Day-12, Nov-27, 2024 (Mangshir 12, 2081)

Example: Use the property of continuous function to evaluate $\lim_{x \to 1}$

$$\lim_{x\to 1} \arcsin\left(\frac{1-\sqrt{x}}{(1-x)}\right)$$

Here, arcsin is continuous in it's domain [using theorem 3]

$\lim_{x\to 1} \frac{(1-\sqrt{x})}{(1-x)}$ exists so by using theorem 4 we have,

$$\Rightarrow \lim_{x\to 1} \arcsin \frac{(1-\sqrt{x})}{(1-x)} = \arcsin\left(\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x}\right)$$

Theorem 4 says,

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$$

$$\Rightarrow f(b)$$

$$\Rightarrow \arcsin\left[\lim_{x\to 1} \frac{\cancel{1-\sqrt{x}}}{\cancel{(1-\sqrt{x})}(1+\sqrt{x})}\right.$$

$$\Rightarrow \arcsin\left(\lim_{x\to 1} \frac{1}{1+\sqrt{x}}\right)$$

$$\Rightarrow \arcsin \frac{1}{2}$$

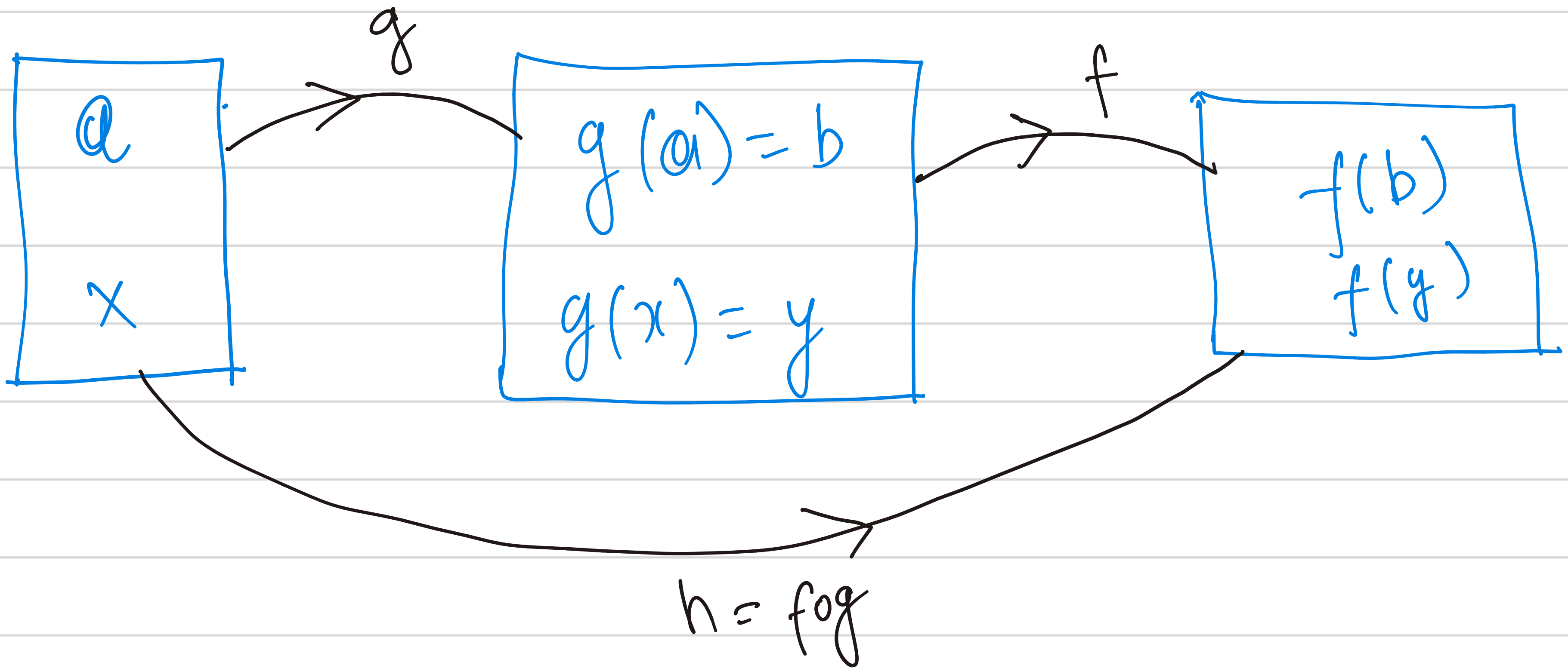
$$\left[\lim_{x \to 1} \arcsin \frac{(1-\sqrt{x})}{(1-x)} = \frac{\pi}{6}\right]$$

Theorem 5: If 'g' is continuous at 'a' and 'f' is continuous at g(a), then the composite function fog given by $(f o g)(x) = f(g(x))$ is continuous at a. $[(f o g)(x) = f(g(x))]$.

