New Edition

# BASIC MATHEMATICS

## GRADE : XI

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# BASIC MATHEMATICS

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# BASIC MATHEMATICS

A Textbook for

**GRADE : XI**

Bishal Gautam Dinesh-gautam
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# Letter of Approval

This is to certify that the Curriculum and Training Division of Higher Secondary Education Board (HSEB) has approved the book entitled **"Higher Secondary Level Basic Mathematics"** written by **Mr. D.R. Bajracharya** et.al. as a reference book for Grade XI.

R.C. panday

(Ram Chandra Panday)

Acting Joint Secretary

Curriculum & Training Division

**Joint Secretary**

# Preface to Third Revised Edition

The present book is the continuity of our "Higher Secondary Level Basic Mathematics Vol I" in accordance with the new syllabus of Higher Secondary Education Board at +2 level. Knowing the fact that the foundation of the mathematics depends upon the clear concept of the subject matters, we have tried our best to present them in an easy, clear, lucid and systematic manner. As far as possible, all the topics included in the syllabus have been dealt with in detail and the concepts and the theories are fully explained with suitable examples to illustrate them.

Some of the new topics such as logic, mathematical induction and the sketching of curves have been introduced. These new topics have been written carefully with a large number of examples. Also some of the topics like sequence and series, circle and the derivatives as the rate measure of Class XII the previous syllabus have now introduced in class XI. These topics are revised in order to make them suitable for the students of class XI. The exercises have been well graded. We feel immense success if the book is found useful both for the teachers and the students, as it was earlier.

We must thank Mr. Ananda Krishna Shrestha the proprietor of Sukunda Pustak Bhawan, for his strong determination and dedication to publish the book and present it in time.

Any suggestion for the improvement of the book will be highly appreciated and thankfully acknowledged.

April 14, 2010
Baishakh 1, 2067

**The Authors**

# Preface to Second Revised Edition

The books on Higher Secondary Level Basic Mathematics Vol. I and Vol. II were revised according to the Higher Secondary Education Board curriculum of Mathematics in the year 2055, Volume I was meant for grade XI and Volume II was meant for grade XII of +2. Since then it has undergone many reprints without any change. The books were widely used in all the +2 colleges as text books.

We received many useful suggestions regarding the contents, errors and omissions, etc. from teachers of different colleges all over the country. Accordingly we have revised the whole contents of volume I, reorganized some chapters, completely re-written many chapters. With due acknowledgement and thanks for the suggestions made available to us we have added many examples solved for illustration; some exercises have also been added in almost all the chapters.

We have tried to give this edition of the book a totally new look with a new outfit in harmony with the changing trend so as to give the students skills to tackle problems of all types besides imparting them factual mathematical concepts in clear and simple language.

We express our gratitude to all the teachers and the students who assisted us by pointing out the shortcomings, omissions, errors and misprints, etc. in the previous issues. We believe this edition of the book will be more useful to the students and the teachers alike.

We thank Mr. Ananda Krishna Shrestha of Sukunda Pustak Bhawan for his untiring efforts in organizing many meetings of the authors for the revision and re-writing works, and for his determination and dedication to publish the book and bring it to the readers in time. We also thank Mr. Kiran Shakya for his appreciable workmanship in computer typing, graphics, figure designing and overall set up to bring forth the book in this get up.

Baisakh 30, 2062

The Authors

# Preface to Revised Edition

Hig Secondary Level Mathematics curriculum has been revised and updated recently. Although Higher Secondary Level Edition of our books Basic Mathematics Vol. I and Vol. II could serve the changed situation, the warmth with which the teachers and the taught received our book make us feel that the teachers and the taught should not have even the least problem in teaching and learning due to the latest changes. We, therefore, tried to rewrite and present this new edition with addition of some new chapters, deletion of some chapters, revision and reordering of the remaining ones. We have also indicated, wherever possible, the year and years in which a particular theoretical or numerical problem has been asked in the HSEB and PCL examinations. We hope that inclusion of mathematics curriculum would be helpful to our students.

Before we conclude, we would like to express our gratitude to the teachers and the taught who helped us through valuable suggestion and critical appreciation in bringing out this edition. We hope that we shall be getting similar cooperation in the future also.

**Authors**

Asadh, 2055
June, 1998

# Preface to Higher Secondary Edition

Basic knowledge of mathematics expected from a post-ool student (Higher Secondary Level, Intermediate or Proficiency Certate Level) remains almost the same although minor variations from iutions to institutions do occur. As far as the contents and extent of ouurses are concerned, we can, without any prejudice, claim that they can compared with the contents and extent of the courses of well-known instion even outside the SAARC countries. Teaching materials prepared accingto the prescribed courses are qualitatively in no way inferior to simi maerials published elsewhere.

Our books Basic Mathematics Vol. I and Vol. II could and d seve, at least to a certain extent of satisfaction, the teachers and taught in is respect for the last one decade. Approval of our books as Text Booksby Higher Secondary Education Board is definitely encouraging to us.

We express our sincere appreciation to HSEB for this.

We, therefore, felt it as our responsibility to revise, rearange, rite and rewrite with some additions and some omissions in orderto meethe requirements of the Higher Secondary Level Students of Nepal. With he publication of Higher Secondary Level Edition' of our books — Bac Mathematics Vol. I and Vol. II, we hope that the teachers and taught will e free from the trouble of lurking through both volumes of our books. S, separate books for Grade XI and Grade XII could be more meaningful an purposeful. We have tried our best. Some mistakes and ome deficiencie might have crept in. We shall feel honoured if the readers are kind enough to bring such errors to our notice.

September 1, 1994
Bhadra 16, 2051

**Authors**

# Higher Secondary School Curriculum
# (MATHEMATICS : I)
# Grade : XI
# (2066)

Teaching hours : 150 Full Marks : 100

## Course Contents

### Unit 1 Sets, Real Number System and Logic 10 hrs

**Sets:** Sets and set operations. Theorems based on set operations.

**Real Number System:** Real numbers, Field axioms, Order axioms, Interval, Absolute value, Geometrical representation of the real numbers

**Logic:** Introduction, Statements, Logical connectives, Truth tables, Basic laws of logic.

### Unit 2 Relations, Functions and Graphs 10 hrs

**Relations:** Ordered pairs, Cartesian product, Geometric representation of cartesian product, Relation, Domain and range of a relation, Inverse of a relation.

**Functions:** Definition, Domain and range of a function, Functions defined as mapping, Inverse function, Composite function, Special functions (Identity, Constant, Absolute value, Greatest integer), Algebraic (Linear, Quardratic and Cubic), Trigonometric, Exponential, Logarithmic functions and their graphs.

### Unit 3 Curve Sketching 10 hrs

Odd and even functions, Periodicity of a function, Symmetry (about x-axis, y-axis and origin) of elementary functions, Monotonocity of a function, Sketching graphs of polynomial functions $\left(\frac{1}{x}, \frac{x^2 - a^2}{x - a}, \frac{1}{x + a}, x^2, x^3\right)$. Trigonometric, Exponential, Logarithmic functions (simple cases only)

### Unit 4 Trigonometry 12 hrs

Inverse circular functions, Trigonometric equations and general values, Properties of a triangle (sine law, cosine law, tangent law, projection law, half angle laws), The area of a triangle, Solution of a triangle (simple cases)

## Unit 5 Sequence and Series and Mathematical Induction — 10 hrs

**Sequence and Series:** Sequence and series (Arithmetic, Geometric, Harmonic), Properties of arithmetic, geometric and harmonic sequences, A.M., G.M. and H.M. Relation among A.M., G.M. and H.M. Sum of infinite geometric series.

**Mathematical Induction:** Sum of first $n$ natural numbers, Sum of the squares of first $n$ natural numbers, Sum of cubes of first $n$ natural numbers, Intuition and induction, Principles of mathematical induction

## Unit 6 Matrices and Determinants — 8 hrs

Matrices and operation on matrices (Review), Transpose of a matrix and its properties, Minors and cofactors, Adjoint of a matrix, Determinant of a square matrix, Properties of determinants (without proof) upto $3 \times 3$, Inverse matrix.

## Unit 7 System of Linear Equations — 10 hrs

Consistency of system of linear equations, Solution of a system of linear equations by Crammer's rule, Matrix method (Row-equivalent and Inverse) upto three variables

## Unit 8 Complex Numbers — 10 hrs

Definition of a complex number, Imaginary unit, Algebra of complex numbers, Geometric representation of a complex number, conjugate and absolute value (modulus) of complex numbers and their properties, Square roots of a complex number, Polar form of a complex number, Product and quotients of two complex numbers, De-Moivre's theorem and its application in finding the roots of a complex number, Properties of cube roots of unity

## Unit 9 Polynomial Equations — 10 hrs

Polynomial function and polynomial equations, Fundamental theorem of algebra (without proof), Quadratic equation, Nature of the roots of a quadratic equation, Relation between roots and coefficients, Formation of a quadratic equation, Symmetric functions of roots, one or both roots common.

## Unit 10 Coordinate Geometry — 10 hrs

**Straight Line:** Review of various forms of equation of straight lines, Angle between two straight lines, Condition of parallelism and perpendicularity, Length of perpendicular from a given point to a given line, Bisectors of the angles between two straight lines.

**Pair of Lines:** General equation of second degree in $x$ and $y$, Condition for representing a pair of lines, Homogeneous second degree equation in $x$ and $y$, Angle between pair of lines, Bisectors of the angles between pair of lines.

**Unit 11 Circle** **12 hrs**

Equation of a circle in various forms (centre at origin, centre at any point, general equation of a circle with a given diameter), Condition of tangency of a line at a point to the circle, Tangent and normal to a circle.

**Unit 12 Limits and Continuity** **10 hrs**

Limits of a function, Indeterminate forms, Algebraic properties of limits (without proof), Theorems on limits of algebraic, trigonometric exponential and logarithmic functions $\left(\lim_{x \to a} \frac{x^n - a^n}{x - a}, \lim_{x \to 0} \sin x, \lim_{x \to 0} \frac{\sin x}{x}, \lim_{x \to 0} \frac{e^x - 1}{x}, \lim_{x \to 0} \frac{\log(1 + x)}{x}\right)$, Continuity of a function, Types of discontinuity, Graphs of discontinuous function.

**Unit 13 The Derivatives** **10 hrs**

Derivative of a function, Derivatives of algebraic, trigonometric, exponential and logarithmic functions by definition (simple forms), Rules of differentiations, Derivatives of parametric and implicit functions, Higher order derivatives

**Unit 14 Application of Derivatives** **8 hrs**

Geometric interpretation of derivative, Monotonocity of a function, Interval of monotonocity, Extrema of a function, Concavity, Points of inflection, Derivative as a rate measure

**Unit 15 Antiderivatives and Its Applications** **10 hrs**

Antiderivative, Integration using basic integrals, Integration by substitution and by parts method, The definite integral, The definite integral as an area under the given curve, Area between two curves

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CHAPTER 1

# Sets, Real Number System and Logic

## 1.1 Sets

In this section, we review the set and set operations. So, we give only the definition of different types of sets and some relations which we use in other sections.

### a) Set

The word "set" is known to carry the same meaning as the words **collection**, **class** and **aggregate**. We make no attempt to define it. However a **set** may be thought of as a well-defined list or collection of material objects such as books and pens or conceptual objects such as numbers and points. Each object of a set is called an **element** or **member** of the set.

The collection of the vowels a, e, i, o and u of the English alphabet constitute a set. Here each of the vowels is an element or member of the set.

### b) Notations

Sets are usually denoted by capital letters such as A, B, C, ...... X, Y, Z and the elements of the set by the small letters such as a, b, c, ... x, y, z.

When we talk about a set, we always have to be sure whether an object "is an element of" or "is a member of" or "belongs to" the set. The symbol $\in$ (epsilon) is used to denote "belongs to" or "is an element of" or "is the member of" whereas $\notin$ is used to "does not belong to" or "is not the element of". The symbol $\Rightarrow$ is used for implies and $\Leftrightarrow$ for implies and implied by.

***Example :***

i) If A is the set of first natural numbers, then $1 \in A$ but $0 \notin A$.

ii) $x^2 = 4 \Rightarrow x = \pm 2$.

### Specification of a set

A set can be specified or described in several ways. But only the following two ways are mainly used.

### i) Tabular form

In this method, the elements are listed without repetition, separate the elements by commas and enclose them in braces { }. This method is known as the roster method or listing or tabular form.

Example : A = {a, e, i, o, u} is an example of a set in the tabular form.

### ii) Set builder's form

Sometimes, a specification of a set by tabular form may be inconvinient or even impossible. So, in such a situation, we specify a set by stating the property which an element of the set satisfies. Thus if A is a set of elements satisfying the property P then

$$A = \{x : x \text{ satisfies } P\}$$

Here x represents an arbitrary element of the set A.

Example : A = {x : x is a vowel}

## Special Kinds of Sets

Some special kinds of sets are defined below:

### a) Empty set

A set having no element is called the **empty set** or **null set** or **void set**. It is denoted by the Greek letter ø (phi) or { }.

Example : M = {x : x is a male student in a girl's campus} is an empty set.

### b) Finite set

A set containing a finite number of elements is known as a finite set.

A = {x : x is a month of a year} and B = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} are the examples of finite sets.

### c) Infinite set

A set which is not finite, is known as an infinite set.

Example: A = {x : x is an integer}; B = {x : x is a point on a line} are the examples of infinite set.

## 1.2 Relation between Sets

A set may have one or more elements common with another set. Also, two sets may not have elements common to them. Depending upon the various possibilities, we have the following relations between the sets.

### a) Subset

A set A is said to be a subset of the set B if every element of A is also an element of B. This relation is denoted by $A \subseteq B$. This is read as A is contained in B or B contains A. Here B is also known as the superset of A and we write as $B \supseteq A$.

Symbolically, $A \subseteq B$ is defined as $x \in A \Rightarrow x \in B$.

If every element of set A is also an element of set B but there is at least one element of B which is not an element of A, then A is known as the **proper subset** of B.

This relation is denoted by $A \subset B$.

$A = \{x : x \text{ is a letter in English alphabet}\}$

$B = \{x : x \text{ is a vowel}\}$

Then, $B \subset A$.

### b) Equal sets

Two sets A and B are said to be equal or identical or same if they have the same elements. They are denoted by $A = B$.

Thus if $A \subseteq B$ and $B \subseteq A$, then $A = B$.

Also if $x \in A \Rightarrow x \in B$ and $x \in B \Rightarrow x \in A$, then $A = B$.

Example :

a) $A = \{a, e, i, o, u\}$

$B = \{x : x \text{ is a vowel}\}$

then $A = B$

b) If $A = \{1, 2, 3\}$ and $B = \{3, 2, 1\}$ then $A = B$

Because in a set, the order of occurrence of the elements is immaterial.

### c) Intersecting sets

Two sets A and B are said to be intersecting if they have atleast one element in common.

Example : If $A = \{s, u, n\}$ and $B = \{m, o, n\}$ then A and B are intersecting sets because the element n is common.

### d) Disjoint Sets

Two sets A and B are said to be disjoint if they have no elements in common.

Example : $A = \{s, u, n\}$ and $B = \{e, a, r, t, h\}$ are disjoint sets because there is no element common between A and B.

### e) Power set

The collection or the set of all possible subsets of any set S is called the power set of S. It is denoted by P(S) or $2^S$.

Example : If S = {1, 2} then its subsets are ø, {1}, {2}, {1, 2}.

So, the power set of S is

$$2^S = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$$

### f) Equivalent sets

Two sets A and B are said to be equivalent if they have the same number of elements. They are denoted by A ~ B.

Example : A = {p, q, r} and B = {a, b, c} are the equivalent sets because they have same number of elements.

### g) Universal set

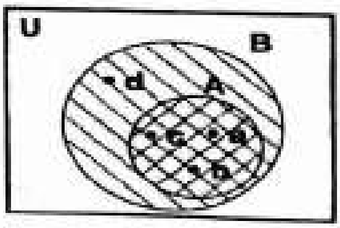
A fixed set is said to be a universal set if all the sets under discussion are the subsets of the fixed set. The universal set is denoted by U.

Example : For the sets of people of different countries, the set of people in the world is the universal set.

### Venn Diagram

The diagrammatic representation of sets, set relations and set operations is known as a venn diagram. It consists of a universal set U represented by a rectangle, subsets of U by the closed curves and the elements of the sets by the points within the closed curve.

Example : If A = {a, b, c} and B = {a, b, c, d} then $A \subset B$. Its venn-diagram is presented below:




$A \subset B$

## 1.3 Operation on Sets

Two given sets can be combined together to produce a new set. The various methods to produce a new set with the help of two given sets are the operations on sets. The following are the operations on sets.

### a) Union

The union of two sets A and B is defined as the set of all elements which belong to A or B or both. In symbol, we denote it by $A \cup B$ and read as 'A union B' or 'A cup B'.

Symbolically, $A \cup B = \{x : x \in A \text{ or } x \in B\}$

Example :

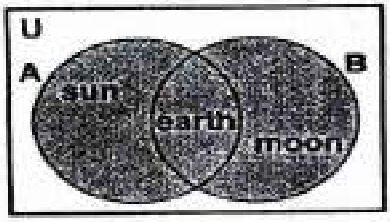
Let $A = \{\text{sun, earth}\}$

$B = \{\text{earth, moon}\}$

Then $A \cup B = \{\text{sun, earth}\} \cup \{\text{earth, moon}\}$

$= \{\text{sun, earth, moon}\}$

The venn diagram of $A \cup B$ is given below :




$A \cup B$

The following relations are very essential to remember that $x \in A \cup B \Rightarrow x \in A$ or $x \in B$ but $x \notin A \cup B \Rightarrow x \notin A$ and $x \notin B$.

### b) Intersection

The intersection of two sets A and B is the set of all elements belonging to both sets A and B. It is denoted by $A \cap B$ and is read as 'A intersection B' or 'A cap B'.

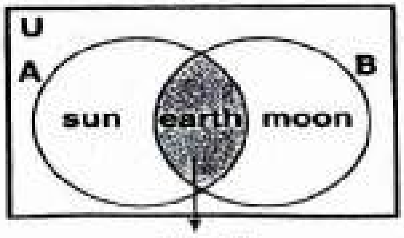
Symbolically, $A \cap B = \{x : x \in A \text{ and } x \in B\}$

Example :

Let $A = \{\text{sun, earth}\}$, $B = \{\text{earth, moon}\}$

Then $A \cap B = \{\text{sun, earth}\} \cap \{\text{earth, moon}\} = \{\text{earth}\}$

The venn diagram of $A \cap B$ is given below:




$A \cap B$

The following relations are very essential to remember that $x \in A \cap B \Rightarrow x \in A$ and $x \in B$ but $x \notin A \cap B \Rightarrow x \notin A$ or $x \notin B$.

### c) Difference

The difference of two sets A and B is the set of all elements of A but not belonging to B. We denote it by A – B and read as 'A difference B'.

Symbolically, $A - B = \{x : x \in A \text{ and } x \notin B\}$

Similarly, $B - A = \{x : x \in B \text{ and } x \notin A\}$

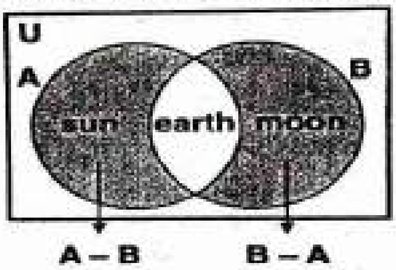
Example :

Let $A = \{\text{sun, earth}\}, B = \{\text{earth, moon}\}$

Then $A - B = \{\text{sun, earth}\} - \{\text{earth, moon}\} = \{\text{sun}\}$

and $B - A = \{\text{earth, moon}\} - \{\text{sun, earth}\} = \{\text{moon}\}$

The venn diagram of A – B and B – A is given below:




### d) Complement

The complement of a set A is the set of all elements in the universal set U that do not belong to A. We denote it by $\overline{A}$ and read as 'A bar'. The symbols A′ and $A^C$ are also used to denote the complement of A.

Symbolically, $\overline{A} = \{x : x \in U \text{ and } x \notin A\}$

$= \{x : x \notin A\}$

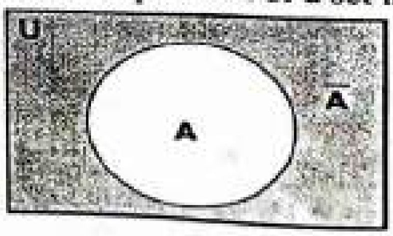
Example :

Let $U = \{x : x \text{ is a student of class XI}\}$

$A = \{x : x \text{ is a male student of Class XI}\}$

$\overline{A} = U - A = \{x : x \text{ is a female student of class XI}\}$

The venn diagram of the complement of a set is given below:



In case of a complement of a set, the following relations are to be remembered.

$$x \in \overline{A} \quad \Rightarrow \quad x \notin A$$

and $$x \in A \quad \Rightarrow \quad x \notin \overline{A}$$

**e) Symmetric difference**

The union of the differences A – B and B – A of two sets A and B is called the symmetric difference of A and B. We denote it by A Δ B and read as 'A delta B'.

Symbolically, $A \Delta B = (A - B) \cup (B - A)$
$= \{x : x \in A - B \text{ or } x \in B - A\}$

Thus, $x \in A \Delta B \Rightarrow x \in A$ or $x \in B$ but $x \notin A \cap B$.

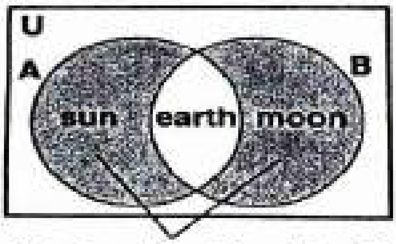
Example :

Let $A = \{\text{sun, earth}\}$, $B = \{\text{earth, moon}\}$

Then, $A - B = \{\text{sun}\}$, $B - A = \{\text{moon}\}$

$A \Delta B = (A - B) \cup (B - A)$
$= \{\text{sun}\} \cup \{\text{moon}\}$
$= \{\text{sun, moon}\}$

The venn diagram of A Δ B is given below:




**A Δ B = (A – B) ∪ (B – A)**

## 1.4 Cardinal number of a finite set

The number of distinct elements in a finite set is known as the cardinal number of the set. The cardinal number of a finite set A is denoted by n(A) or #(A).

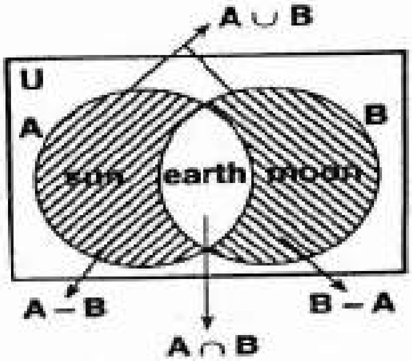
Two sets A and B are said to have the same cardinality or cardinal number if they are equivalent (i.e. there is a one-to-one correspondance between them.)

Example :

If $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4, 5\}$ then $n(A) = 3$ and $n(B) = 5$.

## Some important results from venn diagram
## (Cardinal number of union of two sets)

Let A and B be two non-empty sets. Again let n(A), n(B), $n(A \cup B)$ and $n(A \cap B)$ be the cardinal numbers of the sets A, B, $A \cup B$ and $A \cap B$ respectively.




Then,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

If A and B are disjoint sets then

$$A \cap B = \emptyset, \quad \text{so} \quad n(A \cap B) = 0$$

$\therefore$ for disjoint sets, $n(A \cup B) = n(A) + n(B)$

Also, $\quad n(A - B) = n_0(A) = n(A) - n(A \cap B)$

and, $\quad n(B - A) = n_0(B) = n(B) - n(A \cap B)$

In the same way, if A, B and C are three sets, then

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$

If A, B and C are disjoint sets, then

$$n(A \cup B \cup C) = n(A) + n(B) + n(C)$$

## Worked Out Examples

***Example 1***

Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9}, A = {1, 2, 3}, B = {1, 2, 3, 4, 5}, C = {3, 4, 5} and D = {6, 7}

Perform the following indicated operations:

a) $A \cup B$   b) $A \cap C$   c) $\overline{A \cup B}$

d) $A - C$   e) $(A - C) \cap C$   f) $A \Delta D$

***Solution :***

a) $A \cup B = \{1, 2, 3\} \cup \{1, 2, 3, 4, 5\}$
$= \{1, 2, 3, 4, 5\}$

b) $A \cap C = \{1, 2, 3\} \cap \{3, 4, 5\} = \{3\}$

c) $\overline{A \cup B} = U - (A \cup B)$
$= \{1, 2, 3, 4, 5, 6, 7, 8, 9\} - \{1, 2, 3, 4, 5\}$
$= \{6, 7, 8, 9\}$

d) $A - C = \{1, 2, 3\} - \{3, 4, 5\} = \{1, 2\}$

e) $(A - C) \cap C = \{1, 2\} \cap \{3, 4, 5\} = \phi$

f) $A - D = \{1, 2, 3\} - \{6, 7\} = \{1, 2, 3\}$
$D - A = \{6, 7\} - \{1, 2, 3\} = \{6, 7\}$
$A \Delta D = (A - D) \cup (D - A)$
$= \{1, 2, 3\} \cup \{6, 7\}$
$= \{1, 2, 3, 6, 7\}$

***Example 2***

Given $U = \{1, 2, 3, \ldots\ldots\ldots\ldots, 15\}$, $A = \{x : x \geq 8\}$, $B = \{x : x \leq 4\}$ and $C = \{x : 4 < x < 12\}$. Find $A \cap C$, $B \cup C$, $(A \cup C) - B$ and $(A \cup B) - (A \cap B)$.

***Solution :***

$U = \{1, 2, 3, 4, \ldots\ldots, 15\}$
$A = \{x : x \geq 8\} = \{8, 9, 10, \ldots, 15\}$
$B = \{x : x \leq 4\} = \{1, 2, 3, 4\}$
$C = \{x : 4 < x < 12\} = \{5, 6, 7, \ldots, 11\}$

Then,

$A \cap C = \{8, 9, 10, \ldots, 15\} \cap \{5, 6, 7, \ldots, 11\}$
$= \{8, 9, 10, 11\}$

$B \cup C = \{1, 2, 3, 4\} \cup \{5, 6, 7, \ldots, 11\}$
$= \{1, 2, 3, 4, 5, 6, \ldots, 10, 11\}$

$A \cup C = \{8, 9, 10, \ldots, 15\} \cup \{5, 6, 7, \ldots, 11\}$
$= \{5, 6, 7, 8, 9, \ldots, 14, 15\}$

$(A \cup C) - B = \{5, 6, 7, 8, 9, \ldots 15\} - \{1, 2, 3, 4\}$
$= \{5, 6, 7, 8, 9, \ldots, 14, 15\}$

$A \cup B = \{8, 9, 10, \ldots, 15\} \cup \{1, 2, 3, 4\}$
$= \{1, 2, 3, 4, 8, 9, 10, \ldots, 15\}$

$A \cap B = \{8, 9, 10, \ldots, 15\} \cap \{1, 2, 3, 4\}$
$= \phi$
$(A \cup B) - (A \cap B) = \{1, 2, 3, 4, 8, 9, 10, \ldots, 15\} - \phi$
$= \{1, 2, 3, 4, 8, 9, 10, \ldots, 15\}$

***Example 3***

Make a list of subsets of $\{1, 2, 3\}$

***Solution :***

The required subsets of the given set are
$\phi$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{2, 3\}$, $\{1, 3\}$ and $\{1, 2, 3\}$.

***Example 4***

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 2\}$, $B = \{1, 2, 3, 4, 5\}$ and $C = \{2, 3, 4, 5\}$. Verify the following relations

a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

b) $\overline{A \cup B} = \overline{A} \cap \overline{B}$

c) $A - (B \cup C) = (A - B) - C$

***Solution :***

a) $B \cap C = \{1, 2, 3, 4, 5\} \cap \{2, 3, 4, 5\}$
$= \{2, 3, 4, 5\}$

Now, $A \cup (B \cap C) = \{1, 2\} \cup \{2, 3, 4, 5\}$
$= \{1, 2, 3, 4, 5\}$

Again, $A \cup B = \{1, 2\} \cup \{1, 2, 3, 4, 5\}$
$= \{1, 2, 3, 4, 5\}$

and $A \cup C = \{1, 2\} \cup \{2, 3, 4, 5\}$
$= \{1, 2, 3, 4, 5\}$

Now, $(A \cup B) \cap (A \cup C)$
$= \{1, 2, 3, 4, 5\} \cap \{1, 2, 3, 4, 5\}$
$= \{1, 2, 3, 4, 5\}$

Hence $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

b) $A \cup B = \{1, 2, 3, 4, 5\}$

$\overline{A \cup B} = U - (A \cup B)$
$= \{1, 2, 3, 4, \ldots 9\} - \{1, 2, 3, 4, 5\}$
$= \{6, 7, 8, 9\}$

$\overline{A} = U - A \quad = \{3, 4, 5, 6, 7, 8, 9\}$

$\overline{B} = U - B \quad = \{6, 7, 8, 9\}$

Now, $\overline{A} \cap \overline{B} = \{3, 4, 5, 6, 7, 8, 9\} \cap \{6, 7, 8, 9\}$

$= \{6, 7, 8, 9\}$

Hence $\overline{A \cup B} = \overline{A} \cap \overline{B}$

c) $B \cup C = \{1, 2, 3, 4, 5\} \cup \{2, 3, 4, 5\} = \{1, 2, 3, 4, 5\}$

Now, $A - (B \cup C) = \{1, 2\} - \{1, 2, 3, 4, 5\} = \phi$

Again $A - B = \{1, 2\} - \{1, 2, 3, 4, 5\} = \phi$

$(A - B) - C = \phi - \{2, 3, 4, 5\} = \phi$

Hence $A - (B \cup C) = (A - B) - C$

***Example 5***

In a certain village of Nepal, all people speak Nepali or Tharu or both languages. If 90% of the people speak Nepali and 20% Tharu language, how many speak (a) both languages b) Nepali languages only c) Tharu language only.

***Solution :***

Let total no. of people in the village $n(N \cup T) = 100$

Then no. of people speaking Nepali language $= n(N) = 90$

and the no. of people speaking Tharu language $= n(T) = 20$

No. of people speaking with languages $n(N \cap T) = ?$

a) $n(N \cup T) = n(N) + n(T) - n(N \cap T)$

$100 = 90 + 20 - n(N \cap T)$

$\therefore \quad n(N \cap T) = 110 - 100 = 10$

$\therefore$ 10% of the people speak both languages

b) $n_0(N)$ = no. of people speaking Nepali language only

$= n(N) - n(N \cap T)$

$= 90 - 10 = 80$

80% of the people speak Nepali language only.

c) $n_0(T)$ = no. of people speaking Tharu language only

$= n(T) - n(N \cap T)$

$= 20 - 10 = 10$

$\therefore$ 10% of the people speak Tharu language only.

***Example 6***

If $n(U) = 360$, $n(A) = 240$, $n(B) = 160$, find the maximum value of $n(A \cup B)$ and the minimum value of $n(A \cap B)$. When will the value of $n(A \cap B)$ be maximum and find its value.

***Solution :***

$$
\begin{aligned}
n(A \cup B) &= n(A) + n(B) - n(A \cap B)\\
&= 240 + 160 - n(A \cap B)\\
&= 400 - n(A \cap B)
\end{aligned}
$$

Since $n(A \cap B) \geq 0$, so $n(A \cup B) \leq 400$

But $n(U) = 360$, so $n(A \cup B) \not> 360$

$\therefore$ the maximum value of $n(A \cup B) = 360$.

When $n(A \cup B)$ is maximum, $n(A \cap B)$ will be minimum.

$\therefore$ min. value of $n(A \cap B)$

$$
\begin{aligned}
&= n(A) + n(B) - \text{Max. value of } n(A \cup B)\\
&= 240 - 160 - 360 = 40
\end{aligned}
$$

Again, $n(A \cap B)$ will be maximum when $B \subset A$.

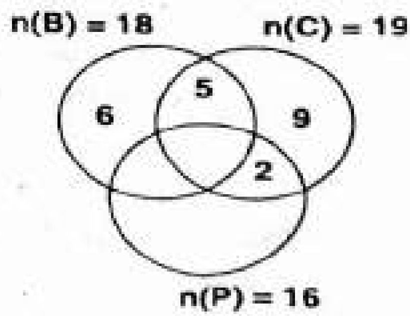
$\therefore$ the max. value of $n(A \cap B) = n(B) = 160$

***Example 7***

In a group of students 18 read Biology, 19 read Chemistry and 16 read Physics. Six read Biology only, 9 read Chemistry only, 5 read Biology and Chemistry only and 2 read Chemistry and Physics only.

a) How many read all three subjects?
b) How many read Biology and Physics only?
c) How many read Physics only?
d) How many students are there all together?

***Solution :***




No. of students reading Biology $= n(B) = 18$

No. of students reading Chemistry $= n(C) = 19$

No. of students reading Physics $= n(P) = 16$

No. of students reading Biology only $= n_0(B) = 6$

No. of students reading Chemistry only $= n_0(C) = 9$

No. of students reading Biology and Chemistry only = $n_0(B \cap C) = 5$

No. of students reading Chemistry and Physics only = $n_0(C \cap P) = 2$

a) No. of students reading all three subjects

$= n(B \cap C \cap P)$

$= n(C) - n_0(C) - n_0(B \cap C) - n_0(C \cap P)$

$= 19 - 9 - 5 - 2 = 3$

b) No. of students reading Biology and Physics only

$= n_0(B \cap P)$

$= n(B) - n_0(B) - n_0(B \cap C) - n(B \cap C \cap P)$

$= 18 - 6 - 5 - 3$

$= 4$

c) No. of students reading Physics only

$= n_0(P)$

$= n(P) - n_0(C \cap P) - n_0(B \cap C \cap P) - n_0(B \cap P)$

$= 16 - 2 - 3 - 4$

$= 7$

d) Total no. of students

$= 6 + 5 + 9 + 2 + 3 + 4 + 7$

$= 36$

## EXERCISE 1.1

1. If U = {1, 2, 3, 4, ..., 9, 10}, A = {1, 2, 3, 4}, B = {3, 5, 7, 8} and C = {1, 2, 7, 8}, find

   a) $A \cup B$ b) $A \cap C$ c) $(A - B) \cap C$

   d) $\overline{A \cup C}$ e) $\overline{B} \cup \overline{C}$ f) $(A \cup B) - C$

2. Let U = {a, b, c, d, ..., i, j, k}, A = {b, c, d, e}, B = {d, e, f, g, h, i}, C = {a, e, i, o, u}, D = {b, d, j, k}

   Perform the following indicated operations

   a) $A \cup B$ b) $A \cap C$ c) $A - B$

   d) $A - C$ e) $(A - C) \cap C$ f) $A \Delta D$

   g) $(A \cup B) - C$ h) $A \cap \overline{B}$ i) $\overline{A - B}$

3. a) Given the sets,
   $U = \{x : x \text{ is a positive integer less than } 12\}$, $A = \{3, 5, 7, 9\}$, $B = \{1, 2, 3, 8, 9\}$, $C = \{1, 4, 7, 10\}$, find
   $\overline{(A \cup B)}$, $(A - B) \cup C$, $(A - C) \cap B$.

   b) Given $U = \{x : x \text{ is a natural number upto } 20\}$, $A = \{x : x \geq 6\}$, $B = \{x : x \leq 8\}$ and $C = \{x : 10 < x < 15\}$, find $B \cup C$, $A \cap B$, $A - C$ and $\overline{A \cup B}$.

4. a) If $A = \{x : x = 2n + 1, n \leq 5, n \in N\}$ and $B = \{x : x = 3n - 2, n \leq 4, n \in N\}$, find $A \cup B$, $A \cap B$ and $B - A$.

   b) If A is the set of all multiples of 3 less than 20, B is the set of multiples of 6 less or equal to 30 and U is the set of all natural numbers, find $A \cap B$ and $A - B$.

   c) If $U = \{x : -1 \leq x - 2 \leq 7\}$, $A = \{x : x \text{ is a prime number}\}$ and $B = \{x : x \text{ is an odd number}\}$, find
   i) the set of elements which are either prime or odd.
   ii) the set of elements which are prime as well as odd
   iii) the set of elements which are prime but not odd

5. Let $U = \{a, b, c, d, e, f, g, h\}$, $A = \{a, b, c, d\}$, $B = \{c, d, e, f\}$ and $C = \{d, e, f, g, h\}$. Verify the following relations
   a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
   c) $\overline{A \cup B} = \overline{A} \cap \overline{B}$
   d) $A - (B \cup C) = (A - B) \cap (A - C)$

6. a) If $n(U) = 100$, $n(A) = 75$, $n(B) = 40$, $n(A \cup B) = 80$, find
   $n(A \cap B)$, $n(A - B)$, $n(\overline{A \cup B})$.

   b) If $n(A) = 37$, $n(B) = 50$, and $A \subset B$, find $n(A \cup B)$, $n(A \cap B)$.

7. 32 students play basketball and 25 students play volley ball. It is found that 13 students play both games. Find the number of students playing atleast one game.

8. In a certain village of Nepal, all people speak Nepali or Newari or both. If 60% of the people speak Nepali and 50% speak Newari, how many speak both languages?

9. Of a group of 120 students, 90 take Mathematics and 72 take Statistics. If 10 students take neither of the two, how many take both?

10. In a group of 65 persons, 20 drink tea but not coffee, 38 drink tea and 15 take neither of the two. Find
    a) how many drink tea and coffee both.
    b) how many drink coffee but not tea.

11. In an examination, 27% of the students failed in Mathematics and 31% failed in Statistics. If 6% of the students failed in both subjects, find the percentage of the students
    a) failed in examination.
    b) passed in both subjects?

12. In a group of students, 12 read Mathematics, 15 read Statistics, 11 read Physics, 4 read mathematics only, 7 read Statistics only, 3 read Statistics and Physics only and 1 read mathematics and Statistics only.
    a) How many read all three subjects?
    b) How many read Mathematics and Physics only?
    c) How many read Physics only?
    d) How many students are there all together?

13. A survey of 500 television viewers produced the following information:

    285 watch football, 195 watch hockey, 115 watch basketball, 45 watch football and basketball, 70 watch football and hockey, 50 watch hockey and basket ball, 50 do not watch any of the three games. How many watch all three games? How many watch exactly one of the three games?

### *Answers*

1. a) {1, 2, 3, 4, 5, 7, 8} b) {1, 2} c) {1, 2} d) {5, 6, 9, 10}
   e) {1, 2, 3, 4, 5, 6, 9, 10} f) {3, 4, 5}
2. a) {b, c, d, e, f, g, h, i} b) {e} c) {b, c} d) {b, c, d} e) ø
   f) {c, e, j, k} g) {b, c, d, f, g, h} h) {b, c}
   i) {a, d, e, f, g, h, i, j, k}
3. a) {4, 6, 10, 11}, {1, 4, 5, 7, 10}, {3, 9}
   b) {1, 2, 3, 4, ... 8, 11, 12, 13, 14}, {6, 7, 8},
   {6, 7, 8, 9, 10, 15, 16, 17, 18, 19, 20}, ø
4. a) {1, 3, 4, 5, 7, 9, 10, 11}, {7}, {1, 4, 10}
   b) {6, 12, 18}, {3, 9, 15}
   c) i) {1, 2, 3, 5, 7, 9} ii) {3, 5, 7} iii) {2}
6. a) 35, 40, 20 b) 50, 37
7. 44 8. 10% 9. 52 10. a) 18 b) 12
11. a) 52% b) 48% 12. a) 4 b) 3 c) 1 d) 23 13. 20, 325

## 1.5 Laws of Algebra of Sets

Combinations of various relations and operations defined on sets give rise to a number of interesting and useful results. Some of them are so fundamental that they are considered as the basic laws of set algebra. We discuss them under the following four categories.

### Properties of Inclusion and Equality Relations

**Theorem 1**

Let A, B and C be subsets of a universal set U. Then

i) $A \subseteq B, B \subseteq C \Rightarrow A \subseteq C$

ii) $A = B \Rightarrow B = A$

iii) $A = B, B = C \Rightarrow A = C$

iv) $A \subseteq \phi \Rightarrow A = \phi$

**Proof:**

i) Let x be an element of A.

Then, $x \in A \Rightarrow x \in B$ $(\because A \subseteq B)$

$\Rightarrow x \in C$ $(\because B \subseteq C)$

$\therefore A \subseteq C$

ii) $A = B \Rightarrow A \subseteq B$ and $B \subseteq A$

$\Rightarrow B \subseteq A$ and $A \subseteq B$

$\Rightarrow B = A$

iii) Let x be an element of A

Then $x \in A \Rightarrow x \in B$ $(\because A = B)$

$\Rightarrow x \in C$ $(\because B = C)$

$\therefore A \subseteq C$

Again let y be an element of C.

Then, $y \in C \Rightarrow y \in B$ $(\because C = B)$

$\Rightarrow y \in A$ $(\because B = A)$

$\therefore C \subseteq A$

Since $A \subseteq C$ and $C \subseteq A$, so, $A = C$.

iv) Since $\phi \subseteq A$

By given $A \subseteq \phi$

$\therefore A = \phi$

**b) Properties of Unions**

**Theorem 2**

Let A, B and C be the subsets of a universal set U. Then

i) $A \cup A = A$

ii) $A \cup \phi = \phi$

iii) $A \cup U = U$

iv) $A \cup B = \phi \Rightarrow A = \phi$ and $B = \phi$

v) $A \cup B = B \cup A$

vi) $(A \cup B) \cup C = A \cup (B \cup C)$

**Proof:**

i) $A \cup A = \{x : x \in A \text{ or } x \in A\}$
$= \{x : x \in A\} = A$

ii) $A \cup \phi = \{x : x \in A \text{ or } x \in \phi\}$
$= \{x : x \in A\} = A$

iii) $A \cup U = \{x : x \in A \text{ or } x \in U\}$
$= \{x : x \in U\} \quad (\because A \subset U)$
$= U$

iv) Since, $A \subseteq A \cup B$
$\Rightarrow A \subseteq \phi \quad (\because A \cup B = \phi)$
But $\phi \subseteq A$
$\therefore A = \phi$
Again since, $B \subseteq A \cup B$
$\Rightarrow B \subseteq \phi$
But $\phi \subseteq B$
$\therefore B = \phi$

v) $A \cup B = \{x : x \in A \text{ or } x \in B\}$
$= \{x : x \in B \text{ or } x \in A\}$
$= B \cup A$

vi) $(A \cup B) \cup C = \{x : x \in A \cup B \text{ or } x \in C\}$
$= \{x : (x \in A \text{ or } x \in B) \text{ or } x \in C\}$
$= \{x : x \in A \text{ or } (x \in B \text{ or } x \in C)\}$
$= \{x : x \in A \text{ or } x \in B \cup C\}$
$= A \cup (B \cup C)$

### c) Properties of Intersections

**Theorem 3**

Let A, B and C be the subsets of a universal set U. Then,

i) $A \cap A = A$

ii) $A \cap \phi = \phi$

iii) $A \cap U = A$

iv) $A \cap B = B \cap A$

v) $(A \cap B) \cap C = A \cap (B \cap C)$

**Proof:**

i) $A \cap A = \{x : x \in A \text{ and } x \in A\}$
$= \{x : x \in A\} = A$

ii) $A \cap \phi = \{x : x \in A \text{ and } x \in \phi\}$
$= \{x : x \in \phi\} = \phi$

iii) $A \cap U = \{x : x \in A \text{ and } x \in U\}$
$= \{x : x \in A\} \quad (\because A \subset U)$
$= A$

iv) $A \cap B = \{x : x \in A \text{ and } x \in B\}$
$= \{x : x \in B \text{ and } x \in A\}$
$= B \cap A$

v) $(A \cap B) \cap C = \{x : x \in A \cap B \text{ and } x \in C\}$
$= \{x : (x \in A \text{ and } x \in B) \text{ and } x \in C\}$
$= \{x : x \in A \text{ and } (x \in B \text{ and } x \in C)\}$
$= \{x : x \in A \text{ and } x \in B \cap C\}$
$= A \cap (B \cap C)$

### d) Miscellaneous Properties

Let A, B and C be the subsets of a universal set U. Then,

I. i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

II. i) $A \cup \overline{A} = U$
ii) $A \cap \overline{A} = \phi$
iii) $\overline{\overline{A}} = A$

III. De-Morgan's Law

i) $\overline{A \cup B} = \overline{A} \cap \overline{B}$

ii) $\overline{A \cap B} = \overline{A} \cup \overline{B}$

IV. i) $A - (B \cap C) = (A - B) \cup (A - C)$

ii) $A - (B \cup C) = (A - B) \cap (A - C)$

iii) $A \cap (B - C) = (A \cap B) - (A \cap C)$

**Proofs:**

I. i) $A \cup (B \cap C) = \{x : x \in A \text{ or } x \in B \cap C\}$
$= \{x : x \in A \text{ or } (x \in B \text{ and } x \in C)\}$
$= \{x : (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C)\}$
$= \{x : x \in A \cup B \text{ and } x \in A \cup C\}$
$= (A \cup B) \cap (A \cup C)$

ii) $A \cap (B \cup C) = \{x : x \in A \text{ and } x \in B \cup C\}$
$= \{x : x \in A \text{ and } (x \in B \text{ or } x \in C)\}$
$= \{x : (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)\}$
$= \{x : x \in A \cap B \text{ or } x \in A \cap C\}$
$= (A \cap B) \cup (A \cap C)$

II. i) $A \cup \overline{A} = \{x : x \in A \text{ or } x \in \overline{A}\}$
$= \{x : x \in U\} = U$

ii) $A \cap \overline{A} = \{x : x \in A \text{ and } x \in \overline{A}\}$
$= \{x : x \in A \text{ and } x \notin A\}$
$= \{x : x \in \emptyset\}$
$= \emptyset$

iii) $\overline{\overline{A}} = \{x : x \notin \overline{A}\}$
$= \{x : x \in A\} = A$

III. i) $\overline{A \cup B} = \{x : x \notin A \cup B\}$
$= \{x : x \notin A \text{ and } x \notin B\}$
$= \{x : x \in \overline{A} \text{ and } x \in \overline{B}\}$
$= \overline{A} \cap \overline{B}$

ii) $\overline{A \cap B} = \{x : x \notin A \cap B\}$

$= \{x : x \notin A \text{ or } x \notin B\}$

$= \{x : x \in \overline{A} \text{ or } x \in \overline{B}\}$

$= \overline{A} \cup \overline{B}$

IV. i) $A - (B \cap C) = \{x : x \in A \text{ and } x \notin B \cap C\}$

$= \{x : x \in A \text{ and } (x \notin B \text{ or } x \notin C)\}$

$= \{x : (x \in A \text{ and } x \notin B) \text{ or } (x \in A \text{ and } x \notin C)\}$

$= \{x : x \in A - B \text{ or } x \in A - C\}$

$= (A - B) \cup (A - C)$

ii) $A - (B \cup C) = \{x : x \in A \text{ and } x \notin B \cup C\}$

$= \{x : x \in A \text{ and } (x \notin B \text{ and } x \notin C)\}$

$= \{x : (x \in A \text{ and } x \notin B) \text{ and } (x \in A \text{ and } x \notin C)\}$

$= \{x : x \in A - B \text{ and } x \in A - C\}$

$= (A - B) \cap (A - C)$

iii) $A \cap (B - C) = \{x : x \in A \text{ and } x \in B - C\}$

$= \{x : x \in A \text{ and } (x \in B \text{ and } x \notin C)\}$

$= \{x : (x \in A \text{ and } x \in B) \text{ and } (x \in A \text{ and } x \notin C)\}$

$= \{x : x \in A \cap B \text{ and } x \notin A \cap C\}$

$= (A \cap B) - (A \cap C)$

We now summarize the above properties as the "Laws of Algebra of sets". The laws of algebra of sets are presented in the following form.

Let A, B and C be the subsets of a universal set U.

**1. Inclusion Laws**

i) $A \subseteq A$

ii) $A \subseteq B, B \subseteq A \Rightarrow A = B$

iii) $A \subseteq B, B \subseteq C \Rightarrow A \subseteq C$

iv) $\emptyset \subseteq A$

**2. Identity Laws**

i) $A \cup \emptyset = A$

ii) $A \cap \emptyset = \emptyset$

iii) $A \cup U = U$

iv) $A \cap U = A$

**3. Idempotent Laws**

i) $A \cup A = A$

ii) $A \cap A = A$

**4. Commutative Laws**

i) $A \cup B = B \cup A$

ii) $A \cap B = B \cap A$

**5. Associative Laws**

i) $A \cup (B \cup C) = (A \cup B) \cup C$

ii) $A \cap (B \cap C) = (A \cap B) \cap C$

**6. Distributive Laws**

i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

**7. Complement Laws**

i) $\overline{U} = \phi$

ii) $\overline{\phi} = U$

iii) $\overline{\overline{A}} = A$

iv) $A \cup \overline{A} = U$

v) $A \cap \overline{A} = \phi$

**8. De-Morgan's Laws**

i) $\overline{A \cup B} = \overline{A} \cap \overline{B}$

ii) $\overline{A \cap B} = \overline{A} \cup \overline{B}$

**9. Difference Laws**

i) $A - (B \cup C) = (A - B) \cap (A - C)$

ii) $A - (B \cap C) = (A - B) \cup (A - C)$

## Worked Out Examples

***Example 1***

Prove that $A \cap B \subseteq A$.

***Solution :***

Let x be an element of $A \cap B$.

Then, $x \in A \cap B \Rightarrow x \in A$ and $x \in B$

$\Rightarrow x \in A$

$\therefore A \cap B \subseteq A$

***Example 2***

Prove that $A - B \subseteq \overline{B}$

***Solution :***

Let x be an element of A – B

Then, $x \in A - B \Rightarrow x \in A$ and $x \notin B$

$\Rightarrow x \notin \overline{A}$ and $x \in \overline{B}$

$\Rightarrow x \in \overline{B}$

$\therefore A - B \subseteq \overline{B}$

***Example 3***

Prove that $A - B = A \cap \overline{B}$

***Solution :***

$A - B = \{x : x \in A \text{ and } x \notin B\}$

$= \{x : x \in A \text{ and } x \in \overline{B}\}$

$= A \cap \overline{B}$

***Example 4***

If $A \cap B = \emptyset$, prove that $B \subseteq \overline{A}$.

***Solution :***

Let x be an element of B. Then

$x \in B \Rightarrow x \notin A \quad (\because A \cap B = \emptyset)$

$\Rightarrow x \in \overline{A}$

$\therefore B \subseteq \overline{A}$

***Example 5***

Prove that : $A \Delta B = (A \cup B) - (A \cap B)$

***Solution :***

By the definition $A \Delta B = (A - B) \cup (B - A)$

So, $A \Delta B = \{x : x \in A - B \text{ or } x \in B - A\}$

$= \{x : (x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin A)\}$

$= \{x : (x \in A \text{ or } x \in B) \text{ and } (x \notin A \text{ or } x \notin B)\}$

$= \{x : x \in A \cup B \text{ and } x \notin A \cap B\}$

$= (A \cup B) - (A \cap B)$

## EXERCISE 1.2

1. Prove that

   a) $A \cap B \subseteq B$ b) $A - B \subseteq A$ c) $A - B \subseteq A \cup B$

2. If $A \cap B = \emptyset$, prove that

   a) $A \subseteq \overline{B}$ b) $B \cap \overline{A} = B$ c) $A \cup \overline{B} = \overline{B}$

3. If $A \subseteq B$, prove that

   a) $\overline{B} \subseteq \overline{A}$ b) $A \cup B = B$ c) $A \cap B = A$

4. Prove that

   a) $B - A = B \cap \overline{A}$

   b) $A - \overline{B} = A \cap B$ c) $A - B = \overline{B} - \overline{A}$

   d) $A - (A - B) = A \cap B$ e) $(A - B) \cap B = \emptyset$

5. If A, B and C are the subsets of a universal set U, prove that

   a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

   b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

6. If A, B and C are the subsets of a universal set U, prove that

   a) $(A \cup B)' = A' \cap B'$

   b) $(A \cap B)' = A' \cup B'$

   c) $A - (B \cup C) = (A - B) \cap (A - C) = (A - B) - C$

   d) $A - (B \cap C) = (A - B) \cup (A - C)$

## 1.6 Real Number System

Number is one of the basic concepts of mathematics. Its origin goes back to ancient times. Primitive man got this idea through the process of matching (pairing a set of pebbles with the flock of sheep by laying aside a pebble for each sheep) and considering the order of occurrence. In other words, the set of numbers discovered or invented by human beings in the early days of civilization are the counting number. Now, in this section some acquintances with the concept of number and its extension to the system of real numbers and their properties are presented.

### Natural Numbers

The simplest and the most familiar unending chain of consecutive numbers 1, 2, 3, ......, 10, 11, 12, ...... 101, 102, 103, ...... are known as the natural numbers. The natural numbers are also known as the counting numbers. The set of natural numbers are denoted by N.

$\therefore N = \{1, 2, 3, 4, ...., 10, 11, 12, ...., 101, 102, 103, ...\}$

Since the sum and the product of two natural numbers are again natural numbers, so natural numbers are said to be closed under the operation of addition and mutliplication.

For example: $2, 3 \in N$

Then, $2 + 3 = 5 \in N$ and $2.3 = 6 \in N$.

But the difference of two natural numbers may or may not be a natural number. So, a new number i.e. integers are developed.

### Integers

The set of natural numbers together with their negatives including zero are known as the set of integers. 1, 2, 3, 4, ... are also known as the positive integers and –1, –2, –3, –4, ...... the negative integers. The set of integers are denoted by Z or I.

$\therefore Z = \{..., -3, -2, -1, 0, 1, 2, 3, 4, ...\}$

The sum, difference and the product of two integers are again integers. So integers are said to be closed under the operation of addition, subtraction and multiplication.

For example: $2, 3 \in Z$

$2 + 3 = 5 \in Z$, $2 \times 3 = 6 \in Z$ and $2 - 3 = -1 \in Z$

But the quotient of two integers may or may not be an integer. So, a new number i.e. rational numbers are developed.

## Rational Number

A number in the form of $\frac{p}{q}$ where p and q are integers and $q \neq 0$ is known as a rational number.

The set of rational numbers are denoted by Q.

$\therefore Q = \{x : x = \frac{p}{q}$, p and q are integers and $q \neq 0\}$

A rational number can also be expressed as a terminating decimal or a repeating decimals.

For example: $3, -2, -\frac{3}{2}, \frac{1}{3}$, 0.25, 0.666... etc are the rational numbers.

The sum, difference, product and the quotient of two rational numbers are again rational numbers. So, rational numbers are closed under the operation of addition, subtraction, multiplication and division.

But extraction of root of a rational number may not be a rational number. So, again a new number known as an irrational number is developed.

## Irrational Numbers

Numbers which are not rational are known as irrational. That is, a number which cannot be expressed in the form of $\frac{p}{q}$ where p and q are integers and $q \neq 0$, is known as an irrational number. The set of irrational number is denoted by $\overline{Q}$ (the complement of Q).

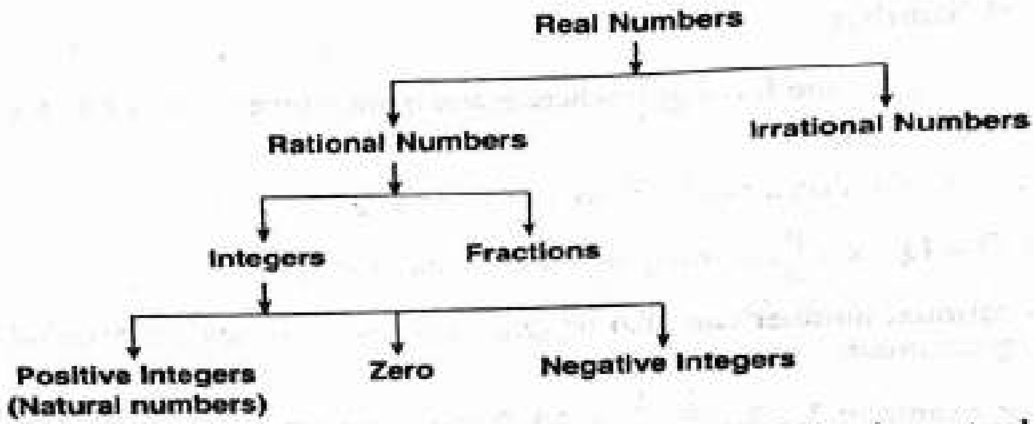
Irrational numbers also have the non-terminating and non-repeating decimals. $\sqrt{2}, \sqrt[3]{4}, \pi$, e are the examples of irrational numbers.

## Real Numbers

The union of the set of rational and irrational numbers is known as the set of real numbers. So, by the set of real numbers, we have the set of natural numbers, the set of integers, the set of rational numbers and the set of irrational numbers.

The set of real numbers are denoted by R. Thus the set of rational and irrational numbers taken together, form a new system of numbers known as the real number system.

A diagram showing the family of real numbers is presented below :




The family members of the real numbers have the following set relation.

$$N \subset Z \subset Q \subset R$$

## 1.7 Field Axioms

Let R, the set of real numbers together with two binary operations + and ( . ) known as the operations of addition and multiplication, satisfy the following properties or axioms.

### Addition Axiom

**i) Closure property:**

The sum of two real numbers is a real number. This property is known as the closure property. Thus, if $a, b \in R$ then $a + b \in R$.

**ii) Commutative property:**

The sum of two real numbers is the same in whatever order they are added. That is, if $a, b \in R$ then $a + b = b + a$

**iii) Associative property:**

The sum of any two of the three real numbers with the third will be the same. That is, if $a, b, c \in R$ then

$$a + (b + c) = (a + b) + c$$

**iv) Additive identity**

For every $a \in R$, there is a real number 0 such that

$$a + 0 = 0 + a = a$$

Here 0 is known as the additive identity.

**v) Additive inverse:**

For every $a \in R$, there exists $-a \in R$ such that

$$a + (-a) = (-a) + a = 0$$

Here $-a$ is known as the additive inverse of $a$.

**Multiplicative Axioms**

**i) Closure property:**

The product of two real numbers is a real number. That is if $a, b \in R$ then $a.b \in R$.

**ii) Commutative property:**

The product of two real numbers is the same whatever in order they are multiplied. That is

$$a, b \in R, \text{ then } a.b = b.a$$

**iii) Associative property:**

The product of any two of the three real numbers multiplied by a third is the same. That is if $a, b, c \in R$, then

$$a(bc) = (ab)c$$

**iv) Multiplicative identity**

For every $a \in R$, there exists $1 \in R$ such that

$$1.a = a.1 = a$$

Here 1 is known as the multiplicative identity.

**v) Multiplicative inverse**

For every $a \in R$ $(a \neq 0)$, there exists $a^{-1} \in R$ such that

$$a.a^{-1} = a^{-1}.a = 1$$

Here $a^{-1}$ is known as the multiplicative inverse of $a$.

**vi) Distributive property**

For the real numbers $a, b, c$ the product of $a$ and $b + c$ is same as the sum of the products $ab$ and $ac$.

That is, if $a, b, c \in R$, then

$$a(b + c) = ab + ac$$

The set R, together with two binary operations called addition (+) and multiplication (.) and satisfying the above axioms (called field axiom) constitute the field.

## 1.8 Order Axioms

Besides the field axioms, the real numbers satify the following order axioms also.

A real number $a$ is positive if $a > 0$ and negative if $a < 0$. The product of two positive numbers is positive. That is, if $a, b \in R$, $a > 0$, $b > 0$ then $ab > 0$.

**The axiom of trichotomy (Trichotomy property)**

If $a$ and $b$ are two real numbers then one and only one of the following relations holds

$$a < b, a = b, a > b$$

**The axiom of transitivity (Transitivity property)**

If $a$, $b$ and $c$ are three real numbers such that $a > b$ and $b > c$ then $a > c$.

i.e. If $a, b, c \in \mathbb{R}$, then $a > b, b > c \Rightarrow a > c$

Also, $a < b$ and $b < c \Rightarrow a < c$

**The axiom of addition (Addition property)**

Let $a$, $b$ and $c$ be three real numbers.

If $a > b$ then $a + c > b + c$

Also if $a < b$ then $a + c < b + c$

**The axiom of multiplication (Multiplicative property)**

Let $a$, $b$ and $c$ be three real numbers.

If $a > b$ then $ac > bc$ when $c > 0$

If $a > b$ then $ac < bc$ when $c < 0$

Again if $a > b$ then $\frac{a}{c} > \frac{b}{c}$ when $c > 0$

and if $a > b$ then $\frac{a}{c} < \frac{b}{c}$ when $c < 0$

Similarly if $a < b$ then $ac < bc$ if $c > 0$

and if $a < b$ then $ac > bc$ if $c < 0$

**The axiom of density**

If $a$ and $b$ are two real numbers such that $a < b$, then there exists a real number $c$ such that

$$a < c < b.$$

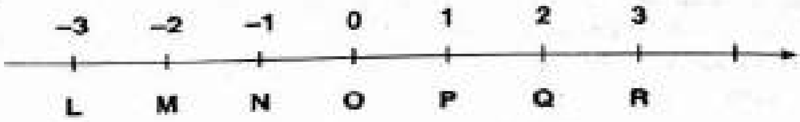
## 1.9 Representation of a number in a real line

### a) Representation of a rational number in a real line

The set of real numbers can be beautifully represented by means of the points on a straight line which is called a real line.

We first see how a number line or a real line can be constructed.

We begin with a straight line and fix a point O, say, on it. The line is generally taken parallel to the base–margin of a copy (i.e., *horizontal*). The point O is called the origin. The direction to the right of O is taken to be *positive* and that to the left is taken to be *negative*. A point P in the positive direction is then fixed. The distance of P from O or length OP is chosen as a *unit length*.




Points are laid off at equal intervals of length OP on either side of O.

We then assign the numbers 1, 2, 3, ... respectively to the points P, Q, R,... on the positive side and the numbers −1, −2, −3, ... to the points N, M, L, ... on the left side of O. In this way, we have set up a *one-to-one correspondence* between the set of numbers (i.e., integers or positive and negative whole numbers including the number zero) and the set of points so far laid off on the number line. We may further divide and subdivide each segment by introducing more and more points.

Such points can be used to represent numbers such as

$$\ldots, -\frac{3}{1}, -\frac{5}{2}, -\frac{2}{1}, -\frac{3}{2}, -1, -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{3}{2}, \frac{2}{1}, \frac{5}{2}, \frac{3}{1}, \ldots$$

called *rational numbers* (i.e., *ratio-numbers*) or fractions.

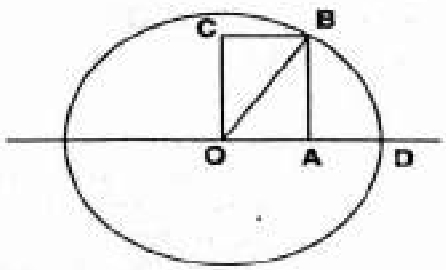
## b) Representation of an irrational number in a real line

An irrational number can be represented by a point on the real line. For example, $\sqrt{2}$ is an irrational number. We represent this irrational number in a real line.

To present a point corresponding to $\sqrt{2}$ on a real line, we construct a square OABC of unit length. Join OB. Then

$$\begin{aligned} OB &= \sqrt{OA^2 + AB^2} \\ &= \sqrt{1+1} = \sqrt{2} \end{aligned}$$

Now, we construct a circle with O as centre and OB as radius cutting the real line at D and E. Then OD = OB = $\sqrt{2}$. Hence D and E represent $\sqrt{2}$ and $-\sqrt{2}$ on the real line.



## c) Inequality

Suppose $a$ and $b$ are two real numbers. The number $a$ is said to be greater than $b$ if $a - b$ is positive and we write it symbolically as $a > b$. Again the number $a$ is said to be **less than** the number $b$, if $b - a$ is positive. In symbols, we write it as

$$a < b.$$

We may sometimes write it as $b > a$ also. Here, the symbol '>' stands for '**is greater than**'.

In case $b - a = 0$, we say $a$ and $b$ are **equal**. We write this as $b = a$.

An important property of the set of real numbers is

"*If a and b are two given numbers, then one and only one of the following order relations*

$$a < b,\ a = b,\ a > b$$

*holds good.*"

This property is known as the **order property** of real numbers. Each of the two relations $a < b$ and $a > b$ is known as an **inequality**.

If $a$ is **less than** or **equal to** $b$, we write it as $a \le b$, and if $a$ **is greater than** or **equal to** $b$, we write it as $a \ge b$.

| Symbol | Meaning |
|---|---|
| $<$ | Less than |
| $>$ | Greater than |
| $\le$ | Less than or equal to |
| $\ge$ | Greater than or equal to |

In a number line, a number representing a point on the left side is said to be *less than* the number representing a point on the right. We may also say that a number representing a point on the right side *is greater than* the number representing a point on the left.

## d) Dense set

Another important property of the set of real numbers is
" *Between any two given real numbers, there exists a real number.*"

This property of real number is described by saying that
" The set of real numbers is **dense**."

## e) Interval

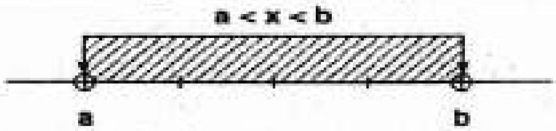
Let $a$ and $b$ be two numbers on the real line. Then the set of points on the real line between $a$ and $b$ is known as an interval. $a$ and $b$ are known as the end points of the interval. An interval is denoted by I. An interval may or may not include the end points. So, we get the following four different types of intervals

**i) Open-interval:**

An interval not containing the end points $a$ and $b$ is known as an open interval. It is denoted by $(a, b)$.

Symbolically, $(a, b) = \{x : a < x < b\}$

The graph of the above open interval $(a, b)$ is shown below:




**ii) Closed interval:**

An interval containing both the end points $a$ and $b$ is known as a closed interval. It is denoted by $[a, b]$

Symbolically, $[a, b] = \{x : a \leq x \leq b\}$

The graph of the closed interval $[a, b]$ is shown below :


**iii) Left open interval:**

An interval not containing the end point $a$ and containing the end point $b$ is known as a left open interval. It is denoted by $(a, b]$.

Symbolically, $(a, b] = \{x : a < x \leq b\}$

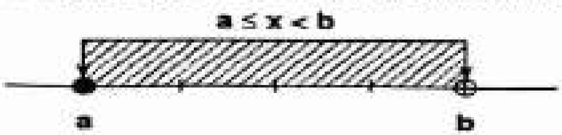
The graph of the left open interval is shown below:


**iv) Right open interval:**

An interval containing the end point $a$ and not containing the end point $b$ is known as the right open interval. It is denoted by $[a, b)$.

Symbolically, $[a, b) = \{x : a \leq x < b\}$

The graph of the right open interval $[a, b)$ is shown below:



## 1.10 Absolute Value

Let $x$ denote any real number. The **absolute value** (or **modulus** or **numerical value**) of $x$, written as $|x|$, is a non-negative real number defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Clearly, $|x| \geq 0$. Geometrically speaking, the absolute value of $x$ is the *distance* of the point $x$ on the real line from the origin. i.e., the point O. Moreover, the distance between any two points $a$ and $b$ on the real line is $|a - b| = |b - a|$.

***Examples:***

$$|-3| = -(-3) = 3,\ |3| = 3, |0| = 0, |\pi| = \pi.$$

Some simple properties of absolute values are discussed below:

1. Let $x$ be any real number. Then
   - i) $|x| \geq 0$
   - ii) $|x| \geq x$ and $|x| \geq -x$
   - iii) $-|x| \leq x \leq |x|$

***Proofs.***

Let $x \in R$

i) If $x = 0$, then $|x| = 0$
If $x > 0$, $|x| = x > 0$
If $x < 0$, $|x| = -x > 0$
$\therefore$ for all $x \in R$, $|x| \geq 0$

ii) If $x \geq 0$, then $x \geq -x$
and $|x| = x$
$\therefore$ $|x| \geq -x$
Again if $x \leq 0$, then $-x \geq x$
But $|x| = -x$
$\therefore$ $|x| \geq x$

iii) $|x| \geq x \Rightarrow x \leq |x|$
Again, $|x| \geq -x \Rightarrow x \geq -|x|$
Combining the two results

$$-|x| \leq x \leq |x|$$

2. For any two real numbers $x$ and $y$,
   a) $|x+y| \le |x| + |y|$ (Triangle Inequality)
   b) $|x-y| \ge |x| - |y|$.

***Proofs:***

a) $|x+y| \le |x| + |y|$ (Triangle Inequality)

**First Proof:**

Let $x + y \ge 0$. Then

$$|x+y| = x + y \le |x| + |y|, \quad (\because x \le |x| \text{ and } y \le |y|)$$

Let $x + y < 0$. Then

$$|x+y| = -(x+y) = (-x) + (-y) \le |x| + |y|,$$
$$(\because -x \le |x| \text{ and } -y \le |y|)$$

This completes the proof.

