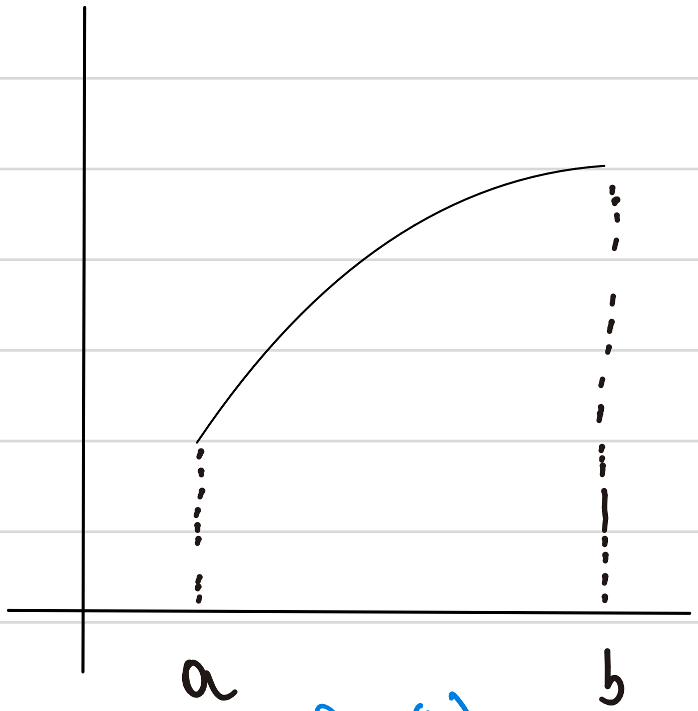


Fig (i)
Continuous ?

