#Day-11, 26 Nov, 2024 (Mangshir 11, 2081)

Example: use a graph to find a number δ such that if

$$|x - \pi/4| < \delta \quad \text{then } |\tan x - 1| < 0.2.$$

Soln: Here the limit of $f(x) = \tan x$.

$L = 1$ and $a = \pi/4$, $\epsilon = 0.2$, $\delta = ?$

Since,

$$|\tan x - 1| < 0.2$$

$$\Rightarrow 0.8 < \tan x < 1.2$$

Now, the point of intersection of $y = 0.8$ and $y = \tan x$ is

$\Rightarrow x = \tan^{-1}(0.8)$

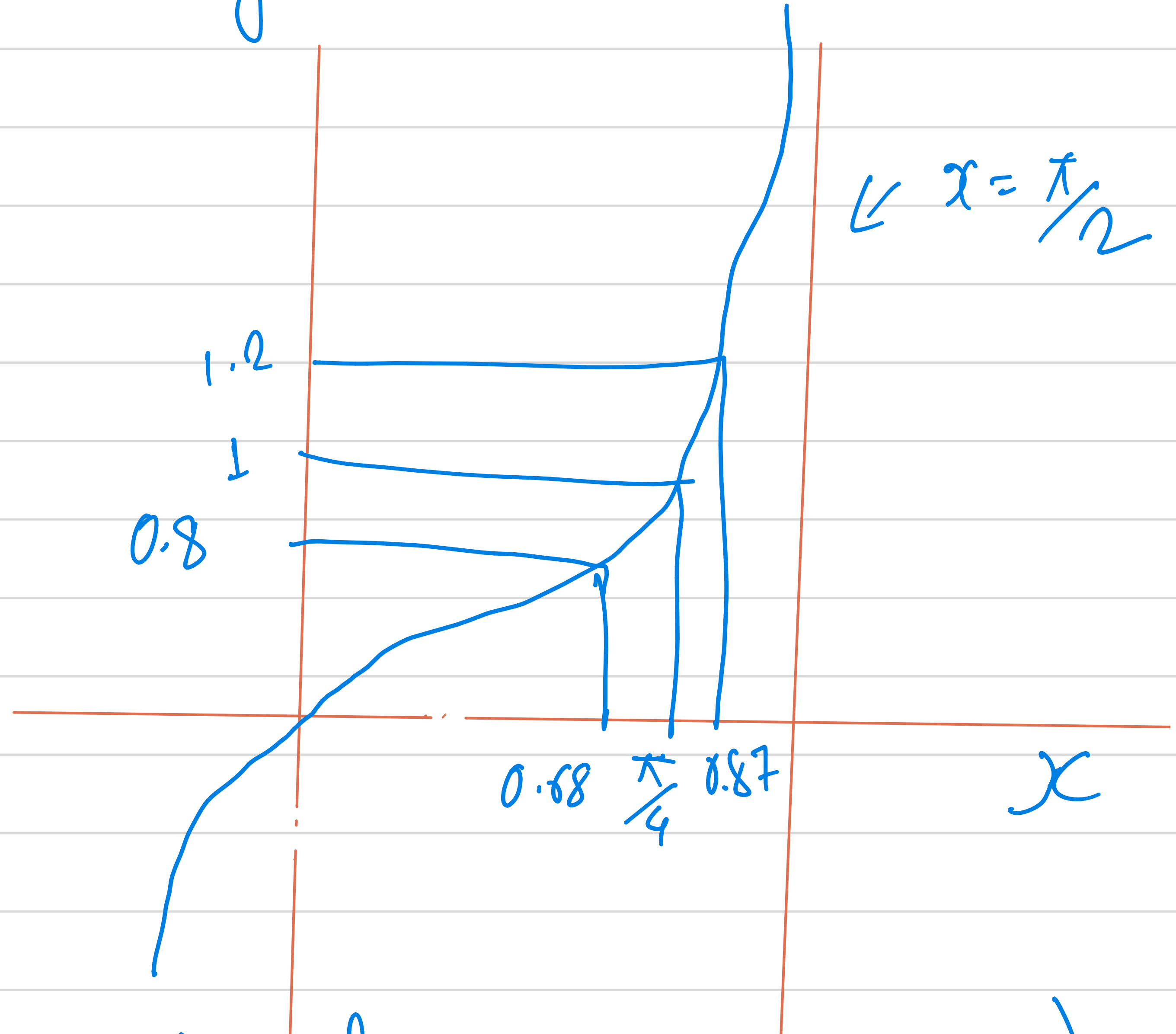
$\Rightarrow \boxed{x \Rightarrow 0.674}$

