Day - 54, Jan - 23, 2025 (Magh 10, 2081 BS.)

LagRange Multipliers :

In 1755 , Lagrange developed a method to solve max-min problems in geometry which is known as Lagrange multipliers.

Constrained Maxima and Minima !

Example 1: find the point closet to the origin on the hyperbolic cylinder $x^2 - z^2 - 1 = 0$.

Sol^n: Given the hyperbolic cylinder

$$f(x,y,z) = x^2 - z^2 - 1 = 0$$

Imagine a small sphere centered at origin that just touches the cylinder. At each point of contact, the sphere and the cylinder have the same tangent plane and normal lines.

Suppose the sphere is,

$$g(x, y, z) = x^2 + y^2 + z^2 - a^2 = 0$$

Then the gradient of f and g will be parallel at where the surface touch. At the point (any point) of contact we may get,

$$\nabla f = \lambda \nabla g$$

Where λ is some scalar value.

i.e $2x\vec{i} - 2z\vec{k} = \lambda(2x\vec{i} + 2y\vec{j} + 2z\vec{k})$

This Implies,

$$2x = 2\lambda x, \quad 0 = 2\lambda y, \quad -2z = 2\lambda z.$$

Clearly, no point is origin of the surface, so let $x \neq 0$ then the first equality gives $\lambda = 1$. By second equality we have $y = 0$.

And the third equality gives,

$$-2z = 2z \qquad (\lambda = 1)$$

$$\Rightarrow z = 0$$

$\therefore$ We conclude the points all have the coordinates of the form $(x, 0, 0)$.

On the surface $x^2 - z^2 = 1$, the points $(x, 0, 0)$ gives the value of x is,

$$x^2 - 0 = 1$$

$\Rightarrow x = \pm 1$

$\therefore$ The points $(\pm 1, 0, 0)$ are on the cylinder closet to the origin.

The method of solving problem in above example, is the method of Lagrange Multipliers in which the scalar value 'λ' is called a Lagrange's Multiplier.

Method of Lagrange's Multiplier:

Suppose that $f(x,y,z)$ and $g(x,y,z)$ are differentiable. To find the local maximum and minimum values of f subject to the constraint $g(x,y,z) = 0$. We find scalar λ such that

$$\nabla f = \lambda \nabla g \quad \text{and} \quad g(x,y,z) = 0.$$

And in the case of two variables, we find the appropriate equations,

$$\nabla f = \lambda \nabla g \quad \text{and} \quad g(x,y) = 0.$$

Q. Find the greatest and smallest value that the function $f(x,y) = xy$ takes on the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$.

Here,

$$f(x,y) = xy$$

and let,

$$g(x,y) = x^2 + 4y^2 - 8 = 0$$

$$\left[\frac{x^2}{8} + \frac{y^2}{2} = 1 \Rightarrow x^2 + 4y^2 - 8 = 0 \right]$$

$$\nabla f = y\vec{i} + x\vec{j} \quad \text{and} \quad \nabla g = 2x\vec{i} + 8y\vec{j}$$

This gives,

$$y = 2\lambda x \quad \text{and} \quad x = 8\lambda y$$

from this we have,

$$y = 2\lambda(8\lambda y)$$

from this we have,

$$y(1 - 16\lambda^2) = 0.$$

this gives $y = 0$ or $\lambda = \pm \frac{1}{4}$.

<u>Case 1:</u> If $y = 0$ then $x = 0$. But $(0,0)$ does not lie on the surface of given ellipse. So, $y \neq 0$

<u>Case 2:</u> If $y = 0$ then we should have $\lambda = \pm \frac{1}{4}$ and therefore $x = \pm 2y$.

Then the equation $g(x,y) = 0$ gives,

$$(\pm 2y)^2 + 4y^2 - 8 = 0$$

$$\Rightarrow 8y^2 - 8 = 0$$

$$\Rightarrow y = \pm 1.$$

Thus, the function $f(x,y)$ takes extreme values at the points $(\pm 2, \pm 1)$

And the extreme value is,

i) at $(2,1)$ $\quad f(2,1) = (2)(1) \Rightarrow 2$

ii) at $(-2,1)$ $\quad f(-2,1) \Rightarrow (-2)(1) \Rightarrow -2$

iii) at $(2,-1)$ $\quad f(2,-1) \Rightarrow (2)(-1) \Rightarrow -2$

iv) at $(-2,-1)$ $\quad f(-2,-1) \Rightarrow (-2)(-1) \Rightarrow +2$

Hence, extreme values are $f = 2$ and $f = -2$

Lagrange's Multipliers with Two Constraints:

Method of Process:

Let $f(x,y,z)$ be a function with two constraints $g_1(x,y,z)=0$ and $g_2(x,y,z)=0$ in which all the functions are differentiable and ∇g_1 is not parallel to ∇g_2.

To find the extrema of f subject to $g_1=0$, $g_2=0$ we introduce two Lagrange multipliers λ and μ such that -

$$\nabla f = \lambda \nabla g_1 + \mu g_2,$$

$$g_1 = 0 \quad \text{and} \quad g_2 = 0$$

Note that for the method function should be of three variables.

