Day-27, Dec-15, 2024 (Mangshir-30, 2081)

Implicit Differentiation

Sometimes the functions may define implictly by a relation between two different variables (like between x and y) as –

$$x^2 + y^2 = 6xy$$

In such case, we can use the method of implicit differentiation. This consists the derivative of both sides with respect to 'x' and the solve the result for final solution of $\frac{dy}{dx}$.

Example: If $x^2+y^2=6xy$ and then find $\frac{dy}{dx}$.

Let,

$$x^2+y^2=6xy$$

Differentiating w.r.t. x then,

$$2x+2y\frac{dy}{dx}=6\left(x.\frac{dy}{dx}+y\right)$$

$$\Rightarrow (2y-6x).\frac{dy}{dx}=6y-2x$$

$$\Rightarrow \frac{dy}{dx}=\frac{6y-2x}{2y-6x}$$

$$\boxed{\frac{dy}{dx}\Rightarrow\frac{3y-x}{y-3x}}$$

Example: Differentiate $y = x^{\sqrt{x}}$.

Let $y = x^{\sqrt{x}}$ then

$$\ln(y) = \sqrt{x}\ \ln(x)$$

Differentiate it with respect to 'x' then

$$\left(\frac{1}{y}\right) y' = \sqrt{x}\left(\frac{1}{x}\right) + \ln(x)\left(\frac{1}{2}\right)x^{-1/2}\cdot(1)$$

$$\Rightarrow y' = y\left(\frac{1}{\sqrt{x}} + \frac{\ln(x)}{2\sqrt{x}}\right)$$

$$\Rightarrow \left(x^{\sqrt{x}}\right)\left(\frac{2+\ln(x)}{2\sqrt{x}}\right)$$

$\rightarrow y = x^{\sqrt{x}}$

$\rightarrow \ln y = \ln \left(x^{\sqrt{x}} \right)$

$\rightarrow \ln y = \sqrt{x} \ln x$

$\rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{d}{dx} \left(\sqrt{x} \ln x \right)$

$\rightarrow u = \sqrt{x}$ and $v = \ln x$

$$\frac{d}{dx}(uv) = u'v + uv'$$

$u = \sqrt{x} = x^{1/2}$ so $u' = \frac{1}{2}x^{-1/2}$

$v = \ln x$, so, $v' = 1/x$

So, Substituting:

$$\frac{d}{dx}\left(\sqrt{x}\ln x\right) = \underbrace{\frac{1}{2}x^{-1/2}\ln x + \sqrt{x}\cdot 1/x}_{(u'v \;+\; uv')}$$

$$\frac{dy}{dx} = y\left(\frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}}\right)$$

Mean Value Theorem

→ Derivative of any constant function is zero.

→ More Complicated function whose derivative is always zero?

Mean Value theorem answers the question if two functions have identical derivatives over are related or not?

Extreme Value Theorem:

If 'f' is continuous on a closed interval $[a, b]$, then 'f' attains an absolute maximum value $f(c)$ and an absolute minimum $f(d)$ at some points 'c' and 'd' in $[a, b]$.

Fermat's Theorem:

If 'f' has a local maximum (or minimum) at 'c' and if 'f' is differentiable at c then $f'(c) = 0$.

Rolle's Theorem:

Let f be a function that satisfies the following three Hypothesis.

i) f is continuous on the closed interval $[a, b]$.

