Day-33, Dec-21, 2024 (Poush-6, 2081 B.S.)

Geometric Descriptions of R^2 and R^n

Consider a rectangular coordinate system in a plane. Since each point in the plane is determined by an ordered pairs of numbers. This means the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ in R^2 is a point in R^2.

Vector in R^2 is a set of all points in the plane $\begin{bmatrix} x \\ y \end{bmatrix}$.

Similar the vector R^3 is the set of all points in Space because R^3 contains vector

$\mathbb{R}^3$ entries $u = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ since space contains three entries.

As similar, the vector of $\mathbb{R}^n$ is the set of all points in n-dimensional space where each vector represented as

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

Algebraic Properties of $\mathbb{R}^n$

For all u, v and w in $\mathbb{R}^n$ and all scalars c and d,

i) $u+v = v+u$

ii) $c(u+v) = cu + cv$

iii) $(u+v)+w = u+(v+w)$

iv) $(c+d)u = cu + du$

v) $u+0 = 0+u = u$

vi) $c(du) = d(cu) = (cd)u$

vii) $u + (-u) = u - u = 0$

viii) $1 \cdot u = u \cdot 1 = u$.

Definition (Linear Combinations)

Given vectors $u_1, u_2, u_3 \cdots u_n$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, c_3 \cdots c_n$ and if the vector y in $\mathbb{R}^n$ is defined by

$$\left[y = c_1 u_1 + c_2 u_2 + \cdots + c_n u_n \right]$$

Then y is called a Linear Combination of $u_1, u_2 \cdots u_n$ with $c_1, c_2 \cdots c_n$.

In such case, the scalars $c_1, c_2 \cdots c_n$ are called weights for the combinations -

$$y = c_1\vec{u_1} + c_2\vec{u_2} + c_3\vec{u_3} + \cdots + c_n\vec{u_n}$$

where c_1 is scalar and called a weight.

Note: $y \in \mathbb{R}^n$ is linear combination of vectors $u_1, u_2, u_3 \cdots u_n$ in $\mathbb{R}^n$. if linear system with Augmented matrix $[u_1\ u_2 \cdots u_n\ y]$ represents consistence system i.e. equation $x_1u_1 + x_2u_2 + \cdots + x_nu_n = y$ is consistence.

Definition (Subset of R^n Spanned by Vectors)

If $u_1, u_2, u_3 \dots u_n$ are in R^n then the set of all linear combinations of the vectors is denoted by Span $\{u_1, u_2 \cdots u_n\}$ and is called subset of R^n spanned (or generated) by vectors $u_1, u_2, u_3 \dots u_n$.

Thus, the span $\{u_1, u_2 \dots u_n\}$ can be written with weights $c_1, c_2 \dots c_n$ as

$$c_1u_1 + c_2u_2 + \dots + c_nu_n \quad \text{———— eqn (i)}$$

Span $\{v_1, v_2 \dots v_n\}$ contain zero vector also, because-

$$0 = 0v_1 + 0v_2 + \dots + 0v_n.$$

Example: Let $u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, show that $\begin{bmatrix} h \\ k \end{bmatrix}$ is the Span $\{u, v\}$ for all h and k.

Given, $u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $b = \begin{bmatrix} h \\ k \end{bmatrix}$

The Augmented matrix form of u, v and b is,

$$[u \quad v \quad b] = \begin{bmatrix} 2 & 2 & h \\ -1 & 1 & k \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 2 & h \\ 0 & 4 & 2k+h \end{bmatrix} \quad \because R_2 \to 2R_2 + R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & h/2 \\ 0 & 1 & (2k+h)/4 \end{bmatrix} \rightarrow R_2 \rightarrow R_2/4$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & (h-2k)/4 \\ 0 & 1 & (h+2k)/4 \end{bmatrix} \because R_1 \rightarrow R_1 - R_2$$

$\therefore$ This shows that the solution of the Augmented Matrix exists for all values of h and k. So, b is span $\{u, v\}$.

<u>Note</u>: If the augmented matrix form of linear combination is <u>inconsistent then the vector b does not span the given vectors</u>.

Linear and One-dimensional -

$$c_1 \vec{v} \quad \text{with } c_1 \text{ scalar } \vec{v} \neq [0] \quad \text{— eqn (i)}$$

This eqn (i) represents line through origin —

$\therefore$ Geometrical Representation of span $\{v\}$ is a line in $\mathbb{R}^3$ containing origin —

Similarly if u, v are non-zero vectors in $\mathbb{R}^3$ then span $\{u, v\}$ be a set of all vectors in the form

$$c_1 \vec{u} + c_2 \vec{v} \quad \text{———} \quad \text{eqn (ii)}$$

with c_1 and c_2 scalars and $\vec{u}$ and $\vec{v}$

and is two-dimensional — represents a plane containing the origin.

Example: Let $a_1 = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$, $a_2 = \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}$ and $b = \begin{bmatrix} 4 \\ 1 \\ h \end{bmatrix}$

For what value of h is b in the plane spanned by a_1 and a_2?

Here —

if $x_1 a_1 + x_2 a_2 = b$ has solution, then b is the plane spanned by a_1 and a_2.

Let b is in the plane spanned by a_1 and a_2. So, the eqn

$$Ax = b$$

has solution $A = [a_1 \ a_2]$

So if we solve above example $h = -17$ for $h = -17$ the vector b is in the plane spanned by a_1 and a_2.

