Day-46, Jan-15, 2025 (Magh 02, 2081 B.S.)

Improper Integral

The definite integral $\int_a^b f(x) \cdot dx$, it is assumed that either

i) a or b or both are infinite or

ii) $f(x)$ becomes infinite at some interior point of the interval $[a, b]$. The integral of these types are called improper integral.

Type I: Improper Integral.

The Integrals

i) $$\int_{a}^{\infty} f(x) \cdot dx = \lim_{h \to \infty} \int_{a}^{h} f(x) \cdot dx$$

ii) $$\int_{-\infty}^{b} f(x) \cdot dx = \lim_{h \to \infty} \int_{h}^{b} f(x) \cdot dx$$

iii) $$\int_{-\infty}^{\infty} f(x) \cdot dx = \int_{-\infty}^{c} f(x) \cdot dx + \int_{c}^{\infty} f(x) \cdot dx$$ where c is

any real number are called Improper Integral diverges.

Useful formula:

i) $$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

ii) $$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$$

iii) $$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|$$

iv) $$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$$

v) $$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|x+\sqrt{x^2+a^2}\right|$$

$$\int_a^b f(x)\,dx$$ Where b or a or both can be infinite or infinite within a certain range.

Example: Determine whether the Integral $\int_1^{\infty} \frac{1}{x} \cdot dx$ is divergent or Convergent?

We have, $$\int_1^{\infty} \frac{1}{x} \cdot dx = \lim_{h \to \infty} \int_1^{h} \frac{1}{x} \cdot dx$$

$$\Rightarrow \lim_{h \to \infty} \left[\ln |x| \right]_1^h$$

$$\Rightarrow \lim_{h \to \infty} \left[\ln |x| - \ln |1| \right]$$

$$\Rightarrow \ln |\infty| - \ln 1$$

$$\Rightarrow \infty$$

Example: $\displaystyle\int_{-\infty}^{0} x\, e^{x} \cdot dx$

We have,

$$\int_{-\infty}^{0} x\, e^{x} \cdot dx = \lim_{t \to -\infty} \int_{t}^{0} x\, e^{x} \cdot dx$$

$$\Rightarrow \lim_{t \to -\infty} \left[\left[x e^{x} \right]_{t}^{0} - \int_{t}^{0} e^{x} \cdot dx \right]$$

$$\Rightarrow \lim_{t \to -\infty} \left(-t e^{t} - 1 + e^{t} \right) \qquad \left[\lim_{t \to \infty} t e^{t} = \lim_{t \to -\infty} \frac{1}{-e^{-t}} = \lim_{t \to -\infty} \frac{1}{-e^{-t}} = 0 \text{ using} \right.$$

(L - Hospital Rule)

$$\Rightarrow -0 - 1 + 0 \quad \Rightarrow -1$$

Example: $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

Here,
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{1} \frac{dx}{1+x^2} + \int_{1}^{\infty} \frac{dx}{1+x^2}$$

$$\Rightarrow I_1 + I_2$$

Since,
$$I_1 = \int_{-\infty}^{1} \frac{dx}{1+x^2} \qquad \left[\frac{1}{1+x^2} = \tan^{-1} x \right]$$

$$\Rightarrow \lim_{h \to -\infty} \left[\tan^{-1} x \right]_h^1$$

$$\Rightarrow \lim_{h \to -\infty} \tan^{-1}(1) - \tan^{-1}(h)$$

$$\Rightarrow \frac{\pi}{4} - \left(\tan^{-1}(-\infty)\right)$$

$$\Rightarrow \pi/4 - \left(-\pi/2\right)$$

$$\Rightarrow \pi/4 + \pi/2$$

Again,

$$I_2 = \int_1^{\infty} \frac{dx}{1+x^2}$$

$$\Rightarrow \lim_{h\to\infty} \int_1^{h} \frac{dx}{1+x^2}$$

$$\Rightarrow \lim_{h\to\infty} \left[\tan^{-1}x\right]_1^{h}$$

$$\Rightarrow \lim_{h \to \infty} \left(\tan^{-1} h - \tan^{-1} 1 \right)$$

$$\Rightarrow \tan^{-1} \infty - \pi/4$$

$$\Rightarrow \pi/2 - \pi/4$$

from I_1 and I_2 we get

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = I_1 + I_2$$

$$\Rightarrow \cancel{\pi/4} + \pi/2 + \pi/2 - \cancel{\pi/4}$$

$$\left[\int_{-\infty}^{\infty} \frac{dx}{1+x^2} \Rightarrow \pi \right]$$

Example: For what value of p is the integral $\int_1^{\infty} \frac{1}{x^p} \cdot dx$ convergent ?