**Second Proof:**

For all $x, y \in$ R

$$-|x| \le x \le |x| \quad \ldots\ldots \text{(i)}$$

and $-|y| \le y \le |y| \quad \ldots\ldots$ (ii)

Adding (i) and (ii)

$$-(|x| + |y|) \le x + y \le |x| + |y|$$

$$\Rightarrow \quad |x+y| \le |x| + |y|$$

**Third Proof:**

$$|x+y|^2 = (x+y)^2$$
$$= x^2 + 2xy + y^2$$
$$= |x|^2 + 2xy + |y|^2$$
$$\le |x|^2 + 2|x||y| + |y|^2 \quad (\because |x| \ge x \text{ and } |y| \ge y)$$
$$= (|x| + |y|)^2$$

$$\therefore |x+y| \le |x| + |y|$$

b) Let $x - y = z$. Then, $x = y + z$ and

$$|x| = |y+z| \le |y| + |z| = |y| + |x-y|$$

Hence, by transposition, we get

$$|x-y| \ge |x| - |y|.$$

3. For any two real numbers $x$ and $y$,
   a) $|xy| = |x||y|$.
   b) $|x/y| = |x|/|y|$, $y \ne 0$

***Proofs:***

a) For $x, y \in R$

$$|xy|^2 = (xy)^2$$
$$= x^2y^2$$
$$= |x|^2 |y|^2$$
$$\therefore \quad |xy| = |x||y|$$

b) For $x, y \in R, (y \neq 0)$

$$\left|\frac{x}{y}\right|^2 = \left(\frac{x}{y}\right)^2$$
$$= \frac{x^2}{y^2}$$
$$= \frac{|x|^2}{|y|^2}$$
$$\therefore \quad \left|\frac{x}{y}\right| = \frac{|x|}{|y|}$$

4. If $x \in R$ and $a$ be any positive real number then $|x| < a \Rightarrow -a < x < a$ and conversely.

VVI ***Proof:***

For all $x \in R, |x| \geq x$

Given, $|x| < a$

$\therefore x \leq |x| < a$

$\Rightarrow \quad x < a \quad \ldots\ldots$ (i)

Again for all $x \in R, |x| \geq -x$

Given $|x| < a$

$\therefore \quad -x \leq |x| < a$

$\Rightarrow \quad -x < a$

$\Rightarrow \quad x > -a \quad \ldots\ldots$ (ii)

Combining (i) and (ii)

$-a < x < a$

Conversely i.e. If $-a < x < a$, then $|x| < a$

Firstly, $x < a$

If $x \geq 0$, then $|x| = x$

$\therefore \quad |x| = x < a \quad \ldots\ldots$ (i)

Again, $-a < x$

$\Rightarrow \quad x > -a$

$\Rightarrow \quad -x < a$

But for $x < 0$, $|x| = -x$

$\therefore \quad |x| = -x < a \quad \ldots\ldots$ (ii)

For all $x \in R$, (From (i) and (ii))

$|x| < a$

Note : For $x \in R$, $a > 0$

$|x| \leq a \Rightarrow -a \leq x \leq a$

and conversely.

## Worked Out Examples

***Example 1***

Evaluate:

a) $|2 - 5|$ b) $|-3| - |-5|$.

***Solution:***

a) $|2 - 5| = |-3| = 3$

b) $|-3| - |-5| = 3 - 5 = -2$.

***Example 2***

Let $x = 3$, $y = -4$. Verify each of the following :

a) $|x + y| \leq |x| + |y|$ b) $|x - y| \geq |x| - |y|$

***Solution :***

a) $|x + y| = |3 - 4| = |-1| = 1$

and $|x| + |y| = |3| + |-4| = 3 + 4 = 7$

Hence $|x + y| < |x| + |y|$

b) $|x - y| = |3 + 4| = 7$

and $|x| - |y| = |3| - |-4| = 3 - 4 = -1$

Hence $|x - y| > |x| - |y|$

Hence the results are verified.

***Example 3***

Rewrite $-1 \leq x - 3 \leq 5$ in the form:

a) $a \leq x \leq b$ b) $[a, b]$ c) $|x - a| \leq c$.

***Solutions:***

a) $1 \le x - 3 \le 5$

Adding 3 to each term, we get

$1 + 3 \le x - 3 + 3 \le 5 + 3$

or, $4 \le x \le 8$.

b) Since $a \le x \le b \Rightarrow x \in [a, b]$,

$1 \le x - 3 \le 5$

$\Rightarrow 4 \le x \le 8$

$\Rightarrow x \in [4, 8]$.

c) Since $-c \le x - a \le c \Rightarrow |x - a| \le c$;

$1 \le x - 3 \le 5$

$\Rightarrow 1 - 3 \le x - 3 - 3 \le 5 - 3$

$\Rightarrow -2 \le x - 6 \le 2$

$\Rightarrow |x - 6| \le 2$.

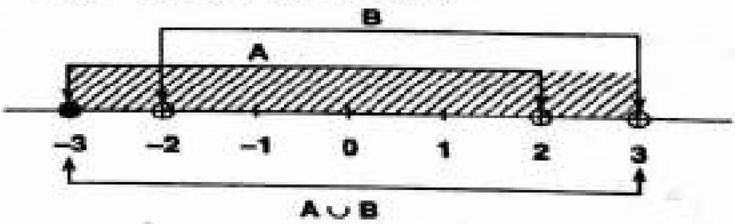
***Example 4***

Let $A = [-3, 2)$ and $B = (-2, 3)$. Perform the indicated operations:

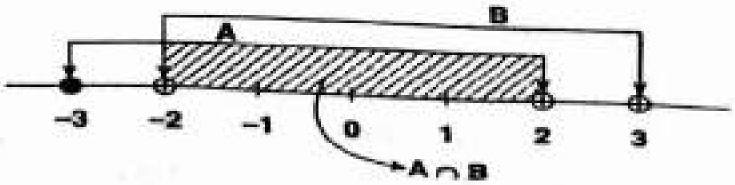
a) $A \cup B$ b) $A \cap B$ c) $A - B$

***Solutions:***

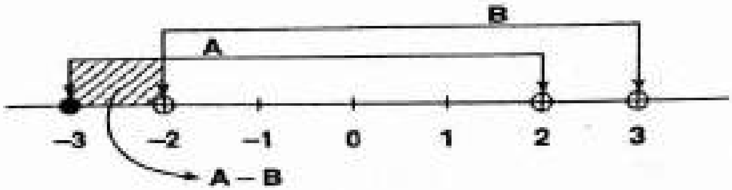
a) $A \cup B = [-3, 2) \cup (-2, 3)$
$= \{x : -3 \le x < 2\} \cup \{x : -2 < x < 3\}$
$= \{x : -3 \le x < 3\} = [-3, 3)$




b) $A \cap B = [-3, 2) \cap (-2, 3)$
$= \{x : -3 \le x < 2\} \cap \{x : -2 < x < 3\}$
$= \{x : -2 < x < 2\}$
$= (-2, 2)$.



c) $A - B = [-3, 2) - (-2, 3)$
$= \{x : -3 \le x < 2\} - \{x : -2 < x < 3\}$
$= \{x : -3 \le x \le -2\} = [-3, -2]$.




**Example 5**

Write the following with $x$ between the inequality signs:

a) $|x| < 3$ b) $|3x + 2| \le 1$.

**Solutions:**

a) $|x| < 3 \Rightarrow -3 < x < 3$.

b) $|3x + 2| \le 1 \Rightarrow -1 \le 3x + 2 \le 1$

$\Rightarrow -3 \le 3x \le -1$ (adding $-2$ to each term)

$\Rightarrow -1 \le x \le -\frac{1}{3}$ (dividing each term by 3)

**Example 6**

Rewrite the following by using the absolute value sign :

a) $-2 < x < 2$ b) $-7 < x < 1$

c) $-6 \le x \le 1$

**Solution :**

a) $-2 < x < 2 \Rightarrow |x| < 2$

b) $-7 < x < 1 \Rightarrow -7 + 3 < x + 3 < 1 + 3$ (Adding 3 to each side)

$\Rightarrow -4 < x + 3 < 4$

$\Rightarrow |x + 3| < 4$

c) $-6 \le x \le 1 \Rightarrow -12 \le 2x \le 2$ (Multiplying each side by 2)

$\Rightarrow -12 + 5 \le 2x + 5 \le 2 + 5$ (Adding 5 to each side)

$\Rightarrow -7 \le 2x + 5 \le 7$

$\Rightarrow |2x + 5| \le 7$

**Example 7**

Solve the following inequality and draw its graph

$|2x + 3| < 5$

***Solution :***

$$|2x + 3| < 5$$

$$\Rightarrow \quad -5 < 2x + 3 < 5$$

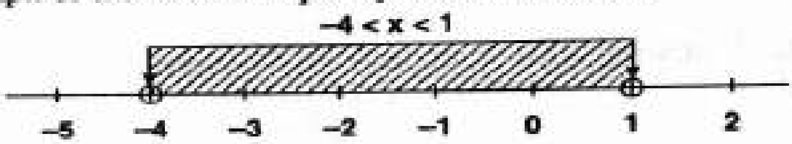
$$\Rightarrow \quad -5 - 3 < 2x + 3 - 3 < 5 - 3$$

$$\Rightarrow \quad -8 < 2x < 2$$

$$\Rightarrow \quad -4 < x < 1$$

$\therefore$ the solution is $\{x : -4 < x < 1\}$

The graph of the above inequality is shown below:




***Example 8***

Solve the inequality : $|2x - 1| \geq 3$ and draw its graph.

***Solution :***

If $(2x - 1) > 0$, then

$$(2x - 1) \geq 3$$

$$\Rightarrow \quad 2x - 1 + 1 \geq 3 + 1$$

$$\Rightarrow \quad 2x \geq 4$$

$$\Rightarrow \quad x \geq 2$$

$$\Rightarrow \quad x \in [2, \infty)$$

Again if $(2x - 1) < 0$, then

$$-(2x - 1) \geq 3$$

$$\Rightarrow \quad 2x - 1 \leq -3$$

$$\Rightarrow \quad 2x - 1 + 1 \leq -3 + 1$$

$$\Rightarrow \quad 2x \leq -2$$

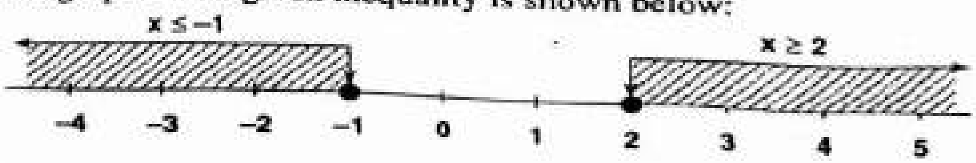
$$\Rightarrow \quad x \leq -1$$

$$\Rightarrow \quad x \in (-\infty, -1]$$

$\therefore$ the required solution of the given inequality is $\{x : x \leq -1 \text{ or } x \geq 2\}$

i.e. $x \in (-\infty, -1] \cup [2, \infty)$

The graph of the given inequality is shown below:



***Example 9***

Let $a, b, c \in$ R. Using the properties of real numbers

i) if $a < b$ prove that $a - c < b - c$

ii) if $a < b$ and $b < c$, prove that $a < c$.

iii) if $ab = 1$ prove that $b = \frac{1}{a}$

***Solution :***

i) $a < b \Rightarrow b - a > 0$ ...... (i)

Now, $(b - c) - (a - c)$

$= b - c - a + c$

$= b - a > 0$ (by (i))

$\therefore$ $b - c > a - c$

i.e. $a - c < b - c$

ii) $a < b \Rightarrow b - a > 0$ ...... (i)

$b < c \Rightarrow c - b > 0$ ...... (ii)

Now, $c - a = (c - b) + (b - a) > 0$ by (i) and (ii)

$\therefore$ $c > a$

i.e. $a < c$

iii)

| | | |
|---|---|---|
| | $a.b = 1$ | Given |
| $\Rightarrow$ | $a^{-1}(ab) = a^{-1}.1$ | For $a \neq 0$, multiplicative axiom |
| $\Rightarrow$ | $(a^{-1}a)b = a^{-1}.1$ | Associative property |
| $\Rightarrow$ | $1.b = a^{-1}.1$ | Multiplicative inverse |
| $\Rightarrow$ | $b = a^{-1}$ | Multiplicative identity |

i.e. $b = \frac{1}{a}$

***Example 10***

Prove that $\sqrt{2}$ is an irrational number.

***Solution :***

If possible, let $\sqrt{2}$ be a rational number.

Then, $\sqrt{2} = \frac{p}{q}$ where $p$ and $q$ are integers with no common factor and $q \neq 0$.

$\Rightarrow$ $p^2 = 2q^2$

This shows that $p^2$ is even and hence $p$ is even.

Let $p = 2r$ where $r$ is an integer.

Then, $4r^2 = 2q^2$

or, $q^2 = 2r^2$

This shows that $q^2$ is even and hence $q$ is even.

$\therefore$ $p$ and $q$ are even and have a common factor which is against our supposition.

Hence $\sqrt{2}$ is an irrational number.

***Example 11***

Solve : $x(x - 1)(x + 3) \le 0$

***Solution:***

The corresponding equation of the given inequation is

$$x(x - 1)(x + 3) = 0$$

$\therefore$ $x = -3, 0, 1$

Let us see the following possible intervals and the sign of $x(x - 1)(x + 3)$ in these intervals

| Interval | Sign of | | | |
|---|---|---|---|---|
| | $x$ | $x - 1$ | $x + 3$ | $x(x - 1)(x + 3)$ |
| $-\infty$ to $-3$ | $-$ | $-$ | $-$ | $-$ |
| $-3$ to $0$ | $-$ | $-$ | $+$ | $+$ |
| $0$ to $1$ | $+$ | $-$ | $+$ | $-$ |
| $1$ to $\infty$ | $+$ | $+$ | $+$ | $+$ |

From the above table, the possible intervals are $(-\infty, -3]$ and $[0, 1]$

$\therefore$ the required solution is $x \in (-\infty, -3] \cup [0, 1]$

## EXERCISE 1.3

1. Evaluate:

a) $|-2| + 4$

b) $|-5| + |-2| - 3$

c) $2 + |-3| - |-5|$

d) $|3 - |-5||$

2. Let (i) $x = 2, y = 3$ (ii) $x = 2, y = -3$ verify each of the followings:
   a) $|x + y| \le |x| + |y|$
   b) $|x - y| \ge |x| - |y|$
   c) $|xy| = |x| . |y|$
   d) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$
3. Solve the following inequalities
   a) $x - 1 > 2$
   b) $x - 3 \le 5$
   c) $-1 < x - 2 < 3$
   d) $-3 \le 2x - 1 \le 5$
   e) $x^2 - 2x > 0$
   f) $6 + 5x - x^2 \ge 0$
   g) $\frac{x(x+2)}{x-1} \le 0$
4. a) Let A = [–3, 1) and B = [–2, 4]. Perform the indicated operations
   i) $A \cup B$ ii) $A \cap B$ iii) $A - B$ iv) $B - A$
   b) If A = (–1, 4) and B = [3, 5), find $A \cup B$, $A \cap B$ and $A - B$.
5. Write the following without using absolute value sign
   a) $|x| < 4$
   b) $|x - 3| < 2$
   c) $|2x + 1| \le 3$
   d) $|2x - 1| \le 5$
6. Rewrite the following inequalities using abolute value sign
   a) $-5 < x < 7$
   b) $-3 \le x \le -1$
   c) $-3 < x < 4$
   d) $-4 \le x \le -1$
7. Solve the following inequalities
   a) $|x + 2| < 4$
   b) $|x - 1| \le 2$
   c) $|2x + 3| \le 1$
   d) $|x - 1| > 1$
   e) $|2x + 1| \ge 3$

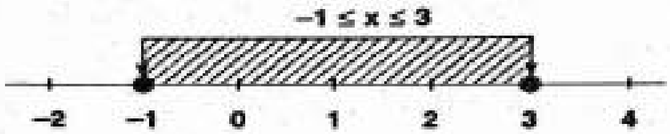
   Also, draw the graphs of the above inequalities.
8. Using the properties of real numbers, prove that
   a) $a + b = b + c \Rightarrow a = c$
   b) $ac = bc \Rightarrow a = b$
   c) $a < b \Rightarrow a + c < b + c$
   d) $a < b$ and $c < d \Rightarrow a + c < b + d$
   e) $a > b$ and $c < 0 \Rightarrow ac < bc$
   f) $a > b$ and $c > 0 \Rightarrow \frac{a}{c} > \frac{b}{c}$

## Answers

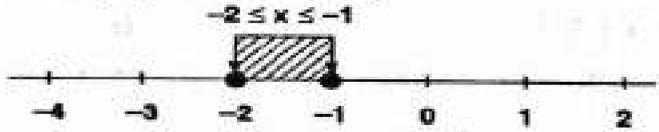
1. a) 6 b) 4 c) 0 d) 2
3. a) $x > 3$ b) $x \le 8$ c) $1 < x < 5$ d) $-1 \le x \le 3$
   e) $x \in (-\infty, 0) \cup (2, \infty)$ f) $x \in [-1, 6]$ g) $x \in (-\infty, -2] \cup [0, 1)$

4. a) i) $[-3, 4]$; ii) $[-2, 1)$; iii) $[-3, -2)$; iv) $[1, 4]$
   b) $(-1, 5)$, $[3, 4)$, $(-1, 3)$
5. a) $-4 < x < 4$ b) $1 < x < 5$ c) $-2 \le x \le 1$ d) $-2 \le x \le 3$
6. a) $|x - 1| < 6$ b) $|x + 2| \le 1$ c) $|2x - 1| < 7$ d) $|2x + 5| \le 3$
7. a) $\{x : -6 < x < 2\}$ (i.e. $-6 < x < 2$)


b) $\{x : -1 \le x \le 3\}$ (i.e. $-1 \le x \le 3$)

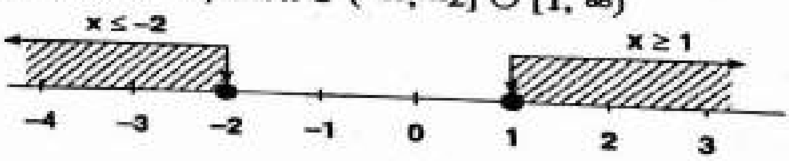



c) $\{x : -2 \le x \le -1\}$ (i.e. $-2 \le x \le -1$)




d) $\{x : x < 0 \text{ or } x > 2\}$ i.e. $x \in (-\infty, 0) \cup (2, \infty)$


e) $\{x : x \le -2 \text{ or } x \ge 1\}$ i.e. $x \in (-\infty, -2] \cup [1, \infty)$




## 1.11 Logic

The word logic is derived from the word "logos' which means reason. Man learnt to talk or give reason or argue systematically either on the basis of his experience and activities or on the basis of the idea or supposition or assumption he makes. Of these two ways the first one is known as "inductive process of reasoning" and the second one as "deductive process of reasoning".

## Concept of Logic

The dictionary meaning of logic is the science of reasoning. In mathematics, we deal with various theorems and formulae. The different procedures used in giving the proofs of various theorems and formulae are based on sound reasoning. The study of such procedures based on sound reasoning is known as a logic. Logic tells us the truth and the falsity of the particular statement. Logic is the process by which we arrive at a conclusion from the given statement with a valid reason.

In logic, we use symbols for words, statements and their relations to get the required result. Hence, logic is known as mathematical logic or symbolic logic.

## Statements

An assertion expressed in words or symbols, which is either true or false but not both at the same time, is known as a statement. Some examples of statements are given below:

i) Water is essential for health.
ii) $2 + 4 = 6$
iii) A quadrilateral has three sides.

(i), (ii) and (iii) are statements as (i) and (ii) are true but (iii) is false.

The sentences of the following type are not the statements because they do not declare the truth or falsity.

a) knock at the door.
b) What is your name?
c) How beautiful your country is !

In the context of logic, statements cannot be imperative, interrogative and exclamatory.

A sentence whose **truth** or **falsity** can be decided only after filling the gap in the sentence or substituting the value of the variable is known as an open sentence, otherwise it is known as closed.

All the sentences considered above ((i), (ii) and (iii)) are closed and hence are statements. The examples of open sentences are

i) ...... is the son of Dasharath
ii) $x + 3 = 5$

These are not the statements.

There are two types of statements : Simple and Compound.

## Simple Statement

A statement which declares only one thing is known as a simple statement. That is, sentence that cannot be divided into two or more sentences, is known as a simple statement.

i) Laxmi Prasad Devkota is a great poet.

ii) $2 \times 3 = 6$

(i) and (ii) are the examples of simple statements.

In mathematical logic, simple statements are denoted by the letters : $p$, $q$, $r$, etc.

### Compound Statement

A combination of two or more simple statements is known as a compound statement. Each simple statement is known as a component of the compound statement. The examples of compound statements are as follows.

i) Nepal is in Asia and Mt. Everest is the highest peak in the world.

ii) $3 - 2 = 1$ and $5 > 6$.

Compound statements are constructed from simple statements by means of logical connectives given below in § 1.12.

### Truth value and truth table

A truth or the falsity of a statement is known as its truth value. T or F is the truth value of a statement according as it is true or false.

The truth value of a simple statement depends upon the truth or falsity of the given statement. But in a compound statement, its truth value depends not only on the truth or falsity of the component statements but also on the connectives (defined below) with which the component statements are combined.

A table presenting the truth values of the component statements together with the truth values of their compound statement, is known as the truth table. The truth table consists of a number of rows and column. Some of the initial columns contain the possible truth values of the component statements and then the truth values of the compound statements formed from the given simple statements using suitable connective.

## 1.12 Logical Connectives

Compound statements are made from the simple statements by using the words or phrase like "and", "or", "If ...... then" and "If and only if" and they are known as logical connectives or simply connectives.

### 1. Conjunction

Two simple statements combined by the word "and" (or equivalent word) to form a compound statement, is known as conjunction of the given statements. The symbol used for the conjunction is ^.

If $p$ and $q$ are two simple statements, then their conjunction is symbolized by $p \wedge q$. In case of conjunction of $p$ and $q$, if $p$ is true and $q$ is true then $p \wedge q$ is true. Otherwise $p \wedge q$ is false.

***Example:***

$p$ : Pabitra is an engineer, $q$ : Sumitra is a doctor

their conjunction is "Pabitra is an engineer and Sumitra is a doctor" and this compound statement is symbolized by $p \wedge q$.

The truth table of the conjunction of the statements $p$ and $q$ is presented below:

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

## 2. Disjunction

Two simple statements combined by the word "or" (or an equivalent word), to form a compound statement, is known as disjunction of the given statements. The symbol used for the disjunction is $\vee$.

If $p$ and $q$ are the two simple statements then disjunction of $p$ and $q$ is symbolized by $p \vee q$.

In case of disjunction of $p$ and $q$, if $p$ is true or $q$ is true or both $p$ and $q$ are true then $p \vee q$ is true, otherwise $p \vee q$ is false.

***Example***

$p$ : Ananda is smart $q$ : Arun is handsome;

their disjunction is "Ananda is smart or Arun is handsome" and this compound statement is symbolized by $p \vee q$.

The truth table of the disjunction of $p$ and $q$ is presented below:

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

## 3. Negation

A statement which denies the given statement is known as the negation of a given statement.

The negation of a given statement is obtained by inserting the word "not" in the given statement or by adding "It is not true" or "It is not the case that" at the beginning of the given statement.

The symbol used for the negation is ~. If $p$ is the given statement, then its negation is symbolized by $\sim p$.

In case of negation, if $p$ is true, then $\sim p$ is false and if $p$ is false then $\sim p$ is true.

***Example:***

$p$: Krishna wears spectacle. Then, $\sim p$ is "Krishna does not wear spectacle."

The truth table for the negation of a statement $p$ is presented below:

| $p$ | $\sim p$ |
|---|---|
| T | F |
| F | T |

The negation of the words "all", "some", "some ... not" and "no" are "some ... not", "no", "all" and some respectively.

Example : $p$ : All students are laborious

$\sim p$ : Some students are not laborious

## 4. Conditional (Implication)

Two simple statements combined by the word "If ...... then" to form a compound statement, is known as the conditional of the given statements. The symbol used for the conditional (or implication) is $\Rightarrow$.

The conditional of the simple statements $p$ and $q$ is symbolized by $p \Rightarrow q$. Here $p$ is known as the antecedent and $q$, the consequent.

In case of conditional, the conditional $p \Rightarrow q$ is false when $p$ is true but $q$ is false, otherwise it is true.

***Example:***

$p$ : ABC is a triangle.

$q$ : the sum of three angles is 180°.

The conditional of $p$ and $q$ is "If ABC is a triangle then the sum of three angles is 180°" and this statement is symbolized by $p \Rightarrow q$.

The truth table of the conditional $p \Rightarrow q$ is presented below:

| $p$ | $q$ | $p \Rightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

## Biconditional (Equivalence)

Two simple statements combined with the word "if and only if" (abbreviated as iff) to form a compound statement, is known as a biconditional. The symbol used for biconditional is $\Leftrightarrow$.

The biconditional of the statement $p$ and $q$ is symbolized by $p \Leftrightarrow q$ which means $p \Rightarrow q$ and $q \Rightarrow p$.

In case of biconditional, $p \Leftrightarrow q$ is true when both $p$ and $q$ have the same truth value and is false when $p$ and $q$ have different truth values.

***Example:***

$p$ : Two triangles are congruent

$q$ : corresponding sides of two triangles are equal.

Then the biconditional of $p$ and $q$ is "Two triangles are congruent if and only if their corresponding sides are equal." *i.e.* $p \Leftrightarrow q$. This biconditional contains the following two cases:

i) If two triangles are congruent then their corresponding sides are equal i.e. $p \Rightarrow q$.

ii) If the corresponding sides of two triangles are equal, then they are congruent. i.e. $q \Rightarrow p$.

$p \Leftrightarrow q$ is same as $p \Rightarrow q$ and $q \Rightarrow p$ i.e. $(p \Rightarrow q) \wedge (q \Rightarrow p)$

The truth table of the biconditional of $p$ and $q$ is presented below

| $p$ | $q$ | $p \Leftrightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

Following table gives the truth values of the compound statements formed with different connectives when the different truth values of the component statements are given.

| $p$ | $q$ | $p \wedge q$ | $p \vee q$ | $\sim p$ | $p \Rightarrow q$ | $p \Leftrightarrow q$ |
|---|---|---|---|---|---|---|
| T | T | T | T | F | T | T |
| T | F | F | T | F | F | F |
| F | T | F | T | T | T | F |
| F | F | F | F | T | T | T |

Before discussing the laws of logic we have the following definitions

**Tautology:** A compound statement which is always true, whatever may be the truth values of its components, is known as a tautology.

The statement that "It is a rainy day or it is not a rainy day" is a tautology.

**Contradiction:** A compound statement which is always false, whatever may be the truth values of its components is known as a contradiction.

The statement that "I like coffee and I don't like coffee" is a contradiction.

**Converse:** If $p$ and $q$ are two simple statements, then the conditional $q \Rightarrow p$ is said to be the converse of the conditional $p \Rightarrow q$.

Given statement: If $a > 0$, then $-a < 0$.

Converse: If $-a < 0$, then $a > 0$.

**Inverse:** If $p$ and $q$ are two statements, then the conditional $\sim p \Rightarrow \sim q$ is said to be the inverse of the conditional $p \Rightarrow q$.

Given statement: If $a > 0$ then $-a < 0$.

Inverse: If $a \not> 0$ then $-a \not< 0$.

**Contrapositive:** If $p$ and $q$ are two statements, then the conditional $\sim q \Rightarrow \sim p$ is said to be the contrapositive of the conditional $p \Rightarrow q$.

Given statement: If $a > 0$ then $-a < 0$.

Contrapositive: If $-a \not< 0$ then $a \not> 0$.

**Logically Equivalent:** Two statements $S_1$ and $S_2$ are said to be logically equivalent if both have same truth values in the columns of the truth table. They are denoted by $S_1 \equiv S_2$.

Let us see the following example. Let $p$ be a statement. $t$ and $c$ are the tautology and the contraction.

| $p$ | $t$ | $c$ | $p \vee t$ | $p \wedge c$ |
|---|---|---|---|---|
| T | T | F | T | F |
| F | T | F | T | F |

From the above table,

$$p \vee t \equiv t \quad \text{and} \quad p \wedge c \equiv c$$

## 1.13 Laws of Logic

The following are the laws of logic:

Let $p$, $q$ and $r$ be any three statements.

**a) Law of Excluded middle**

Only one statement $p$ or $\sim p$ is true.

A statement cannot both be true and false at the same time.

**b) Law of Tautology**

The statement $p \vee \sim p$ is a tautology.

The disjunction of a statement and its negation is a tautology.

**c) Law of Contradiction**

The statement $p \wedge \sim p$ is a contradiction.

The conjunction of a statement and its negation is a contradiction.

**d) Law of Involution**

$$\sim(\sim p) \equiv p$$

The negation of negation of a statement is logically equivalent to a given statement. The law is also known as the law of **double negation**.

**e) Law of Syllogism**

If $p \Rightarrow q$ and $q \Rightarrow r$ then $p \Rightarrow r$. That is

$$(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$$

This can be verified to be a tautology.

**f) Law of Contraposition**

$$(p \Rightarrow q) \equiv ((\sim q) \Rightarrow (\sim p))$$

The conditional and its contrapositive are logically equivalent.

**g) Inverse Law**

$$(\sim p) \Rightarrow (\sim q) \equiv q \Rightarrow p$$

The inverse and the converse of a conditional are logically equivalent.

We verify some of the above laws using truth table and some of them are left to verify.

**The first law (a)** is obviously true by the definition of negation of a statement.

**Verification of law (b)**

To verify the law (b), we present below the truth table of $p \vee \sim p$.

| $p$ | $\sim p$ | $p \vee \sim p$ |
|---|---|---|
| T | F | T |
| F | T | T |

The truth values of $p \vee \sim p$ are all true for different truth values of $p$ and $\sim p$. Hence $p \vee \sim p$ is a tautology.

Similarly, we can verify law (c) easily.

**Verification of law (d)**

The following table gives the truth values of $\sim(\sim p)$

| $p$ | $\sim p$ | $\sim(\sim p)$ |
|---|---|---|
| T | F | T |
| F | T | F |

From the first and third columns of the above table, $p$ and $\sim(\sim p)$ are logically equivalent i.e. $\sim(\sim p) \equiv p$

**Verification of law (f) and (g)**

The following table presents the truth values of $p \Rightarrow q$, $q \Rightarrow p$, $\sim q \Rightarrow \sim p$ and $\sim p \Rightarrow \sim q$.

| $p$ | $q$ | $p \Rightarrow q$ | $q \Rightarrow p$ | $\sim p$ | $\sim q$ | $\sim p \Rightarrow \sim q$ | $\sim q \Rightarrow \sim p$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | F | F | T | T |
| T | F | F | T | F | T | T | F |
| F | T | T | F | T | F | F | T |
| F | F | T | T | T | T | T | T |

From the above truth table, $(p \Rightarrow q)$ and $(\sim q \Rightarrow \sim p)$ are logically equivalent.

Similarly, $(q \Rightarrow p)$ and $(\sim p \Rightarrow \sim q)$ are logically equivalent.

Besides the above laws, we have the following laws as well

**1. Idempotent law:**

a) $p \wedge p \equiv p$

b) $p \vee p \equiv p$

**2. Commutative law**

a) $p \wedge q \equiv q \wedge p$

b) $p \vee q \equiv q \vee p$

**3. Associative law**

a) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

b) $p \vee (q \vee r) \equiv (p \vee q) \vee r$

**4. Distributive law**

a) $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

b) $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

**5. De-Morgan's law**

a) $\sim(p \wedge q) \equiv (\sim p \vee \sim q)$

b) $\sim(p \vee q) \equiv (\sim p \wedge \sim q)$

## Worked Out Examples

***Example 1***

Which of the following is a statement? Find the truth value if it is a statement.

a) The earth moves round the sun. b) Study mathematics.

c) What is your mobile number? d) $2 + 6 = 5$

***Solution :***

a) It is a declarative sentence. So, it is a statement. Its truth value is T.

b) It is an imperative sentence, so it is not a statement.

c) It is an interrogative sentence, so it is not a statement.

d) It is declarative, so it is a statement. Its truth value is F.

***Example 2***

Write the following statements in symbolic form:

a) Newton is a mathematician and Lekha Nath is a poet.

b) If base angles of a triangle are equal, then the triangle is an isosceles.

Find the truth values of each of the above compound statements.

***Solution :***

a) $p$ : Newton's is a mathematician.

$q$ : Lekha Nath is a poet.

The symbolic form of the given statement is $p \wedge q$.

The truth value of $p$ is T and that of $q$ is T.

$\therefore$ the truth value of $p \wedge q$ is also T.

b) $p$ : Base angles of a triangle are equal

$q$ : the triangle is isosceles

The symbolic form of the given statement is $p \Rightarrow q$.

The truth value if $p$ is T and that of $q$ is T. So, the truth value of $p \Rightarrow q$ is T.

***Example 3***

If $p$ : Chandra uses computer

$q$ : Ram sells mobile

express each of the following statements in words

a) $p \wedge {\sim}q$ b) ${\sim}q$ c) ${\sim}p \wedge q$ d) ${\sim}(p \vee q)$

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***Solution :***

a) Chandra uses computer and Ram sells mobile.
b) Ram does not sell mobile.
c) Chandra does not use computer and Ram sells mobile.
d) It is false that Chandra uses computer or Ram sells mobile.

***Example 4***

If $p$ : Suman is rich and $q$ : Ramita is happy

write each of the following statements in symbolic form.

a) Suman is rich or Ramita is unhappy
b) Suman is not rich but Ramita is happy
c) Neither Suman is rich nor Ramita is happy

***Solution :***

$p$ : Suman is rich; $q$ : Ramita is happy

$\sim p$ : Suman is not rich $\sim q$ : Ramita is not happy

a) Suman is rich or Ramita is unhappy in symbolic form is

$$p \vee \sim q$$

b) Suman is not rich but Ramita is happy in symbolic form is

$$\sim p \wedge q$$

c) Neither Suman is rich nor Ramita is happy in symbolic form is

$$\sim p \wedge \sim q$$

***Example 5***

Find the truth value and the negation of each of the following statements

a) $3 \times 2 = 6$ or 9 is divisible by 3.
b) 5 is a prime number and $4 - 1 = 6$

***Solution:***

a) $p : 3 \times 2 = 6$ (T);

$q$ : 9 is divisible by 3 (T)

Since $p$ is true and $q$ is true, so the given statement represented by $p \vee q$ is also true.

$\therefore$ the truth value of the given statement is T.

Negation: The negation of the given statement is

"$3 \times 2 \neq 6$ and 9 is not divisible by 3"

b) $p$ : 5 is a prime number (T);

$q : 4 - 1 = 6$ (F)

Since $p$ is true and $q$ is false, so the given statement represented by $p \wedge q$ is false.

$\therefore$ the truth value of the given statement is F.

Negation: The negation of the given statement is

"5 is not a prime number or $4 - 1 \neq 6$."

**Example 6**

If the simple statements $p, q, r$ and $s$ are true, false, true and false respectively, find the truth value of the compound statement $(p \wedge q) \vee (r \wedge s)$

**Solution :**

The truth value is presented in the following table.

| $p$ | $q$ | $r$ | $s$ | $p \wedge q$ | $r \wedge s$ | $(p \wedge q) \vee (r \wedge s)$ |
|---|---|---|---|---|---|---|
| T | F | T | F | F | F | F |

$\therefore$ the truth value of $(p \wedge q) \vee (r \wedge s) = T$

**Example 7**

Form truth table for each of the following compound statements

a) $p \vee (\sim q)$ b) $p \Rightarrow ((\sim p) \wedge (\sim q))$

**Solution :**

The truth tables are given below:

a)

| $p$ | $q$ | $\sim q$ | $p \vee \sim q$ |
|---|---|---|---|
| T | T | F | T |
| T | F | T | T |
| F | T | F | F |
| F | F | T | T |

b)

| $p$ | $q$ | $\sim p$ | $\sim q$ | $\sim p \wedge \sim q$ | $p \Rightarrow ((\sim p) \wedge (\sim q))$ |
|---|---|---|---|---|---|
| T | T | F | F | F | F |
| T | F | F | T | F | F |
| F | T | T | F | F | T |
| F | F | T | T | T | T |

**Example 8**

Let $p$ and $q$ be any two statements, prove that

a) $(p \wedge q) \equiv (q \wedge p)$

b) $\sim(p \vee q) \equiv (\sim p \wedge \sim q)$

***Solution :***

We prove the above relations using the truth values of $p$ and $q$.

a)

| $p$ | $q$ | $p \wedge q$ | $q \wedge p$ |
|---|---|---|---|
| T | T | T | T |
| T | F | F | F |
| F | T | F | F |
| F | F | F | F |

From the above truth table, $(p \wedge q) \equiv (q \wedge p)$

b)

| $p$ | $q$ | $p \vee q$ | $\sim(p \vee q)$ | $\sim p$ | $\sim q$ | $\sim p \wedge \sim q$ |
|---|---|---|---|---|---|---|
| T | T | T | F | F | F | F |
| T | F | T | F | F | T | F |
| F | T | T | F | T | F | F |
| F | F | F | T | T | T | T |

From the above table,

$$\sim(p \vee q) \equiv (\sim p \wedge \sim q)$$

***Example 9***

Find the converse, inverse and contrapositive of each of the following compound statements. Also, express each of them into a symbolic form:

a) If $a > b$ then $a - b > 0$

b) All scientists are brilliant.

Also find the negation of each of the given statement.

***Solution:***

a) Given statement : If $a > b$ then $a - b > 0$

$p : a > b$ and $q : a - b > 0$

Symbolic form of the given statement is $p \Rightarrow q$.

*Converse*: If $a - b > 0$ then $a > b$

Symbolic form : $q \Rightarrow p$

*Inverse*: If $a \not> b$ then $a - b \not> 0$

Symbolic form: $\sim p \Rightarrow \sim q$

*Contrapositive*: If $a - b \not> 0$ then $a \not> b$

Symbolic form: $\sim q \Rightarrow \sim p$

*Negation of given statement*:

$a > b$ and $a - b \not> 0$

b) Re-statement of the given statement: If a person is a scientist, then he/she is brilliant.

$p$ : A person is a scientist

$q$ : He/she is brilliant

*Converse*: If a person is brilliant then he/she is a scientist

Symbolic form: $q \Rightarrow p$

*Inverse*: If a person is not a scientist, then he is not brilliant.

Symbolic form: $\sim p \Rightarrow \sim q$

*Contrapositive*: If a person is not brilliant then he/she is not a scientist.

Symbolic form: $\sim q \Rightarrow \sim p$

*Negation of given statement*:

Some scientists are not brilliant.

## EXERCISE 1.4

1. Which of the following sentences are the statements? Find the truth values of those sentences which are statements.
   a) Kathmandu is the capital of Nepal.
   b) Nepal exports oil. c) $3 + 5 = 8$
   d) Where do you live? NO e) Understand logic. NO
   f) Oh! how beautiful the scene is? NO
2. Let $p$ : demand is increasing and $q$ : supply is decreasing

   Express each of the following statements into words

   a) $\sim p$ b) $\sim q$ c) $p \wedge q$ d) $p \vee q$

   e) $\sim p \wedge q$ f) $p \vee \sim q$ g) $\sim p \wedge \sim q$ h) $\sim(p \wedge q)$
3. Express each of the following statements into symbolic form
   a) $p$ : temperature is increasing; $q$ : length is expanding

      Temperature is increasing and length is expanding.
   b) $p$ : pressure is decreasing; $q$ : volume is increasing

      Pressure is decreasing or volume is increasing.
   c) $p$ : demand is increasing; $q$ : price is increasing

      Neither demand is increasing nor price is decreasing (i.e. not increasing)
   d) $p$ : Pradeep is bold; $q$ : Sandeep is handsome

      It is false that Pradeep is bold or Sandeep is handsome.

4. Construct truth tables for the following compound statements
 a) $(\sim p) \wedge q$ b) $(\sim p) \vee (\sim q)$ c) $\sim(p \wedge q)$
 d) $\sim[p \vee (\sim q)]$ e) $(p \Rightarrow q) \wedge (q \Rightarrow p)$ f) $(\sim p \wedge q) \Rightarrow (p \vee q)$

5. Let $p$, $q$, $r$ and $s$ be four simple statements. If $p$ is true, $q$ is false, $r$ is true and $s$ is false, find the truth values of the following compound statements.
 a) $p \wedge q$ b) $p \vee (\sim q)$ c) $(\sim p) \wedge (\sim q)$
 d) $q \vee (p \wedge s)$ e) $\sim(\sim p)$ f) $(p \vee q) \wedge (r \vee s)$

6. If $p$ and $q$ are any two statements, prove that
 a) $p \wedge (\sim p) = c$ where $c$ is a contradiction.
 b) $(p \vee q) \equiv (q \vee p)$
 c) $\sim(p \vee (\sim q)) \equiv (\sim p) \wedge q$
 d) $\sim((\sim p) \wedge q) \equiv p \vee (\sim q)$

7. Find the negation of each of the following statements:
 a) Light travels in a straight line.
 b) Rivers can be used to produce electricity.
 c) $x > 0$
 d) Some students are weak in mathematics.
 e) All teachers are labourous.

8. Find the truth value and the negation of each of the following statements.
 a) $3 + 2 = 5$ or 6 is a multiple of 5.
 b) 8 is a prime number and 4 is even.
 c) If $3 > 0$ then $4 + 6 = 10$
 d) If 2 is odd or 3 is a natural number then $2 + 3 = 8$
 e) If $2 \times 3 = 5 \Rightarrow 3 > 1$ then 6 is even.
 f) A triangle ABC is right angled at B if and only if $AB^2 + BC^2 = AC^2$

9. Some statements are given below:
 a) If 3 is a natural number then $\frac{1}{3}$ is a rational number.
 b) $x^2 = 4$ whenever $x = 2$
 c) If the battery is low then the mobile does not work well.
 Find the i) antecedent and consequent
 ii) converse, inverse and contrapositive,
 iii) negation
 of each of the given statements.

10. If p and q be the statements, prove that
   a) $p \vee \sim(p \wedge q)$ is a tautology
   b) $(p \wedge q) \Rightarrow (p \vee q)$ is a tautology
   c) $\sim(p \vee q) \wedge q$ is a contradiction.
   d) $(p \wedge q) \wedge \sim(p \vee q)$ is a contradiction

## Answers

1. a) Statement; T b) Statement; F c) Statement, T
   d) Interrogative sentence, so not a statement
   e) Imperative sentence so not a statement
   f) Exclamatory sentence, so not a statement
2. a) Demand is not increasing
   b) Supply is not decreasing
   c) Demand is increasing and supply is decreasing
   d) Demand is increasing or supply is decreasing
   e) Demand is not increasing and supply is decreasing
   f) Demand is increasing or supply is not decreasing
   g) Neither demand is increasing nor supply is decreasing
   h) It is false that demand is increasing and supply is decreasing.
3. a) $p \wedge q$ b) $p \vee q$ c) $\sim p \wedge \sim q$ d) $\sim(p \vee q)$
4. a)

| $p$ | $q$ | $\sim p$ | $\sim p \wedge q$ |
|---|---|---|---|
| T | T | F | F |
| T | F | F | F |
| F | T | T | T |
| F | F | T | F |

b)

| $p$ | $q$ | $\sim p$ | $\sim q$ | $(\sim p) \vee (\sim q)$ |
|---|---|---|---|---|
| T | T | F | F | F |
| T | F | F | T | T |
| F | T | T | F | T |
| F | F | T | T | T |

c)

| $p$ | $q$ | $p \wedge q$ | $\sim(p \wedge q)$ |
|---|---|---|---|
| T | T | T | F |
| T | F | F | T |
| F | T | F | T |
| F | F | F | T |

d)

| $p$ | $q$ | $\sim q$ | $p \vee (\sim q)$ | $\sim(p \vee (\sim q))$ |
|---|---|---|---|---|
| T | T | F | T | F |
| T | F | T | T | F |
| F | T | F | F | T |
| F | F | T | T | F |

e)

| $p$ | $q$ | $p \Rightarrow q$ | $q \Rightarrow p$ | $(p \Rightarrow q) \wedge (q \Rightarrow p)$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | T | F |
| F | T | T | F | F |
| F | F | T | T | T |

f)

| $p$ | $q$ | $\sim p$ | $(\sim p) \wedge q$ | $p \vee q$ | $(\sim p \wedge q) \Rightarrow p \vee q$ |
|---|---|---|---|---|---|
| T | T | F | F | T | T |
| T | F | F | F | T | T |
| F | T | T | T | T | T |
| F | F | T | F | F | T |

5. a) F b) T c) F d) F e) T f) T

7. a) Light does not travel in a straight line.
   b) Rivers cannot be used to produce electricity.
   c) $x \leq 0$
   d) No student is weak in mathematics.
   e) Some teachers are not labourous.

8. a) T; $3 + 2 \neq 5$ and 6 is not a multiple of 5.
   b) F; 8 is not a prime number or 4 is odd.
   c) T; $3 > 0$ and $4 + 6 \neq 10$
   d) F; 2 is odd or 3 is a natural number and $2 + 3 \neq 8$
   e) T; $2 \times 3 = 5 \Rightarrow 3 > 1$ and 6 is odd.
   f) T; A triangle is right angled at B and $AB^2 + BC^2 \neq AC^2$

9. i)

| Antecedent | Consequent |
|---|---|
| a) 3 is a natural number. | $\frac{1}{3}$ is a rational number |
| b) $x = 2$ | $x^2 = 4$ |
| c) The battery is low. | The mobile does not work well. |

ii) a) *Converse*: If $\frac{1}{3}$ is a rational number then 3 is a natural number.

*Inverse*: If 3 is not a natural number then $\frac{1}{3}$ is not a rational number.

*Contrapositive*: If $\frac{1}{3}$ is not a rational number then 3 is not a natural number.

b) *Converse*: If $x^2 = 4$ then $x = 2$

*Inverse*: If $x \neq 2$ then $x^2 \neq 4$

*Contrapositive*: If $x^2 \neq 4$ then $x \neq 2$.

c) *Converse*: If the mobile does not work well, then the battery is low.

*Inverse*: If the battery is not low then the mobile works well.

*Contrapositive*: If the mobile works well then the battery is not low.

iii) a) 3 is a natural number and $\frac{1}{3}$ is not a rational number.

b) $x = 2$ and $x^2 \neq 4$

c) The battery is low and the mobile works well.

CHAPTER 2

# Relations, Functions and Graphs

## 2.1 Ordered Pairs and Cartesian Products

### a) Ordered Pair

A *pair* consists of two elements. Some examples of pair are

(Nepal, Kathmandu), {Sita, Ram}, (3, 4), $\{a, b\}$, $\{\{a\}, \{a, b\}\}$.

Two pairs such as {Nepal, Kathmandu} and {Kathmandu, Nepal} have the same elements. So is the case with (3, 4) and (4, 3). In each case the elements are distinct. But in the case (3, 3) the two elements are identical. Note that we have used two types of brackets. We have used the curly brackets { } or braces to denote sets. In such cases, we do not worry about the order of occurrence of the elements; but we do not allow the same element to be repeated. In case, we need to account for the order in which they occur and we allow any element to be repeated, we shall use parentheses ( , ) with a comma between the two elements enclosed between them. We then say that such a pair of elements is **ordered**.

A pair having one element as the first and the other as the second is called an **ordered pair**. An ordered pair having $a$ as the first element and $b$ as the second element is denoted by

$$(a, b).$$

An ordered pair $(a, b)$ is generally not the same as the ordered pair $(b, a)$. But, this will happen so when the two elements are **identical**. Thus, (3, 4) is different from (4, 3); but (3, 3) is the same as (3, 3).

Two ordered pairs $(a, b)$ and $(c, d)$ are said to be **equal** *if and only if* $a = c$ and $b = d$.

***Examples:***

1. The ordered pairs (1, 3) and (1,3) are equal, but the ordered pairs (1, 3) and (3, 1) are not equal.
2. The ordered pairs $(a, a)$ and $(a, a)$ are equal.

### b) Cartesian Product

Given two sets A and B, the set of all ordered pairs $(a, b)$ such that $a \in A$ and $b \in B$ is called the **Cartesian product** of A and B, and is denoted by $A \times B$. It is read "A cross B".

In the set-builder notation, we have

$$A \times B = \{(a, b): a \in A, b \in B\}.$$

***Examples:***

1. The Cartesian product of $A = \{1, 2, 3\}$ and $B = \{2, 4\}$ is
$$A \times B = \{(1, 2), (1, 4), (2, 2), (2, 4), (3, 2), (3, 4)\}.$$
It is different from the Cartesian product of $B = \{2, 4\}$ and $A = \{1, 2, 3\}$, i.e., $B \times A = \{(2, 1), (2, 2), (2, 3), (4, 1), (4, 2), (4, 3)\}$.

2. The Cartesian product of $A = \{H, T\}$ with itself, also known as the Cartesian product on A is
$$A \times A = \{(H, H), (H, T), (T, H), (T, T)\}.$$

**Note:**
1. **In general, $A \times B \neq B \times A$.**
2. **If $m$ is the number of elements in A and $n$ is the number of elements in B, then the number of elements in $A \times B$ or $B \times A$ is $mn$.**
3. **If R is the set of real numbers, then the Cartesian product of R on R, i.e. $R \times R$ or $R^2$ is the set $\{(x, y): x \in R, y \in R\}$.**
**This Cartesian product is represented by the entire Cartesian coordinate plane.**

## 2.2 Relations

Any subset of a Cartesian product $A \times B$ of two sets A and B is called a **relation**. A relation from a set A to a set B is denoted by $x\Re y$, if $x \in A$ and $y \in B$, or simply by $\Re$ if $(x, y) \in \Re$.

In particular, a relation from a set A to itself is called a relation **on** A. Relations may be expressed in various ways. Here are some examples:

***Examples:***

1. By specifying or displaying ordered pairs:
Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$, then
$$A \times B = \{(1,1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4)\}.$$
$\Re = \{(1, 1), (2, 2), (3,3)\} \subset A \times B$, is a relation from A to B.

2. By standard description or by using a rule or formula:
The relation $\Re = \{(1,1), (2,2), (3,3)\} \subset A \times B$ can be described by
$$\Re = \{(x, y): x = y\} \subset A \times B.$$

A second example of this type is

$$\mathfrak{R}_1 = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\} \subset A \times B.$$

This can be described by

$$\mathfrak{R}_1 = \{(x, y): x < y\} \subset A \times B.$$

3. The relations $\mathfrak{R}$ and $\mathfrak{R}_1$ may be presented in the form of a table as shown below:

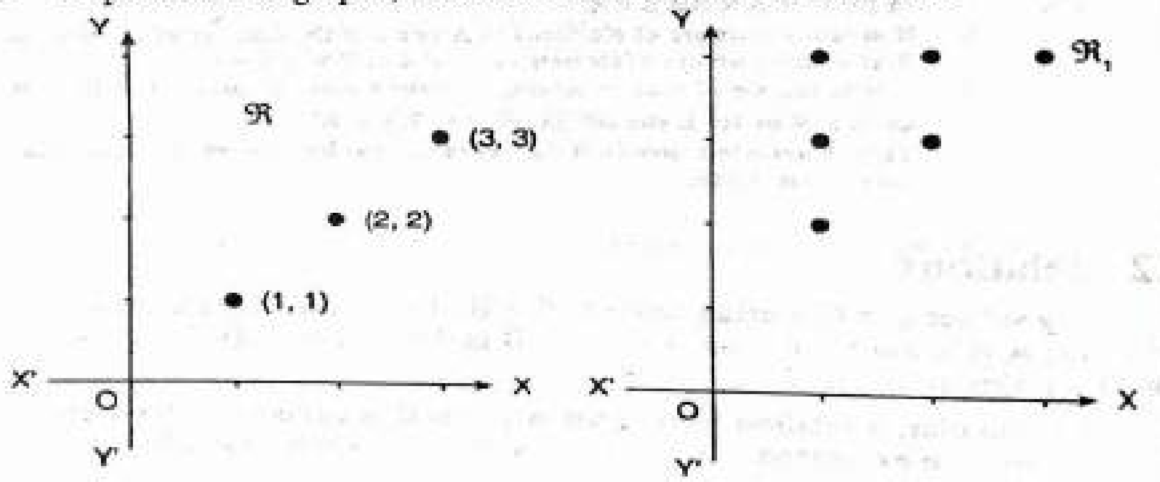
a) $\mathfrak{R}$

| | | | |
|---|---|---|---|
| $x$ | 1 | 2 | 3 |
| $y$ | 1 | 2 | 3 |

b) $\mathfrak{R}_1$

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ | 1 | 1 | 1 | 2 | 2 | 3 |
| $y$ | 2 | 3 | 4 | 3 | 4 | 4 |

4. Represented as graphs, the above relations would look like:




**a) Domain and Range**

The **domain** of a relation $\mathfrak{R}$ is the set of all first members of the pairs $(x, y)$ of $\mathfrak{R}$. It is denoted by Dom($\mathfrak{R}$).

Symbolically, $\text{Dom}(\mathfrak{R}) = \{x : (x, y) \in \mathfrak{R} \text{ for some } y \in B\}$

In the above examples, the domains are

$\text{Dom}(\mathfrak{R}) = \{1, 2, 3\}$ and $\text{Dom}(\mathfrak{R}_1) = \{1, 2, 3\}$.

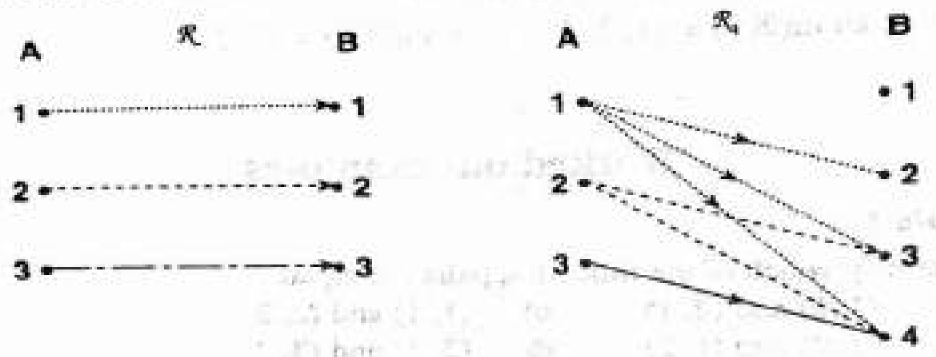
The **range** of a relation $\mathfrak{R}$ is the set of all second members of the pairs $(x, y)$ of $\mathfrak{R}$. It is denoted by Ran($\mathfrak{R}$).

Symbolically, $\text{Ran}(\mathfrak{R}) = \{y : (x, y) \in \mathfrak{R} \text{ for some } x \in A\}$

In the above examples,

$\text{Ran}(\mathfrak{R}) = \{1, 2, 3\}$ and $\text{Ran}(\mathfrak{R}_1) = \{2, 3, 4\}$.

The above relations are shown below by drawing arrow diagrams:




## b) Inverse Relation

Since a relation is a subset of a Cartesian product, we can think of a set formed by interchanging the first and second members of a relation. This implies that every relation from a set A to a set B has a relation from B to A. Given a relation

$$\mathfrak{R} = \{(x, y): x \in A, y \in B\} \subset A \times B,$$

we can define a relation of the form

$$\{(y, x): y \in B, x \in A\} \subset B \times A.$$

Such a relation is denoted by $\mathfrak{R}^{-1}$, read "script R inverse", and is called **inverse relation from B to A of** $\mathfrak{R}$.

***Examples:***

1. Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$ be two sets. Then,

$A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$,

and $B \times A = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$.

a) If $\mathfrak{R} = \{(1, a), (2, b)\}$, then $\mathfrak{R}^{-1} = \{(a, 1), (b, 2)\}$. Here,

$\text{Dom}(\mathfrak{R}) = \{1, 2\}$ and $\text{Ran}(\mathfrak{R}) = \{a, b\}$;

but $\text{Dom}(\mathfrak{R}^{-1}) = \{a, b\}$ and $\text{Ran}(\mathfrak{R}^{-1}) = \{1, 2\}$.

b) If $\mathfrak{R} = \{(1, a),(1, b),(3, a)\}$, then $\mathfrak{R}^{-1} = \{(a, 1),(b, 1),(a, 3)\}$. Here,

$\text{Dom}(\mathfrak{R}) = \{1, 3\}$ and $\text{Ran}(\mathfrak{R}) = \{a, b\}$;

but $\text{Dom}(\mathfrak{R}^{-1}) = \{a, b\}$ and $\text{Ran}(\mathfrak{R}^{-1}) = \{1, 3\}$.

2. Let $A = \{1, 2, 3\}$. Then,

$A \times A = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3, 3)\}$.

Here, the elements (1, 1), (2, 2) and (3, 3) are called the diagonal elements of A × A.

If $\Re = \{(1,1),(2,2),(3,3)\}$, then $\Re^{-1} = \{(1,1),(2,2),(3,3)\}$. Here,

$\text{Dom}(\Re) = \{1, 2, 3\} = \text{Ran}(\Re)$.

Also, $\text{Dom}(\Re^{-1}) = \{1, 2, 3\}$ and $\text{Ran}(\Re^{-1}) = \{1, 2, 3\}$.

## Worked out examples

***Example 1***

Identify which of the following pairs are equal:

a) (1, 2) and (2, 1) b) (1, 1) and (2, 2)

c) (1, 2) and (1, 2) d) (3, 3) and (3, 3).

***Solutions:***

a) Unequal b) Unequal

c) Equal d) Equal.

***Example 2***

If the ordered pairs $(x + y, 1)$ and $(2, 2x - y)$ are equal, find $x$ and $y$.

***Solution:***

Since $(x + y, 1)$ and $(2, 2x - y)$ are equal,

$$x + y = 2 \quad \text{and} \quad 1 = 2x - y.$$

Solving the simultaneous equations, we get

$$x = 1 \quad \text{and} \quad y = 1.$$

***Example 3***

If $A = \{1, 2, 3\}$ and $B = \{a, b\}$, find $A \times B$.

***Solution:***

Here, $A = \{1, 2, 3\}$ and $B = \{a, b\}$.

Hence, the required Cartesian product of A and B is

$$A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

***Example 4***

Let $A = \{a, b\}$, $B = \{c, d\}$ and $C = \{d, e\}$. Find

a) $A \times (B \cup C)$ b) $A \times (B \cap C)$ c) $(A \times B) \cup (A \times C)$

***Solutions:***

Here, $A = \{a, b\}$, $B = \{c, d\}$ and $C = \{d, e\}$.

a) $B \cup C = \{c, d, e\}$. So,
$A \times (B \cup C) = \{a, b\} \times \{c, d, e\}$.
$= \{(a, c), (a, d), (a, e), (b, c), (b, d), (b, e)\}$.

b) $B \cap C = \{d\}$. So,
$A \times (B \cap C) = \{a, b\} \times \{d\} = \{(a, d), (b, d)\}$.

c) $A \times B = \{a, b\} \times \{c, d\}$,
$= \{(a, c), (a, d), (b, c), (b, d)\}$.
$A \times C = \{a, b\} \times \{d, e\}$,
$= \{(a, d), (a, e), (b, d), (b, e)\}$.
Hence,
$(A \times B) \cup (A \times C)$
$= \{(a, c), (a, d), (b, c), (b, d)\} \cup \{(a, d), (a, e), (b, d), (b, e)\}$.
$= \{(a, c), (a, d), (a, e), (b, c), (b, d), (b, e)\}$.

***Example 5***

Let $A = \{a, b\}$, $B = \{c, d\}$ and $C = \{d, e\}$. Verify that
$$A \times (B \cup C) = (A \times B) \cup (A \times C).$$

***Solution:***

From 4(a) above, we have
$A \times (B \cup C) = \{a, b\} \times \{c, d, e\}$,
$= \{(a, c), (a, d), (a, e), (b, c), (b, d), (b, e)\}$.

From 4(c) above, we have
$(A \times B) \cup (A \times C)$
$= \{(a, c), (a, d), (b, c), (b, d)\} \cup \{(a, d), (a, e), (b, d), (b, e)\}$.
$= \{(a, c), (a, d), (a, e), (b, c), (b, d), (b, e)\}$.

Hence, $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

***Example 6***

Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 3, 5\}$. Find the relation $\Re$ from A to B determined by the condition "$x < y$".

***Solution:***

Here, $A \times B = \{(1, 1), (1, 3), (1, 5), (2, 1), (2, 3), (2, 5), (3, 1), (3, 3), (3, 5), (4, 1), (4, 3), (4, 5)\}$.

Then $\Re = \{(x, y): x < y\}$ is the set
$\{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)\}$.

***Example 7***

Find the domain, range and inverse of the following relation:
$$\Re = \{(1, 2), (2, 4), (3, 6), (4, 8)\}.$$

***Solutions:***

$\text{Dom}(\Re) = \{1, 2, 3, 4\}$, $\text{Ran}(\Re) = \{2, 4, 6, 8\}$.

Inverse of $\Re = \Re^{-1} = \{(2, 1), (4, 2), (6, 3), (8, 4)\}$.

## EXERCISE 2.1

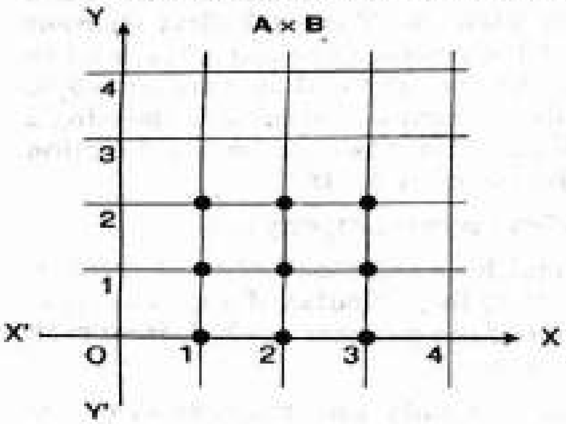
1. Identify which of the following pairs are equal:
   a) $(1, 3)$ and $(3, 1)$ b) $(a, b)$ and $(a, b)$
   c) $(1, a)$ and $(1, x)$ d) $(x, x)$ and $(y, y)$.
2. Find $x$ and $y$, if
   a) $(x + y, 3) = (1, x - y)$ b) $(x + 2y, 3) = (-1, 2x - y)$
   c) $(x - 2, y + 1) = (1, 0)$ d) $(2x - 1, -3) = (1, y + 3)$.
3. If $A = \{1, 2, 3\}$ and $B = \{a, b\}$ find $A \times A$, $B \times B$ and $B \times A$.
4. Let $A = \{a, b\}$, $B = \{b, c\}$ and $C = \{c, d\}$. Find
   a) $A \times (B \cup C)$ b) $A \times (B \cap C)$
   c) $(A \times B) \cup (A \times C)$ d) $(A \times B) \cap (A \times C)$.
5. Let $A = \{a, b\}$, $B = \{c, d\}$ and $C = \{d, e\}$. Verify that
   a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
   b) $A \times (B \cap C) = (A \times B) \cap (A \times C)$.
6. Find the cartesian product $A \times B$ of the following sets
   a) $A = \{x : x = 1, 2, 3\}$, $B = \{y : y = 3 - x\}$
   b) $A = \{x : x = 0, 1, 2\}$, $B = \{y : y = x^2\}$
   Represent the elements of $A \times B$ graphically.
7. a) Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 3, 5\}$. Find the relation R from set A to set B determined by the condition
   (i) $x > y$ (ii) $x \le y$ (iii) $y = x^2$
   b) Let $A = \{1, 2, 3, 4\}$. Find the relation on A satisfying the condition (i) $y = 2x$ (ii) $x + y = 6$ (iii) $x + y \le 4$ (iv) $x - y \ge 1$
8. Find the domain, range and the inverse of the following relations
   a) $R_1 = \{(1, 2), (2, 3), (3, 4), (4, 5)\}$
   b) $R_2 = \{(1, 3), (2, 5), (3, 7), (4, 9)\}$
   c) $R_3 = \{(a, b) (b, a) (b, c), (c, b), (c, a) (a, c)\}$
   d) $R_4 = \{(a, b) (c, b) (d, b), (e, b)\}$

9. a) Let A = {2, 3, 4} and B = {2, 3, 4, 6}. Find a relation from set A to set B determined by the condition that $x$ divides $y$. Also, find the domain and range of the relation.

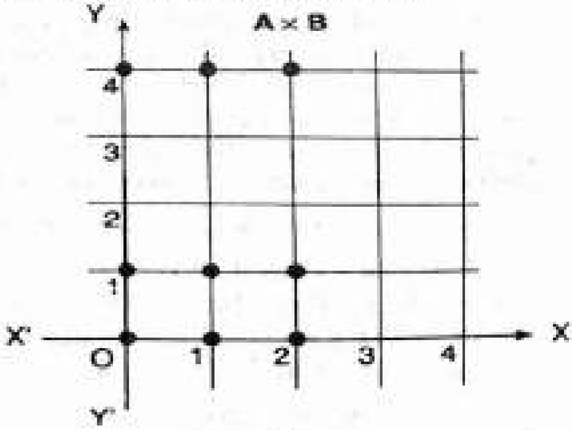
   b) Let A = {3, 4, 5, 6} and the relation is defined as R = {$(x, y)$ : $x, y \in$ A and $x + y = 7$}. Express R as a set of ordered pairs. Find the domain, range and $R^{-1}$.

## *Answers*

1. b
2. a) $x = 2, y = -1$ b) $x = 1, y = -1$
   c) $x = 3, y = -1$ d) $x = 1, y = -6$
3. A × A = {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}
   B × B = {$(a, a)$, $(a, b)$, $(b, a)$, $(b, b)$}
   B × A = {$(a, 1)$, $(a, 2)$, $(a, 3)$, $(b, 1)$, $(b, 2)$, $(b, 3)$}
4. a) {$(a, b)$, $(a, c)$, $(a, d)$, $(b, b)$, $(b, c)$, $(b, d)$}
   b) {$(a, c)$, $(b, c)$}
   c) {$(a, b)$, $(a, c)$, $(a, d)$, $(b, b)$, $(b, c)$, $(b, d)$}
   d) {$(a, c)$, $(b, c)$}
6. a) {(1, 2), (1, 1), (1, 0), (2, 2), (2, 1), (2, 0), (3, 2), (3, 1), (3, 0)}
   b) {(0, 0), (0, 1), (0, 4), (1, 0), (1, 1), (1, 4), (2, 0), (2, 1), (2, 4)}



(a)



(b)

7. a) i) {(2, 1), (3, 1), (4, 1), (4, 3)}
   ii) {(1, 1), (1, 3), (1, 5), (2, 3), (2, 5), (3, 3), (3, 5), (4, 5)}
   iii) {(1, 1)}
   b) i) {(1, 2), (2, 4)}
   ii) {(2, 4), (3, 3), (4, 2)}
   iii) {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1)}
   iv) {(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)}

8. a) Domain (R) = {1, 2, 3, 4}, Range (R) = {2, 3, 4, 5}
   $R^{-1}$ = {(2, 1), (3, 2), (4, 3), (5, 4)}
   b) Domain (R) = {1, 2, 3, 4}, Range (R) = {3, 5, 7, 9}
   $R^{-1}$ = {(3, 1), (5, 2), (7, 3), (9, 4)}
   c) Domain (R) = {a, b, c}, Range (R) = {$a, b, c$}
   $R^{-1}$ = {$(b, a), (a, b), (c, b), (b, c), (a, c), (c, a)$}
   d) Domain (R) = {$a, c, d, e$}, Range (R) = {$b$}
   $R^{-1}$ = {$(b, a), (b, c), (b, d), (b, e)$}
9. a) R = {(2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4)}
   Domain (R) = {2, 3, 4}, Range (R) = {2, 3, 4, 6}
   b) R = {(3, 4), (4, 3)}
   Domain (R) = {3, 4} Range(R) = {3, 4}
   $R^{-1}$ = {(4, 3), (3, 4)}

## 2.3 Function

Relation, as we have seen earlier, is a very broad concept. It manifests itself in everyday life in various forms in a very natural way. To make it more useful and concrete, it is necessary to refine it. For this, we begin with a relation $\Re$ on a set A and an ordered pair $(a, b) \in \Re$. A refinement of $\Re$ can be obtained by assigning **only one second element '$b$' to each first element '$a$'** of a pair $(a, b)$. Such a refinement of the notion of relation leads us to what is known as a **function**. Functions are represented in various ways. Functions can be represented geometrically by means of graphs. In defining a function, we may consider one or more sets. Suppose we have a function defined from a set A to another set B or the same set A. If

$$x \in A, \text{ i.e., } A = \{x : x \text{ satisfies certain property}\},$$

the letter or symbol, $x$ is known as a **variable**. Each member or element of the set A is called the **value** of the variable $x$. In particular, if $x \in A = \{1, 2, 3\}$, then $x$ is a variable and it stands for one of the members 1, 2, 3. Here each of the numbers 1, 2, 3 is a value of the variable $x$.

