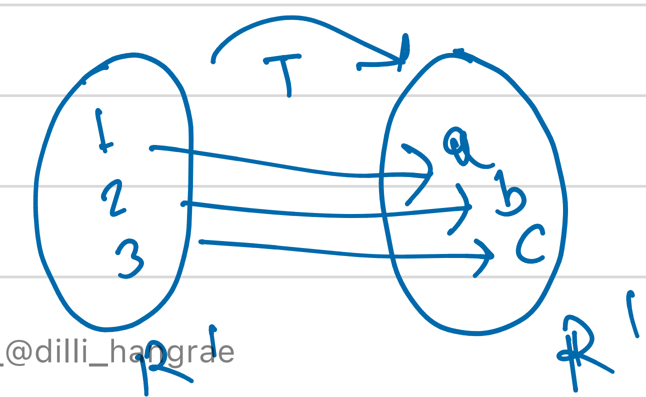
Day-30, Dec-29, 2024 (Poush-14, 2081 B.S.)

Definition (Onto):

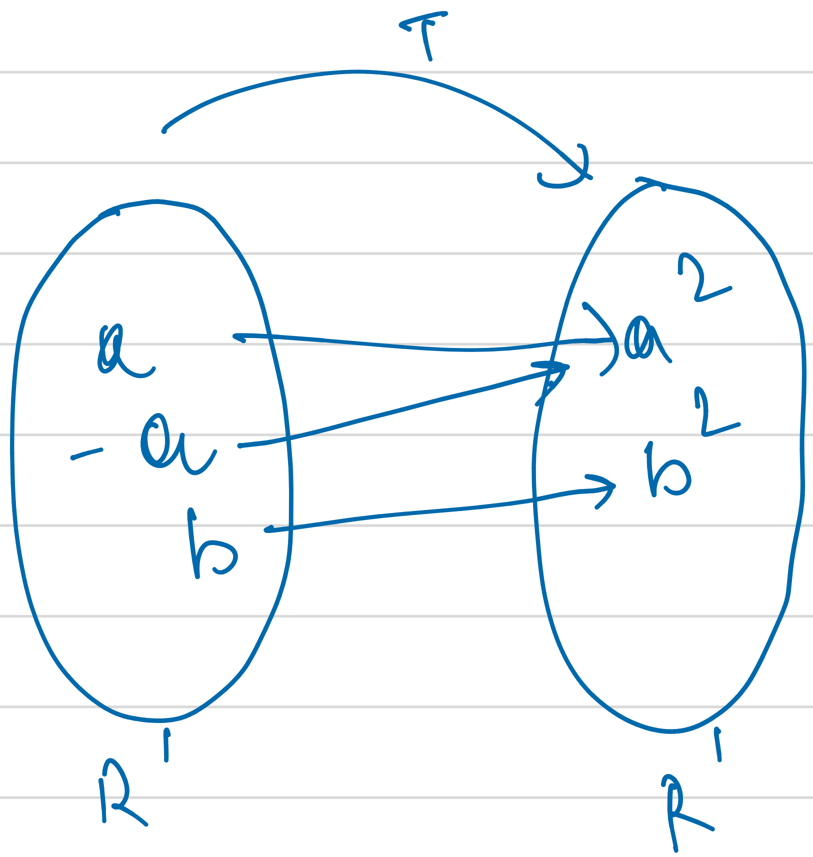
A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is said to be onto $\mathbb{R}^m$ if each b in $\mathbb{R}^m$ is the image of at least one x in $\mathbb{R}^n$.

Definition (One-to-One):