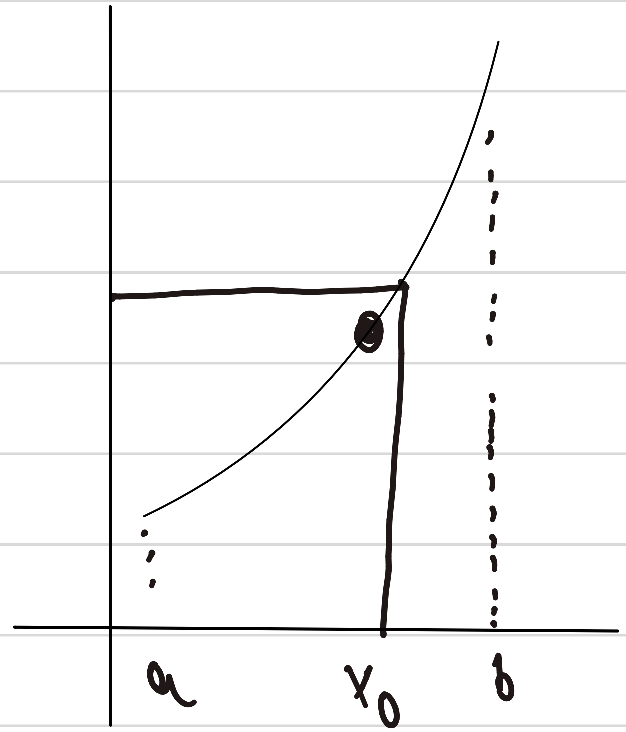


Fig (ii)
Discontinuous

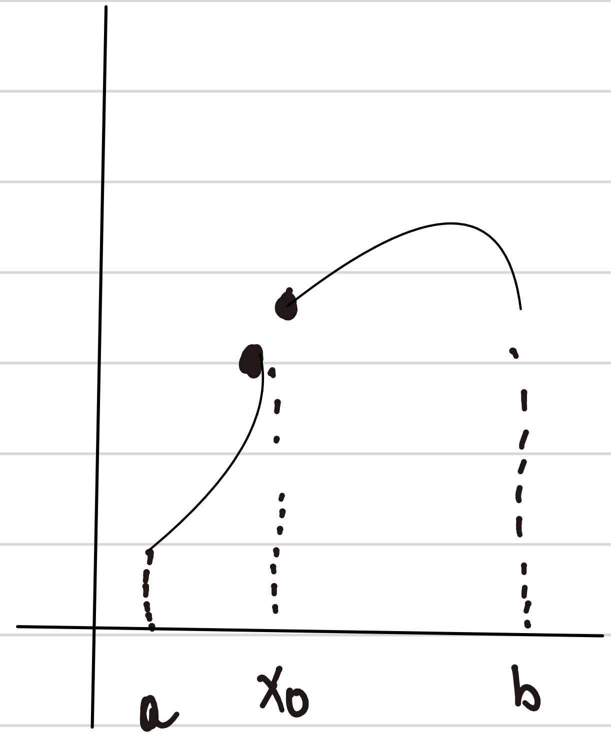


Fig (iii)
Discontinuous

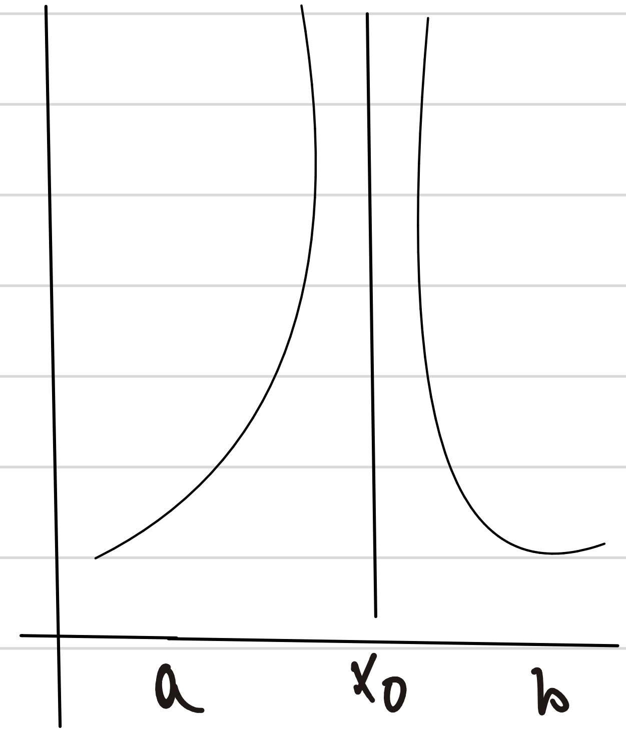


Fig (iv)
Discontinuous

Fig ii) $\lim_{x \to x_0} f(x)$ exists and $f(x_0)$ is also defined, but $\lim_{x \to x_0} f(x) \neq f(x_0)$

Fig iii) $\lim_{x \to x_0} f(x)$ exists but $f(x)$ is defined at $x = x_0$

Fig v) Neither $\lim_{x \to x_0} f(x)$ exists nor the function $f(x)$ is defined at $x = x_0$.

Definition: The function $f(x)$ is said to continuous at the point $x = x_0$ if and only if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

This definition of continuity of the function $f(x)$ at $x = x_0$ implies that

a) $\lim_{x \to x_0} f(x)$ exists i.e. $\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^+} f(x)$ are finite and equal

b) $f(x_0)$ exists

c) $\lim_{x \to x_0} f(x) = f(x_0)$

Hence, $f(x)$ will be continuous at $x = x_0$ if

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = f(x_0)$$

if any of the above conditions is not satisfied then the function $f(x)$ is said to be discontinuous at that point.

Types of discontinuities:

(i) if $\lim_{x \to x_0} f(x)$ does not exist i.e. $\lim_{x \to x_0^-} f(x) \neq \lim_{x \to x_0^+} f(x)$ then $f(x)$ said to be an ordinary discontinuity or a jump.

(ii) if $\lim_{x \to x_0} f(x) \neq f(x_0)$ then the function $f(x)$ is said to have a removable discontinuity at $x = x_0$. This type of discontinuity can be removed redefining the function.

(iii) if $\lim_{x \to x_0} f(x) \to \infty$ or $-\infty$, then $f(x)$ is said to have infinite discontinuities at $x = x_0$.