In case, we have a set C consisting of only one member $c$, i.e., if C = { $c$ } is a singleton set, then the symbol $c$ stands for what is known as a **constant**. In other words, a constant has a *fixed* value.

Consider two sets A and B. Any non-empty subset $\mathcal{R}$ of the Cartesian product A × B is known as a relation from A to B. A special but very important type of relation is that which associates *each element of the set* A *with a unique element of* B. This may therefore be visualized as a *refinement* of the concept of a relation. Let us first see some concrete cases of such a refinement before we give a formal definition of what is known as a **function**.

***Example 1***

Let $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$. From the Cartesian product

$A \times B = \{\mathbf{(1,2)}, (1,4), (1,6), (2,2), \mathbf{(2,4)}, (2,6), (3,2), (3,4), \mathbf{(3,6)}\}$,

let us make the following *selection* (or *refinement*)

$$\{(1, 2), (2, 4), (3, 6)\}.$$

This is a *relation* from A to B. Here each first element is uniquely associated with a second element. The *rule* or *relation* corresponding to the present assignment is

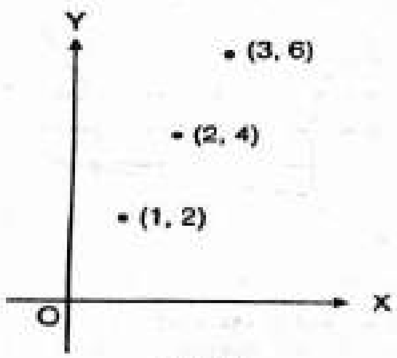
"*To each first element there corresponds a unique second element that is two times the first element.*"

It may represented as a table as shown below:

**Tabular form**

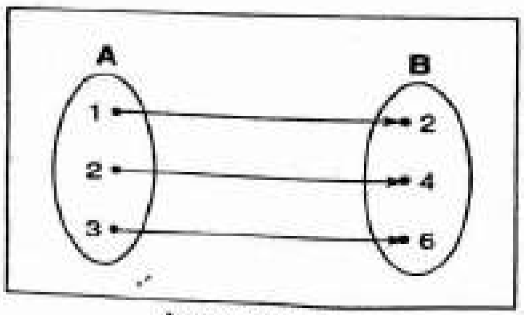
| | | | |
|---|---|---|---|
| $x$ | 1 | 2 | 3 |
| y | 2 | 4 | 6 |

A better way of visualizing it is to use *graph*. In the present case, the graphical representation consists of the above three distinct points as shown below:

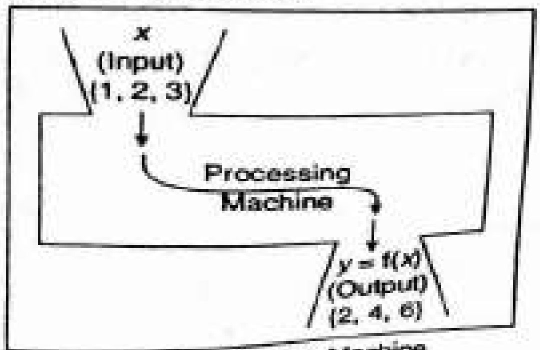



Graph

Two other instructive representations of a function are:




Arrow diagram




Input-Output Machine

***Example 2***

Let $A = \{a, b, c, d\}$ and $B = \{x, y, z, t\}$. Then the Cartesian product of A and B is

$$A \times B = \{(a,x), (a,y), (a,z), (a,t), (b,x), (b,y), (b,z), (b,t), (c,x), (c,y), (c,z), (c,t), (d,x), (d,y), (d,z), (d,t)\}$$

Let us make the following *selection* (or *refinement*)

$$\{(a,x),(b,y), (c,z),(d,z)\}.$$

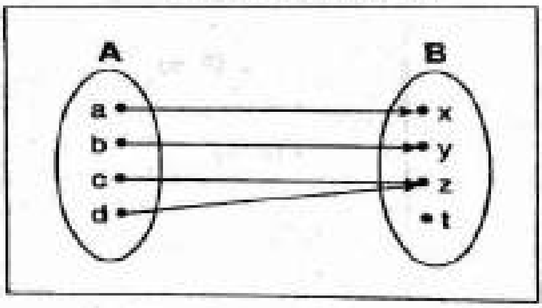
This is a *refined relation* from A to B. Here each first element is uniquely associated with a second element. But there is a slight difference. There are *two first elements* that correspond to *one second element*. The *rule or relation* corresponding to the present assignment is represented as a table as shown below:

**Tabular form**

| $1^{st}$ *element* | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $2^{nd}$ *element* | $x$ | $y$ | $z$ | $z$ |

The following arrow diagram is much more appealing:

**Arrow-diagram**




**Note:** **Here although every element is associated with a unique second element, there is one element of the second set B that is associated with no element of A.**

A **function** from a set A to a set B is a *relation* or *rule* that associates each element of A with a *unique* element of B.

In other words, a **function** from a set A to a set B is a relation $f$ such that for each $a \in A$, $\exists$ a unique $b \in B$ s.t. $(a, b) \in f$.

Here the symbol, " $\exists$ " is read " *there exists* " and " s.t. " stands for " *such that* ".

Also, a function from a set A to set B is a relation in which no two ordered pairs have the same first coordinate.

Symbolically, a **function** $f$ from a set A to a set B is denoted by

$$f : A \to B \quad \text{or} \quad A \xrightarrow{f} B.$$

If $y \in$ B is associated with an element $x$ of A, we write it as

$$y = f(x),$$

which is read " $y$ equals $f$ of $x$ ". Here $f(x)$ is known as the **image of** $f$ at $x$ or **value** of $f$ at $x$.

If $b$ is the unique element of B corresponding to an element $a$ of A under the function $f$, $a$ is called the **pre-image** of $b$ under $f$ and $b$ the **image** of $a$ under $f$. We also write

$$b = f(a).$$

Algebraists are often found to write '$af$' instead of $f(a)$. Quite often, we call $f(a)$ the **value** of $f$ at $a$.

It is to be noted that if $f$ is a function from A to B, then **no** element of A is related to **more than one** element of B; although **more than one** element of A can be related to the **same** element of B. In each case, A is called the **domain** of $f$ and B its **co-domain.**

The subset of B that contains only those elements of B that have pre-images in A is often called the **range** of $f$. Obviously,

$$\text{range of } f \subseteq \text{B}.$$

Thus in the function $f : A \to B$, the set of values of $x \in$ A for which the function is defined, is said to be the domain of the function and is denoted by D($f$).

$\therefore$ D($f$) = $\{x : x \in \text{A for which } f(x) \text{ is defined}\}$

For example : If $y = f(x) = \sqrt{x-1}$ then $f(x)$ is defined for $x - 1 \geq 0$ i.e. $x \geq 1$.

$\therefore$ D($f$) = $\{x : x \geq 1\} = [1, \infty)$

The set of values of $y \in$ B corresponding to each $x \in$ A which runs over the domain of the function is said to be the range of the function and is denoted by R($f$).

$\therefore$ R($f$) = $\{f(x) : x \in D(f)\}$

For example : If $y = f(x) = \sqrt{x - 1}$

then the range of $f$ = R($f$) = $\{y : y \geq 0\} = [0, \infty)$.

In the function $f : A \to B$, A itself is the domain of $f$ and the set of values of $y \in$ B which are the images of the elements of A, is the range of $f$. So, range is, of course a subset of B.

A function is also called a **mapping** or in special contexts, a **transformation** or an **operator**.

The fact that $y$ is the image of an element $x$ under a function $f$ is also indicated by

$$x \mapsto y \text{ or } x \mapsto f(x).$$

In terms of the present terminology and notations, the refined relation from A = {1, 2, 3} to B = {2, 4, 6} and defined by the following **selection** (or **refinement**)

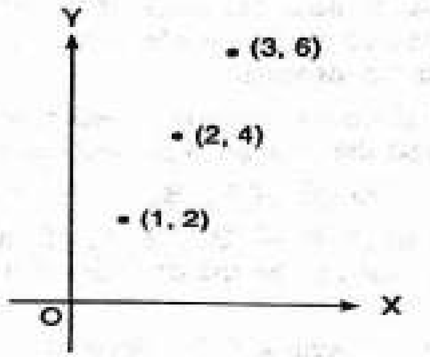
$$\{(1,2), (2,4), (3,6)\},$$

we have the following representations:

**a) Tabular Form:**

| $x$ | 1 | 2 | 3 |
|---|---|---|---|
| $y = f(x)$ | 2 | 4 | 6 |

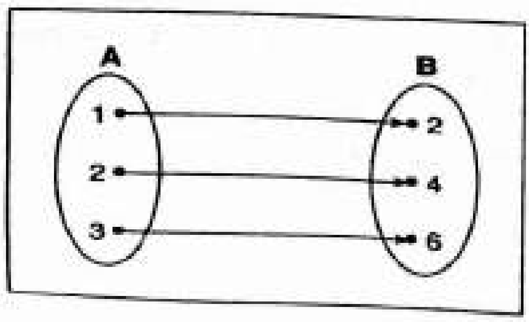
**b) Graph:**



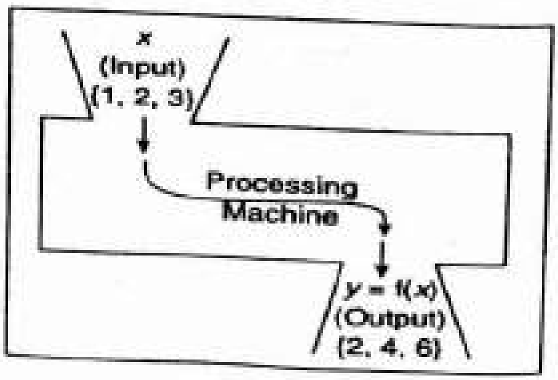

From the given graph, we can examine whether it is of the function or not by a test known as the **vertical line test**. A vertical line is drawn to meet the given graph. If the vertical line cuts the given graph at only one point it is the graph of a function. If the vertical line cuts the graph at two or more points, it is not the graph of a function.

Two other instructive representations of a function are :

**c) Arrow-Diagram:**




**d) Input-output Machine**

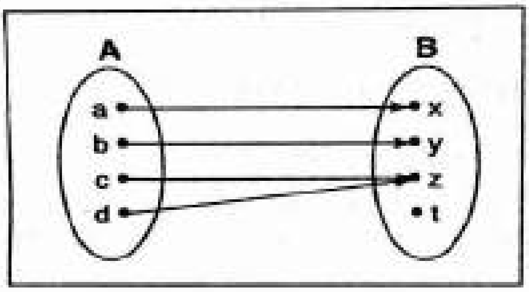


The second example considered earlier is also an example of a function from $A = \{a, b, c, d\}$ and $B = \{x, y, z, t\}$. Here the function $f$ from A to B may be described by

**a) Equations or Formula:**

$$f(a) = x,\ f(b) = y,\ f(c) = z \text{ and } f(d) = z.$$

**b) Arrow-diagram:**




**c) Table:**

| $x$ | $a$ | $b$ | $c$ | d |
|---|---|---|---|---|
| $f(x)$ | $x$ | $y$ | z | z |

Here the domain of definition of $f$ is $\{a, b, c, d\}$ and its range is

$$f(A) = \{x, y, z\}.$$

**Note: Here i)** $f(A) \neq B$ **and ii)** $f(c) = f(d) = z$.

If two functions $f$ and $g : A \to B$ have the same domain $\mathcal{D}$, they are said to be **equal** iff $f(x) = g(x)$ for every $x \in \mathcal{D} \subseteq A$; and in such a case, we write $f = g$.

Now, we have the following definitions

**Function:** A relation is said to be a function from set A to set B if every element of set A associates with a unique element of set B. A function from set A to set B is denoted by $f : A \to B$.

**Domain of the function:** The set A is known as the domain of the function. The domain of the function is denoted by dom $(f)$. Thus,

dom $(f) = \{x : x \in A\}$

**Co-domain of the function:** The set B is known as the co-domain of the function.

**Range of the function:** The set of values of $y = f(x) \in B$ for every $x \in A$ is known as the range of the function $f$. It is denoted by range $(f)$. Thus,

range $(f) = \{y : y \in B,\ y = f(x) \text{ for all } x \in A\}$

**Image:** The element $y \in$ B with which the element $x \in$ A associates, is known as the image of $x$ under $f$. It is also known as the value of $f$ at $x$.

**Pre-image:** The element $x \in$ A which associates with $y \in$ B, is known as the pre-image of $y$ under $f$.

**Equal function:** Two functions $f$ and $g$ are said to be equal i.e. $f = g$ if domain of $f$ = domain of $g$ and $f(x) = g(x)$ for all $x$ belonging to the domain of $f$ (or domain of $g$).

## 2.4 Types of Functions

There are three types of functions that are of special importance.

### a) One-one or Injective Function:

A function $f$ from a set A to another set B i.e. $f : A \rightarrow B$ is said to be **one-one (1-1) or injective if distinct elements (or pre-images) in A have distinct images in B.**

In symbols, for any $x, y \in$ A,

$$x \neq y \Rightarrow f(x) \neq f(y);$$

or, equivalently, $$f(x) = f(y) \Rightarrow x = y.$$

In other words, a function $f$ is said to be **one-one** or **injective** if $(x, f(x)), (y, f(y)) \in f \Rightarrow x = y$.

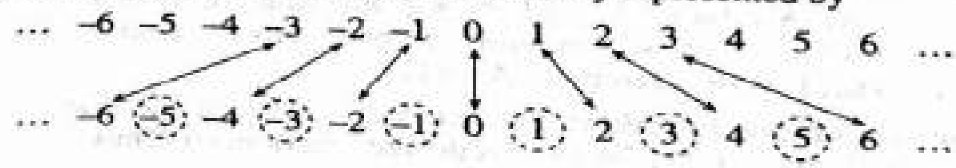
Thus, under one-one function all elements of A are related to different elements of B.

***Examples:***

1. The function

$$f : Z \rightarrow Z, \text{ where } Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

defined by $f(x) = 2x$ and diagrammatically represented by



is one-one, since

$$f(x) = f(y) \Rightarrow 2x = 2y \Rightarrow x = y.$$

Note that $f(Z) = \{\ldots, -4, -2, 0, 2, 4, \ldots\} \subset Z$. In such a case the function is said to be **one-one and into.**

2. The function $f : R \rightarrow R$ (R is the set of real numbers) defined by $f(x) = x^3$, is **one-one,** since the cubes of different real numbers are themselves different.

3. The function $f: N \rightarrow R$ (N is the set of natural numbers and R is the set of real numbers) defined by $f(x) = x^2$, is **one-one and into,** since the squares of different natural numbers are themselves different and $f(N) \subset R$.

4. The function $f: R \rightarrow R$ (R is the set of real numbers) defined by $f(x) = x^2$, is **not one-one,** since $f(3) = f(-3) = 9$. i.e., the *images* of two numbers 3 and −3 have the *same* number 9 as their image.

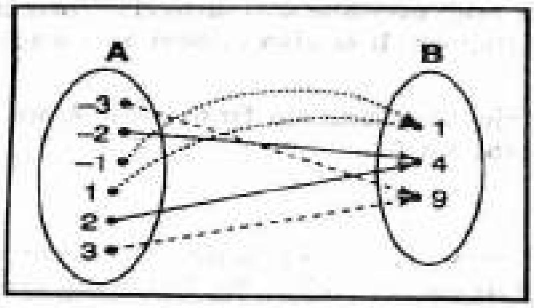
## b) Onto or Surjective Function

A function $f$ from a set A to another set B i.e., $f: A \rightarrow B$ is said to be **onto or surjective,** if **every element** of B is an image of **at least one element** of A, i.e., every element of B has a pre-image or, if $f(A) = B$.

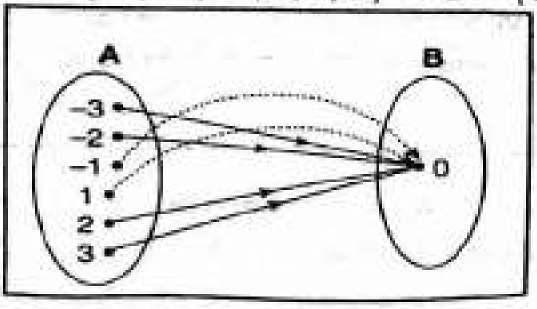
Sometimes such a function becomes a **many-one onto function.**

***Examples:***

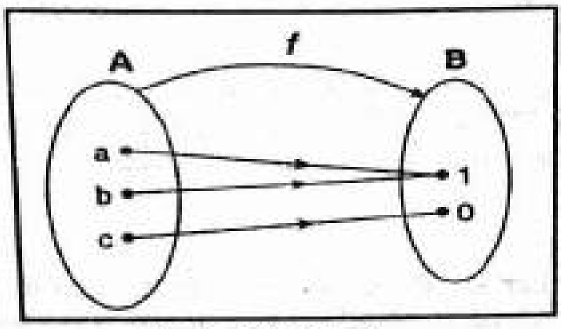
1. If A = {−3, −2, −1, 1, 2, 3} and B = {1, 4, 9}, then the function
$$f: A \rightarrow B,$$
defined by $f(x) = x^2$, is **onto** (moreover, it is **many-one onto**). Here, $f(A) = B$.



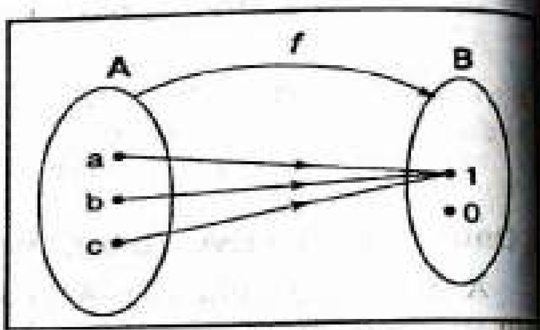

2. A special case of many-one onto or simply onto function is the function $f: A \rightarrow B$ such that $f(x) = c \in B$ for every $x \in A$. Such a function is called a **constant function.** The following figure shows a constant function
$$f: A = \{-3, -2, -1, 1, 2, 3\} \rightarrow B = \{0\}$$



3. The following arrow diagrams show functions defined on the same set $A = \{a, b, c\}$. One of them is onto and the other is into




Onto $f(A) = B$




Into $f(A) \subset B$

4. The function $f : Z \to N$, where $Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$ and $N = \{1, 2, 3, 4, \ldots\}$ defined by $f(x) = x^2$ is **not onto,** since there is **no** $x \in Z$ s.t. $f(x) = x^2 = 2$, although there exist several pairs of values of $x$ each of which has the same image. Obviously, $f(Z) \neq N$, but $f(Z) \subset N$. This is the case of **many-one into function.**

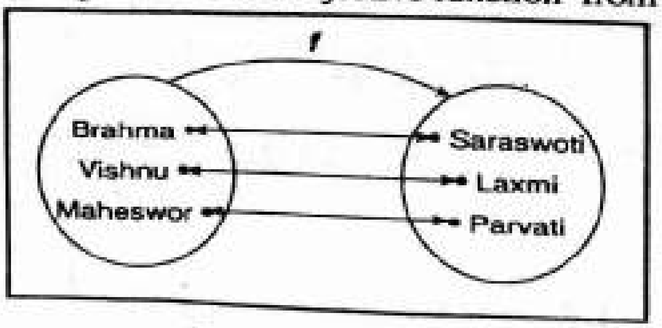
### c) One-one Onto or Bijective Function

A function that is both **one-one** and **onto (i.e., injective and surjective)** is called a **bijective** function. It is also known as a **one-to-one correspondence**.

In particular, a bijective function from a set A to itself is known as a **permutation** or **operator on A.**

***Examples:***

1. Consider two sets A = {Bramha, Vishnu, Maheswor} and B = {Saraswoti, Laxmi, Parvati}. The following arrow diagram shows a one-to-one correspondence or a bijective function from A to B.




2. Consider a function $f : A \to A$ defined by

$$f(x) = x \text{ for all } x \in A$$

Obviously, it is well-defined and is *both one-one and onto*. That is, it is a bijective function. This function is known as the **identity function**.

3. The function $f : \mathrm{N} \to \{3\}$ defined by

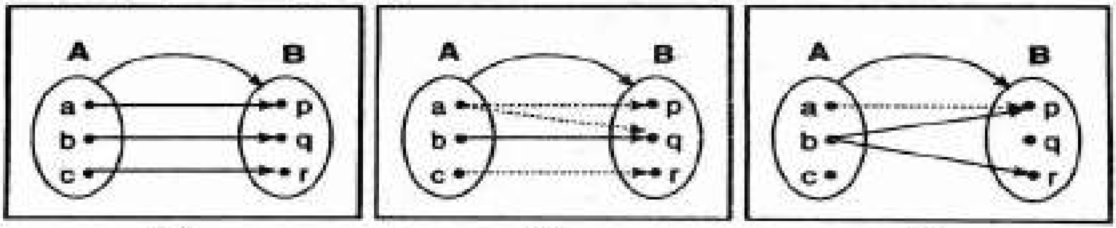
$$f(x) = 3 \text{ for all } x \in \mathrm{N},$$

the set of counting numbers, is obviously well defined. It is **not one-one** but **onto**. So, it is a not a bijective function.

## Worked out examples

***Example 1***

State whether or not each of the diagrams defines a function from $A = \{a, b, c\}$ to $B = \{p, q, r\}$:




(a) (b) (c)

***Solutions:***

a) Yes. To each element of A there corresponds one and only one element of B.

b) No. An element $a$ of A corresponds with two elements $p$ and $q$ of B.

c) No. There is no element of B corresponding to the element $c$ of A.

***Example 2***

Let $f(x) = x + 1$ be a function defined in the closed interval $-1 \le x \le 1$. Find a) $f(-1)$ b) $f(0)$ c) $f(1)$ d) $f(2)$.

***Solutions:***

a) $f(-1) = (-1) + 1 = 0$

b) $f(0) = 0 + 1 = 1$

c) $f(1) = 1 + 1 = 2$

d) $f(2)$ is not defined since 2 does not belong to the domain of definition of $f$ (i.e. $\mathcal{D}(f) = [-1, 1]$).

***Example 3***

Let the function $f : R \to R$ be defined by

$$f(x) = \begin{cases} 3 + 2x & \text{for } -3/2 \le x < 0 \\ 3 - 2x & \text{for } \quad 0 \le x < 3/2 \\ -3 - 2x & \text{for } \quad x \ge 3/2 \end{cases}$$

Find a) $f(-3/2)$ b) $f(0)$ c) $f(3/2)$

d) $\dfrac{f(h) - f(0)}{h}$ for $0 \le h < 3/2$.

(T.U.2051, HSEB 2052)

***Solutions:***

a) Since $-3/2 \in [-3/2, 0)$ or $-3/2 \le x < 0$, we use the first formula

$f(x) = 3 + 2x$.

Hence, $f(-3/2) = 3 + 2(-3/2) = 3 - 3 = 0$.

b) Since $0 \in [0, 3/2)$ or $0 \le x < 3/2$, we use the second formula

$f(x) = 3 - 2x$.

Hence, $f(0) = 3 - 2.0 = 3$.

c) Since $3/2 \in [3/2, \infty)$ or $3/2 \le x$, we use the third formula

$f(x) = -3 - 2x$.

Hence, $f(3/2) = -3 - 2.(3/2) = -6$.

d) For $0 \le h < 3/2$, we have to use the second formula.

Hence, $\dfrac{f(h) - f(0)}{h} = \dfrac{3 - 2h - 3}{h} = \dfrac{-2h}{h}$

$= -2$.

***Example 4***

Let $A = \{0, 1, 2, 3, 4, 5, 6\}$ and a function $f : A \to Q$ is defined by $f(x) = x/2$. Find the range of $f$.

***Solution:***

By given, $f(x) = \frac{x}{2}$ and $A = \{0, 1, 2, 3, 4, 5, 6\}$

When $x = 0, 1, 2, 3, 4, 5, 6$ the values of $f(x)$

i.e. $f(0), f(1), f(2), f(3), f(4), f(5)$ and $f(6)$ are $0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}$ and 3 respectively.

$\therefore$ range of $f = \{f(0), f(1), f(2), f(3), f(4), f(5), f(6)\}$

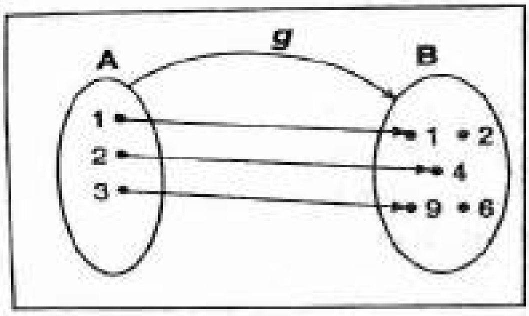
$\therefore f(A) = \left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3\right\}$

***Example 5***

Determine whether the functions $f$ and $g$ defined below are equal or not:

a) $f(x) = x^2$ where $\{x : x = 1, 2, 3\}$

b) Venn-diagram




***Solution:***

The functions $f$ and $g$ are defined by

$$f(x) = x^2 \quad \text{where } \{x : x = 1, 2, 3\} \text{ and the Venn-diagram.}$$

Since $f$ and $g$ have the same domain $\{1, 2, 3\}$ and assign the same image to each element in the domain, so $f = g$.

***Example 6***

Determine whether or not each of the following functions is one-one.

a) Let A be the set of positive integers. A function $f : A \rightarrow A$ is defined by

$$f(x) = 2x + 1;$$

b) Let $A = \{a, e, i, o, u\}$ and $B = \{m, b, s\}$. A function $f : A \rightarrow B$ is defined by

$$\left.\begin{matrix} f(a) \\ f(e) \end{matrix}\right\} = m, \qquad f(i) = b, \qquad \left.\begin{matrix} f(o) \\ f(u) \end{matrix}\right\} = s$$

***Solutions:***

a) Let A be the set of positive integers. A function $f : A \rightarrow A$ defined by $f(x) = 2x + 1$ associates different elements in A with different odd positive integers in A.

So, $f$ is a one-one function as shown below:


**Note:** $f(A) \neq A$.

b) Since $f(a) = f(e) = m$ i.e. $a$ and $e$ have the same image $m$, so $f$ is not the one-one function.

***Example 7***

Let a function $f: A \to A$ be defined by $f(x) = x^3$, where $A = \{-1, 0, 1\}$. Find the range of the function. Is the function one-one, onto or both ?

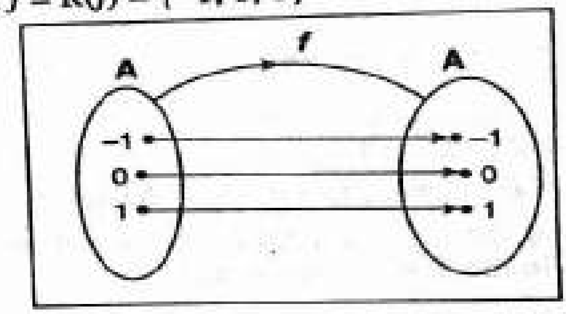
***Solution :***

When $x = -1$, $f(-1) = (-1)^3 = -1$

$x = 0$, $f(0) = 0^3 = 0$

$x = 1$, $f(1) = 1^3 = 1$

$\therefore$ the range of $f = R(f) = \{-1, 0, 1\}$




Since different elements of A (domain) have different images in A (co-domain) so $f$ is one-one function.

Also since, $f(A) = A$, so $f$ is onto. Hence $f$ is one-one and onto both.

***Example 8***

Let $f: R \to R$ be defined by $f(x) = |x - 1|$. Show that $f$ is neither one-one nor onto function.

***Solution :***

$f(x) = |x - 1|$

Consider two element $2, 0 \in R$.

Then, $f(2) = 1$ and $f(0) = 1$

Since $f(2) = f(0) \not\Rightarrow 2 = 0$

so, $f$ is not one-one function.

For all $x \in R$, the set of values of $f(x)$ are the non-negative real numbers. So, range of $f = [0, \infty) \subset R$

$\therefore f$ is not onto.

Hence $f$ is neither one-one nor onto function.

**Example 9**

Let $f: N \to N$ be defined by $f(x) = 3x$ for all $x \in N$ where N is the set of natural numbers. Show that $f$ is one-one but not onto function.

**Solution :**

Let $x_1, x_2 \in N$

then $f(x_1) = f(x_2) \Rightarrow 3x_1 = 3x_2$

$\Rightarrow x_1 = x_2$

$\therefore f$ is one-one function.

Again, $f(x) = 3x$

$\Rightarrow y = 3x$, where y is any element $\in N$

$\Rightarrow x = \frac{1}{3}y$

But, for some $y \in N$, $x \notin N$, so $f$ is not onto.

**Example 10**

Prove that a function $f: R \to R$ defined by $f(x) = 3x - 1$ is one-one and onto (i.e. bijective).

**Solution :**

Let $x_1, x_2 \in R$. Then $f(x_1) = 3x_1 - 1$ and $f(x_2) = 3x_2 - 1$

Now, $f(x_1) = f(x_2) \Rightarrow 3x_1 - 1 = 3x_2 - 1$

$\Rightarrow 3x_1 = 3x_2$

$\Rightarrow x_1 = x_2$

$\therefore f$ is one-one.

Let y be any real number $\in R$. Then,

$$f(x) = y = 3x - 1$$

or, $3x = y + 1$

$$x = \frac{y+1}{3} \in R \quad \text{for all } y \in R.$$

and $$f\left(\frac{y+1}{3}\right) = 3.\left(\frac{y+1}{3}\right) - 1 = y$$

$\therefore$ y is the image of $\frac{y+1}{3}$.

$\therefore f$ is onto.

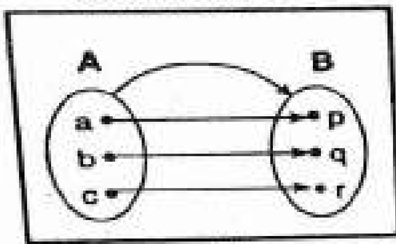
Hence $f$ is one-one and onto (i.e. bijective).

5

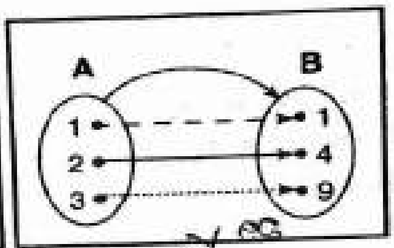
## EXERCISE 2.2

1. State whether or not each of the following diagrams defines a function from A to B:

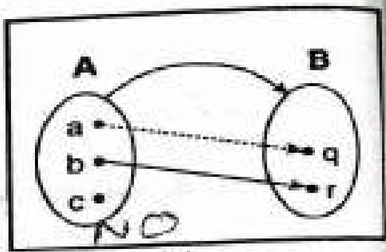
yes




(a)

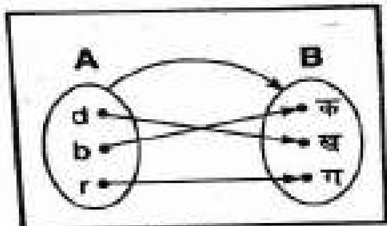



Yes (b)

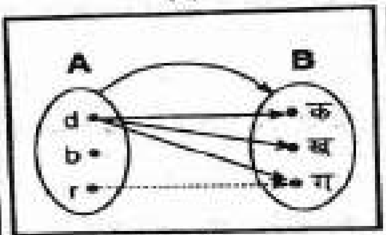



NO

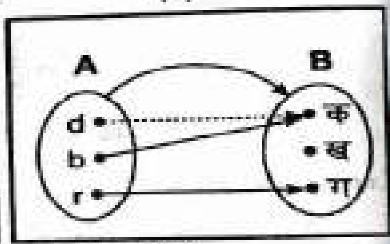
(c)




Y (d)




NO (e)




Yes (f)

2. Let $X = \{a, b, c\}$ and $Y = \{p, r, s\}$. Determine which of the following relations from X to Y are functions. Give reasons for your answers:
   a) $R_1 = \{(a, p)\ (a, r), (b, r), (c, s)\}$
   b) $R_2 = \{(a, p), (b, r)\}$
   c) $R_3 = \{(a, s), (b, s), (c, s)\}$
   d) $R_4 = \{(a, p), (b, p), (c, s)\}$
   e) $R_5 = \{(a, p), (b, r), (c, s)\}$

3. If $f : A \to B$ where A and $B \subset R$, is defined by $f(x) = 1 - x$, find the images of $1, \frac{3}{2}, -1, 2 \in A$.

4. Let i) $f(x) = x + 2$ ii) $f(x) = 2|x| + 3x$ in the interval $-1 \le x \le 2$. Find
   a) $f(-1)$ b) $f(0)$ c) $f(1)$
   d) $f(2)$ e) $f(-2)$ f) $f(3)$.

5. i) Let the function $f : R \to R$ be defined by

$$f(x) = \begin{cases} 3 + 2x & \text{for } -1/2 \le x < 0 \\ 3 - 2x & \text{for } 0 \le x < 1/2 \\ -3 - 2x & \text{for } x \ge 1/2. \end{cases}$$

Find a) $f(-1/2)$ b) $f(0)$ c) $f(1/2)$

d) $\dfrac{f(h) - f(0)}{h}$ for $0 \le h < 1/2$.

ii) Let the function $f : R \rightarrow R$ be defined by

$$f(x) = \begin{cases} 4x - 2 & \text{for } x \geq 1 \\ 2x & \text{for } x < 1. \end{cases}$$

Find a) $f(2)$ b) $f(1)$ c) $f(0)$ d) $f(-1)$

e) $\frac{f(h) - f(1)}{h}$ for $1 < h$.

6. Let $A = \{-1, 0, 2, 4, 6\}$ and a function $f : A \rightarrow R$ is defined by

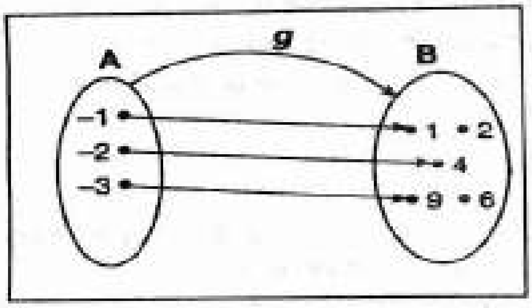
i) $y = f(x) = \frac{x}{x + 2}$ ii) $y = f(x) = \frac{x(x + 1)}{x + 2}$

Find the range of $f$.

7. Determine whether the functions $f$ and $g$ defined below are equal or not

a) $f(x) = x^2$ where $A = \{x : x = -1, -2, -3\}$

b) Venn-diagram




8. Determine whether or not each of the following functions is one-one.

a) Let $A = \{-2, -1, 0, 1, 2\}$ and $B = \{4, 0, 1\}$. A function $f : A \rightarrow B$ is defined by

$$\left.\begin{matrix} f(-2) \\ f(2) \end{matrix}\right\} = 4, \quad f(0) = 0, \quad \left.\begin{matrix} f(-1) \\ f(1) \end{matrix}\right\} = 1$$

b) Let A be the set of positive integers and B the set of squares of positive integers. A function $f : A \rightarrow B$ is defined by $f(x) = x^2$ ;

c) $f : [-2, 2] \rightarrow R$ is defined by $f(x) = x^2$

d) $f : [0, 3] \rightarrow R$ is defined by $f(x) = x^2$

9. Examine whether the following functions are one-one, onto, both or neither.

i) $f : A \rightarrow B$ defined by $f(x) = x^2$ where $A = \{1, -3, 3\}$ and $B = \{1, 9\}$

ii) $f : N \rightarrow N$ defined by $f(x) = 2x$

iii) $f : (-2, 2) \rightarrow R$ defined by $f(x) = x^2$

iv) $f : Q \rightarrow Q$ defined by $f(x) = 6x + 5$

v) $f : R \rightarrow R$ defined by $f(x) = x^3$

10. a) Let a function $f : A \to B$ be defined by $f(x) = \frac{x^2}{6}$ with $A = \{-2, -1, 0, 1, 2\}$ and $B = \{0, \frac{1}{6}, \frac{2}{3}\}$. Find the range of $f$. Is the function $f$ one-one and onto both ?

b) Let a function $f : A \to B$ be defined by $f(x) = \frac{x+1}{2x-1}$ with $A = \{-1, 0, 1, 2, 3, 4\}$ and $B = \{-1, 0, \frac{4}{3}, \frac{5}{7}, 1, 2, 3\}$. Find the range of $f$. Is the function $f$ one-one and onto both? If not, how can the function be made one-one and onto both?

### *Answers*

1. a) Yes, to each element of A there is one and only one element of B
b) Yes, to each element of A there is one and only one element of B
c) No, there is no element of B corresponding to the element c of A
d) Yes, to each element of A there is one and only one element of B
e) No, three elements of B correspond to one element of A.
f) Yes, to each element of A there is one and only one element of B.
2. (c), (d), (e)
3. $0, -\frac{1}{2}, 2, -1$
4. i) a) 1 b) 2 c) 3 d) 4 e & f) $f$ is not defined as $-2$ and 3 do not belong to the domain of definition
ii) a) $-1$ b) 0 c) 5 d) 10 e & f) $f$ is not defined as $-2$ and 3 do not belong to the domain of definition.
5. i) a) 2 b) 3 c) $-4$ d) $-2$
ii) a) 6 b) 2 c) 0 d) $-2$ e) $\frac{4(h-1)}{h}$
6. a) $\{-1, 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}\}$ b) $\{0, \frac{3}{2}, \frac{10}{3}, \frac{21}{4}\}$
7. Since $f(x) = \{(-1, 1), (-2, 4), (-3, 9)\} = g(x)$, the two functions are equal
8. a) No, since $f(-2) = 4 = f(2)$ does not imply $-2 = 2$
b) Yes, $x \neq y$ implies $x^2 \neq y^2$. So $f$ is one-one.
c) No, since $f(-2) = 4 = f(2)$ does not imply $-2 = 2$. That is $f$ is not one-one.
d) Yes, $x \neq y$ implies $x^2 \neq y^2$. So $f$ is one-one.
9. i) onto function ii) one-one function iii) Neither
iv) one-one and onto function v) one-one and onto function
10. a) $R(f) = \{0, \frac{1}{6}, \frac{2}{3}\}$, onto only
b) $R(f) = \{-1, 0, \frac{4}{5}, \frac{5}{7}, 1, 2\}$; No, one-one only

10. a) Let a function $f : A \to B$ be defined by $f(x) = \frac{x^2}{6}$ with $A = \{-2, -1, 0, 1, 2\}$ and $B = \{0, \frac{1}{6}, \frac{2}{3}\}$. Find the range of $f$. Is the function $f$ one-one and onto both ?

b) Let a function $f : A \to B$ be defined by $f(x) = \frac{x+1}{2x-1}$ with $A = \{-1, 0, 1, 2, 3, 4\}$ and $B = \{-1, 0, \frac{4}{5}, \frac{5}{7}, 1, 2, 3\}$. Find the range of $f$. Is the function $f$ one-one and onto both? If not, how can the function be made one-one and onto both?

## *Answers*

1. a) Yes, to each element of A there is one and only one element of B
   b) Yes, to each element of A there is one and only one element of B
   c) No, there is no element of B corresponding to the element c of A
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   e) No, three elements of B correspond to one element of A.
   f) Yes, to each element of A there is one and only one element of B.
2. (c), (d), (e)
3. $0, -\frac{1}{2}, 2, -1$
4. i) a) 1 b) 2 c) 3 d) 4 e & f) $f$ is not defined as $-2$ and 3 do not belong to the domain of definition
   ii) a) $-1$ b) 0 c) 5 d) 10 e & f) $f$ is not defined as $-2$ and 3 do not belong to the domain of definition.
5. i) a) 2 b) 3 c) $-4$ d) $-2$
   ii) a) 6 b) 2 c) 0 d) $-2$ e) $\frac{4(h-1)}{h}$
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9. i) onto function ii) one-one function iii) Neither
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10. a) $R(f) = \{0, \frac{1}{6}, \frac{2}{3}\}$, onto only
    b) $R(f) = \{-1, 0, \frac{4}{5}, \frac{5}{7}, 1, 2\}$; No, one-one only

## 2.5 Inverse Image and Inverse Function

The notions of inverse image of an element and inverse relation can be easily extended to the case of a function also.

### a) Inverse Image of An Element

Given a function

$$f : A \rightarrow B$$

the **inverse image** of an element $y \in B$ with respect to $f$ is defined as the set of elements in A which have $y$ as their image. It is usually denoted by $f^{-1}(y)$ and is read "*f inverse of y*".

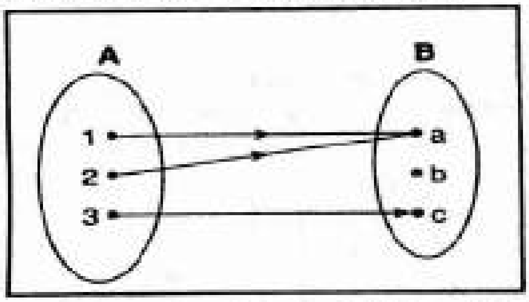
**(Note that $f^{-1} \neq 1/f$)**

In symbols, if a function is defined by $f : A \rightarrow B$, then

$$f^{-1}(y) = \{x \in A : y = f(x)\}.$$

***Examples:***

1. Let $f : A \rightarrow B$ be defined by the arrow diagram




Then, the inverse of $a$ under $f$, i.e., $f^{-1}(a) = \{1, 2\}$; $f^{-1}(b)$ is the null set $\phi$ and $f^{-1}(c) = \{3\}$.

2. Let a function $f : R \rightarrow R$ be defined by $f(x) = x^2$. Then, $f^{-1}(9) = \{-3, 3\}$, since 9 is the image of both $-3$ and 3. We further note that $f^{-1}(-1) = \phi$, since there is no real number whose square is $-1$.

3. Let $f : Z \rightarrow E$, where Z is the set of integers and E is the set of even integers including 0, is defined by

$$f(n) = \begin{cases} n & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Then the inverse image of E is the set S consisting of the even integers. It is to be noted that the inverse image may not exist at all. There may be elements of the first set, which do not belong to the inverse image of any element in the second set.

## b) Inverse Function

Let $f : A \to B$ be a one-one and onto (i.e., injective and surjective) function. Then, since $f$ is onto, corresponding to each element $b \in B$, there is *at least* one element $a \in A$. But, $f$ is one-one, and so $a$ is the only (or unique) element of A corresponding to the element $b \in B$. We thus have a rule which associates each element $b$ of B with a **unique element** $a$ of A, i.e., a function from B to A. We often denote such a function by $f^{-1}$ (read "*ef* " inverse). In other words we have a function of the type:

$$f^{-1} : B \to A.$$

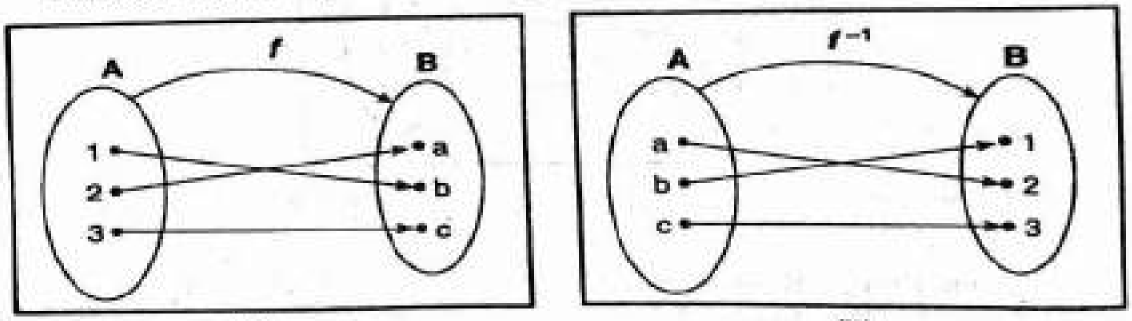
This function is known as the **inverse function** of $f$.

Thus if $f : A \to B$ be one-one and onto, then a function can be defined from B to A such that every element of B associates with a unique element of A, then the function defined from B to A is known as the inverse function of $f$ and is denoted by $f^{-1}$.

In short, when a function $f : A \to B$ is bijective, there exists a function $f^{-1} : B \to A$ called the inverse function of $f$.

***Examples:***

1. Let $f : A \to B$ be a one-one and onto (i.e., injective and surjective) function defined by the following arrow diagram on the left side:




(a) (b)

Obviously, the inverse function of $f$ is well-defined and the arrow diagram on the right represents it.

2. Let a function $f : R \to R$ be defined by $f(x) = x^3$. Then, it is one-one and onto. Hence, $f^{-1}$ is defined. It may be defined by

$$f^{-1}(x) = \sqrt[3]{x}\,. \qquad \text{(T.U. 2053)}$$

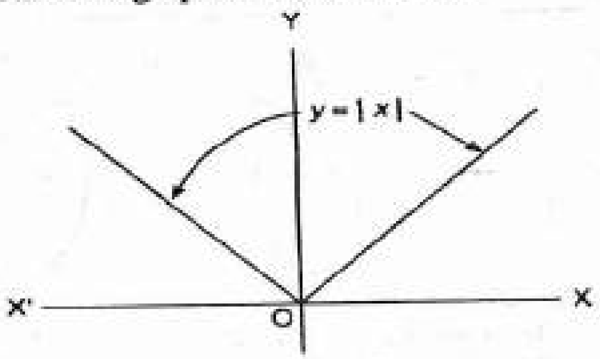
## 2.6 Real-valued Functions and Algebra of Real-valued Functions

A function $f : A \to R$ which associates each element of set A with a unique real number $f(a)$ of set B is called a **real-valued function.**

For example, if $x$ is any real number, the absolute value function defined by

$$f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

is a real-valued function. Its graph is shown below:




In practical life, we often have to deal with two or more real-valued functions. For simplicity, we consider only those real-valued functions that have the same domain D. We can define several algebraic operations for such functions as given below:

Suppose $f : D \to R$ and $g : D \to R$ are two real-valued functions and $k$ is a real number. Then, each of the functions on the left side is defined by the formula on the right:

$(f + k) : D \to R$ by $(f + k)(x) = f(x) + k$

$(|f|) : D \to R$ by $(|f|)(x) = |f(x)|$

$(f^n) : D \to R$ by $(f^n)(x) = (f(x))^n$

$(f \pm g) : D \to R$ by $(f \pm g)(x) = f(x) \pm g(x)$

$(kf) : D \to R$ by $(kf)(x) = k(f(x))$

$(fg) : D \to R$ by $(fg)(x) = f(x)\, g(x)$

$(f/g) : D \to R$ by $(f/g)(x) = f(x)/g(x)$ $(g(x) \neq 0)$

***Examples:***

1. Let $f : R \to R$ be defined by $f(x) = x + 3$. Then,

a) $(f + 3)(x) = f(x) + 3 = (x + 3) + 3 = x + 6$

**Warning:** $(f + 3)(x) \neq f(x) + 3(x)$

b) $(f^2)(x) = (f(x))^2 = (x + 3)^2 = x^2 + 6x + 9$

2. Let $f : R \to R$ and $g : R \to R$ be defined by $f(x) = 3x - 1$ and $g(x) = x^2$. Then,

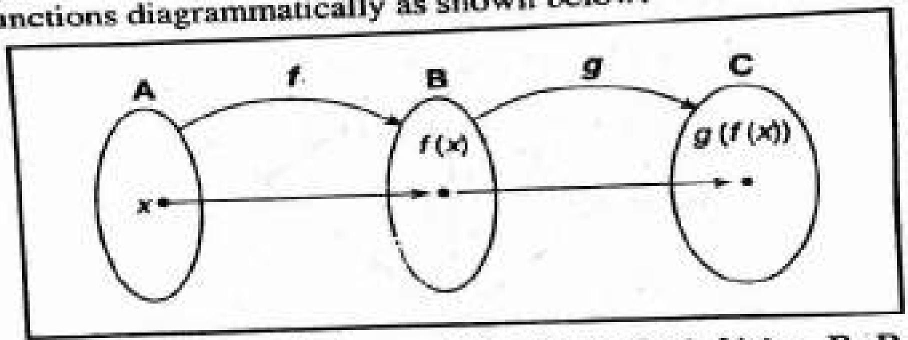
a) $(3f - 2g)(x) = 3f(x) - 2g(x) = 3(3x - 1) - 2x^2 = -2x^2 + 9x - 3$

b) $(fg)(x) = f(x)\, g(x) = (3x - 1)x^2 = 3x^3 - x^2$.

**Warning:** $(fg)(x) \neq f(g(x))$.

## 2.7 Composition of Functions

Let $f : A \to B$ and $g : B \to C$ be any two functions. We can represent the two functions diagrammatically as shown below:




Suppose $x \in A$, then its image (function value) $f(x) \in B$. B being the domain of $g$, we can find the image of $f(x) \in B$ under $g$, that is we can find $g(f(x))$ belonging to C. In other words, we can associate an element $x \in A$ with a unique element $g(f(x)) \in C$. Consequently, we have a function from A to C. This new function is known as the **composite function** of $f$ and $g$ (**not** $g$ and $f$); and it is denoted by

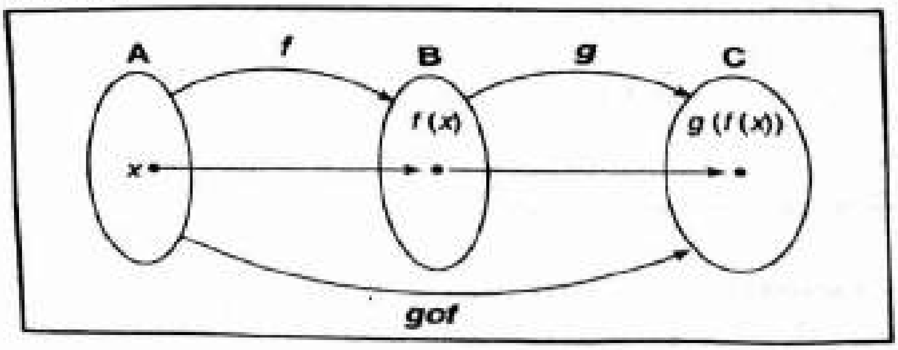
$(gof)$ , (read $g$ ***oh*** $f$) or $(g(f))$.

In short, if $f : A \to B$ and $g : B \to C$ be any two functions, then the **composite function** of $f$ and $g$ (also known as the **product function** or **function of a function**) is the function,

$g \circ f : A \to C$ (read " $g$ ***oh*** $f$ from A to C")

defined by the equation $(g \circ f)(x) = g(f(x))$.

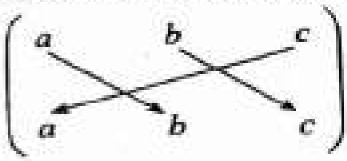
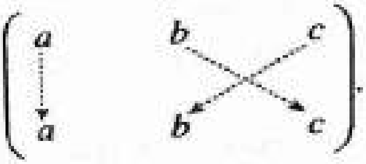
Schematically, the situation described above may be illustrated by the following diagram:



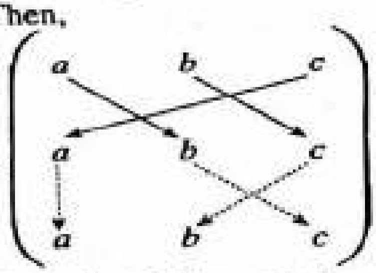
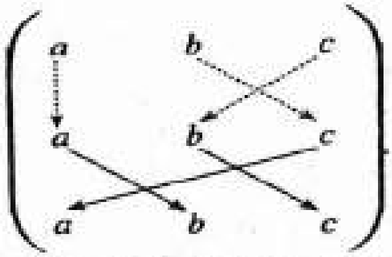
***Examples***

1. Let A, B and C denote the sets of real numbers. Suppose
$$f : A \to B \quad \text{and} \quad g : B \to C$$
are defined by $f(x) = x - 1$ and $g(x) = x^2$. Then,
$$(g \circ f)(x) = g(f(x)) = g(x - 1) = (x - 1)^2.$$

2. Consider the set $S = \{a, b, c\}$. Let $f$ and $g$ be defined by

$f$ :  and $g$ : .

Then,

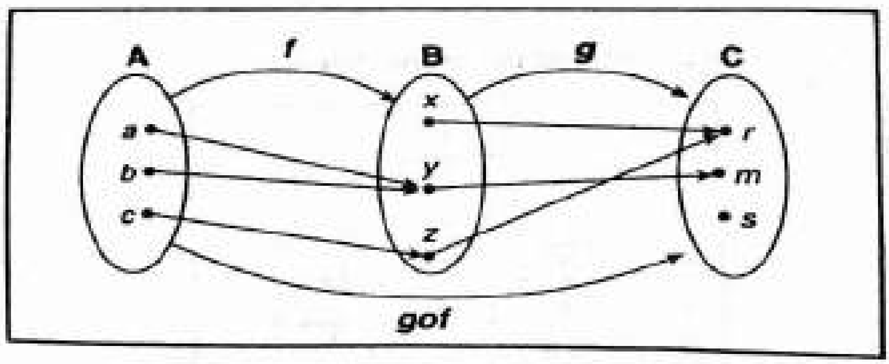
$g \circ f$ :  and $f \circ g$ = .

Here, $(g \circ f)(a) = g(f(a)) = g(b) = c$ and $(f \circ g)(a) = f(g(a)) = f(a) = b$.

This illustrates the fact $f \circ g \neq g \circ f$, in general.

That is, composite functions are not commutative.

3. Suppose $f : A \to B$ and $g : B \to C$ are defined by the following diagram:




Here,
$$(gof)(a) = g(f(a)) = g(y) = m$$
$$(gof)(b) = g(f(b)) = g(y) = m$$
$$(gof)(c) = g(f(c)) = g(z) = r$$

## Properties of Composite Functions

In what follows, we assume that the product or composite functions do exist.

i) If $f : A \to B$, $g : B \to C$ and $h : C \to D$ are given functions, then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

ii) If $f : A \to B$ and $g : B \to C$ and are given functions, then $g \circ f$ is onto or one-one according as each of $f$ and $g$ is onto or one-one.

A brief sketch of the proof of each property mentioned above is as follows:

i) By assumption each side of $h \circ (g \circ f) = (h \circ g) \circ f$ is well defined. The equality of the two sides can easily be seen if we note that

$$(h \circ (g \circ f))(x) = h((g \circ f)(x)) = h(g(f(x)))$$

and $$((h \circ g) \circ f)(x) = (h \circ g)(f(x)) = h(g(f(x)))$$

for any $x \in A$.

Hence, composite function satisfies associative property.

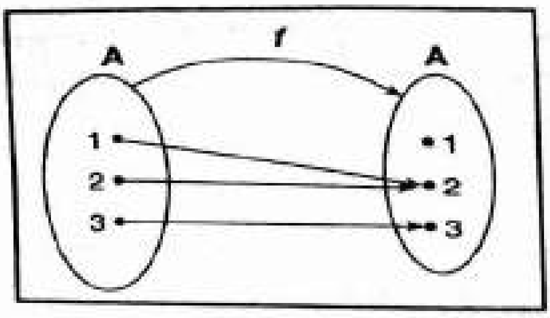
ii) Since $f$ is onto, $f(A) = B$, and so for any $x \in A$, $f(x) \in B$. Since $g$ is onto, $g(B) = C$ for any $f(x) \in B$, $g(f(x)) \in C$. But $g(f(x)) = (g \circ f)(x)$. Thus, $(g \circ f)(A) = C$.

Proof of the second part is left as exercise.

## Worked Out Examples

***Example 1***

Let $f : A \to A$ be defined by the arrow diagram




Find a) $f^{-1}(2)$ b) $f^{-1}(3)$ c) $f^{-1}(1)$ d) $f^{-1}(2,3)$.

***Solutions:***

a) $f^{-1}(2) = \{1, 2\}$ b) $f^{-1}(3) = \{3\}$

c) $f^{-1}(1) = \phi$ d) $f^{-1}(2,3) = \{1, 2, 3\}$.

***Example 2***

If $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$ and $f : A \to B$ is a function such that $f(1) = 4$, $f(2) = 5$ and $f(3) = 6$. Write down $f^{-1} : B \to A$ as a set of ordered pairs.

***Solution :***

$1 \in A$ corresponds with $4 \in B$ $(\because f(1) = 4)$

$2 \in A$ corresponds with $5 \in B$ $(\because f(2) = 5)$

$3 \in A$ corresponds with $6 \in B$ $(\because f(3) = 6)$

The distinct elements of A correspond with distinct elements of B.

$\therefore$ $f$ is one-one.

Since each element of B has at least one pre-image in A, so $f$ is onto.

$\therefore$ $f$ is one-one and onto.

So, $f^{-1}$ exists.

Since, $f = \{(1, 4), (2, 5), (3, 6)\}$, so

$f^{-1} = \{(4, 1), (5, 2), (6, 3)\}$

***Example 3***

Let a function $f : R \to R$ be defined by $y = f(x) = 2x - 3$, $x \in R$. Find a formula that defines the inverse function $f^{-1}$.

***Solution:***

Let $x_1, x_2 \in R$ (domain). Then $f(x_1) = 2x_1 - 3$

and $f(x_2) = 2x_2 - 3$

Now, $f(x_1) = f(x_2)$

$\Rightarrow$ $2x_1 - 3 = 2x_2 - 3$

$\Rightarrow$ $2x_1 = 2x_2$

$\Rightarrow$ $x_1 = x_2$

$\therefore f$ is one-one function.

Again let $k \in R$

Then, $k = 2x - 3$

$\Rightarrow$ $2x = k + 3$

$\Rightarrow$ $x = \dfrac{k+3}{2} \in R$

$\therefore f$ is onto function.

Hence $f$ is one-one and onto function. Since $f$ is one-one and onto function, so $f^{-1}$ exists and $f^{-1} : R \to R$ so that $x$ is the image of $y$ under $f^{-1}$ i.e. $x = f^{-1}(y)$

Solving for $x$ in terms of $y$, we have

$$x = \frac{y+3}{2}$$

or, $f^{-1}(y) = \frac{y+3}{2}$

which is a formula defining the inverse function. But since $y$ is a dummy variable and can be replaced by $x$. So, in terms of $x$, the inverse function is defined by

$$f^{-1}(x) = \frac{x+3}{2}$$

***Example 4***

Let $f : R \to R$ be defined by $f(x) = 2x - 1$ . Find

a) $(f - 3)(x)$ b) $(f^3)(x)$

c) $(3f + 4g)(x)$ d) $(f/g)(x)$ $(x \neq 0, \frac{1}{3})$.

***Solutions:***

a) $(f - 3)(x) = f(x) - 3 = (2x - 1) - 3 = 2x - 4$

b) $(f^3)(x) = (f(x))^3 = (2x - 1)^3$

c) $(3f + 4g)(x) = 3f(x) + 4g(x) = 3(4x^2 - 1) + 4(3x^2 - x) = 24x^2 - 4x - 3$

d) $(f/g)(x) = f(x) / g(x) = (4x^2 - 1)/(3x^2 - x)$.

***Example 5***

a) Let $f : R \to R$ and $g : R \to R$ be defined by $f(x) = x + 1$ and $g(x) = x^3$. Find $(g \circ f)(x)$ and $(f \circ g)(x)$.

b) Let $f : R \to R$ and $g : R \to R$ be defined by $f = \{(5, 2), (6, 3)\}$ and $g = \{(2, 5), (3, 6)\}$, find fog and gof.

***Solutions:***

a) $(g \circ f)(x) = g(f(x)) = g(x + 1) = (x + 1)^3$.
$(f \circ g)(x) = f(g(x)) = g(x^3) = x^3 + 1$.
Clearly, $(g \circ f)(x) \neq (f \circ g)(x)$.

b) **For fog**

$f = \{(5, 2), (6, 3)\}$ and $g = \{(2, 5), (3, 6)\}$

$g(2) = 5$, $g(3) = 6$

and $f(5) = 2$, $f(6) = 3$

Now, $(fog)(2) = f(g(2)) = f(5) = 2$

$(fog)(3) = f(g(3)) = f(6) = 3$

$\therefore$ fog $= \{(2, 2), (3, 3)\}$

For gof

$f(5) = 2, \quad f(6) = 3$

and $g(2) = 5, \quad g(3) = 6$

Now, $(gof)(5) = g(f(5)) = g(2) = 5$

$(gof)(6) = g(f(6)) = g(3) = 6$

$\therefore$ gof $= \{(5, 5), (6, 6)\}$

***Example 6***

Find the domain and the range of the following functions :

a) $y = f(x) = x^2 - 6x + 6$

b) $y = f(x) = \dfrac{1}{x - 1}$

c) $y = f(x) = \dfrac{x^2 - 4}{x - 2}$

d) $y = f(x) = \sqrt{6 - x - x^2}$

***Solution :***

a) $y = f(x) = x^2 - 6x + 6$

The given function is a polynomial of degree two in $x$. $y$ is defined for all $x \in$ R, so domain of $f$ = dom $(f)$ = R = $(-\infty, \infty)$

Again, $y = x^2 - 6x + 6$

$y + 3 = (x - 3)^2$

$y = -3 + (x - 3)^2$

Since $(x - 3)^2 \geq 0$ so for all $x \in$ R, $y \geq -3$

$\therefore$ range of $f$ = R$(f)$ = $[-3, \infty)$

b) $y = \dfrac{1}{x - 1}$

The given function is defined for all values of $x$ except at $x = 1$

$\therefore$ domain of the function = D$(f)$ = R $- \{1\}$

Again, $y = \dfrac{1}{x - 1}$

$\Rightarrow \quad x = \dfrac{1}{y} + 1$

$\Rightarrow \quad x = \dfrac{1 + y}{y}$

$y \neq 0$ for all $x \in$ D$(f)$

$\therefore$ range of $f$ = R$(f)$ = R $- \{0\}$

c) The function will not be defined when $x - 2 = 0$ i.e. $x = 2$.

So, for all values of $x$ except at $x = 2$, $y = f(x)$ exists.

$\therefore$ domain of the function $= D(f) = R - \{2\}$

If $x \neq 2$, then $y = \frac{x^2 - 4}{x - 2} = x + 2$

Since $x = 2$ is not in the domain of the function, so $y = 4$ will not be in the range of the function.

$\therefore$ range of $f = R - \{4\}$

d) $f(x) = \sqrt{6 - x - x^2} = \sqrt{\frac{25}{4} - \left(\frac{1}{4} + x + x^2\right)}$

$$= \sqrt{\left(\frac{5}{2}\right)^2 - \left(x + \frac{1}{2}\right)^2}$$

For $\left(\frac{5}{2}\right)^2 - \left(x + \frac{1}{2}\right)^2 < 0$, y will be imaginary. So, y will be defined only for $\left(\frac{5}{2}\right)^2 - \left(x + \frac{1}{2}\right)^2 \geq 0$

$\Rightarrow \left(x + \frac{1}{2}\right)^2 \leq \left(\frac{5}{2}\right)^2$

$\Rightarrow -\frac{5}{2} \leq x + \frac{1}{2} \leq \frac{5}{2}$

$\Rightarrow -3 \leq x \leq 2$

$\therefore$ the domain of the function $= [-3, 2]$

Again, $y^2 = \left(\frac{5}{2}\right)^2 - \left(x + \frac{1}{2}\right)^2$

$\Rightarrow \left(x + \frac{1}{2}\right)^2 = \left(\frac{5}{2}\right)^2 - y^2$

Since $\left(x + \frac{1}{2}\right)^2 \geq 0$ for all $x \in R$

so, $\left(\frac{5}{2}\right)^2 - y^2 \geq 0$

$\Rightarrow y^2 \leq \left(\frac{5}{2}\right)^2$

Since y is a positive square root,

so, $0 \leq y \leq \frac{5}{2}$

$\therefore$ the range of $f = [0, 5/2]$

**Alternative method:**

If $6 - x - x^2 < 0$, $y$ will be imaginary. So, $y$ will be defined only when

$$6 - x - x^2 \geq 0$$

$$\Rightarrow \quad (x + 3)(2 - x) \geq 0$$

The corresponding equation is

$$(x + 3)(2 - x) = 0$$

$$\therefore \quad x = -3, 2$$

Thus, we may have the following three intervals $(-\infty, -3]$, $[-3, 2]$ and $[2, \infty)$

| Interval | Value of | | |
|---|---|---|---|
| | $x + 3$ | $2 - x$ | $(x + 3)(2 - x)$ |
| $(-\infty, -3]$ | $\leq 0$ | $> 0$ | $\leq 0$ |
| $[-3, 2]$ | $\geq 0$ | $\geq 0$ | $\geq 0$ |
| $[2, \infty)$ | $> 0$ | $\leq 0$ | $\leq 0$ |

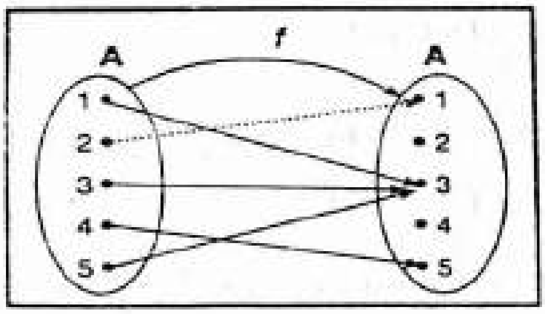
From the above table, the required interval is $[-3, 2]$

$\therefore$ domain of the function = $[-3, 2]$

The range can be obtained as in the above method.

## EXERCISE 2.3

1. Let $f : A \to A$ be defined by the arrow diagram




Find a) $f^{-1}(1)$ b) $f^{-1}(3)$ c) $f^{-1}(5)$
d) $f^{-1}(2)$ e) $f^{-1}(1, 3, 5)$ f) $f^{-1}(2, 4)$.

2. a) If A = {0, 1, 2, 3,}, B = {10, 13, 16, 19} and $f : A \to B$ is a function such that $f(0) = 10$, $f(1) = 13$, $f(2) = 16$, $f(3) = 19$, write down $f^{-1} : B \to A$ as a set of ordered pairs.

b) If $X = \{-1, 1, 2, 4\}$, $Y = \left\{\frac{1}{5}, \frac{2}{5}, \frac{1}{2}, 1\right\}$ and $f : X \to Y$ is a function such that $f(-1) = \frac{1}{5}, f(1) = 1, f(2) = \frac{1}{2}, f(4) = \frac{2}{5}$, write down $f^{-1} : Y \to X$ as a set of ordered pairs.

3. Let a function $f : R \to R$ be defined by
   a) $f(x) = x + 1, x \in R$
   b) $f(x) = 2x + 3, x \in R$
   c) $f(x) = 2x + 5, x \in R$
   d) $f(x) = x^3 + 5, x \in R$
   e) $f(x) = 3x - 2, x \in R$

   Find a formula that defines the inverse function $f^{-1}$.

4. Let $f : R \to R$ and $g : R \to R$ be defined by $f(x) = 4x^3 - 1$ and $g(x) = 3x^3$. Find
   a) $(2f - 5)x$
   b) $(f^2)(x)$
   c) $(3f - 4g)(x)$
   d) $(fg)(x)$
   e) $(f/g)(x)$ $(g(x) \neq 0)$.

5. Let $f : R \to R$ and $g : R \to R$ be defined by
   a) $f(x) = 2x + 1$ and $g(x) = 3x - 1$
   b) $f(x) = 3x^2 - 4$ and $g(x) = 2x - 5$
   c) $f(x) = x^3 - 1$ and $g(x) = x^2$
   d) $f(x) = x^2 + 1$ and $g(x) = x^5$

   Find $(gof)(x)$ and $(fog)(x)$.

