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Matrix Factorizations

→ A factorization of a matrix A is an equation that expresses A as a product of two or more matrices.

$$A = [a_{ij}] \times [a_{kl}]$$

$$\text{or,} \quad A = a_{ij} \times a_{kl} \times a_{mn}$$

where a_{ij} are the matrices.

→ It is an analysis of data.

→ Matrix Multiplication involves a synthesis of data combining the effects of two or more linear transformations into a single matrix).

The LU factorization:

A sequence of equations with all same coefficient matrix:

$$Ax = b_1, \quad Ax = b_2 \quad \dots \quad Ax = b_p$$

where

$A^{-1}b_1 = x$, $A^{-1}b_2 = x$ when A is invertible

one would compute A^{-1} and then compute $A^{-1}b_1$, $A^{-1}b_2$ and so on

(A matrix that has inverse $\Rightarrow$ invertible).

→ Square Matrix is decomposed into two Matrices.

Decomposes $\boxed{\text{Square Matrix}}$ Decomposes

L

A triangular Matrix with ones / 1's in diagonal.

U

An upper triangular Matrix

→ This decomposition helps in Solving the Linear Equations, Compute Determinant and find Matrix Inverse Effectively.

→ Before studying how to construct L and U, we should look at why they are so useful. When $A = LU$ the equation $Ax = b$ can be written as $L(Ux) = b$. Writing y for Ux, we can find x by solving the pair of equations,

$$Ly = b$$
$$Ux = y$$

first solve $Ly = b$ for y, then solve $Ux = y$ for x. Each equation is easy to solve because L and U are triangular.

At first, assume that A is $m \times n$ matrix that can be row reduced to echelon form, without row interchanges. Then

A can be written in the form $A = LU$, where L is $m \times n$ lower triangular matrix with 1's on the diagonal and U is $m \times n$ echelon form of A. Such a factorization is called an LU factorization of A. The matrix L is invertible and is called a unit lower triangular matrix

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{U}$$

fig. An LU factorization

Example:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 3 \\ 8 & 7 & 9 \end{bmatrix}$$

$\rightarrow$ $A = LU$

Here,

$\rightarrow$ $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

$\rightarrow$ Decompose into $A = LU$

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 3 \\ 8 & 7 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \times \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Step: finding the elements L and U we get -

$u_{11} = 2$, $u_{12} = 1$, $u_{13} = 1$

Second Row

$l_{21} u_{11} = a_{21}$ $\qquad$ $a_{21} = l_{21} \cdot u_{11}$

From formula -

$$A_{ij} = \sum_{k=1}^{j} l_{ik} U_{kj}$$

So,

$l_{21} \cdot 2 = 4$ $\qquad$ $\therefore l_{21} = \frac{4}{2} \Rightarrow 2.$

For $j=2$ and $i=2$

$$A_{\underset{i\ j}{22}} = \alpha_{\underset{i}{21}}\, u_{\underset{j}{12}} + \alpha_{\underset{i\ k}{22}} \cdot u_{\underset{k\ j}{22}}$$

$$A_{22} \Rightarrow \alpha_{21} u_{12} + 1 \cdot u_{22}$$

$\therefore A_{22} \Rightarrow 3$ $\qquad l_{21} \Rightarrow 2$

$\therefore A_{23} \Rightarrow 3$ $\qquad u_{12} \Rightarrow 1$ and

Similarly solving all we get $\qquad u_{13} \Rightarrow 1$

So, the final Result is

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Verification,

$$L \times U = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$\Rightarrow \; 1 \times 2 + 0 \times 0 + 0 \times 0 \;,\; 1 \times 1 + 0 \times 1 + 0 \times 0 \;,\; 1 \times 1 + 0 \times 1 + 0 \times 2$

$\Rightarrow \left[2+0+0, \quad 1+0+0, \quad 1+0+0 \right]$

$\Rightarrow \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} \rightarrow$ first Row

Again,

Row 2 : $\begin{bmatrix} 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} \underline{2} & \underline{1} & \underline{1} \\ \underline{0} & \underline{1} & \underline{1} \\ \underline{0} & \underline{0} & \underline{2} \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 2\times 2 + 2\times 0 + 2\times 0, & 2\times 1 + 1\times 1 + 1\times 0, & 2\times 1 + 1\times 1 + 0\times 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & 3 & 3 \end{bmatrix}$$

Again

Row 3: $\begin{bmatrix} 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 8, & 7, & 9 \end{bmatrix}$$

So, find verification

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\left[\text{So. } LU \Rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 3 \\ 8 & 7 & 9 \end{bmatrix} = A \right]$$

Subspace of R^n

→ Sets of vectors in R^n called Subspaces. Subspace gives smaller, manageable "chunks" of a larger space.