Here,

$$\int_1^{\infty} \frac{1}{x^p} \cdot dx$$

$$\Rightarrow \lim_{t \to \infty} \int_1^{t} x^{-p} \cdot dx$$

$$\Rightarrow \lim_{t \to \infty} \left[\frac{x^{-p+1}}{-p+1} \right]$$

$$\Rightarrow \lim_{t \to \infty} \frac{1}{1-p} \left[\frac{1}{t^{p-1}} - 1 \right]$$

If $p > 1$ then $p-1 > 0$ so $t^{p-1} \to \infty$ as $t \to \infty$ so

$$\frac{1}{t^{p-1}} \to 0$$

$$\therefore \int_1^{\infty} \frac{1}{x^p}\cdot dx = \frac{1}{p-1} \text{ if } p > 1$$

if $p < 1$ then $p-1 < 0$ so

$$\therefore \int_1^{\infty} \frac{1}{t^{p-1}} \to \infty \text{ as } t \to \infty$$

For this problem if $p > 1$ then it is convergent else it is divergent.

Type II : Improper Integral

The integrals

i) $\int_a^b f(x) \cdot dx = \lim_{h \to a^+} \int_h^b f(x) \cdot dx$; if $f(x) \to \infty$ as $x = a$.

ii) $\int_a^b f(x) \cdot dx = \lim_{h \to b^-} \int_a^h f(x) \cdot dx$; if $f(x) \to \infty$ as $x = b$.

iii) $\int_a^b f(x) \cdot dx = \int_a^c f(x) \cdot dx + \int_a^b f(x)\, dx$; if $f(x) \to \infty$ as $x = c$

where $a < c < b$.

Determine whether $\int_0^{\pi/2} \sec x \cdot dx$ converges or diverges.

We have, $\int_0^{\pi/2} \sec x \cdot dx = \lim_{t \to \pi/2} \int_0^{t} \sec x \cdot dx$

$\Rightarrow \lim_{t \to \pi/2} \Big[\ln |\sec x + \tan x| \Big]_0^t$

$\Rightarrow t \to \pi/2 \Big[\ln |\sec t + \tan t| - \ln|1| \Big]$

$\Rightarrow \ln \left(\sec \frac{\pi}{2} + \tan \frac{\pi}{2} \right| - \ln 1$

$= \infty$

It is divergent

Comparison Theorem: Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$ then,

a) If $\int_a^{\infty} f(x) \cdot dx$ is convergent, then $\int_a^{\infty} g(x) \cdot dx$ is also convergent

b) If $\int_a^{\infty} g(x) \cdot dx$ is divergent, then $\int_a^{\infty} f(x) \cdot dx$ is also divergent.

Example: Show that $\int_1^{\infty} \frac{1+e^{-x}}{x} \cdot dx$ is divergent.

Since, $\frac{1+e^{-x}}{x} > \frac{1}{x}$ for $x \in [1, \infty)$ and we know that $\int_1^{\infty} \frac{1}{x} \cdot dx$ is divergent and also divergent by Comparison Theorem.

Application of AntiDerivatives:

(1) Area between Two Curves
- Riemann Sum
- Definite Integrals

(2) Area of the Region in Polar form

(3) Volumes of Cylindrical Shells

(4) Approximate Integration

(5) Area of Surface of Revolution

(6) Arc Length

Theorem - Substitution in Definite Integrals

If g' is continuous on the interval $[a, b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x)) \cdot g'(x) \cdot dx = \int_{g(a)}^{g(b)} f(u) \cdot du$$

