Day-36, Dec 25, 2024 (Poush 10, 2081)

Calculus can be applied to more than one or two variables. for an example $V = \pi r^2 h$, we observe the form of function in which V is a function of two variables r and h and π as a constant

Functions (In the case of two variables)

Let 'D' be a set of two variables 'x' and 'y'. A real-valued function f defined on D is a rule that assigns a real value = $w = f(x,y)$ to each element in D.

The value of w is dependent variable of f and x, y are independent variables of f. Sometimes w is called output variable and x, y are input variable.

Definition (In the case of More Variables)

Let 'D' be a set of n-variables $x_1, x_2, x_3 \dots x_n$. A real valued function f defined on D is a rule that assigns a real value $w = f(x_1, x_2, x_3 \dots x_n)$ to each element in D

Example: find the domain and range of function

$$f(x, y) = \sqrt{y - x^2}$$

Let,

$$f(x,y) = \sqrt{y-x^2}$$

Clearly, f is defined for non-negative value of $y-x^2$, that is for $y-x^2 \geq 0 \Rightarrow x^2 \leq y$. This is the parabola having domain $y \geq x^2$.

And the range of f is a set defined by

$$z = y - x^2$$

$$z = -(x^2 - y) \leq \text{region defined by } x^2.$$

Since x^2 is defined on $[0,\infty)$. Therefore range of f is $[0,\infty)$.

Limit of function

If the value of $f(x,y)$ lie arbitrarily close to a fixed real value L as (x,y) sufficiently close to a fixed point (x_0,y_0) then we say that f approaches L as (x,y) approaches (x_0,y_0). In such case, we say L is the limit value of $f(x,y)$

Definition (Limit of a function)

Let $f(x,y)$ be a function of two variables x and y and L be a number. Then we say L is the limit of $f(x,y)$ at point (x_0,y_0) if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L.$$

Example:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x(x-y)\left(\sqrt{x} + \sqrt{y}\right)}{\left(\sqrt{x} - \sqrt{y}\right)\left(\sqrt{x} + \sqrt{y}\right)}$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x\,\cancel{(x-y)}\left(\sqrt{x} + \sqrt{y}\right)}{\cancel{x-y}}$$

$$\Rightarrow 0.$$

Remark: The two path test for non-existence of a limit.

If a function $f(x,y)$ has different limits along two different paths in the domain of f as $(x,y) \to (x_0, y_0)$ then $\lim_{(x,y) \to (x_0, y_0)} f(x,y)$ does not exist.

Show that the function,

$$f(x,y) = \frac{2x^2 y}{x^4 + y^2}$$

has no limit as $(x,y) \to (0,0)$

Solving this we get —

$$\lim_{(x,y)\to(0,0)} f(x,y) \neq \lim_{(x,y)\to(0,0)} f(x,y)$$

$K = 1$

So, the limit does not exist.

Continuity of a function:

A function $f(x,y)$ is continuous at the point (x_0, y_0)

If f is defined at (x_0, y_0), $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists

and $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$.

A function $f(x,y)$ is called continuous in the domain D if $f(x,y)$ is continuous at each point of D.

Let's take an example

$$f(x,y) = \begin{cases} 2xy/x^2+y^2 & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

is continuous at every point except the origin.

→ So, solving this we get -

And for $(x,y) = (0,0)$ we have $f(x,y) = 0$

And for (x,y) we have -

$$f(x,y) = \frac{2xy}{x^2+y^2}$$

Clearly $f(x,y)$ is defined at $(0,0)$.

Set $y = mx$ for $x \neq 0$ then -

$$f(x,y) = \frac{2x\, mx}{x^2 + m^2x^2} \Rightarrow \frac{\cancel{x^2}(2m)}{\cancel{x^2}(1+m^2)}$$

Since y is variable so m has no fixed value

This means $f(x,y)$ may not have a fixed value as $(x,y) \neq (0,0)$.

This means $f(x,y)$ is not necessarily zero at $(x,y) \neq (0,0)$. That is $f(x,y)$ is not continuous at $(x,y) = (0,0)$.

But $f(x,y) = \dfrac{2xy}{x^2+y^2}$ for $(x,y) \neq (0,0)$ that is defined single form function except at $(0,0)$.