Limit of Trigonometric function

$$\lim_{\theta \to \alpha} \sin\theta = \sin\alpha$$

put $\theta \to \alpha + h$ so that $\theta \to \alpha$, $h \to 0$

Now, $$\lim_{\theta \to \alpha} \sin\theta = \lim_{h \to 0} \sin(\alpha + h)$$

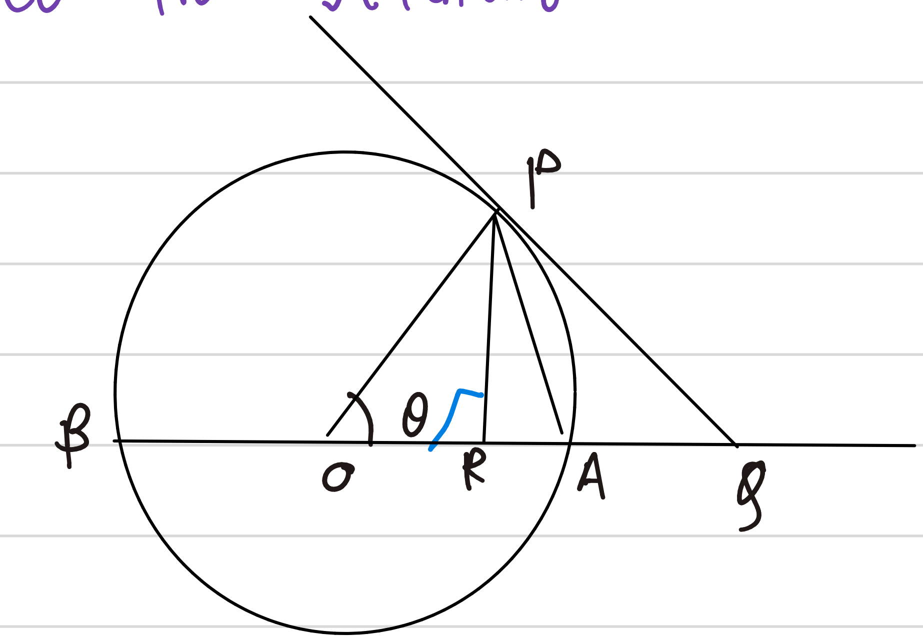
$$\Rightarrow \lim_{h \to 0} [\sin\alpha \cdot \cos h + \cos\alpha \cdot \sin h]$$

$$\Rightarrow \sin\alpha \cdot \lim_{h \to 0} \cos h + \cos\alpha \lim_{h \to 0} \sin h$$

$$\Rightarrow \sin\alpha \cdot 1 + \cos\alpha \cdot 0$$

$$\therefore \lim_{\theta \to \alpha} \sin\theta = \sin\alpha$$

Theorem: $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ where θ is measured in radian.



Here,

AP is an arc which extends an angle θ at the centre O.

PQ = tangent
BA produced Q

Join PA and PR $\perp$ to BA then-

Area of $\Delta OPA \le$ Area of Sector OAP $\le$ Area of ΔOPQ

Area of $\Delta OPA = \frac{1}{2} \cdot OA \cdot PR$

$= \frac{1}{2} \cdot r \cdot r\sin\theta$

[from Δ rule $OA = r$
$PR = r\sin\theta$]

Area of $\Delta OPA = \frac{1}{2} r^2 \sin\theta$

$\therefore$ Area of Sector $OAP = \frac{1}{2} r^2 \theta$

$\therefore$ Area of $\Delta OPQ = \frac{1}{2} \times OP \times PQ$

$= \frac{1}{2} \times r \times r\tan\theta$

$= \frac{1}{2} r^2 \tan\theta$

$\therefore$ Area of $\Delta OPQ = \frac{1}{2} \cdot OP \cdot PQ = \frac{1}{2} r^2 \tan\theta$

$\therefore \frac{1}{2} r^2 \sin\theta \le \frac{1}{2} r^2\theta \le \frac{1}{2} r^2 \tan\theta.$

$\therefore 1 \le \frac{\theta}{\sin\theta} \le \frac{1}{\cos\theta}$

or, $1 \geq \frac{\sin\theta}{\theta} \geq \cos\theta$ [upside-down]

