# Day-42, Jan-11, 2025 ( Poush-27, 2081)

# Anti-Derivatives (Indefinite Integral)

Let $f$ be continuous function defined in an open interval $(a, b)$. The function $F$ is said to be an antiderivative of $f$ on the interval, if the derivative of $F$ is equal to $f$ on the interval, i.e. if,

$$\frac{dF(x)}{dx} = f(x), \quad x \in (a, b)$$

As the derivative of a constant 'c' is zero, $F(x) + c$

is also an anti-derivative of f, whenever the function F is so

Actually, when-

$$\frac{dF(x)}{dx} = f(x)$$

We have,

$$\frac{d\,[F(x)+c]}{dx} = \frac{dF(x)}{dx} + \frac{dc}{dx}$$

$$\Rightarrow f(x) + 0$$

$$\Rightarrow f(x)$$

The converse is also true any two antiderivatives of a function differ a constant. Let $F$ and $G$ be antiderivatives of a function $f$. Then-

$$\frac{d\left[F(x) - G(x)\right]}{dx} = \frac{dF(x)}{dx} - \frac{d\,G(x)}{dx}$$

$$\Rightarrow f(x) - f(x)$$

$$\Rightarrow 0$$

From this it follows that there exists a constant $C$ such that,

$$\left[F(x) - G(x) = C\right]$$

All these go to establish the fact that if $F$ is an antiderivative of $F(x) + c$ gives all the possible antiderivatives of $f$, when $c$ runs through the real numbers

# General form of <u>All Anti-derivatives</u> which we call <u>Indefinite Integral of f</u> denoted by:

$$\int f \, dx \quad \text{or} \quad \int f(x) \cdot dx$$

If $f$ is an antiderivative of $f$, we have

$$\int f(x) \cdot dx = F(x) + C$$

The basic property of the indefinite integral is -

$$\int \{ C_1 f(x) + C_2 g(x) \} dx = C_1 \int f(x) \cdot dx + C_2 \int g(x) \cdot dx + C$$

Where $f$ and $g$ are continuous functions in an interval $(a, b)$ and $C_1$ and $C_2$ are some constants

Integration is the reverse process of differentiation. Thus if $\frac{d}{dx}[f(x)] = F(x)$, then $f(x)$ is called the anti-derivative or integral of $F(x)$.

It is denoted by $\int F(x) \cdot dx = f(x)$

for example: $\frac{d}{dx}(\sin x) = \cos x$ then $\int \cos x \cdot dx = \sin x$

Generally, if $\frac{d}{dx}[f(x)] + C = F(x)$ then, $\int F(x) \cdot dx = f(x) + C$

where $C$ is arbitrary constant.

$\boxed{\text{Anti-Derivative}}$

Evaluation of the Area Under a plane Curve or we can say a limit of a sum when the number of terms in the sum tends to infinity and each term tends to zero. $\frac{d}{dx} f(x)$

→ The inverse of differentiation is Anti-Derivative or Indefinite Integral

$d\,F(x)$ or $\int f(x)\,dx$

Example:

$$\frac{d(\sin x)}{dx} = \cos x$$

$$\therefore \int \cos x \cdot dx = \sin x$$

# Standard Integrals (I) → Directly Related to Standard Differentiation formulae

Eg:

$$\int \frac{1}{a^2+x^2} \cdot dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

Put $x = a\tan\theta$ $\quad dx = a\sec^2\theta \cdot d\theta$

$$a^2 + x^2 \Rightarrow a^2 + a^2 \tan^2\theta$$

$$\Rightarrow a^2(1 + \tan^2\theta)$$

$$\Rightarrow a^2 \sec^2\theta$$

$$\therefore \int \frac{1}{a^2 + x^2} \cdot dx = \int \frac{1}{a^2 \sec^2\theta} \cdot a \sec^2\theta \cdot d\theta$$

$$\Rightarrow \frac{1}{a} \int d\theta$$

$$\Rightarrow \frac{1}{a} \theta + C$$

$$\left[ \therefore \int \frac{1}{a^2 + x^2} \cdot dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C \right]$$

# Standard Integrals (II)

In this section we considered the second set of standard Integrals which can be evaluated by a technique known as Integration by parts. This is a formula which can be deduced as given below:

We know,

$$\frac{d}{dx}(uv_1) = \frac{du}{dx}.v_1 + u.\frac{dv_1}{dx}$$

for any two differentiable functions $u$ and $v_1$ of $x$. Integrating both sides, we get –

We know,

$$\frac{d}{dx}(u \cdot v_1) = \frac{du}{dx} \cdot v_1 + u \cdot \frac{dv_1}{dx}$$

for any two differentiable functions $u$ and $v_1$ of $x$. Integrating both sides, we get -

$$uv_1 = \int \left( \frac{du}{dx} \cdot v_1 \right) \cdot dx + \int \left( u \cdot \frac{dv_1}{dx} \right) \cdot dx$$

Let $\frac{dv_1}{dx} = v$ so that $v_1 = \int v \, dx$. Then find

$$\int (uv) \cdot dx = u \int v \cdot dx - \int \left( \frac{du}{dx} \int v \cdot dx \right) \cdot dx$$

which is the required formula.

Recovering the function $F$ from its known derivative.
If such function $F$ exists it is called an anti-derivative of $f$.

Example: (a) $f(x) = 2x$ (b) $g(x) = \cos x$ (c) $h(x) = 2x + \cos x$

(i) $f(x) = 2x$ $F(x) = x^2$

(b) $g(x) = \cos x$ $G(x) = \sin x$

(c) $h(x) = 2x + \cos x$ $H(x) = x^2 + \sin x$

Example:

$f(x)$ if $f'(x) = e^x + 20(1+x^2)^{-1}$ and $f(0) = 5$

Here,

$$f'(x) = e^x + 20(1+x^2)^{-1}$$

$$f(x) = e^x + 20\tan^{-1}x + C$$

$$\left\{ \begin{array}{l} \frac{d}{dx}e^x = e^x \\ \frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} \end{array} \right.$$

Since $f(0) = 5$

$$f(0) = e^0 + 20\tan^{-1}0 + C$$

$$5 = 1 + C$$

$$\boxed{C = 4}$$

$$\boxed{f(x) = e^x + 20\tan^{-1}x + 4}$$

## Day-43, Jan-12, 2025 ( Poush-28, 2081 B.S)

**Anti-derivative Example:**

$$\frac{d}{dx}(\sin x) = \cos x \quad \text{then} \quad \int \cos x \cdot dx = \sin x$$

Generally,

$$\text{if } \frac{d}{dx}[f(x)] + c = F(x) \quad \text{then} \quad \int F(x) \cdot dx = f(x) + C$$

Where C is arbitrary Constant

# Some formula Related to Integration

i) $$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

ii) $\int (ax+b)^n \cdot dx = \frac{1}{a} \frac{(ax+b)^{n+1}}{n+1} + C \quad [n \neq -1]$

iii) $\int \frac{1}{x} \cdot dx = \log x + C$ , $\int \frac{1}{ax+b} \cdot dx = \frac{1}{a} \log(ax+b) + C$

iv) $\int 1\, dx = x + C$

v) $\int e^x \cdot dx = e^x + C$

vi) $\int e^{ax} \cdot dx = \frac{1}{a} e^{ax} + C$ .

vii) $\int \frac{f'(x)}{f(x)} \cdot dx = \log f(x) + C$

viii) $\int (u \times v) \cdot dx = u \int v\, dx \bullet \int \left[ \frac{du}{dx} \times \int v \cdot dx \right] \cdot dx$

Performances Order $L \to A \to T \to E$

Where $L$ = Logarithmic, $A \Rightarrow$ Algebraic functions,

$T$ = Trigonometric $E \Rightarrow$ Exponential functions.

