# Day-7, Nov-22, 2024 (Mangshir 7, 2081 B.S).

## Important theorem on limit:

1. for all rational values of $n$,

$$\lim_{x \to a} \frac{x^n - a^n}{x - a}$$

$$\Rightarrow n a^{n-1}$$

This proof has 3 cases. (+, -, fraction)

# Exponential (e)

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \Rightarrow \lim_{h \to 0} (1 + h)^{1/h} \Rightarrow e.$$

# Limits of Logarithmic and Exponential functions:

For the limits of logarithmic and exponential functions, we recall the following definition of exponential e.

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

if we put $n = \frac{1}{h}$ so that when $n\to\infty$, $h\to 0$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{h\to 0}(1+h)^{1/h}$$

$$\Rightarrow \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}$$

$$\Rightarrow \lim_{h\to 0}\left(1+\frac{1}{1/h}\right)^{1/h} = \lim_{h\to 0}(1+h)^{1/h} = e$$

## Some Standard Results:

a) $$\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$$

$$\lim_{x \to 0} \frac{\log(1+x)}{x} \Rightarrow \lim_{x \to 0} \frac{1}{x} \log(1+x)$$

$$\Rightarrow \lim_{x \to 0} \log(1+x)^{1/x}$$

$$\Rightarrow \log \left\{ \lim_{x \to 0} (1+x)^{1/x} \right\}$$

$$\Rightarrow \log e$$

$$\Rightarrow 1$$

### What is happening in the reverse step?

We are working with this term:

$$\frac{1}{x} \cdot \log(1 + x).$$

**Recall the logarithmic property:**

For any numbers $a > 0$ and $b$,

$$\log(a^b) = b \cdot \log(a).$$

This is **normally** used to move the exponent $b$ down in front of the logarithm. But here, we use it **in reverse** to push the coefficient $1/x$ into the argument of the logarithm.

**Apply the property in reverse:**

Start with:

$$\frac{1}{x} \cdot \log(1 + x).$$

We recognize that this matches the form $b \cdot \log(a)$, where:

- $a = 1 + x$,
- $b = \frac{1}{x}$.

Using the property **backwards**, this becomes:

$$\log((1 + x)^{1/x}).$$

### Why does this work?

The logarithmic property is all about how multiplication outside the logarithm relates to an exponent inside. Think of it as "absorbing" the $1/x$ into the logarithm by raising the base to that power.

- Before: The $\frac{1}{x}$ is multiplying the logarithm.
- After: The $(1 + x)$ is raised to the power of $\frac{1}{x}$ **inside the logarithm**.

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### Substitute this back into the limit:

Once we rewrite:

$$\frac{\log(1 + x)}{x} = \log((1 + x)^{1/x}),$$

we now work with:

$$\lim_{x \to 0} \frac{\log(1 + x)}{x} = \lim_{x \to 0} \log((1 + x)^{1/x}).$$

From here, it simplifies further because we already know:

$$\lim_{x \to 0} (1 + x)^{1/x} = e.$$

Thus:

$$\lim_{x \to 0} \log((1 + x)^{1/x}) = \log(e) = 1.$$

b) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

Put $e^x - 1 = y$ then $e^x = 1 + y$ and $x = \log(1+y)$

$\Rightarrow e^x - 1 = y$

$\Rightarrow e^x = 1 + y$

$\Rightarrow x = \log(1+y)$ So that when $x \to 0$, $y \to 0$.

Now, $\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{y \to 0} \frac{y}{\log(1+y)}$

$\Rightarrow \lim_{y \to 0} \frac{1}{\frac{1}{y}\log(1+y)}$ $\Rightarrow \frac{1}{\frac{1}{0}\log(1+0)}$ $\Rightarrow \frac{1}{1}$

$\Rightarrow 1$

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \log a$$

put $a^x - 1 = y$ then $a^x = 1 + y$ which implies $x \log a = \log(1+y)$

$$a^x - 1 = y$$

$$a^x = 1 + y \quad \text{or,} \quad x = \log(1+y)$$

So, $x \log a = \log(1+y)$

So, $x = \dfrac{\log(1+y)}{\log a}$ So when $x \to 0$, $y \to 0$

$$\Rightarrow \lim_{y \to 0} \frac{y}{\frac{\log(1+y)}{\log a}}$$ [putting $a^x - 1$ value as $y$]

$$\Rightarrow \log a \cdot \lim_{y \to 0} \frac{1}{\frac{1}{y} \log(1+y)}$$

$$\Rightarrow \log a \cdot 1$$

$$\Rightarrow \log a$$

$$\boxed{\therefore \lim_{x \to 0} \frac{a^x - 1}{x} = \log a}$$

① Types of Discontinuities

→ Jump or ordinary discontinuity at $\lim_{x \to x_0^-} \neq \lim_{x \to x_0^+}$.

→ Removable discontinuity at $x = x_0$.