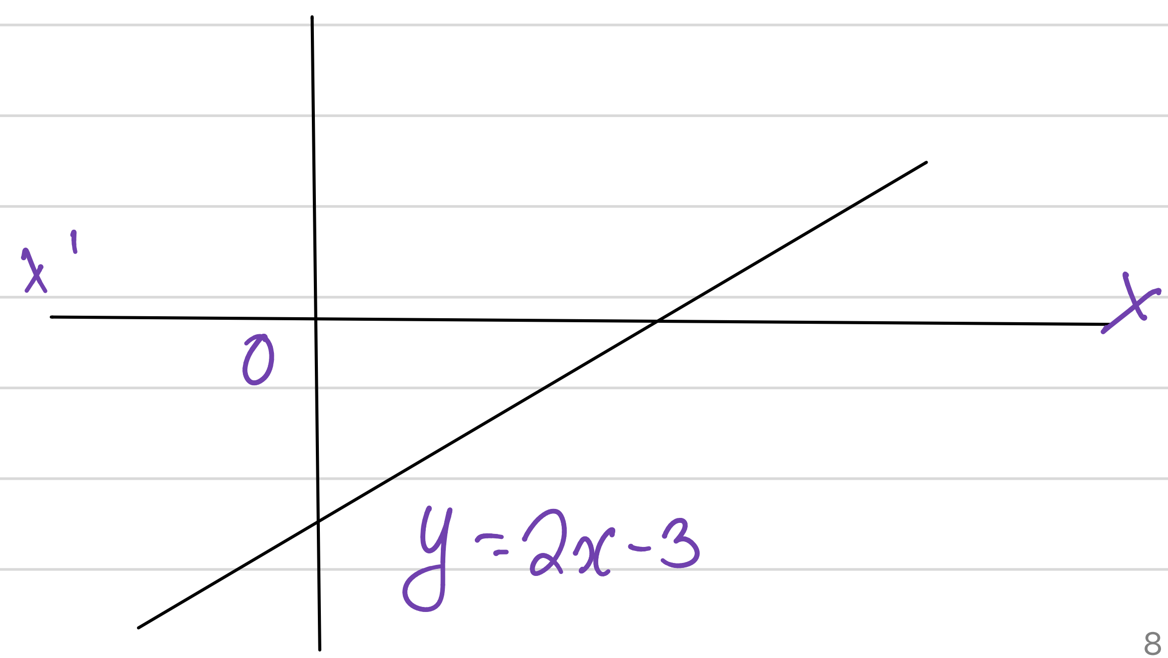
or, $\lim_{\theta \to 0} 1 \geq \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \geq \lim_{\theta \to 0} \cos\theta$