Imagine R^n the entire room, you're standing in. And the Subspace is like corner, or wall, or floor within that room.

→ Works with simpler systems while understanding properties of the full space.

A Subspace of R^n is any set of H in R^n that has 3 properties:

1) The zero vector in H.

2) for each u and v in H, the sum $u+v$ is H.

3) for each u in H and each scalar c, the vector cu is H.

So, A subspace is closed under addition and scalar multiplication.

Dimension and Rank:

Dimension of a matrix refers to its size give as $m \times n$

Where m: Number of rows

n: Number of Columns

for example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

- The matrix A has 2 rows and 3 columns, So its dimension is 2×3.

2. Rank of a Matrix:

Rank of a matrix is the number of linearly Independent rows or linearly Independent Columns. It measures the non-redundant or useful information contained in the matrix.

Key Properties of Rank:

1) Rank $\leq \min(m, n)$: The rank cannot exceed the smaller of the number of rows or columns.

2) Row Rank = Column Rank

3) Full Rank $\rightarrow$ A $m \times n$ matrix has full rank if the rank is $\min(m, n)$.

How to Compute the Rank:

$\rightarrow$ Perform Gaussian Elimination to reduce the matrix to row echelon form (REF) or reduced row echelon form (RREF)

$\rightarrow$ For a square matrix $(m = n)$, the rank is at most n.

Examples

Compute Rank of a Matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Perform Gaussian Elimination:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ REF has 2 non-zero rows so the Rank is 2.

Dimension	Shape	Example	Use cases
1D	n or $1 \times n$	$[1, 2, 3]$ or $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$	Coordinates, features, vectors
2D	$m \times n$	$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$	Grids, tables, images
3D	$n \times m \times p$	3D grid or stack	RGB, time-series, volumetric data

→ Determinants:

→ Determinant is scalar component of matrix.

→ plays pivotal role in the study of eigen value and eigen vector.

→ useful to measure the amount of linear transformation changes the area of a figure.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Then its determinant is defined as

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \Rightarrow a_{11}a_{12} - a_{22}a_{21}$$

Likewise, the 3x3 Matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

and its determinant is

$$\det(A) = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Determinant: For $n \geq 2$ the determinant of an $n \times n$ matrix $A = [a_{ij}]$ of n terms $\rightarrow$

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j})$$

Theorem 1: The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across the ith row as-

$$\det(A) = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}$$

Where

$$C_{ij} = (-1)^{i+j} \det(A_{ij})$$

The cofactor expansion across the jth column is

$$\det(A) = a_{1j} C_{1j} + a_{2j} C_{2j} + \cdots + a_{nj} C_{nj}$$

Where C_{ij} is defined above.

Theorem: If A is triangular matrix, then $\det(A)$ is the product of the entries on the main diagonal to A. That is, if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ 0 & a_{22} & a_{2n} \\ \vdots & \vdots & \vdots \\ 0 & 0 & a_{nn} \end{bmatrix}$$

then $\det(A) = (a_{11})(a_{22})(a_{33}) \cdots (a_{nn})$

Properties of Determinants:

Theorem: Let A be a square matrix

a) If a multiple of one row of A is added to another row to produce a matrix B then $\det(A) = \det(B)$.

b) If two rows of A are interchanged to produce B then $\det(A) = -\det(B)$.

c) If one row of A is multiplied by K (scalar) to produce B then $K\det(A) = \det(B)$.

Theorem: A square matrix A is invertible if and only if $\det(A) \neq 0$.

Theorem: If A is $n \times n$ matrix then $\det(A^T) = \det(A)$.

Theorem: If A and B are $n \times n$ matrices then $\det(AB) = \det(A) \cdot \det(B)$.

