Day-44, Jan-13, 2024 (Poush 29, 2081 B.S.)

function	Particular Anti-Derivative	function	Particular Anti-deriv
$c\,f(x)$	$c\,F(x)$	$\sec^2 x$	$\tan x$
$f(x) + g(x)$	$F(x) + G(x)$	$\sec x.\tan x$	$\sec x$
$x^n \; (n \neq -1)$	$\frac{x^{n+1}}{n+1}$	$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
$1/x$	$\ln\lvert x\rvert$	$1/1+x^2$	$\tan^{-1} x$
e^x	e^x	$\cosh x$	$\sinh x$

$$\frac{d\,F(x)}{dx} = f(x) \;, \quad x \in (a,b)$$

So,
$$\int f(x) \cdot dx = F(x) + C$$

Rectilinear Motion, Area and Distance:

Anti-derivative → useful in analyzing the motion of an object moving in a straight line.

Position function $s = f(t)$
velocity function $v(t) = s'(t)$
where $s = f(t)$ is the anti-derivative of the velocity function

velocity is the anti-derivative of acceleration $a(t) = v'(t)$
If the acceleration and the initial values $s(0)$ and $v(0)$ are known then the position function can be found by antidifferentiating twice.

Example: A Particle moves in a straight line and has acceleration given by $a(t) = 2t+1$. Its initial velocity is $v(0) = -2\text{cm/s}$ and its initial displacement is $s(0) = 3\text{cm}$. find the Position function $s(t)$.

Given,

$$a(t) = 2t+1, \quad v(0) = -2, \quad s(0) = 3$$

Now, we have $a(t) = v'(t)$

$$v(t) = t^2 + t + C$$

Since $v(0) = -2$ so,

$$v(0) = 0^2 + 0 + C \quad \text{or,} \quad \boxed{C = -2}$$

$\therefore v(t) = t^2 + t - 2$

Since, $v(t) = s'(t)$ so

$$s(t) = \frac{t^3}{3} + \frac{t^2}{2} - 2t + D$$

This gives $s(0) = D$ so $D = 3$.

$\therefore$ The required position function is $s(t) = \frac{t^3}{3} + \frac{t^2}{2} - 2t + 3$

Integration of a Product Function "by Parts".

If a given function to be integrated in the product form and if can be integrated either by reducing the integrand into the standard form and substitution, we use the following rule known as the integration by parts.

Let 'u' and 'w' be two differential functions of x. Then using the product rule of differentiation, we have -

$$\frac{d}{dx}(uw) = u \cdot \frac{dw}{dx} + w \cdot \frac{du}{dx}$$

or, $$u \frac{dw}{dx} = \frac{d}{dx}(uw) - w \cdot \frac{du}{dx}$$

Integrating both sides w.r.t 'x' we have,

$$\int \left(u \frac{dw}{dx} \right) \cdot dx = \int \left(\frac{d}{dx} (uw) \right) \cdot dx - \int \left(w \cdot \frac{du}{dx} \right) \cdot dx$$

$$\Rightarrow \int \left(u \cdot \frac{dw}{dx} \right) dx = uw - \int \left(w \cdot \frac{du}{dx} \right) \cdot dx \longrightarrow \text{eqn (i)}$$

Let $\frac{dw}{dx} = v$ then $w = \int v \cdot dx$.

Now, the relation (i) we get the form,

$$\int u \cdot v \cdot dx = u \int v \cdot dx - \int \left(\frac{du}{dx} \int v \cdot dx \right) \cdot dx$$

The formula can be stated as follows:

The integral of the product of two functions

= first function × integral of second − integral (Derivative of first × integral of second)

