Day-1, Oct-16, 2024 (Mangshir 1, 2081 BS)

Limits and Continuity

→ Limits & Continuity are fundamental for the development of calculus.

→ Before the continuity, comes the limit.

Function

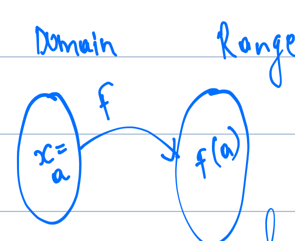
Let X and Y are two non-empty sets. Then function-

$y = f(x)$ or $f: X \to Y$ means a function from X to Y.

→ $f: X \to Y$ is a rule which assigns a unique element of Y to each element of X.

→ The element $f(x)$ of Y is called the Image of 'x' under the function f.

Value of the Function



Domain of f has $x = a$ element then the image $f(a)$ corresponding to $x = a$ is said to be the value of a function at $x = a$.

Example: I $y = f(x) = 6x + 5$ exists or defined at $x = 2$ as

$\Rightarrow 6 \times 2 + 5 \Rightarrow 17$ is a finite number

II $y = f(x) \Rightarrow \dfrac{1}{x - 1}$ is not defined at $x = 1$ as

$f(1) \Rightarrow \dfrac{1}{0}$ is not a finite number.

Hence $f(x)$ doesn't exist or is undefined at $x = 1$.

III Indeterminate form $\left(\frac{0}{0}\right)$

$$y = f(x) = \frac{x^2 - 1}{x - 1} \text{ and } x = 1$$

When $x = 1$, $y = f(0) \Rightarrow \dfrac{0}{0}$ (Indeterminate Form)

Other forms of Indeterminate forms are $\frac{\infty}{\infty}$, $\infty - \infty$, 1^{∞} and 0^{∞}.

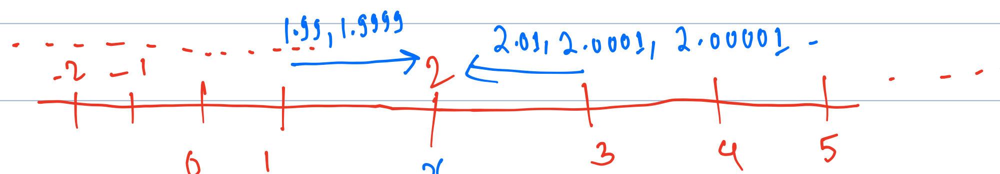
Meaning of $x \to a$

Let 'x' be a variable and 'a' is a constant number. If 'x' takes a value such that the numerical difference between 'x' and 'a' is sufficiently small, then we say that 'x' tends to 'a' and is written as $x \to a$.

Intuitive Idea of $x \to a$

Let 'x' be a variable and if x takes values 1.9, 1.99, 1.999, 1.9999 ...... The idea is x is nearer to 2 but never be 2 with the very small increment so $x \to 2$ x tends to 2.

Similarly if x takes values 2.1, 2.01, 2.001, 2.0001 but due to very small value it is nearer to 2 but never 2 so $x \to 2$.

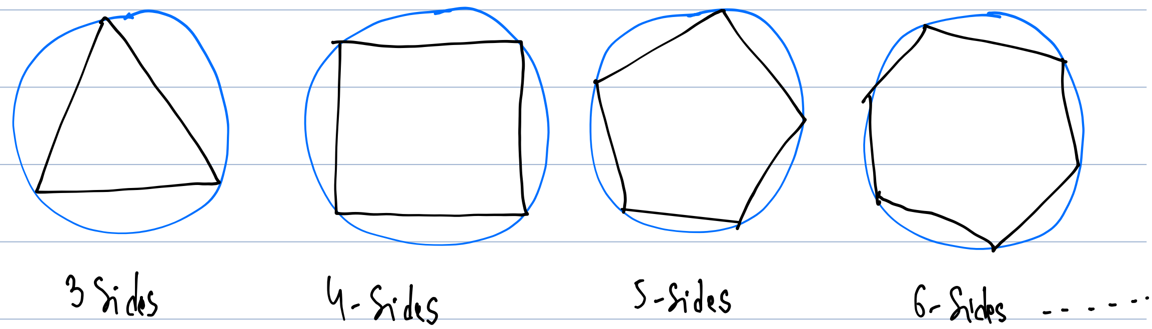


$\boxed{x \to 2}$

Example and Intuitive Idea of Limit.

Take polygon inside the circle, the 'n' sides makes the types of polygon. place polygon inside the circle such that

[Area of polygon inside the circle < Area of Circle.] Area of circle is the limit of a series of areas (or perimeters) of the polygon obtained by giving the sequence of integral values



Area of a regular polygon with 'n' sides inscribed in a finite circle by $A_n, n > 2$. Then the set of ordered pairs (n, A_n) gives the increasing sequence of areas of polygons

$$\left[\therefore f(n) = A_n \right]$$

when $n \to \infty$, $f(n)$ or A_n gets close to the area of the circle,

$\left[\lim_{n \to \infty} f(n) \text{ or } \lim_{n \to \infty} A_n \right]$ i.e. the limiting value of $f(n)$ or A_n.

Limit in the form of Sequence

$0.9,\ 0.99,\ 0.999,\ 0.9999,\ 0.99999 \ldots\ldots\ldots$

So the sequence tends to the limiting value 1

Putting in the notational sequence by definition of the function f by

$f(1) = 0.9,$ 1st term

$f(2) \Rightarrow 0.99,$ 2nd term

$f(3) \Rightarrow 0.999,$ 3rd term

$\vdots$

$f(n) \Rightarrow 0.99\ldots9$ (n 9's) the n'th term.

$\boxed{\lim_{n\to\infty} f(n) = 1}$ can be defined as when 'n' tends to infinity, $f(n)$ becomes almost equal to 1 if only $f(n)$ is 1.

This part can also be concluded with following Sequence-

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots\ldots$$

no matter how much or many terms added it never reaches to the absolute 2 but gets nearer to 2 so, functional notation is -

$$S_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \ldots\ldots \text{ to 'n' terms}$$

$$\Rightarrow 1 + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \ldots \text{ to 'n' terms}$$

$$\Rightarrow \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}}$$

$$\Rightarrow \frac{\frac{2^n - 1}{2^n}}{1/2} \qquad \Rightarrow \frac{2^n - 1}{2^n} \times \frac{2}{1}$$

$$\Rightarrow 2 - \frac{1}{2^{n-1}}$$

$$\Rightarrow \frac{2 \times 2^{n-1} - 1}{2^{n-1}}$$

$$\Rightarrow \frac{4^{n-1} - 1}{2^{n-1}}$$

$$\boxed{\text{So, } S_n = 2 - \frac{1}{2^{n-1}}}$$

when 'n' is sufficiently large, $\frac{1}{2^{n-1}}$ is sufficiently small,

$n \to \infty$ then $S_n \to 2$ i.e. limiting value of S_n is 2.

$$\boxed{\lim_{n \to \infty} S_n = 2}$$

REFERENCES:

D.R. Bajracharya, R.M. Shrestha, M.B. Singh, Y.R Sthapit & B.C. Bajracharya, 2014, Basic Mathematics Grade XI, Sunkunda Pustak Bhawan Bhotahity Kathmandu.

Day 1 of 100DaysOfMaths.