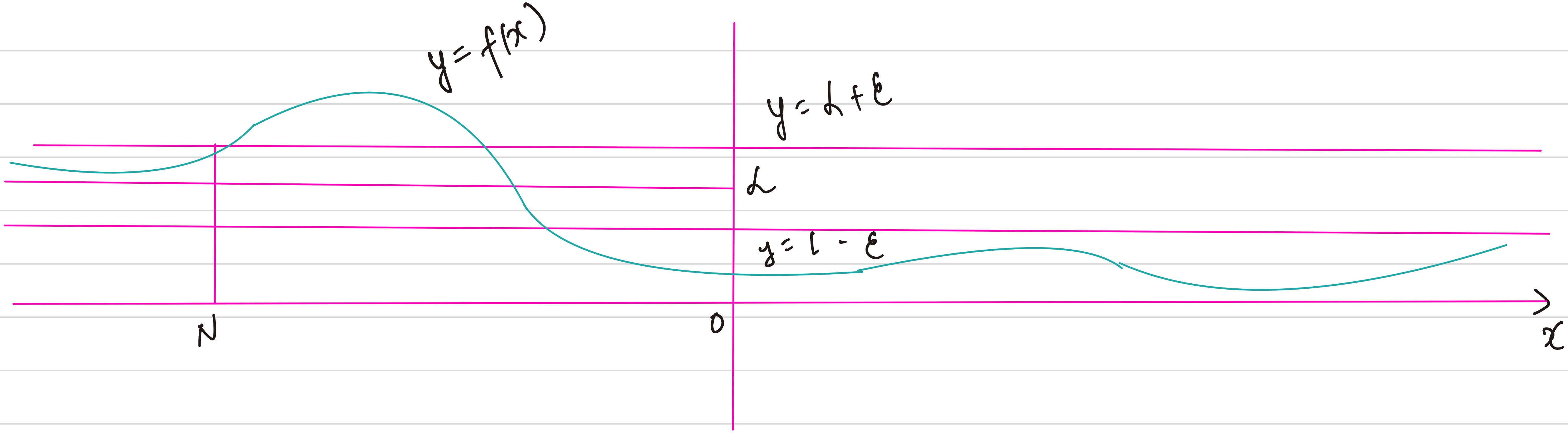
Day-16, Dec-1, 2024 (Mangshir 16, 2081 B.S)

Definition: Let 'f' be a function defined on some interval $(-\infty, \infty)$. Then $\lim\limits_{x \to -\infty} f(x) = L$ means for every $\epsilon > 0$ there is corresponding N sufficiently large in magnitude but negative in sign such that if $x < N$ then $(f(x) - L) < \epsilon$.

if $x < N$ then
$$(f(x) - L) < \epsilon$$

(where N sufficiently large in magnitude but negative in sign.)



Example: Use a graph to find a number N such that if $x > N$ then

$$\left| \frac{3x^2 - x - 2}{5x^2 + 4x + 1} - 0.6 \right| < 0.1.$$

Here, $L = 0.6$ and the problem is -

$$\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = 0.6, \quad N = ?$$

Since, $-0.1 < \dfrac{3x^2 - x - 2}{5x^2 + 4x + 1} - 0.6 < 0.1$

$$\Rightarrow \quad 0.5 < \frac{3x^2 - x - 2}{5x^2 + 4x + 1} < 0.7$$

Solving $y = 0.5$ and $y = \dfrac{3x^2 - x - 2}{5x^2+4x+1}$ we have $x = 6.7$.

To the right of this number it seems that the curve stays between the lines $y = 0.5$ and $y = 0.7$. Rounding to be safe, we can say that –

$$\text{if } x > 7 \text{ then } \left| \frac{3x^2 - x - 2}{5x^2 + 4x + 1} - 0.6 \right| < 0.1 \qquad [\because \underline{x = 6.7}].$$

In other words, for $\varepsilon = 0.1$, we can choose $N = 7$ or any larger number.

Example: Use definition limit at Infinity to prove that $\lim_{x \to \infty} \frac{1}{x} = 0$.

Here,

Given $\varepsilon > 0$, we want to find N such that if $x > N$ then

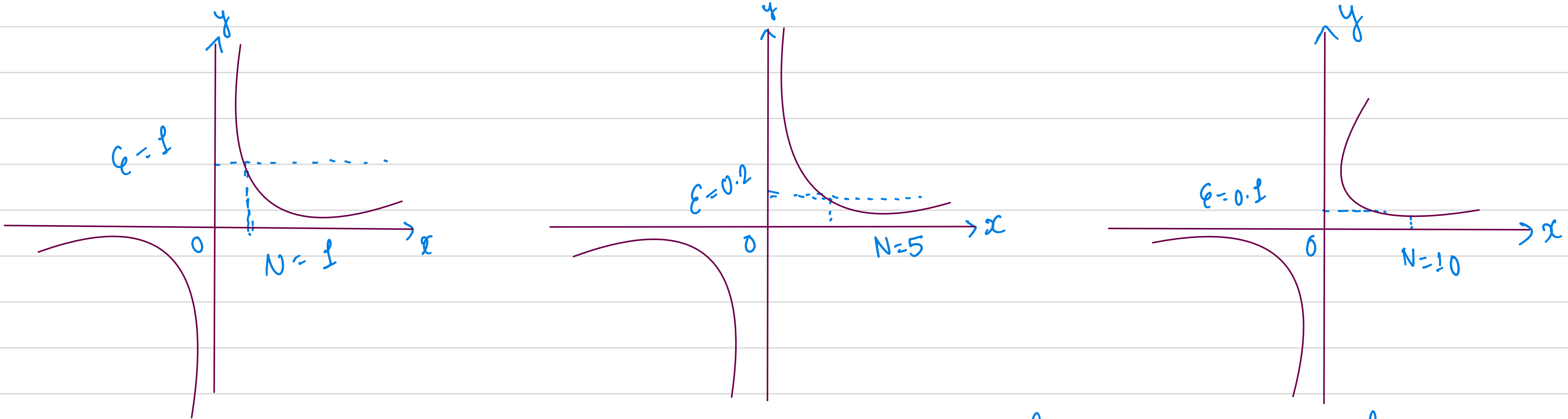
$\left| \frac{1}{x} - 0 \right| < \varepsilon$.