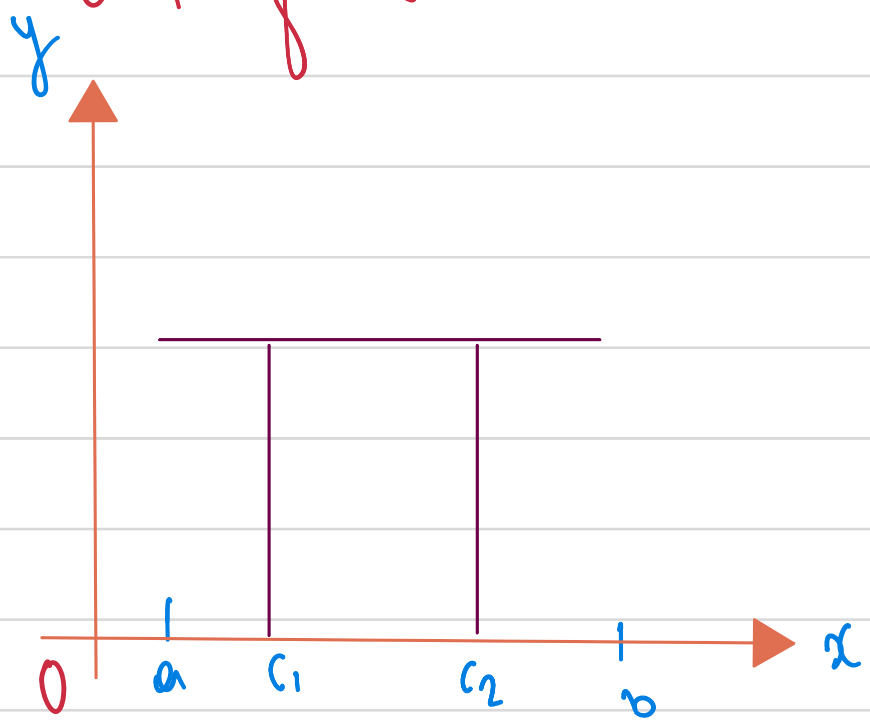
ii) f is differentiable on the open interval (a, b)

