Day-29, Dec-17, 2024 (Poush-2, 2081)

We define a new function $g(x)$ involving $f(x)$ so that $g(x)$ satisfy the Rolle's Theorem in $[a,b]$. So, consider

$$g(x) = f(x) + Ax,$$ where A is constant such that $g(a) = g(b)$.

$$g(a) = g(b) \Rightarrow f(a) + Aa$$
$$\Rightarrow f(b) + Ab$$
$$\Rightarrow Aa - Ab = f(b) - f(a)$$

$$\Rightarrow -A = \frac{f(b) - f(a)}{b-a}$$

Since, Ax is continuous everywhere hence is continuous on $[a,b]$. Also, $f(x)$ is continuous on $[a,b]$ by given condition.

$\therefore$ $f(x) + Ax = g(x)$ also continuous on $[a,b]$. Similarly, $f(x)+Ax = g(x)$ also differentiable on (a,b).

Moreover, we have $g(a) = g(b)$.

Thus, $g(x)$ satisfies all three condition of Rolle's theorem, so by the Rolle's theorem there exists at least a point $c \in (a,b)$

Such that

$$g'(c) = 0$$

$$\Rightarrow f'(c) + A = 0$$

$$\Rightarrow f'(c) = -A$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b-a}$$

This completes the proof.

Geometrical Meaning of Mean Value Theorem:

If a function $f(x)$ is

i) Continuous in the closed interval $[a,b]$;

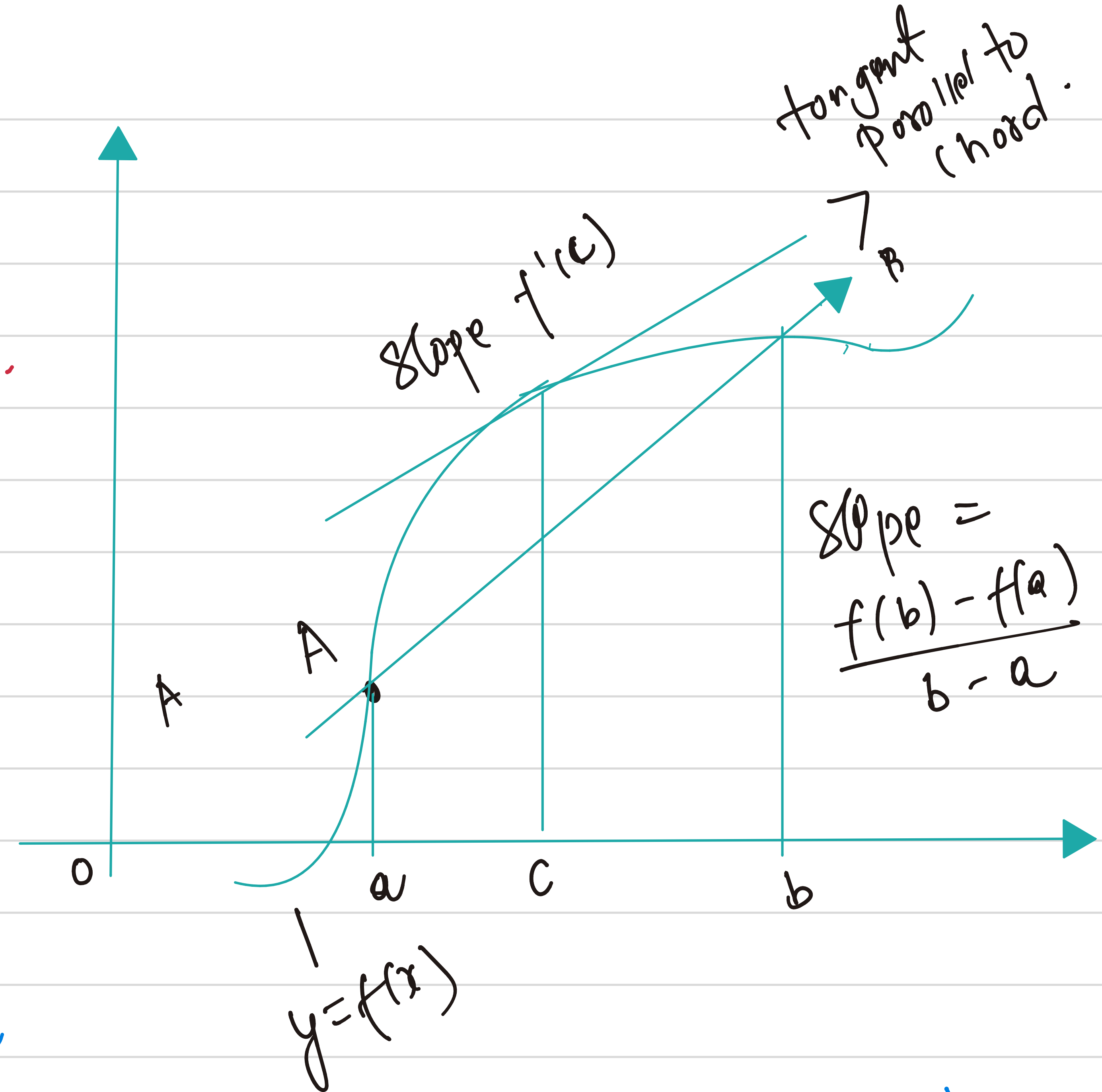
ii) Differentiable in the open interval (a,b).