# Definite Integral as the Limit of a Sum.

Let $f(x)$ be a continuous function defined in the interval $[a, b]$ and let the interval $[a, b]$ be divided into $n$ equal sub intervals each of length $h$ by the points

$a+h$, $a+2h$, ....., $a+(n-1)h$ where $nh = b-a$

Then, the definite integral of $f(x)$ w.r.t. $x$ becomes the limits $a$ and $b$ denoted by $\int_a^b f(x)\cdot dx$ and is defined by

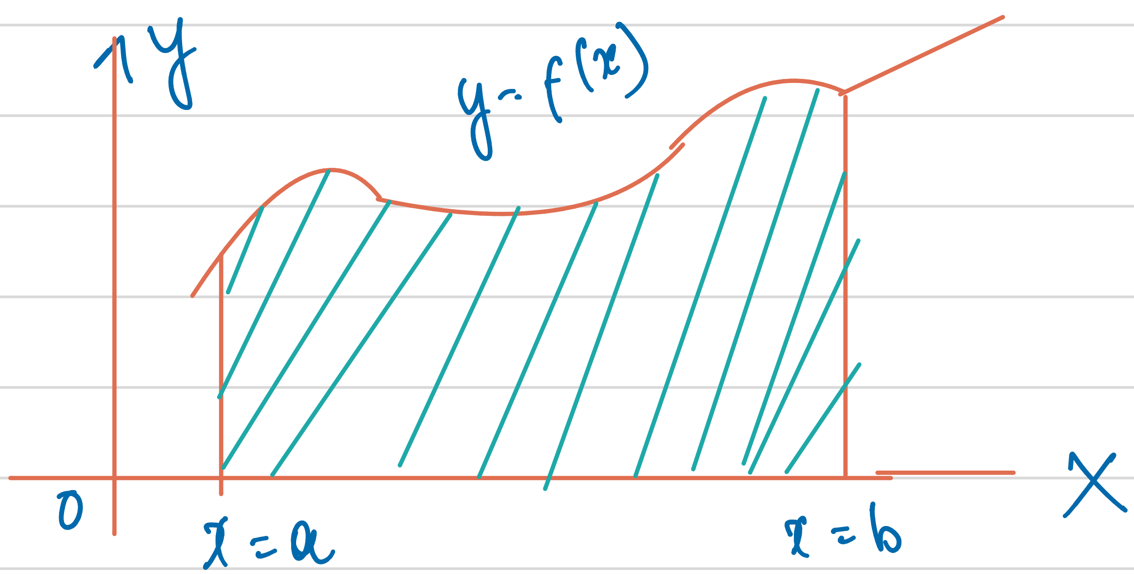
$$\int_a^b f(x)\cdot dx = \lim_{h\to 0} h \sum_{r=1}^{n} f(a+rh)$$

$$\Rightarrow \lim_{h\to 0} h\left[f(a+h) + f(a+2h) + \cdots\cdots + f(a+nh)\right]$$

# Some Important Formulae on Area:

a) The area bounded by the curve $y = f(x)$, the x-axis and the ordinates (co-ordinates $(x, y)$) $x = a$ and $x = b$ is given by -