Here,

$$[(f o g)(x) = f(g(x))]$$

Proof:



Since f is continuous at $g(a) = b$ then for every $\varepsilon > 0$ $\exists\ \delta' > 0$ such that $0 < |y - b| < \delta'$.

$\Rightarrow |g(x) - g(a)| < \delta'$

$\Rightarrow |f(y) - f(b)| < \varepsilon$

$\Rightarrow |f(g(x)) - f(g(b))| < \varepsilon$ ——— eqn (i).

Again g is continuous at a then for every $\delta' > 0$,

$\exists\ \delta > 0$ Such that $0 < |x-a| < \delta$

$$|g(x) - g(a)| < \delta' \text{ ——— eqn (ii)}.$$

using (i) and (ii) we have, $\forall\ \epsilon > 0$, $\exists\ \delta > 0$ such that

$$0 < |x-a| < \delta$$

$$\Rightarrow |f(g(x)) - f(g(a))| < \epsilon$$

$$\Rightarrow |(f \circ g)(x) - (f \circ g)(a)| < \epsilon$$

Which shows that $h = f \circ g$ is also Continuous at 'a'.

Theorem 6: The Intermediate Value Theorem:

Suppose that 'f' is continuous on the closed interval $[a, b]$ and let 'N' be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$ then there exists a number c in (a, b) such that $f(c) = N$.

The intermediate value theorem states that a continuous function takes on every intermediate values between the function values $f(a)$ and $f(b)$. And these are illustrated by the figures 1 and 2.