Verify the mean value theorem for function on $f(x) = 1 - x^2$ on $[0, 2]$.

Since, $f(x) = 1 - x^2$ is continuous on $[0, 2]$ and $f'(x) = -2x$ So, differentiable on $(0, 2)$.

Thus, $f(x) = 1 - x^2$ satisfy the both conditions for mean value theorem. So, there exist $c \in (0, 2)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



$$\Rightarrow -2c = \frac{f(2) - f(0)}{2-0}$$

$$\Rightarrow -2c = \frac{-3-1}{2}$$

$$\Rightarrow -2c = -1$$

$$c = \frac{1}{2} \in (0,2)$$

$\therefore$ Hence, mean value theorem satisfied

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Functions with zero derivatives are constant. OR

If $f'(x) = 0$, $\forall x \in I$; then $f(x) = C$ $\forall x \in I$, where C is constant.

Proof: To show $f(x) = c \ \forall x$ in I (ie. function is constant), where $f'(x) = 0 \ \forall x \in I$, we show

$$f(x_1) = f(x_2) \text{ for any } x_1, x_2 \in I \text{ with } x_1 < x_2.$$

Let $x_1, x_2 \in I$ with $x_1 < x_2$. Then $f(x)$ is differentiable on $[x_1, x_2]$ because given that $f'(x) = 0 \ \forall x \in I$.

Also, $f(x)$ is continuous on $[x_1, x_2]$ because every differentiable function is continuous.

Thus, both condition of mean value theorem is satisfied on $f(x)$, hence $\exists\, c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\Rightarrow 0 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\Rightarrow f(x_1) - f(x_2) = 0$$

$$\Rightarrow f(x_1) = f(x_2).$$

So, $f(x)$ is constant function.

We must careful in applying above theorem,

Let
$$f(x) = \frac{|x|}{x} \Rightarrow \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Then the domain of $f(x)$ is $D = \{x : x \neq 0\}$ and $f'(x) = 0$ for all x in D. Clearly if f is not a constant function on D.

This does not contradict to above theorem because D is not on interval. Notice that f is a constant function in $(0, \infty)$ and also in $(-\infty, 0)$.

\# Suppose that 'f' is continuous on $[a,b]$ and differentiable on (a,b).

i) If $f'(x) > 0$ at each $x \in (a,b)$ then f is increasing on $[a,b]$.

ii) If $f'(x) < 0$ at each $x \in (a,b)$ then f is decreasing on $[a,b]$.

Proof:

Let x_1 and x_2 in $[a,b]$ with $x_1 < x_2$. Here $f(x)$ is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2) so by mean value theorem of $f(x)$ on $[x_1, x_2]$; $\exists\, c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\Rightarrow f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

$$x_1 < x_2 \Rightarrow x_2 - x_1 > 0.$$

i) If $f'(x) > 0$ at each $x \in (a,b) \Rightarrow f'(c) > 0$ then RHS of (A) is greater than 0.

Hence, from (A),

$$f(x_2) - f(x_1) > 0$$

$$\Rightarrow f(x_2) > f(x_1)$$

Thus for $x_1 < x_2$ we find $f(x_2) > f(x_1)$. This means $f(x)$ is increasing.

ii) If is $f'(x) < 0$ at each $x \in (a, b) \Rightarrow f'(c) < 0$ then RHS of (A) is less than 0.

Hence, from (A)

$$f(x_2) - f(x_1) < 0$$

$$\Rightarrow f(x_2) < f(x_1)$$

Thus, for $x_1 < x_2$ we find $f(x_2) < f(x_1)$. This means $f(x)$ is decreasing.

Increasing

$$\left[\underline{f'(x) > 0 \quad \text{and} \quad f'(x) < 0} \right] \rightarrow \text{decreasing.}$$

Example: Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x. How large can $f(2)$ possibly be?

Suppose that,

$$f(0) = -3 \quad \text{and} \quad f'(x) \leq 5 \quad \text{for all values of } x.$$

This implies f is differentiable (and therefore is continuous) everywhere.

In particular, we choose f is defined on $[0, 2]$. Then by Mean value theorem on $[0, 2]$, there exists a number c in $(0, 2)$ such that

$$f'(c) = \frac{f(2) - f(0)}{2 - 0}$$

$$\Rightarrow f(2) = 2f'(c) - 3 \leq 2(5) - 3$$

$$\Rightarrow 10 - 3$$

$$\Rightarrow 7.$$

#. This shows the largest possible values for $f(2)$ is 7.