Multiple Integrals:

Multiple integrals are used to compute areas, masses, volumes, centroids, etc. In this unit, we extend the idea of definite integrals to double integrals and triple integrals.

We can solve problems related to functions of two or three variables just similar to the problems related to function of single variable.

Double Integrals in Rectangular Coordinates.

Let $f(x,y)$ be continuous defined in a rectangular region R given by-

$$R = \{(x,y) \in R^2 ; a \le x \le b, c \le y \le d\}.$$

Let Region R be covered by lines parallel to x-axis and y-axis.

Those lines divide R into small rectangles of area $\Delta A = \Delta x . \Delta y$ and ordered as $\Delta A_1, \Delta A_2, ----, \Delta A_n$.

Let (x_k, y_k) be randomly chosen point in each rectangle having area ΔA_k and form the sum

$$S_n = \sum_{k=1}^{n} f(x_k, y_k) . \Delta A_k \quad \text{—— eqn (i)}$$

If f is continuous ~~throughout~~ throughout R, then limit of sum (i) when $\Delta A \to 0$ exists and is called the double integral of f over R and we write

$$\iint_R f(x,y) . dA \quad \text{or} \quad \iint_R f(x,y) \, dx . dy$$

Thus,

$$\iint_R f(x,y)\cdot dA = \lim_{\Delta A \to 0} \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k \quad \text{—— eqn (ii)}$$

Note that limit in eqn (ii) is not depend on the order in which ΔA_k are numbered and independent of choice of (x_k, y_k) within each ΔA_k.

Properties of Double Integrals:

Double integrals of a continuous function satisfy algebraic properties similar to single integrals which are:

1. for any number K,

$$\iint_R K \cdot f(x,y)\, dA = K \iint_R f(x,y) \cdot dA.$$

2 $$\iint_R [f(x,y) \pm g(x,y)]\, dA = \iint_R f(x,y) \cdot dA \pm \iint_R g(x,y) \cdot dA$$

3. i) If $f(x,y) \geq 0$ on R,

$$\iint_R f(x,y)\, dA \geq 0.$$

ii) If $f(x,y) \geq g(x,y)$ on R,

$$\iint_R f(x,y) \cdot dA \geq \iint_R g(x,y) \cdot dA.$$

4) If R is the union of two non overlapping rectangles R_1 and R_2.

$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\cdot dA + \iint_{R_2} f(x,y)\cdot dA$$

Fubini's Theorem

If $f(x,y)$ is continuous on the rectangular region R: $a \le x \le b$, $c \le y \le d$, then

$$\iint_R f(x,y)\cdot dA = \int_c^d \int_a^b f(x,y)\cdot dx\cdot dy = \int_a^b \int_c^d f(x,y)\cdot dy\cdot dx$$

Fubini's Theorem (Stronger form):

Let $f(x,y)$ be continuous on a region R,

1. If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$ with g_1 and g_2 continuous on $[a,b]$, then

$$\iint_R f(x,y) \cdot dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y) \cdot dy \cdot dx$$

2. If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$ with h_1 and h_2 continuous on $[c,d]$, then

$$\iint_R f(x,y) \cdot dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y) \cdot dx \cdot dy$$

Q. Verify Fubini's theorem for $\iint_R f(x,y)\cdot dA$, where $f(x,y) = 1-6x^2y$

and

$R: 0 \le x \le 2, \quad -1 \le y \le 1.$

Here,

$$\iint_R f(x,y)\cdot dA = \int_{-1}^{1}\int_0^2 (1-6x^2y)\cdot dx\cdot dy$$

$$\Rightarrow \int_{-1}^{1}\left[x - 2x^3y\right]_0^2 \cdot dy$$

$$\Rightarrow \int_{-1}^{1}\left[\underline{(2)-(0)} - 2(2)^3y - 2(0)^3\cdot y\right]\cdot dy$$

$$\Rightarrow \int_{-1}^{1} (2 - 16y) \cdot dy$$

$$\Rightarrow \left[2y - 16y^2 \right]_{-1}^{1}$$

$$\Rightarrow 2\times(1) - 2(-1) - 16(1)^2 - 16(-1)^2$$

$$\Rightarrow 4 - \cancel{16} + \cancel{16}$$

$$\Rightarrow 4$$

Again, Reversing the Order of Integration,

$$\iint_R f(x,y)\cdot dA = \int_0^2\int_{-1}^{1}(1-6x^2y)\cdot dy\cdot dx$$

$$\Rightarrow \int_0^2 \left[y - 3x^2y^2 \right]_{-1}^{1} \cdot dx$$

$$\Rightarrow \int_0^2 \left[(1)-(-1) - 3(1)^2(y)^2 - 3(-1)^2\cdot y^2\right]\cdot dx$$

$$\Rightarrow \int_0^2 [2]\, dx$$

$$\Rightarrow \left[2x\right]_0^2 \Rightarrow 2\times(2) - 2\times 0$$

$$= 4$$