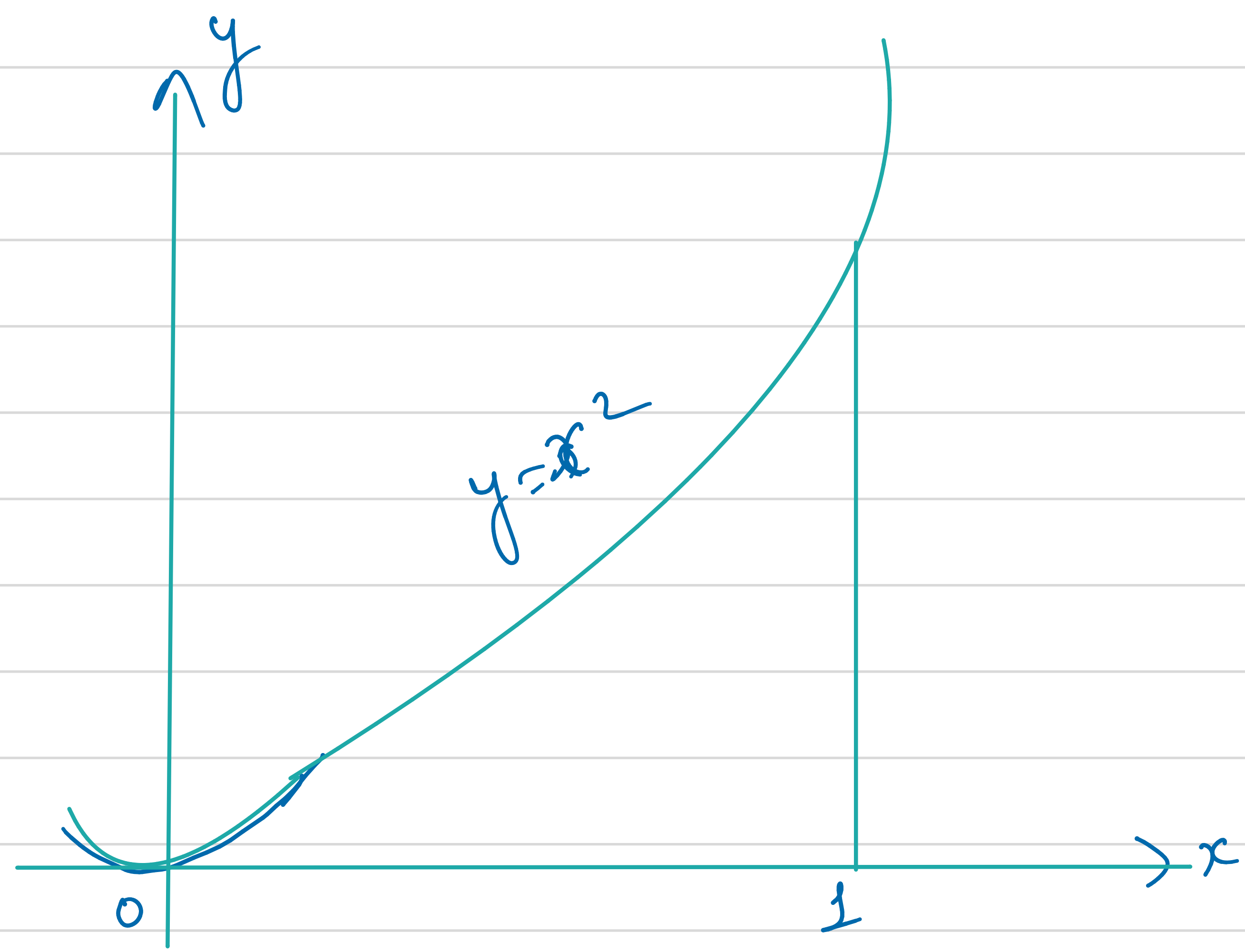
We call this integration by parts. It depends upon the proper choice of the first function (reduces to simple form) and second function should be easily integrable.