6. Let $f : R \to R$ and $g : R \to R$ be defined by
   a) $f = \{(1, 5), (2, 6), (3, 7), (4, 6)\}$ and
      $g = \{(5, 1), (6, 2), (7, 3)\}$
      find $gof$ and $fog$.
   b) $f = \{(1, 2), (3, 5), (4, 1)\}$ and
      $g = \{(2, 3), (5, 1), (1, 3)\}$,
      find $gof$ and $fog$.

7. a) If $f : R \to R$ be defined by $f(x) = \frac{1}{1 - x}$, $(x \neq 1)$, show that

$$(fof)\left(\frac{1}{2}\right) = -1$$

   b) If $f : R \to R$ be defined by $f(x) = \frac{x - 1}{x + 1}$, $(x \neq -1)$ show that

$$(fof)(4) = -\frac{1}{4}.$$

8. Find the domain and the range of the following functions defined in the set of real numbers (i.e. $f : R \to R$)
   a) $y = 3x + 1$
   b) $y = x^2 - 1$
   c) $y = x^3$
   d) $y = -x^2 + 4x - 3$

y = √x-२



Relations, Functions and Graphs

e) $y = \sqrt{x - 2}$ f) $y = \dfrac{1}{x + 1}$

g) $y = \dfrac{x^2 - 16}{x - 4}$ h) $y = \sqrt{x^2 - 2x - 8}$

i) $y = \sqrt{21 - 4x - x^2}$ j) $y = \dfrac{|x - 1|}{x - 1}$

9. Show that $f : R \to R$ defined by $f(x) = cx + d$, where c ($\neq 0$) and d are real numbers, is one to one and onto. Find $f^{-1}$. Also show that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

10. Let $f : R - \{2\} \to R - \{3\}$ be defined by $f(x) = \dfrac{3x}{x - 2}$. Show that $f$ is bijective. Also, find $f^{-1}$.

***Answers***

1. a) {2} b) {1, 3, 5} c) {4} d) $\phi$ e) {1, 2, 3, 4, 5} f) $\phi$
2. a) {(10, 0), (13, 1), (16, 2), (19, 3)}
   b) $\left\{\left(\frac{1}{5}, -1\right), (1, 1), \left(\frac{1}{2}, 2\right), \left(\frac{2}{5}, 4\right)\right\}$
3. a) $f^{-1}(x) = x - 1$ b) $f^{-1}(x) = \dfrac{(x - 3)}{2}$ c) $f^{-1}(x) = \dfrac{(x - 5)}{2}$
   d) $f^{-1}(x) = (x - 5)^{1/3}$ e) $f^{-1}(x) = \dfrac{(x + 2)}{3}$
4. a) $8x^3 - 7$ b) $(4x^3 - 1)^2$ c) $-3$ d) $12x^6 - 3x^3$ e) $\dfrac{4x^3 - 1}{3x^3}$
5. a) $6x + 2$, $6x - 1$ b) $6x^2 - 13$, $12x^2 - 60x + 71$
   c) $(x^3 - 1)^2$, $x^6 - 1$ d) $(x^2 + 1)^5$, $x^{10} + 1$
6. a) $gof$ = {(1, 1), (2, 2), (3, 3), (4, 2)}; $fog$ = {(5, 5), (6, 6), (7, 7)}
   b) $gof$ = {(1, 3), (3, 1), (4, 3)}, $fog$ = {(2, 5), (5, 2), (1, 5)}
8. a) $D(f) = R = (-\infty, \infty)$, $R(f) = (-\infty, \infty) = R$
   b) $D(f) = R = (-\infty, \infty)$, $R(f) = [-1, \infty)$
   c) $D(f) = R = (-\infty, \infty)$, $R(f) = R = (-\infty, \infty)$
   d) $D(f) = R = (-\infty, \infty)$, $R(f) = (-\infty, 1]$ e) $D(f) = [2, \infty)$, $R(f) = [0, \infty)$
   f) $D(f) = R - \{-1\}$, $R(f) = R - \{0\}$ g) $D(f) = R - \{4\}$, $R(f) = R - \{8\}$
   h) $D(f) = (-\infty, -2] \cup [4, \infty)$, $R(f) = [0, \infty)$
   i) $D(f) = [-7, 3]$, $R(f) = [0, 5]$ j) $D(f) = R - \{1\}$, $R(f) = \{-1, 1\}$
9. $\dfrac{x - d}{c}$ 10. $\dfrac{2x}{x - 3}$

## 2.8 Some Simple Algebraic Functions and their graphs

The definition of a function, as we have seen, is very general. In both theoretical and practical investigations, we often have to *specialize* it according to the situation. In other words, we shall quite often have to consider *special functions*. The identity function, the constant function, the linear function, the quadratic function and the cubic function can be taken as the simplest of the special functions that can be categorized as **algebraic functions.**

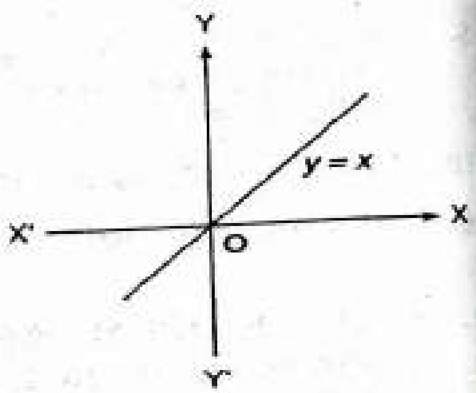
### a) The Identity Function:

Let A be any set. The function $f : A \to A$ defined by

$$y = f(x) = x \text{ for } x \in A$$

is called the **identity function**. It is usually denoted by $I_A$.

If A = R, the set of real numbers, the graph of $I_R$ is as given aside:



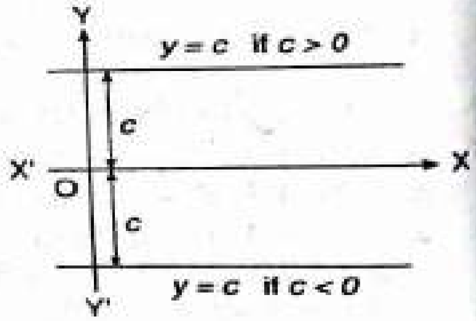

### b) The Constant Function:

Let A be any set and $B = \{c\}$. Then, the function $f : A \to B$ defined by

$$y = f(x) = c \quad \text{for } x \in A$$

is called the **constant function.** In other words, a function is said to be a constant function if all its functional values are the same (i.e., if the range of the function is a singleton set).

If A = R, the set of real numbers and $c$ is a real number, the graph of $y = f(x) = c$ is a straight line parallel to the x-axis at a distance of $c$ units from the x-axis:




### c) The Linear Function:

Let A and B be any two sets. Then, a function $f : A \to B$ defined by

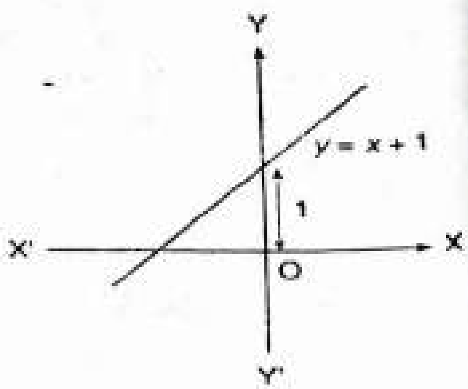
$$y = f(x) = mx + c \quad \text{for } x \in A,$$

where $m$ and $c$ are constants, is called a **linear function.**

If A = B = R, the set of real numbers, the function defined by

$$y = f(x) = x + 1$$

is a linear function. Its graph is a straight as shown in figure given aside:



**d) The Quadratic Function:**

Let A and B be any two sets. Then, a function $f: A \to B$ defined by

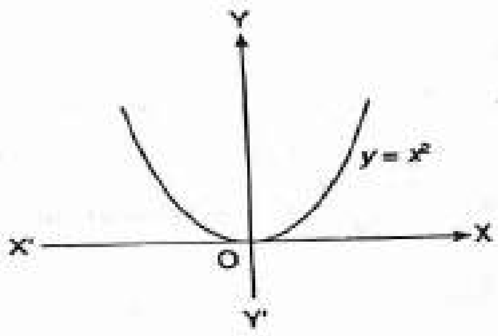
$$y = f(x) = ax^2 + bx + c \quad \text{for } x \in A,$$

where $a$, $b$ and $c$ are constants, is called a **quadratic function.**

If A = B = R, the set of real numbers, the function defined by

$$y = f(x) = x^2$$

is a quadratic function. Its graph is shown as follows:




**e) The Cubic Function:**

Let A and B be any two sets. Then, a function $f: A \to B$ defined by

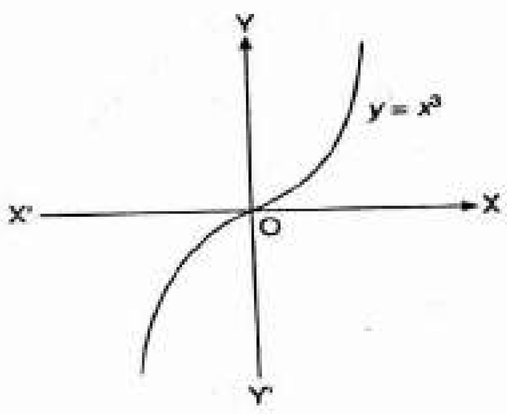
$$y = f(x) = ax^3 + bx^2 + cx + d \quad \text{for } x \in A,$$

where $a$, $b$, $c$ and $d$ are constants, is called a **cubic function.**

If A = B = R, the set of real numbers, the function defined by

$$y = f(x) = x^3$$

is a cubic function. Its graph is shown below:



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### f) The Polynomial Function:

Let A and B be any two sets. Then, a function $f : A \rightarrow B$ defined by $y = f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x^1 + a_0$ for $x \in A$, where $a_n, a_{n-1}, \ldots, a_2, a_1$ and $a_0$ are constants, is called a **polynomial function.**

The constant function, the linear function, the quadratic function and the cubic function described above can be obtained as special cases of the polynomial function by putting $n = 0, 1, 2$ and $3$ respectively.

### Rational Function

A function $f$ defined by $f(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are the polynomials in $x$ and $q(x) \neq 0$ is known as a rational function.

For example: $f(x) = \dfrac{3x^2 - 4x}{x^3 - 3x + 6}$ is a rational function.

### Absolute Value Function

A function $f(x)$ defined by $f(x) = |x|$ where

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

is known as the absolute value function. The domain of the function is the set of real numbers R and the range is the set of non-negative real numbers. That is, domain of $f = D(f) = R$ and the range of $f = R(f) = [0, \infty)$.

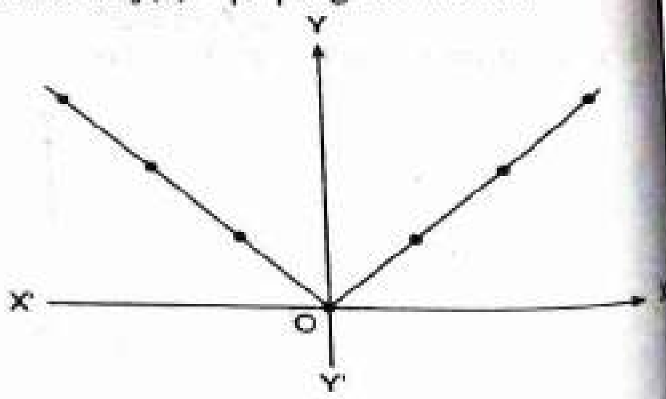
The graph of the absolute value function $f(x) = |x|$ is given below:

When $x \geq 0$, $y = f(x) = x$

| x | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| y | 0 | 1 | 2 | 3 |

Again, when $x < 0$, $y = f(x) = -x$

| x | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| y | 0 | −1 | −2 | −3 |




### Greatest Integer Function

A function $f$ defined by $f(x) = [x]$ where $n \leq x \leq n + 1$, $n$ is the greatest integer, is known as the greatest integer function. The domain of the function is the set of real numbers (R) and the range is the set of integers (Z).

For example: $[2] = 2$ $\quad [2.6] = 2$

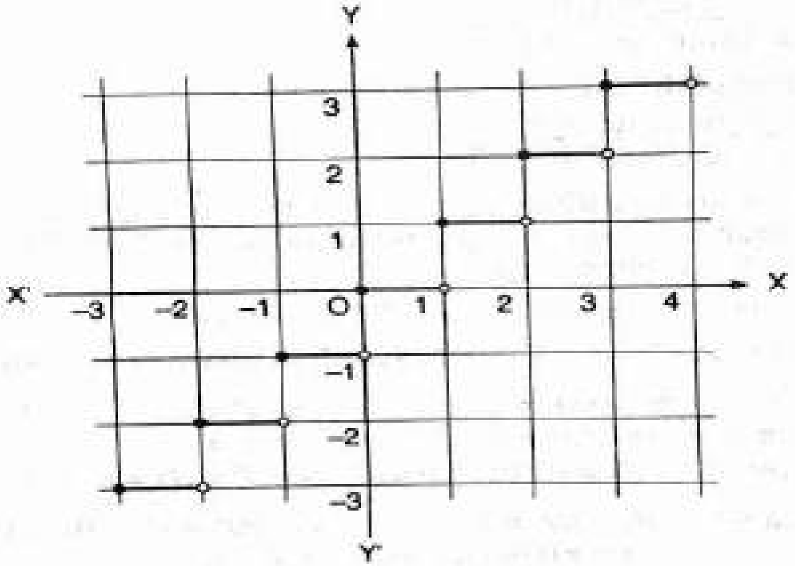
$[0] = 0$, $[-1.2] = -2$ $\quad [-5.6] = -6$

Let us see the graph of the greatest integer function

$y = f(x) = [\,x\,]$

| $x$ | $-3 \le x < -2$ | $-2 \le x < -1$ | $-1 \le x < 0$ | $0 \le x < 1$ | $1 \le x < 2$ | $2 \le x < 3$ | $3 \le x < 4$ |
|---|---|---|---|---|---|---|---|
| $y$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |

The graph of the greatest integer function is given below:




## 2.9 Elementary Transcendental Functions

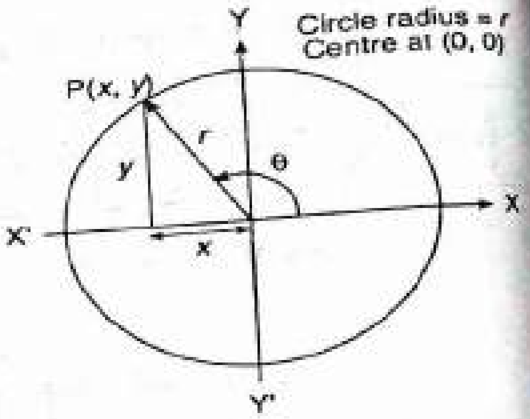
Some functions of common interest are the trigonometric functions, exponential function, logarithmic functions and the hyperbolic functions. They are not algebraic. They are given the special name: **elementary transcendental functions.**

### a) Trigonometric Functions:

An angle θ is placed in the standard position together with a circle of radius $r$ and centre at the origin O. Suppose the terminal arm of the angle θ cuts the circle at the point P($x$, $y$). Then the six **trigonometric functions** (also called **ratios**) are defined by the formulae given below:

sine of the angle θ or sin θ $= \dfrac{\text{y-coordinate of P}}{\text{length of OP}} = \dfrac{y}{r}$

cosine of the angle θ or cos θ $= \dfrac{\text{x-coordinate of P}}{\text{length of OP}} = \dfrac{x}{r}$




tangent of the angle θ or tan θ

$$= \frac{\text{y-coordinate of P}}{\text{x-coordinate of P}} = \frac{y}{x}$$

cosecant of the angle θ or cosec θ

$$= \frac{\text{length of OP}}{\text{y-coordinate of P}} = \frac{r}{y}$$

secant of the angle θ or sec θ

$$= \frac{\text{length of OP}}{\text{x-coordinate of P}} = \frac{r}{x}$$

cotangent of the angle θ or cot θ

$$= \frac{\text{x-coordinate of P}}{\text{y-coordinate of P}} = \frac{x}{y}$$

From the above definition, it follows that the angles having the same initial and terminal arms (i.e., **co-terminal angles**) have the same sine, cosine, tangent, etc. For examples

$\sin 60° = \sin(-300°)$, $\cos 120° = \cos(-240°)$.

The definitions of trigonometric functions given above readily yield:

i) $\sin 0° = y/r = 0/r = 0$, $\cos 0° = x/r = r/r = 1$,
$\tan 0° = y/x = 0/r = 0$, $\cot 0° = x/y = r/0 =$ (undefined),
$\text{cosec } 0° = r/y = r/0$ (undefined), $\sec 0° = r/x = r/r = 1$;

ii) $\sin 90° = y/r = r/r = 1$, $\cos 90° = x/r = 0/r = 0$,
$\tan 90° = y/x = r/0$ (undefined) and so on;

iii) $\sin 180° = y/r = 0/r = 0$, $\cos 180° = x/r = -r/r = -1$,
$\tan 180° = y/x = 0/(-r) = 0$, and so on.

Values of standard angles such as 30°, 45° and 60° can be easily calculated by referring to the following figures in each of which the x- and/or y- coordinates is calculated by using the Pythagorean theorem;

" *The square on the hypotenuse of a right angled triangle is equal to the sum of the squares on the other two sides.*"

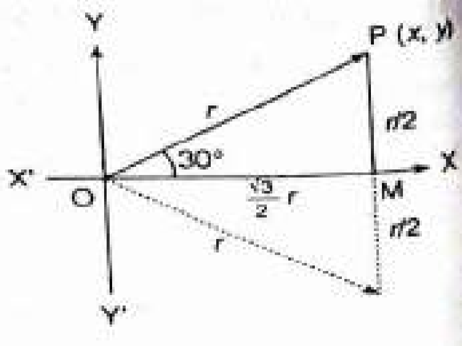
a) *Trigonometric ratios of 30°*

$$\sin 30° = \frac{MP}{OP} = \frac{r/2}{r} = \frac{1}{2}$$

$$\cos 30° = \frac{OM}{OP} = \frac{\sqrt{3}r/2}{r} = \frac{\sqrt{3}}{2}.$$

$$\tan 30° = \frac{MP}{OM} = \frac{r/2}{\sqrt{3}r/2} = \frac{1}{\sqrt{3}}$$

and so on.



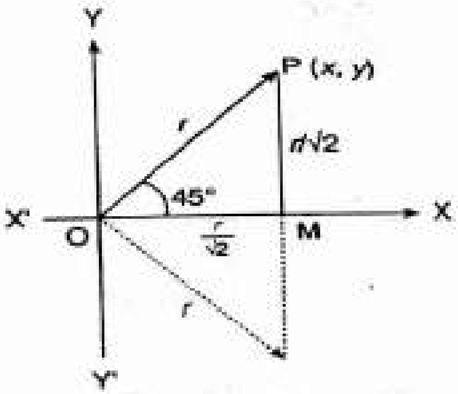
b) *Trigonometric ratios of* 45°

$$\sin 45° = \frac{MP}{OP} = \frac{r/\sqrt{2}}{r} = \frac{1}{\sqrt{2}}$$

$$\cos 45° = \frac{OM}{OP} = \frac{r/\sqrt{2}}{r} = \frac{1}{\sqrt{2}}$$

$$\tan 45° = \frac{MP}{OM} = \frac{r/\sqrt{2}}{r/\sqrt{2}} = 1$$

and so on.



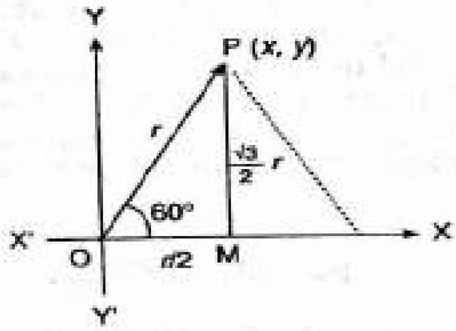

c) *Trigonometric ratios of* 60°

$$\sin 60° = \frac{MP}{OP} = \frac{\sqrt{3}r/2}{r} = \frac{\sqrt{3}}{2}$$

$$\cos 60° = \frac{OM}{OP} = \frac{r/2}{r} = \frac{1}{2}$$

$$\tan 60° = \frac{MP}{OM} = \frac{\sqrt{3}r/2}{r/2} = \sqrt{3}$$

and so on.




The following table lists the value of each trigonometric function for some of the most frequently encountered angles:

| θ | 0° | 30° | 45° | 60° | 90° | 120° | 135° | 150° | 180° | 210° | 225° | 240° | 270° | 300° | 315° | 330° | 360° |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| sin θ | 0 | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{3}}{2}$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | 0 | $\frac{-1}{2}$ | $\frac{-1}{\sqrt{2}}$ | $-\frac{\sqrt{3}}{2}$ | −1 | $-\frac{\sqrt{3}}{2}$ | $-\frac{1}{\sqrt{2}}$ | $-\frac{1}{2}$ | 0 |
| cos θ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | 0 | $\frac{-1}{2}$ | $\frac{-1}{\sqrt{2}}$ | $\frac{-\sqrt{3}}{2}$ | −1 | $-\frac{\sqrt{3}}{2}$ | $-\frac{1}{\sqrt{2}}$ | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{3}}{2}$ | 1 |
| tan θ | 0 | $\frac{1}{\sqrt{3}}$ | 1 | $\sqrt{3}$ | $\infty$ | $-\sqrt{3}$ | −1 | $-\frac{1}{\sqrt{3}}$ | 0 | $\frac{1}{\sqrt{3}}$ | 1 | $\sqrt{3}$ | $-\infty$ | $-\sqrt{3}$ | −1 | $-\frac{1}{\sqrt{3}}$ | 0 |

The graphs of some of the trigonometric functions are shown in the following figures:

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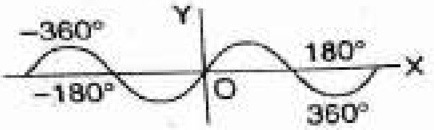
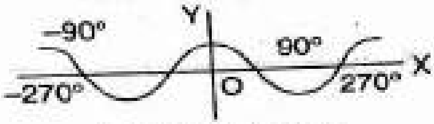
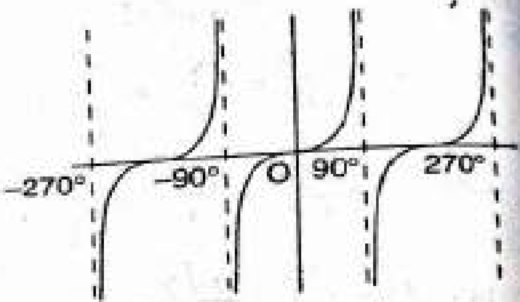



The sine graph




The cosine graph




The tangent graph

## b) Exponential Function:

A number $a$ when multiplied by itself gives a unique number denoted by $a.a$ or $a^2$ (read " *a square* or *a raised to the power* 2"). Here the number 2 is called the **exponent** or **index**, $a$ the **base** and $a^2$ the **second power** of $a$. Repeated multiplication of $a$ by itself $n$ times yields the $n^{th}$ power of $a$, denoted by $a^n$. Here the exponent $n$ of $a$ is a positive integer. If $n$ is a positive integer, the *negative* $n^{th}$ power or $a^{-n}$ is defined by

$$a^{-n} = 1/a^n,\ (a \neq 0).$$

In particular, $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$.

If $x$ is any rational number of the form $p/q$ $(q \neq 0)$, $a^x$ or $a^{p/q}$ is interpreted as the $q^{th}$ root of the $p^{th}$ power of $a$., i.e.,

$$a^{p/q} = (a^p)^{1/q} \qquad (a \neq 0,\ q \neq 0)$$

In particular, for $a = 4$, $q = 2$ and $p = 3$, we have

$$4^{3/2} = (4^3)^{1/2} = (64)^{1/2} = (8^2)^{1/2} = 8$$

This is how we give meaning to expressions with rational exponent or index.

In general, if $a \neq 0$ and $x$ is any rational number, the power $a^x$ (or the $x^{th}$ power of $a$) is a real number and satisfies the following laws of exponents or indices:

i) $a^m.a^n = a^{m+n}$

ii) $(a^m)^n = a^{mn}$, $m$ and $n$ being rational numbers.

A natural question is whether the exponent can be extended to any number. The answer is affirmative. But, unfortunately, we cannot prove at this stage that $a^x$ represents a real number for any real exponent $x$ and $a \neq 0$. We shall therefore assume that $a^x$ is a real number for every real $x$ and $a \neq 0$. We further assume that the laws of indices also hold in such cases. With these assumptions in mind, we are now in a position to define the exponential function.

For every real number $a > 0$, the **exponential function** $f$ with base $a$, is defined by the formula

$$y = f(x) = a^x, x \in \mathbb{R}.$$

We may rewrite this definition in the following standard form:

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $y = f(x) = a^x$, $x \in \mathbb{R}$ for a given real number $a > 0$. Then $f$ is called an **exponential function** with base $a$.

Three typical examples of exponential functions are

$$2^x, \quad 1/2^x \quad \text{and} \quad e^x,$$

where $e$ is an irrational number lying between 2 and 3. Its value to ten places of decimal is

$$e = 2.7182818284\ldots$$

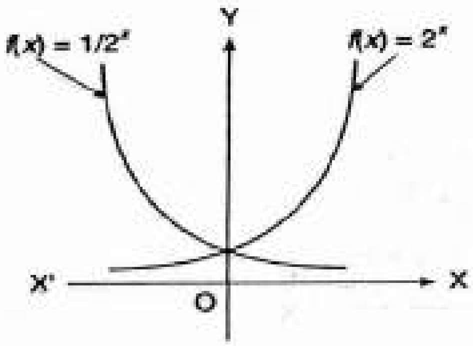
The number $e$ is defined as the value of

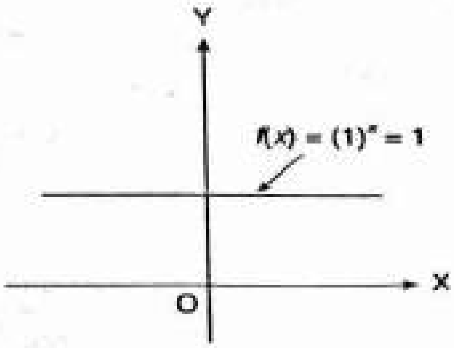
$$\left(1 + \frac{1}{n}\right)^n$$

as $n$ increases indefinitely (or as $n$ becomes larger than any number N however large it may be). In symbols,

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

The graphs of $y = f(x) = 2^x$ and $y = f(x) = 1/2^x$ are given below:




For the typical case $a = 1$, the graph of $y = f(x) = (1)^x = 1$ is a horizontal straight line at a distance of 1 unit from the x-axis.

The graphs of $y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$ can be obtained by plotting the points with the values of $x$ as the $x$-coordinates and the corresponding values of $y$ as the $y$-coordinates.

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## c) The Logarithmic Function:

The logarithmic function may be defined in several ways. We define here as an inverse function. It can be proved that the inverse of an exponential function exists (How?). The inverse of an exponential function is known as a **logarithmic function**.

More precisely, if $y = f(x) = a^x$ defines an exponential function for a given real number $a > 0$, then its inverse, known as the **logarithmic function**, is defined by

$$x = a^y \text{ or } a^{f(x)}.$$

We then say that $y$ or $f(x)$ *is* the **logarithm** of $x$ to the **base** $a$, and is denoted by

$$\log_a x.$$

Thus, the exponential equation

$$x = a^y \text{ or } a^{f(x)}$$

carries the same meaning as the logarithmic equation

$$y = \log_a x.$$

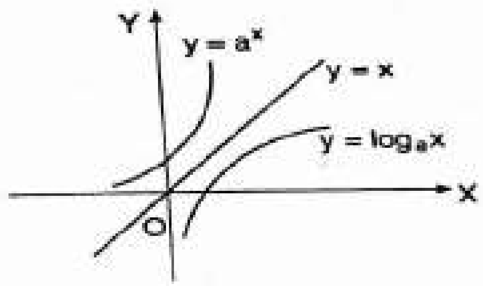
Logarithms are used for speeding up computation when the uses of electronic calculators or computers were not so common as today. Logarithm to the base 10 is known as **common logarithm**; and that to the base $e$ is known as **natural logarithm**. If the base of a logarithm is $e$, it is generally omitted. We often write

$$\boxed{\log_e x = \log x = \ln x}$$

The graphs of

$$y = a^x \text{ and } y = \log_a x$$

are as follows:




Note that one is the reflection of the other on the line $y = x$. An important particular case is

$$\log_a a = 1, \text{ since } a^1 = a.$$

The graphs of $\log_{10} x$ and $\log_e x$ can be obtained by plotting the points with the values of $x$ as the $x$-coordinates and the corresponding values of $y$ as the $y$-coordinates.

Logarithms possess certain properties that are fundamental in many branches of mathematics. Some of them are stated and proved below:

**Theorem 1.**

For any positive numbers $x$, $y$ and $a$,

$$\log_a(xy) = \log_a x + \log_a y.$$

**Proof.** For any positive numbers $x$, $y$ and $a$, put

$$\log_a x = b \text{ and } \log_a y = c.$$

Then by definition $x = a^b$ and $y = a^c$.

So $\quad xy = a^b.a^c = a^{b+c}$.

Hence, $\log_a (xy) = b + c = \log_a x + \log_a y$.

**Theorem 2.**

For any positive number $a$ and $x$,

$$\log_a x^p = p \log_a x ,$$

where $p$ is any real number.

**Proof.** For any positive numbers $a$ and $x$, put $\log_a x = b$.

Then, $x = a^b$ and so $x^p = (a^b)^p = a^{bp}$.

Hence, $\log_a x^p = bp = pb = p \log_a x$.

**Theorem 3.**

For any positive numbers $x$, $y$ and $a$,

$$\log_a(x/y) = \log_a x - \log_a y.$$

**Proof.** For any positive numbers $x$, $y$ and $a$, put

$$\log_a x = b \text{ and } \log_a y = c.$$

Then by definition $x = a^b$ and $y = a^c$.

So $\quad x/y = a^b/a^c = a^{b-c}$.

Hence, $\log_a (x/y) = b - c = \log_a x - \log_a y$.

**Theorem 4.**

For any positive numbers $a$, $b$ and $x$.

$$\log_a x = \log_a b. \log_b x.$$

**Proof.** For any positive numbers $a$, $b$ and $x$, put

$$\log_b x = m.$$

Then by definition $x = b^m$,

Hence, $\log_a x = \log_a b^m = m \log_a b = \log_a b. \log_b x.$

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### d) Hyperbolic Functions:

With the help of an exponential function, a new function is defined which is known as the hyperbolic function. They are so named as the functions are closely related to the conic section i.e. hyperbola.

Certain combinations of exponential functions $e^x$ and $e^{-x}$ play very important roles in pure and applied mathematics. We consider the following six combinations:

a) $\dfrac{e^x - e^{-x}}{2}$ b) $\dfrac{e^x + e^{-x}}{2}$

c) $\dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ d) $\dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

e) $\dfrac{2}{e^x - e^{-x}}$ f) $\dfrac{2}{e^x + e^{-x}}$.

These functions have properties somewhat similar to those of trigonometric functions. Trigonometric functions are related to circle and hence they are also known as **circular functions**. In the same way, the above six combinations are related to a hyperbola; and hence the name hyperbolic functions. They are defined as follows:

a) Hyperbolic sine of $x$ : $\sinh x = \dfrac{e^x - e^{-x}}{2}$

b) Hyperbolic cosine of $x$ : $\cosh x = \dfrac{e^x + e^{-x}}{2}$

c) Hyperbolic tangent of $x$ : $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

d) Hyperbolic cotangent of $x$ : $\coth x = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

e) Hyperbolic cosecant of $x$ : $\text{cosech } x = \dfrac{2}{e^x - e^{-x}}$

f) Hyperbolic secant of $x$ : $\text{sech } x = \dfrac{2}{e^x + e^{-x}}$.

We list below some of the basic identities and formulae related to these functions:

a) $\cosh^2 x - \sinh^2 x = 1$
b) $\tanh^2 x + \text{sech}^2 x = 1$
c) $\coth^2 x - \text{cosech}^2 x = 1$
d) $\sinh(-x) = -\sinh x$
e) $\cosh(-x) = \cosh x$
f) $\sinh 2x = 2 \sinh x \cosh x$
g) $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$
h) $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$
i) $\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$

As an illustration of the technique of proving the above results, we consider the first one. From definition, we have

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \left(\frac{e^x + e^{-x} + e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x} - e^x + e^{-x}}{2}\right)$$

$$= \left(\frac{2e^x}{2}\right)\left(\frac{2e^{-x}}{2}\right) = e^x.e^{-x} = e^0 = 1$$

## Worked out examples

***Example 1***

a) Prove that : $\cosh 2x = 2\cosh^2 x - 1$

b) Prove that :

$$\tanh 2x = \frac{2\tanh x}{1 + \tanh^2 x}.$$

***Solution :***

a) $$\cosh 2x = \frac{e^{2x} + e^{-2x}}{2}$$

$$= \frac{e^{2x} + 2 + e^{-2x} - 2}{2}$$

$$= \frac{(e^x + e^{-x})^2 - 2}{2}$$

$$= 2\left(\frac{e^x + e^{-x}}{2}\right)^2 - 1$$

$$= 2\cosh^2 x - 1$$

b) Here $$\tanh 2x = \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} = \frac{(e^x + e^{-x})(e^x - e^{-x})}{\frac{(e^x - e^{-x})^2 + (e^x + e^{-x})^2}{2}}$$

$e^x + e^{-x}$

$$= \frac{\frac{2(e^x - e^{-x})}{(e^x + e^{-x})}}{1 + \frac{(e^x - e^{-x})^2}{(e^x + e^{-x})^2}}$$

$$= \frac{2\tanh x}{1 + \tanh^2 x}.$$

***Example 2***

Prove that $\log_a \sqrt{a^3\sqrt{a^2}} = 2$

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**Solution :**

$$\log_a\sqrt{a^3\sqrt{a^2}} = \log_a\sqrt{a^3.a}$$
$$= \log_a\sqrt{a^4}$$
$$= \log_a a^2$$
$$= 2\log_a a$$
$$= 2\times 1 = 2$$

***Example 3***

Prove that

a) $\log_a(x^3y^2z) = 3\log_a x + 2\log_a y + \log_a z$

b) $\log a^2/bc + \log b^2/ca + \log c^2/ab = 0$

c) $\log\left(\frac{a+b}{3}\right) = \frac{1}{2}(\log a + \log b)$, if $a^2 + b^2 = 7ab$.

**Solutions:**

a) $$\log_a(x^3y^2z) = \log_a x^3y^2 + \log_a z$$
$$= \log_a x^3 + \log_a y^2 + \log_a z$$
$$= 3\log_a x + 2\log_a y + \log_a z$$

b) Here, $\log a^2/bc + \log b^2/ca + \log c^2/ab$
$$= \log(a^2/bc)(b^2/ca)(c^2/ab)$$
$$= \log\frac{a^2b^2.c^2}{bc.ca.ab}$$
$$= \log 1 = 0\ (a^0 = 1)$$

c) Here, $a^2 + b^2 = 7ab$ or $a^2 + b^2 + 2ab = 9ab$.

So, $(a+b)^2/3^2 = ab$

Taking log of both sides, we get

$$\log[(a+b)^2/3^2] = \log ab.$$

or, $2\log(a+b)/3 = \log a + \log b$

or, $\log\frac{a+b}{3} = \frac{1}{2}(\log a + \log b)$

$(a+b)^2 = ab$

$(a+b)^2/3^2$

***Example 4***

If $f(x) = \log\frac{1+x}{1-x}$ $(-1 < x < 1)$, show that $f(a) + f(b) = f\left(\frac{a+b}{1+ab}\right)$

$(|a| < 1, |b| < 1)$

***Solution :***

Since, $f(x) = \log \frac{1+x}{1-x}$

so $f(a) = \log \frac{1+a}{1-a}$ and $f(b) = \log \frac{1+b}{1-b}$

Now, $f(a) + f(b) = \log \frac{1+a}{1-a} + \log \frac{1+b}{1-b}$

$$= \log \left(\frac{1+a}{1-a} \cdot \frac{1+b}{1-b}\right)$$

$$= \log \frac{1+a+b+ab}{1-a-b+ab}$$

$$= \log \frac{(1+ab)+(a+b)}{(1+ab)-(a+b)}$$

$$= \log \left(\frac{1+\frac{a+b}{1+ab}}{1-\frac{a+b}{1+ab}}\right)$$

$$= f\left(\frac{a+b}{1+ab}\right)$$

***Example 5***

If $x = \log_a bc$, $y = \log_b ca$, $z = \log_c ab$, prove that

$$\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = 1$$

***Solution :***

$x + 1 = \log_a bc + \log_a a = \log_a abc$

$$= \frac{1}{\log_{abc} a} \quad (\because \log_b a \,.\, \log_a b = 1)$$

Similarly, $y + 1 = \frac{1}{\log_{abc} b}$ and $z + 1 = \frac{1}{\log_{abc} c}$

Now, $\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1}$

$= \log_{abc} a + \log_{abc} b + \log_{abc} c$

$= \log_{abc} abc$

$= 1$

## EXERCISE 2.4

1. Prove that
   a) $\log_a (xy^3/z^2) = \log_a x + 3 \log_a y - 2 \log_a z$
   b) $\log_a (2x + 3(\log_a x - \log_a y) = \log_a (2x^4 / y^3)$
   c) $\log_a x^2 - 2 \log_a \sqrt{x} = \log_a x$
   d) $a^{\log_a x} = x$
   e) $\log_a a^x = x$
   f) $(\log a)^2 - (\log b)^2 = \log (ab). \log (a/b)$
   g) $\log(1 + 2 + 3) = \log 1 + \log 2 + \log 3$
   h) $x^{\log y - \log z} . y^{\log z - \log x} . z^{\log x - \log y} = 1$
   i) $(yz)^{\log y - \log z} . (zx)^{\log z - \log x} . (xy)^{\log x - \log y} = 1$
   j) $\log_a \sqrt{a\sqrt{a\sqrt{a^2}}} = 1$
2. If $f(x) = \log \frac{1-x}{1+x}$ $(-1 < x < 1)$, show that $f\left(\frac{2ab}{1 + a^2b^2}\right) = 2f(ab)$ where $|ab| < 1$.
3. If $x = \log_{2a} a$, $y = \log_{3a} 2a$ and $z = \log_{4a} 3a$, prove that
   $$xyz + 1 = 2yz$$
4. If $\frac{\log x}{y - z} = \frac{\log y}{z - x} = \frac{\log z}{x - y}$, prove that $x^x y^y z^z = 1$.

## ADDITIONAL QUESTIONS

1. Let A = $\{a, b, c\}$ and B = $\{x, y, z\}$. Find A × B. Determine (i) a relation from A to B which is not a function (ii) relations from A to B which are functions.
2. A function $f$ is defined on the set of integers as follows :
   $$f(x) = \begin{cases} 1 + x : & 1 \le x < 2 \\ 2x - 1 : & 2 \le x < 4 \\ 3x - 10 : & 4 \le x < 6 \end{cases}$$
   a) Find the domain of $f$
   b) Find the range of $f$
   c) State whether $f$ is a one-one or not.
3. What is the fundamental difference between a relation and a function ? Is $f = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$ a function ? If it is defined by $f(x) = ax + b$, what values should be assigned to $a$ and $b$ ?

4. Let A = $\{a, b, c\}$ and B = $\{m, n, p\}$. Define a function $f : A \to B$ such that it is
   (i) One-one and Onto
   (ii) into function
   (iii) Onto function only.

   Show these functions by diagram. Find the domain and the range of each.

5. What do you mean by a function ? Distinguish between the range and the domain of a function. Let A = $\{-1, -2, 0, 1, 2\}$ and a function $f : A \to B$ be defined by $f(x) = \frac{x^2}{2}$ . Find the range of f. Is the function one-one ? (T.U. 2049)

6. Let a function $f : A \to B$ be defined by $f(x) = \frac{x-1}{x+2}$ with A = $\{-1, 0, 1, 2, 3, 4\}$ and B = $\{-2, 1, -1/2, 0, 1/2, 1/4, 2/5\}$. Find the range of $f$. Is the function $f$ one-one and onto both ? If not, how can you make it one-one and onto both ? (T.U. 2050, 2058 S)

7. Let a function $f : A \to B$ be defined by $f(x) = \frac{x}{2+x}$ with A = $\{-1, 0, 1, 2, 3, 4\}$ and B = $\{-1, 0, 1, 1/3, 1/2, 3/5, 2/3\}$. Find the range of $f$. Is the function $f$ one-one and onto both ? If not, how can you make it one-one and onto both ? (T.U. 2051)

8. Let Q be the set of all rational numbers. Show that the function $f : Q \to Q$ such that $f(x) = 3x + 5$ for all $x \in$ Q is one-one and onto.

***Solution:***

Let $x_1$ and $x_2 \in$ Q. Then

$$f(x_1) = 3x_1 + 5$$

and

$$f(x_2) = 3x_2 + 5$$

$$\begin{aligned} f(x_1) = f(x_2) &\Rightarrow 3x_1 + 5 = 3x_2 + 5 \\ &\Rightarrow 3x_1 = 3x_2 \\ &\Rightarrow x_1 = x_2 \end{aligned}$$

Hence $f$ is one-one

Again, $y = 3x + 5$ or, $3x = y - 5$

$\therefore \quad x = \frac{y-5}{3}$

For each $y \in$ Q, $x \in$ Q, so $f$ is onto

$\therefore f$ is one-one and onto.

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## EXERCISE 2.4

1. Prove that
   a) $\log_a (xy^3/z^2) = \log_a x + 3\log_a y - 2\log_a z$
   b) $\log_a (2x + 3(\log_a x - \log_a y)) = \log_a (2x^4/y^3)$
   c) $\log_a x^2 - 2\log_a \sqrt{x} = \log_a x$

   loga(

   d) $a^{\log_a x} = x$
   e) $\log_a a^x = x$
   f) $(\log a)^2 - (\log b)^2 = \log (ab). \log (a/b)$
   g) $\log(1 + 2 + 3) = \log 1 + \log 2 + \log 3$
   h) $x^{\log y - \log z} . y^{\log z - \log x} . z^{\log x - \log y} = 1$
   i) $(yz)^{\log y - \log z} . (zx)^{\log z - \log x} . (xy)^{\log x - \log y} = 1$
   j) $\log_a \sqrt{a\sqrt{a\sqrt{a^2}}} = 1$

2. If $f(x) = \log \dfrac{1-x}{1+x}$ $(-1 < x < 1)$, show that $f\left(\dfrac{2ab}{1+a^2b^2}\right) = 2f(ab)$ where $|ab| < 1$.

3. If $x = \log_{2a} a$, $y = \log_{3a} 2a$ and $z = \log_{4a} 3a$, prove that
$$xyz + 1 = 2yz$$

4. If $\dfrac{\log x}{y - z} = \dfrac{\log y}{z - x} = \dfrac{\log z}{x - y}$, prove that $x^x y^y z^z = 1$.

## ADDITIONAL QUESTIONS

1. Let $A = \{a, b, c\}$ and $B = \{x, y, z\}$. Find $A \times B$. Determine (i) a relation from A to B which is not a function (ii) relations from A to B which are functions.

2. A function $f$ is defined on the set of integers as follows :
$$f(x) = \begin{cases} 1 + x : & 1 \le x < 2 \\ 2x - 1 : & 2 \le x < 4 \\ 3x - 10 : & 4 \le x < 6 \end{cases}$$
   a) Find the domain of $f$
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4. Let A = $\{a, b, c\}$ and B = $\{m, n, p\}$. Define a function $f : A \rightarrow B$ such that it is
   (i) One-one and Onto
   (ii) into function
   (iii) Onto function only.

   Show these functions by diagram. Find the domain and the range of each.

5. What do you mean by a function ? Distinguish between the range and the domain of a function. Let A = $\{-1, -2, 0, 1, 2\}$ and a function $f : A \rightarrow B$ be defined by $f(x) = \frac{x^2}{2}$ . Find the range of f. Is the function one-one ? (T.U. 2049)

6. Let a function $f : A \rightarrow B$ be defined by $f(x) = \frac{x-1}{x+2}$ with A = $\{-1, 0, 1, 2, 3, 4\}$ and B = $\{-2, 1, -1/2, 0, 1/2, 1/4, 2/5\}$. Find the range of $f$. Is the function $f$ one-one and onto both ? If not, how can you make it one-one and onto both ? (T.U. 2050, 2058 S)

7. Let a function $f : A \rightarrow B$ be defined by $f(x) = \frac{x}{2+x}$ with A = $\{-1, 0, 1, 2, 3, 4\}$ and B = $\{-1, 0, 1, 1/3, 1/2, 3/5, 2/3\}$. Find the range of $f$. Is the function $f$ one-one and onto both ? If not, how can you make it one-one and onto both ? (T.U. 2051)

8. Let Q be the set of all rational numbers. Show that the function $f : Q \rightarrow Q$ such that $f(x) = 3x + 5$ for all $x \in Q$ is one-one and onto.

   ***Solution:***

   Let $x_1$ and $x_2 \in$ Q. Then

   $$f(x_1) = 3x_1 + 5$$

   and $$f(x_2) = 3x_2 + 5$$

   $$f(x_1) = f(x_2) \Rightarrow 3x_1 + 5 = 3x_2 + 5$$
   $$\Rightarrow 3x_1 = 3x_2$$
   $$\Rightarrow x_1 = x_2$$

   Hence $f$ is one-one

   Again, $y = 3x + 5$ or, $3x = y - 5$

   $$\therefore \quad x = \frac{y-5}{3}$$

   For each $y \in$ Q, $x \in$ Q, so $f$ is onto

   $\therefore f$ is one-one and onto.

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9. a) Show that the function $f : [-2, 0] \to R$ defined by $f(x) = x^2$ is one to one but not onto.

   b) Show that the function $f : [0, \infty) \to R$ defined by $f(x) = |x|$ is one to one and onto.

10. Let $R_0$ be the set of non-zero real numbers and $f : R_0 \to R_0$ be defined by $f(x) = 1/x$, $x \in R_0$, show that $f$ is one-one and onto.

11. Define an inverse of a function. What condition makes a function to have its inverse ? Let $f : R \to R$ be defined by $f(x) = 2x - 4$, $x \in R$. Find $f^{-1}$ and show that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$

    **Hint :** $f(x) = 2x - 4 \Rightarrow y = 2x - 4$

    To find an inverse function, we interchange the role of $x$ and $y$. That is,

    $x = 2y - 4$ or, $2y = x + 4$ $\therefore$ $y = \dfrac{x+4}{2}$

    i.e. $f^{-1}(x) = \dfrac{x+4}{2}$

    Now, $f(f^{-1}(x)) = f\left(\dfrac{x+4}{2}\right) = 2.\left(\dfrac{x+4}{2}\right) - 4 = x$

    Again, $f^{-1}(f(x)) = f^{-1}(2x - 4) = \dfrac{2x - 4 + 4}{2} = x$

    $\therefore$ $f(f^{-1}(x)) = f^{-1}(f(x)) = x$

12. Let $f : R \to R$ where R is the real number be defined by $f(x) = x^3$. Show that $f$ is one-one and onto both. Hence show that the inverse function $f^{-1}$ is defined and may be represented by $f^{-1}(x) = \sqrt[3]{x}$. (T.U. 2053)

13. Let $f : R \to R$ be defined by $f(x) = ax + b$, $x \in R$. $a$ and $b$ are also real numbers. Find $f^{-1}$ and show that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

14. Let $f : R \to R$ be defined by $f(x) = 2x$ and $g : R \to R$ be defined by $g(x) = x + 1$, find $(fog)(x)$ and $(gof)(x)$. Are the functions $(fog)(x)$ and $(gof)(x)$ one-one ? (HSEB 2054)

15. Let $f : R \to R$ and $g : R \to R$ be defined by $f(x) = 2x^2 - 3$, $g(x) = 3x + 2$. Determine the following composite functions :

    i) $(fog)(x)$ ii) $(gof)(x)$
    iii) $(fof)(x)$ iv) $(gog)(x)$

16. If $p$ and $q$ are any two statements

    a) prove that $\sim(p \wedge (\sim p))$ is a tautology

    b) prove that $\sim(p \vee (\sim p))$ is a contradicton

    c) prove that $(p \vee q) \vee (\sim p)$ is a tautology

    d) prove that $((\sim q) \wedge p) \wedge q$ is a contradiction

17. If $f : R - \{1\} \to R - \{2\}$ is defined by $f(x) = \frac{2x}{x-1}$ prove that $f$ is one to one and onto function.
18. Is the following argument valid?

    All natural numbers are integers.
    $x$ is not an integer
    
    ---
    Therefore, $x$ is not a natural number.
19. Is the following argument valid?

    Some students of class XI are smart.
    Prakash is a student of class XI.
    
    ---
    Therefore, Prakash is smart.
20. Prove that the functions $f : R \to R$ defined by $f(x) = 2x^3 - 1$ is one to one and onto. Also find $f^{-1}$.
21. If $f, g : R \to R$ be defined by $f(x) = x^3 + 2$ and $g(x) = 4x - 1$, find $(fog)(x)$ and $(gof)(x)$ and show that the composite function is not commutative. [HSEB, 2065]
22. From the functions $f(x) = 2x^3 - 3$ and $g(x) = 3x + 2$ where $x \in R$, determine $(fog)(x)$ and $(gof)(x)$.
    Are $(fog)(x)$ and $(gof)(x)$ one to one? [HSEB, 2061]
23. If $f(x) = x^3 - 1$ and $g(x) = 2x - 3$, compute $(f^{-1}og)(2)$ and $(fog^{-1})(1)$.
24. If $f : R \to R$, $g : R \to R$ and $h : R \to R$ be defined by $f(x) = x + 1$, $g(x) = e^x$ and $h(x) = \log x$, find the value of $(ho(gof))(x)$ at $x = 1$.

### Answers

2. Domain of $f = \{1, 2, 3, 4, 5\}$, Range of $f = \{2, 3, 5\}$; No
3. $a = 2, b = -1$
5. Range of $f = \{0, 1/2, 2\}$, No
6. Range of $f = \{-2, -1/2, 0, 1/4, 2/5, 1/2\}$, No
7. Range of $f = \{-1, 0, 1/3, 1/2, 3/5, 2/3\}$, No
14. $2x + 2$, $2x + 1$; yes
15. (i) $18x^2 + 24x + 5$ (ii) $6x^2 - 7$
    (iii) $8x^4 - 24x^2 + 15$ (iv) $9x + 8$
16. a)

| $p$ | $\sim p$ | $p \wedge (\sim p)$ | $\sim(p \wedge (\sim p))$ |
|---|---|---|---|
| T | F | F | T |
| F | T | F | T |

↑
All true

b)

| $p$ | $\sim p$ | $p \vee (\sim p)$ | $\sim(p \vee (\sim p))$ |
|---|---|---|---|
| T | F | T | F |
| F | T | T | F |

↑ All False

c)

| $p$ | $q$ | $p \vee q$ | $\sim p$ | $(p \vee q) \vee (\sim p)$ |
|---|---|---|---|---|
| T | T | T | F | T |
| T | F | T | F | T |
| F | T | T | T | T |
| F | F | F | T | T |

↑ All true

d)

| $p$ | $q$ | $(\sim q)$ | $(\sim q) \wedge p$ | $((\sim q) \wedge p) \wedge q$ |
|---|---|---|---|---|
| T | T | F | F | F |
| T | F | T | T | F |
| F | T | F | F | F |
| F | F | T | F | F |

↑ All false

20. $\left(\frac{x+1}{2}\right)^{1/3}$ 21. No 23. 2, 7 24. 2

# CHAPTER 3
# Curve Sketching

## 3.1 Introduction

Graph is one of the various ways of representing a function. For the graphical representation of a function $y = f(x)$, we consider different values of $x$ in the domain of $f$ and calculate the corresponding values of $y$ i.e. $f(x)$. Then points are plotted taking the values of $x$ as the x-coordinates and the corresponding values of $y$ as the y-coordinates. Points thus plotted are joined by a scale if the given function is linear. Points are joined freely if the given function is not linear.

Thus the graph of the function $f$ is the set $\{(x, f(x)) : x \in A\}$ which is the subset of $A \times B$. Hence, curve sketching is the representation of the given function in its graphical form. But in this chapter, we sketch the graph of the given function not by taking different points and plotting the points on the paper but is sketched with the help of the different characteristics which the graph satisfies.

## 3.2 Characteristics of Curves

Before sketching the curve of a given function, we must have some idea (or information) about the curve which the given function satisfies. For this, we define some of the following terms which are very essential for the function to present it into its graphical form.

**Origin:** The first point to be remembered is to see whether the curve passes through the origin or not. To see this, put $x = 0$ in the given equation (given function). If the value of $y = 0$ when $x = 0$, then the curve passes through the origin.

**Points on Axes:** In this case, we see where does the curve meet the axes of coordinates. For this, we put $x = 0$ and $y = 0$ successively and thus getting the values of $y$ and $x$ respectively.

**Even function:** A function $f : A \to B$ is said to be an even function if $f(-x) = f(x)$ for all $x \in A$. That is, if $(x, y)$ lies on the curve, then $(-x, y)$ also lies on the same curve.

For example: If $f(x) = x^2$, then $f(-x) = (-x)^2 = x^2 = f(x)$, so $f(x) = x^2$ is an even function.

**Odd function:** A function $f : A \rightarrow B$ is said to be an odd function if $f(-x) = -f(x)$ for all $x \in A$. That is, if $(x, y)$ lies on the curve, then $(-x, -y)$ lies on the same curve.

For example: If $f(x) = x + x^3$, then $f(-x) = -x + (-x)^3 = -x - x^3$
$= -(x + x^3) = -f(x)$

So $f(x) = x + x^3$ is an odd function.

**Symmetry:** A curve represented by the function $y = f(x)$ is **symmetric about y-axis** if no change occurs in $f(x)$ when $x$ is replaced by $-x$ i.e. if $f(-x) = f(x)$. In this case, the parts of the curve lying on either side of y-axis are same.

A curve represented by the equation $f(x, y) = 0$ is **symmetric about x-axis** if no change occurs in the equation when $y$ is replaced by $-y$.

A curve represented by the function $y = f(x)$ is **symmetric about the origin** if $f(-x) = -f(x)$ i.e. if $f(x)$ is an odd function.

**Increasing function:** A function $f(x)$ is said to be increasing in an interval $(a, b)$ if $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in (a, b)$.

**Decreasing function:** A function $f(x)$ is said to be decreasing in an interval $(a, b)$ if $x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in (a, b)$

Increasing and decreasing functions are known as monotonic functions.

For example: i) $y = x^2$, $x \in [0, 3]$ is an increasing function.
ii) $y = 10 - x^2$, $x \in [0, 3]$ is a decreasing function.

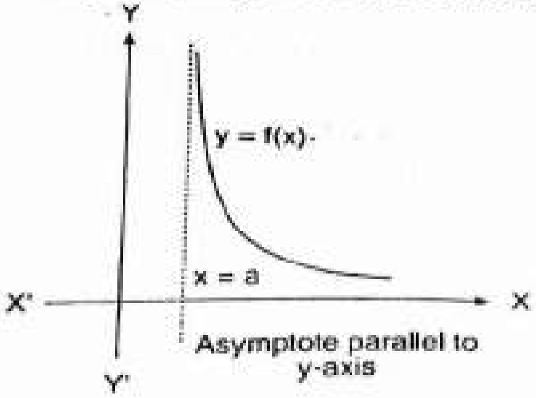
**Periodicity:** A function $f$ which satisfies $f(x + k) = f(x)$ ...... (i) for all $x$ belonging to its domain and $k > 0$ is said to be a periodic function. The smallest value of k is known as the period of the function.

For example: $\sin(x + 2\pi) = \sin x$ and $\cos(x + 2\pi) = \cos x$

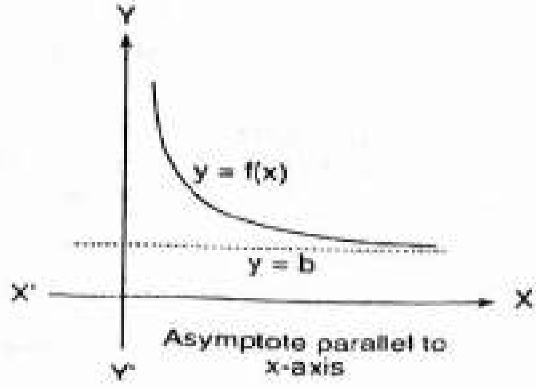
So, $\sin x$ and $\cos x$ are periodic functions with period $2\pi$.

**Asymptote:** A line $x = a$ is said to be an asymptote to the curve $y = f(x)$ if $y = \infty$ when $x = a$ (i.e. $x$ close to $a$ makes $y$ close to $\infty$)

Similarly $y = b$ is said to be an asymptote to the curve $y = f(x)$ if $x = \infty$ when $y = b$ (i.e. $y$ close to $b$ makes $x$ close to $\infty$)




Asymptote parallel to y-axis

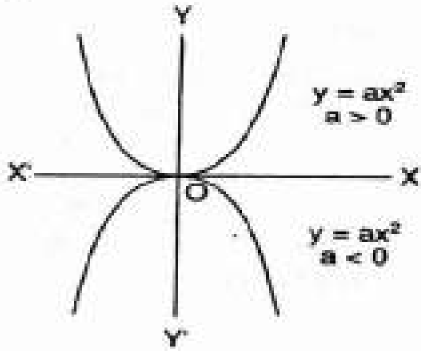



Asymptote parallel to x-axis

For the graph of the function of the type $f(x) = ax^2 + bx + c$; we use the following way:

$$y = ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x\right) + c$$
$$= a\left\{(x)^2 + 2.x.\frac{b}{2a} + \left(\frac{b}{2a}\right)^2\right\} + c - \frac{b^2}{4a}$$
$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}$$
$$\left\{x - \left(-\frac{b}{2a}\right)\right\}^2 = \frac{1}{a}\left\{y - \frac{4ac - b^2}{4a}\right\}$$

Then $\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$ will give the vertex of the parabola. Or, find $x = -\frac{b}{2a}$ and then the vertex of the parabola is at $(x, f(x))$, giving the highest or the lowest point of the curve (i.e. parabola). The curve turns (opens) upward or downward according as $a > 0$ or $a < 0$.




If the given equation is

$$\left(x + \frac{b}{2a}\right)^2 = \frac{1}{a}\left\{y - \frac{4ac - b^2}{4a}\right\}$$

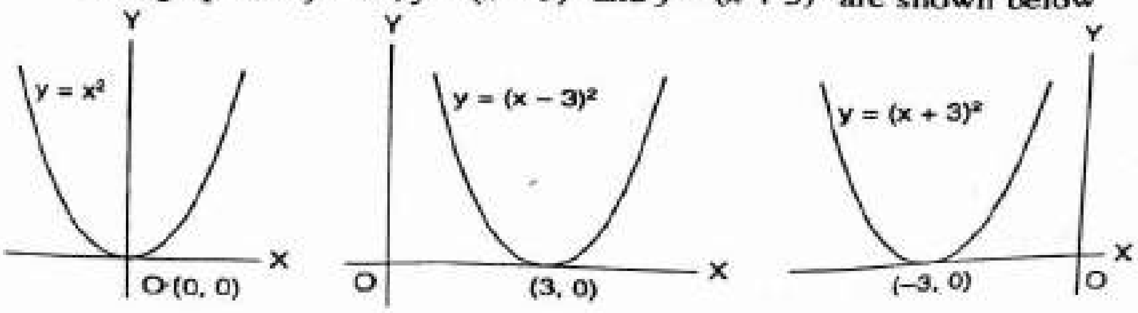
then it is of the form $X^2 = 4Y$ which is symmetric about Y-axis. i.e. $X = 0$ or $x + \frac{b}{2a} = 0$

## 3.3 Transformation of Graphs

**Shifting or translation of graphs**

The graph of the function $y = f(x - a)$ $(a > 0)$ is the graph of $y = f(x)$ translated or shifted by $a$ units to the right. Again the graph of $y = f(x + a)$, $(a > 0)$ is the graph of $y = f(x)$ translated by a units to the left.

The graphs of $y = x^2$, $y = (x - 3)^2$ and $y = (x + 3)^2$ are shown below



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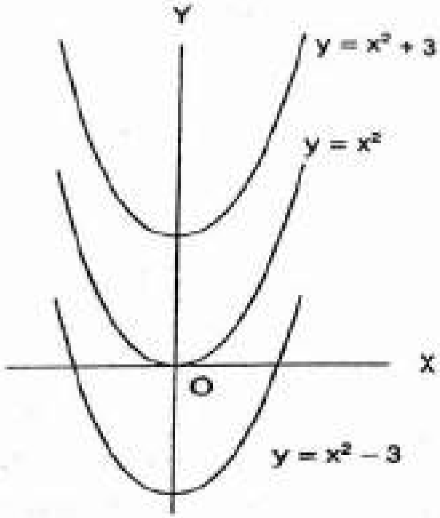
**Shifting or translation of graph upward or downward**

The graph of $y = f(x) + b$ $(b > 0)$ is the graph of $y = f(x)$ translated by $b$ units upward. Again the graph of $y = f(x) - b$ is the graph of $y = f(x)$ translated by $b$ units downward.

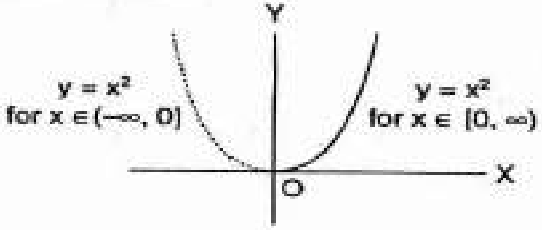
The graphs of $y = x^2$, $y = x^2 + 3$ and $y = x^2 - 3$ are shown aside.

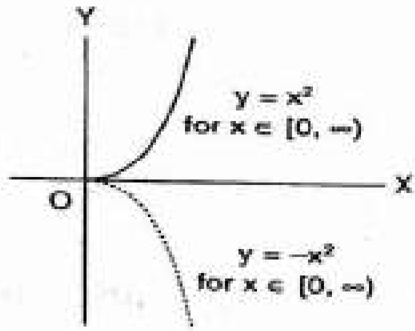



**Reflection:**

The graph of $y = f(-x)$ is the reflection of graph of $y = f(x)$ about y-axis. Again the graph of $y = -f(x)$ is the reflection of graph of $y = f(x)$ about x-axis.

For example: The graphs of $y = x^2$ for $x \in [0, \infty)$ and $y = x^2$ for $x \in (-\infty, 0]$ and also the graphs of $y = x^2$ and $y = -x^2$ for $x \in [0, \infty)$ are shown below.




## Worked Out Examples

***Example 1***

Examine whether the function $f(x) = 3x^2 + \cos x + 1$ is even or odd.

***Solution:***

$$f(-x) = 3(-x)^2 + \cos(-x) + 1$$
$$= 3x^2 + \cos x + 1 = f(x)$$

$\therefore$ $f(x)$ is even function.

***Example 2***

Test the periodicity of the function $f(x) = \sin bx$.

**Solution:**

$$f\left(x+\frac{2\pi}{b}\right) = \sin b\left(x+\frac{2\pi}{b}\right) = \sin(bx+2\pi)$$

$$= \sin bx = f(x)$$

$\therefore f(x) = \sin bx$ is the periodic function, its period being $\frac{2\pi}{b}$.

**Example 3**

Examine the symmetricity of the function $f(x) = e^x + e^{-x}$

**Solution:**

$f(-x) = e^{-x} + e^{-(-x)} = e^{-x} + e^x = f(x)$

$\therefore f(x)$ is symmetric about y-axis.

**Example 4**

Show that $f(x) = 4 - 3x$ is decreasing function for all $x \in R$.

**Solution:**

Let $x_1, x_2 \in R$ and $x_1 > x_2$

Then, $x_1 > x_2 \Rightarrow -3x_1 < -3x_2$

$\Rightarrow 4 - 3x_1 < 4 - 3x_2$

$\Rightarrow f(x_1) < f(x_2)$

$\therefore f(x)$ is decreasing function for all $x \in R$.

**Example 5**

Draw the graph of the function $y = f(x) = 3x + 2$

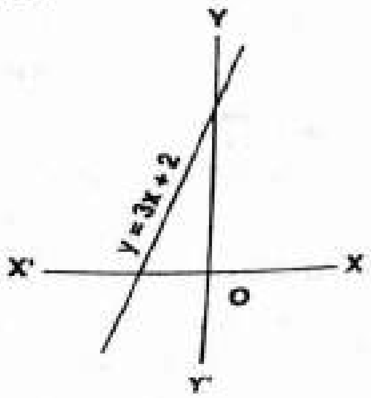
**Solution :**

(i) When $x = 0$, $y = 2 \neq 0$, so the curve does not pass through the origin but cuts the y-axis at (0, 2)

(ii) When $y = 0$, $x = -\frac{2}{3}$, so the curve meets the x-axis at (–2/3, 0)

(iii) When $x > 0$, $y > 2$ and $x < -\frac{2}{3}$, $y < 0$

The graph of the given function with the above characteristics is shown aside.




**Example 6**

Draw the graph of the function $y = f(x) = x^2 - 4x + 3$.

***Solution :***

$y = f(x) = x^2 - 4x + 3$

This function represents a parabola.

(i) When $x = 0$, $y = 3 \neq 0$, so the curve does not pass through the origin but cuts the y-axis at (0, 3)

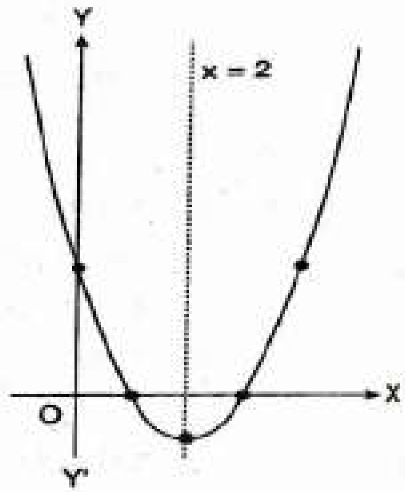
(ii) The curve meets the x-axis at the point where $y = 0$

i.e. $x^2 - 4x + 3 = 0$

or, $(x - 1)(x - 3) = 0$

$\therefore$ $x = 1, 3$

i.e. the curve cuts the x-axis at the points (1, 0) and (3, 0)




(iii) Comparing $y = x^2 - 4x + 3$ with $y = ax^2 + bx + c$, we have

$a = 1, b = -4, c = 3$

$x = -\frac{b}{2a} = -\frac{(-4)}{2 \times 1} = 2$

$y = (2)^2 - 4 \times 2 + 3 = -1$

$\therefore$ the vertex of the parabola = (2, –1)

Also, $y = x^2 - 4x + 3$

$\Rightarrow$ $y = (x - 2)^2 - 1$

$\Rightarrow$ $(x - 2)^2 = y + 1$

which is similar to $X^2 = Y$.

So, the curve is symmetric about Y-axis i.e. $X = 0 \Rightarrow x - 2 = 0$

The graph of the given function is given aside.

***Example 7***

Sketch the graph of $y = x^2$ and hence sketch the graph of

$$y = x^2 - 4x + 3.$$

***Solution:***

Characteristics of the graph of $y = x^2$

(i) The given function is of degree two, hence represents a parabola.

(ii) Comparing $y = x^2$ with $y = ax^2 + bx + c$, we have

$a = 1, b = 0, c = 0$

$a = 1 > 0$, so the parabola turns upwards.

For vertex: $x = -\frac{b}{2a} = 0, y = 0$

$\therefore$ vertex is at (0, 0)

(iii) Domain $(-\infty, \infty)$, Range = $[0, \infty]$

(iv) For $0 < x < 1$, $y = x^2 < x$, so the part of the curve lies below the line $y = x$.

(v) For $x > 1$, $y = x^2 > x$, so the curve lies above the line $y = x$.

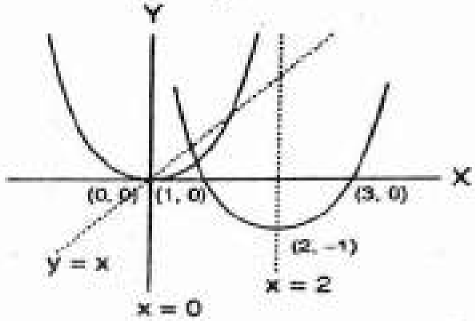
(vi) The curve is symmetric about y-axis.