Again, solving $y = 1.2$ and $y = \tan x$

$y = \tan^{-1}(1.2)$

$\Rightarrow 0.876$

$\boxed{y \Rightarrow 0.88}$



Hence, the values of x varies in the interval $= (0.67, 0.88)$

the points $x = 0.67$ and $x = 0.88$ are not at symmetric distance from $x = \pi/4$

Hence, the value of δ = Smallest of $\left\{ \frac{\pi}{4} - 0.67,\ 0.88 - \frac{\pi}{4} \right\}$

$\Rightarrow \min \{ 0.115,\ 0.094 \}$

$\Rightarrow$ 0.094 or any smaller +ve number.

$$\boxed{\therefore\ \delta = 0.094}$$

More Examples on Right-Hand Limit / Left-Hand Limit:

Example: use the definition: prove $\lim_{x \to 0^+} \sqrt{x} = 0$.

Sol^n: Let 'ϵ' be given positive number. Here $a = 0$ and $L = 0$, we want to find a number δ such that, if $0 < x < \delta$ then $|\sqrt{x} - 0| < \epsilon$

i.e. $0 < x < \delta$ then $\sqrt{x} < \epsilon$.

or Squaring both sides of the inequality.

we have $0 < x < \delta$ then $x < \epsilon^2$ which suggest that $\delta = \epsilon^2$ (or any smaller positive number).

Now, observation (How this δ works for the definition of right hand limit)

Given $\epsilon > 0$, let $\delta = \epsilon^2$ if $0 < x < \delta$, then

$$\sqrt{x} < \sqrt{\delta}$$

$$\Rightarrow \sqrt{\epsilon^2} \Rightarrow \epsilon$$

So,

$$|\sqrt{x} - 0| < \varepsilon$$

$$\therefore \lim_{x \to 0^+} \sqrt{x} = 0$$

Definition Infinite Limits:

Let f be a function defined on some open interval that contains the number 'a', except possibly at itself. Then,

$$\therefore \lim_{x \to a} f(x) = \infty$$

means that for every positive number M there is a positive number δ such that if $0 < |x - a| < \delta$ then $|f(x)| > M$.

Theorem 2: a) Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.

Proof:

Let $p(x) = C_n x^n + C_{n-1} x^{n-1} + \cdots + C_1 x + C_0$

where

$C_0, C_1, C_2, \ldots, C_n$ are constants.

We know that $\lim_{x \to a} C_0 = C_0$

and $\lim_{x \to a} x^m = a^m$, $m = 1, 2, 3, \ldots, n$.

Since the function $f(x) = x^m$, $m = 0, 1, 2, \ldots, n$ is a continuous function. Hence, the new function $g(x) = Cx^m$ is also continuous

at $x = a$, Since $p(x)$, the Sum of continuous functions and constant function (which is always continuous) is continuous at $x = a$
[$\because$ using theorem 1].

Proof (b): Any Rational function is continuous wherever it is defined, that, is it is continous on it's domain.

Proof: A rational function is a function of the form of

$$f(x) = \frac{P(x)}{Q(x)}$$ where P and Q are polynomials

The domain of f is $D = \{x \in R\}$, $Q(x) \neq 0$. Since by part (a) both $p(x)$ and $Q(x)$ are continuous everywhere, and hence the quotient $\frac{p(x)}{Q(x)}$ is also continuous everywhere in their domain (using part (v) of theorem 1).

Example: find $\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$

Here, the given function is rational, so by using theorem 2 (b). it is continuous on it's domain which is $\{x : x \neq 5/3\}$.

Therefore, $\lim_{x \to -2} f(x) = f(-2)$

$$\Rightarrow \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} \Rightarrow -1/11.$$

Theorem 3: The following types of functions are continuous at every number in their domains:

a) polynomials
b) Rational functions
c) Root functions
d) Trigonometric functions
e) Inverse trigonometric functions
f) Exponential functions
g) Logarithmic function

Example 2: Where is the function $f(x) = \frac{\ln x + \tan^{-1} x}{x^2 - 1}$ continuous?

Sol^n: By using theorem 3

$y = \ln x$ is continuous for $x > 0$ and $y = \tan^{-1} x$ is continuous on 'R'. Thus by part (i) of the theorem 1.

Again the denominator $x^2 - 1$ being a polynomial is continuous by using theorem 2(a) everywhere. Now by using part (v) of theorem 1 the quotient $\left(f(x) = \frac{\ln x + \tan^{-1} x}{x^2 - 1} \right)$ is continuous everywhere except where $x^2 - 1 = 0$. Therefore, f is continuous in $(0,1) \cup (1,\infty)$.