Area

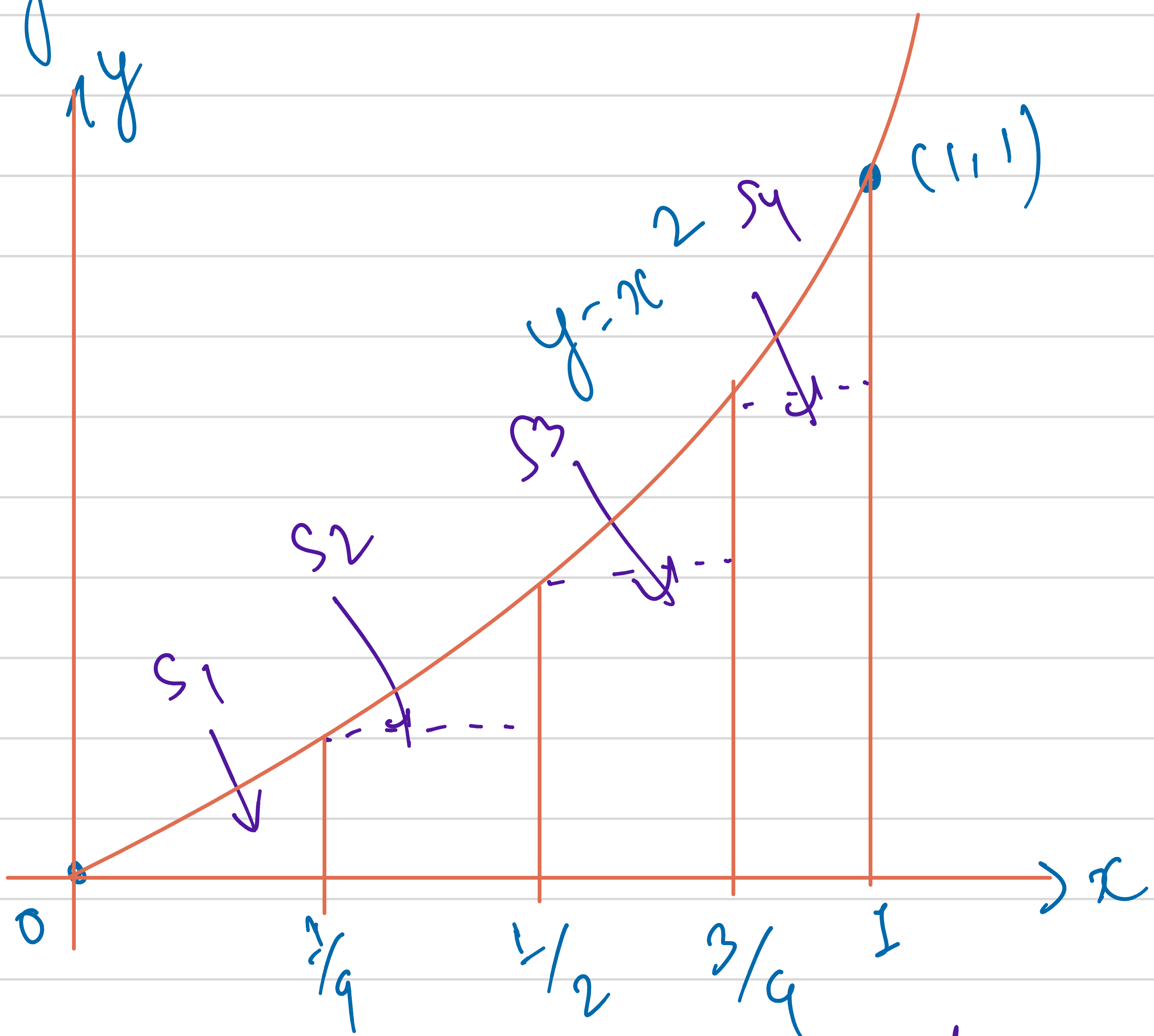
Area of a Region with a Curved boundary can be approximated by Summing the areas of a collection of rectangles.

Using more rectangles can increase the accuracy of the approximation.