For the graph of $y = x^2 - 4x + 3$

$\Rightarrow \quad y = x^2 - 4x + 4 - 1$

$\Rightarrow \quad y = (x-2)^2 - 1$

So the graph of $y = x^2 - 4x + 3$ is the graph of $y = x^2$, translated by 2 units to right and translated below by 1 unit. The graphs of $y = x^2$ and $y = x^2 - 4x + 3$ are given aside.




***Example 8***

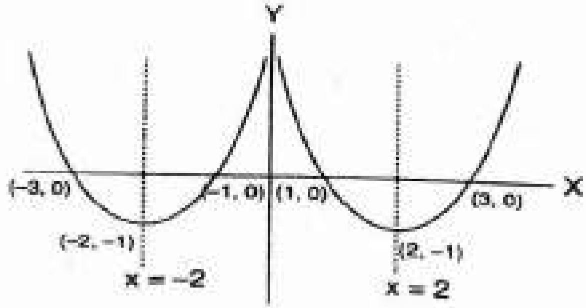
Sketch the graph of $f(x) = x^2 - 4x + 3$ and hence draw the graph of $f(-x) = x^2 + 4x + 3$.

***Solution:***

Write the different characteristics of the graph of the function $f(x) = x^2 - 4x + 3$ and then draw its graph. Again since $f(-x) = (-x)^2 - 4(-x) + 3 = x^2 + 4x + 3$ so its graph is the reflection of the graph of $f(x) = x^2 - 4x + 3$ about y-axis.

$(1, 0) \rightarrow (-1, 0)$, $(3, 0) \rightarrow (-3, 0)$ and $(2, -1) \rightarrow (-2, -1)$

The graphs of $f(x)$ and $f(-x)$ are given below




***Example 9***

Sketch the graph of $y = f(x) = x^2 - 4x + 3$ and hence draw the graph of $-f(x) = -(x^2 - 4x + 3) = -x^2 + 4x - 3$.

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***Solution:***

The graph of $-f(x)$ is the reflection of the graph of $f(x)$ about x-axis.

We have already given the graph of $y = f(x) = x^2 - 4x + 3$ where points on x-axis are $(1, 0)$ and $(3, 0)$ and vertex at $(2, -1)$.

For the graph of $-f(x)$,

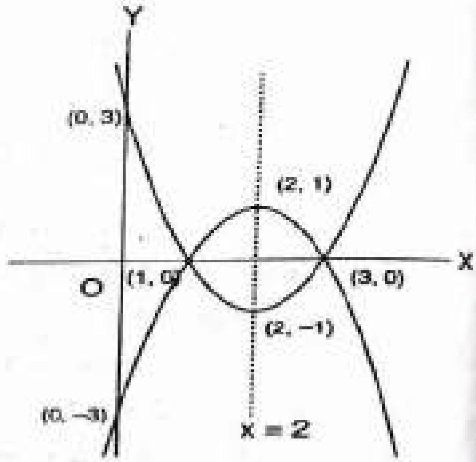
$(1, 0) \rightarrow (1, 0)$,

$(3, 0) \rightarrow (3, 0)$

and $(2, -1) \rightarrow (2, 1)$

(i.e. $(x, y) \rightarrow (x, -y)$)

The graphs of $f(x)$ and $-f(x)$ are given aside.




***Exampel 10***

Draw the graph of the following function $f(x) = x(x - 1)(x - 2)$

***Solution :***

$f(x) = x(x - 1)(x - 2)$

The graph of the above function has the following characteristics:

(i) When $x = 0$, $y = 0$, so the curve passes through the origin.

(ii) The curve meets the x-axis at the point where $y = 0$

i.e. $x(x - 1)(x - 2) = 0$

$\therefore$ $x = 0, 1, 2$

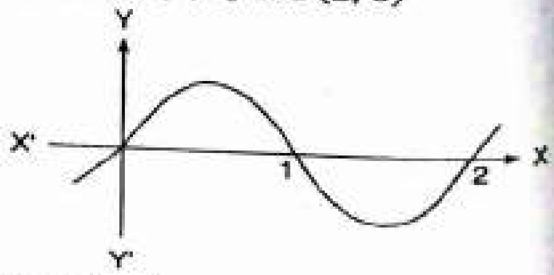
The curve cuts the x-axis at the points $(0, 0)$, $(1, 0)$ and $(2, 0)$

iii) The curve does not meet the y-axis other than the origin.

(iv) When $x > 2$, $y > 0$ i.e. when $x$ increases $y$ will also increase.

(v) When $x < 0$, $y < 0$ i.e. when $x$ decreases, $y$ also decreases.

(vi) When $0 < x < 1$, $y > 0$ so the part of the curve lies above x-axis and when $1 < x < 2$, $y < 0$ so the part of the curve lies below the x-axis.




Now the graph of the given function with above characteristics is given aside.

***Example 11***

Draw the graph of the function $y = \dfrac{1}{x + 2}$

***Solution :***

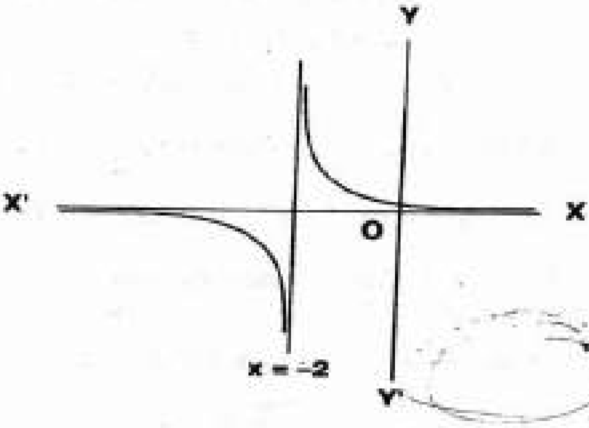
(i) When $x = 0$, $y = \frac{1}{2} \neq 0$ so the curve does not pass through the origin but cuts the y-axis at $\left(0, \frac{1}{2}\right)$

(ii) When $y = 0$, $x = \infty$ i.e. $x$ does not have finite value or the curve does not meet the x-axis at finite distance so $y = 0$ is the asymptote.

(iii) When $x = -2$, $y = \infty$. So $x = -2$ is an asymptote.

Since the given function is of the form $xy = c^2$, so its graph has the curve in the opposite quadrants.

The graph of the function $y = \frac{1}{x+2}$ is given aside.




***Example 12***

Draw the graph of the function

$$y = f(x) = \frac{x^2 - 4}{x - 2}$$

***Solution :***

$$y = f(x) = \frac{x^2 - 4}{x - 2}$$

i) The function is not defined at $x = 2$.

ii) For $x \neq 2$, $y = x + 2$

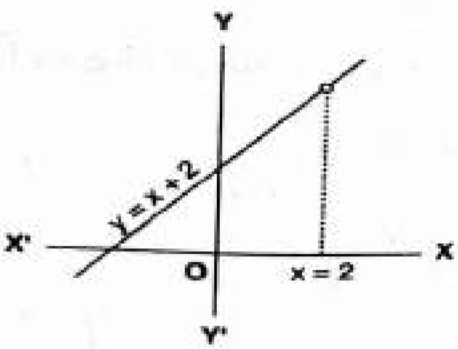
When $y = 0$, $x = -2$, so the curve meets the x-axis at $(-2, 0)$

iii) When $x = 0$, $y = 2$, so the curve meets the y-axis at $(0, 2)$

iv) When $x > 2$, $y > 4$. So, when $x$ increases, $y$ also increases.

v) When $x < -2$, $y < 0$, so when $x$ decreases $y$ also decreases.

The graph of the function with the above characteristics is shown aside.




***Example 13***

Draw the graph of the function $y = \sin x \quad (0 \leq x \leq 2\pi)$

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***Solution :***

$y = \sin x$

(i) When $x = 0$, $y = 0$, so the curve passes through the origin.

(ii) When $y = 0$, $\sin x = 0$

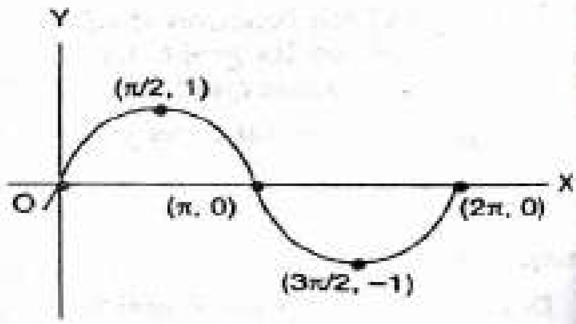
$\sin x = \sin 0$ or $\sin \pi$ or $\sin 2\pi$

$\therefore \quad x = 0, \pi$ or $2\pi$

i.e. the curve cuts the x-axis at three points i.e. at $x = 0$, $x = \pi$ and $x = 2\pi$.

(iii) When $x$ increases from 0 to $\frac{\pi}{2}$, $y$ increases from 0 to 1. Again when $x$ increases from $\frac{\pi}{2}$ to $\pi$, $y$ decreases from 1 to 0.

In the similar manner, the curve decreases from 0 to $-1$ when $x$ runs from $\pi$ to $\frac{3\pi}{2}$ and increases from $-1$ to 0 when $x$ runs from $\frac{3\pi}{2}$ to $2\pi$. For $0 < x < \pi$, $y > 0$, so the part of the curve lies above x-axis. For $\pi < x < 2\pi$, $y < 0$, the part of the curve lies below x-axis.




With the above characteristics, the graph of the given function is given aside.

***Example 14***

Draw the graph of $y = \cos x \left(-\frac{\pi}{2} \le x \le \frac{\pi}{2}\right)$

***Solution :***

$y = \cos x$

(i) When $x = 0$, $y = 1$, so the curve cuts the y-axis at (0, 1).

(ii) When $y = 0$, $\cos x = 0$

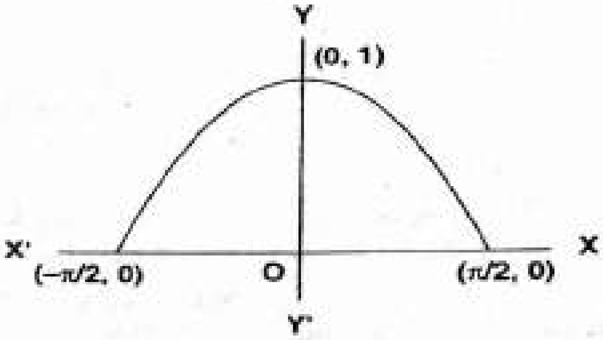
or, $\cos x = \cos\left(\pm \frac{\pi}{2}\right) \qquad \therefore \quad x = -\frac{\pi}{2}, \frac{\pi}{2}$

i.e. the curve meets the x-axis at two points $\left(-\frac{\pi}{2}, 0\right)$ and $\left(\frac{\pi}{2}, 0\right)$

(iii) Since $\cos x$ is an even function, so the curve is symmetric about $y$-axis for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

(iv) For $-\frac{\pi}{2} \le x \le 0$, $y$ increases and for $0 \le x \le \frac{\pi}{2}$, $y$ decreases. At $x = 0$, the curve turns down.

(v) Since for all $x \in [-\pi/2, \pi/2]$, $y$ is positive, so the curve lies above $x$-axis.




With the above characteristics, the graph of $y = \cos x$ is given aside.

***Example 15***

Draw the graph of $y = 2^x$.

***Solution :***

$$y = 2^x$$

The characteristics of the above function are as follows:

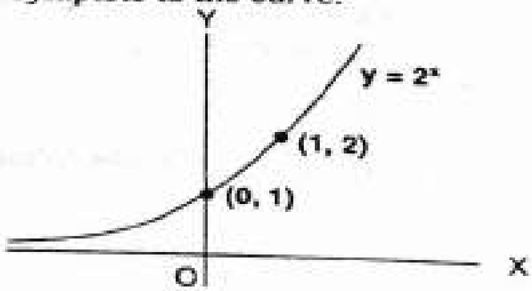
(i) When $x = 0$, $y = 1$, so the curve cuts the $y$-axis at (0, 1).

(ii) When $y = 0$, $x = -\infty$ that is, the curve does not meet the $x$-axis at a finite distance but it approaches the $x$-axis on its negative side at an infinite distance. So, $y = 0$ i.e. x-axis is the asymptote to the curve.

(iii) When $x = 1$, $y = 2$, so the curve passes through (1, 2)

(iv) When $x > 1$, $y > 2$ so when $x$ increases $y$ also increases. Similarly $y$ decreases when $x < 0$.

(v) No part of the curve lies below $x$-axis as no value of $x$ makes $y$ negative.




With the above characteristics of the given function, the graph is given aside.

***Example 16***

Draw the graph of $y = \log_2 x$

***Solution :***

$$y = \log_2 x$$

The characteristics of the graph of the above function are as follows:

(i) When $y = 0$, $x = 1$, so the curve cuts the $x$-axis at the point (1, 0)

(ii) When $x = 2$, $y = 1$, so the curve passes through the point (2, 1).

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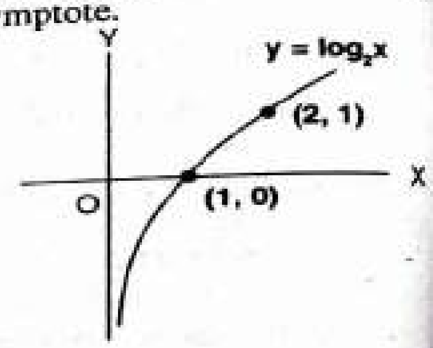
(iii) When $x = 0$, $y = -\infty$, so the curve does not meet the $y$-axis at a finite distance but the curve approaches the $y$-axis at an infinite distance on its negative side. So $x = 0$ i.e. y-axis is the asymptote.

(iv) For $x > 2$, $y > 1$, so when $x$ increases $y$ also increases. Similarly for $x < 1$, $y$ decreases.

(v) No part of the curve lies on the left of y-axis as no value of y makes $x$ negative.

With the above characteristics, the graph of the given function is given aside.




## EXERCISE 3.1

1. Examine whether the following functions are even, odd or neither.
   a) $x \sin x + \cos x$ b) $5x^2 + \sin x$
   c) $\sqrt{1+x^2} - \sqrt{1-x^2}$ d) $\sqrt{1+x} - \sqrt{1-x}$
   e) $10^x - 10^{-x}$ f) $\dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

2. Test the symmetricity of the following functions
   a) $f(x) = x^4 + 3x^2 + 1$ b) $f(x) = \cos x\ (-\pi/2 \le x \le \pi/2)$
   c) $f(x) = x^3 - x$ d) $f(x) = \sqrt{4 - x^2}$

3. Test the periodicity of the following functions and find their periods
   a) $f(x) = \sin 2x$ b) $f(x) = \cos \pi x$
   c) $f(x) = \tan \dfrac{x}{4}$ d) $f(x) = \sin (ax + b),\ a > 0$
   e) $f(x) = \sin x + \cos x$

4. a) Prove that $f(x) = 3 + 5x$ is increasing for all $x \in R$.
   b) Prove that $f(x) = 5 - 4x$ is decreasing for all $x \in R$.

5. Draw the graphs of the following functions:
   a) $y = 3x - 2$ b) $y = x^2 + 2x + 3$
   c) $y = -x^2 + 4x - 3$ d) $y = \dfrac{1}{x}$
   e) $y = \dfrac{1}{x-3}$ f) $y = \dfrac{x^2-1}{x-1}$
   g) $f(x) = (x-1)(x-2)(x-3)$ h) $y = x^2(x-2)$

6. Draw the graphs of the following functions

| | | | |
|---|---|---|---|
| a) | $y = 2^{-x}$ | b) | $y = 3^x$ |
| c) | $y = \left(\frac{1}{3}\right)^x$ | d) | $y = \log_3 x$ |
| e) | $y = \log_{10} x$ | f) | $y = \cos x \ (0 \le x \le 2\pi)$ |
| g) | $y = \sin x \ (-\pi \le x \le \pi)$ | h) | $y = \tan x \ (-\pi \le x \le \pi)$ |

$x^2 - 2\cdot 1\cdot x + 2^2 + x - 2$

$(x-2)^2 + x$

7. a) Sketch the graph of $y = x^2$ and hence draw the graph of

   i) $y = (x-2)^2$ ii) $y = x^2 - 6x + 5$.

   b) Sketch the graph of $y = x^2 - 2x - 3$ and hence draw the graph of
   $y = x^2 + 2x - 3$.

   c) Sketch the graph of $y = x^3$ and hence draw the graph of
   $y = (x-1)^3 - 2$

   d) Sketch the graph of $y = x - x^2$ and hence draw the graph of
   $y = -(x - x^2) = -x + x^2$

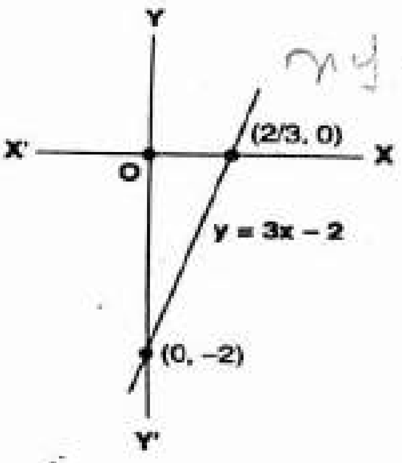
8. Sketch the graph of the following functions

   a) $y = x^2 - x + 2$ b) $y = x^3 - x - 3$

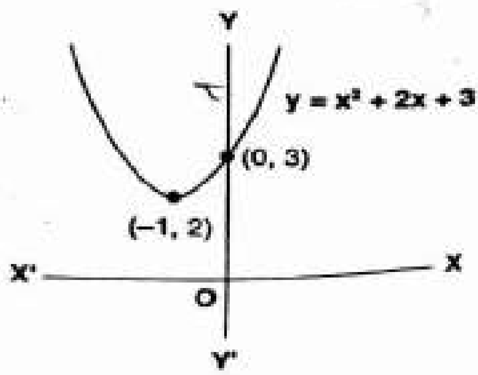
## Answers

1. a) Even b) Neither c) Even d) Odd e) Odd f) Odd
2. a) Symmetric about y-axis b) Symmetric about y-axis
   c) Symmetric about origin d) Symmetric about y-axis
3. a) Periodic, $\pi$ b) Periodic, 2 c) Periodic, $4\pi$
   d) Periodic, $\frac{2\pi}{a}$ e) Periodic, $2\pi$

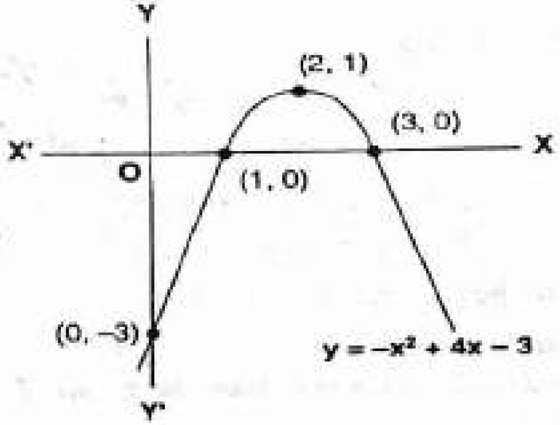
5. a)



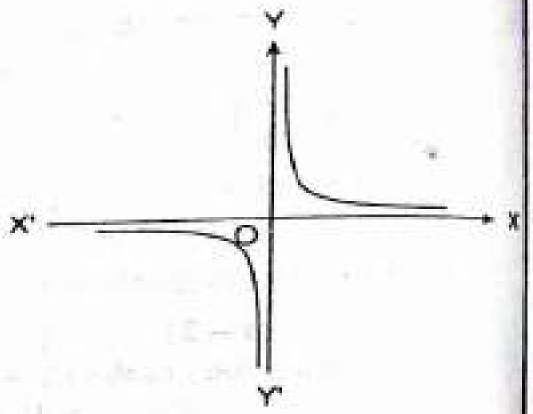

b)



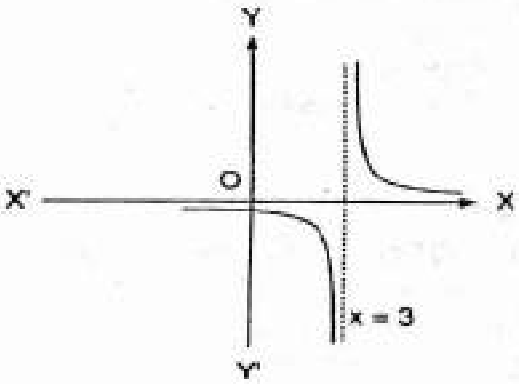
c)



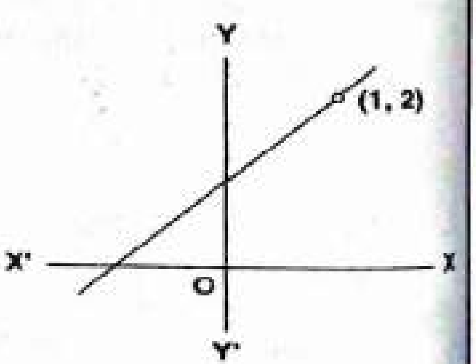

d)



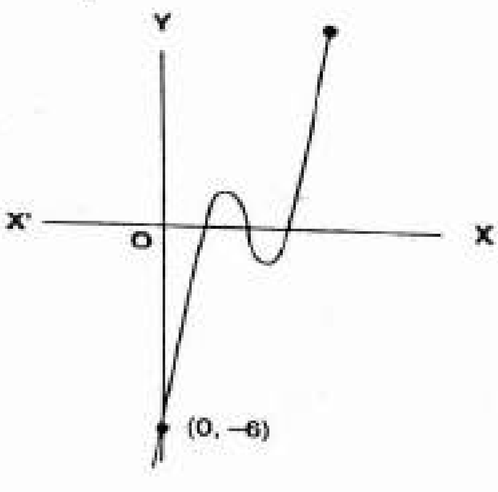

e)



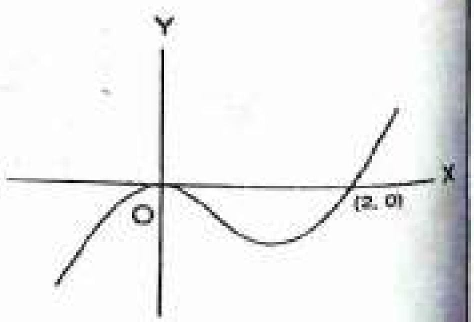

f)



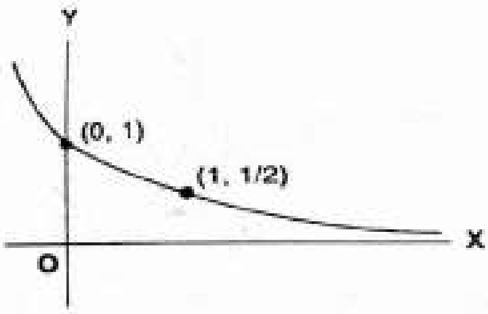

g)



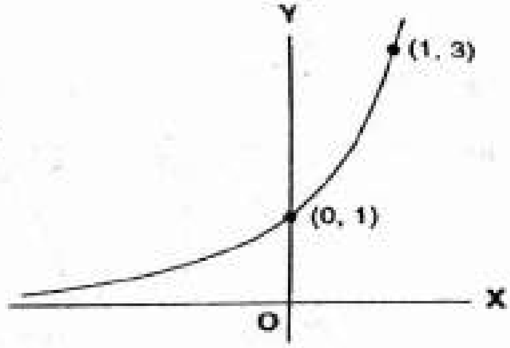

h)



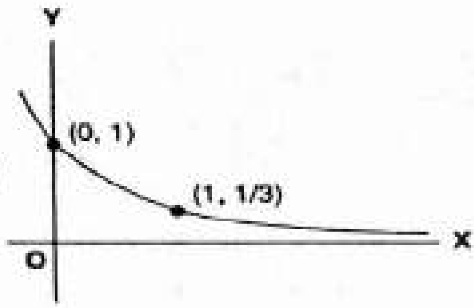
6. a)


Y
(0, 1)
(1, 1/2)
O
X

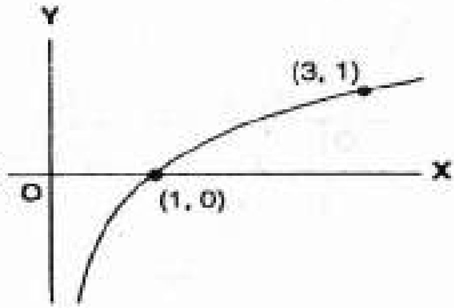
b)


Y
(1, 3)
(0, 1)
O
X

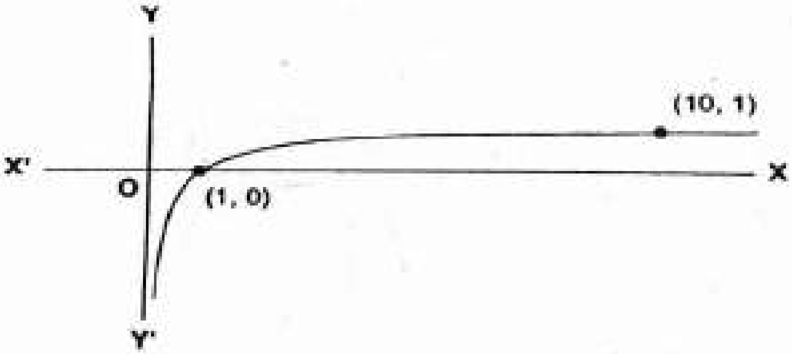
c)


Y
(0, 1)
(1, 1/3)
O
X

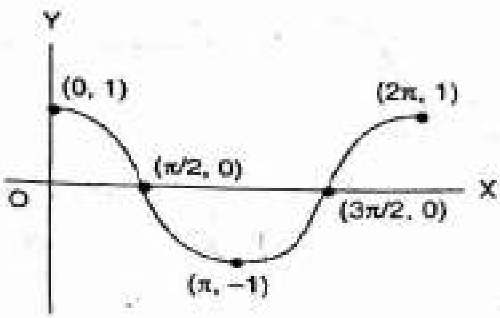
d)


Y
(3, 1)
O
(1, 0)
X

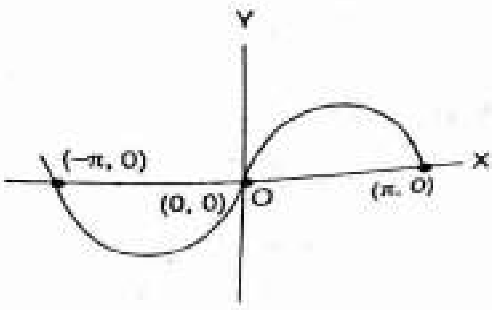
e)


Y
(10, 1)
X'
O
(1, 0)
X
Y'

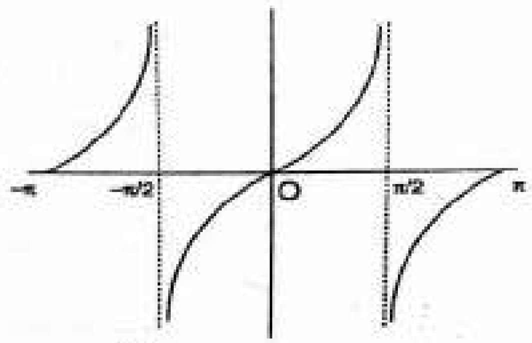
f)


Y
(0, 1)
(2π, 1)
(π/2, 0)
O
X
(3π/2, 0)
(π, −1)

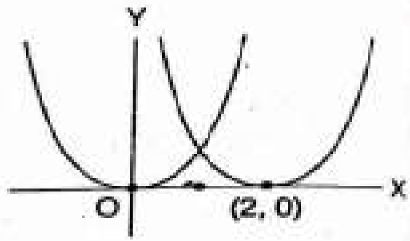
g)


Y
(−π, 0)
X
(0, 0)
O
(π, 0)

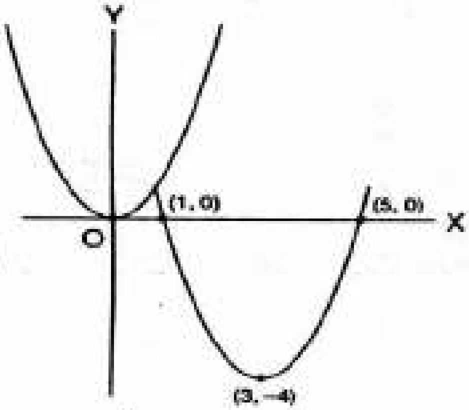
h)



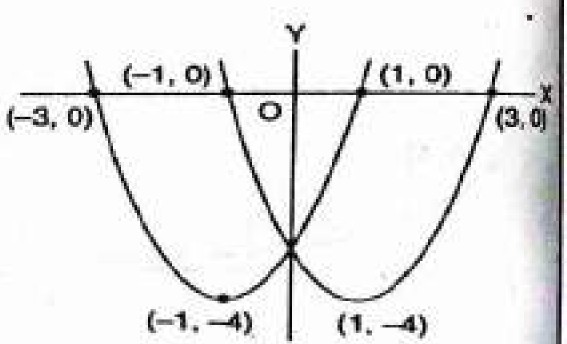

7. a) (i)



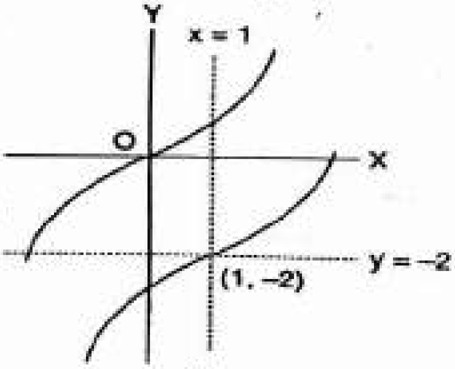

(ii)



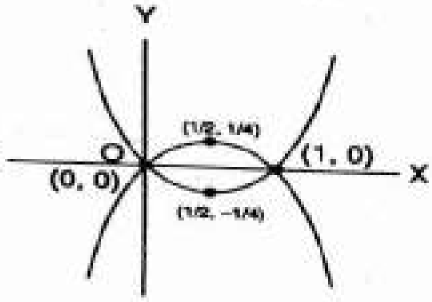

b)



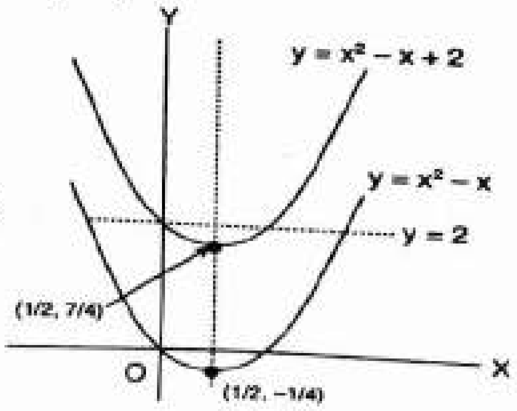

c)



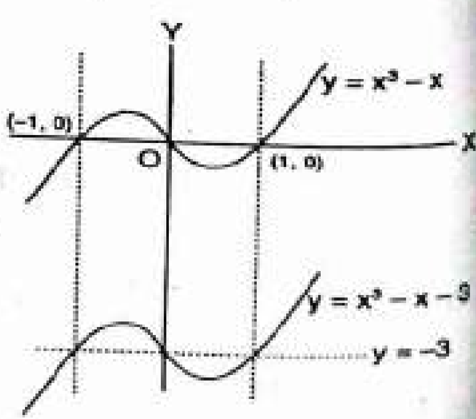

d)




8. a)




b)



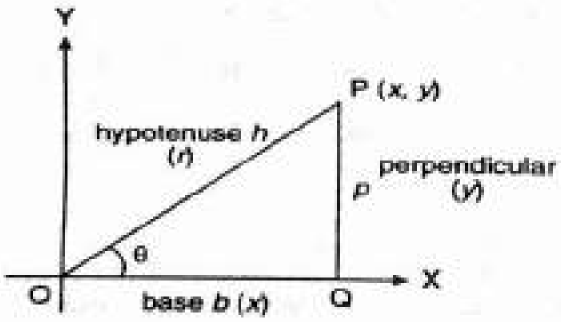
CHAPTER 4

# Trigonometrical Equations and General Values

## 4.1 Review of Some Trigonometrical Functions and Their Relations

### a) Trigonometric ratios of an acute angle

If θ is an acute angle of a right-angled triangle, our definitions of the six trigonometric functions may be put in the form




$$\sin\theta = \frac{PQ}{OP} = \frac{y}{r} = \frac{p}{h} = \frac{\text{perpendicular}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{OQ}{OP} = \frac{x}{r} = \frac{b}{h} = \frac{\text{base (adj. side)}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{PQ}{OQ} = \frac{y}{x} = \frac{p}{b} = \frac{\text{perpendicular}}{\text{base}}$$

$$\cot\theta = \frac{OQ}{PQ} = \frac{x}{y} = \frac{b}{p} = \frac{\text{base}}{\text{perpendicular}}$$

$$\sec\theta = \frac{OP}{OQ} = \frac{r}{x} = \frac{h}{b} = \frac{\text{hypotenuse}}{\text{base}}$$

$$\text{cosec}\,\theta = \frac{OP}{PQ} = \frac{r}{y} = \frac{h}{p} = \frac{\text{hypotenuse}}{\text{perpendicular}}$$

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## b) Some basic properties

For trigonometric angles of any size, positive or negative, the following properties follow directly from the definitions:

### *i)* Arms or side length

For a given angle of reference, each of the six trigonometric ratios– sine, cosine, tangent etc. has the same value whatever be the size of the arms (or triangle).

### *ii)* Elementary identities

The trigonometric ratios — sine, cosine and tangent of an angle are respectively the reciprocals of cosecant, secant and cotangent of the angle, i.e.,

$$\sin\theta = \frac{1}{\operatorname{cosec}\theta}, \qquad \cos\theta = \frac{1}{\sec\theta}, \qquad \tan\theta = \frac{1}{\cot\theta}.$$

$$\text{i.e., } \sin\theta.\operatorname{cosec}\theta = 1, \qquad \cos\theta.\sec\theta = 1, \qquad \tan\theta.\cot\theta = 1.$$

### *iii)* Limit of values

Since hypotenuse is the greatest side in a right-angled triangle, (i.e., the distance $r$ of any point $P(x, y)$ on the terminal arm from the origin is numerically greater than the x-coordinate or y-coordinate)

$\sin\theta = \frac{p}{h} = \frac{y}{r}$ and $\cos\theta = \frac{b}{h} = \frac{x}{r}$ are both *less than* 1.

or, $-1 \le \sin\theta \le 1$ and $-1 \le \cos\theta \le 1$.

### *iv)* Quadrant rule of signs

In the first quadrant (i.e., the given angle is positive and acute), both x-coordinate, y-coordinate and the distance $r$ of the given point $P(x, y)$ from the origin are all positive. Hence, the six trigonometric functions are all positive.

In the second quadrant (i.e., the given angle is positive and obtuse), $y > 0$, $r > 0$ but $x < 0$. Hence,

$$\sin\theta = \frac{y}{r} > 0,\ \cos\theta = \frac{x}{r} < 0 \text{ and } \tan\theta = \frac{y}{x} < 0.$$

Similar consideration gives rise to the following table showing the signs of the various trigonometric functions in the four quadrants:

| Function | Quadrant | | | |
|---|---|---|---|---|
| | I | II | III | IV |
| sin θ | + | + | – | – |
| cos θ | + | – | – | + |
| tan θ | + | – | + | – |

*v) Pythagorian identities*

In any right-angled triangle, using the fact that *the sum of the squares on the two sides forming the right angle is equal to the square on the hypotenuse*, an unknown side can be found whenever two other sides are given.

If P($x$, $y$) denotes any point on the terminal arm of an angle θ placed in the standard position and $r$ its distance from the origin, then by *Pythagorian* theorem, we have

$$h^2 = b^2 + p^2 \qquad \text{or,} \quad r^2 = x^2 + y^2 \qquad (1)$$

a) Dividing both sides by $h^2$ or $r^2$, we get

$$\frac{h^2}{h^2} = \frac{b^2}{h^2} + \frac{p^2}{h^2} \qquad \text{or} \quad \frac{r^2}{r^2} = \frac{x^2}{r^2} + \frac{y^2}{r^2}$$

i.e., $1 = \left(\frac{b}{h}\right)^2 + \left(\frac{p}{h}\right)^2$ or $1 = \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2$

So, $1 = \cos^2\theta + \sin^2\theta$,

i.e. $\sin^2\theta + \cos^2\theta = 1$. (2)

b) Proceeding similarly, we can arrive at

$$\sec^2\theta = 1 + \tan^2\theta \qquad (3)$$

$$\text{cosec}^2\theta = 1 + \cot^2\theta \qquad (4)$$

From the above three basic identities: (2), (3) and (4), we can easily obtain the following derived identities:

$$\sin^2\theta = 1 - \cos^2\theta, \qquad (5)$$

$$\cos^2\theta = 1 - \sin^2\theta, \qquad (6)$$

$$\sec^2\theta - \tan^2\theta = 1, \qquad (7)$$

$$\text{cosec}^2\theta - \cot^2\theta = 1, \qquad (8)$$

$$\sin\theta = \sqrt{1 - \cos^2\theta} \qquad (9)$$

and $\cos\theta = \sqrt{1 - \sin^2\theta}$ (10)

## c) Addition and subtraction formulae:

The formulae that are popularly known as the **addition** (or **sum**) and **subtraction** (or **difference**) **formulae** are:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B,$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B,$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B,$$

and $\cos(A - B) = \cos A \cos B + \sin A \sin B$.

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

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$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\cot(A+B) = \frac{\cot A \cot B - 1}{\cot B + \cot A}$$

$$\cot(A-B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$$

**d) Some trigonometrical formulae deduced from the addition and subtraction formulae**

1. Put $A = 0°$, then

$$\sin(-B) = -\sin B$$
$$\cos(-B) = \cos B$$
$$\tan(-B) = -\tan B$$

2. Put $A = 90°$, then

$$\sin(90° - B) = \cos B$$
$$\cos(90° - B) = \sin B$$
$$\tan(90° - B) = \cot B$$

Also, $\sin(90° + B) = \cos B$

$$\cos(90° + B) = -\sin B$$
$$\tan(90° + B) = -\cot B$$

3. Put $A = 180°$, then

$$\sin(180° - B) = \sin B$$
$$\cos(180° - B) = -\cos B$$
$$\tan(180° - B) = -\tan B$$

Also, $\sin(180° + B) = -\sin B$

$$\cos(180° + B) = \cos B$$
$$\tan(180° + B) = -\tan B$$

**e) Multiple and Sub-multiple angle formulae**

1. **Double angle formula**

Put $B = A$ in addition formula. Then

$$\sin 2A = 2 \sin A \cos B$$

$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= 2\cos^2 A - 1$$
$$= 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

$$\sin 2A = \frac{2 \tan A}{1 + \tan^2 A}$$

and $$\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

Also, $1 - \cos 2A = 2 \sin^2 A$

and $1 + \cos 2A = 2 \cos^2 A$

2. **Half angle formulae:**

Replace A by $\frac{1}{2}$ A in formulae (1)

$$\sin A = 2 \sin \frac{1}{2} A \cos \frac{1}{2} A$$

$$\cos A = \cos^2 \frac{1}{2} A - \sin^2 \frac{1}{2} A$$

$$= 2 \cos^2 \frac{A}{2} - 1$$

$$= 1 - 2 \sin^2 \frac{A}{2}$$

$$\tan A = \frac{2 \tan A/2}{1 + \tan^2 A/2}$$

$$\sin A = \frac{2 \tan A/2}{1 + \tan^2 A/2}$$

$$\cos A = \frac{1 - \tan^2 A/2}{1 + \tan^2 A/2}$$

Also, $1 - \cos A = 2 \sin^2 \frac{A}{2}$

and $1 + \cos A = 2 \cos^2 \frac{A}{2}$

3. **Triple-angle formula**

Put B = 2A in addition formulae.

$$\sin 3A = 3 \sin A - 4 \sin^3 A$$

$$\cos 3A = 4 \cos^3 A - 3 \cos A$$

$$\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$$

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### f) Transformation Formulae

We have the following formulae with the sum and the difference of the addition and subtraction formulae:

$$\sin (A + B) + \sin (A - B) = 2 \sin A \cos B$$
$$\sin (A + B) - \sin (A - B) = 2 \cos A \sin B$$
$$\cos (A + B) + \cos (A - B) = 2 \cos A \cos B$$
$$\cos (A - B) - \cos (A + B) = 2 \sin A \sin B$$

Also, by the replacement of A + B and A − B by C and D respectively, we have the following formulae:

$$\sin C + \sin D = 2 \sin \frac{1}{2} (C + D) \cos \frac{1}{2} (C - D)$$
$$\sin C - \sin D = 2 \cos \frac{1}{2} (C + D) \sin \frac{1}{2} (C - D)$$
$$\cos C + \cos D = 2 \cos \frac{1}{2} (C + D) \cos \frac{1}{2} (C - D)$$
$$\cos C - \cos D = 2 \sin \frac{1}{2} (C + D) \sin \frac{1}{2} (D - C)$$

### g) Conditional Identities

Trigonometrical identities involving certain conditions are known as conditional identities. Generally, the condition will be $A + B + C = \pi$ where A, B and C are the angles of a triangle. Let us see the following results which often need in conditional identities.

If $A + B + C = \pi$, then $A + B = \pi - C$

and $\sin (A + B) = \sin (\pi - C) = \sin C$

$\cos (A + B) = \cos (\pi - C) = - \cos C$

$\tan (A + B) = \tan (\pi - C) = - \tan C$

If $A + B + C = \frac{\pi}{2}$, then $\frac{1}{2} (A + B) = \frac{\pi}{2} - \frac{C}{2}$

and $\sin \frac{1}{2} (A + B) = \sin \left(\frac{\pi}{2} - \frac{C}{2}\right) = \cos \frac{C}{2}$

$\cos \frac{1}{2} (A + B) = \cos \left(\frac{\pi}{2} - \frac{C}{2}\right) = \sin \frac{C}{2}$

$\tan \frac{1}{2} (A + B) = \tan \left(\frac{\pi}{2} - \frac{C}{2}\right) = \cot \frac{C}{2}$ etc.

## Worked out examples

***Example 1***

Prove that

a) $\dfrac{\sec A + \tan A - 1}{\tan A - \sec A + 1} = \dfrac{1 + \sin A}{\cos A}$

b) $\dfrac{1}{\sin A} - \dfrac{1}{\operatorname{cosec} A - \cot A} = \dfrac{1}{\operatorname{cosec} A + \cot A} - \dfrac{1}{\sin A}$

***Solutions:***

a) L.S. $= \dfrac{\sec A + \tan A - 1}{\tan A - \sec A + 1}$

$= \dfrac{\sec A + \tan A - (\sec^2 A - \tan^2 A)}{\tan A - \sec A + 1}$

$= \dfrac{(\sec A + \tan A)(1 - \sec A + \tan A)}{\tan A - \sec A + 1}$

$= \sec A + \tan A$

$= \dfrac{1}{\cos A} + \dfrac{\sin A}{\cos A}$

$= \dfrac{1 + \sin A}{\cos A} =$ R.S.

b) L.S. $= \dfrac{1}{\sin A} - \dfrac{1}{\operatorname{cosec} A - \cot A}$

$= \operatorname{cosec} A - \dfrac{\operatorname{cosec}^2 A - \cot^2 A}{\operatorname{cosec} A - \cot A}$

$= \operatorname{cosec} A - (\operatorname{cosec} A + \cot A)$

$= \operatorname{cosec} A - \cot A - \operatorname{cosec} A$

$= \dfrac{\operatorname{cosec} A - \cot A}{\operatorname{cosec}^2 A - \cot^2 A} - \operatorname{cosec} A$ (Why?)

$= \dfrac{1}{\operatorname{cosec} A + \cot A} - \dfrac{1}{\sin A}$ (Why?)

$=$ R.S.

***Example 2***

Prove that

$\sin A \sin (B - C) + \sin B \sin (C - A) + \sin C \sin (A - B) = 0.$

***Solution.***

First term in the left-side

$= \sin A (\sin B \cos C - \cos B \sin C)$

$= \sin A \sin B \cos C - \sin A \sin C \cos B$

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Second term in the left-side

$= \sin B (\sin C \cos A - \cos C \sin A)$

$= \sin B \sin C \cos A - \sin B \cos C \sin A.$

Similarly, third term in the left-side

$= \sin A \sin C \cos B - \sin C \cos A \sin B$

Adding the three expressions, we get the right side.

***Example 3***

Find the value of $\sin 18°$.

***Solution.***

Let $\theta = 18°$, then $5\theta = 90°$ and $2\theta = 90° - 3\theta$.

So, $\sin 2\theta = \sin (90° - 3\theta) = \cos 3\theta$.

or $2 \sin \theta \cos \theta = \cos \theta (4 \cos^2\theta - 3)$.

Since $\theta = 18°$, $\cos \theta \neq 0$, dividing both sides by $\cos \theta$, we get

$$2 \sin \theta = 4 (1 - \sin^2\theta) - 3 = 1 - 4 \sin^2 \theta$$

or $$4 \sin^2 \theta + 2 \sin \theta - 1 = 0.$$

$$\therefore \quad \sin \theta = \frac{-2 \pm \sqrt{4 + 16}}{2.4} = \frac{\pm \sqrt{5} - 1}{4}$$

Since $\theta$, being equal to $18°$, lies in the first quadrant; and its sine cannot have a negative value.

Therefore $$\sin \theta = \sin 18° = \frac{\sqrt{5} - 1}{4}.$$

***Example 4***

Show that $\cos 40° \cos 100° \cos 160° = 1/8$.

***Solution.***

Left-side $= \cos 40° \cos 100° \cos 160°$

$= \frac{1}{2} \cos 40° (\cos 60° + \cos 260°)$

$= \frac{1}{4} \cos 40° + \frac{1}{2} \cos 40° \cos 260°$

$= \frac{1}{4} \cos 40° + \frac{1}{4} (\cos 220° + \cos 300°)$

$= \frac{1}{4} \cos 40° + \frac{1}{4} \cos (180° + 40°) + \frac{1}{4} \cos (360° - 60°)$

$= \frac{1}{4} \cos 40° - \frac{1}{4} \cos 40° + \frac{1}{4} \cos 60°$

$= 1/8.$

***Example 5***

If $\sin A + \sin B = x$ and $\cos A + \cos B = y$, show that

$$\tan \tfrac{1}{2}(A - B) = \pm\sqrt{\frac{4 - x^2 - y^2}{x^2 + y^2}} \text{ and } \tan \tfrac{1}{2}(A + B) = \frac{x}{y}$$

***Solution.***

Here

$$\sin A + \sin B = 2 \sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B) = x$$

$$\cos A + \cos B = 2 \cos \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B) = y$$

Squaring and adding, we obtain

$$4\left\{\sin^2\tfrac{1}{2}(A + B) + \cos^2\tfrac{1}{2}(A + B)\right\}\cos^2\tfrac{1}{2}(A - B) = x^2 + y^2$$

So, $\quad 4\cos^2\tfrac{1}{2}(A - B) = x^2 + y^2$, and

$$4\sin^2\tfrac{1}{2}(A - B) = 4 - 4\cos^2\tfrac{1}{2}(A - B)$$
$$= 4 - x^2 - y^2$$

Hence $\tan \tfrac{1}{2}(A - B) = \pm\sqrt{\tan^2\tfrac{1}{2}(A - B)}$

$$= \pm\sqrt{\frac{4 - x^2 - y^2}{x^2 + y^2}}.$$

Again, dividing the first by the second, we get

$$\tan \tfrac{1}{2}(A + B) = \frac{x}{y}.$$

***Example 6***

If $A + B + C = \pi$, prove that

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

***Solution.***

Here $B + C = \pi - A$, and so $\tan(B + C) = -\tan A$.

Hence $\dfrac{\tan B + \tan C}{1 - \tan B \tan C} = -\tan A$,

or, $\quad \tan B + \tan C = -\tan A + \tan A \tan B \tan C$

$\therefore \quad \tan A + \tan B + \tan C = \tan A \tan B \tan C.$

***Example 7***

If $A + B + C = \pi$, show that

$$\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \cos B \cos C.$$

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***Solution.***

Here $\cos^2 A + \cos^2 B + \cos^2 C$

$$= \frac{1}{2}(1 + \cos 2A) + \frac{1}{2}(1 + \cos 2B) + \cos^2 C$$

$$= 1 + \frac{1}{2}(\cos 2A + \cos 2B) + \cos^2 C$$

$$= 1 + \frac{1}{2}.\, 2 \cos (A + B) \cos (A - B) + \cos^2 C$$

$$= 1 - [\cos (A - B) + \cos (A + B)] \cos C$$

$$(\because \cos (A + B) = -\cos C)$$

$$= 1 - 2 \cos A \cos B \cos C.$$

Only Q. repeat

## EXERCISE 4.1

1. Prove that

a) $\dfrac{\tan A - \sec A + 1}{\tan A + \sec A - 1} = \dfrac{\cos A}{1 + \sin A}$

b) $\dfrac{1}{\cos A} - \dfrac{1}{\sec A + \tan A} = \dfrac{1}{\sec A - \tan A} - \dfrac{1}{\cos A}$

2. If $\cos A + \sin A = \sqrt{2} \cos A$, show that
$\cos A - \sin A = \sqrt{2} \sin A$

3. Show that

a) $\dfrac{\sin (B - C)}{\sin B \sin C} + \dfrac{\sin (C - A)}{\sin C \sin A} + \dfrac{\sin (A - B)}{\sin A \sin B} = 0$

b) $\cos A + \cos (120° + A) + \cos (120° - A) = 0$

4. a) If $\tan x = k \tan y$, show that
$(k - 1) \sin (x + y) = (k + 1) \sin (x - y)$

b) If $\sin (A + B) = k \sin (A - B)$, prove that
$(k - 1) \cot B = (k + 1) \cot A$.

5. Prove that

a) $\cos 2A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$

b) $\dfrac{1 + \cos 2B}{\sin 2B} = \cot B$

c) $\cos^2 (A - 120°) + \cos^2 A + \cos^2 (A + 120°) = \frac{3}{2}$

6. If $\theta$ and $\phi$ are acute angles and $\cos 2\theta = \dfrac{3\cos 2\phi - 1}{3 - \cos 2\phi}$ show that $\tan\theta = \sqrt{2}\tan\phi$.

7. If $2\tan A = 3\tan B$, prove that $\tan(A - B) = \dfrac{\sin 2B}{5 - \cos 2B}$

8. If $\tan\frac{1}{2}\theta = \sqrt{\dfrac{1-e}{1+e}}\tan\frac{1}{2}\phi$, prove that

$$\cos\phi = \frac{\cos\theta - e}{1 - e\cos\theta}$$

9. Prove that
   a) $8\sin 20^\circ \sin 40^\circ \sin 80^\circ = \sqrt{3}$
   b) $16\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = 1$
   c) $\tan 6^\circ \tan 42^\circ \tan 66^\circ \tan 78^\circ = 1.$

10. If $\sin 2x = 3\sin 2y$, prove that $2\tan(x - y) = \tan(x + y)$.

11. Prove that

$$\left(\frac{\cos A + \cos B}{\sin A - \sin B}\right)^n + \left(\frac{\sin A + \sin B}{\cos A - \cos B}\right)^n = 2\cot^n\tfrac{1}{2}(A - B)$$

or zero according as $n$ is even or odd.

12. If $A + B + C = \pi$, prove that
    a) $\tan\frac{1}{2}B\tan\frac{1}{2}C + \tan\frac{1}{2}C\tan\frac{1}{2}A + \tan\frac{1}{2}A\tan\frac{1}{2}B = 1$
    b) $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$

13. If $A + B + C = \pi$, prove that
    a) $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A\cos B\cos C$
    b) $\sin^2\frac{1}{2}A + \sin^2\frac{1}{2}B + \sin^2\frac{1}{2}C = 1 - 2\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C$

## 4.2 Introduction of Trigonometric Equations

A **trigonometric equation** is an equation involving one or more trigonometric functions of a variable. The equation may be true for one or more values, but not every value of the variable.

By solving a trigonometric equation, we mean to find the angle or angles which satisfy the equation. The values of the variable (angle) are known as the **root(s)** of the equation.

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The simplest type of trigonometric equation is that in which a trigonometric function of a variable (angle) is equal to a constant. The equations $\sin x = \frac{1}{2}$ is such an example. It is satisfied by $x = 30°$, $x = 150°$ and all angles which differ from these by any integral multiple of $360°$, that is, for all integers, the solutions are

$$x = 30° + n.\ 360° \quad \text{and} \quad x = 150° + n.\ 360°.$$

thus there exists an infinite number of roots of the equation $\sin x = \frac{1}{2}$.

The set of all possible solutions of a trigonometric equation form the **general solution** of the equation.

## 4.3 General Solution of the Equations $\sin x = k$, $\cos x = k$ and $\tan x = k$.

Any angle $x$ and each of the angles formed by adding or subtracting any integral multiple of $360°$ to or from the angle $x$, have the same initial and terminal arms, i.e. are *coterminal*. Therefore, any trigonometric function of an angle $x$ has the same value as the same trigonometric function of every angle coterminal with the angle. In particular

$$\sin 30° = \sin (\pm 360° + 30°) = \frac{1}{2} = \sin (n\ 360° + 30°)$$
for $n = 0, \pm 1, \pm 2, \pm 3 \ldots$, etc.

### (a) The General Solution of $\sin x = k$, $(-1 \leq k \leq 1)$.

Let $\theta$ be the angle (preferably the smallest) whose sine is $k$. Then, $\pi - \theta$ is the other angle having the same sine. Since the value of the trigonometric function is unaltered by adding or subtracting an integral multiple of $360°$ to or from $\theta$, we have

$$x = 2\,m\pi + \theta = 2m\pi + (-1)^{2m}\,\theta \qquad (1)$$

and $$x = 2m\pi + \pi - \theta = (2m + 1)\,\pi + (-1)^{2m+1}\,\theta \qquad (2)$$

where $m$ is an integer, as the solution of the equation $\sin x = k$.

Combining (1) and (2), we have $x = n\,\pi + (-1)^n\,\theta$, $\quad n = 0, \pm 1, \pm 2 \cdots$

But $x = n\pi + (-1)^n\,(\pm\theta)$

i.e. $x = n\pi \pm (-1)^n\,\theta$

is same as to write $x = n\pi \pm \theta$.

Because if $n$ is even, $(-1)^n = 1$, so

$$x = n\pi \pm (-1)^n\theta = n\pi \pm \theta$$

and if $n$ is odd, $(-1)^n = -1$, so

$$\begin{aligned} x &= n\pi \pm (-1)^n\theta = n\pi \pm (-1)\theta \\ &= n\pi \pm \theta \end{aligned}$$

$\therefore x = n\pi \pm (-1)^n\theta = n\pi \pm \theta$, for all integral values of $n$.

Cor. 1 If $k = 0$, $\sin x = \sin n\pi$, or $x = n\pi$ for all integers $n$.

Cor. 2 If $\operatorname{cosec} x = k$, $x = n\pi + (-1)^n \theta$, for all integral values of $n$.

Cor. 3 If $\sin x = 1$, $x = 2n\pi + \frac{\pi}{2} = (4n + 1)\frac{\pi}{2}$ for all integers $n$.

Cor. 4 If $\sin x = -1$, $x = 2n\pi - \frac{\pi}{2} = (4n - 1)\frac{\pi}{2}$ for all integers $n$.

**(b) The General Solution of $\cos x = k$, $(-1 \le k \le 1)$.**

Let $\theta$ be a particular angle such that $\cos \theta = k$. Then $-\theta$ is another value of $x$ whose cosine is the same costant $k$. Since all coterminal angles of $\theta$ and $-\theta$ are given by $2n\pi + \theta$ and $2n\pi - \theta$,

$$\cos x = \cos(2n\pi \pm \theta).$$

Hence $\quad x = 2n\pi \pm \theta$, for any integer $n$.

Cor. 1 If $k = 1$, $\cos x = \cos 2n\pi$ and $x = 2n\pi$, for any integer $n$.

Cor. 2 If $\sec x = k$, then $x = 2n\pi \pm \theta$, for any integer $n$.

Cor. 3 If $k = 0$, $\cos x = \cos\frac{\pi}{2}$ and $x = (2n + 1)\frac{\pi}{2}$

Cor. 4 If $k = -1$, $\cos x = \cos \pi$ and $x = (2n + 1)\pi$

**(c) The General Solution of $\tan \theta = k$.**

Let $\theta$ be a particular angle so that $\tan \theta = k$. The other value of $x$ for which $\tan x = k$, is $\pi + \theta$. Thus the coterminal angles are given by

$$2m\pi + \theta \quad \text{and} \quad 2m\pi + \pi + \theta = (2m + 1)\pi + \theta.$$

Also $\tan x = \tan(n\pi + \theta)$, for any integer $n$.

Hence the required general solution is $x = n\pi + \theta$, for any integer $n$.

Cor. 1 If $k = 0$, $x = n\pi$, for any integer $n$.

Cor. 2 If $\cot x = k$, then $x = n\pi + \theta$, for any integer $n$.

## 4.4 Trigonometric Equations in Other Forms

If a given trigonometric equation is not in the simplest form discussed so far, the usual procedure is to derive two or more simple equations which yield all the solutions of the given equation. In order to obtain simple equations from a given equation, the most useful tools are the algebraic operations and trigonometric identities. We give below a few hints that will be of adequate help in solving a trigonometric equation.

(a) Express, if possible, all trigonometric functions in terms of a single trigonometric function of same angle.

(b) Transfer every term to the left.

(c) If quadratic in certain trigonometric function is obtained, use the formula for the solution of a quadratic equation.

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(d) Or factorise the left side, and equate each factor to zero

(e) Use the general values of the last section.

*Remark.* In the process of solving trigonometric equations, algebraic operations such as squaring, cubing, etc. give rise to some additional equations and consequently some additional roots. It is therefore advisable to check which of the roots thus obtained do not satisfy the given equation. Such roots must be discarded. We illustrate this with an example.

Suppose we have to solve the equation

$$\cos x - \sin x = 1 \text{ for } 0° \le x \le 360°.$$

It is usual to write the above equation in the form

$$\cos x = 1 + \sin x .$$

Squaring both sides, we have

$$\cos^2 x = \sin^2 x + 2 \sin x + 1$$

Using $\cos^2 x = 1 - \sin^2 x$ and simplifying, we have

$$2 \sin^2 x + 2 \sin x = 0$$

or, $\sin x (\sin x + 1) = 0.$

Hence, either $\sin x = 0$ or $\sin x = -1$.

That is, $x = 0°$ and $180°$ in the given range if $\sin x = 0$;

and $x = 270°$ in the same range if $\sin x = -1$.

It is easy to see that the roots $x = 0°$ and $x = 270°$ satisfy the given equation where as the value $x = 180°$ does not. Hence $x = 180°$ should be discarded. (Why ?)

## Worked Out Examples

***Example 1.***

Solve $2 \cos^2 x - 5 \cos x + 2 = 0$ for $0° \le x \le 360°$. (T.U. 2048)

***Solution:***

Factoring the left hand side, we get

$$(\cos x - 2)(2 \cos x - 1) = 0.$$

Thus $\cos x - 2 = 0$ and $2 \cos x - 1 = 0$

or $\cos x = 2$ and $\cos x = \frac{1}{2}$

Since $\cos x$ can never be greater than (numerically) 1, $\cos x = 2$ has no solution.

From $\cos x = \frac{1}{2}$, we have $x = 60°$ and $x = 300°$ as the required solution in the given range. (Checking our results is quite simple.)

***Example 2.***

Find all values of $x$ in the interval $0 \le x \le 2\pi$ which satisfy the equation (T.U. 2049)

$$6\cos^2 x + 4\sin^2 x = 5.$$

***Solution:***

The given equation may be written as

$$6\cos^2 x + 4(1 - \cos^2 x) = 5$$

or, $2\cos^2 x = 1$

or, $\cos^2 x = \frac{1}{2}$

Hence $\cos x = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}}$

Clearly, the roots of the simple equation $\cos x = \sqrt{\frac{1}{2}}$ in the given range are $x = 45°$ and $x = 315°$; and those of $\cos x = -\sqrt{\frac{1}{2}}$ in the same range are $x = 135°$ and $x = 225°$ . All these roots satisfy the given equation (Checking is left as exercise).

***Example 3.***

Solve $\cot x + \tan x = 2$.

***Solution***

The given equation is same as

$$\frac{1}{\tan x} + \tan x = 2$$

or, $1 + \tan^2 x = 2\tan x$

or, $\tan^2 x - 2\tan x + 1 = 0$

or, $(\tan x - 1)^2 = 0$

$\therefore$ $\tan x = 1$, (repeated)

$$\tan x = \tan\frac{\pi}{4}$$

Hence $x = n\pi + \frac{1}{4}\pi$, for any integral values of $n$.

Checking is quite simple.

***Example 4.***

Solve : $a\cos x + b\sin x = c$, where $a$, $b$ and $c$ are constants.

***Solution:***

If $a = 0$ or $b = 0$, the equation reduces to simple equation in each case, and the solution can be written down directly. However, we shall assume that

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$a$ and $b$ are not simultaneously (which is obviously the case) equal to zero. For definiteness, let us assume that $a \neq 0$.

We then put

$$\tan \theta = \frac{\text{coefficient of the second term (i.e. } \sin x)}{\text{coefficient of the first term (i.e. } \cos x)} = \frac{b}{a}.$$

Dividing both sides of the given equation by $a$ and using $\tan \theta = \frac{b}{a}$,

we have

$$\cos x + \tan \theta \sin x = \frac{c}{a}$$

or, $\cos x \cos \theta + \sin x \sin \theta = \frac{c}{a} \cos \theta$

or, $\cos (x - \theta) = \frac{c}{\sqrt{a^2 + b^2}}$ since $\cos \theta = \frac{a}{\sqrt{a^2 + b^2}}$

If $c^2 > a^2 + b^2$, no real solution of the equation exists (Why ?)

If $c^2 \leq a^2 + b^2$, we can find an angle $\phi$ such that

$$\cos \phi = \frac{c}{\sqrt{a^2 + b^2}}$$

Therefore $\cos (x - \theta) = \cos \phi$

This gives the following general solutions of the given equation

$$x - \theta = 2n\pi \pm \phi$$

i.e. $x = 2n\pi + \theta \pm \phi$

***Example 5.***

Solve $\sqrt{3} \sin x - \cos x = \sqrt{3}$ for $0 \leq x \leq 2\pi$. (T.U. 051, 054, 055 S)

***Solution.:***

Divide both sides of the equation by

$$\sqrt{(\text{coefficient of } \sin x)^2 + (\text{coefficient of } \cos x)^2}$$

$$= \sqrt{3 + 1} = 2.$$

Then, we have

$$\sin x . \frac{\sqrt{3}}{2} - \cos x . \frac{1}{2} = \frac{\sqrt{3}}{2}$$

$$\sin x \cos 30° - \cos x \sin 30° = \frac{\sqrt{3}}{2} \left(\begin{matrix}\text{using the smallest} \\ \text{positive angle}\end{matrix}\right)$$

$$\sin (x - 30°) = \sin 60°.$$

Hence $x - 30° = n\pi + (-1)^n \frac{\pi}{3}$.

Putting $n = 0$ and $n = 1$,

we get $x = 90°$ and $x = 150°$.

**Example 6.**

Solve the equation $2 \sin 3x - 2 \sin x + 5 \cos 2x = 0.$ (HSEB 2057)

**Solution:**

Since $\sin 3x - \sin x = 2 \cos 2x \sin x$, the given equation reduces to

$$4 \cos 2x \sin x + 5 \cos 2x = 0$$

$$\cos 2x (4 \sin x + 5) = 0$$

Thus $\cos 2x = 0$

or $4 \sin x + 5 = 0.$

Since $\sin x = -\frac{5}{4}$ does not have any solution, the required solution

$$2x = 90°, 270°, 450°, 630°, \text{etc.}$$

or $x = 45°, 135°, 225°, 315°$, etc.

**Example 7.**

Solve $\tan ax = \cot bx.$

**Solution:**

Here $\tan ax = \cot bx = \tan\left(\frac{1}{2}\pi - bx\right)$

Hence $ax = n\pi + \frac{1}{2}\pi - bx$

or $x = \dfrac{2n+1}{a+b} \cdot \dfrac{\pi}{2}.$

**Example 8**

Solve $\cos 3x + \cos x = \cos 2x$

**Solution :**

$\cos 3x + \cos x = \cos 2x$

or, $2 \cos 2x \cos x - \cos 2x = 0$

or, $\cos 2x (2 \cos x - 1) = 0$

Either $\cos 2x = 0$ or, $2 \cos x - 1 = 0$

$2x = (2n+1)\frac{\pi}{2}$ $\cos x = \frac{1}{2}$

$x = (2n+1)\frac{\pi}{4}$ $x = 2n\pi \pm \frac{\pi}{3}$

$\therefore$ $x = (2n+1)\frac{\pi}{4}, 2n\pi \pm \frac{\pi}{3}$

**Example 9**

Solve $\sin^2\theta - 2 \cos\theta + \frac{1}{4} = 0$

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***Solution :***

$$\sin^2\theta - 2\cos\theta + \frac{1}{4} = 0$$

or, $4\sin^2\theta - 8\cos\theta + 1 = 0$

or, $4 - 4\cos^2 ) - 8\cos\theta + 1 = 0$

or, $-4\cos^2\theta - 8\cos\theta + 5 = 0$

or, $4\cos^2\theta + 8\cos\theta - 5 = 0$

or, $(2\cos\theta + 5)(2\cos\theta - 1) = 0$

Either $2\cos\theta + 5 = 0$

$$\cos\theta = -\frac{5}{2}$$

Since $\cos\theta \nless -1$, so

$\cos\theta = -\frac{5}{2}$ is not possible.

or, $2\cos\theta - 1 = 0$

$$\cos\theta = \frac{1}{2}$$

or, $\cos\theta = \cos\frac{\pi}{3}$

$\therefore$ $\theta = 2n\pi \pm \frac{\pi}{3}$

***Example 10***

Solve $2\tan 3x\cos 2x + 1 = \tan 3x + 2\cos 2x$

***Solution :***

$$2\tan 3x\cos 2x + 1 = \tan 3x + 2\cos 2x$$

or, $2\tan 3x\cos 2x + 1 - \tan 3x - 2\cos 2x = 0$

or, $\tan 3x(2\cos 2x - 1) - 1(2\cos 2x - 1) = 0$

or, $(2\cos 2x - 1)(\tan 3x - 1) = 0$

Either $2\cos 2x - 1 = 0$

$$\cos 2x = \frac{1}{2} = \cos\frac{\pi}{3}$$

$$2x = 2n\pi \pm \frac{\pi}{3}$$

$\therefore$ $x = n\pi \pm \frac{\pi}{6}$

or, $\tan 3x - 1 = 0$

$$\tan 3x = 1 = \tan\frac{\pi}{4}$$

$$3x = n\pi + \frac{\pi}{4} = (4n+1)\frac{\pi}{4}$$

$$x = (4n+1)\frac{\pi}{12}$$

$$\therefore\ x = n\pi \pm \frac{\pi}{6},\ (4n+1)\frac{\pi}{12}$$

***Example 11***

Solve $\sin\theta - \sqrt{3}\cos\theta = 2 \qquad (-2\pi < \theta < 2\pi)$

***Solution :***

$$\sin\theta - \sqrt{3}\cos\theta = 2$$

Dividing each term by 2,

$$\frac{1}{2}\sin\theta - \frac{\sqrt{3}}{2}\cos\theta = 1$$

or, $\sin\theta\cos\frac{\pi}{3} - \cos\theta\sin\frac{\pi}{3} = 1$

or, $\sin\left(\theta - \frac{\pi}{3}\right) = 1$

$\therefore \quad \theta - \frac{\pi}{3} = (4n+1)\frac{\pi}{2}$

$\therefore \quad \theta = (4n+1)\frac{\pi}{2} + \frac{\pi}{3}$

When n = 0, $\theta = \frac{\pi}{2} + \frac{\pi}{3} = \frac{5\pi}{6}$

$n = 1, \quad \theta = \frac{5\pi}{2} + \frac{\pi}{3} > 2\pi$

$n = -1, \quad \theta = -\frac{3\pi}{2} + \frac{\pi}{3} = -\frac{7\pi}{6}$

$n = -2, \quad \theta = -\frac{7\pi}{2} + \frac{\pi}{3} = -\frac{19\pi}{6} < -2\pi$

$\therefore$ the values of $\theta$ lying between $-2\pi$ and $2\pi$ are $-\frac{7\pi}{6}, \frac{5\pi}{6}$.