Example: Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

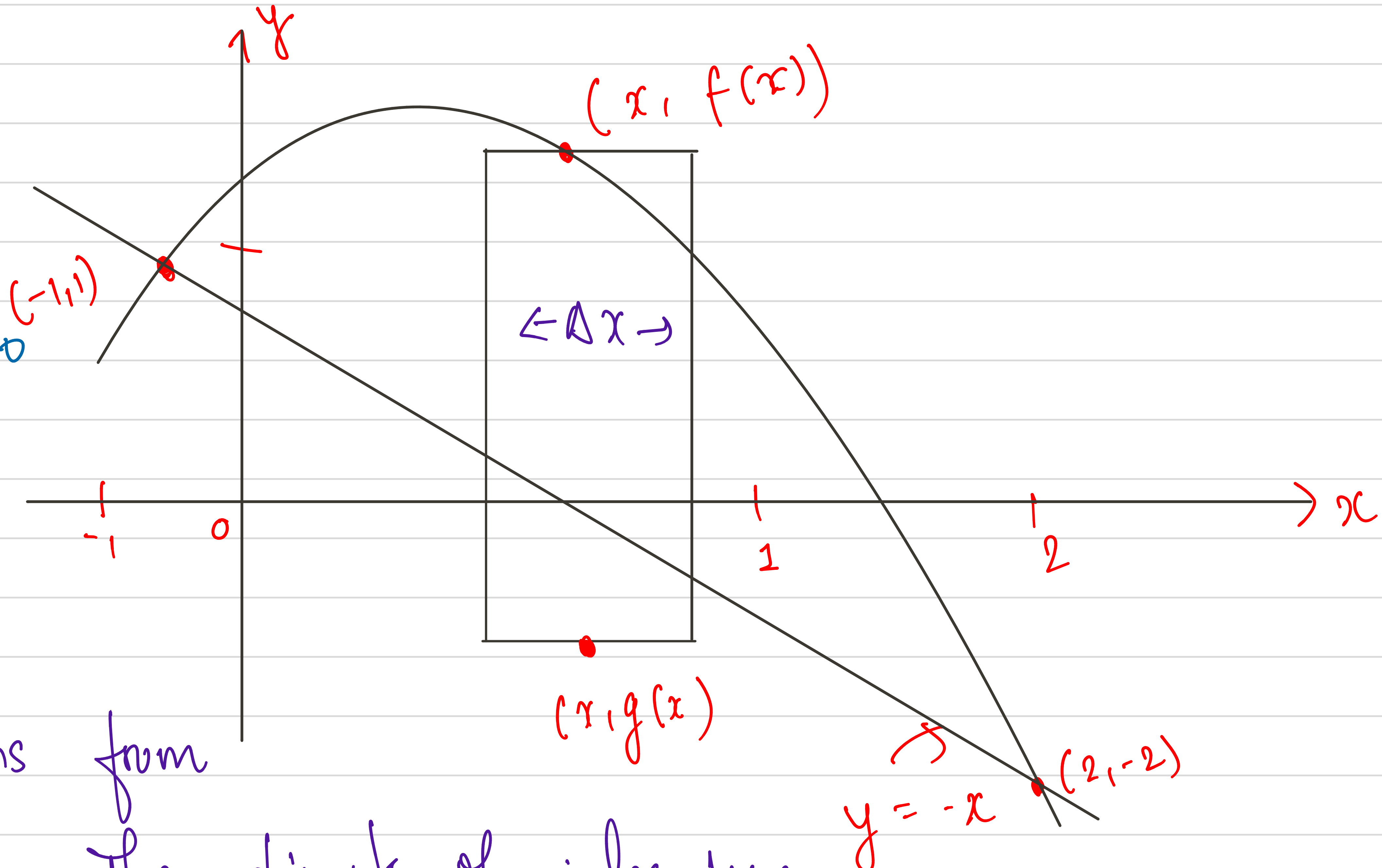
Here, first we sketch the two curves, the limits of integration are found by solving $y = 2 - x^2$ and $y = -x$ simultaneously for x.

$2 - x^2 = -x$

or, $2 - x^2 + x = 0$

or, $(x+1)(x-2) = 0$

$$\boxed{\begin{aligned} x &= -1 \\ x &= 2 \end{aligned}}$$



The region runs from $x = -1$ to $x = 2$. The limits of integration are $a = -1$, $b = +2$. The area between the curves is

$$A = \int_a^b [f(x) - g(x)] \cdot dx$$

$$\Rightarrow \int_{-1}^{2} \left[(2 - x^2) - (-x) \right] \cdot dx$$

$$\Rightarrow \int_{-1}^{2} \left[(2 + x - x^2) \right] \cdot dx$$

$$\Rightarrow \left[2x + \frac{x^2}{2} - \frac{x^3}{3} \right]_{-1}^{2}$$

$$\Rightarrow \left[2(2) + \frac{(2)^2}{2} - \frac{(2)^3}{3} \right] - \left[2 \times (-1) + \frac{(-1)^2}{2} - \frac{(-1)^3}{3} \right]$$

$$\Rightarrow \left[4 + 4/2 - 8/3 \right] - \left[-2 + 1/2 + 1/3 \right] \Rightarrow 9/2$$

Area of the Region in Polar form:

Theorem: Area of region $0 \le r_1(\theta) \le r(\theta) \le r_2(\theta)$,

$$\alpha \le \theta \le \beta,$$

$$A = \int_{\alpha}^{\beta} \frac{1}{2}\left(r_2^2 - r_1^2\right) \cdot d\theta$$