$1 \geq \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \geq 1$

$\therefore \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.$

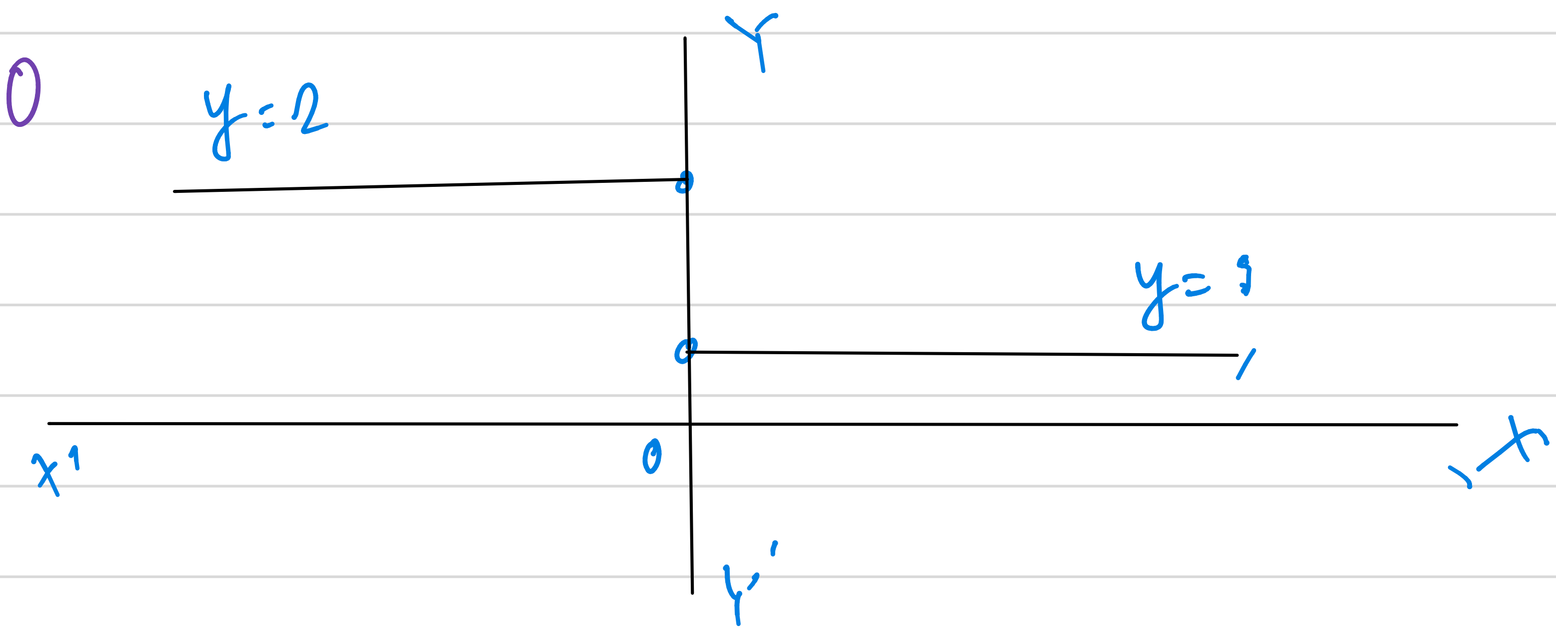
Examples of Continuities Graph

$y = 2x - 3$ is Continuous at every point.



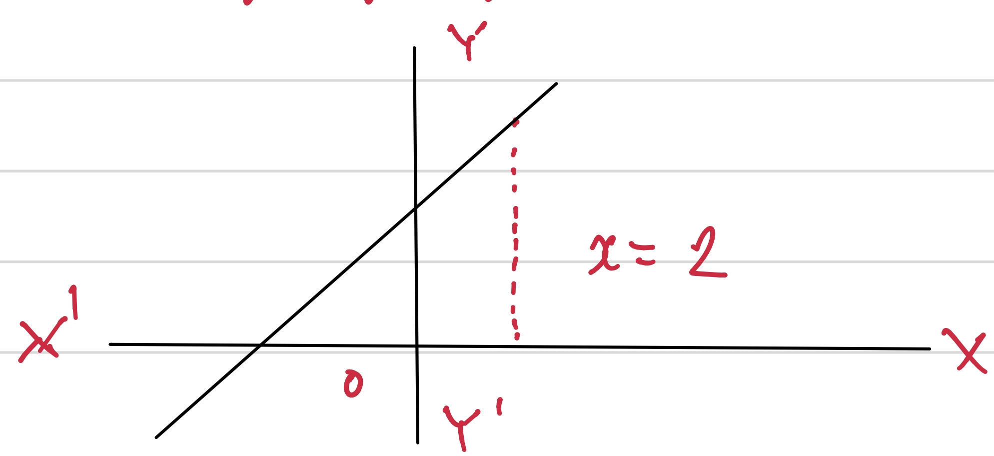
Example of discontinuities at $x=0$

$$f(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 2 & \text{for } x < 0 \end{cases}$$