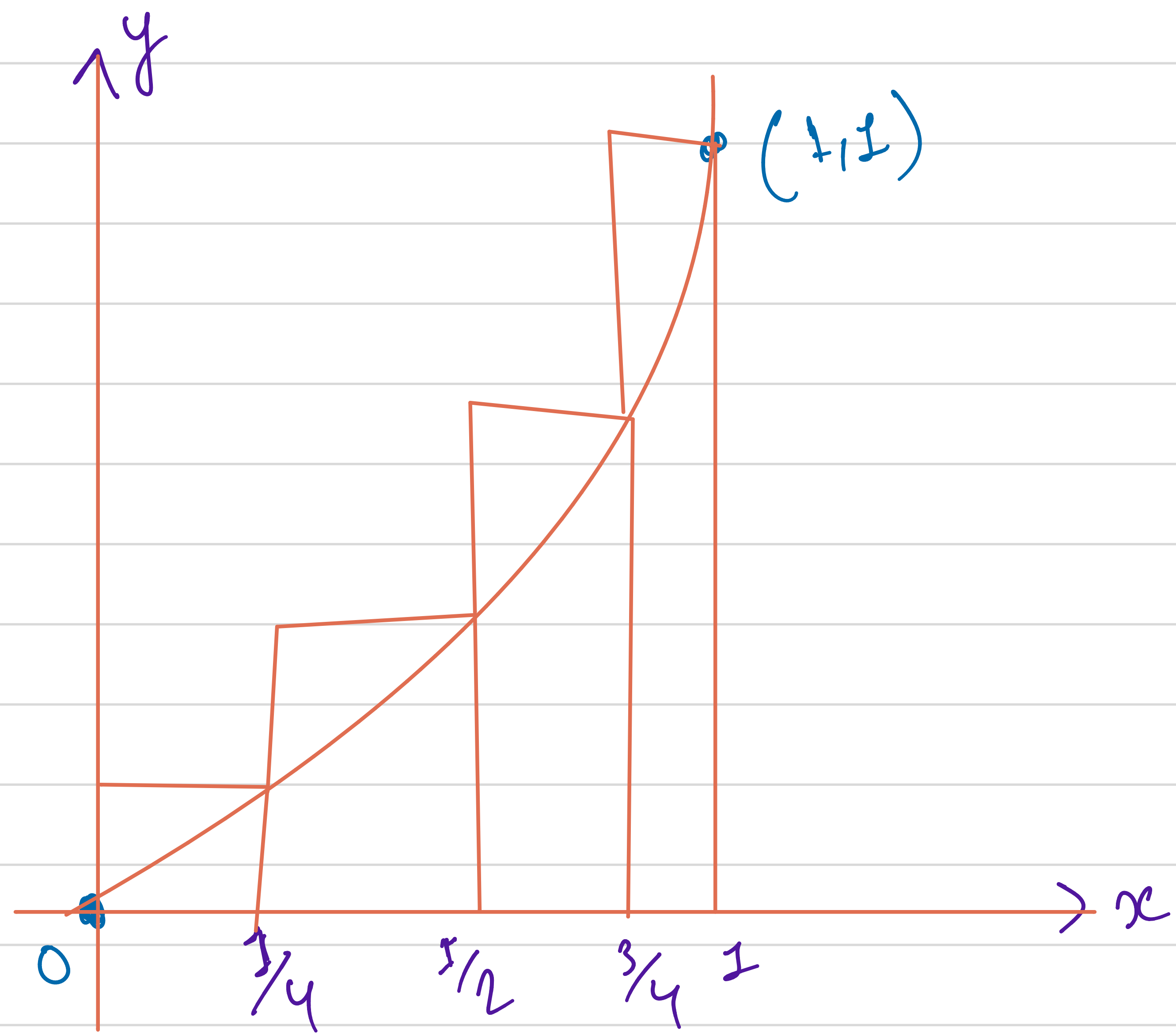
Area of S must be Somewhere between 0 and 1 because S is contained in a Square with side length 1.



Suppose we divide into four strips S_1, S_2, S_3 and S_4 by drawing the vertical lines $x = 1/4$, $x = 1/2$ and $x = 3/4$ as in figure.



a) using left end points



b) using right endpoints

The heights of these rectangles are the values of the $f(x) = x^2$ at the right end points of the subintervals $\left[0, \frac{1}{4}\right], \left[\frac{1}{4}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{3}{4}\right]$ and $\left[\frac{3}{4}, 1\right]$.