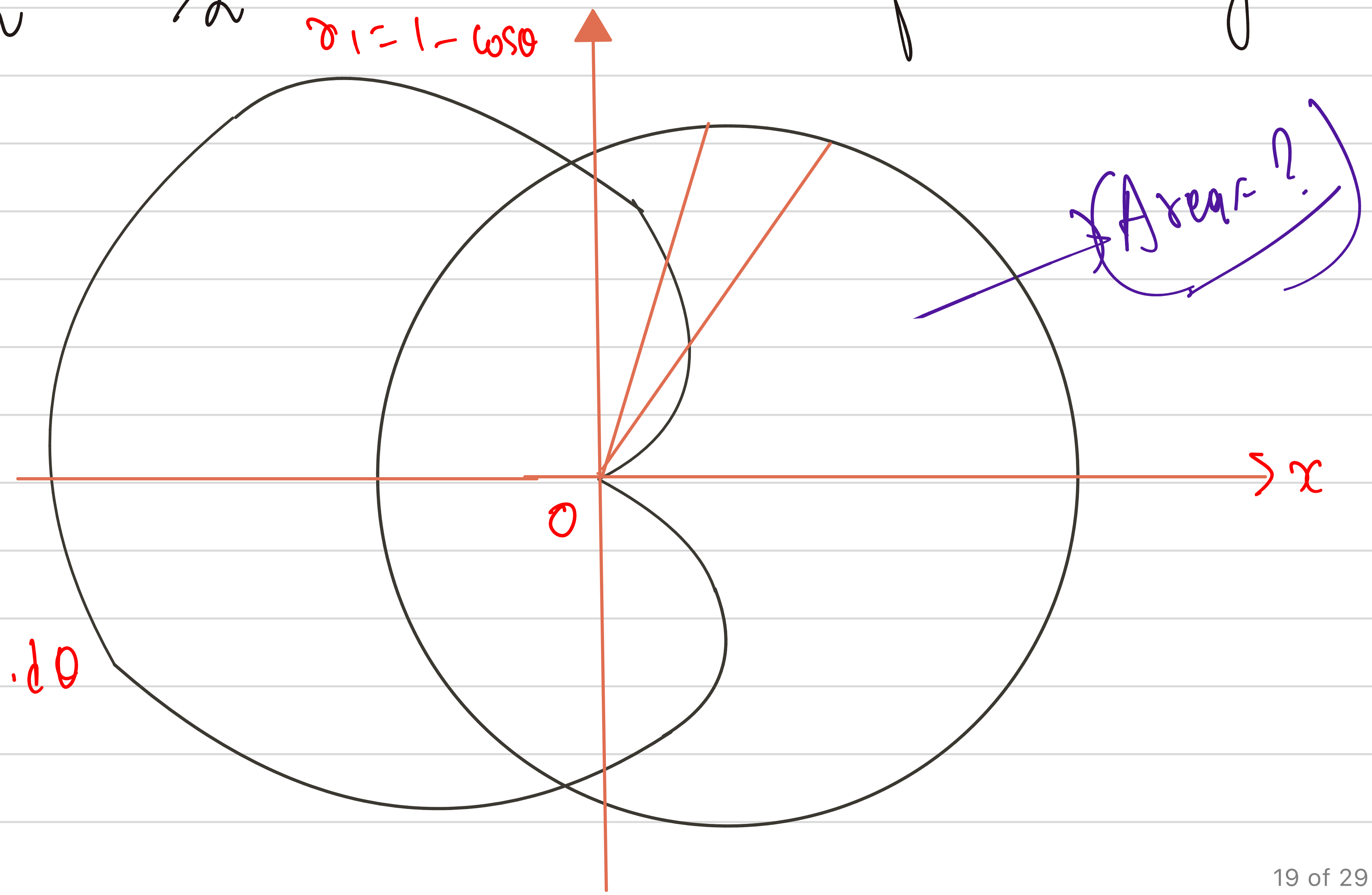
Example: Find the area of the region that lies inside the circle $r = 1$ and outside the Cardioid $r = 1 - \cos\theta$.