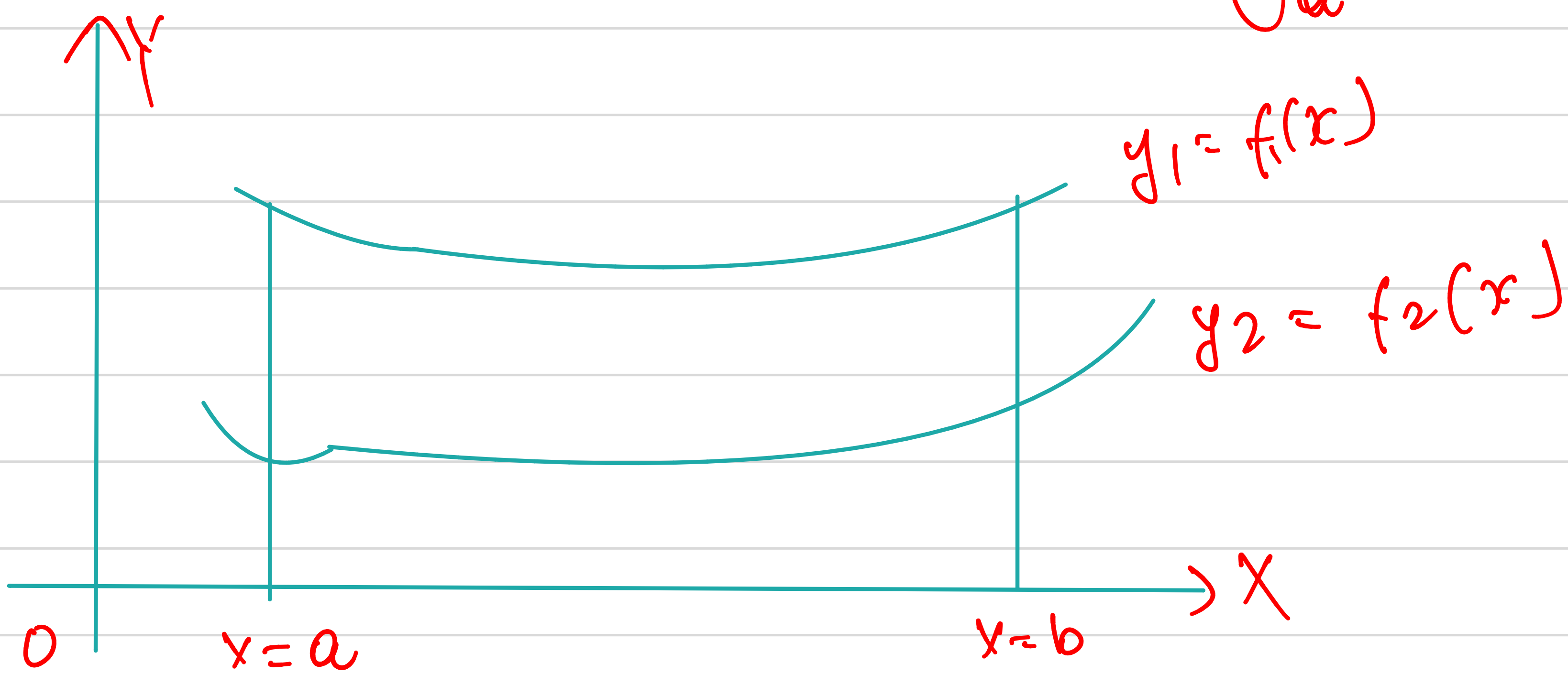
$$\int_a^b f(x) \cdot dx \quad \text{or} \quad \int_a^b y \cdot dx$$