Cramer's Rule and Linear Transformations

Cramer's Rule is one powerful tool for solving process of a system of linear equations that has wide applications.

Theorem: (Cramer's Rule)

Let 'A' be an invertible n×n matrix. For any 'b' in R^n, the unique solution 'x' of $Ax = b$ has entries given by

$$x_i = \frac{\det(A_i(b))}{\det(A)} \quad \text{for } i = 1, 2, 3 \ldots n.$$

—

A formula for A^{-1}

Theorem: An Inverse formula,

Let 'A' be an invertible $n \times n$ matrix then,

$$A^{-1} = \frac{1}{\det(A)} \cdot adj(A)$$

Proof: A is invertible then $|A| \neq 0$. So, A^{-1} exists,
Since for any square matrix A, we have -

$$A \cdot adj(A) = Adj(A) \cdot A$$

$$\left[A \cdot adj(A) = |A| \cdot I \right]$$

Therefore,

$$A \cdot \frac{\text{Adj } A}{|A|} = \frac{\text{Adj } A}{|A|} \cdot A$$

$$= I$$

$$\Rightarrow A^{-1} = \frac{\text{Adj} \cdot A}{|A|}$$

Vector Spaces

Since Linear Algebra is the study of vector spaces and the functions of vector spaces (Linear transformations).

We'll start with Definitions

Definition (Field):

Let F be a non-empty set of objects such that the addition and multiplication is defined on it and satisfy the following conditions:

1) F is an abelian group under addition and multiplication
2) F satisfied the distributive property from the left and right

i.e. $a(b+c) = ab + bc$

$(a+c)b = ab + cb$ for all $a, b, c \in F$.

Then F is called field.

Examples:

→ The set of real numbers is a field.

→ The set of rational numbers is a field.

→ The set of complex numbers is a field.

→ The set of integers is not a field. Since there is no multiplicative inverse of every non-zero integers.

In particular $3 \in \mathbb{Z}$ but $1/3 \notin \mathbb{Z}$.

[Abelian Group → fundamental algebraic structures]

also called commutative group whose order of operation doesn't matter

Vector Space:

Let V be a non-empty set of vectors and 'K' be the field. Then 'V' is said to be a vector space over the field K. If a sum of any two elements of V is again in V and multiplication of any elements of V by an element of K is again in V and satisfy the conditions.

Intuitive: A vector space is like a "playground" for vectors where we can add vectors or stretch / shrink them using numbers (scalars) and the results will be still within the same playground.

So, a vector space is a set of vectors where you can add vectors together and multiply them by scalars, following certain rules.

The conditions that need to be satisfied are:

a) <u>Commutative law:</u> $v_1 + v_2 = v_2 + v_1$ for all $v_1, v_2 \in V$

b) <u>Associative law:</u> $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$ for all $v_1, v_2, v_3 \in V$

c) <u>Existence of Additive Identity:</u> for all $v \in V$, there exists an element $0 \in V$ (called additive identity) such that

$$0 + v = v + 0$$
$$\Rightarrow v$$

d) Existence of additive Inverse:

for all $v \in V$, there exists $-v \in V$ (called additive inverse of v) such that,

$$v + (-v) = (-v) + v$$
$$\Rightarrow 0 \quad \text{for all } v \in V.$$

e) Associative law of vector Multiplication by Scalar:

$$(ab)v = a(bv)$$
$$\Rightarrow b(av) \quad \text{for all } a, b \in K, \; v \in V$$

f) Distributive Law

i) $(a+b)v = av + bv$

ii) $a(v_1 + v_2) = av_1 + av_2$ for all $v_1, v_2 \in V$, $a, b \in K$.

Existence of Multiplicative Identity: $1 \in K$ such that

$1 \cdot v = v \cdot 1 \ \ \forall v$ for all $v \in V$.

A non-empty set V is called a vector space over field K. If V forms an additive abelian group and scalar multiplication of any element of V by any element of K, and sum of two elements of V is again in V satisfies above conditions.

REFERENCE: Binod Prasad Dhakal, ph.D, Ramesh Gautam et. al, 2076 B.S, Mathematics - II B.Sc.CSIT Second Semester, New edition, KEC publication and Distribution Pvt.Ltd, Kathmandu, Nepal.