Soln; To determine the boundaries and find the limit of integration, we sketch the graph as shown in figure -

The outer curve is $r_2 = 1$, the inner curve is $r_1 = 1 - \cos\theta$ and θ runs from $-\pi/2$ to $\pi/2$. Thus area of the region is given by -

$$A = \int_{\alpha}^{\beta} \frac{1}{2}(r_2^2 - r_1^2) \cdot d\theta$$

$$\Rightarrow \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left[1 - (1-\cos\theta)^2\right] \cdot d\theta$$



$$\Rightarrow +\frac{1}{2}\int_{-\pi/2}^{\pi/2}\left[\left(1-1+2\cos\theta-\cos^2\theta\right)\right].d\theta$$

$$\Rightarrow \frac{1}{2}\int_{-\pi/2}^{\pi/2}\left(2\cos\theta-\frac{1+\cos 2\theta}{2}\right).d\theta$$

$$\Rightarrow \frac{1}{2}\left[2\sin\theta-\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right]_{-\pi/2}^{\pi/2}$$

$$\Rightarrow \frac{1}{2}\left[2\sin\frac{\pi}{2}-\frac{\pi}{4}-\frac{1}{4}\left(\sin 2\cdot\frac{\pi}{2}\right)-\left\{2\sin\left(-\frac{\pi}{2}\right)+\frac{\pi}{4}\right.\right.$$

$$\left.\left.-\frac{1}{4}\left(\sin\left(2\cdot-\frac{\pi}{2}\right)\right)\right\}\right]$$

$$\Rightarrow \frac{1}{2}\left[2\sin\frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{4}\left(\sin 2\cdot\frac{\pi}{2}\right) - \left\{2\sin\left(-\frac{\pi}{2}\right) + \frac{\pi}{4} - \frac{1}{4}\left(\sin 2\cdot -\frac{\pi}{2}\right)\right.\right.$$

$$\Rightarrow \frac{1}{2}\left[2 - \frac{\pi}{4} - 0 + 2 - \frac{\pi}{4} - 0\right]$$

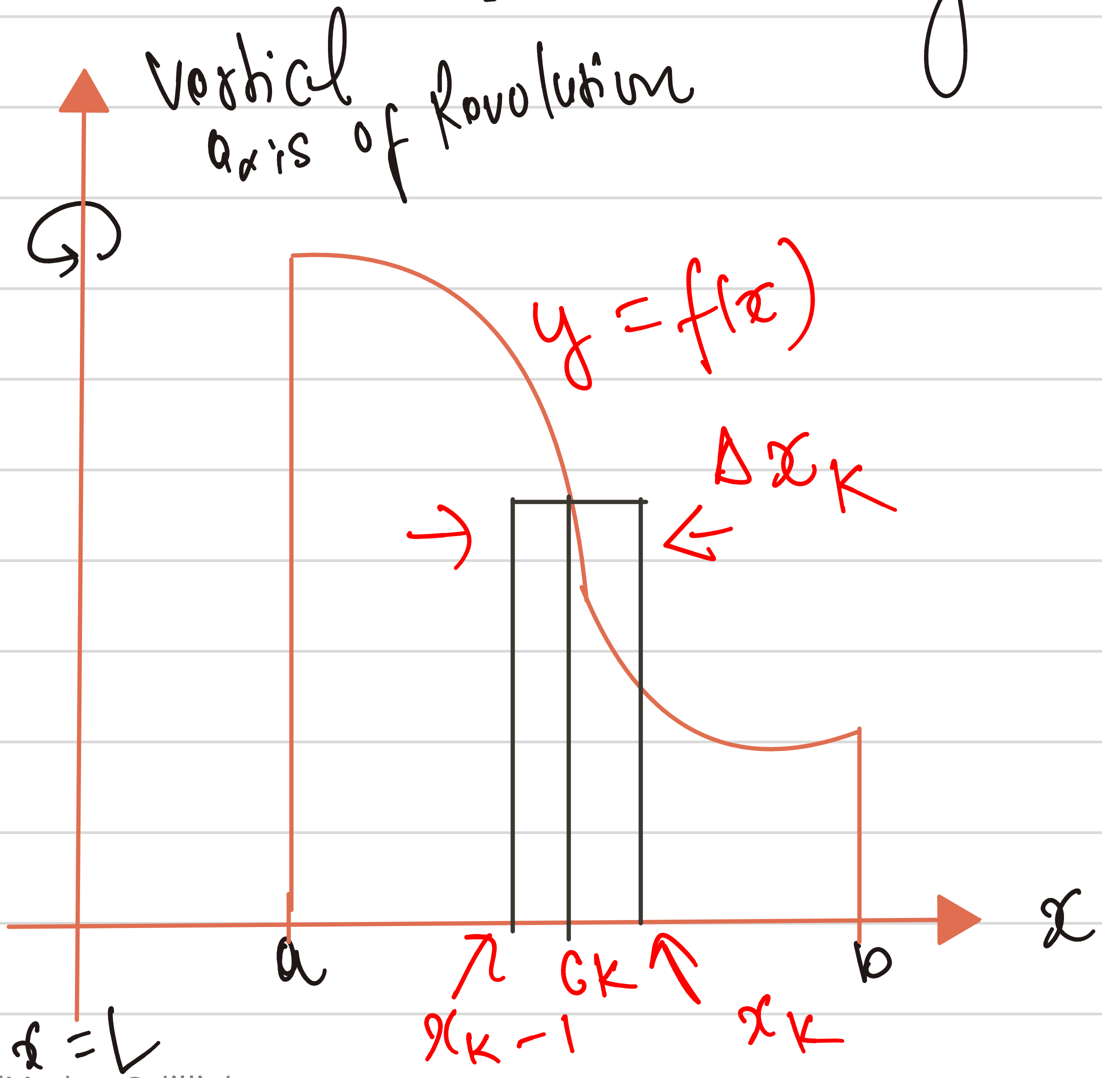
$$\Rightarrow \frac{1}{2}\left(-\frac{2\pi}{4} + 4\right)$$

