Day-39, Dec 28, 2024 (Poush-13, 2081 B.S.)

Jacobian Matrix

The Jacobian Matrix is a matrix of all first-order partial derivatives of a vector-valued function. It provides valuable information about how a vector-valued function changes in response to small changes in its input variables.

for a function $F: \mathbb{R}^n \to \mathbb{R}^m$ defined as:

$$F(x) = \begin{bmatrix} f_1(x_1, x_2, \cdots, x_n) \\ f_2(x_1, x_2, \cdots, x_n) \\ \vdots \\ f_m(x_1, x_2, \cdots, x_n) \end{bmatrix}$$

Where $f_1, f_2 \cdots f_m$ are the components of the function and $X = (x_1, x_2, x_3 \cdots x_n)$, the Jacobian Matrix describes the rate of change of each component of the output of a multivariable function with respect to each variable in an input.

$f: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x,y) = (x^2+y^2, 2xy)$

Jacobian Matrix is

$$[J(x,y)] = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x & 2y \\ 2y & 2x \end{bmatrix}$$

Dimension: If F maps from $\mathbb{R}^n$ to $\mathbb{R}^m$, the Jacobian Matrix will be an $m \times n$ matrix.

$$J(F) \Rightarrow \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

for a function $F(x) = \begin{bmatrix} x_1^2 + x_2 \\ x_1 x_2 \end{bmatrix}$ $\quad J(F) = \begin{bmatrix} \frac{\partial}{\partial x_1}(x_1^2+x_2) & \frac{\partial}{\partial x_2}(x_1^2+x_2) \\ \frac{\partial}{\partial x_1}(x_1 x_2) & \frac{\partial}{\partial x_2}(x_1 x_2) \end{bmatrix}$

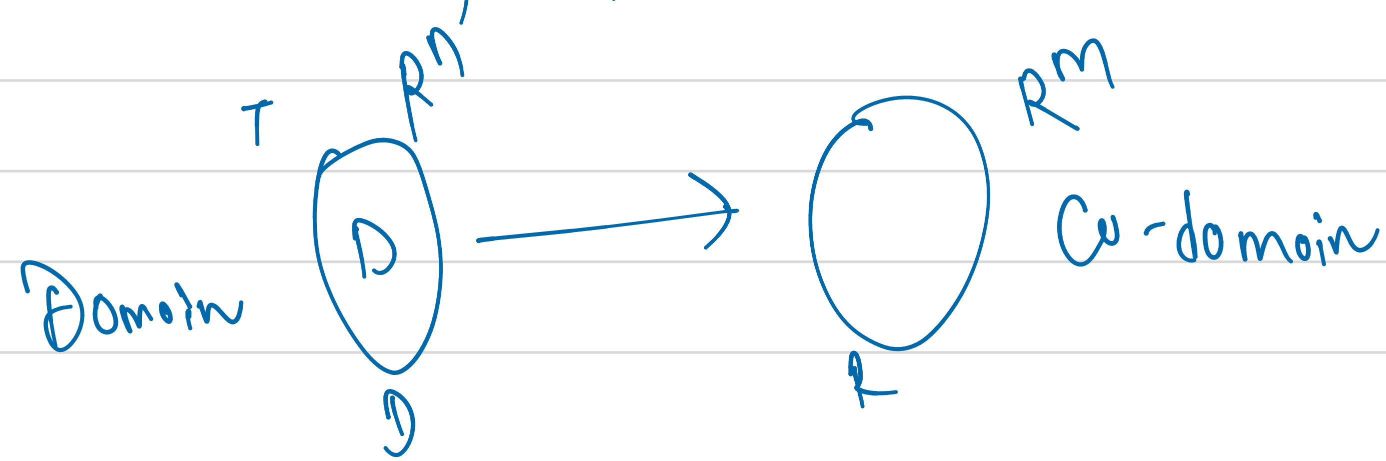
$$J(F) \Rightarrow \begin{bmatrix} 2x_1 & 1 \\ x_2 & x_1 \end{bmatrix}$$

Matrix Transformations

Matrix equation $Ax = b$ every value of x associates with A that gives the value b of particular Ax. Such association is a transformation.