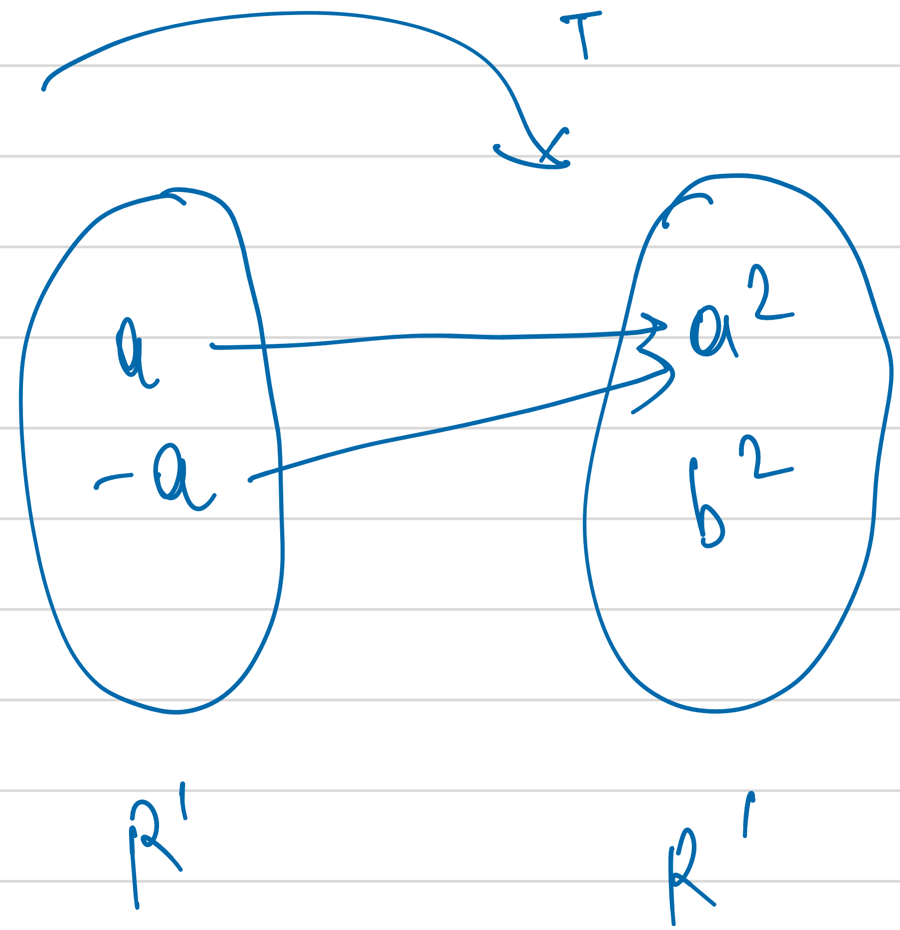
A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is said to be one-to-one if each b in $\mathbb{R}^m$ is the image of at most one x in $\mathbb{R}^n$.



$\Rightarrow$ T is one-to-one and onto



T is onto but not one-to-one.



T is not onto

Theorem: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(x) = 0$ has <u>only the trivial solution</u>.

Proof: Let $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is linear Transformation

Suppose that T is one-to-one. Then for any x in $\mathbb{R}^n$,

$$T(x) = 0 = T(0)$$

$$\Rightarrow x = 0 \qquad [\because \text{being } T \text{ is one-to-one}]$$

This means the equation $T(x)=0$

Conversely, Suppose that the equation $T(x)=0$ has only the trivial solution And, we wish to show T is one-to-one,

Take, $\quad T(u) = T(v)$ for some u, v in $\mathbb{R}^n$.

$$\Rightarrow T(u) - T(v) = 0$$

$\Rightarrow T(u-v) = 0$, being T is linear

Since $T(x) = 0$ has only trivial solution. So, we should have,

$$T(u-v) = 0$$

$$\Rightarrow u - v = 0$$

$$\Rightarrow u = v$$

Thus, $T(u) = T(v)$

$\Rightarrow u = v$ [$\therefore$ This means T is one-to-one.]

#Theorem: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for then,

a) T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$,

b) T is one-to-one if and only if the columns of A are linearly independent.

Proof: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T.

a) Let T is onto $\iff$ for each $b \in \mathbb{R}^m$ $\exists$ $x \in \mathbb{R}^n$ such that $T(x) = b$.

$\Leftrightarrow$ for each $b \in R^m$ $Ax = b$ has solution, where A is $m \times n$ matrix

$\Leftrightarrow$ Column of A span R^m.

b) Let T is one to one $\Leftrightarrow$ equation $T(x) = 0$ has only the trivial solution

$\Leftrightarrow$ Equation $Ax = 0$ has only trivial solution

$\Leftrightarrow$ Column of A are linearly independent

Example: Consider the linear transformation $T: R^2 \to R^2$ defined by the matrix A.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

This transformation maps any vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in R^2$ to a new vector in R^2 as follows:

$$T(x) = Ax = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \end{pmatrix}$$

Verifying if the transformation is one-to-one.

$Ax = 0$ such that non-zero vectors x.

To check,

$$Ax = 0$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This results in the system of equations

$x_1 + 2x_2 = 0$ —— eqn (1)

$3x_1 + 4x_2 = 0$ —— eqn (11)

Solving we get $x_1 = 0$ and $x_2 = 0$ [verified]

Homogeneous Linear System:

A Linear Systems is called homogeneous if it can be written in the form $Ax = 0$ where A is $m \times n$ matrix and 0 be a null matrix of order $m \times 1$.

Definition (Trivial and Non-trivial Solution of Homogeneous Linear System)

Let $Ax = 0$ be a homogeneous Linear Systems. The equations $Ax = 0$ always has one solution $x = 0$ where 0 is zero vector (null vector), Such solution is called trivial solution And, the non-zero solution of the equation $Ax=0$ is called non-trivial solution.

The equation will have non-trivial solution if and only if the equation has at least one free variable.

Example:

$$x_1 - 3x_2 + 7x_3 = 0$$
$$-2x_1 + x_2 - 4x_3 = 0$$
$$x_1 + 2x_2 + 9x_3 = 0$$

gives

$$\begin{bmatrix} 1 & -3 & 7 & 0 \\ 0 & -5 & 10 & 0 \\ 0 & 0 & 12 & 0 \end{bmatrix}$$

System has trivial solution because it has no free variable.