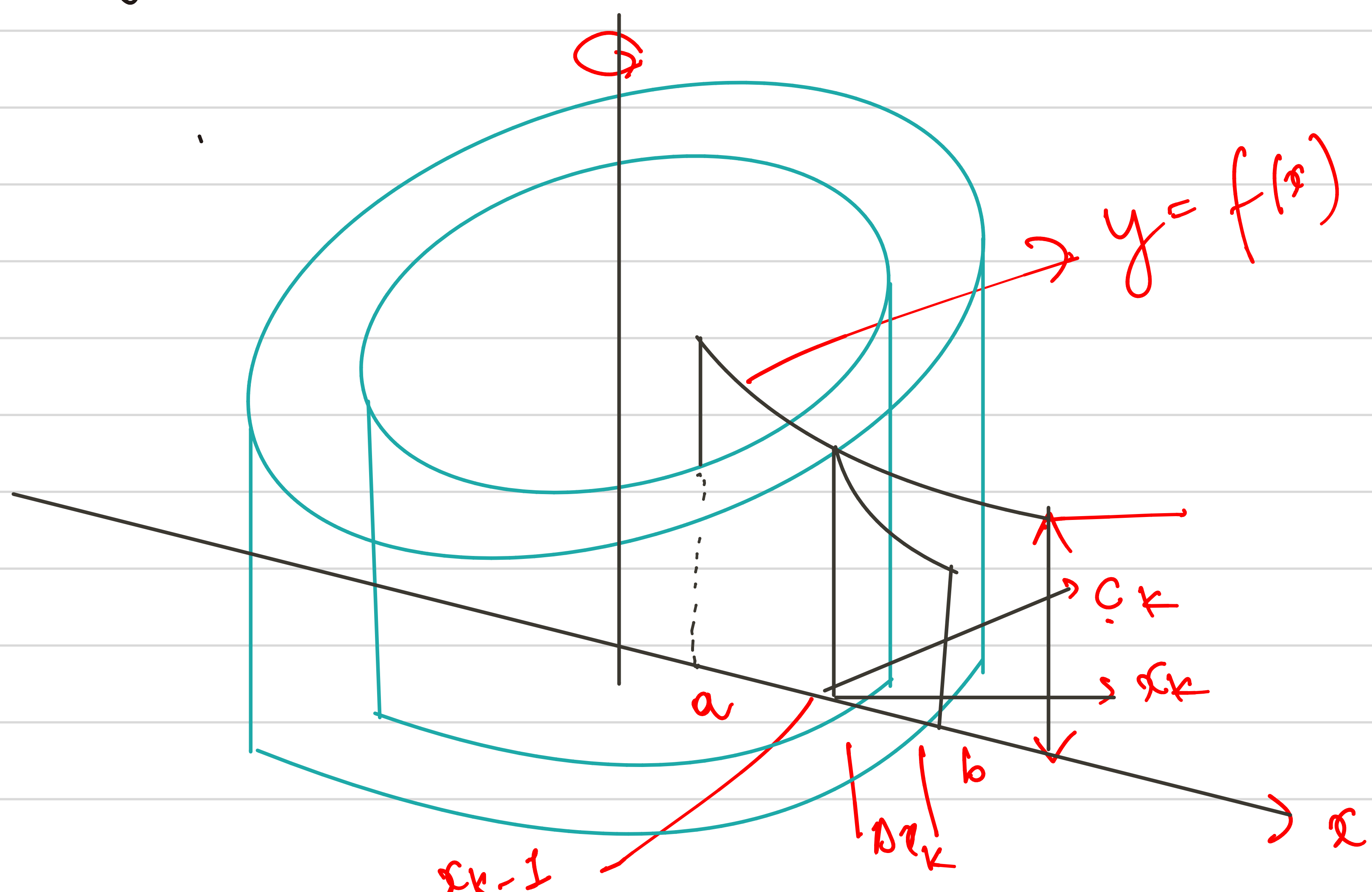
$$\Rightarrow -\frac{\pi}{4} + 2$$

$$\left[A \Rightarrow 2 - \frac{\pi}{4}\right]$$

Volume of Cylindrical Shells:

The volume of the solid generated by revolving the region between x-axis and the graph of $y = f(x) \geq 0$; $a \leq x \leq b$ about a vertical line $x = L \leq a$ is given by –





Here,

$$\Delta V_k = 2\pi\left(c_k - L\right) f(c_k) \cdot \Delta x_k .$$

Using Riemann Sum,

$$V = \int_a^b 2\pi\left(x - L\right) f(x) \cdot dx$$

about $L = 0$,

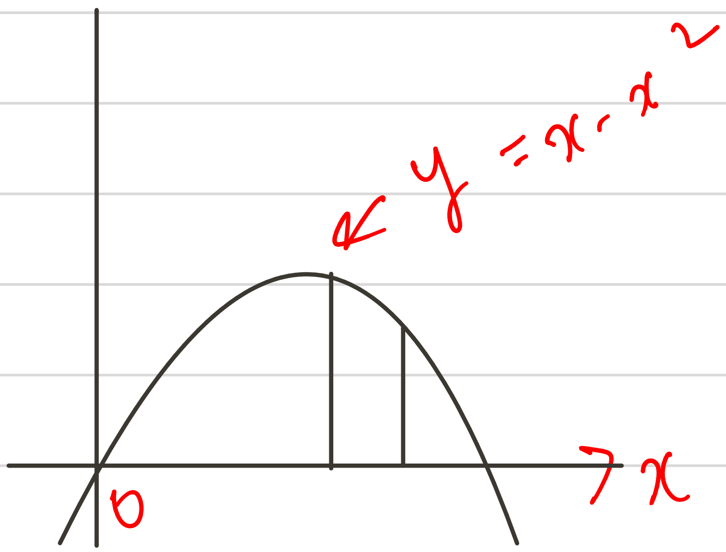
$$V = \int_a^b 2\pi x \, f(x) \cdot dx$$

