Day-23, Dec-8, 2024 (Mangshir-23, 2081 BS.)

Derivatives of Inverse Circular functions (Comprehensive)

i) $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

ii) $\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$

iii) $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

iv) $\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$

v) $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$

vi) $\frac{d}{dx}(\operatorname{cosec}^{-1}x) = \frac{-1}{x\sqrt{x^2-1}}$

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Example:

Let $y = f(x) = x^2$

Let Δx be a small increment in x and Δy be the corresponding small increment in y. Then

$$y + \Delta y = (x + \Delta x)^2$$

$$\Delta y \Rightarrow x^2 + 2x \cdot \Delta x + \Delta x^2 - y$$

$$\Rightarrow \cancel{x^2} + 2x \cdot \Delta x + \Delta x^2 - \cancel{x^2}$$

$$\Delta y = 2x \cdot \Delta x + (\Delta x)^2$$

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x) \Rightarrow 2x$$

Example: Let $y = f(x) = x^3$.

Let Δx be a small increment in 'x' and Δy be the Corresponding Small increment in y. Then

$$y + \Delta y = (x + \Delta x)^3$$

$$\Delta y \Rightarrow \cancel{x^3} + 3x^2 \Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - \cancel{x^3}$$

$$\Delta y \Rightarrow 3x^2 \cdot \Delta x + 3x(\Delta x)^2 + (\Delta x)^3$$

$$\frac{\Delta y}{\Delta x} = 3x^2 + 3x \cdot \Delta x + \Delta x^2$$

$$\therefore \frac{dy}{dx} = 3x^2 + 3x \cdot \Delta x + \Delta x^2$$

$$\Rightarrow \lim_{\Delta x \to 0} \left[(3x^2 + 3x \cdot \Delta x + (\Delta x)^2 \right]$$

$$\Rightarrow 3x^2.$$

Thus, we see that,

the derivative of $x = \frac{dx}{dx} = x^{1-1} \Rightarrow 1$

the derivative of $x^2 = \frac{dx^2}{x} \Rightarrow 2x^{2-1} \Rightarrow 2x$

the derivative of $x^3 = \frac{dx^3}{x} \Rightarrow 3x^{3-1} \Rightarrow 3x$

Then these results, we can conclude that

the derivative of $x^n = \frac{dx^n}{dx} \Rightarrow nx^{n-1}$

This formula holds not for only natural numbers, it holds for any number.

Derivative of logarithmic functions

i) $\dfrac{d}{dx}(\log x) = \dfrac{1}{x}$

ii) $\dfrac{d}{dx}(e^{x}) = e^{x}$

iii) $\dfrac{d}{dx}(a^{x}) = a^{x} \log_{e} a$

iv) $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x} \log_a e$

Properties of logarithmic function

i) $\log_a(x, y) = \log_a x + \log_a y$

ii) $\log_a \dfrac{x}{y} = \log_a x + \log_a y$

iii) $\log_a x^{n} \Rightarrow n \log_a x$

iv) $\log_a a = 1$

v) $\log_a m = \log_a b \times \log_b m$

vi) $\log_a 1 = 0.$

Example: Let $y = f(x) = c$, a constant

Then, $y + \Delta y = c$

$$\Delta y = c - y$$
$$\Rightarrow c - c$$
$$\Rightarrow 0.$$

$$\frac{\Delta y}{\Delta x} = 0.$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \Rightarrow 0.$$

Derivative of x^n From first Principles (Using Limit Theorem)

Let $y = x^n$ —— eqn (i).

Let Δx be a small increment in x and Δy, a corresponding increment in y. Then,

$$y + \Delta y = (x + \Delta x)^n \quad \text{--- eqn (ii)}$$

Subtracting eqn (i) from eqn (ii) we get

$$\Delta y = (x + \Delta x)^n - x^n$$

or, $$\frac{\Delta y}{\Delta x} \Rightarrow \frac{(x + \Delta x)^n - x^n}{\Delta x}$$

Now, $$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{(x + \Delta x)^n - x^n}{(x + \Delta x) - x} \qquad \left(\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1} \right)$$

$$\Rightarrow \lim_{(x + \Delta x) \to x} \frac{(x + \Delta x)^n - x^n}{(x + \Delta x) - x} \Rightarrow nx^{n-1}$$

<u>Example</u>: find from first principles the derivative of $(ax+b)^n$.

Solution :

$$\text{Let } y = (ax+b)^n$$

Let Δx and Δy be the small increment in 'x' and 'y' respectively.

Therefore

$$y + \Delta y = \{ a(x+\Delta x) + b \}^n$$

$$\Delta y \Rightarrow (ax + a\Delta x + b)^n - (ax+b)^n$$

$$\frac{\Delta y}{\Delta x} \Rightarrow \frac{(ax + a\Delta x + b)^n - (ax+b)^n}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \Rightarrow \lim_{\Delta x \to 0} \frac{(ax + a\Delta x + b)^n - (ax+b)^n}{\Delta x}$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{(ax + a\Delta x + b)^n - (ax+b)^n}{(ax + a\Delta x + b) - (ax+b)} \cdot a$$

$$\Rightarrow \lim_{(ax+a\Delta x+b)\to (ax+b)} \frac{(ax+a\Delta x+b)^n-(ax+b)^n}{(ax+a\Delta x+b)-(ax+b)}\cdot a$$

$$\Rightarrow n\cdot(ax+b)^{n-1}\cdot a \quad \left(\because \lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}\right)$$

$$\boxed{\frac{dy}{dx} \Rightarrow n\cdot a(ax+b)^{n-1}}$$

Find out the derivative of $\frac{1}{\sqrt{x+2}}$

Let δx be a small increment in x and Δy be the corresponding small increment in y. Then-

$$y + \Delta y \Rightarrow \frac{1}{\sqrt{(x + \Delta x + 2)}}$$

$$\Delta y \Rightarrow \frac{1}{(\sqrt{x + \Delta x + 2}} - \frac{1}{\sqrt{x + 2}}$$

$$\Rightarrow \frac{\sqrt{(x+2)} - \sqrt{(x + \Delta x + 2)}}{\sqrt{(x + \Delta x + 2)}\ \sqrt{(x+2)}}$$

$$\Rightarrow \frac{\left[\sqrt{x+2} - \sqrt{x + \Delta x + 2}\right]\left[\sqrt{x+2} + \sqrt{x + \Delta x + 2}\right]}{\sqrt{x + \Delta x + 2}\ \sqrt{x+2}\ \left[\sqrt{x+2} + \sqrt{x + \Delta x + 2}\right]}$$

$$\Rightarrow \frac{x + 2 - x - \Delta x - 2}{\sqrt{x + \Delta x + 2}\ \sqrt{x+2}\ \left[\sqrt{x+2} + \sqrt{x + \Delta x + 2}\right]}$$

$$\Rightarrow \frac{-\Delta x}{\sqrt{x+\Delta x+2}\ \sqrt{x+2}\ \left[\sqrt{x+2}+\sqrt{x+\Delta x+2}\right]}$$

$$\Rightarrow \frac{-1}{\sqrt{x+\Delta x+2}\ \sqrt{x+2}\ .\left[\sqrt{x+2}+\sqrt{x+\Delta x+2}\right]}$$

$$\therefore \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{-1}{\sqrt{x+\Delta x+2}\ .\sqrt{x+2}\ .\left[\sqrt{x+2}+\sqrt{x+\Delta x+2}\right]}$$

$$= \frac{-1}{2(x+2)^{3/2}}$$