Day-13, Nov-28, 2024 (Mongshir 13, 2081 BS.)

Example: Use the definition of continuity and the properties of limits to show that the function is continuous on the given interval

a. $f(x) = \dfrac{2x+3}{x-2}$ $\quad (2, \infty)$

Sol^n; $f(x)$ is a continuous at $x=a$ if and only if $\lim\limits_{x \to a} f(x) = f(a)$

$$\lim_{x \to a} f(x) = \lim_{x \to a} \frac{2x+3}{x-2} \quad \text{(division law of limit)}$$

$$= \frac{\lim\limits_{x \to a} 2x+3}{\lim\limits_{x \to a} x-2}$$

$$\Rightarrow \frac{2\lim\limits_{x\to a} x + \lim\limits_{x\to a} 3}{\lim\limits_{x\to a} x - \lim\limits_{x\to a} 2}$$ [$\because$ constant law of limit]

$$\Rightarrow \frac{2a+3}{a-2}$$ [$\because \lim\limits_{x\to a} x = a, \lim\limits_{x\to a} c = c$]

[$\lim\limits_{x\to a} f(x) = f(a)$]

Hence $f(x)$ is continuous for all except $a = 2$ or $a \neq 2$.

(b) $f(x) = 2\sqrt{3-x}$, $(-\infty, 3]$

Polynomial and root functions are continuous over their domain.

So, the polynomial $3-x$ is Continuous for all real numbers. So, it is Continuous and also ≥ 0 for the given interval.

The domains of root functions are the values of 'x' for which the radicand is ≥ 0. The composition of $3-x$ and the root function will also be continuous by theorem 5.

Again, 2 can be consider as constant function which is also continuous. Since the product of two continuous functions is also continuous.

Hence $f(x) = 2\sqrt{3-x}$ is continuous in the interval $(-\infty, 3]$.

Example: Explain using theorem 1, 2, 3, 4, 5 why the function is continuous at every number in its domain. State the domain.

(1) $f(x) = \dfrac{2x^2 - x - 1}{x^2 + 1}$

This function is rational so it is continuous on its domain; theorem 3 states that all rational functions are continuous on their domain.

the domain $f(x)$ is every value for which $x^2+1 \neq 0$, x^2+1 is always greater than 0, so the domain of $f(x)$ is $(-\infty, \infty)$.

(b) $G(x) = \dfrac{x^2+1}{2x^2-x-1}$

By using the similar argument of theorem 2 the functions x^2+1 and $2x^2-x-1$ are continuous in $(-\infty, \infty)$ but the quotient function $G(x) = \dfrac{x^2+1}{2x^2-x-1}$ is discontinuous at the points for which $2x^2-x-1=0$.

ie. $2x^2 - 2x + x - 1 = 0$.

or, $2x(x-1) + (x-1) = 0$

or, $x = -\frac{1}{2}, 1$.

Hence the domain of $G(x)$ is $D = (-\infty, -\frac{1}{2}) \cup (-\frac{1}{2}, 1) \cup (1, \infty)$

$\therefore$ The function $G(x)$ is continuous in the domain D.

c) $R(t) = \dfrac{e^{\sin t}}{2+\cos\pi t}$

Here $e^{\sin t}$ being an exponential function is continuous everywhere in $(-\infty, \infty)$ (using theorem 3). The function $2+\cos\pi t$ is the sum of two continuous functions is continuous in $(-\infty, \infty)$. Hence the quotient function $R(t) = \dfrac{e^{\sin t}}{2+\cos\pi t}$ is continuous in its domain (using theorem1)

Now, the domain of $R(t)$

The function $R(t)$ is not defined for

$2+\cos\pi t = 0$

Since $-1 < \cos\pi t < 1$ i.e. $\cos\pi t$ never gives -2 and hence $2+\cos\pi t$ cannot be zero in $\mathbb{R}$. Hence $R(t)$ is continuous in it's domain $D = (-\infty, \infty)$.

d. $B(x) = \dfrac{\tan x}{\sqrt{4-x^2}}$

Here, $B(x) = \dfrac{\tan x}{\sqrt{4-x^2}}$

Both $\tan x$ and $\sqrt{4-x^2}$ are continuous in their domain using theorem 3. Again by using theorem (1) the quotient $\dfrac{\tan x}{\sqrt{4-x^2}}$ of two continuous functions is also continuous for $x \neq \pm 2$ including their domain.