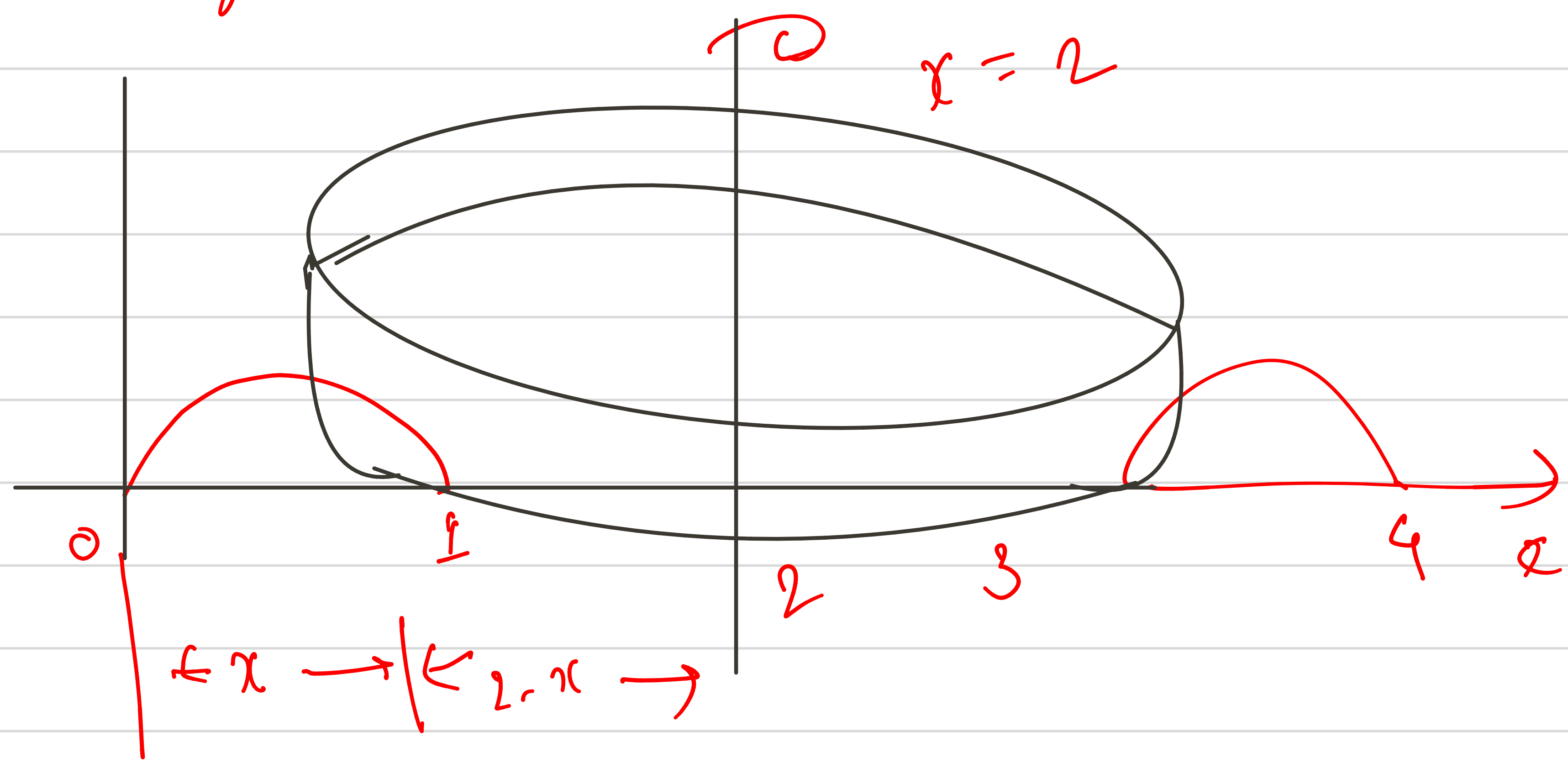
for Simplicity,

$$V = \int_a^b 2\pi \left(\text{radius of shell}\right)\left(\text{height of shell}\right) \cdot dx$$

$$V = \int_a^b \left(\text{Circumference}\right)\left(\text{height}\right)\left(\text{thickness}\right).$$

Example: Find the volume of the solid obtained by rotating about the line $x = 2$. It has radius $2 - x$, circumference $2\pi(2-x)$, and height $x - x^2$.





The volume of the given solid is

$$V = \int_0^1 2\pi(2-x)(x-x^2) \cdot dx$$

$$\Rightarrow 2\pi \int_0^1 (x^3 - 3x^2 + 2x) \cdot dx$$

$$\Rightarrow 2\pi \left[\frac{x^4}{4} - x^3 + x^2 \right]_0^1$$

$$\boxed{V = \pi/2}$$

Arch Length:

Finding the length of the curve whose graph is continuous defined over an interval I, we have the formula following the concept of Partition on I:

The Length of a Curve

1. Let $y = f(x)$ is continuous first order derivative. Then the total length of curve from $x = a$ to $x = b$ is given by

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, . \, dx$$

Note if $\frac{dy}{dx}$ at $x = a$ or $x = b$ doesn't exist then we observe $\frac{dx}{dy}$. Then the total length curve from $y = a$ to $y = b$ is given by —

$$L = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, . \, dy$$

2. If $x = f(t)$ and $y = g(t)$ be two continuous function of t, $a \le t \le b$, then

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot dt$$

3. If $r = f(\theta)$ continuous first order derivative for $a \le \theta \le b$ and if the point $P(r, \theta)$

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta$$

Example: Find the length of the curve -

$$y = \frac{4\sqrt{2}}{3} x^{3/2} - 1, \quad 0 \leq x \leq 1.$$

Sol^n: Using $a = 0$, $b = 1$ and

$$y = \frac{4\sqrt{2}}{3} x^{3/2} - 1$$

$x = 1, \; y = 0.89$

$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3} \cdot \frac{3}{2} x^{1/2}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(2\sqrt{2x}^{\,1/2}\right)^2$$

$$\Rightarrow 8x$$

The length of the curve over $x = 0$ to $x = 1$ is

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$$

$$\Rightarrow \int_0^1 \sqrt{(1 + 8x)} \cdot dx \qquad \left[\int u^{1/2} \cdot du = \frac{2}{3} u^{3/2}\right]$$

$$\Rightarrow \left[\frac{2}{3} \cdot \frac{1}{8} (1 + 8x)^{3/2}\right]_0^1$$

$$\Rightarrow \frac{13}{6}$$

$$\boxed{L \Rightarrow 2.17}$$