Matrix Equation [$Ax = b$]

Definition: If 'A' is an $m \times n$ matrix with columns $a_1, a_2 \ldots a_n$ and if x is in $\mathbb{R}^n$ then Ax is defined as

$$Ax = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\Rightarrow a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

This product Ax is possible only if the total number entries of A and x are same.

Rewrite the following System of linear equation as the form $Ax = b$.

$$x_1 + 2x_2 - x_3 = 4$$

$$-5x_2 + 3x_3 = 1.$$

$$\left[\begin{array}{ccc:c} 1 & 2 & -1 & 4 \\ 0 & -5 & 3 & 1 \end{array}\right]$$

$$Ax = b$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Thus the solution exists for x_1 and x_2.

Theorem If A is an $m \times n$ matrix with columns $a_1, a_2 \dots a_n$ and b is in $\mathbb{R}^m$ the matrix equation $Ax = b$ has the same solution set as the vector equation.

$$x_1 a_1 + x_2 a_2 + \dots + x_n a_n = b$$

which in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} a_1 & a_2 & \dots & a_n & b \end{bmatrix}.$$

$x_1 a_1 + x_2 a_2 + \dots + x_n a_n = b$ $\qquad [Ax = b]$

b is in $\mathbb{R}^m$ $\quad$ A is $m \times n$ then the same solution set as $\begin{bmatrix} a_1 & a_2 & \dots & a_n & b \end{bmatrix}$.

The theorem pointed that a problem can be represented in 3 ways:

i) as a matrix equation
ii) as a vector equation
iii) as a system of linear equation

So we can switch from one formulation of problem into another whenever it is convenient.

Existence of Solutions:

$Ax = b$ has solution if and only if b is a linear combination of the columns of A. Note that if $[0 \; 0 \; 0 \; 1]$ then the system is inconsistent being the last column is an pivot column

Theorem: Let A be an $m \times n$ matrix. Then the following statements are equivalent.

a) For each b in $\mathbb{R}^m$, the equation $Ax = b$ has a solution

b) Each b in $\mathbb{R}^m$ is a linear combination of the columns of A.

c) The columns of A span $\mathbb{R}^m$.

d) 'A' has a pivot position in every row. [only for the Coefficient of matrix 'A' but not for the Augment matrix $Ax = b$).

Let $A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix}$

i) Does the eqn $Ax = b$ has a solution for each $b \in R^4$.

ii) Does column of A span R^4.

iii) Can each $b \in R^4$ is linear combination of column of A

Solution,

All above equations answer will given by knowing 'A' has pivot position in every row or not ?

Here, solving we get - $\begin{bmatrix} \boxed{1} & 3 & 0 & 3 \\ 0 & \boxed{2} & -1 & 4 \\ 0 & 0 & \boxed{0} & 0 \\ 0 & 0 & 0 & \boxed{5} \end{bmatrix}$

3rd Row has no Pivot $\square$

So, i) $Ax = b$ has no solution for each $b \in \mathbb{R}^4$
ii) Column of A doesn't Span $\mathbb{R}^4$.
iii) Each $b \in \mathbb{R}^4$ can't be the linear Combination of column of A.

Theorem 5: If A is an $m \times n$ matrix, u and v are vectors in $\mathbb{R}^n$ and c is scalar then

i) $A(u+v) = Au + Av$ ii) $A(cu) = c(Au)$.

Let $A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}$ $v_1 = \begin{bmatrix} 0 \\ -3 \\ 8 \end{bmatrix}$ $v_2 = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix}$

and $c = 5$

$$Av_1 = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 8 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0+9-32 \\ 0-6+48 \\ 0+3-64 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -23 \\ 42 \\ -61 \end{bmatrix}$$

$$Av_2 = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 4+3-20 \\ -12-2+30 \\ 20+1-40 \end{bmatrix} \Rightarrow \begin{bmatrix} -13 \\ 16 \\ -19 \end{bmatrix}$$

Also, $v_1 + v_2 = \begin{bmatrix} 0 \\ -3 \\ 8 \end{bmatrix} + \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 4 \\ -4 \\ 13 \end{bmatrix}$$

then $A(v_1 + v_2) = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix} \begin{bmatrix} 4 \\ -4 \\ 13 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 4 + 12 - 52 \\ -12 - 8 + 78 \\ 20 + 4 - 104 \end{bmatrix} \Rightarrow \begin{bmatrix} -36 \\ 58 \\ -80 \end{bmatrix}$$

Now,

$$Av_1 + Av_2 = \begin{bmatrix} -23 \\ 42 \\ -61 \end{bmatrix} + \begin{bmatrix} -13 \\ 16 \\ 19 \end{bmatrix}$$

$$= \begin{bmatrix} -36 \\ 58 \\ -80 \end{bmatrix}$$

ii) $$c(Av_1) = 5 \begin{bmatrix} -23 \\ 42 \\ -61 \end{bmatrix} = \begin{bmatrix} -115 \\ 210 \\ -305 \end{bmatrix}$$

$$A(cv_1) = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix} \left(5 \begin{bmatrix} 0 \\ -3 \\ 8 \end{bmatrix} \right)$$

$$= \begin{bmatrix} -115 \\ 210 \\ -305 \end{bmatrix}$$

$$\left[A(cv_1) = c(Av_1) \right]$$