→ Infinity

Example 1:

$$\lim_{x\to 0} \frac{\tan x}{x} = 1$$

$$\left[\tan = \frac{\sin}{\cos}\right]$$

$$= \lim_{x\to 0} \left(\frac{\sin x}{x} \cdot \frac{1}{\cos x}\right)$$

$$= 1 \cdot \frac{1}{\cos 0}$$

$$\Rightarrow 1$$

Example 2:

$$\lim_{x\to 0} \frac{1-\cos 3x}{3x^2} = \frac{3}{2}$$

$$\frac{1-\cos 3\times 0}{0} \Rightarrow \frac{0}{0}$$

[Using trigonometric formula $1-\cos 3x = 2\sin^2\frac{3x}{2}$]

So, $$\lim_{x\to 0} \frac{2\sin^2\frac{3x}{2}}{3x^2}$$

$$\Rightarrow \frac{2}{3} \lim_{x \to 0} \left( \frac{\sin . \frac{3x}{2}}{x} \right)^2$$

$$\Rightarrow \frac{2}{3} \lim_{x \to 0} \left( \frac{\sin . \frac{3x}{2}}{\frac{3x}{2}} \cdot \frac{3}{2} \right)^2$$

$$\Rightarrow \frac{2}{3} \left( 1 \cdot \frac{3}{2} \right)^2$$

$$\Rightarrow \frac{3}{2}$$

Example 3: $\lim_{x \to 0} \frac{e^{3x} - 1}{x \cdot 5^x}$

$$\Rightarrow \lim_{x \to 0} \frac{e^{3x} - 1}{3x} \cdot 3 \cdot \frac{1}{5^x} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$\Rightarrow \left(\lim_{x \to 0} \frac{e^{3x} - 1}{3x} \cdot 3\right) \left(\lim_{x \to 0} \frac{1}{5^x}\right)$$

$$\Rightarrow \left(\frac{e^{3\times 0} - 1}{0} \cdot 3\right) \left(\frac{1}{5^0}\right)$$

$$\Rightarrow \left(\frac{e^0 - 1}{0}\ 3\right) (1)$$

$$\Rightarrow \frac{1-1}{0} \cdot 3 \cdot 1 \qquad \Rightarrow 3$$

Example 4: $\lim_{x \to a} \dfrac{\sqrt{x} - \sqrt{a}}{\tan(x-a)}$

Here,

$$\lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{\tan(x-a)}$$

$$\Rightarrow \lim_{x \to a} \left\{ \frac{\sqrt{x} - \sqrt{a}}{\tan(x-a)} \cdot \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}} \right\}$$

$$\Rightarrow \lim_{x \to a} \left\{ \frac{(x-a)\cos(x-a)}{\sin(x-a)} \cdot \frac{1}{\sqrt{x} + \sqrt{a}} \right\}$$

You're absolutely right! Let me carefully explain that part of the derivation where the expression involves $\cos(x-a)$ and $\sin(x-a)$.

The step in question involves the term:

$$\frac{x-a}{\tan(x-a)}.$$

Now, let's break it down step by step.

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**1. The relationship between $\tan$, $\sin$, and $\cos$**

We know that:

$$\tan(x-a) = \frac{\sin(x-a)}{\cos(x-a)}.$$

So, the expression $\frac{x-a}{\tan(x-a)}$ becomes:

$$\frac{x-a}{\tan(x-a)} = \frac{x-a}{\frac{\sin(x-a)}{\cos(x-a)}}.$$

Simplify this fraction by multiplying by the reciprocal of $\frac{\sin(x-a)}{\cos(x-a)}$:

$$\frac{x-a}{\tan(x-a)} = \frac{(x-a)\cdot\cos(x-a)}{\sin(x-a)}.$$

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$$\frac{x-a}{\tan(x-a)} = \frac{(x-a)\cdot\cos(x-a)}{\sin(x-a)}.$$

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**2. Why write it this way?**

This step is useful because the limit of $\frac{\sin(x-a)}{x-a}$ as $x \to a$ is a standard limit result:

$$\lim_{x\to a}\frac{\sin(x-a)}{x-a} = 1.$$

So, rewriting the term in this way helps simplify the limit. Specifically, we can now separate the terms as:

$$\lim_{x\to a}\frac{(x-a)\cdot\cos(x-a)}{\sin(x-a)} = \lim_{x\to a}\cos(x-a)\cdot\lim_{x\to a}\frac{x-a}{\sin(x-a)}.$$

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**3. What happens to $\frac{x-a}{\sin(x-a)}$?**

Using the same standard limit property:

$$\lim_{x\to a}\frac{x-a}{\sin(x-a)} = 1.$$

Thus, this simplifies to:

$$\lim_{x\to a}\frac{x-a}{\tan(x-a)} = \cos(x-a).$$

$$\Rightarrow \lim_{(x-a)\to 0} \frac{1}{\frac{\sin(x-a)}{x-a}} \quad \lim_{(x-a)\to 0} \cos(x-a) \times \lim_{x\to a} \frac{1}{\sqrt{x}+\sqrt{a}}$$

$$\Rightarrow \frac{1}{1}\cdot 1 \cdot \frac{1}{\sqrt{a}+\sqrt{a}}$$

$$\Rightarrow \frac{1}{2\sqrt{a}} \qquad \therefore \lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{\tan(x-a)} = \frac{1}{2\sqrt{a}}$$

Example 5: $\lim_{x \to 0} \frac{a^x - b^x}{x}$

$$\lim_{x \to 0} \frac{a^x - b^x}{x}$$

$$= \lim_{x \to 0} \frac{(a^x - 1) - (b^x - 1)}{x}$$

$$\Rightarrow \lim_{x \to 0} \frac{a^x - 1}{x} - \lim_{x \to 0} \frac{b^x - 1}{x}$$

$$\Rightarrow \log a - \log b$$

$$\Rightarrow \log\left(\frac{a}{b}\right)$$

Example 2:

$$\lim_{x \to 1} \frac{\log x}{x-1}$$

put $x - 1 = y$

then $x = 1 + y$ so that when $x \to 1$, $y \to 0$

$$\lim_{x \to 1} \frac{\log x}{x-1}$$

$$\Rightarrow \lim_{y \to 0} \frac{\log(1+y)}{y}$$

$$\Rightarrow 1$$