Day-22, Dec-7, 2024 (Mangshir-22, 2081)

Relation Between Continuity and differentiability

Now let us see what we can expect of a function $f(x)$, if $f'(x)$ for $x = a$ exists, i.e. if $f'(a)$ is finite.

We can write,

$$\lim_{h \to 0} f(a+h) - f(a) = \lim_{h \to 0} \{f(a+h) - f(a)\}$$

$$\Rightarrow \lim_{h \to 0} \left\{ \frac{f(a+h) - f(a)}{h} \cdot h \right\}$$

$$\Rightarrow \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \cdot \lim_{h \to 0} h$$

$$\Rightarrow f'(a) \cdot 0 \quad \Rightarrow 0$$

or, $\lim_{h \to 0} f(a+h) = f(a)$

$\therefore f(x)$ is continuous at $x = a$.

Thus we have seen that the differentiability of a function at a point implies the continuity of the function at that point.

But the converse is not necessarily true i.e. the continuity of a function at a point does not generally imply the differentiability of the function at that point

Differentiable functions are continuous but the continuous function cannot be differentiable.

Consider the function $f(x)$ defined as

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

We have seen that this function is continuous at $x = 0$,

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$\Rightarrow \lim_{h \to 0} \frac{f(h)}{h} \qquad \left[\because f(0) = 0 \right]$$

$$\Rightarrow \lim_{h \to 0} \frac{h \sin 1/h}{h}$$

$$\Rightarrow \lim_{h \to 0} \sin 1/h$$ which is not defined.

So $f'(0)$ does not exist. Now what we have seen can be stated as -

"The continuity of a function at a point is the necessary but not the sufficient condition for the existence of the derivative of the function at that point."

Fundamental formula on Differentiation:

1) Power rule: $\frac{d}{dx}(x^n) = nx^{n-1}$

2) Sum Rule (or difference Rule): If 'u' and 'v' are the two functions of x, then $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$

3) <u>The product Rule:</u>

If 'u' and 'v' are the two functions of x, then

$$\frac{d}{dx}(u \times v) = u \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

4) <u>The general power rule:</u> If 'u' is the function of x, then

$$\frac{d}{dx}(u^n) = n u^{n-1} \cdot \frac{du}{dx}$$

5) <u>The quotient Rule:</u> If 'u' and 'v' are the two functions of x then,

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \cdot \frac{du}{dx} \cdot u \frac{dv}{dx}}{v2}$$

6) <u>The chain Rule:</u> if $y = g(u)$ where u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

7) $\frac{d}{dx}(K) = 0$, where K is constant

8) $\frac{d}{dx}(Ku) = K \cdot \frac{du}{dx}$ where K is constant and u is a function of x

9) <u>Implicit function:</u> A function which is of the type $f(x,y)=0$, is called the implicit function. The rules of differentiation of implicit function is given below:

Step (i): Differentiating both sides of the given implicit function w.r.t. 'x'

Step (ii): Solve the remaining function for $\frac{dy}{dx}$.

Now let us calculate the derivatives of the function x then deduce the derivative of x^n

i) Let $y = f(x) = x$

Let Δx be a small increment in 'x' and Δy be the corresponding small increment in y. Then –

$$y + \Delta y = x + \Delta x \qquad [y = x]$$

$$\Delta y = x + \Delta x - y$$

$$\Rightarrow x + \Delta x - x$$

$$[\Delta y \Rightarrow \Delta x]$$

or, $\dfrac{\Delta y}{\Delta x} = 1$

$$\therefore \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 1$$

Derivatives of Trigonometrical functions:

i) $\frac{d}{dx}(\sin x) = \cos x$

ii) $\frac{d}{dx}\cos x = -\sin x$

iii) $\frac{d}{dx}(\tan x) = \sec^2 x$

iv) $\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$

v) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x$

vi) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x.$

REFERENCE:

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