***Example 12***

Solve $\tan\left(\theta + \frac{\pi}{3}\right) + \tan\left(\theta + \frac{2\pi}{3}\right) = 4$

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***Solution :***

$$\tan\left(\theta + \frac{\pi}{3}\right) + \tan\left(\theta + \frac{2\pi}{3}\right) = 4$$

or, $$\frac{\tan\theta + \tan\pi/3}{1 - \tan\theta\tan\pi/3} + \frac{\tan\theta + \tan 2\pi/3}{1 - \tan\theta\tan 2\pi/3} = 4$$

or, $$\frac{\tan\theta + \sqrt{3}}{1 - \sqrt{3}\tan\theta} + \frac{\tan\theta - \sqrt{3}}{1 + \sqrt{3}\tan\theta} = 4$$

or, $$\frac{(\tan\theta+\sqrt{3})(1+\sqrt{3}\tan\theta) + (\tan\theta-\sqrt{3})(1-\sqrt{3}\tan\theta)}{1 - 3\tan^2\theta} = 4$$

or, $$\frac{8\tan\theta}{1 - 3\tan^2\theta} = 4$$

or, $2\tan\theta = 1 - 3\tan^2\theta$

or, $3\tan^2\theta + 2\tan\theta - 1 = 0$

or, $(\tan\theta + 1)(3\tan\theta - 1) = 0$

Either $\tan\theta + 1 = 0$

$$\tan\theta = -1 = \tan\left(-\frac{\pi}{4}\right)$$

$\therefore \quad \theta = n\pi - \frac{\pi}{4}$

or, $3\tan\theta - 1 = 0$

$\tan\theta = \frac{1}{3}$

$\tan\theta = \tan\alpha$ where $\tan\alpha = \frac{1}{3}$

$\therefore \quad \theta = n\pi + \alpha$

$\therefore \quad \theta = n\pi - \frac{\pi}{4},\ n\pi + \alpha$ where $\alpha = \tan^{-1}\frac{1}{3}$

## EXERCISE 4.2

Solve the following equations (Exs. 1 to 10):

1. (a) $4\cos^2 x = 1$ (b) $\cos 2x - \sin x = 0$
   (c) $2\sin 2x = \sin x$ (d) $\sin 2x + \sin x = 0$
2. (a) $\sin^2 x - \cos x = 1$ (b) $\cos^2 x - \sin x + 5 = 0$
   (c) $4\cos x + \sec x - 4 = 0$ (d) $2\sin x + \cot x - \operatorname{cosec} x = 0$.

3. (a) $2\cos^2 x + 4\sin^2 x = 3$ (b) $7\sin^2 x + 3\cos^2 x = 4$.
   (c) $\tan x + \cot x = 2\ \text{cosec}\ x$ (d) $\tan^2 x = \sec x + 1$ (HSEB 2058)
4. (a) $\sin x + \sqrt{3}\cos x = \sqrt{2}$ (T.U. 2049 H) (HSEB 2055)
   (b) $\sqrt{3}\sin x - \cos x = \sqrt{2}$ $(0 \le x \le \pi)$
   (c) $\cos x + \sqrt{3}\sin x = \sqrt{2}$ (T.U. 2053)
   (d) $\sin x + \cos x = \sqrt{2}$ $(-2\pi \le x \le 2\pi)$
5. (a) $\sin 3x + \sin x = 0$ (b) $\sin 3x + \sin x = \sin 2x$
   (c) $\cos 3x - \cos x = 0$ (d) $\tan 2x + \tan x = 0$ $(-\pi/2 \le x \le \pi/2)$
6. (a) $2\cos^2 x + \sin x\cos x - \sin^2 x = 0$
   (b) $\sin 2x - 4\sin x - \cos x + 2 = 0$
7. (a) $\sin 9\theta = \sin\theta$ (b) $\tan 5\theta = \cot 2\theta$
   (c) $\tan 2x - \cot x = 0$ (d) $\tan m\theta + \cot n\theta = 0$
8. (a) $\cos\theta + \cos 2\theta + \cos 3\theta = 0$ (T.U. 2056 S)
   (b) $\cos\theta - \sin 3\theta = \cos 2\theta$
   (c) $\tan\theta + \tan 2\theta = \tan 3\theta$ (T.U. 2057 S)
   (d) $2\sin x\sin 3x = 1$ (T.U. 2052 H)
9. (a) $\cot^2 x - \text{cosec}\ x - 1 = 0$ (b) $2\cos x + 1 = \sin x$
   (c) $\sec x\tan x = \sqrt{2}$ (d) $4\sin^4 x - \cos^2 2x = 0$
10. (a) $\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta = 0$
    (b) $\tan\theta + \tan 2\theta + \tan 3\theta = 0$.
11. Find all the solutions of $\tan\theta - 3\cot\theta = 2\tan 3\theta$ that lie between 0° and 360°.
12. Find the solution of the equations (general solution not required)

$$\tan x + \tan y = 2$$
$$2\cos x\cos y = 1$$

13. (i) $\tan\theta + \tan 2\theta + \tan 3\theta = \tan\theta\tan 2\theta\tan 3\theta$ Ag No 8
    (ii) $\tan\theta + \tan 2\theta + \tan\theta.\tan 2\theta = 1$
    (iii) $\tan\left(\frac{\pi}{4} + \theta\right) + \tan\left(\frac{\pi}{4} - \theta\right) = 4$
14. (i) $\sin 2x.\tan x + 1 = \sin 2x + \tan x$ (T.U. 2051 H, 058 S)
    (ii) $2\sin x.\tan x + 1 = \tan x + 2\sin x$ (T.U. 2053 H)
15. (i) $\cos x + \sin x = \cos 2x + \sin 2x$
    (ii) $\cos x - \sin x = \cos\alpha + \sin\alpha$

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### *Answers*

1. (a) $n\pi \pm \frac{\pi}{3}$ (b) $(4n+1)\frac{\pi}{6}, (4n-1)\frac{\pi}{2}$
   (c) $n\pi, 2n\pi \pm (\cos^{-1}\frac{1}{4})$ (d) $n\pi, (6n \pm 2)\frac{\pi}{3}$
2. (a) $(4n \pm 1)\frac{\pi}{2}, (2n \pm 1)\pi$ (b) No solution
   (c) $(6n \pm 1)\frac{\pi}{3}$ (d) $(6n \pm 2)\frac{\pi}{3}$
3. (a) $n\pi \pm \frac{\pi}{4}$ (b) $n\pi \pm \frac{\pi}{6}$
   (c) $2n\pi \pm \frac{\pi}{3}$ (d) $(2n+1)\pi, 2n\pi \pm \frac{\pi}{3}$
4. (a) $2n\pi + \frac{\pi}{6} \pm \frac{\pi}{4}$ (b) $\frac{5\pi}{12}, \frac{11\pi}{12}$
   (c) $2n\pi + \frac{\pi}{3} \pm \frac{\pi}{4}$ (d) $-\frac{7\pi}{4}, \frac{\pi}{4}$
5. (a) $n\pi, (2n \pm 1)\frac{\pi}{2}$ $\left[\text{or } \frac{n\pi}{2}, (4n \pm 1)\frac{\pi}{2}\right]$
   (b) $\frac{n\pi}{2}, (6n \pm 1)\frac{\pi}{3}$ (c) $n\pi, \frac{n\pi}{2}$ (d) $-\frac{\pi}{3}, 0, \frac{\pi}{3}$
6. (a) $n\pi - \frac{\pi}{4}, n\pi + \theta$ where $\theta = \tan^{-1} 2$. (b) $n\pi + (-1)^n\frac{\pi}{6}$
7. (a) $\frac{n\pi}{4}, (4n \pm 1)\frac{\pi}{10}$ (b) $(4n \pm 1)\frac{\pi}{14}$ or $(2n+1)\frac{\pi}{14}$
   (c) $(4n \pm 1)\frac{\pi}{6}$ or $(2n+1)\frac{\pi}{6}$ (d) $(4n \pm 1)\frac{\pi}{2(m-n)}$ or $\frac{(2n+1)\pi}{2(m-n)}$
8. (a) $(4n \pm 1)\frac{\pi}{4}, (6n \pm 2)\frac{\pi}{3}$ (b) $\frac{2n\pi}{3}, (4n+1)\frac{\pi}{4}, (4n-1)\frac{\pi}{2}$
   (c) $n\pi, \frac{n\pi}{2}, \frac{n\pi}{3}$ (d) $(4n \pm 1)\frac{\pi}{4}, (6n \pm 1)\frac{\pi}{6}$
9. (a) $n\pi + (-1)^n\frac{\pi}{6}, (4n-1)\frac{\pi}{2}$
   (b) $(4n+1)\frac{\pi}{2}, 2n\pi + \theta, \theta = \cos^{-1}\left(-\frac{4}{5}\right)$
   (c) $n\pi + (-1)^n\frac{\pi}{4}$ (d) $(6n \pm 1)\frac{\pi}{6}$
10. (a) $(4n \pm 1)\frac{\pi}{2}, (4n \pm 1)\frac{\pi}{4}, (4n \pm 1)\frac{\pi}{8}$
    (b) $\frac{n\pi}{3}, n\pi \pm \theta, \theta = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$
11. $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$
12. $\frac{\pi}{4}, \frac{\pi}{4}$ 13. (i) $\frac{n\pi}{3}$ (ii) $(4n+1)\frac{\pi}{12}$ (iii) $n\pi \pm \frac{\pi}{6}$
14. (i) $n\pi + \frac{\pi}{4}$ (ii) $n\pi + \frac{\pi}{4}, n\pi + (-1)^n\frac{\pi}{6}$
15. (i) $2n\pi, (4n+1)\frac{\pi}{6}$ (ii) $2n\pi - \alpha, 2n\pi - \frac{\pi}{2} + \alpha$

## ADDITIONAL QUESTIONS

1. What does the trigonometrical equation mean ? What is its solution ? What does the general value of the trigonometrical equation mean ?

2. Solve the following equations :
   (i) $\sin 2x + \sin 4x + \sin 6x = 0$
   (ii) $2 \sin^2 x + \sin^2 2x = 2$
   (iii) $2 \sin^2 x + 3 \cos x = 0 \quad (0 < x < 2\pi)$
   (iv) $\sin^2\theta - \cos\theta = 1/4 \quad (0 \le \theta \le 2\pi)$
   (v) $2 \tan\theta - \cot\theta = -1$
   (vi) $\tan^2\theta + (1 - \sqrt{3}) \tan\theta - \sqrt{3} = 0$

3. Solve the following equations :
   (i) $\sin\theta + \cos\theta = \dfrac{1}{\sqrt{2}} \quad (0 \le \theta \le 2\pi)$
   (ii) $\cos\theta - \sin\theta = \dfrac{1}{\sqrt{2}} \quad (-2\pi \le \theta \le 2\pi)$
   (iii) $\tan\theta + \sec\theta = \sqrt{3}$ (T.U. 2050)
   (iv) $\sqrt{2} \sec\theta + \tan\theta = 1$
   (v) $\sqrt{3} \cos\theta + \sin\theta = 1 \quad (-2\pi < \theta < 2\pi)$ (T.U. 2052)
   (vi) $\cos\theta + \sqrt{3} \sin\theta = 2 \quad (-2\pi \le \theta \le 2\pi)$

4. Find the general values of $x$ satisfying the following pair of equations :
   (i) $\sin x = \dfrac{1}{2}, \quad \cos x = -\dfrac{\sqrt{3}}{2}$
   (ii) $\sin x = -\dfrac{1}{\sqrt{2}}, \quad \tan x = 1$
   (iii) $\sin x = -\dfrac{\sqrt{3}}{2}, \quad \cos x = \dfrac{1}{2}$
   (iv) $\operatorname{cosec} x = -\dfrac{2}{\sqrt{3}}, \quad \tan x = -\sqrt{3}$

   (Hint for (iii) : $\sin x = -\frac{\sqrt{3}}{2}$ and $\cos x = \frac{1}{2}$
   Since sine of an angle is negative and cosine of the same angle is positive, so the angle must lie in the fourth quadrant.
   $\therefore \quad x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$ satisfies both equations.
   Since $\sin x$ and $\cos x$ both are the periodic function of $2\pi$, so the required general value is $x = 2n\pi + \frac{5\pi}{3}$ .)

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### Answers

2. (i) $\frac{1}{4}n\pi$, $n\pi \pm \pi/3$ (ii) $n\pi + \pi/2$, $n\pi \pm \pi/4$ (iii) $\frac{2\pi}{3}$, $\frac{4\pi}{3}$

(iv) $\frac{\pi}{3}$, $\frac{5\pi}{3}$ (v) $n\pi - \frac{\pi}{4}$, $n\pi + \alpha$, $\alpha = \tan^{-1}\frac{1}{2}$

(vi) $n\pi + \frac{\pi}{3}$, $n\pi - \frac{\pi}{4}$

3. (i) $\frac{7\pi}{12}$, $\frac{23\pi}{12}$ (ii) $-\frac{23\pi}{12}$, $-\frac{7\pi}{12}$, $\frac{\pi}{12}$, $\frac{17\pi}{12}$ (iii) $2n\pi \pm \frac{\pi}{3} - \frac{\pi}{6}$

(iv) $2n\pi - \frac{\pi}{4}$ (v) $-\frac{3\pi}{2}$, $-\frac{\pi}{6}$, $\frac{\pi}{2}$, $\frac{11\pi}{6}$ (vi) $-\frac{5\pi}{3}$, $\frac{\pi}{3}$

4. (i) $2n\pi + \frac{5\pi}{6}$ (ii) $2n\pi + \frac{5\pi}{4}$ (iv) $2n\pi + \frac{5\pi}{3}$

# CHAPTER 5

# Inverse Circular Functions

## 5.1 Introduction

Trigonometric functions defined by

$$y = \sin x, \qquad y = \cos x, \qquad y = \tan x. \text{ etc.,}$$

are known as the sine, cosine, tangent, etc. functions respectively. Here each $x$ corresponds to a unique $y$. Expressed as a set of ordered pairs $(x, y)$ of real numbers, each of them would look like

$$f = \{ (x, y) : y = \sin x\}, \qquad g = \{ (x, y) : y = \cos x\},$$
$$h = \{ (x, y) : y = \tan x\}, \text{ etc.}$$

On interchanging the roles of $x$ and $y$, they will look like

$$F = \{ (x, y) : x = \sin y \}, \qquad G = \{ (x, y) : x = \cos y \},$$
$$H = \{ (x, y) : x = \tan y \}, \text{ etc.}$$

In each of these cases, each $x$ corresponds to more than one $y$. For instance, if $x = \frac{1}{2}$, then $y$ may be 30°, 150°, 360° + 30°, etc. so, none of them represents a function. But, on suitably restricting the values of $y$ in these cases (or the values of $x$ in the former cases), we can get only one value of $y$ corresponding to each value of $x$. We then get well-defined functions. In the next section, we make this point more clear with special reference to the sine function.

## 5.2 Inverse Circular Functions

The circular function defined by $y = \sin x$, i.e.,

$$f = \{ (x, y): y = \sin x \}$$

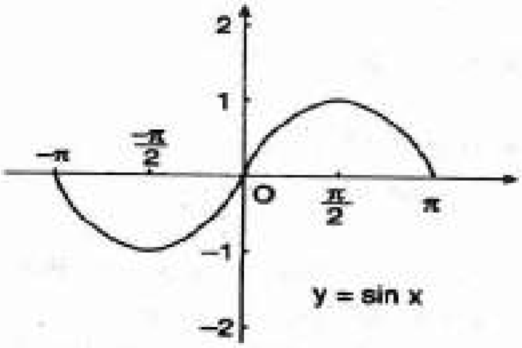
is the sine function. Here the domain of the function $f$ is the set of real numbers, and its range is the set of real numbers between $-1$ and $1$ inclusive. We thus have

$$\text{function} = f$$
$$\text{domain of } f = \{x : x \in \mathbb{R}\}$$
$$\text{range of } f = \{y : -1 \le y \le 1 \}.$$

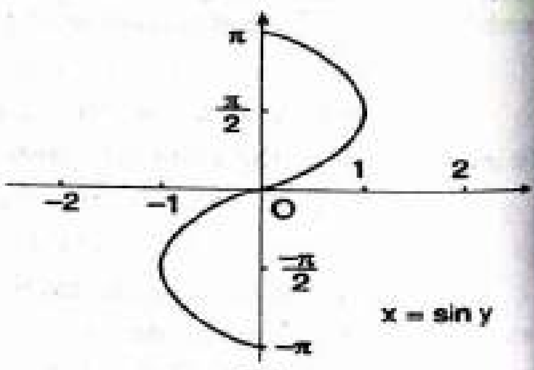
Interchanging the roles of $x$ and $y$, we have

$$F = \{ (x, y) : x = \sin y \}.$$

In this case, the domain of F is the set of real numbers between -1 and 1 inclusive; and its range is the set of all real numbers, several of which may correspond to one $x$. This can be seen more clearly from the graphs of $y = \sin x$ and $x = \sin y$. The graphs of $y = \sin x$ winds along the x-axis (Fig. a) and that of $x = \sin y$ (is of the same form but winds along the y-axis (Fig. b). In fact, one is the mirror image of the other (i.e. reflection) on the line $y = x$ . Note that a horizontal line lying between $y = -1$ and $y = 1$ intersect the graph of $y = \sin x$ at several points. This shows that there are many values of $x$ which correspond to a certain value of $y$. But one can also see that the length of the longest interval on the x-axis for which a horizontal line can intersect the curve $y = \sin x$ in at most one point in $\pi$. For convenience, this interval (in fact, the domain of the function) is chosen to be an interval $-\frac{1}{2}\pi \le y \le \frac{1}{2}\pi$ (the heavy part of the graph). Moreover, this choice gives a full range of values (i.e. $-1 \le y \le 1$) of the function defined by $y = \sin x$.




(a)




(b)

On other hand, a vertical line lying between $x = -1$ and $x = 1$ intersects the graph of $x = \sin y$ at more than one point. This shows that $x = \sin y$, as it stands, does not define a function. But, on restricting the value of $y$ so that $-\frac{1}{2}\pi \le y \le \frac{1}{2}\pi$ (i.e. , restricting to the solid part of the curve only), each value of $x$ between $-1$ and 1 inclusive corresponds to exactly one value of $y$. To indicate this restriction in $x = \sin y$, it is customary to write (capitalise s in sin)

$$x = \sin y$$

and the value of $y$ so obtained is called the principal value (p.v.).

Other commonly used notations for this are

$$y = \sin^{-1} x, \text{ (read 'inverse sine of } x\text{')}$$

and $y = \text{Arc} \sin x$, (from the Latin phrase *arcs cuisus sinus x est* –the arc whose sine is $x$)

Note that $x = \sin y$, $y = \text{arc} \sin x$ and $y = \sin^{-1} x$ are three equivalent notations for the same thing. Also, it should be remembered that

$$\sin^{-1}x \text{ NEVER MEANS } \frac{1}{\sin x}.$$

In short, we have the following working definitions of the inverse sine, the inverse cosine, the inverse tangent, the inverse cotangent, the inverse secant and the inverse cosecant being omitted as they are of less common use.

(a) The *inverse sine function* is defined by

$$\sin^{-1} x = y \text{ or equivalently } x = \sin y$$

provided that $-1 \leq x \leq 1$ and $-\frac{1}{2}\pi \leq y \leq \frac{1}{2}\pi$.

(b) The *inverse cosine function* is defined by

$$\cos^{-1} x = y \text{ or equivalently } x = \cos y$$

provided that $-1 \leq x \leq 1$ and $0 \leq y \leq \pi$.

(c) The *inverse tangent function* is defined by

$$\tan^{-1} x = y \text{ or equivalently } x = \tan y$$

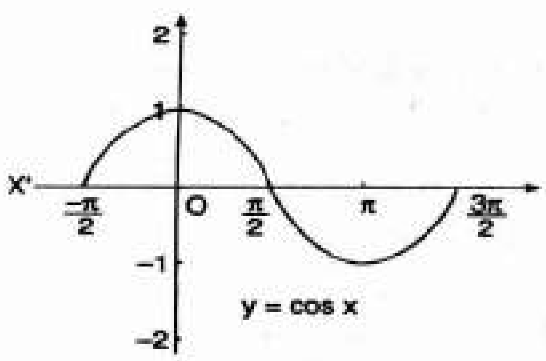
provided that $-\infty < x < \infty$ and $-\frac{1}{2}\pi < y < \frac{1}{2}\pi$.

and (d) The inverse cotangent function is defined by

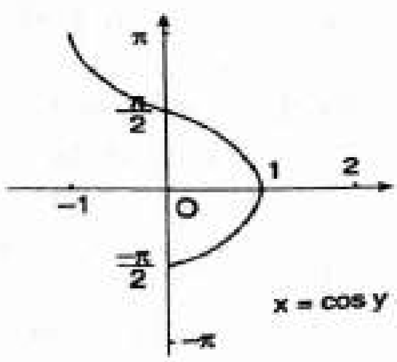
$$\cot^{-1} x = y \text{ or equivalently } x = \cot y$$

provided that $-\infty < x < \infty$ and $0 < y < \pi$.

The graphs of these inverses together with the corresponding original functions are shown in the figures below.




(a)




(b)

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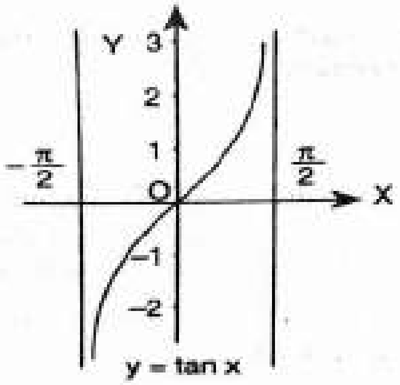
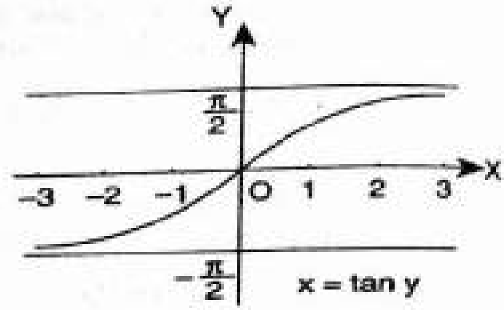



(a)




(b)

For emphasis and convenience, we list here the definitions of principal inverse trigonometric functions, together with corresponding domains and ranges.

| Function | Notation for Principal Inverse Function | Defining Equation | Domain | Range |
|---|---|---|---|---|
| sine | $\sin^{-1}$ | $y = \sin^{-1}x$ | $-1 \le x \le 1$ | $-\frac{1}{2}\pi \le y \le \frac{1}{2}\pi$ |
| cosine | $\cos^{-1}$ | $y = \cos^{-1}x$ | $-1 \le x \le 1$ | $0 \le y \le \pi$ |
| tangent | $\tan^{-1}$ | $y = \tan^{-1}x$ | $-\infty < x < \infty$ | $-\frac{1}{2}\pi < y < \frac{1}{2}\pi$ |
| cotangent | $\cot^{-1}$ | $y = \cot^{-1}x$ | $-\infty < x < \infty$ | $0 < y < \pi$ |

## 5.3 Some Useful Results

We shall now establish some elementary but useful results involving trigonometric and inverse trigonometric functions.

(a) For a given angle $\theta$,

(i) $\theta = \sin^{-1} \sin \theta$ (ii) $\theta = \sin \sin^{-1} \theta$ , etc.

Let $\sin \theta = x$ , then, by definition, $\theta = \sin^{-1}x$.

Hence $\theta = \sin^{-1} \sin \theta$ (since $x = \sin \theta$)

Again, let $y = \sin^{-1} \theta$ , then $\theta = \sin y$, and

$\theta = \sin \sin^{-1} \theta$ (since $y = \sin^{-1} \theta$ )

The rest follows similarly.

(b) For a given numerical value $x$,

(i) $\operatorname{cosec}^{-1} x = \sin^{-1} \frac{1}{x}$ (ii) $\cot^{-1} x = \tan^{-1} \frac{1}{x}$

(iii) $\sin^{-1} x = \operatorname{cosec}^{-1} \frac{1}{x}$, etc.

We shall prove the first one only and the rest will be left as exercises;

Let $\theta = \operatorname{cosec}^{-1} x$, then, by definition, $\operatorname{cosec} \theta = x$ and $\sin \theta = \frac{1}{x}$,

and hence

$$\theta = \sin^{-1} \frac{1}{x}.$$

Thus $\quad \operatorname{cosec}^{-1} x = \sin^{-1} \frac{1}{x}$.

(c) Expressions of a given inverse trigonometric function in terms of the remaining inverse trigonometric functions.

A few simple cases of the results that we are going to discuss here has already been given in (b) above. We shall consider a few similar but a bit more general relations.

(i) $\sin^{-1} x = \cos^{-1} \sqrt{1 - x^2}$ (ii) $\sin^{-1} x = \tan^{-1} \frac{x}{\sqrt{1 - x^2}}$

(HSEB 2050)

(iii) $\sin^{-1} x = \cot^{-1} \frac{\sqrt{1 - x^2}}{x}$ (iv) $\sin^{-1} x = \sec^{-1} \frac{1}{\sqrt{1 - x^2}}$.

To prove these, we put $\theta = \sin^{-1} x$ then $\sin \theta = x$, and

(i) $\cos \theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - x^2}$, i.e. $\theta = \cos^{-1} \sqrt{1 - x^2}$.

(ii) $\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{x}{\sqrt{1 - x^2}}$, i.e. $\theta = \tan^{-1} \frac{x}{\sqrt{1 - x^2}}$

The remaining two also follow similarly, and are left as exercises. Combining the above results, we find that

$$\sin^{-1} x = \cos^{-1} \sqrt{1 - x^2} = \tan^{-1} \frac{x}{\sqrt{1 - x^2}} = \cot^{-1} \frac{\sqrt{1 - x^2}}{x}$$

$$= \sec^{-1} \frac{1}{\sqrt{1 - x^2}} = \operatorname{cosec}^{-1} \frac{1}{x}.$$

(d) For a given numerical value $x$

(i) $\sin^{-1} x + \cos^{-1} x = \frac{1}{2} \pi$

(ii) $\tan^{-1} x + \cot^{-1} x = \frac{1}{2} \pi$

(iii) $\operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{1}{2} \pi$ .

Suppose $\sin^{-1} x = \theta$, then $x = \sin \theta$, and so $\cos (\frac{1}{2} \pi - \theta) = x$

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This gives $\cos^{-1}x = \frac{\pi}{2} - \theta$; and hence

$$\sin^{-1}x + \cos^{-1}x = \frac{1}{2}\pi.$$

The other results also follow similarly.

(e) For a given numerical value $x$,

i) $\sin^{-1}(-x) = -\sin^{-1}x$ ii) $\cos^{-1}(-x) = \pi - \cos^{-1}x$

iii) $\tan^{-1}(-x) = -\tan^{-1}x$ iv) $\cot^{-1}(-x) = \pi - \cot^{-1}x$

Let $\sin^{-1}x = \theta$, then $x = \sin\theta$

So, $-x = -\sin\theta$ or, $-x = \sin(-\theta)$ or, $\sin^{-1}(-x) = -\theta$

$$\therefore \sin^{-1}(-x) = -\sin^{-1}x$$

Again, if $\cos^{-1}x = \theta$, then $x = \cos\theta$

So, $-x = -\cos\theta$ or, $-x = \cos(\pi - \theta)$ or, $\cos^{-1}(-x) = \pi - \theta$

$$\therefore \cos^{-1}(-x) = \pi - \cos^{-1}x.$$

The other results follow similarly.

We shall now illustrate the procedure of handling with problem involving inverse trigonometric functions.

## Worked Out Examples

***Example 1***

Use Trigonometric table if necessary to evaluate

(a) $\sin^{-1}(-1)$ (b) $\sin(\cos^{-1}\frac{2}{3})$ (c) Arc tan (tan 60°)

***Solution.***

Using trigonometric tables, we have

(a) $\sin^{-1}(-1) = -\frac{1}{2}\pi$ and

(b) Let $\cos^{-1}\frac{2}{3} = x$, then

$\cos x = \frac{2}{3}$ and $\sin(\cos^{-1}\frac{2}{3}) = \sin x = \sqrt{1 - \cos^2 x}$

$$= \sqrt{1 - \frac{4}{9}} = \frac{\sqrt{5}}{3}$$

(c) Arc tan (tan 60°) = 60°.

***Example 2***

Find the value of $\cos\left(\sin^{-1}\frac{1}{4} + \cos^{-1}\frac{1}{2}\right)$

**Solution.**

Let $x = \sin^{-1}\frac{1}{4}$ and $y = \cos^{-1}\frac{1}{2}$, then

$\sin x = \frac{1}{4}$ and $\cos y = \frac{1}{2}$, and hence

$\cos x = \sqrt{1 - \frac{1}{16}} = \sqrt{\frac{15}{16}}$ and $\sin y = \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}}$.

Therefore, $\cos\left(\sin^{-1}\frac{1}{4} + \cos^{-1}\frac{1}{2}\right)$

$$= \cos(x + y)$$
$$= \cos x \cos y - \sin x \sin y$$
$$= \frac{\sqrt{15}}{4} \cdot \frac{1}{2} - \frac{1}{4} \cdot \frac{\sqrt{3}}{2}$$
$$= \frac{\sqrt{15} - \sqrt{3}}{8}$$

**Example 3**

Prove that $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x + y}{1 - xy}$.

**Solution.**

Let $\tan^{-1} x = A$ and $\tan^{-1} y = B$, then

$x = \tan A$ and $y = \tan B$

Now, $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$$= \frac{x + y}{1 - xy}$$

Hence $A + B = \tan^{-1} \frac{x + y}{1 - xy}$.

i.e. $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x + y}{1 - xy}$

**Example 4**

Show that $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1} \{x\sqrt{1 - y^2} \pm y\sqrt{1 - x^2}\}$

**Solution.**

Let $\sin^{-1} x = A$ and $\sin^{-1} y = B$, then

$\sin A = x$ and $\sin B = y$ and

$\cos A = \sqrt{1 - x^2}$ and $\cos B = \sqrt{1 - y^2}$

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Now, since $\sin(A + B) = \sin A \cos B + \cos A \sin B$

$$= x\sqrt{1-y^2} + y\sqrt{1-x^2}\}.$$

$$A + B = \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}$$

i.e. $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}$

The second case follows similarly.

***Example 5***

Prove that

$$2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2} = \cos^{-1}\frac{1-x^2}{1+x^2} = \tan^{-1}\frac{2x}{1-x^2}.$$

***Solution.***

Let $x = \tan\theta$, then $2\tan^{-1}x = 2\tan^{-1}\tan\theta = 2\theta$,

$$\sin^{-1}\frac{2x}{1+x^2} = \sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta} = \sin^{-1}(\sin 2\theta) = 2\theta,$$

$$\cos^{-1}\frac{1-x^2}{1+x^2} = \cos^{-1}\frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos^{-1}(\cos 2\theta) = 2\theta$$

$$\tan^{-1}\frac{2x}{1-x^2} = \tan^{-1}\frac{2\tan\theta}{1-\tan^2\theta} = \tan^{-1}(\tan 2\theta) = 2\theta.$$

Combining the above results, we get the required result.

***Example 6***

Solve $\sin^{-1}\frac{2a}{1+a^2} + \sin^{-1}\frac{2b}{1+b^2} = 2\tan^{-1}x$ (HSEB 2056)

***Solution.***

Let $a = \tan\theta$ $\therefore$ $\theta = \tan^{-1}a$

Then, $\sin^{-1}\frac{2a}{1+a^2} = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$

$$= \sin^{-1}(\sin 2\theta)$$

$$= 2\theta = 2\tan^{-1}a$$

$$\therefore \quad \sin^{-1}\frac{2a}{1+a^2} = 2\tan^{-1}a,$$

similarly $\sin^{-1}\frac{2b}{1+b^2} = 2\tan^{-1}b$

$\therefore$ the given equation becomes

$$2\tan^{-1}a + 2\tan^{-1}b = 2\tan^{-1}x$$

or $\quad \tan^{-1} \dfrac{a+b}{1-ab} = \tan^{-1} x.$

Hence $\quad x = \dfrac{a+b}{1-ab}.$

***Example 7***

Find the value of $\cos \tan^{-1} \sin \cot^{-1} x$. (H.S.E.B. 2057)

***Solution.***

Let $\cot^{-1} x = \theta$, then $\cot \theta = x$ and $\sin \theta = \dfrac{1}{\sqrt{1+x^2}}$

Again, suppose

$$\tan^{-1} \sin \cot^{-1} x = \tan^{-1} \sin \theta = \phi$$

then $\quad \tan \phi = \sin \cot^{-1} x = \sin \theta = \dfrac{1}{\sqrt{1+x^2}}.$

Hence $\cos \tan^{-1} \sin \cot^{-1} x = \cos \phi$

$$= \frac{1}{\sqrt{1+\tan^2\phi}} = \frac{1}{\sqrt{1+\dfrac{1}{1+x^2}}}$$

$$= \sqrt{\frac{1+x^2}{2+x^2}}$$

***Example 8***

Prove that $\cos (3 \cos^{-1}x) = 4x^3 - 3x$

***Solution :***

Let $\cos^{-1}x = \theta$ then $x = \cos \theta$

Now, $\cos (3 \cos^{-1}x) = \cos 3\theta$

$$= 4 \cos^3\theta - 3 \cos \theta$$

$$= 4x^3 - 3x$$

***Example 9***

Prove that $\quad \tan^{-1} \left(\dfrac{1+\cos x}{\sin x}\right) = \dfrac{\pi}{2} - \dfrac{x}{2}$

***Solution :***

$$\tan^{-1} \left(\frac{1+\cos x}{\sin x}\right) = \tan^{-1} \left(\frac{2 \cos^2 x/2}{2 \sin x/2 \cos x/2}\right)$$

$$= \tan^{-1} (\cot x/2)$$

$$= \tan^{-1} (\tan (\pi/2 - x/2))$$

$$= \frac{\pi}{2} - \frac{x}{2}$$

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***Example 10***

Prove that $\cot^{-1}x - \cot^{-1}z = \cot^{-1}\frac{xy+1}{y-x} + \cot^{-1}\frac{yz+1}{z-y}$

***Solution :***

$$\cot^{-1}x - \cot^{-1}z = \cot^{-1}x - \cot^{-1}y + \cot^{-1}y - \cot^{-1}z$$
$$= \cot^{-1}\frac{xy+1}{y-x} + \cot^{-1}\frac{yz+1}{z-y}$$

***Example 11***

Prove that $\sin^{-1}\frac{4}{5} + 2\tan^{-1}\frac{1}{3} = \frac{\pi}{2}$

***Solution :***

Since, $\sin^{-1}x = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$, so

$$\sin^{-1}\frac{4}{5} = \tan^{-1}\frac{4/5}{\sqrt{1-\left(\frac{4}{5}\right)^2}} = \tan^{-1}\frac{4}{3}$$

Again since, $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$, so

$$2\tan^{-1}\frac{1}{3} = \tan^{-1}\frac{2\times 1/3}{1-\left(\frac{1}{3}\right)^2} = \tan^{-1}\frac{3}{4}$$

Now, $\sin^{-1}\frac{4}{5} + 2\tan^{-1}\frac{1}{3} = \tan^{-1}\frac{4}{3} + \tan^{-1}\frac{3}{4}$

$$= \tan^{-1}\frac{4}{3} + \cot^{-1}\frac{4}{3} \quad \left(\because \tan^{-1}x = \cot^{-1}\frac{1}{x}\right)$$
$$= \frac{\pi}{2} \quad \left(\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\right)$$

***Example 12***

If $\cos^{-1}x + \cos^{-1}y = \frac{\pi}{2}$, prove that $x^2 + y^2 = 1$

***Solution :***

$\cos^{-1}x + \cos^{-1}y = \frac{\pi}{2}$

or, $\cos^{-1}x = \frac{\pi}{2} - \cos^{-1}y$

or, $\cos^{-1}x = \sin^{-1}y$

or, $\cos^{-1}x = \cos^{-1}\sqrt{1-y^2}$

or, $x = \sqrt{1 - y^2}$

$x^2 + y^2 = 1$

## EXERCISE 5.1

1. Evaluate the following without using tables:

(a) $\sin^{-1} 1$ (b) $\sin^{-1} (-\frac{1}{2})$. (c) $\cos^{-1} \left(-\frac{\sqrt{3}}{2}\right)$

(d) $\tan^{-1} 1$ (e) Arccot $(-1)$ (f) Arctan $\left(-\frac{1}{\sqrt{3}}\right)$

2. Express each of the following in terms of $x$:

(a) $\cos \tan^{-1} x$ (b) $\sin (\cot^{-1}x)$ (c) $\tan$ (Arccot $x$)

(d) $\cos \sin^{-1} x$ (e) $\tan (2 \tan^{-1} x)$ (f) $\sin (2 \tan^{-1} x)$

(g) $\cos (2 \cot^{-1} x)$ (h) cot (2 Arccot $x$)

3. Evaluate each of the following, using tables if necessary;

(a) $\sin \left(\cos^{-1} \frac{3}{5}\right)$ (b) cos (Arccos $\frac{2}{3}$) (c) Arctan $\left(\tan \frac{\pi}{6}\right)$

(d) $\sin \left(\tan^{-1} \frac{3}{4}\right)$ (e) $\sin \left(2 \cos^{-1} \frac{1}{2}\right)$ (f) $\sin^{-1} \left(2 \cos \frac{\pi}{3}\right)$

4. Prove each of the following:

(a) $2 \cos^{-1}x = \cos^{-1}(2x^2 - 1)$ (b) $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$

(c) $3 \tan^{-1}x = \tan^{-1}\frac{3x - x^3}{1 - 3x^2}$ (d) sin (Arccos $t$) = cos (Arcsin $t$)

(e) cos (2 Arccos $t$) = $2t^2 - 1$

(f) $\sin (2 \sin^{-1} x) = 2x \sqrt{1 - x^2}$ (HSEB 2057)

(g) $\cos (\sin^{-1} u + \cos^{-1} v) = v\sqrt{1 - u^2} - u\sqrt{1 - v^2}$ (T.U. 2049)

(h) $\tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{6}{17}$

(i) $\tan^{-1} a - \tan^{-1} c = \tan^{-1} \frac{a - b}{1 + ab} + \tan^{-1} \frac{b - c}{1 + bc}$

(j) tan (Arctan $u$ – Arctan $v$) $= \frac{u - v}{1 + uv}$

(k) $\tan (2 \tan^{-1} x) = 2 \tan (\tan^{-1} x + \tan^{-1} x^3)$ (HSEB 2052)

(l) $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1} \frac{x + y + z - xyz}{1 - yz - zx - xy}$ (HSEB 2055)

(m) $\sin^{-1} \frac{4}{5} + \sin^{-1} \frac{5}{13} + \sin^{-1} \frac{16}{65} = \frac{1}{2} \pi$ (HSEB 2053)

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(n) Prove that $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\sin^{-1}\left(\frac{2\sqrt{x}}{1+x}\right)$ (T.U. 2055 S, HSEB 2054)

5. Find the value of each of the following:

(a) $\cos\left(\sin^{-1}\frac{4}{5} + \tan^{-1}\frac{5}{12}\right)$ (HSEB 2054, 2055)

(b) $\sin\left(\cos^{-1}\frac{1}{2} + \sin^{-1}\frac{3}{5}\right)$ (T.U. 2053 H)

(c) $\tan\left(\text{Arccos}\,\frac{4}{5} - \text{Arcsin}\,\frac{12}{13}\right)$

(d) $\tan(\tan^{-1}x - \tan^{-1}2y)$ (HSEB 2053)

(e) $\tan^{-1}3 + \tan^{-1}\frac{1}{3}$

(f) $\text{Arcsin}\,t - \text{Arccos}\,(-t)$

6. Solve each of the following equation:

(a) $\cos^{-1}x - \sin^{-1}x = 0$

(b) $\tan^{-1}x - \cot^{-1}x = 0$

(c) $\sin^{-1}\frac{1}{2}x = \cos^{-1}x$ (HSEB 2052)

(d) $\cos^{-1}x = \cos^{-1}\frac{1}{2x}$ (HSEB 2051)

(e) $\tan^{-1}2x = 2\tan^{-1}x$

(f) $\cos^{-1}x + \cos^{-1}2x = \frac{1}{2}\pi$

(g) $\sin^{-1}x + \cos^{-1}(1-x) = \frac{1}{2}\pi$

(h) $\sin^{-1}\frac{2a}{1+a^2} - \cos^{-1}\frac{1-b^2}{1+b^2} = 2\tan^{-1}x$ (HSEB 2053)

(i) $\tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = \tan^{-1}1$

(j) $\tan^{-1}2x + \tan^{-1}3x = \frac{1}{4}\pi$

(k) $3\tan^{-1}\frac{1}{2+\sqrt{3}} - \tan^{-1}\frac{1}{x} = \tan^{-1}\frac{1}{3}$

7. Prove that

(a) $x + y + z = xyz$, if $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$ (T.U. 2050)

(b) $xy + yz + zx = 1$, if $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \frac{\pi}{2}$ (T.U. 2048, 2050)

8. If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$, show that
$$x^2 + y^2 + z^2 + 2xyz = 1.$$ (T.U. 2056 S)

9. Prove that:

(i) $\tan^{-1}\frac{3}{5} + \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{27}{11}$

(ii) $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} = \frac{1}{4}\pi$

(iii) $2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \frac{1}{4}\pi$ (HSEB 2051)

(iv) $4(\cot^{-1}3 + \operatorname{cosec}^{-1}\sqrt{5}) = \pi$ (HSEB 2058) (T.U. 052H, 058S)

(v) $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$

$= 2\left(\tan^{-1}1 + \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}\right)$

(T.U. 2051, 54)

10. Prove that:

(i) $\sin^{-1}\sqrt{\frac{x-b}{a-b}} = \cos^{-1}\sqrt{\frac{a-x}{a-b}} = \tan^{-1}\sqrt{\frac{x-b}{a-x}}$ (T.U. 2057 S)

(ii) $\cos^{-1}x = 2\sin^{-1}\sqrt{\frac{1-x}{2}} = 2\cos^{-1}\sqrt{\frac{1+x}{2}}$

11. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$, prove that

$x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz.$ (T.U. 2052)

12. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{\pi}{2}$, prove that $x^2 + y^2 + z^2 + 2xyz = 1$.

Cos = x
Sin

x Sin = Co
Sin 0 = 1
Co  $\pi$

## Answers

1. (a) $\frac{1}{2}\pi$ (b) $-\pi/6$ (c) $\frac{5\pi}{6}$ (d) $\frac{1}{4}\pi$ (e) $\frac{3\pi}{4}$ (f) $-\frac{\pi}{6}$

2. (a) $1/\sqrt{1+x^2}$ (b) $1/\sqrt{1+x^2}$ (c) $1/x$
(d) $\sqrt{1-x^2}$ (e) $2x/(1-x^2)$ (f) $2x/(1+x^2)$
(g) $\frac{x^2-1}{x^2+1}$ (h) $(x^2-1)/2x$

3. (a) 4/5 (b) 2/3 (c) $\pi/6$
(d) 3/5 (e) $\sqrt{3}/2$ (f) $\frac{1}{2}\pi$

5. (a) 16/65 (b) $\frac{4\sqrt{3}+3}{10}$ (c) $-\frac{33}{56}$
(d) $\frac{x-2y}{1+2xy}$ (e) $\frac{1}{2}\pi$ (f) $-\frac{\pi}{2}$

6. (a) $x = 1/\sqrt{2}$ (b) $x = 1$ (c) $x = 2/\sqrt{5}$
(d) $x = \pm 1/\sqrt{2}$ (e) $x = 0$ (f) $x = \pm\frac{1}{\sqrt{5}}$
(g) $x = \frac{1}{2}$ (h) $x = \frac{a-b}{1+ab}$ (i) $x = \pm 1/\sqrt{2}$
(j) $x = -1$ or $1/6$ (k) $x = 2$

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## ADDITIONAL QUESTIONS

1. What is the domain of the inverse function $\tan^{-1}x$ ? Find $\sin\left(\cos^{-1}\frac{1}{3}\right)$. Prove that $\sin^{-1}x = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$ (HSEB 2050)

2. Explain what the inverse circular function is ? What are the domains and ranges of $\sin^{-1}x$, $\cos^{-1}x$ and $\tan^{-1}x$ so that they are functions of $x$ ?

3. Prove that : $\tan^{-1}\frac{a-b}{1+ab} + \tan^{-1}\frac{b-c}{1+bc} + \tan^{-1}\frac{c-a}{1+ca}$
$$= \tan^{-1}\frac{a^2-b^2}{1+a^2b^2} + \tan^{-1}\frac{b^2-c^2}{1+b^2c^2} + \tan^{-1}\frac{c^2-a^2}{1+c^2a^2}$$

4. Prove that :
   (i) $\cot^{-1}\frac{ab+1}{a-b} + \cot^{-1}\frac{bc+1}{b-c} + \cot^{-1}\frac{ca+1}{c-a} = 0$
   (ii) $\cot^{-1}1 + \cot^{-1}2 + \cot^{-1}3 = \pi/2$
   (iii) $\sec^2(\tan^{-1}2) + \mathrm{cosec}^2(\cot^{-1}3) = 15$
   (iv) $\tan\left(\frac{1}{2}\sin^{-1}\frac{2x}{1+x^2} + \frac{1}{2}\cos^{-1}\frac{1-y^2}{1+y^2}\right) = \frac{x+y}{1-xy}$

5. Prove that
   (i) $\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right) = \frac{\pi}{4} - x$
   (ii) $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right) = \frac{\pi}{4} - \frac{x}{2}$
   (iii) $\tan^{-1}\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}} = \frac{\pi}{4} + \frac{1}{2}\cos^{-1}x^2$

6. (i) If $\sec^{-1}x = \mathrm{cosec}^{-1}y$, show that $\frac{1}{x^2} + \frac{1}{y^2} = 1$
   (ii) If $\cos^{-1}x + \cos^{-1}y = \theta$, show that
   $$x^2 - 2xy\cos\theta + y^2 = \sin^2\theta$$
   (iii) If $\tan^{-1}\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}} = \alpha$, prove that $x^2 = \sin 2\alpha$.

**Answers**

1. $\{x : -\infty < x < \infty\}$, $2\sqrt{2}/3$
2. $\sin^{-1}x$: Domain = $\{x : -1 \le x \le 1\}$, Range = $\{y : -\pi/2 \le y \le \pi/2\}$
   $\cos^{-1}x$: Domain = $\{x : -1 \le x \le 1\}$, Range = $\{y : 0 \le y \le \pi\}$
   $\tan^{-1}x$: Domain = $\{x : -\infty < x < \infty\}$, Range = $\{y : -\pi/2 < y < \pi/2\}$

CHAPTER 6

# Properties of Triangle

## 6.1 Introduction

A triangle consists of three sides and three angles. They are collectively known as the six **elements** of the triangle. Various relations between the sides and angles are known. There are relations connecting the area, sides, angles, circum-radius, ex-radius, in-radius etc. Throughout our discussion, we denote the angles BAC, CBA, ACB of a triangle ABC by A, B, C respectively; and the lengths of the sides BC, CA, AB by $a, b, c$ respectively. The circum-radius and the area of the triangle are denoted by R and $\Delta$ respectively. The ex-radii corresponding to the angles A, B, C will be denoted by $r_1, r_2, r_3$. The two more symbols that we shall use often are $s$ and $r$. Here $2s$ is the perimeter $(= a+b+c)$ of the triangle and $r$ the in-radius.

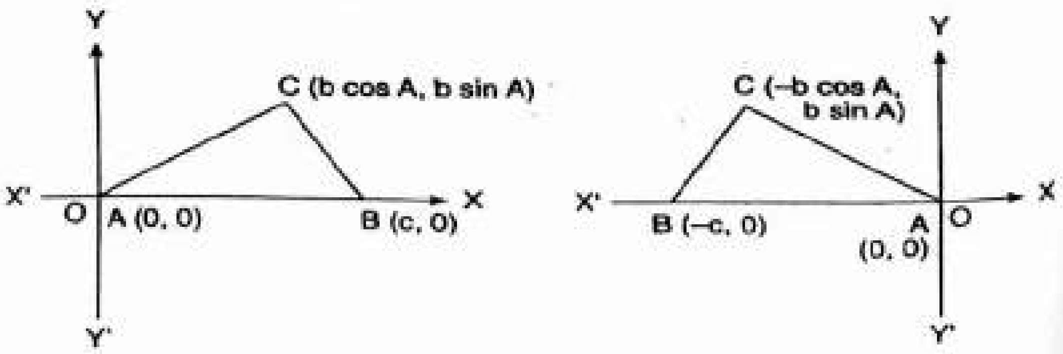
## 6.2 The Cosine Law

In any triangle ABC,

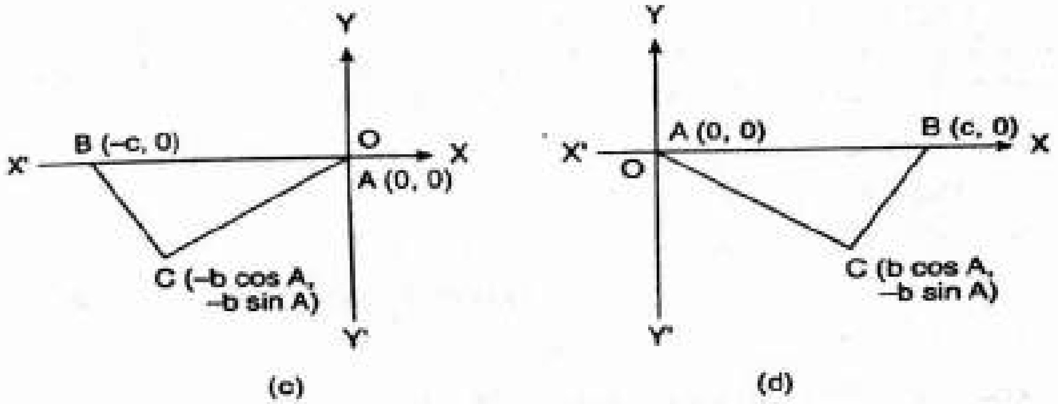
$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc} \quad \text{or} \quad a^2 = b^2 + c^2 - 2bc \cos A,$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2\,ca} \quad \text{or} \quad b^2 = c^2 + a^2 - 2ca \cos B,$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2\,ab} \quad \text{or} \quad c^2 = a^2 + b^2 - 2ab \cos C.$$




(a) (b)




To prove the first of these formulae, we place the triangle ABC in the standard position with the vertex A at the origin and the side AB along the positive x- axis. Then, the coordinates of three vertices A, B and C are clearly (0, 0), ($c$, 0) and ($b$ cos A, $\pm$ $b$ sin A) respectively. The positive sign is to be taken if the vertex C is above the x-axis and the negative sign if it is below the x-axis. Now, using the distance formula, we have

$$BC^2 = (b\cos A - c)^2 + (\pm b\sin A - 0)^2$$

or $$a^2 = b^2(\cos^2 A + \sin^2 A) + c^2 - 2bc\cos A$$

Hence $$a^2 = b^2 + c^2 - 2bc\cos A.$$

That is, $$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

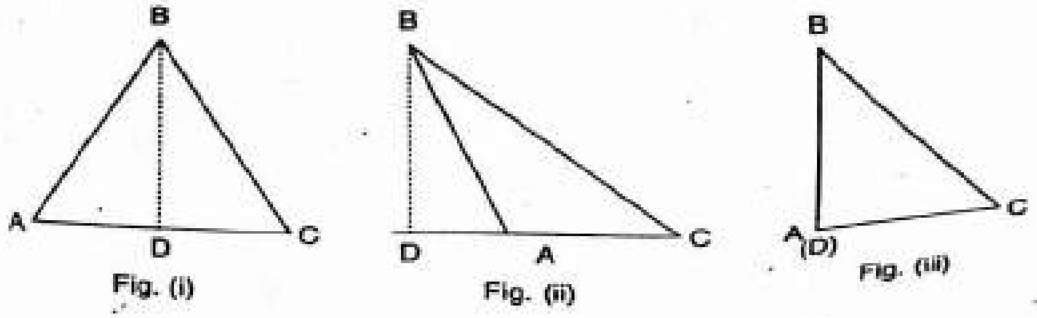
Similarly, the same formula can be proved if the triangle ABC lies in the second or third quadrant in which the coordinates of A, B and C are (0, 0), ($-c$, 0) and ($-b$ cos A, $\pm$ $b$ sin A) respectively.

The other formulae follow likewise.

In particular, if A is right angle, we have, by Pythagorean theorem, $a^2 = b^2 + c^2$, which may be written in the form

$$a^2 = b^2 + c^2 - 2bc\cos A, \quad \text{since } A = 90^\circ,\ \cos 90^\circ = 0.$$

**Alternative Proof**



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Let ABC be a triangle, acute angled in fig. (i), obtuse angled in fig. (ii) and right angled in fig. (iii). From B, draw BD perpendicular to CA produced if necessary (in fig. (ii)). In figure (iii), A and D coincide.

When A is acute angled (fig. (i)), we have from geometry

$$BC^2 = CA^2 + AB^2 - 2.CA.DA$$

$$\Rightarrow \quad BC^2 = CA^2 + AB^2 - 2.CA.AB \cos A$$

(From triangle ADB, DA = AB cos A)

$$\Rightarrow \quad a^2 = b^2 + c^2 - 2bc \cos A$$

When A is obtuse angled (fig. (ii)), we have

$$BC^2 = CA^2 + AB^2 + 2.CA.AD$$

$$\Rightarrow \quad BC^2 = CA^2 + AB^2 + 2.CA.AB \cos (\pi - A)$$

(From triangle ADB, AD = AB cos $(\pi - A)$)

$$\Rightarrow \quad a^2 = b^2 + c^2 - 2bc \cos A$$

When A is right angled (fig. (iii)), then by Pythagorean theorem

$$BC^2 = CA^2 + AB^2$$

$$\Rightarrow \quad BC^2 = CA^2 + AB^2 - 2.CA.AB \cos A \quad (\because \cos A = \cos 90° = 0)$$

$$\Rightarrow \quad a^2 = b^2 + c^2 - 2bc \cos A$$

Hence for all values of A, we have

$$a^2 = b^2 + c^2 - 2bc \cos A$$

That is, $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

In a similar manner, the remaining two relations can be established.

## 6.3 The Sine Law

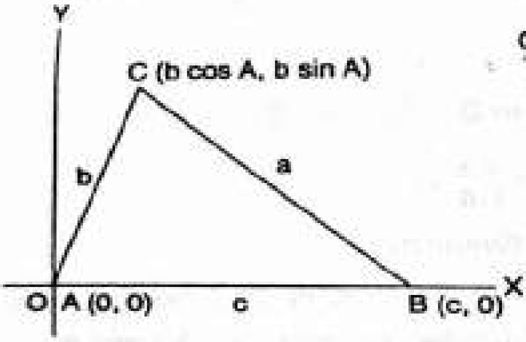
Let us consider a triangle ABC placed in the standard position with the vertex A at the origin and the side AB along the positive x-axis. The vertex of the triangle may be below or above the x-axis. If it is below the x-axis, the angle BAC, or simply A, will be negative; and if it is above the x-axis, it will be positive. The vertex C will lie on the first or fourth quadrant if A is acute; and on the second or third quadrant if it is obtuse. Denoting the sides of the triangle ABC opposite to the angles A, B and C by $a$, $b$ and $c$ respectively, we notice that

(a) the coordinates of A are (0, 0),

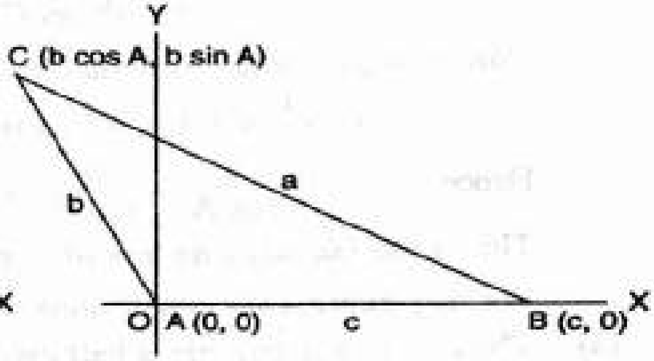
(b) the coordinates of B are $(c, 0)$, and

(c) the coordinates of C are $(b \cos A, b \sin A)$, if C is above x-axis
$(b \cos A, -b \sin A)$, if C is below x-axis.

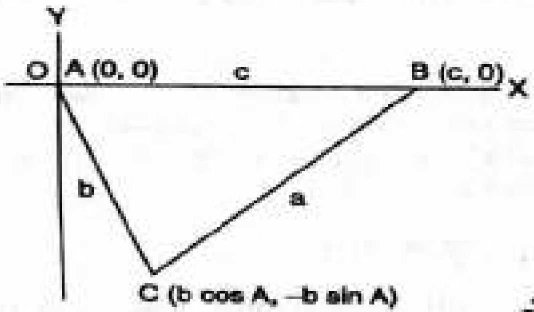
The four cases are shown in the following figures.



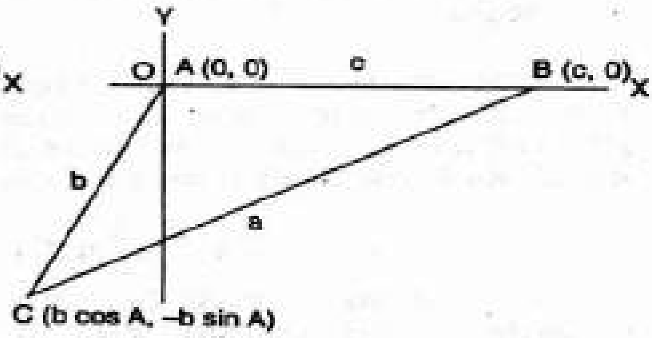

(a)




(b)




(c)




(d)

In each of the above cases, the area of the triangle ABC is given by

$$\text{Area of } \Delta\, ABC = \frac{1}{2} \text{ base} \times \text{altitude}$$

$$= \frac{1}{2} AB \times \text{ordinate of } C$$

$$= \frac{1}{2} c\, (\pm\, b \sin A)$$

$$= \pm \frac{1}{2} bc \sin A.$$

Since the area of a triangle is a positive quantity, the positive sign is to be chosen if A is positive; and the negative sign if A is negative.

Since the angle under consideration is an angle of a triangle, we can, without loss of generality, assume A or any other angle to be positive. We thus have

$$\Delta = \frac{1}{2} bc \sin A.$$

Similarly, placing the other angles B and C in the standard position, we can prove that

$$\Delta = \frac{1}{2} ca \sin B$$
$$= \frac{1}{2} ab \sin C.$$

Combining the three formulae, we obtain

$$\Delta = \frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B = \frac{1}{2} ab \sin C.$$

Hence $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

This is the famous **sine law** of trigonometry.

We may express the conclusions of this section in the following ways.

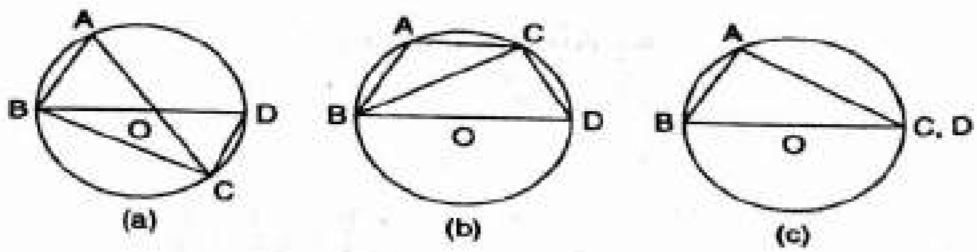
(a) The area of any triangle is half the product of any two sides and the sine of the angle between them and

(b) The sides of a triangle are proportional to the sines of the opposite angles.

Here we shall give a second proof of the law of sines. This proof will eventually lead us to relations connecting the radius of the circum-circle (i.e. circum-radius R) with the sine of an angle and its opposite side. In fact, $a = 2R \sin A$, $b = 2R \sin B$ and $c = 2R \sin C$

or $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

To prove this we denote the circum-centre of the triangle ABC by O. If the angle A is acute or obtuse (Fig. a, b below), join BO and produce it to meet the circumference at D. Join DC. Then, circum–radius = OB = R and BD = 2R and $\angle BCD = 90°$, being an angle in a semi-circle. If A is acute (Fig. a), $\angle BDC = \angle BAC = A$, lying in the same segment of the circle; and they are supplementary when A is obtuse, since A and D lie on opposite sides.




So, $\angle BDC = 180° - \angle BAC = 180° - A.$

Lastly, if A is a right angle, then D coincides with C (Fig. c). In the first two cases, $\sin BDC = \sin \angle BAC = \sin A$ or $\sin (180° - A)$ and hence

$$BC = BD \sin BDC \text{ (since } \angle BCD = 90°)$$

or $$a = 2R \sin A.$$

In the third case, when A = 90°, BC = 2R = 2R sin A.

or $\quad a = 2R \sin A$.

Hence, the result.

**Alternative Proof**

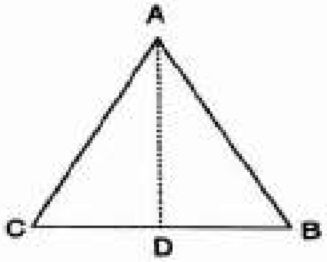



Fig. (i)

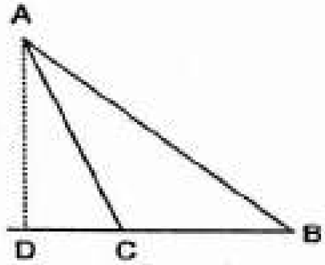



Fig. (ii)

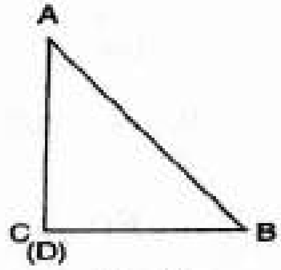



Fig. (iii)

Let ABC be a triangle, acute angled in fig. (i), obtuse angled in fig. (ii) and right angled in fig. (iii). From A, draw AD perpendicular to BC, produced if necessary (fig. (ii). In fig. (iii), C and D coincide.

When C is acute angled (fig. (i)), we have from triangle ABD,

$$AD = AB \sin B = c \sin B$$

Again from triangle ADC,

$$AD = AC \sin C = b \sin C$$

$\therefore \quad b \sin C = c \sin B$

i.e. $\quad \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

When C is obtuse angled (fig. (ii)), then from triangle ABD,

$$AD = AB \sin B = c \sin B$$

Again from triangle ADC,

$$AD = AC \sin (\pi - C) = b \sin c$$

$\therefore \quad b \sin C = c \sin B$

i.e. $\quad \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

When C is right angled, from triangle ABD,

$$AD = AB \sin B = c \sin B$$

Also $\quad AD = b = b \sin C$

$\therefore \quad b \sin C = c \sin B$

i.e. $\quad \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

GK

From the above three cases,

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

Similarly, if a perpendicular is drawn from B on CA, we have

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

Hence for all cases,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

(For right angle at C, $\frac{a}{\sin A} = \frac{b}{\sin B} = c = \frac{c}{\sin C}$)

## 6.4 The Projection Law

In any triangle ABC,

$b \cos C + c \cos B = a$, $\quad c \cos A + a \cos C = b$

and $a \cos B + b \cos A = c$.

We know $b = 2R \sin B$, $c = 2R \sin C$; and $a = 2R \sin A$.

So, $b \cos C + c \cos B = 2R (\sin B \cos C + \cos B \sin C)$

$$= 2R \sin (B + C)$$

$$= 2R \sin A \qquad (\because B + C = 180° - A)$$

$$= a.$$

The other results follow similarly.

The above formulae express the algebraic sum of the projections of any two sides on the third side in terms of the third side.

## 6.5 The Half-angle Formulae

In any triangle ABC,

$$\sin \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{bc}}, \qquad \sin \tfrac{1}{2} B = \sqrt{\frac{(s-c)(s-a)}{ca}}$$

$$\cos \tfrac{1}{2} A = \sqrt{\frac{s(s-a)}{bc}}, \qquad \cos \tfrac{1}{2} B = \sqrt{\frac{s(s-b)}{ca}}$$

$$\tan \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \qquad \tan \tfrac{1}{2} B = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$$

$$\sin \tfrac{1}{2} C = \sqrt{\frac{(s-a)(s-b)}{ab}} \qquad \cos \tfrac{1}{2} C = \sqrt{\frac{s(s-c)}{ab}}$$

$$\tan \tfrac{1}{2} C = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

We know, $2bc \cos A = b^2 + c^2 - a^2$,

and so, $2bc\,(1 + \cos A) = b^2 + c^2 + 2bc - a^2 = (b + c)^2 - a^2$

or $2bc.\ 2\cos^2 \frac{1}{2} A = (b + c - a)\,(b + c + a)$

or $4bc \cos^2 \frac{A}{2} = (2s - 2a).\ 2s\ (\because a + b + c = 2s)$

Hence, $\cos \frac{A}{2} = \sqrt{\frac{s\,(s - a)}{bc}}$.

The positive sign is chosen since the angle $\frac{1}{2}$ A is acute.

Again, since $-2bc \cos A = a^2 - (b^2 + c^2)$

$2bc\,(1 - \cos A) = a^2 - (b^2 + c^2 - 2bc) = a^2 - (b - c)^2$

$2bc.\ 2\sin^2 \frac{1}{2} A = (a - b + c)\,(a + b - c)$

or $2bc.\ 2\sin^2 \frac{1}{2} A = (2s - 2b)\,(2s - 2c)\quad (\because a + b + c = 2s)$

Hence $\sin \frac{1}{2} A = \sqrt{\frac{(s - b)\,(s - c)}{bc}}$,

the positive sign is chosen since $\frac{1}{2}$ A is acute.

Lastly, dividing $\sin \frac{1}{2} A$ by $\cos \frac{1}{2} A$, we arrive at

$$\tan \frac{1}{2} A = \sqrt{\frac{(s - b)\,(s - c)}{s\,(s - a)}}$$

The remaining results follow similarly.

## 6.6 The Tangent Law

In any triangle ABC,

$$\tan \frac{1}{2}\,(B - C) = \frac{b - c}{b + c} \cot \frac{1}{2} A,$$

$$\tan \frac{1}{2}\,(C - A) = \frac{c - a}{c + a} \cot \frac{1}{2} B$$

and $$\tan \frac{1}{2}\,(A - B) = \frac{a - b}{a + b} \cot \frac{1}{2} C.$$

We know $b - c = 2R\,(\sin B - \sin C)$

$= 4R \cos \frac{1}{2}(B + C) \sin \frac{1}{2}(B - C)$

and $b + c = 2R\,(\sin B + \sin C)$

$= 4R \sin \frac{1}{2}(B + C) \cos \frac{1}{2}(B - C)$

Hence $$\frac{b - c}{b + c} = \frac{\cos \frac{1}{2}(B + C) \sin \frac{1}{2}(B - C)}{\sin \frac{1}{2}(B + C) \cos \frac{1}{2}(B - C)}$$

GK

$$= \cot \tfrac{1}{2}(B + C) \tan \tfrac{1}{2}(B - C)$$

or $\quad \tan \frac{1}{2}(B - C) = \dfrac{b - c}{b + c} \tan \frac{1}{2}(B + C)$

$$= \frac{b - c}{b + c} \tan (90° - \tfrac{1}{2} A)$$

$\therefore \quad \tan \frac{1}{2}(B - C) = \dfrac{b - c}{b + c} \cot \frac{1}{2} A. \quad (\because \frac{1}{2}(B + C) = 90° - \frac{1}{2} A.)$

The remaining results follow similarly.