Definition (Transformation or function or Mapping):

A transformation T from R^n to R^m (noted as $T: R^n \to R^m$) is a rule that assigns each vector x in R^n to a vector $T(x)$ in R^m, In such condition, R^n is domain T and R^m is Co-domain of T



Definition: (Domain, Co-Domain, Image and Range of a Transformation)

→ Let $T: R^n \rightarrow R^m$ is a transformation. Then the set R^n is domain and R^m is Co-domain of T.

→ And, for x in R^n, the value $T(x)$ in R^m is called Image of x under the transformation 'T'.

→ The set of all such images of 'x' under T (ie. $T(x)$) is called the range of T.

Example: Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ be the given matrix and define $T: R^2 \rightarrow R^2$ by $T(x) = Ax$. Find the images under T of

$u = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $v = \begin{bmatrix} a \\ b \end{bmatrix}$.

Soln; let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and the transformation $T: R^2 \to R^2$

defined by $T(x) = Ax$.

Also, let,

$u = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $v = \begin{bmatrix} a \\ b \end{bmatrix}$

$T(u) = Au$

$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 \times 1 + 0 \times (-3) \\ 0 \times 1 + 2(-3) \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2 \\ -6 \end{bmatrix}$

$$T(v) = Av$$

$$\Rightarrow \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 \times a + 0 \times b \\ 0 \times a + 2 \times b \end{bmatrix} \Rightarrow \begin{bmatrix} 2a \\ 2b \end{bmatrix}$$

Thus, the images of u and v under T are $\begin{bmatrix} 2 \\ -8 \end{bmatrix}$ and $\begin{bmatrix} 2a \\ 2b \end{bmatrix}$.

Matrix Transformation:

In the above example T is a transformation that transforms a matrix u (or v) to $T(u)$ or $(T(v))$ which is a matrix.

Definition: Matrix Transformation : for each $x \in R^n$, $T(x)$ is computed as $Ax \in R^m$, where A is $m \times n$ matrix behaves as transformation operator

Example: Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $C = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ and define a transformation $T : R^2 \to R^3$ by $T(x) = Ax$ so that

a) find $T(u)$

b) find x in R^2 whose image under T is b.

c) Is there more than one 'x' whose image under T is b?

d) Determine if C is in the range of T.

Let,

$$A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}, \quad u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} \text{ and } c = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

Given that the transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ by $T(x) = Ax$.

$(3 \times 2 \qquad 2 \times 1)$

a) $$T(u) = Au = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 \times 2 + (-3) \times (-1) \\ 3 \times 2 + 5 \times (-1) \\ -1 \times 2 + 7 \times (-1) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5 \\ +1 \\ -9 \end{bmatrix}$$

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Suppose x in R^2 whose Image under T is b. Then,

$T(x) = b \Rightarrow Ax = b$

$$\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$$

(or, $x_1 - 3x_2$?) → No this is not way

We change the $Ax = b$ into Augmented and make echelon form-

$$\sim \begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{bmatrix}$$

Solving we get – $\begin{bmatrix} 1 & 0 & 1.5 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{bmatrix}$

So, $x = \begin{bmatrix} 1.5 \\ -0.5 \end{bmatrix}$ in R^2 whose Image under T is b.

ie

(b) $\begin{bmatrix} 1 & 0 & 1.5 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} x_1 = 1.5 \\ x_2 = -0.5 \end{bmatrix}$$

Now, we can infer

(c) In the above solution x has no free variable, so the solution 'x' is unique. This means there is exactly one 'x' in $\mathbb{R}$ whose image under T is 'b'.

(d) From (c), there is exactly one range b of T. So, c is not a range of T

If we place value of b by c then we get inconsistent matrix of $Ax = c$. This implies c is not a range of T.

Shear Transformation:

A transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x) = Ax$ is called a shear transformation.

Contraction and Dilation:

A transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x) = rx$ for some scalar r. Then 'T' is called contraction when $0 \le r \le 1$ and T is called dilation when $r > 1$.

The Matrix of Linear Transformations:

A transform $x \to Ax$ has the properties.

$$A(u+v) = Au + Av$$

and $A(cu) = cA(u)$

for any u, v in $\mathbb{R}^n$ and for any scalar c.

This concept leads the idea of linearity.