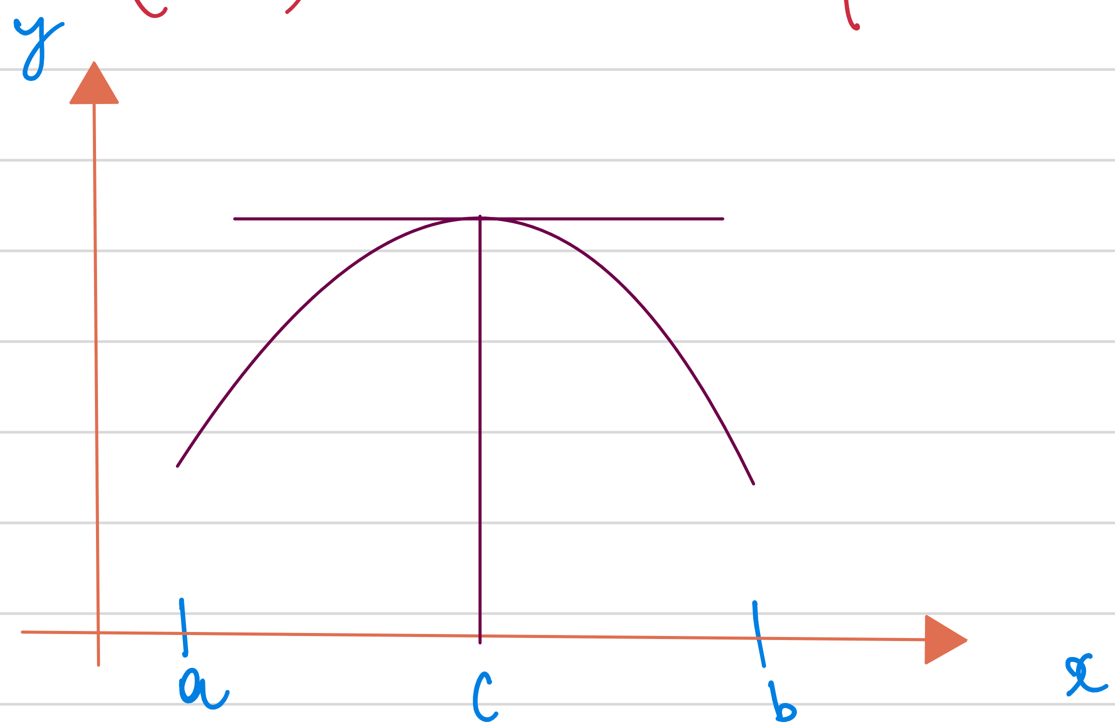
iii) $f(a) = f(b)$

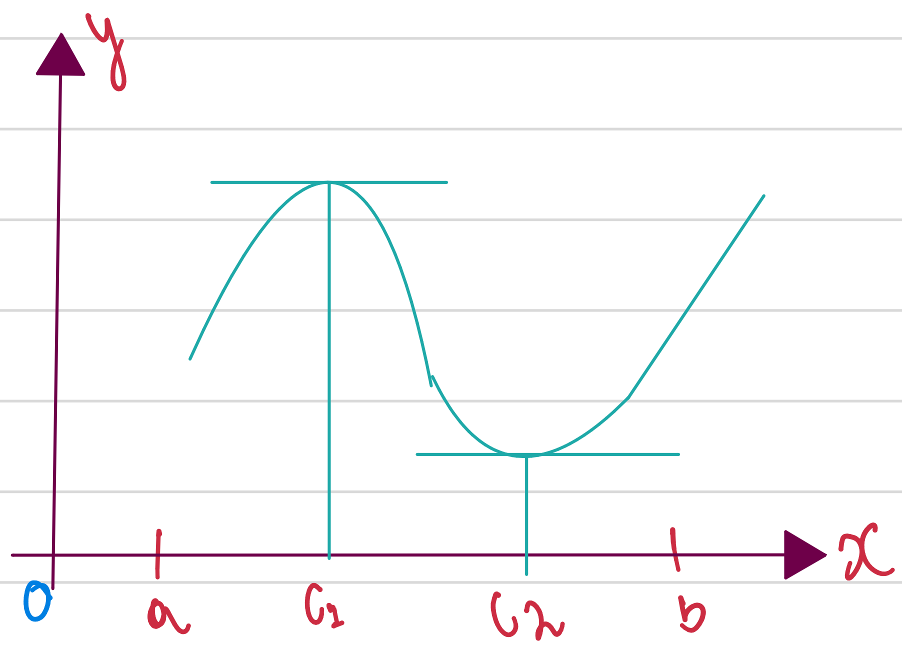
Closed interval $[a, b] = [2, 5]$ numbers between 2 and 5 including themselves

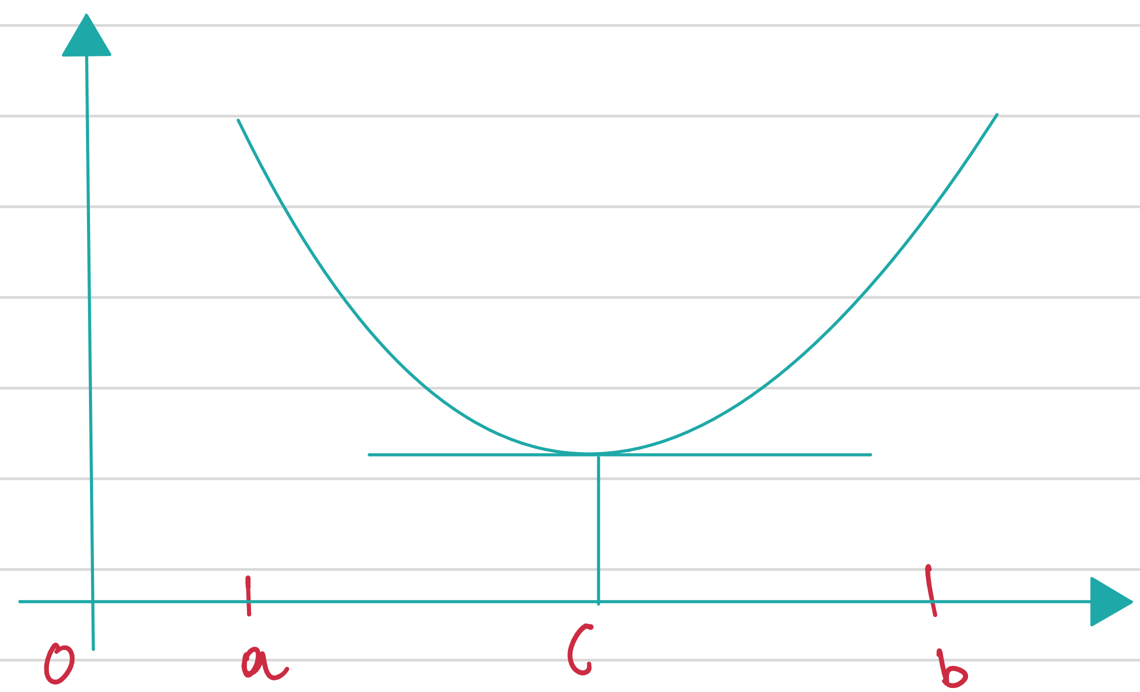
Open interval $(a, b) = (2, 5)$ numbers between 2 and 5 but not themselves

So, if there is a number c in (a,b) such that $f'(c)=0$.









