Day-3, Nov-18, 2024. (Mangshir 03, 2081 B.S.)

Important Theorem on Limit:

1. For all rational values of 'n'

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} \Rightarrow n a^{n-1}$$

The proof of this theorem consists of the following 3 cases:

Case I: When 'n' is positive integer

By actual division $\frac{x^n - a^n}{x - a} = x^{n-1} + x^{n-2} \cdot a + x^{n-3} \cdot a^2 + x^{n-4} \cdot a^3 + \ldots \ldots + a^{n-1}$

Now, $\lim_{x \to a} \dfrac{x^n - a^n}{x - a} = \lim_{x \to a} \left[x^{n-1} + x^{n-2} \cdot a + x^{n-3} \cdot a^2 + x^{n-4} \cdot a^3 + \cdots + a^{n-1}\right]$

$$\Rightarrow a^{n-1} + a^{n-1} + a^{n-1} + \cdots + a^{n-1}$$

$$\left[\lim_{x \to a} \frac{x^n - a^n}{x - a} \Rightarrow n a^{n-1}\right]$$

Case II: when n is -ve | negative integer.

Let $n = -m$ where m is a +ve integer. positive integer

Then, $\lim_{x \to a} \dfrac{x^n - a^n}{x - a} = \lim_{x \to a} \dfrac{x^{-m} - a^{-m}}{x - a}$

$$\Rightarrow \lim_{x \to a} \frac{\frac{1}{x^m} - \frac{1}{a^m}}{x - a}$$

$$\Rightarrow \lim_{x \to a} \frac{\dfrac{a^m - x^m}{x^m \cdot a^m}}{x-a}$$

$$\Rightarrow \lim_{x \to a} \frac{a^m - x^m}{x^m \cdot a^m (x-a)}$$

$$\Rightarrow \lim_{x \to a} \left[-\frac{x^m - a^m}{x-a} \times \frac{1}{a^m \cdot x^m} \right]$$

$$\Rightarrow -\left(\lim_{x \to a} \frac{x^m - a^m}{x-a} \right) \left(\lim_{x \to a} \frac{1}{x^m \cdot a^m} \right)$$

Using case I

$$\Rightarrow -m \cdot a^{m-1} \cdot \frac{1}{a^m \cdot a^m}$$

$$\Rightarrow \quad -m \cdot a^{m-1} \cdot \frac{1}{a^m \cdot a^m}$$

Using Exponent properties –

$$\frac{1}{a^m \cdot a^m}$$

$$\Rightarrow \quad \frac{1}{a^{2m}}$$

$$\Rightarrow \quad -m \cdot a^{m-1} \cdot \frac{1}{a^{2m}}$$

$$\Rightarrow \quad -m \cdot \frac{a^{m-1}}{a^{2m}}$$

$$\Rightarrow \quad -m \cdot a^{(m-1)-2m}$$

$$\Rightarrow -m \cdot a^{m-1-2m}$$

$$\Rightarrow -m \cdot a^{-m-1}$$

$$\Rightarrow \left[-m \cdot a^{-(m+1)}\right] - \text{for -ve.}$$

$$\Rightarrow \left[n a^{n-1}\right] - \text{for +ve}$$

But if we $n = -m$ then

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} \Rightarrow n a^{n-1}$$

Case III: When n is rational function

Let $n = p/q$ where p and q are integers and $q \neq 0$.

Then, $\lim_{x \to a} \frac{x^n - a^n}{x - a} \Rightarrow \lim_{x \to a} \frac{x^{p/q} - a^{p/q}}{x - a}$

$$\Rightarrow \lim_{x \to a} \frac{\left(x^{1/q}\right)^p - \left(a^{1/q}\right)^p}{x - a}$$

put $x^{1/q} = y$ and $a^{1/q} = b$ so that $x = y^q$ and $a = b^q$

When $x \to a$, $y \to b$

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{y \to b} \frac{y^p - b^p}{y^q - b^q}$$

$$\Rightarrow \lim_{y \to b} \frac{\frac{y^p - b^p}{y-b}}{\frac{y^q - b^q}{y-b}}$$

$$\Rightarrow \frac{\lim_{y \to b} \frac{y^p - b^p}{y-b}}{\lim_{y \to b} \frac{y^q - b^q}{y-b}}$$ $\longrightarrow$ dividing both by $y-b$

$$\Rightarrow \frac{p \cdot b^{p-1}}{q \cdot b^{q-1}}$$

Using Case I for two separate $\frac{y^p - b^p}{y-b}$ and $\frac{y^q - b^q}{y-b}$

$$\Rightarrow \frac{p}{q} \, b^{p - \cancel{1} - q + \cancel{1}}$$

$\Rightarrow \frac{p}{q} \cdot b^{p-q}$

$\Rightarrow \frac{p}{q} \cdot b^{\left(\frac{q}{q} - 1\right)}$

$\Rightarrow \frac{p}{q} \cdot (b^q)^{p/q - 1}$ Since $n = p/q$ and $a = b^q$ then

$\Rightarrow n \cdot a^{n-1}$

So, for all rational values of n,

$$\lim_{x \to a} \frac{x^n - a^n}{x - a}$$

$\Rightarrow n a^{n-1}$

Limits of Algebraic Functions:

Example 1:

$f(x) = 3x - 2$ where $x \to 3$

$$\lim_{x \to 3} f(x) \Rightarrow 3 \times 3 - 2$$

$$\Rightarrow 7$$

Ex 2.