In Computing this limit we may assume that $\underline{x > 0}$. Then $\underline{1/x < \varepsilon}$

$\frac{1}{x} < \varepsilon \iff x > 1/\varepsilon$. Lets choose $N = 1/\varepsilon$ so

if $x > N = 1/\varepsilon$ then $\left| 1/x - 0 \right| = 1/x < \varepsilon$

from the definition $\lim_{x \to \infty} \frac{1}{x} = 0$.



Definition: Let f be a function defined on some interval (a, ∞) then $\lim_{x \to \infty} f(x) = \infty$ means that for every positive number M there corresponds a positive N such that if $x > N$ then $f(x) > M$.

Definition: Let f be a function define on some interval $(-\infty, a)$. Then $\lim_{x \to \infty} f(x) = -\infty$ means that for every negative number M there corresponds a negative number N such that if $x < N$ then $f(x) < M$. In this case we write $\lim_{x \to -\infty} f(x) = -\infty$.

Slant (Oblique) Asymptote:

Let $f(x) = \dfrac{p(x)}{d(x)}$ be a rational function. The function $f(x)$ will have a horizontal asymptote - if the degree of 'p' is strictly less than that of 'd', then the x-axis will be the horizontal asymptote.

The geometrical condition that can be expressed analytically by saying $f(x) \to 0$ as $x \to \infty$ and as $x \to -\infty$.

if the degree of numerator in a rational function is exactly one degree greater than the dominator's degree. it is called Slant Asymptote.

$$p(x) = d(x) \cdot q(x) + r(x)$$ where $q(x)$ is a quotient $r(x)$ is a remainder

If the degree of 'p' is greater than or equal to the degree of 'd', then long division use to obtain $p(x) = d(x) \cdot q(x) + r(x)$ where $q(x)$ is a quotient

and $r(x)$ is a remainder such that

$\deg r(x) < \deg d(x)$.

Now, we have $f(x) = q(x) + \dfrac{r(x)}{d(x)}$.

In the special case where the degree of p is one more than the degree of d, the quotient is a linear function, whose graph is a non-horizontal and non-vertical line in the plane. That line is called oblique or slant asymptote to the curve of the particular rational function.

Example: Find the Asymptote of the following curves.

1) $$y = \frac{x^2 - 3x/2}{2x+1} = \frac{p(x)}{d(x)}$$

Here deg. $p(x)$ is one more than degree of $d(x)$ so it has the slant asymptote –

Since, $$\frac{2x^2-3x}{4x+2} = \frac{2x^2-3x}{4x+2} \Rightarrow \frac{x(\cancel{2x+1})-4x}{2(\cancel{2x+1})}$$

$$\Rightarrow \frac{x-4x}{2}$$

Alternatively, $$\frac{2x^2-3x}{4x+2} = \frac{x}{2} - 1 + \frac{2}{4x+2}$$

Taking $x \to \pm\infty$ we have $\frac{2}{4x+2} \to 0$

and hence $\frac{2x^2-3x}{4x+2}$ and $\frac{x}{2} - 1$ are very close to each other

as $x \to \pm\infty$ and hence $y = \frac{x}{2} - 1$ is an oblique (slant) asymptote

of the curve $\frac{x^2 - 3x/2}{2x+1}$.

In addition, since $\lim_{x \to -1/2} \frac{x^2 - 3x/2}{2x+1} = \infty$

Hence the line $x = -1/2$ is a vertical Asymptote

Example: $y = \frac{(1-x)^3}{x^2} = \frac{p(x)}{d(x)}$

Here the degree of $p(x)$ is exactly 1 more of degree of $d(x)$ so it has slant asymptotes, Since $\frac{(1-x)^3}{x^2} = (3-x) + \frac{1-3x}{x^2}$.

Taking $x \to \pm\infty$, $\frac{1-3x}{x^2} \to 0$ Hence $y = \frac{(1-x)^3}{x^2}$ approaches with the straight line $y = 3-x$ as $x \to \pm\infty$.

Hence, $y = 3-x$ is an oblique asymptote more over we can See that for $\lim_{x \to 0} \frac{(1-x)^3}{x^2} = \infty$ Hence $x=0$. y-axis is an vertical asymptote of the curve.