Now $\tan x$ has domain of all real numbers except $\frac{\pi}{2} + n\pi$, where n is any integer.

$\sqrt{4-x^2}$ has domain $[-2,2]$ since any numbers outside $[-2,2]$ will make $4-x^2$ negative. Hence the domain of $B(x)$ is the combination of the portions of their individuals domains that write for both functions, and also excluding values that make the denominator 0.

So we must exclude -2 and 2. Also $\pm\frac{\pi}{2} \neq \pm 1.57$ falls within $(-2,2)$. So, we exclude those numbers. Hence $D = \left(-2, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, 2\right)$.

Example: Show that function

$$f(x) = 1 - \left(\sqrt{1-x^2}\right) \text{ is continuous on the interval } [-1,1].$$

Solⁿ; Let $a \in (-1, 1)$ i.e. $-1 < a < 1$ then

$$\lim_{x \to a} f(x) = \lim_{x \to a} \left(1 - \sqrt{1-x^2}\right)$$

$$\Rightarrow \lim_{x \to a} 1 - \lim_{x \to a} \sqrt{1-x^2}$$

$$\Rightarrow 1 - \sqrt{1-a^2}$$

$\Rightarrow f(a)$ which shows that $f(x)$ is continuous at $x = a \in (-1, 1)$. for which the end points i.e. $x = -1$.

$$\lim_{x \to -1^+} f(x) = 1 - \sqrt{1-(1)^2}$$

$$\Rightarrow 1$$

$$\Rightarrow f(1)$$

which Shows that $f(x)$ is continuous from right at the left end point $x = -1$. Similarly for the end point $x = 1$.

$$\lim_{x \to 1^-} f(x) = 1 - \sqrt{1-1}$$

$$\Rightarrow 1$$

$$\Rightarrow f(1)$$

which shows that $f(x)$ is continuous from the left at $x = 1$.

Hence, $f(x)$ is continuous at $[-1, 1]$.

Example: Evaluate $\lim_{x \to 3} \frac{x^2 + 5x + 6}{2x - 1}$

Here, both $x^2 + 5x + 6$ and $2x - 1$ are continuous everywhere in their domain and hence the quotient function $f(x)$ is also continuous in it's domain $(x \neq 1/2)$.

Example: Let $f(x) = \begin{cases} \dfrac{2x^2 - 5x - 3}{x - 3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$ $a = 3$. Check the continuity of $f(x)$ at $x = 3$. [Using theorem 2 (a) and theorem 1 (v)]

Here,

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x^2 - 5x - 3}{x - 3}$$

$$= \lim_{x \to 3} \frac{x^2 - 6x + x - 3}{x - 3}$$

$$\Rightarrow \lim_{x \to 3} \frac{2x(x-3)+1(x-3)}{x-3}$$

$$\Rightarrow \lim_{x \to 3} \frac{(2x+1)(x-3)}{x-3}$$

$$\Rightarrow \lim_{x \to 3} 2x+1$$

$$\Rightarrow 7$$

$\therefore$ functional value $f(3) = 6$. Hence $f(x)$ is discontinuous at $x = 3$. This type of discontinuous is called removable discontinuous. Because we can make the function continuous by redefining $f(x) = 7$ for $x = 3$.

Example: Where are the following Continuous?

a) $h(x) = \sin(x^2)$

b) $f(x) = \ln(1 + \cos x)$

Sol^n; Let $g(x) = x^2$, $f(x) = \sin x$. Here both $f(x)$ and $g(x)$ are continuous everywhere in $\mathbb{R}$ and hence their composite function $\sin(x^2)$ is also continuous in $(-\infty, \infty)$.

b) Since $f(x) = \ln x$ is continuous in its domain and $g(x) = 1 + \cos x$ is also continuous, then $\ln(1 + \cos x)$ is the composite of two continuous function wherever it is defined

Break the composite function, use tricks to see the continuity.