$f(x) = 3x^2 - 5x + 6$ when $x \to 2$

$$f(x) = 3(2)^2 - 5(2) + 6$$

$$\Rightarrow 8$$

Ex 3.

$$\lim_{x \to 3} \frac{4x - 5}{2x + 3}$$

$$\Rightarrow 7/9$$

Ex 4.

$$\lim_{x \to 0} \frac{4x - 5}{2x + 3}$$

$$\Rightarrow -5/3$$

$$\lim_{x \to 0} \frac{5x^2 + 3x}{x}$$

$\Rightarrow \frac{0}{0}$ takes indeterminate form so.

$$\frac{5x^2 + 3x}{x}$$

$$\Rightarrow \lim_{x \to 0} \frac{\cancel{x}(5x+3)}{\cancel{x}}$$

$$\Rightarrow \lim_{x \to 0} 5x + 3$$

$$\Rightarrow 0 + 3$$

$$\boxed{\Rightarrow 3}$$

So, the aim is to remove the indeterminate form.

Extra Bonus.

Linear Regression Model

$$f_{w,b}(x) = w.x + b$$

Cost function. $J(w,b) = \frac{1}{2m}\sum_{i=1}^{m}\left(f_{w,b}\left(x^{(i)}\right) - y^{(i)}\right)^2$

Gradient Descent Algorithm

Repeat until convergence [

$$w = w - \alpha . \frac{\partial}{\partial w} J(w,b) \rightarrow \frac{1}{m}\sum_{i=1}^{m}\left(f_{w,b}\left(x^{(i)}\right) - y^{(i)}\right)x^{(i)}$$

$$b = b - \alpha . \frac{\partial}{\partial b} J(w,b) \longrightarrow \frac{1}{m}\sum_{i=1}^{m}\left(f_{w,b}\left(x^{(i)}\right) - y^{(i)}\right)$$

]

$$\frac{\partial}{\partial w} J(w,b) = \frac{\partial}{\partial w} \frac{1}{2m} \sum_{i=1}^{m} \left(f_{w,b}(x^{(i)}) - y^{(i)} \right)^2$$

$$\Rightarrow \frac{\partial}{\partial w} \cdot \frac{1}{2m} \sum_{i=1}^{m} \left(f\,w x^{(i)} + b - y^{(i)} \right)^2$$

$$\Rightarrow \frac{1}{\cancel{2}m} \sum_{i=1}^{m} \left(w x^{(i)} + b - y^{(i)} \right) \cancel{2} x^{(i)}$$

$$\boxed{\frac{\partial}{\partial w} J(w,b) \Rightarrow \frac{1}{m} \sum_{i=1}^{m} \left(f_{(w,b)} x^{(i)} - y^{(i)} \right) x^{(i)}} \rightarrow \text{Cost function.}$$

↳ for linear Regression

$$\frac{\partial}{\partial b}(w,b) = \frac{\partial}{\partial b} \cdot \frac{1}{2m} \sum_{i=1}^{m} \left(f_{w,b}(x^{(i)}) - y^{(i)} \right)^2$$

$$\Rightarrow \frac{\partial}{\partial b} \cdot \frac{1}{2m} \sum_{i=1}^{m} \left(w x^{(i)} + b - y^{(i)} \right)^2$$

$$\Rightarrow \frac{1}{\cancel{2}m} \sum_{i=1}^{m} \left(w x^{(i)} + b - y^{(i)} \right) \cancel{2}$$

$$\boxed{\frac{\partial}{\partial b}(w,b) \Rightarrow \frac{1}{m} \sum_{i=1}^{m} \left(f_{w,b}(x^{(i)}) - y^{(i)} \right)}$$

$\rightarrow$ bias. for Linear Regression

Gradient Descent Algorithm.

repeat until convergence {

$$w = w - \alpha \frac{1}{m} \sum_{i=1}^{m} \left(f_{w,b}(x^{(i)}) - y^{(i)} \right) x^{(i)}$$

$$b = b - \alpha \frac{1}{m} \sum_{i=1}^{m} \left(f_{w,b}(x^{(i)}) - y^{(i)} \right)$$