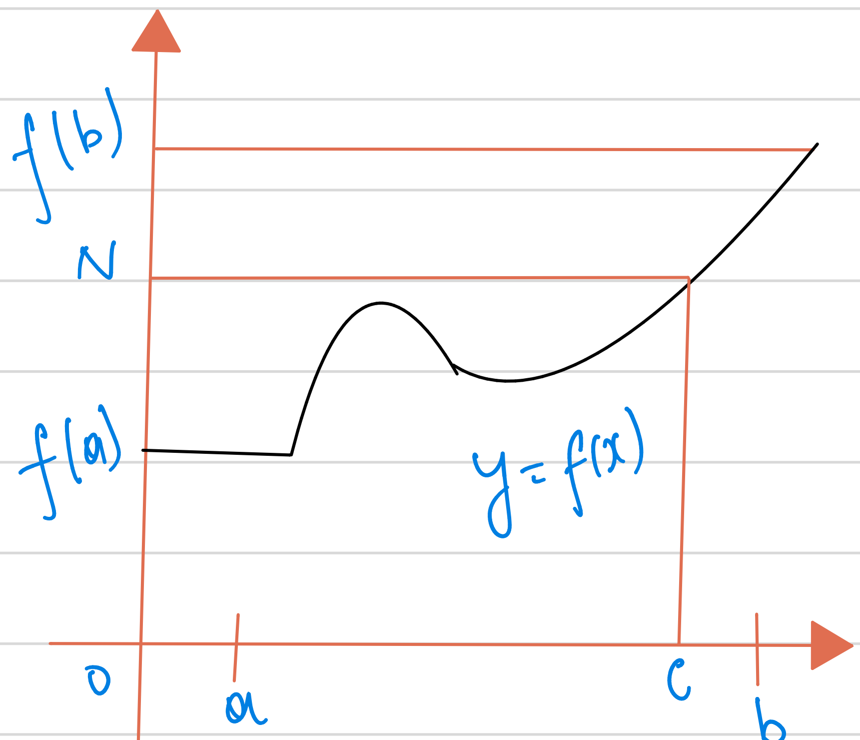


Fig 1

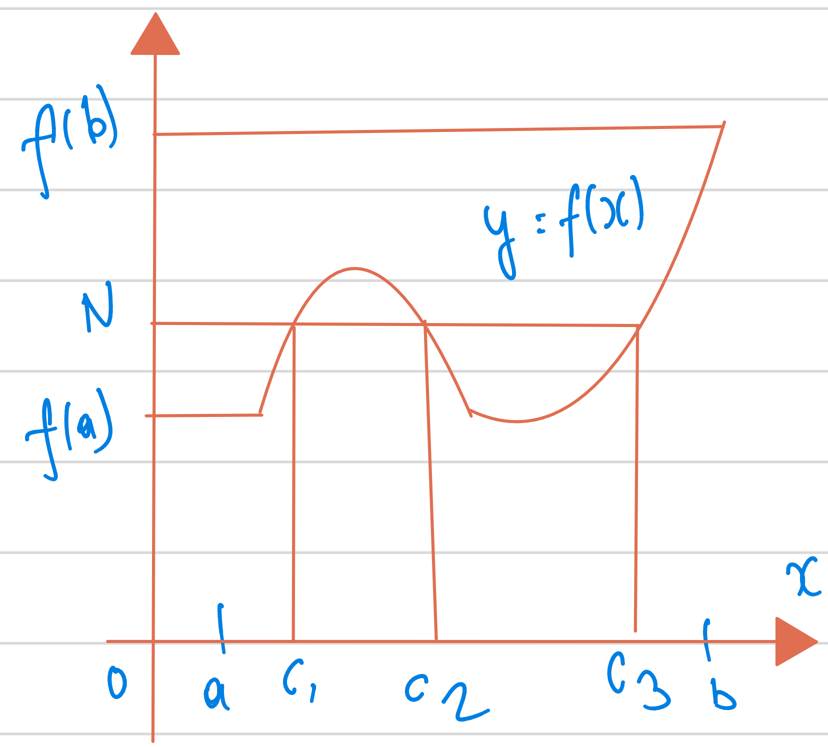


fig 2

→ In the fig(1) values N can be taken on once and more than once in fig(2).

Remarks:

if we think of a continuous function as a function whose graph has no hole or break then it is easy to believe that the Intermediate value theorem is true.