Each Rectangle has width $\frac{1}{4}$ and the heights are $\left(\frac{1}{4}\right)^2, \left(\frac{1}{2}\right)^2, \left(\frac{3}{4}\right)^2$ and 1^2. If we let R_4 be the sum of the areas of these approximating rectangles, we get-

$$R_4 = \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot 1^2 \quad \left(R_4 \Rightarrow 0.46875\right)$$

$$L_4 \Rightarrow \frac{1}{4}\cdot(0)^2 + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2$$

$$L_4 \Rightarrow 0.21875$$

So,

$$L_4 < A < R_4$$

ie. $0.21875 < A < 0.46785$

So one possible answer to the question is to say that the true area of S lies somewhere between 0.27875 and 0.46785. Using more strips gives the better estimation.

Definition: The area A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles.

$$A = \lim_{n \to \infty} R_n$$

$$\Rightarrow \lim_{n \to \infty} \left[f(x_1)\,\Delta x + f(x_2)\,\Delta x + \dots + f(x_n) \cdot \Delta x \right]$$

$$\boxed{A \Rightarrow \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x} \rightarrow \text{END POINTS}$$

It can be proved that the limit in definition always exists, since we are assuming that f is continuous. It can be shown that we get the same value if we use diff endpoints.

$$A = \lim_{n \to \infty} L_n$$

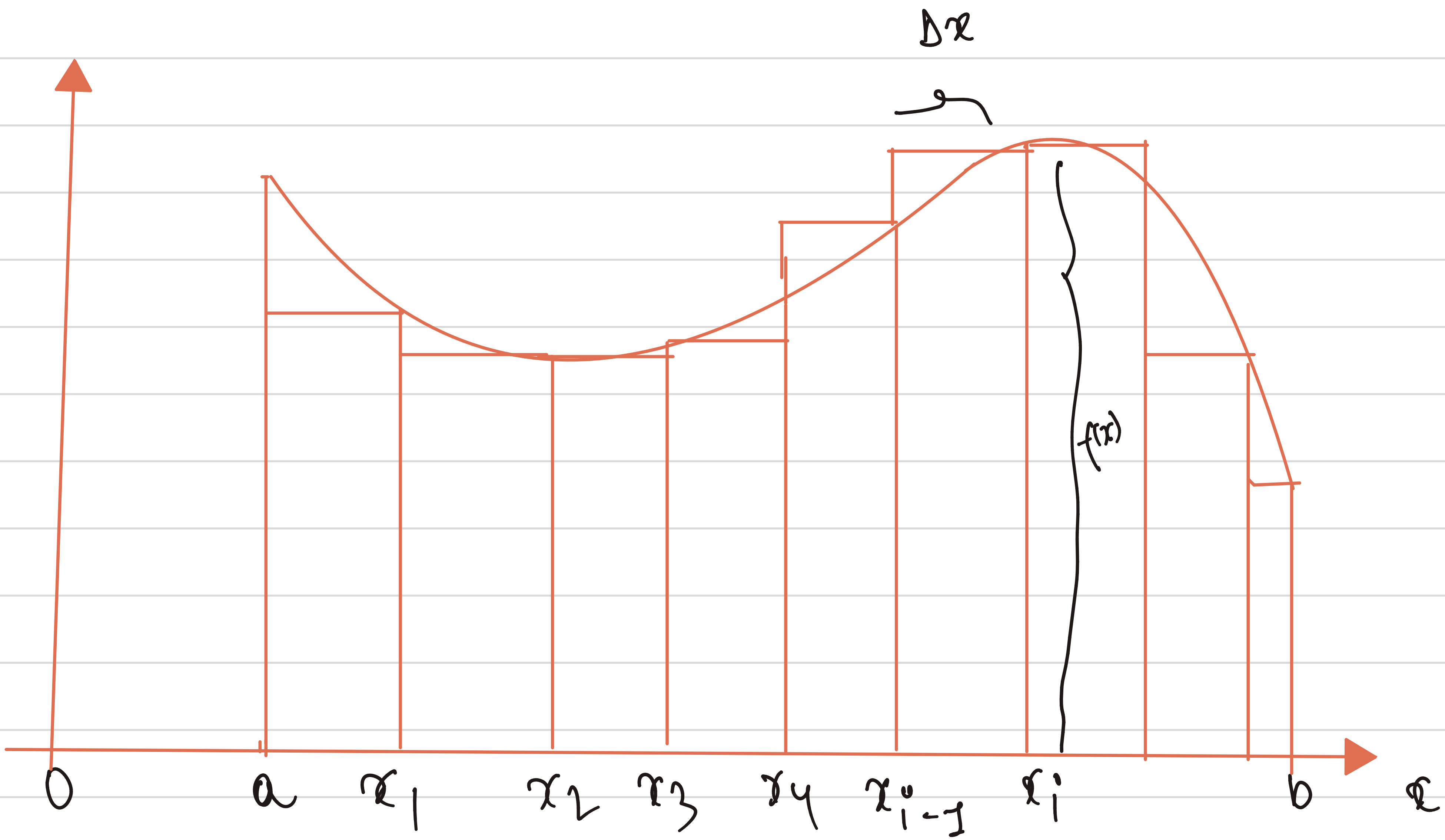
$$= \lim_{n \to \infty} \left[f(x_0)\,\Delta x + f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_{n-1}) \cdot \Delta x \right]$$