## 6.7 The Area of a Triangle

In order to express the area of a triangle in terms of the sides of a triangle, we shall appeal to the formula

$$\Delta = \tfrac{1}{2} bc \sin A = \tfrac{1}{2} ca \sin B = \tfrac{1}{2} ab \sin C$$

and the half-angle formulae

$$\sin \tfrac{1}{2} A = \sqrt{\frac{(s - b)(s - c)}{bc}}, \qquad \cos \tfrac{1}{2} A = \sqrt{\frac{s(s - a)}{bc}}, \text{ etc.}$$

We shall give three different representations for $\Delta$,

(i) $\quad \Delta = \frac{1}{2} bc \sin A = \frac{1}{2} bc . 2 \sin \frac{1}{2} A \cos \frac{1}{2} A$

$$= bc \sqrt{\frac{(s - b)(s - c)}{bc}} . \sqrt{\frac{s(s - a)}{bc}}$$

$$= bc \sqrt{\frac{s(s - a)(s - b)(s - c)}{bc.bc}}$$

$$= \sqrt{s(s - a)(s - b)(s - c)}.$$

(ii) $\quad \Delta = \frac{1}{4} \sqrt{(a + b + c)(b + c - a)(c + a - b)(a + b - c)}$

$$(\because 2s = a + b + c)$$

$$= \tfrac{1}{4} \sqrt{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4}.$$

(iii) $\quad \Delta = \frac{1}{2} bc \sin A = \frac{1}{2} bc . \dfrac{a}{2R} . \left(\text{since } \sin A = \dfrac{a}{2R}\right)$

$$= \frac{abc}{4R}$$

and also, $R = \dfrac{abc}{4\Delta}$ (T.U. 2049)

Cor. In any triangle

(i) $\sin A = \frac{2\Delta}{bc}$, etc.

(ii) $\tan \frac{1}{2} A = \frac{(s-b)(s-c)}{\Delta}$, etc.

(iii) $\cot \frac{1}{2} A = \frac{s(s-a)}{\Delta}$,

and (iv) $\tan A = \frac{\sin A}{\cos A} = \frac{abc}{R} \cdot \frac{1}{b^2 + c^2 - a^2} = \frac{4\Delta}{b^2 + c^2 - a^2}$

***Example 1.***

In any triangle, prove that

(i) $a^2 + b^2 + c^2 - 2(bc \cos A + ca \cos B + ab \cos C) = 0$

(ii) $a^2 (\sin^2 B - \sin^2 C) + b^2 (\sin^2 C - \sin^2 A) + c^2 (\sin^2 A - \sin^2 B) = 0$

(iii) $a \cos B \cos C + b \cos C \cos A + c \cos A \cos B$

$$= \frac{1}{2}(a \cos A + b \cos B + c \cos C) = \frac{abc}{4R^2}.$$

(iv) $a^3 \cos (B - C) + b^3 \cos (C - A) + c^3 \cos (A - B) = 3abc$.

(v) $\frac{b+c}{a} = \frac{\sin(\frac{1}{2} A + B)}{\sin \frac{1}{2} A}$.

(vi) $1 - \tan \frac{1}{2} A \tan \frac{1}{2} B = \frac{2c}{(a+b+c)}$.

***Solution.***

(i) Using the cosine law, we have

L.S. $= a^2 + b^2 + c^2 - (b^2 + c^2 - a^2 + c^2 + a^2 - b^2 + a^2 + b^2 - c^2)$

$= 2a^2 - 2a^2 + 2b^2 - 2b^2 + 2c^2 - 2c^2 = 0$.

(ii) Using the sine law, we have

L.S. $= b^2 \sin^2 A - c^2 \sin^2 A + c^2 \sin^2 B - b^2 \sin^2 A$

$+ c^2 \sin^2 A - c^2 \sin^2 B$

$= 0$ (since $a \sin B = b \sin A$, etc.)

(iii) Using the projection law: $b \cos C + c \cos B = a$, we have

L.S. $= \frac{1}{2}(a \cos B \cos C + b \cos C \cos A + b \cos C \cos A$

$+ c \cos A \cos B + c \cos A \cos B + a \cos B \cos C)$

$= \frac{1}{2}(c \cos C + b \cos B + a \cos A)$.

GK

Again, using $c = 2R \sin C$, etc. and $\sin 2A = 2 \sin A \cos A$, and $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$, we get

$$\text{L.S.} = \frac{1}{2} . R (\sin 2A + \sin 2B + \sin 2C)$$
$$= 2R. \sin A \sin B \sin C$$
$$= 2R \frac{a}{2R}. \frac{b}{2R}. \frac{c}{2R} = \frac{abc}{4R^2}.$$

(iv) First term $= a^3 \cos (B - C) = 2R\, a^2 \sin A \cos (B - C)$ $\quad (\because a = 2R \sin A)$

$$= 2R\, a^2 \sin (B + C) \cos (B - C) \quad (\because A = 180° - (B + C))$$
$$= a^2 R (\sin 2B + \sin 2C)$$
$$= a^2 (2R \sin B \cos B + 2R \sin C \cos C)$$
$$= a^2 (b \cos B + c \cos C).$$

Proceeding similarly we may find the second and thrid terms.

Hence L.S. $= a^2 (b \cos B + c \cos C) + b^2 (c \cos C + a \cos A) + c^2 (a \cos A + b \cos B)$

$$= ab (a \cos B + b \cos A) + bc (b \cos C + c \cos B) + ca (c \cos A + a \cos C)$$
$$= abc + abc + abc = 3abc.$$

(v) We know $\quad b + c = 2R (\sin B + \sin C)$

$$= 4R \sin \tfrac{1}{2} (B + C) \cos \tfrac{1}{2} (B - C)$$

and $\quad a = 2R \sin A = 2R \sin (B + C)$

$$= 4R \sin \tfrac{1}{2} (B + C) \cos \tfrac{1}{2} (B + C).$$

Dividing the first by the second, we get

$$\frac{b + c}{a} = \frac{\cos \frac{1}{2} (B - C)}{\cos \frac{1}{2} (B + C)} = \frac{\cos \frac{1}{2} (B - 180° + A + B)}{\cos \frac{1}{2} (180° - A)}$$
$$= \frac{\sin (\frac{1}{2} A + B)}{\sin \frac{1}{2} A}.$$

(vi) Since $\tan \frac{1}{2} A = \frac{\Delta}{s (s - a)}$ and $\tan \frac{1}{2} B = \frac{\Delta}{s (s - b)}$, we get

$$\text{L.S.} = 1 - \frac{\Delta}{s (s - a)}. \frac{\Delta}{s (s - b)}$$
$$= 1 - \frac{s (s - a) (s - b) (s - c)}{s^2 (s - a) (s - b)}$$

$$= 1 - \frac{2(s-c)}{2s}$$

$$= \frac{2c}{a+b+c}.$$

***Example 2***

In any triangle, prove that

$$\frac{b - c\cos A}{a - c\cos B} = \frac{\sin A}{\sin B}$$

***Solution :***

$$\text{L.S.} = \frac{b - c\cos A}{a - c\cos B} = \frac{a\cos C + c\cos A - c\cos A}{b\cos C + c\cos B - c\cos B}$$

$$= \frac{a\cos C}{b\cos C}$$

$$= \frac{a}{b} = \frac{2R\sin A}{2R\sin B} = \frac{\sin A}{\sin B}$$

***Example 3***

If $c^4 - 2(a^2 + b^2)c^2 + a^4 + a^2b^2 + b^4 = 0$, prove $C = 60°$ or $120°$.

***Solution.***

Since $c^4 - 2(a^2 + b^2)c^2 + a^4 + a^2b^2 + b^4 = 0$,

$(c^2)^2 - 2c^2(a^2 + b^2) + (a^2 + b^2)^2 = a^2b^2$,

or $(a^2 + b^2 - c^2)^2 = a^2b^2$.

$\therefore$ $(2ab\cos C)^2 = (ab)^2$, i.e. $\cos^2 C = \frac{1}{4}$

Thus $\cos C = \pm\frac{1}{2}$, and $C = 60°$ or $120°$

***Example 4***

In any triangle ABC, prove that the sides are in A.P. if

$$a\cos^2\tfrac{1}{2}C + c\cos^2\tfrac{1}{2}A = \tfrac{3}{2}b.$$

***Solution.***

$$\text{L.S.} = \frac{1}{2}a\left(2\cos^2\frac{C}{2}\right) + \frac{1}{2}c\left(2\cos^2\frac{A}{2}\right)$$

$$= \frac{1}{2}a(1 + \cos C) + \frac{1}{2}c(1 + \cos A)$$

$$= \frac{1}{2}(a + a\cos C) + \frac{1}{2}(c + c\cos A) \qquad (\because 2\cos^2\tfrac{1}{2}A = 1 + \cos A)$$

$$= \frac{1}{2}(a + c) + \frac{1}{2}(a\cos C + c\cos A)$$

$= \frac{1}{2}(a+c) + \frac{1}{2}b$

Thus $\frac{1}{2}(a+c) + \frac{1}{2}b = \frac{3}{2}b$

or $\frac{1}{2}(a+c) = b.$

This shows that $b$ is the A.M. of $a$ and $c$. That is, $a, b, c$ are in A.P.

**Example 5**

Prove that : $(a+b+c)\left(\tan\frac{A}{2} + \tan\frac{B}{2}\right) = 2c\cot\frac{C}{2}$

**Solution :**

We know that $\tan\frac{1}{2}A = \frac{(s-b)(s-c)}{\Delta}$ and $\cot\frac{C}{2} = \frac{s(s-c)}{\Delta}$

Now, $(a+b+c)(\tan\frac{1}{2}A + \tan\frac{1}{2}B)$

$= 2s\left[\frac{(s-b)(s-c)}{\Delta} + \frac{(s-c)(s-a)}{\Delta}\right]$

$= \frac{2s(s-c)}{\Delta}(s-b+s-a)$

$= \frac{2s(s-c)}{\Delta}.(2s-a-b)$

$= \frac{2s(s-c)}{\Delta}(a+b+c-a-b)$

$= 2\cot\frac{C}{2}c = 2c\cot\frac{C}{2}$

**Example 6**

Prove that $4(bc\cos^2\frac{1}{2}A + ca\cos^2\frac{1}{2}B + ab\cos^2\frac{1}{2}C) = (a+b+c)^2$

**Solution :**

$4(bc\cos^2\frac{1}{2}A + ca\cos^2\frac{1}{2}B + ab\cos^2\frac{1}{2}C)$

$= 2bc(2\cos^2\frac{1}{2}A) + 2ca(2\cos^2\frac{1}{2}B) + 2ab(2\cos^2\frac{1}{2}C)$

$= 2bc(1+\cos A) + 2ca(1+\cos B) + 2ab(1+\cos C)$

$= 2bc + 2ca + 2ab + 2bc\cos A + 2ca\cos B + 2ab\cos C$

$= 2bc + 2ca + 2ab + b^2 + c^2 - a^2 + c^2 + a^2 - b^2 + a^2 + b^2 - c^2$

$(\because \cos A = \frac{b^2+c^2-a^2}{2bc}$ etc.)

$= a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

$= (a+b+c)^2$

***Example 7***

If $\frac{\sin(A-B)}{\sin(A+B)} = \frac{a^2-b^2}{a^2+b^2}$, prove that the triangle ABC is either isosceles or right angled.

***Solution :***

$$\frac{\sin(A-B)}{\sin(A+B)} = \frac{a^2-b^2}{a^2+b^2}$$

or, $$\frac{\sin(A+B)\sin(A-B)}{\sin^2(A+B)} = \frac{a^2-b^2}{a^2+b^2}$$

or, $$\frac{\sin^2A-\sin^2B}{\sin^2(\pi-C)} = \frac{a^2-b^2}{a^2+b^2}$$

($\because$ $\sin(A+B)\sin(A-B) = \sin^2A - \sin^2B$ and $A+B+C=\pi$)

or, $$\frac{\sin^2A-\sin^2B}{\sin^2C} = \frac{a^2-b^2}{a^2+b^2}$$

or, $$\frac{a^2-b^2}{c^2} = \frac{a^2-b^2}{a^2+b^2} \qquad (\because \sin A = \frac{a}{2R} \text{ etc.})$$

or, $(a^2+b^2)(a^2-b^2) = c^2(a^2-b^2)$

or, $(a^2+b^2)(a^2-b^2) - c^2(a^2-b^2) = 0$

or, $(a^2-b^2)(a^2+b^2-c^2) = 0$

Either $a^2-b^2=0$

$a=b$ which shows that the triangle is an isosceles.

or, $a^2+b^2-c^2=0$

or, $a^2+b^2=c^2$ which shows that the triangle is a right angled triangle.

Hence, the triangle is either isosceles or right angled.

***Example 8***

In triangle ABC, if $a=3$, $b=4$ and $c=5$, find R and $\cos\frac{A}{2}$.

***Solution :***

$$s = \frac{a+b+c}{2} = \frac{3+4+5}{2} = 6$$

and $$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$
$$= \sqrt{6(6-3)(6-4)(6-5)}$$
$$= \sqrt{6.3.2.1}$$
$$= 6$$

GK

$$R = \frac{abc}{4\Delta} = \frac{3.4.5}{4.6} = \frac{5}{2}$$

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} = \sqrt{\frac{6(6-3)}{4.5}} = \frac{3}{\sqrt{10}}$$

## EXERCISE 6.1

In any triangle, prove that (Exs. 1 – 5)

1. (a) $a(b\cos C - c\cos B) = b^2 - c^2$.

   (b) $\frac{\cos A}{a} + \frac{a}{bc} = \frac{\cos B}{b} + \frac{b}{ca} = \frac{\cos C}{c} + \frac{c}{ab}$.

   (c) $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2+b^2+c^2}{2abc}$.

2. (a) $a^3(\sin^3 B - \sin^3 C) + b^3(\sin^3 C - \sin^3 A) + c^3(\sin^3 A - \sin^3 B) = 0$.

   (b) $\frac{a^2\sin(B-C)}{\sin A} + \frac{b^2\sin(C-A)}{\sin B} + \frac{c^2\sin(A-B)}{\sin C} = 0$. (T.U. 2052)

   (c) $\frac{a\sin(B-C)}{b^2-c^2} = \frac{b\sin(C-A)}{c^2-a^2} = \frac{c\sin(A-B)}{a^2-b^2}$ (T.U. 051H)

3. (a) $(b+c)\cos A + (c+a)\cos B + (a+b)\cos C = a+b+c$.

   (b) $b^2\sin 2C + c^2\sin 2B = 2ab\sin C$. (T.U. 2051 H)

   (c) $c^2\cos^2 B + b^2\cos^2 C + bc\cos(B-C) = \frac{1}{2}(a^2+b^2+c^2)$ (T.U. 2058 S)

   (d) $\frac{c - b\cos A}{b - c\cos A} = \frac{\cos B}{\cos C}$ (HSEB 2064)

   (e) $\frac{a - b\cos C}{c - b\cos A} = \frac{\sin C}{\sin A}$

4. (a) $a^3\sin(B-C) + b^3\sin(C-A) + c^3\sin(A-B) = 0$

   (b) $\frac{b^2-c^2}{a^2}\sin 2A + \frac{c^2-a^2}{b^2}\sin 2B + \frac{a^2-b^2}{c^2}\sin 2C = 0$

   (c) $\sin(A+B) : \sin(A-B) = c^2 : (a^2-b^2)$

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(d) $(a^2 - b^2 + c^2) \tan B = (a^2 + b^2 - c^2) \tan C = (b^2 + c^2 - a^2) \tan A$

(e) $\dfrac{\cos B - \cos C}{\cos A + 1} = \dfrac{c - b}{a}$

(f) $\dfrac{a \sin A + b \sin B + c \sin C}{a \cos A + b \cos B + c \cos C} = \dfrac{R}{abc} (a^2 + b^2 + c^2)$

5. (a) $\dfrac{b - c}{a} \cos \frac{1}{2} A = \sin \frac{1}{2} (B - C)$

(b) $b \cos^2 \frac{1}{2} A + a \cos^2 \frac{1}{2} B = \frac{1}{2} (a + b + c)$

(c) $a (\cos B - \cos C) = 2 (c - b) \cos^2 \frac{1}{2} A$

(d) $\dfrac{b - c}{a} \cos^2 \frac{1}{2} A + \dfrac{c - a}{b} \cos^2 \frac{1}{2} B + \dfrac{a - b}{c} \cos^2 \frac{1}{2} C = 0$

(e) $bc \cos^2 \frac{1}{2} A + ca \cos^2 \frac{1}{2} B + ab \cos^2 \frac{1}{2} C = s^2$

(f) $\tan^2 \frac{1}{2} A \tan^2 \frac{1}{2} B \tan^2 \frac{1}{2} C = \left(\dfrac{s - a}{s}\right) \left(\dfrac{s - b}{s}\right) \left(\dfrac{s - c}{s}\right)$

(g) $(b + c - a) (\cot \frac{1}{2} B + \cot \frac{1}{2} C) = 2a \cot \frac{1}{2} A$.

6. If $a^4 + b^4 + c^4 = 2c^2 (a^2 + b^2)$, prove that $C = 45°$ or $135°$.
(T.U. 2051) (HSEB 2056)

7. If $(a + b + c) (b + c - a) = 3bc$, show that $A = 60°$.

8. If $\dfrac{1}{a + c} + \dfrac{1}{b + c} = \dfrac{3}{a + b + c}$, show that $C = 60°$.
(T.U. 2054, HSEB 2053)

9. If the cosines of two of the angles of a triangle are proportional to the opposite sides, prove that the triangle is isosceles.

10. If $(\cos A + 2 \cos C) : (\cos A + 2 \cos B) = \sin B : \sin C$, prove that the triangle is either isosceles or right-angled.

11. If $2 \cos A = \sin B : \sin C$, show that the triangle is isosceles.

12. Prove that $a^2, b^2, c^2$ are in A.P.
if $\sin A : \sin C = \sin (A - B) : \sin (B - C)$.

13. In any triangle, prove that

(a) $a^2 \cot A + b^2 \cot B + c^2 \cot C = 4\Delta$.

(b) $(a \sin A + b \sin B + c \sin C)^2$
$= (a^2 + b^2 + c^2) (\sin^2 A + \sin^2 B + \sin^2 C)$.

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(c) $\sin A + \sin B + \sin C = \frac{s}{R}$

(d) $a \cos B \cos C + b \cos C \cos A + c \cos A \cos B = \frac{\Delta}{R}$

14. If $8R^2 = a^2 + b^2 + c^2$, prove that the triangle is right angled. (T.U. 2057S)

15. If $a = 10, b = 8, c = 6$, find $s$, $\Delta$, R and $\sin \frac{B}{2}$.

### *Answers*

15. 12, 24, 5, $\frac{1}{\sqrt{5}}$

## ADDITIONAL QUESTIONS (I)

1. i) State and establish sine law. (T.U. 2055 S)

   ii) State sine law. Use this law to prove the projection law $a = b \cos C + c \cos B$ and tangent law $\tan \frac{B - C}{2} = \frac{b - c}{b + c} \cot \frac{A}{2}$ (T.U. 057 S) (HSEB 2050)

2. In any triangle ABC, prove that
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$ (HSEB 2058)

3. In any triangle, prove that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$, where R is the radius of the circum circle. (HSEB 2051, 2052, 2055)

4. In any triangle ABC, prove that $\cos \frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$, write the similar results for $\sin \frac{B}{2}$ and $\tan \frac{B}{2}$. (HSEB 2054)

5. Prove that : $\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$ (T.U. 2056 S) (HSEB 2057)

6. In any triangle ABC, prove that

   (i) $\frac{c - a \cos B}{b - a \cos C} = \frac{\sin B}{\sin C}$

(ii) $\dfrac{\cos^2 B - \cos^2 C}{b + c} + \dfrac{\cos^2 C - \cos^2 A}{c + a} + \dfrac{\cos^2 A - \cos^2 B}{a + b} = 0$

(iii) $\dfrac{a^2 \sin (B - C)}{\sin B + \sin C} + \dfrac{b^2 \sin (C - A)}{\sin C + \sin A} + \dfrac{c^2 \sin (A - B)}{\sin A + \sin B} = 0$

(iv) $\tan A/2 \tan B/2 = \dfrac{a + b - c}{a + b + c}$

(v) $(b - c) \cot A/2 + (c - a) \cot B/2 + (a - b) \cot C/2 = 0$

(vi) $(a - b)^2 \cos^2 \dfrac{C}{2} + (a + b)^2 \sin^2 \dfrac{C}{2} = c^2$

7. (i) If the angles of a triangle are to one another as 1 : 2 : 3, prove that the corresponding sides are $1 : \sqrt{3} : 2$.

(ii) If the sides of a triangle are 7 cms, $4\sqrt{3}$ cms, $\sqrt{13}$ cms, prove that the smallest angle is 30°.

(iii) The angle of a triangle ABC are in A.P., and it is being given that $b : c = \sqrt{3} : \sqrt{2}$, find $\angle A$.

(iv) In triangle ABC, if $\angle A = 30°$, $b : c = 2 : \sqrt{3}$, find $\angle B$.

(v) If $a = 2b$, $A = 3B$, find the angles of a triangle.

8. If $\dfrac{1}{a + b} + \dfrac{1}{b + c} = \dfrac{3}{a + b + c}$, show that $\angle B = 60°$.

9. (i) If in a triangle ABC, $A = 60°$, show that

$$b + c = 2a \cos \frac{B - C}{2}$$

(ii) In a triangle ABC, if $\angle B = 60°$, prove that

$$(a + b + c)(a - b + c) = 3ca.$$

10. (i) If in any triangle, $a : b : c = 2 : 3 : 4$ and $s = 27$, find the area of the triangle.

(ii) In a triangle ABC, $a = 5$, $b = 6$ and $C = 30°$, show that its area is 7.5 sq. units.

(iii) In a triangle ABC, if $a = \sqrt{2}$, $b = \sqrt{3}$ and $c = \sqrt{5}$, show that its area is $\frac{1}{2}\sqrt{6}$ sq. units.

11. In any triangle ABC, if $a = 13$ cms, $b = 14$ cms, $c = 15$ cms, find $\sin A/2$, $\cos A/2$, $\sin A$, $\cos A$ and $\tan A/2$.

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12. (i) If $a, b, c$ are in A.P., prove that $3 \tan \frac{A}{2} \tan \frac{C}{2} = 1$

(ii) If $b, a, c$ are in A.P., prove that $\cos \frac{B-C}{2} = 2 \sin \frac{A}{2}$.

13. If $b - a = mc$, prove that

$$\cot \frac{1}{2}(B - A) = \frac{1 + m \cos B}{m \sin B}.$$

### *Answers*

7. (iii) 75° (iv) 90° (v) 90°, 30°, 60° 10. (i) $27\sqrt{15}$ sq. units

11. $\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, \frac{4}{5}, \frac{3}{5}, \frac{1}{2}$

CHAPTER 7

# Solution of Triangle

## 7.1 Introduction

A triangle has three sides and three angles. They are called the six elements of a triangle. To solve a triangle means to find the remaining elements when three other elements are known.

## 7.2 Solutions of Triangles

Various relations connecting the sides and the trigonometrical ratios of the angles of a triangle have been established in chapter 7. With the help of those relations, it is possible to determine the magnitudes of three parts of the triangle when the other three parts out of the six elements of a triangle are given, except in one case in which each of the three given parts is an angle. In this case only the ratios of the sides can be determined, for there exist innumerable triangles with the same set of angles. Thus, to solve a triangle means to determine the unknown elements of a triangle from its known elements.

In solving a triangle, the following different cases arise.

i) Three angles given

ii) Three sides given

iii) Two angles and one side given

iv) Two sides and the included angle given

v) Two sides and an angle opposite to one of them given

Note : The right-angled and oblique triangles are also included in the above cases.

**Case I: Given Three Angles**

No unique solution is possible in this case. The sine rule,

$$a : b : c = \sin A : \sin B : \sin C$$

gives the ratio of the sides but not their actual lengths. Only the shape of the triangle is determined but not the size.

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**Case II : Given Three Sides**

Applying any one of the formula, the angle A can be determined :

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

$$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

and $\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

After finding the angle A, we can determine the angle B from the relation

$$\sin A : \sin B = a : b$$

and then the angle C may be found from the relation

$$C = 180° - (A + B).$$

**Case III : Given Two Angles and A Side**

Let A and C be two given angles and $a$ the given side. The sum of the three angles being two right angles, the third angle can be obtained from the relation

$$B = 180° - (A + C)$$

To find the other two sides $b$ and $c$, we use the formula

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

**Case IV : Given Two Sides and Included Angle**

Let the sides $b$, $c$ and the angle A be given. Then we may find '$a$' from the relation

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Then from $\frac{a}{\sin A} = \frac{b}{\sin B}$, we can determine B.

Lastly, using $C = 180° - (A + B)$, we can find C.

**Alternatively,** we may first find B and C before we find a, thus

$$\tan \frac{1}{2}(B - C) = \frac{b-c}{b+c} \cot \frac{1}{2} A$$

From this formula, we can find $\frac{1}{2}$ (B – C).

Also, we know that $\frac{1}{2}(B + C) = 90° - \frac{1}{2} A$

Thus we have two equations which give us the values of B and C.

Lastly, using $\frac{a}{\sin A} = \frac{c}{\sin C}$, we can find c.

The third side can be obtained by using Sine Rule

$$a : b : c = \sin A : \sin B : \sin C$$

**Case V : Given Two Sides and An Opposite Angle**

Let $b$, $c$ and B be the given parts. Then, from the formula

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

we can find C. When C is thus found, $A = 180° - (B + C)$, from which A can be found. Also from $\frac{a}{\sin A} = \frac{b}{\sin B}$, $a$ can be found.

Thus the solution of the triangles depends on the possibility of the determination of c from $\sin C = \frac{c \sin B}{b}$

Three cases arise :

**Case I :** $c \sin B > b$. In this case sin C is greater than 1 which is impossible and therefore no triangle is possible.

**Case II :** $c \sin B = b$. In this case $\sin C = 1$ and so $C = 90°$. Thus $A = 90° - B$. Thus a right angled triangle is the required solution. Also, from $c^2 = a^2 + b^2$, or $a = \sqrt{c^2 - b^2}$ we can find '$a$'.

**Case III :** If $c \sin B < b$, then there are two values of C, one acute and the other obtuse (i.e. two supplementary values are possible). Three sub-cases arises:

**Sub-case (a) :** If $c < b$, we have $C < B$, hence C must be acute and only one triangle is possible.

**Sub-case (b) :** If $c = b$, then $C = B$; hence C can not be obtuse, otherwise there will be two obtuse angles in a triangle. So, only one triangle is possible.

**Sub-case (c) :** If $c > b$, then the angle C is not restricted and it can have either acute or obtuse value; As long as B is less than 90°, C can have either acute or obtuse value; hence *two triangles are possible with given parts.*

This is usually known as the '**Ambiguous Case**' in the solution of triangle.

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## Worked out examples

***Example 1***

If two angles of a triangle are 75° and 60°, find the ratio of the sides.

***Solution:***

If A = 75° and B = 60°, then

the third angle C = 180° − (75° + 60°) = 45°

Using sine law,

$$\begin{aligned} a : b : c &= \sin A : \sin B : \sin C \\ &= \sin 75° : \sin 60° : \sin 45° \\ &= \frac{\sqrt{3}+1}{2\sqrt{2}} : \frac{\sqrt{3}}{2} : \frac{1}{\sqrt{2}} \\ &= \sqrt{3}+1 : \sqrt{6} : 2 \end{aligned}$$

***Example 2***

Given $a = \sqrt{6}$, $b = 2$, $c = \sqrt{3} - 1$, solve the triangle. (HSEB 2057)

***Solution:***

Here, $a = \sqrt{6}$, $b = 2$, $c = \sqrt{3} - 1$

Now, $$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{6 + 4 - (\sqrt{3} - 1)^2}{2\sqrt{6}.2}$$

$$= \frac{6 + 2\sqrt{3}}{4\sqrt{6}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$\therefore$ C = 15°

Again, $$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{4 + (\sqrt{3} - 1)^2 - 6}{2.2.(\sqrt{3} - 1)}$$

$$= \frac{2 - 2\sqrt{3}}{4(\sqrt{3} - 1)} = -\frac{1}{2}$$

$\therefore$ A = 120°

Now, B = 180° − (A + C)

= 180° − (15° + 120°) = 45°

***Example 3***

If A = 30°, B = 45°, $a = 6\sqrt{2}$, solve the triangle.

***Solution:***

Here $A = 30°$, $B = 45°$, $a = 6\sqrt{2}$

$$\frac{a}{\sin A} = \frac{b}{\sin B} \qquad \text{or,} \quad \frac{6\sqrt{2}}{\sin 30°} = \frac{b}{\sin 45°}$$

$$\frac{6\sqrt{2}}{1/2} = \frac{b}{1/\sqrt{2}} \qquad \therefore \quad b = \frac{12\sqrt{2}}{\sqrt{2}} = 12$$

$C = 180° - (A + B) = 180° - (30° + 45°) = 105°$

Also, $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$ or, $\dfrac{6\sqrt{2}}{\sin 30°} = \dfrac{c}{\sin 105°}$

$$\therefore c = \frac{6\sqrt{2}\sin 105°}{\sin 30°} = \frac{6\sqrt{2}\,(\sqrt{3} + 1)/2\sqrt{2}}{1/2} = 6(\sqrt{3} + 1)$$

***Example 4***

In a triangle ABC, $C = 30°$, $b = \sqrt{3}$ and $a = 1$. Find the other angles and the sides.

***Solution:***

Here, $a = 1$, $b = \sqrt{3}$, $C = 30°$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \qquad \text{or, } \cos 30° = \frac{1 + 3 - c^2}{2.1.\sqrt{3}}$$

$$\frac{\sqrt{3}}{2} = \frac{4 - c^2}{2\sqrt{3}} \qquad \text{or,} \quad 3 = 4 - c^2$$

$$c^2 = 4 - 3 = 1 \qquad \therefore \quad c = 1$$

Using sine law,

$$\frac{a}{\sin A} = \frac{c}{\sin C} \qquad \text{or, } \frac{1}{\sin A} = \frac{1}{\sin 30°}$$

$\therefore \quad A = 30°$

Since, $A + B + C = 180°$

so, $B = 180° - (A + C)$

$= 180° - (30° + 30°) = 120°$

***Example 5***

If $a = 2$, $b = \sqrt{6}$, $A = 45°$, solve the triangle.

***Solution:***

Since, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$

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so, $\sin B = \frac{b \sin A}{a} = \frac{\sqrt{6} \sin 45°}{2} = \frac{\sqrt{6} \times 1/\sqrt{2}}{2} = \frac{\sqrt{3}}{2}$

$\therefore$ $B = 60°$ or $120°$

and accordingly, $C = 75°$ or $15°$

When $B = 60°, C = 75°$

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$c = \frac{b \sin C}{\sin B} = \frac{\sqrt{6} \sin 75°}{\sin 60°} = \frac{\sqrt{6}\ \frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}}{2}} = \sqrt{3} + 1$$

When $B = 120°, C = 15°$

$$c = \frac{b \sin C}{\sin B} = \frac{\sqrt{6} \sin 15°}{\sin 120°} = \frac{\sqrt{6} \times \frac{\sqrt{3}-1}{2\sqrt{2}}}{\frac{\sqrt{3}}{2}} = \sqrt{3} - 1$$

$\therefore$ $B = 60°, C = 75°, c = \sqrt{3} + 1$

or, $B = 120°, C = 15°, c = \sqrt{3} - 1$

## EXERCISE 7.1

1. i) The angles of a triangle are 105° and 15°, find the ratio of its sides.
   ii) If $A = 45°, B = 60°$, show that $a : c = 2 : (\sqrt{3} + 1)$.
   iii) If one angle of a triangle is 60° and the ratio of the other two is 1 : 3, find all the angles and the ratio of the sides.
   iv) The angles of a triangle are in the ratio 2 : 3 : 7. Prove that the sides are in the ratio of $\sqrt{2} : 2 : (\sqrt{3} + 1)$.
   v) If $\cos A = \frac{4}{5}$, $\cos B = \frac{3}{5}$, find $a : b : c$.

2. i) Given $a = 2, b = \sqrt{2}, c = \sqrt{3} + 1$, solve the triangle. (HSEB 2058)
   ii) If $a = 3 + \sqrt{3}, b = 2\sqrt{3}, c = \sqrt{6}$, then solve the triangle.

iii) If three sides of a triangle are proportional to $2 : \sqrt{6} : \sqrt{3} + 1$ find all the angles. (HSEB 2056)

3. i) If $C = 30°$, $B = 45°$, $c = 6\sqrt{2}$, solve the triangle.

ii) If $A = 60°$, $B = 75°$, $a = 2\sqrt{3}$, solve the triangle.

iii) If $A = 30°$, $B = 45°$, $b = 2$, then solve the triangle.

4. i) If $a = 2$, $b = 4$, $C = 60°$, find A and B.

ii) Two sides of a triangle are $\sqrt{3} + 1$ and $\sqrt{3} - 1$ and the included angle is 60°, solve the triangle.

iii) Given $a = \sqrt{57}$, $A = 60°$, $\Delta = 2\sqrt{3}$, find $b$ and $c$.

5. Given

i) $a = 3$, $b = 3\sqrt{3}$, $A = 30°$

ii) $a = 2$, $b = \sqrt{3} + 1$, $A = 45°$

iii) $A = 30°$, $a = 6$, $b = 4$ (Take $\sin 19°30' = \frac{1}{3}$ and $\sin 49°30' = \frac{3}{4}$).

Solve the triangle. Find in which case the solution is ambiguous, in which case there is only one solution.

### Answers

1. (i) $\sqrt{3} + 1 : \sqrt{6} : \sqrt{3} - 1$ (iii) 30°, 60°, 90°; $1 : \sqrt{3} : 2$
   (v) 3 : 4 : 5
2. (i) 45°, 30°, 105° (ii) 105°, 45°, 30° (iii) 45°, 60°, 75°
3. (i) 105°, 12, $6(\sqrt{3} + 1)$ (ii) 45°, $\sqrt{6} + \sqrt{2}$, $2\sqrt{2}$
   (iii) 105°, $\sqrt{2}$, $\sqrt{3} + 1$
4. (i) 30°, 90° (ii) $\sqrt{6}$, 15°, 105° (iii) 8, 1
5. (i) Ambiguous = two solutions :
   $B = 60°$, $C = 90°$, $c = 6$ or $B = 120°$, $C = 30°$, $c = 3$
   (ii) Ambiguous : two solutons :
   $B = 75°$, $C = 60°$, $a = \sqrt{6}$ or, $B = 105°$, $C = 30°$, $a = \sqrt{2}$
   (iii) $B = 19° 30'$, $C = 130° 30'$, $c = 9$ only one solution.

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## CHAPTER 8

# Sequence & Series and Mathematical Induction

## 8.1 Ordered Sets

The idea of a sequence originates in the process of counting in a very natural way. During counting, we begin with the number 1. The number is followed by (succeeded by) the number 2, 2 by 3, 3 by 4, 4 by 5, and so on. This is how we arrive at the succession of counting numbers beginning with the number 1. One of the conventional ways of representing the succession of counting numbers is to list the first few numbers separated by commas and then write dots at the end to indicate the remaining numbers. Thus we have the symbolic representation

$$1, 2, 3, 4, 5, \ldots\ldots \ \ldots\ldots,$$

in which the numbers are in the given order: 1 is followed by 2, 2 by 3, 3 by 4 and so on. Some other examples which contain the same idea are

(a) Set of even numbers : 2, 4, 6, 8, 10, ... ... ... ...

(b) Set of positive integral powers of 2 :

$$2, 2^2, 2^3, 2^4, 2^5, \ldots \ldots \ldots \ldots$$

(c) Set of reciprocals of positive integers :

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \ldots \ldots \ldots$$

These are possibly some of the simplest and best known examples which gave rise to the following definition of a sequence:

*A sequence is an ordered set of numbers.*

This is a classical definition of sequence. Its modern version is based on the definition of a function and is indeed defined as a special kind of function.

## 8.2 Sequences

In order to obtain an up-to-date definition of a sequence, we begin with some examples which will provide some kind of motivation. Consider the correspondence indicated below:

| | | | | | | |
|---|---|---|---|---|---|---|
| (a) Positive integers : | 1 | 2 | 3 | 4 | 5 | ... ... |
| | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | |
| Even numbers : | 2 | 4 | 6 | 8 | 10 | ... ... |
| (b) Positive integers : | 1 | 2 | 3 | 4 | 5 | ... ... |
| Positive integral | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | |
| powers of 2 : | 2 | $2^2$ | $2^3$ | $2^4$ | $2^5$ | ... ... |
| (c) Positive integers : | 1 | 2 | 3 | 4 | 5 | ... ... |
| | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | |
| Reciprocals of positive integers : | 1 | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{1}{5}$ | ... ... |

In each case the numbers in the first row are the positive integers and those in the second the numbers obtainable directly from them in an obvious manner. Moreover, the numbers in

(a) 2, 4, 6, 8, 10, ... ... ...

(b) 2, $2^2$, $2^3$, $2^4$, $2^5$, ... ... ...

(c) $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \ldots \ldots$

are in the same natural order as the set of positive integers. This means that the idea of a sequence as an ordered set of numbers is contained in such a correspondence between the set of positive integers and some other set. But the correspondence indicated above are of such a nature that corresponding to each positive integer, there is one and only one number in the second set. Since such a correspondence defines a function it is justifiable to consider a sequence as a special case of a function. A definition of sequence to this effect is:

*A sequence is a function defined on the set Z = {1, 2, 3, 4, 5, ...} of positive integers.*

We denote the function (sequence) by the symbol $f$.

Then its values at 1, 2, 3, 4, 5, ... ..., $n$, ... ... are $f(1)$, $f(2)$, $f(3)$, $f(4)$, ... ..., $f(n)$, ... ... respectively. These values make up the range:

$$R_f = \{f(1), f(2), f(3), \ldots \ldots, f(n), \ldots \ldots\}.$$

of the sequence $f$. The elements of the range, i.e., $f(1)$, $f(2)$ ... ... are called the *terms* of the sequence $f$.

As a set of ordered pairs, the above sequence (function) would look like the set :

$$\begin{aligned} f &= \{(1, f(1)), (2, f(2)), (3, f(3)), \ldots \ldots, (n, f(n)), \ldots \ldots\} \\ &= \{(n, f(n)) : n = 1, 2, 3, \ldots\} \end{aligned}$$

A notation such as this is cumbersome. It is therefore customary to denote the sequence by the simpler notation $\{f(n)\}$ or $\{f_n\}$ or sometimes by $f_1$,

$f_2, f_3$ ... ...... only. The notation $(f_n)$, rather than $\{f_n\}$, is preferred by some authors to emphasize the fact that in a sequence we are concerned not only with the set of values of the sequence but also with the order in which they appear; rearrangement of non-equal terms results in a different sequence. For our purpose, we adopt the conventional notation $\{f_n\}$ or $f_1, f_2, f_3, \ldots$ with the reservation that the terms are in a definite order. We also call the terms $f_1, f_2, f_3, \ldots \ldots$, the first term, second term, third term, ... ... . The term $f_n$ is called the $n^{th}$ term or the *general term* of the sequence. It is sometimes denoted by $t_n$. Other conventional notations for sequences are $\{x_n\}$ and $\{a_n\}$, where $x_n$ and $a_n$ denote the respective general terms.

An important point to be noted is that we have considered the entire set of positive integers as the domain of definition of the sequence. If the domain consists of a finite number of elements only, the sequence is said to be *finite*, otherwise the sequence is said to be *infinite*.

***Example 1.***

Let $f : N \rightarrow R$ be defined by $f(n) = f_n = 1/n$.

Then the sequence thus defined is the set

$$\{(1,1), (2, 1/2), (3, 1/3), \ldots \ldots, (n, 1/n), \ldots \ldots\}$$

This is usually abbreviated in the form $\{1/n\}$.

Its terms are $1, 1/2, 1/3, \ldots \ldots, 1/n, \ldots \ldots$

***Example 2.***

Let $f : N \rightarrow R$ be defined by $f_n = 2 - 1/2^{n-1}$.

In the conventional notation, this sequence is $\{f_n\} = \{2 - 1/2^{n-1}\}$ or

$$1, 3/2, 7/4, 15/8, \ldots \ldots$$

Some other examples of sequences written in the conventional style are

(a) $2, 4, 6, 8, 10, \ldots \ldots, 2n, \ldots \ldots$

(b) $1, 3, 5, 7, 9, \ldots \ldots, 2n - 1, \ldots \ldots$

(c) $1, 2, 4, 8, 16, \ldots \ldots, 2^{n-1}, \ldots \ldots$

and (d) $1, x, x^2, x^3, \ldots \ldots \ldots, x^{n-1}, \ldots \ldots$

In the above examples, the terms of the sequences are specified by a rule conveyed by the general term. Another way of specifying the terms *(values)* of a sequence is by means of *recursion formula*. That is, we may specify the first value (term) and give a rule for finding $f_n$ $(n \geq 2)$ once $f_{n-1}$ is known. As a simple example, let us see how the population of a country which is steadily increasing every year can be specified by a formula.

Suppose P is the population of a country which is increasing at the rate of $p$ % every year. If $P_n$ is the total population in $n$ years, the sequence $\{P_n\}$ or $P_1, P_2, P_3, \ldots \ldots$ is determined by the recursive formula

$$P_1 = P + \frac{p}{100}P = P\left(1 + \frac{p}{100}\right).$$

and for $n \geq 2$, $P_n = P_{n-1} + \frac{p}{100}\ P_{n-1} = P_{n-1}\left(1 + \frac{p}{100}\right)$.

The first three terms of the sequence are

$$P_1 = P + \left(\frac{p}{100}\right)P = P\left(1 + \frac{p}{100}\right)$$

$$P_2 = P_1 + \left(\frac{p}{100}\right)P_1 = P_1\left(1 + \frac{p}{100}\right)$$

$$= P\left(1 + \frac{p}{100}\right)^2$$

$$P_3 = P_2 + \left(\frac{p}{100}\right)P_2 = P\left(1 + \frac{p}{100}\right)^3.$$

It is apparent that this pattern will continue and that for any n,

$$P_n = P\left(1 + \frac{p}{100}\right)^n.$$

### A simple application of this formula

The population of Nepal in 2035 is 11000000 and is believed to be increasing at the rate of 2·2% every year. If the same rate continues, the population of Nepal in 2040 B.S. will be

$$P_5 = 11000000\left(1 + \frac{2\cdot2}{100}\right)^5$$

$$= 11000000\ (1\cdot022)^5$$

$$= 11800000 \text{ (nearest to lakh).}$$

## 8.3 Types of Sequences

Various types of sequences are known. We shall be concerned with a few specific types of sequences only. They are arithmetic sequences, geometric sequences and harmonic sequences.

### a) Arithmetic sequences

A sequence of numbers in which a certain number can be added to (or subtracted from) any term to get the next term is called an *arithmetic sequence* (or *arithmetic progression*). In other words, *a sequence in which the difference between successive terms always has the same value is called an arithmetic sequence*. The arithmetic sequence is also known as the arithmetic progression. As an abbreviation, the arithmetic sequence and the arithmetic progression are respectively written as A.S. and A.P. respectively.

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For examples,

(a) 1, 3, 5, 7, 9 ... ...
(b) 1, 4, 7, 10, ... ...
(c) 10, 20, 30, ... ...
(d) –9, –6, –3, 0, 3, ... ...
(e) $a + 2b$, $3a + b$, $5a$, $7a - b$, ... ...

**b) Geometric sequences**

A sequence of numbers in which every term after the first may be obtained by multiplying the preceding term by a certain number is called a *geometric sequence* (or *geometric progression*). In other words, *a geometric sequence is a sequence of numbers in which the ratio of any term to its preceding term is always the same.* The geometric sequence is also known as the geometric progression. Geometric sequence and geometric progression are written as G.S. and G.P. respectively. Examples of such sequences are

(a) 1, 2, 4, 8, 16, ... ...
(b) 3, –9, 27, –81, ... ...
(c) 1, $x^2$, $x^4$, $x^6$, ... ... and so on.

**c) Harmonic sequences**

*A sequence is said to be a harmonic sequence if the reciprocals of its terms form an arithmetic sequence.* The harmonic sequence is also known as the harmonic progression. Harmonic sequence and harmonic progression are also written as H.S. and H.P. respectively. Some examples of harmonic sequence (harmonic progression) are

(a) 1, 1/2, 1/3, 1/4, ... ...
(b) 1, 1/3, 1/5, 1/7, ... ...
(c) 1/2, 1/4, 1/6, 1/8, ... ... and so on.

## 8.4 Series

A sequence may be finite or infinite. Given a finite sequence such as

1, 2, ... ..., 5 or 1, 1/2, 1/4, ..., 1/16,

we can form sums such as

1 + 2 + ... ... + 5 or 1 + 1/2 + 1/4 + ... ... + 1/16.

From elementary arithmetic, we know that

$$1 + 2 + \dots \dots + 5 = 15$$

and

$$1 + 1/2 + 1/4 + \dots \dots + 1/16 = 31/16.$$

The expressions 1 + 2 + ...... + 5 and 1 + 1/2 + ... ... + 1/16 are called the *series* associated with the sequences 1, 2, ... ... ... ..., 5 and 1, 1/2, 1/4, ... ... ..., 1/16 respectively.

In general, if $a_1, a_2, \dots \dots a_n$ denotes a finite sequence, the symbolic expression $a_1 + a_2 + a_3 + \dots \dots + a_n$ is called *finite series* associated with the sequence.

In other words, the expression $a_1 + a_2 + a_3 + \dots \dots + a_n$ represents the sum of the $n$ terms of the sequence $\{a_n\}$.

A short hand notation for the sum $a_1 + a_2 + a_3 + \dots \dots + a_n$ is

$$\sum_{k=1}^{n} a_k$$

where $\Sigma$ is the capital Greek letter sigma, and $\sum_{k=1}^{n} a_k$ is read, '*the sum of $a_k$ as k goes from 1 to n*'.

The symbol $k$ is called the variable of summation or a *dummy suffix* (why ?). Moreover, the sum is denoted by the symbol $S_n$. Thus

$$S_n = \sum_{k=1}^{n} a_k$$

As in the case of a sequence, we do have terms in a series. But these are same as the terms of the associated sequence.

If the sequence is an infinite sequence $a_1, a_2, a_3, \dots \dots, a_n, \dots \dots$, then the symbolic expression $a_1 + a_2 + a_3 + \dots \dots$

or $$S_n = \sum_{n=1}^{\infty} a_n$$

is called an *infinite series*. As a simple example of an infinite series, we have

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + \dots \dots \dots$$

associated with the infinite sequence of the natural numbers $\{n\}$. There is an important difference between the finite and infinite series. In former case, elementary arithmetic provides us with a *unique number* called the *sum* of the series. But in the later case, no process of elementary arithmetic will yield such a sum (Why ?). We shall come to this point sometime later.

## 8.5 Arithmetic Sequences and Series

An arithmetic sequence has the property that the difference between two successive terms (any term subtracted from the following term) is always the same. This difference is called the *common difference*. We shall use the following notations for terms and expressions involved in an arithmetic sequence and series.

$a_1$ = the first term,

$a_2$ = the second term,

$a_n$ = the $n^{th}$ term,

$n$ = the number of terms from $a_1$ up to and including $a_n$;

$d$ = the common difference,
$S_n$ = the sum of the first $n$ terms.

In terms of these notation, the arithmetic sequence and associated arithmetic series may be written in the following way :

Arithmetic sequence : $a_1, (a_1 + d), (a_1 + 2d), \ldots \ldots, (a_n - d), a_n$

Arithmetic series : $a_1 + (a_1 + d) + (a_1 + 2d) + \ldots \ldots + (a_n - d) + a_n$.

We now give some formulae connecting the various terms mentioned above.

**Formula 1.**

The common difference $d = a_{k+1} - a_k,\ k \geq 1$.

**Formula 2.**

The $n$th term of an arithmetic sequence is given by

$$a_n = a_1 + (n-1)d.$$

**Formula 3.**

The sum of the first $n$ terms of an arithmetic series is given by

$$S_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}[2a_1 + (n-1)d]$$

## Properties of Arithmetic Sequence

The following are the properties of an A. S.

(i) If each term of an A.S. be *increased* or *decreased* by a *constant* number, the resulting numbers are in A.S. with the same c. d. as before.

(ii) If each term of an A.S. be *multiplied* or *divided* by a *constant* number, the resulting numbers are in A.S. with a common difference equal to that of the given A.S. multiplied or divided by the corresponding constant number.

So, if $a, b, c, d, \ldots \ldots \ldots$ be an A. S. and $k$ a constant number not equal to zero, then

(i) $a + k,\ b + k,\ c + k,\ d + k, \ldots \ldots \ldots$

(ii) $a - k,\ b - k,\ c - k,\ d - k, \ldots \ldots \ldots$

(iii) $ak, bk, ck, dk, \ldots \ldots \ldots$

(iv) $\frac{a}{k}, \frac{b}{k}, \frac{c}{k}, \frac{d}{k}, \ldots \ldots \ldots$

are each an A.S.

## 8.6 Geometric Sequence and Series

A geometric sequence has the property that the ratio of any term (except the first) to its preceding term is always the same. This ratio is called the *common ratio* and is denoted by $r$. We follow the same notations as used in the case of an arithmetic sequence and series. Here also, we give some formulae connecting the terms, number of terms, sums, common ratio etc.

**Formula 1.**

The common ratio $\quad r = a_{k+1}/a_k \ , \ k \geq 1$

**Formula 2.**

In a geometric sequence, the $n^{\text{th}}$ term is given by

$$a_n = a_1 r^{n-1}$$

**Formula 3.**

The sum of the first $n$ terms of a geometric series is given by

$$S_n = \frac{a_1(1 - r^n)}{1 - r} \quad (r < 1) \qquad S_n = \frac{a_1(r^n - 1)}{r - 1} \quad (r > 1)$$

and also, $S_n = \dfrac{a_1 - a_n \cdot r}{1 - r}$

### Properties of Geometric Sequence

The following are the properties of a G.S.

(i) If each term of a G.S. is *multiplied* or *divided* by a *constant number*, the resulting sequence is also a G.S.

(ii) If each term of a G.S is raised to a *constant power*, the resulting sequence is again a G.S.

Thus if $a, b, c, d, \ldots \ldots\ldots \ldots$ be a G.S. and $k$ a constant number not equal to zero, then

(i) $ka, kb, kc, kd, \ \ldots \ldots\ldots \ldots\ldots$ is a G.S.

(ii) $\frac{a}{k}, \frac{b}{k}, \frac{c}{k}, \frac{d}{k}, \ \ldots \ldots\ldots \ldots\ldots$ is a G.S.

(iii) $a^k, b^k, c^k, d^k, \ \ldots \ldots\ldots \ldots\ldots$ is a G.S.

## 8.7 Harmonic Sequence and Series

A sequence of numbers will form a harmonic sequence (H.S.) if the sequence of numbers formed by taking the reciprocals of the terms forms an A.S. There are no formulae connecting the terms, number of terms, sum in

H.S. Since to every H.S. there always corresponds an A.S., the terms of an H.S. are determined with reference to the corresponding A.S. So, we follow the same notation as in the case of A.S.

### Properties of Harmonic Sequence

If each term of an H. S. be multiplied or divided by a constant quantity, the sequence of the resulting numbers is also an H. S. So, if $a, b, c, d, \ldots \ldots$ be an H. S., and $k$ a constant not equal to zero, then

(i) $ka, kb, kc, kd, \ldots \ldots\ldots \ldots$ is also an H. S.

(ii) $\frac{a}{k}, \frac{b}{k}, \frac{c}{k}, \frac{d}{k}, \ldots \ldots\ldots$ is again an H. S.

## 8.8 Means

A finite sequence consisting of more than two terms has one or more terms in between the first and the last terms. These terms are called the *arithmetic, geometric or harmonic means* according as the sequences are arithmetic, geometric or harmonic. In other words, we have the following definitions :

(a) Any term in between the first and last terms of an arithmetic sequence is called an *arithmetic mean* (A. M.).

(b) Any term in between the first and last terms of a geometric sequence is called a *geometric mean* (G. M.).

(c) Any term in between the first and last terms of a harmonic sequence is called a *harmonic mean* (H. M.).

### Formula for the Means

**Formula 1.**

Given any two numbers $a$ and $b$, the A.M., G.M. and H.M between them are given by

(a) $\text{A. M.} = \text{A} = \frac{a+b}{2}$

(b) $\text{G. M.} = \text{G} = \sqrt{ab}$

(c) $\text{H. M.} = \text{H} = \frac{2ab}{a+b}$

**Proof.**

(a) If A is the single A. M. between $a$ and $b$ , then $a$, A, $b$ form an A.S. Now by the definition of an A. S. ,

$$A - a = b - A$$

or, $2A = a + b$

$\therefore$ $A = \dfrac{a + b}{2}$

(b) If G is the single G. M. between $a$ and $b$, then $a$, G, $b$ form a G. S. Now by the definition of a G. S.,

$$\frac{G}{a} = \frac{b}{G}$$

or, $G^2 = ab$

$\therefore$ $G = \sqrt{ab}$

(c) If H be a single H. M. between $a$ and $b$, then $a$, H, $b$ are in H. P.

i.e. $\dfrac{1}{a}, \dfrac{1}{H}, \dfrac{1}{b}$ are in A. P.

Hence, $\dfrac{1}{H} - \dfrac{1}{a} = \dfrac{1}{b} - \dfrac{1}{H}$

or, $\dfrac{2}{H} = \dfrac{1}{a} + \dfrac{1}{b}$

or, $\dfrac{2}{H} = \dfrac{a + b}{ab}$

$\therefore$ $H = \dfrac{2ab}{a + b}$

**Formula 2.**

Given any two numbers $a$ and $b$, the $n$ A. M.'s between them are given by

$$a + d, a + 2d, a + 3d, \dots \dots \dots \dots, a + nd,$$

where $d = \dfrac{b - a}{n + 1}$

**Proof.**

Let $m_1, m_2, m_3, \dots \dots \dots \dots, m_n$ be $n$ A. M.'s to be inserted between $a$ and $b$. Then,

$$a, m_1, m_2, m_3, \dots \dots\dots \dots, m_n, b \text{ are in A. P.}$$

The number of terms in the above A. S. is $n + 2$ of which the first term is $a$ and the last term $b$, the $(n + 2)^{th}$ term of an A. S.

If $d$ is the common difference, then

$$b = a + (n + 2 - 1)\, d$$

or, $b - a = (n + 1)\, d$

$\therefore \quad d = \dfrac{b - a}{(n + 1)}$

Now, $m_1 = t_2 = a + d$

$m_2 = t_3 = a + 2d$

$m_3 = t_4 = a + 3d$

... ... ... ... ... ...

... ... ... ... ... ...

$m_n = t_{n+1} = a + nd$

**Formula 3.**

Given any two numbers $a$ and $b$, the $n$ G. M. 's between them are given by $ar, ar^2, ar^3, \ldots \ldots\ldots \ldots\ldots, ar^n$, where $r = \left(\dfrac{b}{a}\right)^{1/(n+1)}$

**Proof**

Let $G_1, G_2, G_3, \ldots \ldots\ldots \ldots, G_n$ be $n$ G. M. 's between $a$ and $b$ then,

$a, G_1, G_2, G_3, \ldots \ldots\ldots \ldots, G_n, b$ form a G. S.

The number of terms in the above G. S. is $(n + 2)$ of which the first term is $a$ and the last term $b$, the $(n + 2)^{th}$ term of a G. S.

If r be the common ratio, then

$b = ar^{n+2-1}$

or, $b = ar^{n+1}$

or, $r = \left(\dfrac{b}{a}\right)^{1/(n+1)}$

Now, $G_1 = t_2 = ar$

$G_2 = t_3 = ar^2$

$G_3 = t_4 = ar^3$

... ...... ......

... ...... ......

$G_n = t_{n+1} = ar^n$

**Formula 4.**

The A.M., G.M. and H.M. between any two unequal positive numbers satisfy the following relations

(a) $(G.M.)^2 = (A.M.) \times (H.M.)$

(b) A. M. > G. M. > H. M.

**Proof**

Let $a$ and $b$ be two unequal positive numbers. Then,

$$\text{A. M.} = \frac{a+b}{2}$$

$$\text{G. M.} = \sqrt{ab} \quad \text{( only positive square root is taken )}$$

and, $$\text{H. M.} = \frac{2ab}{a+b}$$

To prove the first part, we have

$$\text{A.M.} \times \text{H.M.} = \frac{a+b}{2} \times \frac{2ab}{a+b}$$

$$= ab = (\sqrt{ab})^2$$

$$= (\text{G.M.})^2$$

This result shows that G.M. is again the geometric mean between A.M. and H.M.

To prove the second part, consider

$$\text{A.M.} - \text{G.M.} = \frac{a+b}{2} - \sqrt{ab}$$

$$= \frac{a+b-2\sqrt{ab}}{2}$$

$$= \frac{1}{2}(\sqrt{a} - \sqrt{b})^2$$

which is always positive (why ?)

Hence, A. M. > G. M.

Again, (A. M.) × (H. M.) = (G. M.) × (G. M.)

or, $$\frac{\text{A. M.}}{\text{G. M.}} = \frac{\text{G. M.}}{\text{H. M.}}$$

Since, A. M. > G. M.

we have G. M. > H. M.

Combining the two, we have

A. M. > G. M. > H. M.

This result shows that A.M., G.M. and H.M. are in decreasing order of magnitudes.

## Worked Out Examples

**Example 1.**

If $a^2$, $b^2$, $c^2$, are in A.P., prove that $b+c$, $c+a$, $a+b$ are in H. P.

**Solution :**

Here, $b+c,\ c+a,\ a+b$ will be in H. P.

if $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$, are in A. P.

i.e. if $\frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a}$

or, $\frac{b+c-c-a}{(c+a)(b+c)} = \frac{c+a-a-b}{(a+b)(c+a)}$

or, $\frac{b-a}{(c+a)(b+c)} = \frac{c-b}{(a+b)(c+a)}$

or, $b^2 - a^2 = c^2 - b^2$

i.e. if $a^2, b^2, c^2$, are in A. P. which is given.

**Example 2.**

The G. M. and H. M. between two numbers are respectively 9 and 5·4. Find the numbers.

**Solution :**

Let $a$ and $b$ be two numbers. Then

$$\text{G. M.} = \sqrt{ab}$$

$$9 = \sqrt{ab}$$

$$\therefore \quad ab = 81 \ \ldots\ \ldots\ldots \ldots\text{(i)}$$

Again, $\text{H. M.} = \frac{2ab}{a+b}$

$$5{\cdot}4 = \frac{2ab}{a+b}$$

or, $5{\cdot}4 = \frac{2 \times 81}{a+b}$ (From (i) )

$$\therefore \quad a+b = 30$$

$$\therefore \quad b = 30 - a$$

Substituting the value of $b$ in (i)

$$a(30-a) = 81$$

or, $-a^2 + 30a - 81 = 0$

$\therefore \quad a^2 - 30a + 81 = 0$

$(a-3)(a-27) = 0$

$\therefore \quad a = 3 \text{ or } 27$

When $a = 3$, $\quad b = 30 - a = 30 - 3 = 27$

When $a = 27$, $\quad b = 30 - a = 30 - 27 = 3$

$\therefore$ the required numbers are 3, 27 or 27, 3.

**Example 3.**

If the three consecutive terms of a G. S. be increased by their middle term, then prove that the resulting terms will be in H. S.

**Solution :**

Let $a, b$ and $c$ be three terms of a G. S.

Then, $\quad \dfrac{b}{a} = \dfrac{c}{b}$

$\therefore \quad b^2 = ac \ldots \ldots\ldots \ldots(i)$

When the middle term $b$ be added to each of the terms $a, b, c$, then we prove that the resulting terms will be in H. P. For this, we shall show that, $a+b, b+b, b+c$ i.e. $a+b, 2b, b+c$ will be in H. P.

This holds

if, $\quad \dfrac{1}{a+b}, \dfrac{1}{2b}, \dfrac{1}{b+c},$ are in A. P.

if, $\quad \dfrac{1}{2b} - \dfrac{1}{a+b} = \dfrac{1}{b+c} - \dfrac{1}{2b}$

or, $\quad \dfrac{a+b-2b}{2b(a+b)} = \dfrac{2b-b-c}{2b(b+c)}$

or, $\quad \dfrac{a-b}{a+b} = \dfrac{b-c}{b+c}$

or, $\quad 2ac = 2b^2$

or, $\quad b^2 = ac$ which is true by (i)

**Example 4.**

If $a, b, c$, be in H. P., prove that $\dfrac{b+a}{b-a} + \dfrac{b+c}{b-c} = 2$

**Solution :**

Since $a, b, c$ are in H. P.,

we have $\quad b = \dfrac{2ac}{a+c}$

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$$\text{Now, } \frac{b+a}{b-a} + \frac{b+c}{b-c}$$

$$= \frac{\frac{2ac}{a+c} + a}{\frac{2ac}{a+c} - a} + \frac{\frac{2ac}{a+c} + c}{\frac{2ac}{a+c} - c}$$

$$= \frac{2ac + a^2 + ac}{2ac - a^2 - ac} + \frac{2ac + ac + c^2}{2ac - ac - c^2}$$

$$= \frac{3c + a}{c - a} - \frac{3a + c}{c - a}$$

$$= \frac{3c + a - 3a - c}{c - a}$$

$$= \frac{2(c - a)}{c - a} = 2$$

***Example 5.***

Show that if three quantities form any two of the three sequences A.S., G.S. and H.S., then they also form the remaining third sequence.

***Solution :***

Let $a$, $b$ and $c$ be three quantities which form A. S. and G. S.

Now, we prove that the three quantities also form H. S.

Since, $a, b, c$ form an A. S.

so, $\quad 2b = a + c \quad \ldots\ldots\ldots$(i)

Again since $a, b, c$ form a G. S. ,

so, $\quad b^2 = ac \quad \ldots\ldots\ldots$(ii)

Multiplying (i) by b and using (ii) we have

$$2ac = b\,(a + c)$$

$$\therefore \quad b = \frac{2ac}{a+c}$$

$\therefore \quad a, b, c$ form a H. S.

Similarly, we can show that $a, b, c$ form an A. S. when they form G. S. and H. S. ; and G. S. when they form A. S. and H. S.

***Example 6***

If H be the H.M. between $a$ and $b$, prove that $(H - 2a)(H - 2b) = H^2$.

***Solution :***

Since H is the H.M., so

$$H = \frac{2ab}{a+b} \qquad \ldots\ldots \text{(i)}$$

Now, $(H - 2a)(H - 2b)$

$$= \left(\frac{2ab}{a+b} - 2a\right)\left(\frac{2ab}{a+b} - 2b\right)$$

$$= 2a.2b\left(\frac{b}{a+b} - 1\right).\left(\frac{a}{a+b} - 1\right)$$

$$= 4ab.\left(-\frac{a}{a+b}\right).\left(-\frac{b}{a+b}\right)$$

$$= \frac{4a^2b^2}{(a+b)^2} = \left(\frac{2ab}{a+b}\right)^2$$

$$= H^2$$

**Example 7**

If one A.M. 'A' and the two G.M.'s $G_1$ and $G_2$ are inserted between two given positive numbers, prove that

$$\frac{G_1^2}{G_2} + \frac{G_2^2}{G_1} = 2A$$

**Solution :**

Let $a$ and $b$ be two given positive numbers. Then,

$$A = \frac{a+b}{2} \qquad \ldots\ldots \text{(i)}$$

$a, G_1, G_2, b$ are in G.P.

So, $\frac{G_1}{a} = \frac{G_2}{G_1} = \frac{b}{G_2}$

which gives,

$$G_1^2 = aG_2 \qquad \Rightarrow \qquad \frac{G_1^2}{G_2} = a$$

and

$$G_2^2 = bG_1 \qquad \Rightarrow \qquad \frac{G_2^2}{G_1} = b$$

Adding the two results,

$$\frac{G_1^2}{G_2} + \frac{G_2^2}{G_1} = a + b$$

or,

$$\frac{G_1^2}{G_2} + \frac{G_2^2}{G_1} = 2A \qquad \text{(from (i))}$$

***Example 8***

If $a, b, c$ are in A.P., $a, x, b$ are in A.P. and $b, y, c$ are in A.P. prove that $\frac{1}{x} + \frac{1}{y} = \frac{2}{b}$

***Solution :***

Since, $a, b, c$ are in G.P., so

$$b^2 = ac \qquad \text{...... (i)}$$

$a, x, b$ are in A.P., so

$$x = \frac{a+b}{2} \qquad \text{...... (ii)}$$

Again, $b, y, c$ are in A.P., so

$$y = \frac{b+c}{2} \qquad \text{...... (iii)}$$

$$\text{Now, } \frac{1}{x} + \frac{1}{y} = \frac{2}{a+b} + \frac{2}{b+c}$$

$$= \frac{2(a+2b+c)}{(a+b)(b+c)}$$

$$= \frac{2(a+2b+c)}{ab+2b^2+bc} \qquad (\because b^2 = ac)$$

$$= \frac{2(a+2b+c)}{b(a+2b+c)}$$

$$= \frac{2}{b}$$

## EXERCISE 8.1

1. a) Prove that $x, y, z$ are in A.P., G.P. or H.P. according as

$$\frac{x-y}{y-z} = \frac{x}{x} \text{ or } \frac{x}{y} \text{ or } \frac{x}{z} \text{ respectively.}$$

b) If $\frac{1}{2}(x+y)$, $y$ and $\frac{1}{2}(y+z)$ be in H.P., show that $x, y, z$ are in G.P.

c) If G is the geometric mean between $a$ and $b$, show that

$$\frac{1}{G^2 - a^2} + \frac{1}{G^2 - b^2} = \frac{1}{G^2}$$

2. a) If H be the harmonic mean between $a$ and $b$, prove that

$$\frac{1}{H-a}+\frac{1}{H-b}=\frac{1}{a}+\frac{1}{b}$$

b) If A be the arithmetic mean and H, the H.M. between two quantities $a$ and $b$, show that

$$\frac{a-A}{a-H}\times\frac{b-A}{b-H}=\frac{A}{H}$$

3. a) If $x$ be the A.M. between $y$ and $z$, $y$ the G.M. between $z$ and $x$, prove tht $z$ will be the H.M. between $x$ and $y$.

b) If $a, b, c$ be in A.P., $b, c, a$ in H.P., prove that $c, a, b$ are in G.P.

c) If $a, b, c$ be in A.P., $b, c, d$ in G.P. and $c, d, e$ in H.P., prove that $a, c, e$ are in G.P.

4. Show that $b^2$ is greater than, equal to or less than $ac$, according as $a, b, c$ are in A.P., G.P. or H.P.

5. a) Find two numbers whose arithmetic mean is 25 and geometric mean is 20.

b) The A.M. between two numbers exceeds their G.M. by 2 and the G.M. exceeds the H.M. by 1.6. Find the numbers.

6. The sum of three numbers in A.P. is 36. When the numbers are increased by 1, 4, 43 respectively, the resulting numbers are in G.P. Find the numbers.

7. a) If $a^x = b^y = c^z$ and $a, b, c$ are in G.P., prove that $x, y, z$ are in H.P.

b) If one G.M. 'G' and two A.M.'s $p$ and $q$ are inserted between two given positive numbers, prove that

$$G^2 = (2p-q)(2q-p)$$

8. If $a, b, c$ are in H.P., prove that

i) $\frac{bc}{b+c}, \frac{ca}{c+a}, \frac{ab}{a+b}$ are in H.P.

ii) $2a-b, b, 2c-b$ are in G.P.

iii) $a(b+c), b(c+a), c(a+b)$ are in A.P.

### *Answers*

5. a) 40, 10 or 10, 40 b) 4, 16 or 16, 4

6. a) 3, 12, 21 or 63, 12, –39

## 8.9 Infinite Series

Finite series may be added to obtain a certain number called the *sum* of the series. For example

$$1 + 2 + 3 + 4 + 5 = 15$$

$$1 + 2 + 3 + \dots \dots + 100 = 5050$$

and so on.

Let us now see what happens when we proceed in the same manner in the case of an infinite series. Consider the infinite series

$$1 + 2 + 3 + \dots \dots \dots \dots + 100 + \dots \dots \dots \dots$$

We may add up to 5 terms to get $S_5 = 15$

... ... ... ... ... ... 100 terms to get $S_{100} = 5050$

... ... ... ... ... ... $n$ terms ... ... ... $S_n = \frac{n}{2}(n + 1)$

and so on.

The process will never come to an end. We may choose a very large number of terms, say, 10000000000000000; but the series will not be exhausted. For there is another term, namely the 10000000000000001st term.