b) The area enclosed by the two given curves –

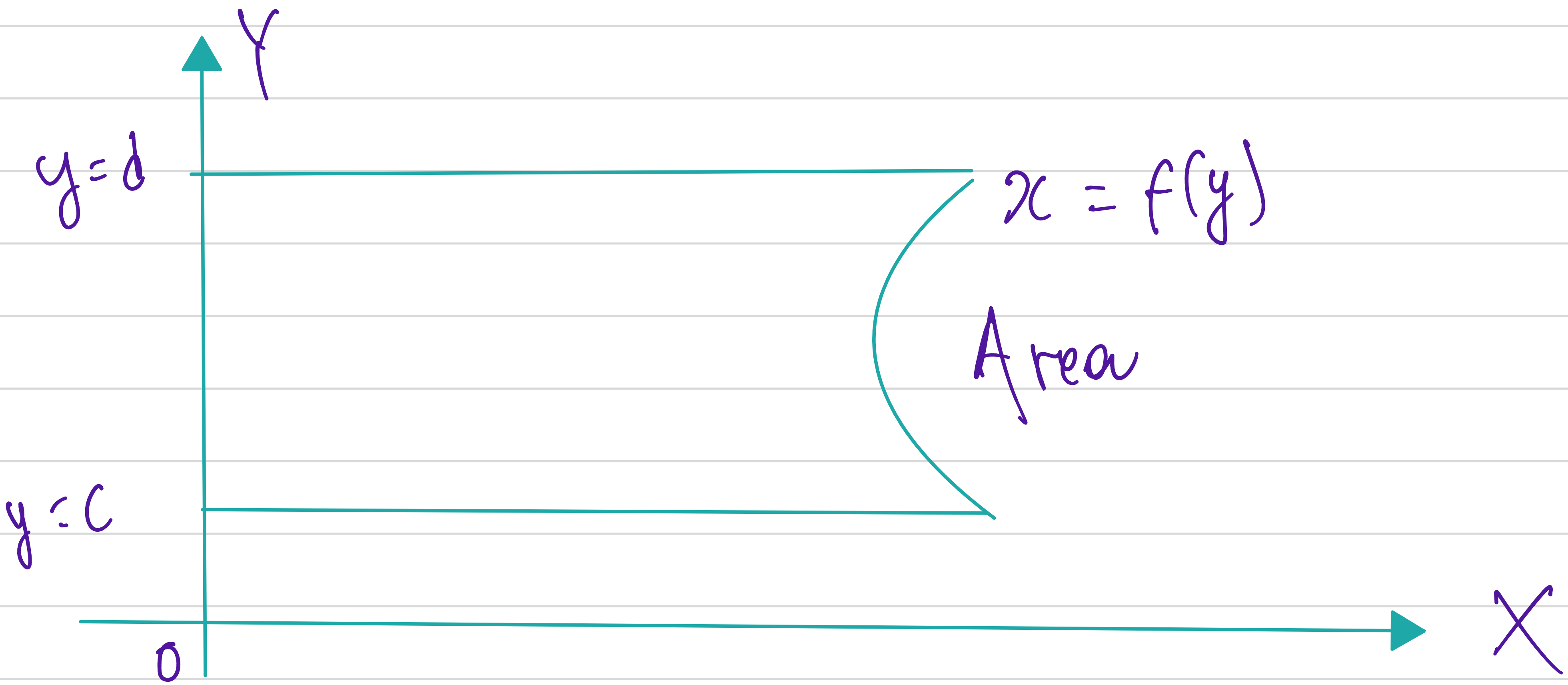
$y_1 = f_1(x)$ and $y_2 = f_2(x)$ is

$$\int_a^b [f_1(x) - f_2(x)] \cdot dx \quad \text{or} \quad \int_a^b (y_1 - y_2) \cdot dx$$



(c) The area bounded by the curve $x = f(y)$, the y-axis and two abscissa $y=c$ and $y=d$ is given by -

$$\int_c^d f(y)\cdot dy \quad \text{or} \quad \int_c^d x\, dy$$



Given Curve $y = x^3$ and the line $x = y$.

Now, $y = x^3$ ——— eqn (i)

$x = y$ ——— eqn (ii)

$x^3 = x$

$x(x^2 - 1) = 0$

So, $x = 0$

$x = 1$ are the ordinates point at which the given curves intersect.

The Required Area $= \int_0^1 [y_2 - y_1] \cdot dx$ when $y_2 = y = x$.

and $y_1 = y = x^3$

$$\Rightarrow \int_0^1 (x - x^3) \cdot dx .$$

$$\Rightarrow \left[\frac{x^2}{2}\right]_0^1 \cdot \left[\frac{x^4}{4}\right]_0^1$$

$$\Rightarrow \frac{1}{2}(1 - 0) \cdot \frac{1}{4}(1 - 0)$$

$$\Rightarrow \frac{1}{2} \cdot \frac{1}{4}$$

$$\Rightarrow 1/4$$

# Indefinite Integral

If the derivative of $F(x)$ is $f(x)$ then we say that the anti-derivative or integral $f(x)$ is $F(x)$ and we write —

$$\int f(x) \cdot dx = F(x) + C$$

Thus, $\frac{d}{dx}[F(x) + c] = f(x)$ then

$$\int f(x) \cdot dx = F(x) + C$$

Clearly, the different values of C will give different integrals. Thus, the given function may have indefinite number of integrals

For Second Part:

Integrals

**Definite Integrals**

→ doesnot contain Arbitrary Constant C

Hence, it has unique value

**Indefinite Integrals**

→ Contains arbitrary Constant C

for last Point:

The integral $\int_a^b f(x) \cdot dx$ is called definite integral of $f(x)$ between 'a' and 'b' if $\frac{d}{dx}[F(x)] = f(x)$, then $\int_a^b f(x) \cdot dx = F(b) - F(a)$.

# Anti-Derivative of a same function may differ by a constant.

We Know,

$$\frac{d}{dx}(x^3+3) = 3x^2, \quad \frac{d}{dx}(x^3+100) = 3x^2 \text{ and}$$

$$\frac{d}{dx}(x^3+c) = 3x^2$$

Those show that,

$$\int 3x^2 \cdot dx = x^3+3 \text{ and } \int 3x^2 \cdot dx = x^3+100$$

and $\int 3x^2 \cdot dx = x^3+C$ are the same integrals of the same function $3x^2$.

This shows that anti-derivative of a same function is differ by a constant.