$$\left[A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_{i-1}) \cdot \Delta x \right] \rightarrow \text{diff End points.}$$

More General Expression for the area of S is

$$A = \lim_{n \to \infty} \left[f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots\cdots + f(x_n^*)\,\Delta x \right]$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x$$



from the figure the width of the interval $[a,b]$ is $b-a$,
So the width of each of the 'n' strips is

$$\Delta x = \frac{b-a}{n}$$

These strips divide the interval $[a, b]$ into n subintervals,

$$[x_0, x_1] \; [x_1, x_2] \; [x_2, x_3] \; \ldots\ldots \; [x_{n-1}, x_n]$$

where $x_0 = a$ and $x_n = b$. The right endpoints of the subintervals are

$$x_1 = a + \Delta x$$
$$x_2 = a + 2\Delta x$$
$$x_3 = a + 3\Delta x$$
$$\vdots$$

So for ith strip S_i with width Δx and height $f(x_i)$. Then the area of ith rectangle $f(x_i) \cdot \Delta x$, S is sum of the areas of these rectangles

$$\left[R_n = f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x \right]$$

Definite Integral:

The expression $\int_a^b f(x) \cdot dx$ called the definite integral of $f(x)$ from a to b. Here, a is said to be lower limit and b is the upper limit.

Definite Integral has definite value.

Now, deriving the formula if f is a continuous function and $F(x) = \int_a^x f(t) \cdot dt$ then

$$\frac{d}{dx} F(x) = f(x)$$

This theorem establishes the relation between the two basic concepts of the calculus the derivative and the definite integral. Proving the corollary

Corollary: If f is continuous on $[a, b]$ and ϕ is an anti-derivative of f.

$$\int_a^b f(x) \cdot dx = \phi(b) - \phi(a)$$

Proof: Let $F(x) = \int_a^x f(t) \cdot dt$

Obviously, we get $F(a) = 0$. Now, as F and ϕ are anti-derivative of the function f, they differ only by constant. So,

$$F(x) = \phi(x) + c \quad \text{for some constant } c.$$

$$\therefore \; F(a) = \phi(a) + c$$

$$0 = \phi(a) + c$$

$$\left[\phi(a) = -c \right]$$

$$F(x) = \phi(x) - \phi(a)$$

and $F(b) = \phi(b) - \phi(a)$

But we have,

$$F(b) = \int_a^b f(t) \cdot dt .$$

$$\therefore \int_a^b f(t) \cdot dt = \phi(b) - \phi(a)$$

This theorem is known as the fundamental theorem of integral calculus. Thus in evaluating a definite integral $\int_a^b f(x) \cdot dx$.