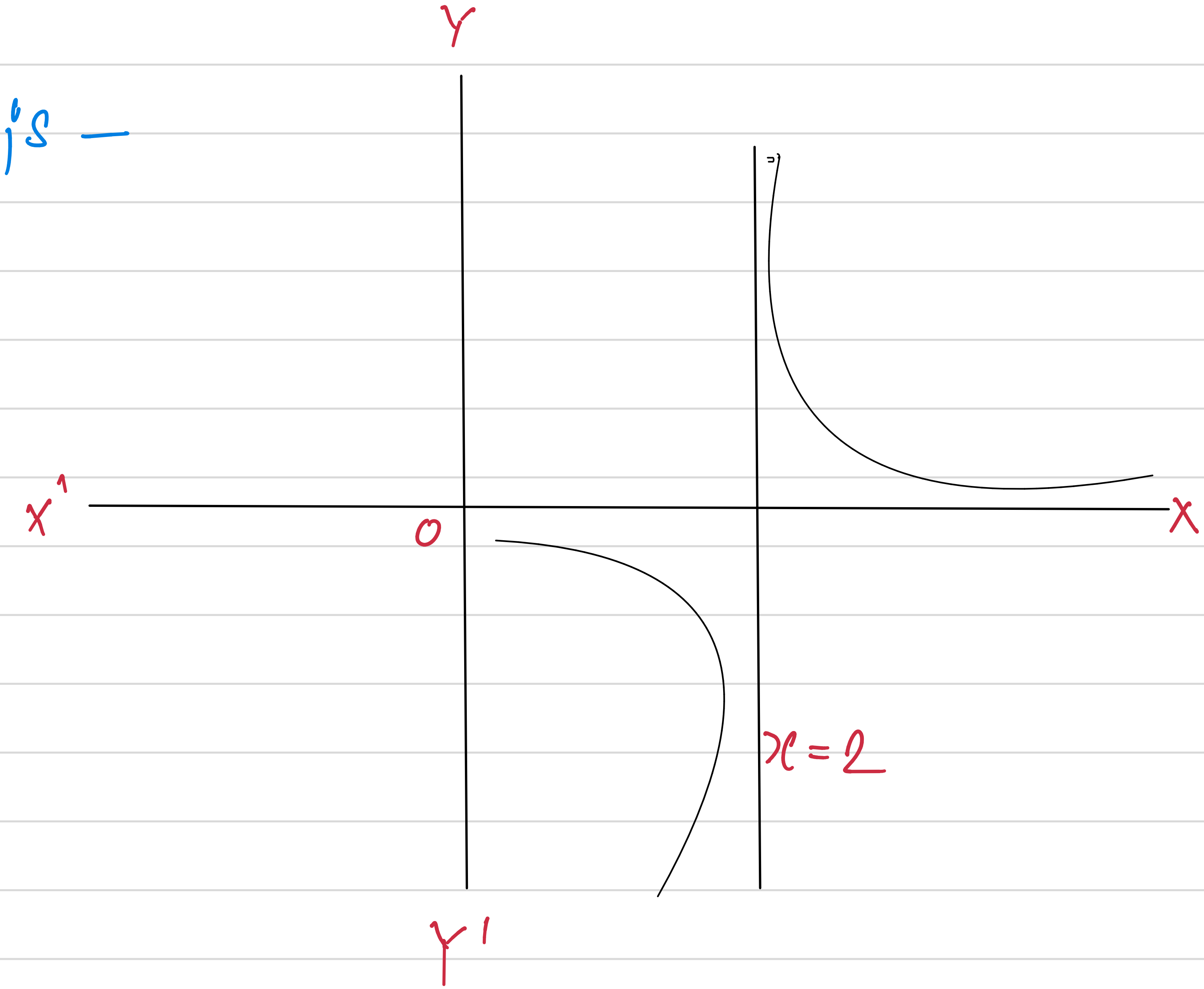


$f(x)$ discontinuous at $x=0$ so there is a jump at $x=0$.

Example of Indeterminate form $\left(\frac{0}{0}\right)$ $y = f(x) = \frac{x^2-4}{x-2}$ is $\frac{0}{0}$ at a given $f(2) = \frac{0}{0}$. So at $f(2)$ does not exist and discontinuities at $x=2$.



Graph of $y = f(x) = \frac{1}{x-2}$ is —



Here $\lim\limits_{x \to 2^-} \frac{1}{x-2} \to -\infty$

and $\lim\limits_{x \to 2^+} \frac{1}{x-2} \to \infty$

at $f(x) = 2$.

REFERENCE:

D.R. Bajracharya et. al, 2014, Basic Mathematics Grade XI (3rd-Edition), Sukunda Pustak Bhawan, Kathmandu.