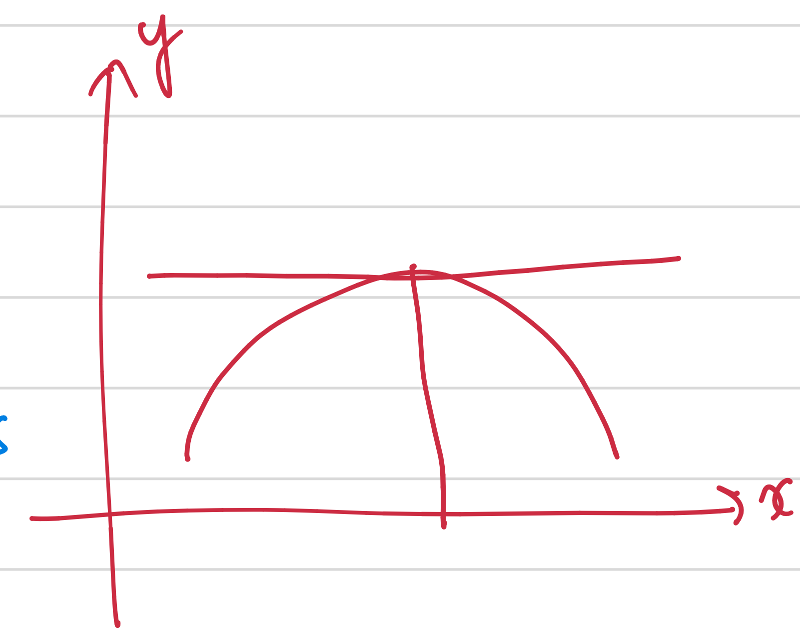
Let f satisfies the given Hypothesis (i) (ii) and (iii).

There are three cases.

Case 1: $f(x) = K$ (K is is a constant) →

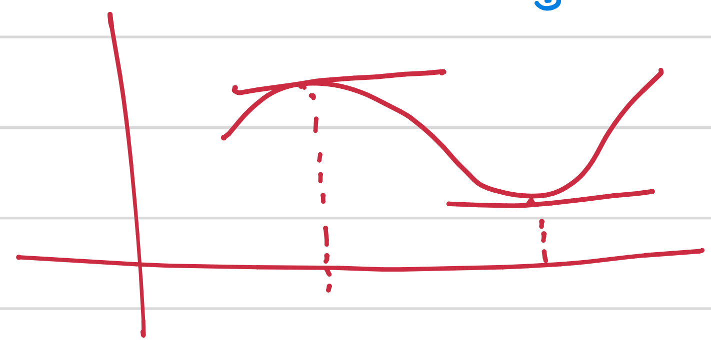
if $f(x) = k$ (k is a constant) for x in $[a, b]$ then $f'(x) = 0$, for x in (a, b). So, at $x = c$ for any x in (a, b), $f'(c) = 0$.

Case 2: If $f(x) > f(a)$ for some x in (a, b) then by extreme value theorem, f has a maximum value somewhere in $[a, b]$ as hypothesis



i) Since by iii) $f(a) = f(b)$, so f has maximum value somewhere in (a, b). Let 'f' has local maximum at c in (a, b)

Since by Hypothesis (ii) f is differentiable at c. Therefore by Fermat theorem, $f'(c) = 0$.



Case III: $f(x) < f(a)$ for some x in (a, b)

Let $f(x) < f(a)$ for some x in (a, b) then by the Extreme Value theorem f has a minimum value in $[a, b]$. Since $f(a) = f(b)$, Let f attains its minimum value at c in (a, b). Therefore by Fermat's Theorem $f'(c) = 0$.

Thus, in either case, if f satisfies the Hypothesis (i) (ii) (iii) then $f'(c) = 0$ for some c in (a, b).