Partial Derivatives: (∂)

Partial Differentiation

Let $z = f(x,y)$ be a function of two variables x and y in Region R. If y is held constant then $f(x,y)$ becomes a function of x alone and its derivative (if exists) with respect to x, is called the partial derivative of $f(x,y)$ with respect to x.

Similarly if x is held constant then the derivative (if exists) of $f(x,y)$ with respect to y, is called the partial derivative of $f(x,y)$ with respect to y.

Thus, the partial derivative of a function (if exists) is equal as number of variables occur in the function

Definition (Partial Derivative of Two variables)

The partial derivative (if exists) of a function $f(x,y)$ with respect to x at a point (x_0, y_0) is

$$\frac{\partial f}{\partial x} = f_x = \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

Provided that the limit exists.

Similarly, the partial derivative of $f(x,y)$ with respect to y at (x_0, y_0) is

$$\frac{\partial f}{\partial y} = f_y = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

provided that the limit exists.

Example: find the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(4, -5)$ if -

$$f(x,y) = x^2 + 3xy + y - 1$$

Let,

$$f(x,y) = x^2 + 3xy + y - 1$$

finding $\frac{\partial f}{\partial x}$ Suppose y is constant value and differentiate f with respect to x, we get -

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3xy + y - 1)$$

$$\Rightarrow 2x + 3y$$

At $\frac{\partial f}{\partial x}(-4,2) \Rightarrow 2(-4) + 3 \times 2 \Rightarrow -8 + 6 \Rightarrow -2$

Example $\frac{\partial f}{\partial y}$ Keeping x - Constant and differentiate f w.r.t y we get -

$$\Rightarrow \frac{\partial f}{\partial y} \Rightarrow \frac{\partial (x^2 + 3xy + y - 1)}{\partial y}$$

$$\Rightarrow 3x + 1$$

At $\frac{\partial f}{\partial y}(-4, 2) \Rightarrow 3(-4) + 1 \Rightarrow -11$

So, $f_x(-4, 2) = \frac{\partial f}{\partial x}\Big|_{(-4,2)} \Rightarrow -2$ and $f_y(-4, 2) = \frac{\partial f}{\partial y}\Big|_{(-4,2)} \Rightarrow -11$

find $\frac{\partial f}{\partial y}$ if $f(x,y) = y \sin xy$.

$$f(x,y) = y \sin xy$$

Here to find $\frac{\partial f}{\partial y}$. for this, letting x as constant value and then opening derivative of f w.r.t to 'x' we get –

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y \sin xy)$$

$$= y \frac{\partial}{\partial y} \sin xy + \sin xy \frac{\partial}{\partial y}(y)$$

$$f_y \Rightarrow xy\cos xy + \sin xy$$

$$f_y = xy\cos xy + \sin xy$$

We have used Chain Rule -

$$\frac{\partial f}{\partial y} = y \cdot \frac{\partial}{\partial y}(\sin(xy)) + \sin xy \, \frac{\partial}{\partial y}(y)$$

$$\Downarrow$$

$$\frac{\partial}{\partial y}(u \cdot v) = u \cdot \frac{\partial v}{\partial y} + v \frac{\partial u}{\partial y} \qquad \left[\begin{matrix} u = y \\ v = \sin xy \end{matrix}\right]$$

Example: $\frac{\partial z}{\partial x}$ if $yz - \ln(z) = x+y$ where z is a function of x and y and the partial derivative exists.

Here,

$$yz - \ln(z) = x+y$$

Keeping y as a constant value and treating z as a differentiable function of x differentiate the above equation w.r.t x, partially then

$$\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial x}(\ln(z)) = \frac{\partial}{\partial x}(x+y)$$

$$\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial x}(\ln(z)) = \frac{\partial}{\partial x}(x+y)$$

or, $$\frac{\partial z}{\partial x} \cdot y - \frac{1}{z} \frac{\partial z}{\partial x} = 1$$

$$\left(y - \frac{1}{z}\right) \frac{\partial z}{\partial x} = 1$$

$$\boxed{\text{or, } \quad \frac{\partial z}{\partial x} = \frac{z}{yz-1}} \quad \#$$