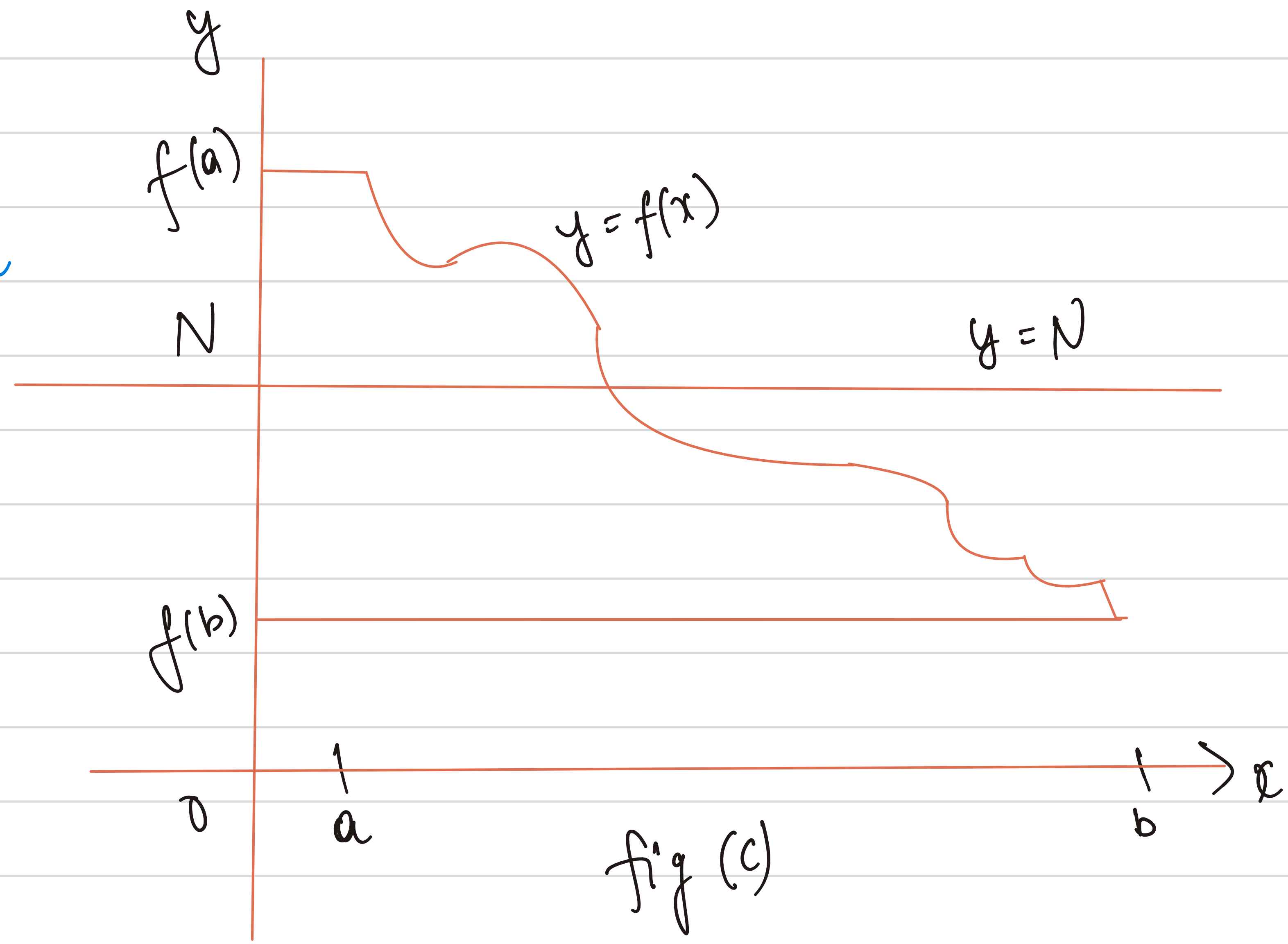


fig (c)

In geometric terms it says that if any horizontal line y = N is given between y = f(a) and y = f(b) as in fig (c), then the graph of 'f' can't jump over the line. It must intersect y = N somewhere.

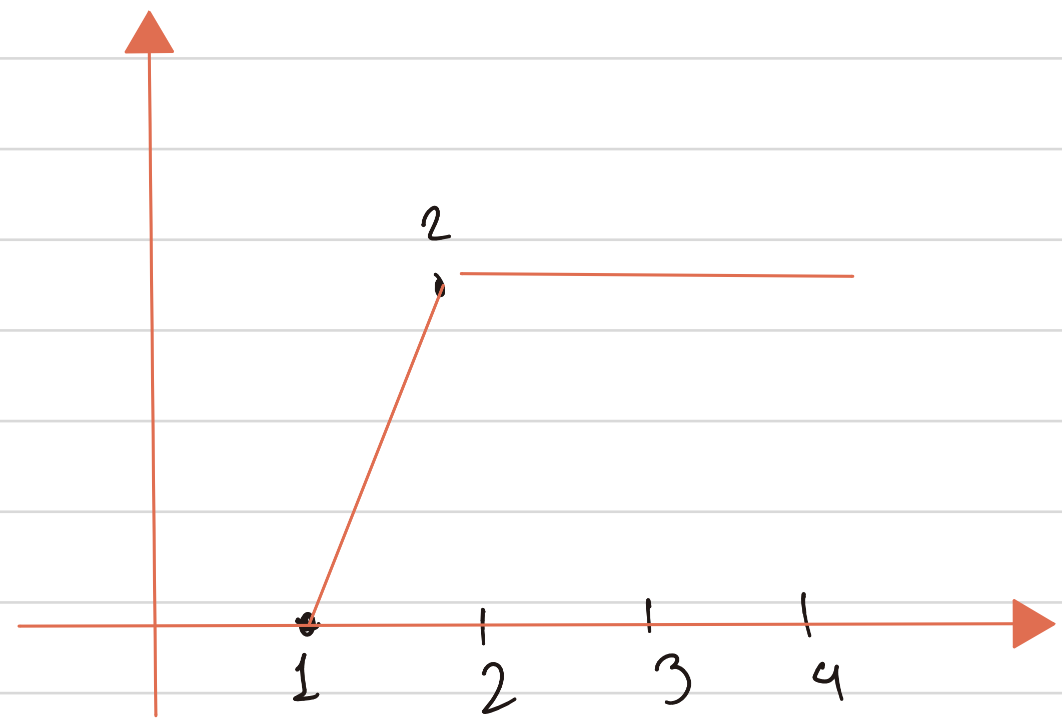
Example: Show that the hypothesis of intermediate value theorem is essential to hold the theorem.