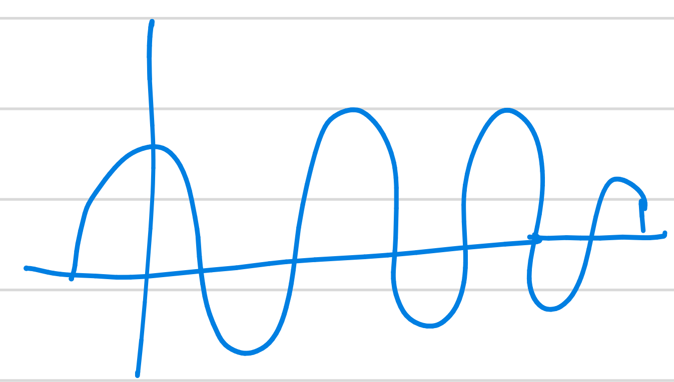
}

Update 'w' and 'b' simultaneously.

Cost Error function → Convex function

only have Single Global Minima.

Non-Convex function have multiple Minima.



Q. $\lim_{x \to a} \dfrac{x^5 - a^5}{x^4 - a^4}$

$\Rightarrow \lim_{x \to a} \dfrac{(x-a)(x^4 + x^3a + x^2a^2 + xa^3 + a^4)}{(x-a)(x^3 + x^2a + xa^2 + a^3)}$

$\Rightarrow \lim_{x \to a} \dfrac{x^5 + \cancel{x^4a} + \cancel{x^3a^2} + \cancel{x^2a^3} + \cancel{xa^4} - \cancel{ax^4} - \cancel{x^3a^2} - \cancel{x^2a^3} - a^4x - a^5}{x^4 + \cancel{x^3a} + \cancel{x^2a^2} + \cancel{xa^3} - \cancel{ax^3} - \cancel{x^2a^2} - \cancel{xa^3} - a^4}$

$\Rightarrow \lim_{x \to a} \dfrac{\cancel{(x-a)}(x^4 + x^3a + x^2a^2 + xa^3 + a^4)}{\cancel{(x-a)}(x^3 + x^2a + xa^2 + a^3)}$

$$\Rightarrow \lim_{x \to a} \frac{(x^4 + x^3 a + x^2 a^2 + x a^3 + a^4)}{(x^3 + x^2 a + x a^2 + a^3)}$$

$$\Rightarrow \frac{a^4 + a^3 a + a^2 a^2 + a a^3 + a^4}{a^3 + a^2 a + a a^2 + a^3}$$

$$\Rightarrow \frac{5a^4}{4a^3} \Rightarrow \frac{5a}{4}$$

<u>Ex.</u> $$\lim_{x \to a} \frac{x^{1/3} - a^{1/3}}{x^{1/2} - a^{1/2}}$$

It also takes $\frac{0}{0}$ form when $x = a$ But,

$$\Rightarrow \lim_{x\to a} \frac{(x^{1/6})^2 - (a^{1/6})^2}{(x^{1/6})^3 - (a^{1/6})^3}$$

$$\Rightarrow \lim_{x\to a} \frac{(x^{1/6} - a^{1/6})(x^{1/6} + a^{1/6})}{(x^{1/6} - a^{1/6})(x^{2/6} + x^{1/6}a^{1/6} + a^{2/6})}$$

$$\Rightarrow \lim_{x\to a} \frac{x^{1/6} + a^{1/6}}{(x^{2/6} + x^{1/6}a^{1/6} + a^{2/6})}$$

$$\Rightarrow \frac{a^{1/6} + a^{1/6}}{(a^{2/6} + a^{1/6} \cdot a^{1/6} + a^{2/6})}$$

$$\Rightarrow \frac{2a^{1/6}}{3a^{1/6}}$$

Q. 3. $$\lim_{x \to a} \frac{\sqrt{x+a} - \sqrt{3x-a}}{x-a}$$

$$\Rightarrow \lim_{x \to a} \frac{(\sqrt{x+a} - \sqrt{3x-a})(\sqrt{x+a} + \sqrt{3x-a})}{(x-a)(\sqrt{x+a} + \sqrt{3x-a})}$$

$$\Rightarrow \lim_{x \to a} \frac{x+a-3x+a}{(x-a)(\sqrt{x+a} + \sqrt{3x-a})}$$

$$\Rightarrow \lim_{x \to a} \frac{-2(x-a)}{(x-a)\left(\sqrt{x+a}+\sqrt{3x-a}\right)}$$

$$\Rightarrow \frac{-2\cancel{(x-a)}}{\cancel{(x-a)}\left(\sqrt{a+a}+\sqrt{3a-a}\right)}$$

$$\Rightarrow \frac{-2}{\left(\sqrt{2a}+\sqrt{2a}\right)}$$

$$\Rightarrow \frac{-2}{2\sqrt{2a}} \qquad \Rightarrow \frac{-1}{\sqrt{2a}}$$