Definition (Linearity of Transformations):

A transformation (or mapping) T is linear if

$T(u+v) = T(u) + T(v)$ for any u, v in domain of T.

$T(cu) = cT(u)$ for any 'u' in domain of T and for any scalar c.

The single equivalent condition for linearity of T is for all $\alpha, \beta \in F$, u, v domain of T, is $T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$.

Note: If $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear then $T(0)=0$, if $T(0) \neq 0$ then T is not linear.

Theorem 1: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear Transformation. Then there exists a unique matrix A **Such** that $T(x) = Ax$ for all x in $\mathbb{R}^n$, In fact A is $m \times n$ matrix whose jth column is the vector $T(e_j)$ where e_j is the jth column of

The identity matrix in R^n: $A = [\,T(e_1) \ \dots \ T(e_n)\,]$

Proof: Let

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \Rightarrow x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \dots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$\left[\, x \Rightarrow x_1 e_1 + x_2 e_2 + \dots + x_n e_n \,\right]$$

Let T is linear. So,

$$T(x) \Rightarrow T(x_1 e_1 + x_2 e_2 + \dots + x_n e_n)$$

$$\Rightarrow x_1 T(e_1) + x_2 T(e_2) + \dots + x_n T(e_n)$$

$$\Rightarrow \begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\Rightarrow Ax.$$

where $A = \begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{bmatrix}$ is called standard matrix for linear transformation T.

If possible suppose that A is not unique. Then there is a matrix B such that $T(x) = Bx$ for all x in $\mathbb{R}^n$.

$$\therefore \quad Ax = Bx \quad \forall x \in \mathbb{R}^n.$$

$$\Rightarrow (A-B)x = 0 \quad \forall x \in \mathbb{R}^n$$

Let we choose, $x = e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

then $Cx = 0$

or, $Ce_1 = 0$

$$\Rightarrow \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{m1} & C_{m2} & \cdots & C_{mn} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \\ 0 \end{bmatrix}$$

$C_{11} = 0$

Similarly $C_{ij} = 0$ $\forall$ $i = 1, 2, \ldots n$ and $j = 1, 2, \ldots m$.

$\therefore C = 0$

$A - B = 0$

$\Rightarrow A = B$

This means the matrix A is unique.

Prove that contraction map is linear transformation:

We know that map $T : R^2 \to R^2$ defined by $T(x) = rx$ where $0 \leq r \leq 1$ is called contraction map.

Let $u, v \in R^2$ and c and d are scalar. Then -

$$T(cu+dv) = \gamma(cu+dv)$$

$$\Rightarrow \gamma cu + d\gamma v$$

$$\Rightarrow c(\gamma u) + d(\gamma v)$$

$$\Rightarrow c\,T(u) + d\,T(v)$$

$(\because T \text{ is linear})$

Show that the transformation T defined by

$$T(x_1, x_2) = (2x_1 - 3x_2,\ x_1 + 4,\ 5x_2)$$

Let T is a transformation, defined by

$$T(x_1, x_2) = (2x_1 - 3x_2, x_1 + 4, 5x_2)$$

Now,

$$T(u+v) = T(u_1 + v_1, u_2 + v_2)$$

$$\Rightarrow 2(u_1 + v_1) - 3(u_1 + v_2),$$

$$(u_1 + v_1) + 4, 5(u_2 + v_2))$$

$$\Rightarrow (2u_1 + 2v_1 - 3u_1 - 3v_2,$$

$$u_1 + v_1 + 4, 5u_2 + 5v_2)$$

and

$$T(u) + T(v) = T(u_1, u_2) + T(v_1 v_2)$$

$\Rightarrow (2u_1 - 3u_2, u_1 + 4, 5u_2) + (2v_1 - 3v_2, v_1 + 4, 5v_2)$

$\Rightarrow (2u_1 + 2v_1 - 3u_2 - 3v_2, u_1 + v_1 + 8,$
$5u_2 + 5v_2)$

$T(u) + T(v) \neq T(u, v)$

So, T is not linear Transformation. for this

$T(x_1, x_2) = (2x_1 - 3x_2 + x_1 + 4, 5x_2)$

$T(0, 0) = (0, 4, 0)$
$\neq (0, 0, 0)$

$T(0) \neq 0$ so T is not linear.