Here, consider a function $f(x) = \begin{cases} 2x-2 & \text{for } 1 \leq x \leq 2 \\ 3 & \text{for } 2 \leq x \leq 4 \end{cases}$

$$LHL = \lim_{x \to 2^-} f(x) = 2$$

$$RHL = \lim_{x \to 2^+} f(x) = 3$$

Then $LHL \neq RHL$.



Hence, $\lim_{x \to 2} f(x)$ does not exist, then $f(x)$ is discontinuous at $x = 2$. So, it is discontinuous in $[1, 4]$.

There is no $c \in [1, 4]$ for $f(x)$ lying between 2 and 3.

[or for any $y_0 = f(c)$ lying between 2 and 3 there is no $c \in (1, 4)$].

Example: Show that there is a root of the equation $4x^3 - 6x + 3x - 2 = 0$ between 1 and 2.

Here given, $f(x) = 4x^3 - 6x + 3x - 2$

We are looking for a solutions of the given equation, that is, a number between 1 and 2 such that $f(c) = 0$. Therefore, we take $a = 1$, $b = 2$ and $N = 0$ in the intermediate value theorem. We have,

$$f(a) = f(1) = 4 - 6 + 3 - 2 = -1 < 0$$

and $$f(b) = f(2) = 32 - 24 + 6 - 2 = 12 > 0$$

Thus, $f(1) < 0 < f(2)$; i.e. $N = 0$ is a number between $f(1)$ and $f(2)$

By Intermediate theorem there exists a number $c \in (1, 2)$.

Such that $f(c) = N = 0$, which shows that $x = c$ is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$.

Revision of Linear Regression with Mathematics:

Problem statement: predicting the next potential cities where business has higher profits

load dataset
train, test
prediction → cost (looping
total_cost / 2

1. load dataset
2. Train, testing (Split the dataset)
3. Make prediction

$$f_{w,b}(x^{(i)}) = wx^{(i)} + b$$

and cost for each

$$cost^{(i)} = (f_{wb} - y^{(i)})^2$$

Step 3 is for-loop

4. Returning the total cost for all $J(w,b) = \frac{1}{2} \sum_{i=0}^{m-1} cost^{(i)}$

total_cost = total_cost / 2m where m is input x.shape[0].

(5) Reducing the loss using gradient descent -

repeat until convergence {

$$b := b - \alpha \frac{\partial J(w,b)}{\partial b}$$

$$w := w - \alpha \frac{\partial J(w,b)}{\partial w}$$

}

where parameters w,b are both updated simultaneously and where,

$$\frac{\partial J(w,b)}{\partial b} = \frac{1}{m} \sum_{i=0}^{m-1} \left(f_{w,b}(x^{(i)}) - y^{(i)} \right)$$

$$\frac{\partial J(w,b)}{\partial w} = \frac{1}{m} \sum_{i=0}^{m-1} \left(f_{w,b}(x^{(i)}) - y^{(i)} \right) x^{(i)}$$

or simply, $\frac{\partial J(w)}{\partial w}$, $\frac{\partial J(w)}{\partial b}$ $\rightarrow$ m is the number of training examples -

1. Linear Regression –

$$f_{w,b}(x) = wx + b$$

2. Compute-cost :

$$J(w,b) = \frac{1}{2m} \sum_{i=0}^{m-1} \left(f_{w,b}(x^{(i)}) - y^{(i)}\right)^2$$

→ where $y^{(i)}$ is the Actual output

→ m is the number of training examples

→ $f_{w,b}(x^{(i)})$ → model's prediction –

3. $f_{w,b}(x^{(i)}) = wx^{(i)} + b$ → $Cost = (f_{wb} - y^{(i)})^2$

$$J(w,b) = \frac{1}{2m} \sum_{i=0}^{m-1} Cost^{(i)}$$

4. Repeat for Gradient Descent until Convergence:

$$\{ \quad b := b - \alpha \frac{\partial J(w,b)}{\partial b}$$

$$w := w - \alpha \frac{\partial J(w,b)}{\partial w} \quad \}$$

where parameters w, b are both updated simultaneously.

$$\frac{\partial J(w,b)}{\partial b} = \frac{1}{m} \sum_{i=0}^{m-1} \left(f_{w,b}(x^{(i)}) - y^{(i)} \right)$$

$$\frac{\partial J(w,b)}{\partial w} = \frac{1}{m} \sum_{i=0}^{m-1} \left(f_{w,b}(x^{(i)}) - y^{(i)} \right) (x)^{(i)}$$