In other words, having chosen a number N, as *large as we please* (*however large it may be*), there are still values of $n$ (i.e., the number of terms) greater than N. This situation is mathematically described by the statement : '*n tends to infinity*' or '*n approaches infinity.*' In symbols we write $n \to \infty$.

Another important feature of the above example is that the larger the number of terms (i.e. the value of $n$) , the greater is the value of the sum $S_n$; and actually approaches (or tends) to *infinity*.

Summarising, we may say that as *n tends to infinity* (or as the number of terms approaches infinity), *the sum of these terms* also *approaches infinity*. In symbols,

$$S_n \to \infty \quad \text{as} \quad n \to \infty.$$

This is read. '$S_n$ *tends to infinity as n tends to infinity*'.

We now consider an interesting example illustrating the consequence of employing indiscriminately the ordinary process of addition. Let

$$1 - 1 + 1 - 1 + 1 - \dots \dots$$

be an infinite series. Consider the sum of this series as indicated below :

| | | |
|---|---|---|
| $1$ | $= 1$ | 1 term |
| $1 - 1$ | $= 0$ | 2 terms |
| $1 - 1 + 1$ | $= 1$ | 3 terms |
| $1 - 1 + 1 - 1$ | $= 0$ | 4 terms |

$1 - 1 + 1 - 1 + 1 \quad = 1 \quad$ 5 terms

... ... ... ... ... ... ... ... ... ...

$1 - 1 + 1 \ldots \ldots \quad = 1 \quad$ if $n$ is odd

$\quad\quad\quad\quad\quad\quad = 0 \quad$ if $n$ is even

... ... ... ... ... ... ... ... ... ... ... ... ... ... ... ...

On the other hand, if we write

$S_\infty = 1 - 1 + 1 - 1 + 1 - 1 + \ldots \ldots \ldots \ldots,$

we have

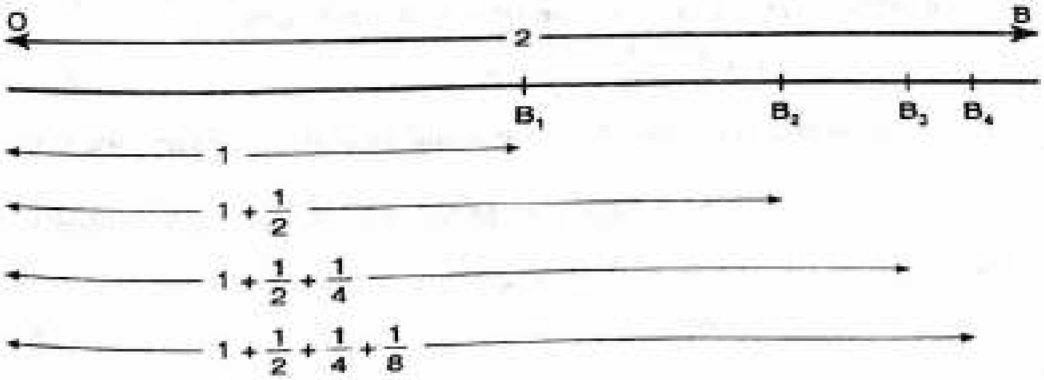
$$S_\infty = 1 - (1 - 1 + 1 - 1 + \ldots \ldots) = 1 - S_\infty$$

or $\quad 2S_\infty = 1$

Hence $\quad S_\infty = \frac{1}{2}$

As a third and final example, we consider a geometrical problem. Consider a line-segment OB of 2 units length. Bisect it at $B_1$. Then $OB_1 = 1$ and $B_1B = OB - OB_1 = 2 - 1 = 1$. Leave the left-hand half as it is; and bisect the right-hand half (i.e. $B_1B$) at $B_2$ so that

$$OB_2 = OB_1 + B_1B_2 = 1 + \frac{1}{2} = \frac{3}{2} \quad \text{and} \quad B_2B = OB - OB_2 = 2 - \frac{3}{2} = \frac{1}{2}$$




Again leave the portion $B_1B_2$ as it is ; bisect the portion $B_2B$ at $B_3$ so that

$$OB_3 = OB_1 + B_1B_2 + B_2B_3 = 1 + \frac{1}{2} + \frac{1}{4} = 1 + \frac{1}{2} + \frac{1}{2^2} = \frac{7}{4}$$

and

$$B_3B = OB - OB_3 = 2 - \frac{7}{4} = \frac{1}{4} = \frac{1}{2^2}$$

Repeating the above procedure of continued bisection of the right-hand half $n$ times, we arrive at

$$OB_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots\dots + \frac{1}{2^{n-1}}$$

and $$B_nB = \frac{1}{2^{n-1}}$$

The above construction shows that the point of bisection moves closer and closer to the end point B as the number of points of bisection increases; but never reaches the point B. In other words, the length of the left-hand segment becomes larger and larger and the segment $B_nB$ becomes smaller and smaller. (Note that the left-hand segment can never be larger than the whole segment OB and the length of the right-hand segment never be negative). To be more precise, we can make the length of the right-hand segment *as small as we please* by choosing a *sufficiently large value of n*. For instance, if we would like to make the length of the right-hand segment $B_nB = \frac{1}{2^{n-1}}$ smaller than, say $\frac{1}{1000000000000000}$ we may choose n to be 64 or any number greater than 64 (How ?). The same reason tells us that the left-hand segment

$$OB_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\dots + \frac{1}{2^{n-1}}$$

can be made *as close to the real number 2 as we please* ; but never *equal to* 2. It is in this sense that we say the sum of the infinite series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{2^3} + \dots\dots$$

is 2. This is obviously different from the usual sum.In summary, we may say that

(a) $B_n$ approaches B as the number '$n$' of bisections *approaches* infinity,

(b) the length of the segment $B_nB = \frac{1}{2^{n-1}}$ *approaches* zero as $n$ *tends to* (approaches) infinity and

(c) the sum
$$S_n = OB_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{2^3} + \dots\dots + \frac{1}{2^{n-1}}$$
approaches the value 2 as $n$ approaches infinity.

Symbolically, we write

(a) $B_n \to B$ as $n \to \infty$

(b) $B_nB = \frac{1}{2^{n-1}} \to 0$ as $n \to \infty$, and

(c) $OB_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{2^3} + \ldots\ldots + \frac{1}{2^{n-1}}$

$$= \frac{1 - 1\cdot\left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} \to 2 \text{ as } n \to \infty.$$

In general, *an infinite series is said to have a sum if the sum of the first n terms $S_n$ gets closer and closer to some real number as the number of terms gets larger and larger.*

Of the two sequences and series we have considered, the arithmetic series has no such sum. The geometric series may have a sum. A geometric series will have a sum if and only if the *numerical value of the common ratio r* is less than 1, i.e. $|r| < 1$. We shall assume this result. With this assumption, we now derive a formula for the sum of an infinite geometric series :

$$a_1 + a_1r + a_1r^2 + \ldots\ldots + a_1r^{n-1} + \ldots\ldots$$

From formula 3 of Art. 9.6, the sum to $n$ terms of a G.S. with first term $a$ and common ratio $r$ is

$$S = \frac{a(1 - r^n)}{1 - r} = \frac{a - ar^n}{1 - r}$$

$$= \frac{a}{1 - r} - \frac{ar^n}{1 - r}$$

If $|r| < 1$, $n \to \infty$, $r^n \to 0$ (How ?)

Hence, $S_\infty = \frac{a}{1 - r}$

## Worked Out Examples

**Example 1**

Use geometrical series to express $0.\dot{5}$ as a rational number.

**Solution :**

$0.\dot{5} = 0.5555\ldots\ldots\ldots$

$= 0.5 + 0.05 + 0.005 + 0.0005 + \ldots\ldots\ldots$

$= \frac{0.5}{1 - 0.10} = \frac{0.5}{0.90} = \frac{5}{9}$

**Example 2**

Prove that : $2^{1/2}.2^{1/4}.2^{1/8}\ldots\ldots\ldots = 2$

GK

***Solution :***

$2^{1/2}.2^{1/4}.2^{1/8}$ .........

$= 2^{1/2 + 1/4 + 1/8 + ........}$

But $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ......$ is an infinite geometric series of first term $\frac{1}{2}$ and common ratio $\frac{1}{2}$.

$$\therefore \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...... = \frac{\frac{1}{2}}{1 - \frac{1}{2}} \qquad \left(S_\infty = \frac{a}{1-r}\right)$$

$$= 1$$

$$\therefore 2^{1/2}.2^{1/4}.2^{1/8}...... = 2^{1/2 + 1/4 + 1/8 + ......} = 2^1 = 2$$

***Example 3***

If $|x| < 1$, $y = x + x^2 + x^3 + ...... \infty$, prove that $x = \frac{y}{1+y}$

***Solution :***

$y = x + x^2 + x^3 + ...... \infty$

The series on the R.H.S. is an infinite geometric series with first term $x$ and the common ratio is also $x$. So,

$$y = \frac{x}{1-x} \qquad \left(\because S_\infty = \frac{a}{1-r}\right)$$

$\Rightarrow \quad y - xy = x$

$\Rightarrow \quad y = x(1 + y)$

$\therefore \quad x = \frac{y}{1+y}$

## EXERCISE 8.2

1. Decide which infinite series have sums :
   a) $1 + 2 + 2^2 + 2^3 + .........$
   b) $1 + (-1) + 1 + (-1) + .........$
   c) $1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + ......$

d) $3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \dots\dots\dots$

e) $0.6 + 0.06 + 0.006 + \dots\dots\dots$

2. Find the sum of each of the following geometric series :

a) $16 + 8 + 4 + 2 + \dots\dots$

b) $5 + \frac{5}{2} + \frac{5}{4} + \frac{5}{8} + \dots\dots$

c) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots\dots$

d) $4^{-1} + 4^{-2} + 4^{-3} + \dots\dots$

e) $3 + \sqrt{3} + 1 + \dots\dots$

3. The sum to infinity of a geometric series is 15, and the first term is 3. Find the common ratio.

4. The sum of an infinite number of terms in G. S. is 15, and the sum of their squares is 45 ; find the series.

5. Prove that $3^{1/3} . 3^{1/9} . 3^{1/27} \dots\dots = \sqrt{3}$.

6. If $x = \frac{1}{1+y}$, $y > 0$, prove that $x + x^2 + x^3 + \dots\dots \infty = \frac{1}{y}$

***Answer***

1. (c), (d), (e). 2 (a) 32 (b) 10 (c) $\frac{2}{3}$ (d) $\frac{1}{3}$ (e) $\frac{9+3\sqrt{3}}{2}$

3. $\frac{4}{5}$ 4. $5 + \frac{10}{3} + \frac{20}{9} + \dots\dots$

## 8.10 Sum of the Natural Numbers

The numbers 1, 2, 3, ... ... ... are said to be the natural numbers. Now we derive some of the formulae for the sum of the first $n$ natural numbers, the sum of the squares of first $n$ natural numbers and the sum of the cubes of first $n$ natural numbers.

### (i) Sum of the first $n$ natural numbers

The first $n$ natural numbers are 1, 2, 3, ... ...... ...., $n$

Let $S_n = 1 + 2 + 3 + \dots \dots\dots \dots + n$

$= \frac{n}{2} [2 \cdot 1 + (n-1) \cdot 1]$ $[S_n = \frac{n}{2} [2a + (n-1)d]]$

$$= \frac{n(n+1)}{2}$$

$$\therefore \quad S_n = \frac{n(n+1)}{2}$$

**(ii) Sum of the first $n$ even natural numbers**

The first $n$ even natural numbers are $2, 4, 6, \ldots \ldots\ldots n$ terms

Let $\quad S_n = 2 + 4 + 6 + \ldots \ldots\ldots \ldots$ to $n$ terms

$$= 2(1 + 2 + 3 + \ldots \ldots\ldots \text{ to } n \text{ terms})$$

$$= 2 \cdot \frac{n(n+1)}{2}$$

$$= n(n+1)$$

**(iii) Sum of the first $n$ odd natural numbers**

The first $n$ odd natural numbers are $1, 3, 5, \ldots \ldots \ldots\ldots, n$ terms

Let $\quad S_n = 1 + 3 + 5 + \ldots \ldots \ldots$ to $n$ terms

$$= \frac{n}{2}\,[2\cdot 1 + (n-1)\cdot 2] \qquad [\,S_n = \frac{n}{2}\,\{2a + (n-1)\,d\}]$$

$$= n(1 + n - 1) = n^2$$

**(iv) Sum of the squares of the first $n$ natural numbers**

Let $\quad S_n = 1^2 + 2^2 + 3^2 + \ldots \ldots\ldots + n^2,$

We know $r^3 - (r-1)^3 = 3r^2 - 3r + 1$

This is an identity and is true for all values of $r$. Putting $r$ equal to $1, 2, 3, \ldots \ldots\ldots, n$ respectively

we have

$$1^3 - 0^3 = 3\cdot 1^2 - 3\cdot 1 + 1$$
$$2^3 - 1^3 = 3\cdot 2^2 - 3\cdot 2 + 1$$
$$3^3 - 2^3 = 3\cdot 3^2 - 3\cdot 3 + 1$$
$$4^3 - 3^3 = 3\cdot 4^2 - 3\cdot 4 + 1$$
$$\ldots \ldots\ldots \quad \ldots \ldots\ldots \quad \ldots \ldots\ldots$$
$$n^3 - (n-1)^3 = 3n^2 - 3n + 1$$

Adding we get,

$$n^3 = 3(1^2 + 2^2 + 3^2 + \ldots + n^2) - 3(1 + 2 + 3 + \ldots + n) + n$$

or, $\quad n^3 = 3S_n - 3\cdot \frac{1}{2}\, n(n+1) + n$

or, $\quad 3S_n = n^3 - n + \frac{3n(n+1)}{2}$

$$= n(n^2 - 1) + \frac{3n(n+1)}{2}$$
$$= n(n+1)(n-1) + \frac{3n(n+1)}{2}$$
$$= \frac{1}{2}\ n(n+1)(2n+1)$$
$$\therefore \quad S_n = \frac{n(n+1)(2n+1)}{6}$$

**(v) Sum of the cubes of the first $n$ natural numbers**

Let $\quad S_n = 1^3 + 2^3 + 3^3 + \ldots \ldots\ldots + n^3$

We know, $r^4 - (r-1)^4 = 4r^3 - 6r^2 + 4r - 1$

Putting $r = 1, 2, 3, \ldots \ldots\ldots, n$ in succession, we have

$$1^4 - 0^4 = 4\cdot1^3 - 6\cdot1^2 + 4\cdot1 - 1$$
$$2^4 - 1^4 = 4\cdot2^3 - 6\cdot2^2 + 4\cdot2 - 1$$
$$3^4 - 2^4 = 4\cdot3^3 - 6\cdot3^2 + 4\cdot3 - 1$$
$$4^4 - 3^4 = 4\cdot4^3 - 6\cdot4^2 + 4\cdot4 - 1$$
$$\ldots \ldots\ldots \ldots\ldots \ldots\ldots \ldots\ldots \ldots\ldots \ldots$$
$$n^4 - (n-1)^4 = 4\cdot n^3 - 6\cdot n^2 + 4\cdot n - 1$$

Adding $n^4 = 4(1^3 + 2^3 + 3^3 + \ldots + n^3) - 6(1^2 + 2^2 + \ldots + n^2) + 4(1 + 2 + 3 + \ldots \ldots\ldots + n) - n$

$$n^4 = 4S_n - 6\,\frac{n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2} - n$$

or, $\quad 4S_n = n^4 + n + n(n+1)(2n+1) - 2n(n+1)$

$$= n(n^3 + 1) + n(n+1)(2n+1) - 2n(n+1)$$
$$= n(n+1)(n^2 - n + 1) + n(n+1)(2n+1) - 2n(n+1)$$
$$= n(n+1)(n^2 - n + 1 + 2n + 1 - 2)$$
$$= n(n+1)(n^2 + n)$$
$$= n^2(n+1)^2$$

or, $\quad S_n = \dfrac{n^2(n+1)^2}{4}$

$$\therefore \quad S_n = \Sigma n^3 = \frac{n^2(n+1)^2}{4} = \left(\frac{n(n+1)}{2}\right)^2$$

*Cor.* $\quad 1^3 + 2^3 + 3^3 + \ldots\ldots\ldots + n^3 = \left(\dfrac{n(n+1)}{2}\right)^2$

and $\because \quad 1 + 2 + 3 + \ldots\ldots\ldots + n = \dfrac{n(n+1)}{2}$.

it is clear that the sum of the cubes of the first $n$ natural numbers is the square of the sum of the first $n$ natural numbers.

## Sum of the series using the method of successive differences

Sometimes, the terms of the given series may not be in A.S. or G.S. But the difference of the successive terms of the given series will be in A.S. or G.S. and then we can easily find the $n$th term and the corresponding sum of $n$ terms as well.

Let us see the following example

$$1 + 5 + 12 + 23 + \ldots\ldots$$

The terms of the above series are neither in A.S. nor in G.S.

Let $\quad S = 1 + 5 + 12 + 22 + \ldots\ldots + t_n$

Also, $\quad S = \quad 1 + 5 + 12 + \ldots\ldots + t_{n-1} + t_n$

$$0 = 1 + 4 + 7 + 10 + \ldots\ldots\ n^{\text{th}} \text{ term} - t_n$$

$\therefore \quad t_n = 1 + 4 + 7 + 10 + \ldots\ldots$

Thus, we see that the terms after getting the difference of the successive terms are in A.S. Now we can find the $n^{\text{th}}$ term and the corresponding sum easily.

## Arithmetico-Geometric Series

The terms of an A.S. are

$$a, a + d, a + 2d, \ldots\ldots$$

The terms of a G.S. are

$$1, r, r^2, \ldots\ldots$$

A series of the type

$$a.1 + (a + d)r + (a + 2d)r^2 + \ldots\ldots$$

whose each term is the product of the corresponding terms of an A.S. and a G.S. is known as the arithmetico-geometric series.

The following example will illustrate the method of finding the sum of the above type of series.

Sum to $n$ terms of the following series :

$$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \ldots\ldots$$

The given series may be written as follows:

$$1.\frac{1}{2} + 2.\frac{1}{4} + 3.\frac{1}{8} + 4.\frac{1}{16} + \ldots\ldots$$

Clearly, the above series is the arithmetico-geometric series. The common ratio of the G.S.

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\ldots \text{ is } \frac{1}{2}.$$

Let $S = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \ldots\ldots + \frac{n}{2^n}$

Then, $\frac{1}{2}S = \quad \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \ldots\ldots + \frac{n-1}{2^n} + \frac{n}{2^{n+1}}$

$$- \qquad - \quad - \quad - \qquad - \qquad - \qquad -$$

$$\frac{1}{2}S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots\ldots \frac{1}{2^n} - \frac{n}{2^{n+1}}$$

$$\frac{1}{2}S = \frac{\frac{1}{2}\left\{1 - \left(\frac{1}{2}\right)^n\right\}}{1 - \frac{1}{2}} - \frac{n}{2^{n+1}} \qquad \left(\because S_n = \frac{a(1-r^n)}{1-r}\right)$$

$$\frac{1}{2}S = 1 - \frac{1}{2^n} - \frac{n}{2^{n+1}}$$

$$\therefore \quad S = 2 - \frac{1}{2^{n-1}} - \frac{n}{2^n}$$

## Worked Out Examples

***Example 1***

Sum to $n$ terms the series : $2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + \ldots \ldots\ldots$

**Solution :**

Here, the $r^{th}$ term of 2, 3, 4, ... ...... $= r + 1$

and the $r^{th}$ term of 3, 4, 5, ... ...... $= r + 2$

Hence, the $r^{th}$ term of the given series is

$$t_r = (r+1)(r+2) = r^2 + 3r + 2$$

Thus, sum of $n$ terms of the given series

$$\sum_{r=1}^{n} t_r = \sum_{r=1}^{n} (r^2 + 3r + 2)$$

$$= \sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r + 2n$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2} + 2n$$

$$= \frac{n(n^2+6n+11)}{3}.$$

***Example 2***

Find the $n^{th}$ term and then the sum of the first $n$ terms of the series

$$1 + 3 + 6 + 10 + \dots\dots\dots$$

***Solution :***

Let $t_n$ be the $n^{th}$ term and $S_n$ the sum of the first $n$ terms of

$$1 + 3 + 6 + 10 + \dots\dots\dots$$

then, $S_n = 1 + 3 + 6 + 10 + \dots\dots\dots + t_{n-1} + t_n$

Also, $S_n = \quad 1 + 3 + 6 + \dots\dots\dots + t_{n-2} + t_{n-1} + t_n$

$- \quad - \quad - \quad - \quad - \quad - \quad - \quad -$

Subtraction yields, $0 = 1 + 2 + 3 + \dots\dots\dots + (t_n - t_{n-1}) - t_n$

or, $t_n = 1 + 2 + 3 + \dots\dots\dots$ to $n$ terms

$$= \frac{n(n+1)}{2}$$

$$= \frac{1}{2}n^2 + \frac{1}{2}n$$

Hence, $S_n = \frac{1}{2}\Sigma n^2 + \frac{1}{2}\Sigma n$

$$= \frac{1}{2}(1^2 + 2^2 + 3^2 + \dots\dots + n^2) + \frac{1}{2}(1 + 2 + 3 + \dots\dots + n)$$

$$= \frac{1}{2}\frac{n(n+1)(2n+1)}{6} + \frac{1}{2}\frac{n(n+1)}{2}$$

$$= \frac{1}{4}n(n+1)\left\{\frac{(2n+1)}{3} + 1\right\}$$

$$= \frac{1}{4}n(n+1)\frac{(2n+1+3)}{3}$$

$$= \frac{n(n+1)(n+2)}{6}.$$

***Example 3***

Sum to $n$ terms of the following series :

$$1 + (1+2) + (1+2+3) + \dots\dots$$

**Solution :**

$t_n = n^{th}$ term of the given series

$= 1 + 2 + 3 + \ldots\ldots + n = \frac{n(n+1)}{2}$

Now, sum to $n$ term of given series is

$\Sigma t_n = \frac{1}{2}\Sigma n^2 + \frac{1}{2}\Sigma n$

$S_n = \frac{1}{2}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{1}{2}\cdot\frac{n(n+1)}{2}$

$= \frac{1}{4}n(n+1)\left\{\frac{2n+1}{3} + 1\right\}$

$= \frac{1}{12}n(n+1)\cdot 2(n+2)$

$= \frac{1}{6}n(n+1)(n+2)$

**Example 4**

Find the sum to $n$ terms of the following series

$4 + 44 + 444 + \ldots\ldots\ldots$

**Solution :**

Let S be the required sum. Then,

$S = 4 + 44 + 444 + \ldots\ldots\ldots\ n$ terms

$= \frac{4}{9}(9 + 99 + 999 + \ldots\ldots\ldots\ n \text{ terms})$

$= \frac{4}{9}\{(10-1) + (100-1) + (1000-1) + \ldots\ldots\}$

$= \frac{4}{9}\{(10 + 10^2 + 100^3 + \ldots\ldots\ n \text{ terms}) - (1 + 1 + 1 + \ldots\ldots\ n \text{ terms})\}$

$= \frac{4}{9}\left\{\frac{10(10^n-1)}{10-1} - n\right\} \qquad \left(\because S_n = \frac{a(r^n-1)}{r-1}\right)$

$= \frac{40}{81}(10^n - 1) - \frac{4}{9}n$

**Example 5**

Sum $1 + \frac{3}{2} + \frac{5}{4} + \frac{7}{8} + \ldots\ \ldots\ \ldots$ to infinity.

***Solution :***

Let $\qquad S = 1 + \frac{3}{2} + \frac{5}{4} + \frac{7}{8} + \ldots \ldots \ldots$

then, $\qquad \frac{1}{2} S = \qquad \frac{1}{2} + \frac{3}{4} + \frac{5}{8} + \ldots \ldots\ldots$

By subtraction, we get

$$\left(1 - \frac{1}{2}\right) S = 1 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \ldots\ldots$$

$$= 1 + \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \ldots\right)$$

or, $\qquad \frac{1}{2} S = 1 + \frac{1}{1 - \frac{1}{2}}$

or, $\qquad \frac{1}{2} S = 1 + 2 = 3$

$\therefore \qquad S = 6.$

***Example 6***

Sum to $n$ terms of the series : $1 + 2a + 3a^2 + 4a^3 + \ldots \ldots\ldots$

***Solution :***

Let $\quad S_n = 1 + 2a + 3a^2 + 4a^3 + \ldots \ldots\ldots + (n-1)\, a^{n-2} + na^{n-1}$

then, $aS_n = \qquad a + 2a^2 + 3a^3 + \ldots \ldots\ldots + (n-1)\, a^{n-1} + na^n$

By subtraction, we get

$$(1 - a)\, S_n = 1 + a + a^2 + a^3 + \ldots \ldots\ldots + a^{n-1} - na^n$$

$$= \frac{1 - a^n}{1 - a} - na^n$$

$$\therefore \qquad S_n = \frac{1 - a^n}{(1 - a)^2} - \frac{na^n}{1 - a}.$$

2: 1 + 3 + 5

## EXERCISE 8.3

1. Find the $n^{th}$ term and then the sum of the first $n$ terms of each of the following series.
   a) $1 + 4 + 9 + 16 + \ldots\ldots$
   b) $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \ldots\ldots$
   c) $2 + 6 + 12 + 20 + \ldots\ldots$
   d) $1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \ldots\ldots$
   e) $1 + (1 + 3) + (1 + 3 + 5) + \ldots\ldots\ldots$
   f) $3 + 7 + 13 + 21 + 31 + \ldots\ldots$
2. Find the general term and then the sum of first $n$ terms :
   $$1 \cdot n + 2 \cdot (n - 1) + 3 \cdot (n - 2) + \ldots\ \ldots\ldots$$
3. Sum to $n$ terms the following series
   a) $1^2 + 3^2 + 5^2 + \ldots\ \ldots\ldots$
   b) $1^3 + 3^3 + 5^3 + \ldots\ \ldots\ldots$
   c) $1^2 \cdot 1 + 2^2 \cdot 3 + 3^2 \cdot 5 + \ldots\ \ldots$
   d) $3.1^2 + 4.2^2 + 5.3^2 + \ldots\ldots$
4. Sum to $n$ terms the following series
   a) $(x + a) + (x^2 + 2a) + (x^3 + 3a)$ , $\ldots\ \ldots\ldots$
   b) $2 + 22 + 222 + 2222 + \ldots\ldots\ldots$
   c) $0.3 + 0.33 + 0.333 + 0.3333 + \ldots\ldots\ldots$ 0.3
5. Sum to infinity the following series :
   a) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots\ \ldots\ldots$
   b) $16 - 8 + 4 - \ldots\ \ldots\ldots\ \ldots$
   c) $\frac{3}{5} + \frac{4}{5^2} + \frac{3}{5^3} + \frac{4}{5^4} + \frac{3}{5^5} + \frac{4}{5^6} + \ldots\ \ldots\ldots\ \ldots\ldots$
   d) $\frac{a}{x} + \frac{b}{x^2} + \frac{a}{x^3} + \frac{b}{x^4} + \ldots\ \ldots\ldots\ \ldots\ldots$ $(|x| > 1)$
   e) $1 + \frac{3}{4} + \frac{7}{16} + \frac{15}{64} + \ldots\ \ldots\ldots$
6. Sum to infinity :
   (a) $1 + 3x + 5x^2 + 7x^3 + \ldots\ \ldots\ldots$ to $\infty$ $(-1 < x < 1)$
   (b) $1 - 5a + 9a^2 - 13a^3 + \ldots\ \ldots\ldots$ to $\infty$ $(-1 < a < 1)$

### Answer

1. (a) $n^2, \frac{n(n+1)(2n+1)}{6}$ (b) $n(n+2), \frac{n(n+1)(2n+7)}{6}$
   (c) $n(n+1), \frac{n(n+1)(n+2)}{3}$ (d) $4n^2 - 1, \frac{n}{3}(4n^2 + 6n - 1)$
   (e) $n^2, \frac{1}{6}n(n+1)(2n+1)$ f) $n^2 + n + 1, \frac{n}{3}(n^2 + 3n + 5)$

2. $r(n-r+1); \frac{n(n+1)(n+2)}{6}$

3. (a) $\frac{1}{3}n(4n^2-1)$ (b) $n^2(2n^2-1)$ (c) $\frac{1}{6}n(n+1)(3n^2+n-1)$

(d) $n^3+2n^2, \frac{1}{12}n(n+1)(3n^2+11n+4)$

4. (a) $\frac{x(x^n-1)}{x-1}+\frac{n(n+1)a}{2}$ (b) $\frac{20}{81}(10^n-1)-\frac{2n}{9}$

(c) $\frac{1}{3}n-\frac{1}{27}\left(1-\frac{1}{10^n}\right)$

5. (a) $\frac{2}{3}$ (b) $\frac{32}{3}$ (c) $\frac{19}{24}$ (d) $\frac{ax+b}{x^2-1}$ (e) $\frac{8}{3}$

6. (a) $\frac{1+x}{(1-x)^2}$ (b) $\frac{1-3a}{(1+a)^2}$

## ADDITIONAL QUESTIONS

1. If $a, b, c$ are in H.P., prove that

$$\frac{a}{a-b}=\frac{a+c}{a-c}$$

2. If $a, b, c$ are in A.P. $x$ is the G.M. between $a$ and $b$, $y$ is the G.M. between $b$ and $c$, prove that $b^2$ is the A.M. between $x^2$ and $y^2$.

3. If the A.M. of two positive numbers $a$ and $b$ be twice their G.M., show that $a : b = (2+\sqrt{3}) : (2-\sqrt{3})$.

4. A rubber ball is dropped from a height of 16 feet. At each rebound it rises to a height which is $\frac{3}{4}$th of the height of the previous fall. What is the total distance through which the ball will have moved before it finally comes to rest ? (Hint : Use an infinite geometric series and calculate the distance covered in the upward as well as the downward direction).

5. The side of a square are each 16 cm. A second square is inscribed by joining the mid points of the sides, successively. In the second square we repeat the process, inscribing the third square. If this process is continued indefinitely, what is the sum of the perimeters of all the squares. (Use an infinite geometric series).

6. Find the $n^{th}$ term and then the sum of the first $n$ terms of each of the following series.

a) $5+7+13+31+85+\ldots\ldots$

b) $1^2.2+2^2.3+3^2.4+\ldots\ldots$

7. Sum to $n$ terms of the following series:

   a) $1 + \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + \dots \dots\dots$

   b) $1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \dots \dots\dots$

8. If a, b, c are in A.P. and x, y, z are in G.P. prove that

$$x^{b-c} \cdot y^{c-a} \cdot z^{a-b} = 1.$$

9. If p, q, r are in A.P., prove that $q^{th}$ term is the G.M. between $p^{th}$ and $r^{th}$ terms of the same G.P.

**_Answer_**

4. 112 ft. 5. 64 $(2 + \sqrt{2})$ cms.

6. (a) $4 + 3^{n-1}$, $4n + \frac{1}{2}(3^n - 1)$ (b) $n^3 + n$, $\frac{1}{12}n(n+1)(n+2)(3n+1)$

7. (a) $4 - \frac{n}{2^{n-1}} - \frac{1}{2^{n-2}}$ (b) $\frac{35}{16} - \frac{12n+7}{16.5^{n-1}}$

## 8.11 Mathematical Induction

We know that the product of two consecutive natural numbers is an even number. Thus if $n$ and $n + 1$ be two consecutive natural numbers, then their product $n(n + 1)$ is an even number. This is the general result. For $n = 5$, the product $= 5.6 = 30$ which is an even number. This is the particular result obtained from the general one. This type of process of getting the particular result from the general one is known as the method of deduction.

Again, let us see the following two digit numbers.

24, 32, 40, 68

Each of the above two digit numbers is exactly divisible by 2. But from the above results, we cannot conclude that all two digits numbers are exactly divisible by 2 as 31 is also a two digit number but not divisible by 2.

If the above results are to be shown true, we would have to present its validity either by verifying the above type of results for all two digit numbers or by using some mathematical process.

The process of getting the general result from the particular one is known as the method of induction.

Thus there are two ways of proving the results. One is the direct method known as the deductive method in which the results are proved using the established axioms, definitions or the theorems already proved. But second is the indirect method known as the inductive method where the results are proved by making observations or experiments and drawing conclusion on their basis.

### Mathematial Induction

Many mathematical theorems or formulae which are complicated to prove by direct method, are proved easily by indirect method known as the mathematical induction. Its meaning will be clear from the following explanation.

Firstly, we prove that theorem or formulae for $n = 1$. When the theorem is true for $n = 1$, we shall prove that it is also true for $n = 1 + 1 = 2$. In the same way, we prove that it is also true for $2 + 1 = 3$ and so on. Then, we conclude that the theorem or result is true for all values of $n \in N$.

The most important word used in this section is the "statement". In this section, the result or the formula to be proved is termed as "statement". A statement involving the natural number is denoted by P($n$) where $n \in N$.

For example: The sum of two consecutive natural numbers is odd. This is a statement. So, it is denoted by P($n$) : the sum of two consecutive natural numbers is odd.

## 8.12 Principle of Mathematical Induction

The principle of mathematical induction states that if P(n) be the statement and if

i) P(1) is true

ii) P(k + 1) is true whenever P(k) is true

then P($n$) is true for all $n \in$ N..

### Working rules for the use of Principle of Mathematical Induction

In solving a problem with the use of principle of mathematical induction, the following steps are to be used.

1. Denote the given statement (i.e. the result to be proved) by P($n$).
2. Show that the given statement is true for $n = 1$ i.e. P(1) is true.
3. Assume that the given statement is true for $n$ = m i.e. assume P(m) is true.
4. Show that the statement is true for $n$ = m + 1 when it is true for $n$ = m i.e. show that P(m + 1) is true when P(m) is true.
5. Draw a conclusion that the statement is true for all $n \in$ N.

## Worked Out Examples

***Example 1***

Let P($n$) be the statement "$n(n + 1)$ is divisible by 2".

Are P(1), P(2) and P(3) true?

**Solution :**

P($n$) : $n(n + 1)$ is divisible by 2

P(1) : $1(1 + 1) = 1.2 = 2$

P(2) : $2(2 + 1) = 2.3 = 6$

P(3) : $3.(3 + 1) = 3.4 = 12$

Each of P(1), P(2) and P(3) is divisible by 2. Hence P(1), P(2) and P(3) are true.

***Example 2***

Prove by the principle of mathematical induction that

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n + 1)(2n + 1)}{6}$$

**Solution :**

Let P(n) be the given statement. Then,

$$P(n) : 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n + 1)(2n + 1)}{6}$$

**When n = 1 :** L.H.S. $= 1^2 = 1$,

$$\text{R.H.S.} = \frac{1(1 + 1)(2.1 + 1)}{6} = 1$$

$\therefore$ L.H.S. = R.H.S. i.e. P(1) is true.

Let P(m) be true for $m \in N$. Then

$$P(m) : 1^2 + 2^2 + 3^2 + \ldots + m^2 = \frac{m(m + 1)(2m + 1)}{6} \qquad \ldots\ldots (i)$$

Now, we shall show that P(m + 1) is true when P(m) is true. For this, we add $(m + 1)^2$ on both sides of (i)

Now, $1^2 + 2^2 + 3^2 + \ldots + m^2 + (m + 1)^2$

$$= \frac{m(m + 1)(2m + 1)}{6} + (m + 1)^2$$

$$= (m + 1)\left\{\frac{m(2m + 1)}{6} + (m + 1)\right\}$$

$$= (m + 1)\left(\frac{2m^2 + m + 6m + 6}{6}\right)$$

GX

$$= \frac{(m + 1)(2m^2 + 7m + 6)}{6}$$

$$= \frac{(m + 1)(m + 2)(2m + 3)}{6}$$

$$= \frac{(m + 1)((m + 1) + 1)(2(m + 1) + 1)}{6}$$

which shows that P(m + 1) is true whenever P(m) is true. Hence by the principle of mathematical induction P(n) is true for all $n \in N$.

***Example 3***

Prove by the method of induction that

$$1.3 + 2.4 + 3.5 + \ldots + n.(n + 2) = \frac{n(n + 1)(2n + 7)}{6}$$

***Solution :***

Let P(n) be the given statement. Then,

$$P(n) : 1.3 + 2.4 + 3.5 + \ldots + n.(n + 2) = \frac{n(n + 1)(2n + 7)}{6}$$

**When n = 1 :**

L.H.S. = 1.3 = 3

$$\text{R.H.S.} = \frac{1.(1 + 1)(2.1 + 7)}{6} = 3$$

$\therefore$ L.H.S. = R.H.S. i.e. P(1) is true

Let P(m) be true for $m \in N$. That is,

$$P(m) : 1.3 + 2.4 + 3.5 + \ldots + m.(m + 2) = \frac{m(m + 1)(2m + 7)}{6} \quad \ldots\ldots \text{(i)}$$

Now, we shall show that P(m + 1) is true when P(m) is true. For this, we add (m + 1) (m + 3) on both sides of (i).

Now, $1.3 + 2.4 + 3.5 + \ldots + m.(m + 2) + (m + 1)(m + 3)$

$$= \frac{m(m + 1)(2m + 7)}{6} + (m + 1)(m + 3) \quad \text{(From (i))}$$

$$= (m + 1)\left\{\frac{m(2m + 7)}{6} + m + 3\right\}$$

$$= \frac{(m + 1)(2m^2 + 7m + 6m + 18)}{6}$$

$$= \frac{(m + 1)(2m^2 + 13m + 18)}{6}$$

$$= \frac{(m+1)(m+2)(2m+9)}{6}$$

$$= \frac{(m+1)(\overline{m+1}+1)\ (2(m+1)+7)}{6}$$

which shows that P(m + 1) is true whenever P(m) is true. Hence, by the principle of mathematical induction P(n) is true for all $n \in N$.

***Example 4***

Applying principle of mathematical induction, prove that

$n(n + 1)(2n + 1)$ is divisible by 6 for all $n \in N$.

***Solution :***

Let P(n) be the given statement. Then

$P(n) : n(n + 1)(2n + 1)$ is divisible by 6.

**When n = 1 :** $n(n + 1)(2n + 1) = 1(1 + 1)(2.1 + 1) = 6$ which is divisible by 6. So, P(1) is true.

Let us assume that P(m) is true for $m \in N$. Then,

$P(m) : m(m + 1)(2m + 1)$ is divisible by 6 ...... (i)

Now, we shall show that P(m + 1) is true when P(m) is true

i.e. we show that $(m + 1)(m + 2)\ (2(m + 1) + 1)$ is divisible by 6

Now, $(m + 1)(m + 2)(2(m + 1) + 1)$

$= (m + 1)(m + 2)(2m + 1 + 2)$

$= \{(m + 1)m + (m + 1)2\}\ \{(2m + 1) + 2\}$

$= m(m + 1)(2m + 1) + 2m(m + 1)$
$\qquad + 2(m + 1)(2m + 1) + 4(m + 1)$

$= m(m + 1)(2m + 1) + 2(m + 1)\{m + 2m + 1 + 2\}$

$= m(m + 1)(2m + 1) + 2(m + 1)(3m + 3)$

$= m(m + 1)(2m + 1) + 6(m + 1)^2$

which is divisible by 6 as the first term is divisible by 6 by (i) and the second term is the multiple of 6.

This shows that P(m + 1) is true whenever P(m) is true. Hence, by the principle of mathemtical induction, P(n) is true for all $n \in N$.

***Example 5***

Prove by induction method that

$2^{3n} - 1$ is divisible by 7

***Solution :***

Let P(n) be the given statement. Then,

$P(n) : 2^{3n} - 1$ is divisible by 7

**When n = 1 :**

$2^{3n} - 1 = 2^3 - 1 = 8 - 1 = 7$ which is divisible by 7.

$\therefore$ P(1) is true.

Let us suppose that P(m) is true for $m \in N$.

i.e. $\quad P(m) : 2^{3m} - 1$ is divisible by 7 ...... (i)

Now, we shall show that P(m + 1) is true when P(m) is true.

i.e. we show that $2^{3(m+1)} - 1$ is divisible by 7

Now, $2^{3(m+1)} - 1 = 2^{3m+3} - 1$

$= 2^{3m}\, 2^3 - 1 = 2^{3m}.8 - 1$

$= 2^{3m}\, 8 - 8 + 8 - 1$

$= 8(2^{3m} - 1) + 7$

which is divisible by 7 as the first term is divisible by 7 by (i)

This relation shows that P(m + 1) is true whenever P(m) is true. Hence, by the principle of mathematical induction P(n) is true for all $n \in N$.

## EXERCISE 8.4

1. a) If P($n$) is the statement "$(n + 1)(n + 2)$ is an odd number". Find P(1), P(2) and P(3). Are they true?

   b) If P($n$) is the statement "$n^3 + n$ is divisible by 2". Write P(1) and P(2). Are they true ?

   c) If P($n$) is the statement "$n^2 \geq 2^n$" show that P(1) is false and P(2), P(3) are true.

2. Using the principle of mathematical induction, show the following statements for all natural numbers (n) :

   i) $1 + 2 + 3 + \dots + n = \dfrac{n(n+1)}{2}$

   ii) $1 + 3 + 5 + \dots + (2n - 1) = n^2$

   iii) $2 + 4 + 6 + \dots + 2n = n(n + 1)$

   iv) $2 + 5 + 8 + \dots + (3n - 1) = \dfrac{n(3n+1)}{2}$

   v) $1^2 + 3^2 + 5^2 + \dots + (2n - 1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$

   vi) $1^3 + 2^3 + 3^3 + \dots + n^3 = \dfrac{n^2(n+1)^2}{4}$

$((K+1)+2]$



$(K+1)$

3. Applying the principle of mathematical induction, show the following statements for all natural numbers (n) :

   i) $1.2 + 2.3 + 3.4 + \dots + n.(n + 1) = \frac{1}{3} n(n + 1)(n + 2)$

   ii) $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots\dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

   iii) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$

   iv) $2 + 2^2 + 2^3 + \dots + 2^n = 2(2^n - 1)$

4. Prove by the method of induction that

   i) $n^2 + n$ is an even number

   ii) $n^3 + 2n$ is divisible by 3

   iii) $n(n + 1)(n + 2)$ is a multiple of 6

5. Using principle of mathematical induction prove that

   i) $3^{2n} - 1$ is divisible by 8

   ii) $x^n - y^n$ is divisible by $x - y$

[5+(1+k)]



CHAPTER 9

# Matrices and Determinants

## 9.1 Review of Matrices

### Matrix Notation, Order of a Matrix

A rectangular array of numbers arranged in rows (horizontal lines) and columns (vertical lines) enclosed between round (or square) brackets is called a *matrix*. For example, here are some matrices,

i. $\begin{pmatrix} 1 & 4 & 2 \\ 3 & 6 & 5 \end{pmatrix}$

ii. $\begin{pmatrix} 3 & 4 \\ 5 & 6 \\ 1 & 2 \end{pmatrix}$

Each member in the array is called an entry or an element of the matrix.

The *order* or the *size* of the matrix is given by the number of rows followed by the number of columns. In example (i) there are two rows and three columns and so the matrix is said to be of order $2 \times 3$. In example (ii) the order of the matrix is $3 \times 2$.

The matrices are usually denoted by capital letters such as A, B, C,... etc. The elements are denoted by the corresponding small letters with double subscript to indicate their position in the matrix. If A is a matrix, the element in row $i$ and column $j$ of A is denoted by $a_{ij}$.

Thus, a general $2 \times 3$ matrix may be written as

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

In short, a matrix A of order $m \times n$ is written as $A = (a_{ij})$, $i = 1, 2, \ldots, m$ and $j = 1, 2, 3, \ldots, n$.

## 9.2 Some Special Types of Matrices

### i) Row Matrix

A matrix having only one row is called *a row matrix*. For example, the matrices (1, 2), (2, −1, 6) and ( $a_{11}, a_{12}, \ldots\ldots a_{1n}$) are row matrices of order $1 \times 2$, $1 \times 3$ and $1 \times n$ respectively.

### ii) Column Matrix

A matrix having only one column is called *a column matrix*. For example, the matrices

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}$$

are column matrices of order $2 \times 1$, $3 \times 1$ and $m \times 1$ respectively

### iii) Square Matrix

A matrix having the same number of rows and columns is called *a square matrix*. For example, the matrices

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \text{and} \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \\ 2 & 0 & 3 \end{pmatrix}$$

are square matrices of order $2 \times 2$, $3 \times 3$.

### iv) Rectangular Matrix

A matrix which is not a square matrix is called *a rectangular matrix*.

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 4 \end{pmatrix} \text{and} \begin{pmatrix} 1 & 3 \\ -2 & 3 \\ 3 & 0 \end{pmatrix}$$

are rectangular matrices of order $2 \times 3$, $3 \times 2$.

### v) Diagonal Matrix

A square matrix having all non-diagonal elements zero is called *a diagonal matrix*. For example, the square matrices

GX

$$\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \text{and} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are diagonal matrices of order 2 and 3.

### vi) Scalar Matrix

A diagonal matrix having all the diagonal elements equal is called a *scalar matrix*. For example, the matrices

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

are scalar matrices of order 2 and 3 respectively.

### vii) Unit Matrix or Identity Matrix

A diagonal matrix having all the diagonal elements equal to 1 is called a *unit matrix*. For example, the diagonal matrices

$$I_1 = (1), \qquad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are unit matrices of order 1, 2 and 3 respectively.

### viii) Zero Matrix or Null Matrix

A matrix having each of the elements zero is called a *zero matrix*. A zero matrix of order $m \times n$ is denoted by $0_{mn}$ or simply by 0. For example, the matrices

$$0_{2\times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad 0_{2\times 3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

are zero matrices.

### ix) Triangular Matrix

A square matrix having all the elements below the diagonal are zero is called an *upper triangular matrix*. Thus a square matrix $A = (a_{ij})$ is an upper triangular matrix if $a_{ij} = 0$ for all $i > j$. For example, the square matrices

$$\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & -3 \end{pmatrix} \text{ and } \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$$

are upper triangular matrices.

A square matrix having all the elements above the diagonal are zero is called a *lower triangular matrix*. Thus a square matrix $A = (a_{ij})$ is a lower triangular matrix if $a_{ij} = 0$ for all $i < j$. For example, the square matrices

$$\begin{pmatrix} 5 & 0 \\ 3 & \sqrt{2} \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ -3 & -1 & -2 \end{pmatrix}, \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

are lower triangular matrices.

### x) Submatrix of Matrix

Any matrix obtained by omitting some rows and columns from a given matrix A is called a *submatrix* of A.

For example, the matrix

$$\begin{pmatrix} 1 & 2 & 4 \\ 3 & 1 & 0 \end{pmatrix} \text{ is a submatrix of} \begin{pmatrix} 1 & 2 & 4 & 3 \\ 3 & 1 & 0 & 5 \\ 1 & 0 & 2 & 4 \end{pmatrix}$$

## 9.3 Equality of Matrices

***Definition.*** Two matrices A and B are said to be *equal* and written as $A = B$, if

i) they are of the same order

ii) their corresponding elements are equal.

Thus the two matrices of order $2 \times 3$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$$

are equal if

$$a_{11} = b_{11}, \quad a_{12} = b_{12}, \quad a_{13} = b_{13}$$
$$a_{21} = b_{21}, \quad a_{22} = b_{22}, \quad a_{23} = b_{23}$$

## 9.4 Addition of Matrices

*Definition.* If A and B are two matrices of the same order, the sum of A and B denoted by A + B is the matrix obtained by adding corresponding elements of A and B.

The matrix A + B is of the same order as each of the matrices A and B is. Thus if

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \quad \text{and } B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$$

$$\text{Then } A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \end{pmatrix}$$

For example, if

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -3 & 4 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 4 & -7 \end{pmatrix}$$

then

$$A + B = \begin{pmatrix} 1+1 & 2+(-2) & 3+3 \\ -3+4 & 4+4 & 1+(-7) \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 6 \\ 1 & 8 & -6 \end{pmatrix}$$

## 9.5 Multiplication of a Matrix by a Scalar (Real Number)

Let $A = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$. From the definition

$$A + A = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 2\times3 & 2\times4 \\ 2\times1 & 2\times2 \end{pmatrix}$$

$$\text{and } A + A + A = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 9 & 12 \\ 3 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 3\times3 & 3\times4 \\ 3\times1 & 3\times2 \end{pmatrix}$$

We now write

$$2\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2\times3 & 2\times4 \\ 2\times1 & 2\times2 \end{pmatrix}$$

and $$3\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3\times3 & 3\times4 \\ 3\times1 & 3\times2 \end{pmatrix}.$$

*Definition*. If A is any matrix and $k$ is any constant or a scalar, then $k$A the matrix obtained by multiplying each element of A by $k$ is called the *scalar multiple* of A by $k$.

For example, if $A = \begin{pmatrix} 1 & -3 & 4 \\ 2 & -1 & 0 \end{pmatrix}$

then

$$2A = \begin{pmatrix} 2 & -6 & 8 \\ 4 & -2 & 0 \end{pmatrix}$$

## 9.6 Algebraic Properties of Matrix Addition

We now state some properties of matrix addition. If A, B, C are any matrices of the same order, 0 is the zero matrix of the same order and $c$, $k$ are any scalars, then the following laws hold.

i) A + B = B + A (commutative)

ii) (A + B) + C = A + (B + C) ( associative)

iii) A + 0 = 0 + A = A (existence of identity element)

iv) A + (-A) = 0 (existence of additive inverse)

v) $c$(A + B) = $c$A + $c$B (scalar Multiplication distributes addition of scalars)

vi) ($c$ + $k$) A = $c$A + $k$A (multiplication with a matrix distributes addition of scalars)

vii) $c$($k$A) = ($ck$)A ( associative law of scalar multiplication )

## 9.7 Difference of Two Matrices

*Definition.* If A and B are two matrices of the same order, then the *difference* of the matrix B from A is defined to be the matrix obtained by adding the additive inverse of B i.e. – B to the matrix A. This difference is denoted by A – B. Then A – B = A + (–B).

For example, if

$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & -1 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \end{pmatrix}$

SK

Then

$$A - B = A + (-B) = \begin{pmatrix} 1 & 2 & 3 \\ 4 & -1 & -2 \end{pmatrix} + \begin{pmatrix} 1 & -2 & 3 \\ -2 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 6 \\ 2 & -1 & -3 \end{pmatrix}$$

## 9.8 Cancellation Laws of Addition

*Theorem.*

If A, B and C are the matrices of the same order, then

(i) $A + B = A + C \Rightarrow B = C$

(ii) $B + A = C + A \Rightarrow B = C$

***Proof :***

(i) We have

$A + B = A + C$

$\Rightarrow (-A) + (A + B) = (-A) + (A + C)$

$\Rightarrow (-A + A) + B = (-A + A) + C$ (By Associative law of addition)

$\Rightarrow O + B = O + C$ (as '$-A$' is the additive inverse of A)

$\Rightarrow B = C$ (as O is the additive identity)

(ii) Left as an exercise.

## 9.9 Matrix Multiplication

Before we write the definition of matrix multiplication, let us consider the following example.

Let $A = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}$ and $B = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$

Since A is a $3 \times \boxed{2}$ matrix and B is a $\boxed{2} \times 2$ matrix, the number of columns in A is the same as the numbers of rows in B. In such a case only we can define the product AB.

The entire product AB is defined as follows

$$AB = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \\ ep + fr & eq + fs \end{pmatrix}_{3 \times 2}$$

$$3 \times \boxed{2} \qquad \boxed{2} \times 2$$

same

order of AB

So the product AB is a matrix of order $3 \times 2$, if the orders of A and B are $3 \times \boxed{2}$ and $\boxed{2} \times 2$ respectively.

*Definition.* If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then the product AB is an $m \times p$ matrix. The element in row $i$ and column $j$ of AB is obtained by multiplication of the elements in the $i^{th}$ row of A by corresponding elements of $j^{th}$ column of B and addition of resulting products.

## 9.10 Some Properties of Matrix Multiplication

We now state some properties of multiplication of matrices as follows:

i) $AB \neq BA$ (not commutative for all matrices A and B)

ii) $AI = IA = A$ (existence of multiplicative identify element)

iii) $A.(BC) = (AB).C$ (associative law)

iv) $A(B + C) = AB + AC$ (distributive law)

v) If A is a square matrix, then A. A is defined and is written as $A.A = A^2$.

Similarly A.A.A ... n times = $A^n$. ( Note : A square matrix is called an *idempotent matrix*, if $A^2 = A$ and an *involutory*, if $A^2 = I$.)

## 9.11 Symmetric Matrix

*Definition.* A square matrix $A = (a_{ij})$ is called a *symmetric matrix*, if its $(i, j)^{th}$ element is equal to its $(j, i)^{th}$ element , i.e. if $a_{ij} = a_{ji}$ , for all $i, j$.

For example,

$$\begin{pmatrix} 1 & 3 & 7 \\ 3 & 0 & -6 \\ 7 & -6 & 8 \end{pmatrix}$$

is a symmetric matrix of order 3. Here $a_{12} = a_{21} = 3$, $a_{13} = a_{31} = 7$, $a_{23} = a_{32} = -6$. There is no restriction on elements on the principal diagonal.

## 9.12 Skew-symmetric Matrix

*Definition.* A square matrix $A = (a_{ij})$ is called a *skew-symmetric matrix*, if its $(i, j)^{th}$ element is the negative of its $(j, i)^{th}$ element, i.e. if $a_{ij} = -a_{ji}$ for all $i, j$. (Note that if $j = i$, then $a_{ii} = -a_{ii}$ or $2a_{ii} = 0$ or $a_{ii} = 0$. This shows that in a skew-symmetric matrix, each element in the principal diagonal is zero.)

For example

$$A = \begin{pmatrix} 0 & 5 & -7 \\ -5 & 0 & 3 \\ 7 & -3 & 0 \end{pmatrix}$$

is a skew - symmetric matrix. Here

$a_{12} = -a_{21} = 5$, $a_{13} = -a_{31} = -7$, $a_{23} = -a_{32} = 3$, $a_{11} = 0$, $a_{22} = 0$, $a_{33} = 0$.

## Worked Out Examples

***Example 1***

Construct a $3 \times 3$ matrix A whose elements $a_{ij}$ are given by

$$a_{ij} = 3i + 2j$$

***Solution.***

It is given that $a_{ij} = 3i + 2j$

Thus

$a_{11} = 3.1 + 2.1 = 5$, $a_{12} = 3.1 + 2.2 = 7$, $a_{13} = 3.1 + 2.3 = 9$

$a_{21} = 3.2 + 2.1 = 8$, $a_{22} = 3.2 + 2.2 = 10$, $a_{23} = 3.2 + 2.3 = 12$

$a_{31} = 3.3 + 2.1 = 11$, $a_{32} = 3.3 + 2.2 = 13$, $a_{33} = 3.3 + 2.3 = 15$

Hence the required matrix A is given by

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 5 & 7 & 9 \\ 8 & 10 & 12 \\ 11 & 13 & 15 \end{pmatrix}$$

***Example 2***

Find the product AB and BA and show that $AB \neq BA$.

$$A = \begin{pmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{pmatrix}$$

***Solution :***

$$\text{Let } A = \begin{pmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{pmatrix} \qquad \text{(T.U.2050 H)}$$

Here A is a $2 \times 3$ matrix and B a $3 \times 2$ matrix. The product AB is defined and it is a $2 \times 2$ matrix. The product BA is also defined and it is a $3 \times 3$ matrix. Since the order of AB and BA are not the same, $AB \neq BA$.

$$AB = \begin{pmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -4 \\ 10 & 3 \end{pmatrix}$$

and

$$BA = \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{pmatrix} = \begin{pmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{pmatrix}$$

Thus $AB \neq BA$.

***Example 3***

If $A = \begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix}$, show that $(A - 2I)(A - 3I) = O$ where I and O are unit matrix and the zero matrix of order 2.

***Solution :***

$$A - 2I = \begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4-2 & 2-0 \\ -1-0 & 1-2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ -1 & -1 \end{pmatrix}$$

$$A - 3I = \begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 4-3 & 2-0 \\ -1-0 & 1-3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}$$

Now, (A – 2I) (A – 3I)

$$=\begin{pmatrix} 2 & 2 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}$$

$$=\begin{pmatrix} 2-2 & 4-4 \\ -1+1 & -2+2 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\therefore$ (A – 2I) (A – 3I) = 0

## EXERCISE 9.1

1. Construct a $3 \times 3$ matrix whose elements $a_{ij}$ are given by
   i) $a_{ij} = i + 2j$ (HSEB 2051) ii) $a_{ij} = 3j - 2i$ (T.U. 2048 H)
   iii) $a_{ij} = 2ij$ iv) $a_{ij} = i^j$

2. Find $x, y, z$, if $\begin{pmatrix} x + y & z - x \\ y + 2z & x \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 8 & 1 \end{pmatrix}$.

3. Let $A = \begin{pmatrix} -1 & 4 & -3 \\ 2 & 2 & 2 \\ 3 & 0 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 3 & -5 & -3 \\ -1 & 3 & -4 \\ 2 & 4 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & 0 & -2 \\ -3 & 4 & 1 \\ 6 & -1 & 3 \end{pmatrix}$

   Determine if possible
   i) 2A + B ii) 3A – 4B iii) (A + B) – A
   iv) A + (B + C) v) (A + B) + C

4. If $A = \begin{pmatrix} 3 & 2 \\ 1 & 5 \end{pmatrix}$, find the matrix X such that

   $A - 3X = \begin{pmatrix} 3 & 5 \\ -8 & 2 \end{pmatrix}$. (T.U. 2052H)

5. Find the matrices A and B, if

   i) $A + B = \begin{pmatrix} -1 & 3 & -2 \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix}$, $A - B = \begin{pmatrix} 3 & -1 & 2 \\ 2 & 3 & -3 \\ 3 & 1 & 2 \end{pmatrix}$.

   ii) $2A + B = \begin{pmatrix} 3 & 4 & 5 \\ 1 & 2 & 0 \end{pmatrix}$, $A - 2B = \begin{pmatrix} -1 & 2 & -5 \\ 8 & 6 & 5 \end{pmatrix}$

6. If $A = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}$, show that $AB \neq BA$.

7. If $A = \begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix}$, find a matrix X such that $AX = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

8. If $A = \begin{pmatrix} -1 & 2 & 3 \\ 4 & 2 & 0 \\ -3 & 2 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -5 \\ 0 & 4 \\ -5 & 1 \end{pmatrix}$

and $C = \begin{pmatrix} 3 & 5 & -3 \\ 0 & 2 & 4 \end{pmatrix}$, compute the following, if possible

i) AB ii) BC iii) CA iv) BA v) CB vi) AC

9. If $A = \begin{pmatrix} 1 & -2 & 3 \\ -1 & 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 \\ 3 & 1 \\ 1 & 2 \end{pmatrix}$, find the matrix $AB - 3I$ where I is the unit matrix of order 2. (T.U. 2051H)

10. If $A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$, show that $A^2 - 2A - 5I = 0$, where 0 is the $2 \times 2$ null matrix. (T.U. 2058)

11. If $A = \begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix}$, prove that

a) $A^2 - A - 6I = 0$  b) $(A - 3I)(A + 2I) = 0$

12. If $A = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$ and $f(x) = x^2 - 5x + 7$, find $f(A)$.

### Answers

1. (i) $\begin{pmatrix} 3 & 5 & 7 \\ 4 & 6 & 8 \\ 5 & 7 & 9 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 4 & 7 \\ -1 & 2 & 5 \\ -3 & 0 & 3 \end{pmatrix}$ (iii) $\begin{pmatrix} 2 & 4 & 6 \\ 4 & 8 & 12 \\ 6 & 12 & 18 \end{pmatrix}$

(iv) $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 27 \end{pmatrix}$

2. $x = 1, y = 2, z = 3$

3. (i) $\begin{pmatrix} 1 & 3 & -9 \\ 3 & 7 & 0 \\ 8 & 4 & 10 \end{pmatrix}$ (ii) $\begin{pmatrix} -15 & 32 & 3 \\ 10 & -6 & 22 \\ 1 & -16 & 15 \end{pmatrix}$ (iii) $\begin{pmatrix} 3 & -5 & -3 \\ -1 & 3 & -4 \\ 2 & 4 & 0 \end{pmatrix}$

(iv) $\begin{pmatrix} 4 & -1 & -8 \\ -2 & 9 & -1 \\ 11 & 3 & 8 \end{pmatrix}$ (v) $\begin{pmatrix} 4 & -1 & -8 \\ -2 & 9 & -1 \\ 11 & 3 & 8 \end{pmatrix}$

4. $\begin{pmatrix} 0 & -1 \\ 3 & 1 \end{pmatrix}$

5. i) $A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & -1 \\ 2 & 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 2 & -2 \\ 0 & -1 & 2 \\ -1 & -1 & 0 \end{pmatrix}$

ii) $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 3 \\ -3 & -2 & -2 \end{pmatrix}$

7. $X = \begin{pmatrix} 1/4 & 1/2 \\ 2/5 & 4/5 \end{pmatrix}$

8. (i) $\begin{pmatrix} -17 & 16 \\ 8 & -12 \\ -31 & 28 \end{pmatrix}$ (ii) $\begin{pmatrix} 6 & 0 & -26 \\ 0 & 8 & 16 \\ -15 & -23 & 19 \end{pmatrix}$ (iii) $\begin{pmatrix} 26 & 10 & -6 \\ -4 & 12 & 20 \end{pmatrix}$

(iv) Impossible (v) $\begin{pmatrix} 21 & 2 \\ -20 & 12 \end{pmatrix}$ (vi) impossible

9. $\begin{pmatrix} -4 & 7 \\ 5 & -2 \end{pmatrix}$ 12. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

## 9.13 Determinant

*Determinant* originally appeared in the study of linear equations. We shall, however, associate the notion of the determinant with matrices. For this purpose, we have to consider square matrices of order 2, 3, ...,$n$, i.e. $2 \times 2$, $3 \times 3$, ...,$n \times n$ matrices.

*Definitions. Determinant* of a $1\times1$ matrix $A = (a_{11})$ is defined to be the number $a_{11}$. In symbol, we write

$$\det(A) \text{ or } |A| = a_{11}$$

Determinant of a $2\times2$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is defined to be the number $a_{11}a_{22} - a_{12}a_{21}$. In symbols, we write

$$\det(A) \text{ or } |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$
$$= a_{11}a_{22} - a_{12}a_{21}$$

The numbers $a_{11}, a_{12}, a_{21}, a_{22}$ are the elements of the determinant. Since it has two rows and two columns, it is called the determinant of order 2.

***Example 1.***

(i) If $A = (2)$, then $\det(A) = 2$

(ii) If $A = (-3)$, then $\det(A) = -3$.

(iii) If $A = \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}$, then $\det(A) = \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} = 2 - (-2) = 4$.

We now proceed to define the determinant of a $3 \times 3$ matrix. This definition requires concepts of the minor and cofactor of an element in the matrix.

## 9.14 Minors and Cofactors

*Definition.* Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

be a $3 \times 3$ matrix. The determinant of $2 \times 2$ matrix formed by omitting the $i^{th}$ row and $j^{th}$ column of A is called *the minor* of the element $a_{ij}$ and it is denoted by number $M_{ij}$. Thus

$$M_{11} = \text{minor of } a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$M_{12} = \text{minor of } a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$M_{13} = \text{minor of } a_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

and so on.

*Definition.* The *cofactor* $A_{ij}$ of the $i^{th}$ row and $j^{th}$ column element $a_{ij}$ of the 3×3 matrix A is the number $A_{ij} = (-1)^{i+j} M_{ij}$, where $i = 1,2,3$ and $j = 1,2,3$.

***Example 2.***

$$\text{Let } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 0 & 4 \\ 3 & 2 & 1 \end{pmatrix}.$$

So

$a_{11} = 1,\quad a_{12} = 2,\quad a_{13} = -3$

$a_{21} = 2,\quad a_{22} = 0,\quad a_{23} = 4$

$a_{31} = 3,\quad a_{32} = 2,\quad a_{33} = 1$

Hence

$$A_{11} = \text{cofactor of } a_{11} = (-1)^{1+1} M_{11} = +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 0 & 4 \\ 2 & 1 \end{vmatrix} = -8$$

$$A_{12} = \text{cofactor of } a_{12} = (-1)^{1+2} M_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -\begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix} = 10$$

$$A_{13} = \text{cofactor of } a_{13} = (-1)^{1+3} M_{13} = +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 3 & 2 \end{vmatrix} = 4$$

$$A_{21} = \text{cofactor of } a_{21} = (-1)^{2+1} M_{21} = -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} 2 & -3 \\ 2 & 1 \end{vmatrix} = -8$$

$$A_{22} = \text{cofactor of } a_{22} = (-1)^{2+2} M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = \begin{vmatrix} 1 & -3 \\ 3 & 1 \end{vmatrix} = 10$$

$$A_{23} = \text{cofactor of } a_{23} = (-1)^{2+3} M_{23} = -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = -\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = 4$$

$$A_{31} = \text{cofactor of } a_{31} = (-1)^{3+1} M_{31} = +\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = \begin{vmatrix} 2 & -3 \\ 0 & 4 \end{vmatrix} = 8$$

$$A_{32} = \text{cofactor of } a_{32} = (-1)^{3+2} M_{32} = -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = -\begin{vmatrix} 1 & -3 \\ 2 & 4 \end{vmatrix} = -10$$

$$A_{33} = \text{cofactor of } a_{33} = (-1)^{3+3} M_{33} = +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} = -4$$

*Defination.*

*Determinant* of a 3×3 matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

may now be defined by

$$\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \qquad \text{.........(1)}$$

where $a_{11}, a_{12}, a_{13}$ are the elements on the first row and $A_{11}, A_{12}, A_{13}$ are their corresponding cofactors.

or $$\det(A) = a_{11}(-1)^{1+1} M_{11} + a_{12}(-1)^{1+2} M_{12} + a_{13}(-1)^{1+3} M_{13} \qquad \text{.........(2)}$$

where $M_{11}, M_{12}, M_{13}$ are the minors of the elements $a_{11}, a_{12}, a_{13}$ respectively. So

$$\det(A) = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \qquad \text{...(3)}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \qquad \text{...(4)}$$

*Remarks*

i) The expression (1) is the sum of the products of the elements of first row and their corresponding cofactors.

ii) (4) is the expression in terms of the elements.

iii) This determinant may also be expanded in terms of the products of elements of any row (or column) and thier corresponding cofactors.

***Example 3.***

Let A be as in Ex. 2. Expanding the det (A) as the sum of the products of the elements of the first row and corresponding cofactors, we get

$$\det(A) = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}$$

$$= 1\begin{vmatrix} 0 & 4 \\ 2 & 1 \end{vmatrix} + 2(-1)\begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix} + (-3)\begin{vmatrix} 2 & 0 \\ 3 & 2 \end{vmatrix}$$

$$= 1(0-8) - 2(2-12) - 3(4-0)$$

$$= 0$$

Expanding the same det (A) in terms of the elements of the second column of A, we have

$$\det(A) = a_{12}\,A_{12} + a_{22}\,A_{22} + a_{32}\,A_{32}$$

$$= 2(-1)\begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & -3 \\ 3 & 1 \end{vmatrix} + 2(-1)\begin{vmatrix} 1 & -3 \\ 2 & 4 \end{vmatrix}$$

$$= -2(2-12) + 0 - 2(4+6)$$

$$= 20 - 20$$

$$= 0.$$

**Note** that a mechanical rule for finding the value of a third order determinant is as indicated below.

The value of a third order determinant can also be evaluated by the rule of Sarrus

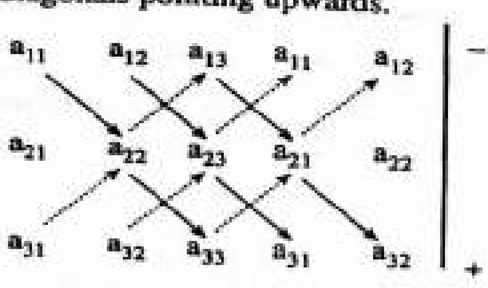
$$\text{Let } |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The rule of Sarrus is

i) Write the three columns of | A | and repeat the first two columns again.

ii) Draw arrows as shown in the figure below.

iii) Form the algebraic sum of the products of the elements of the diagonals pointing downwards, subtract the algebraic sum of the products of the elements of the diagonals pointing upwards.




$$= (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) - (a_{31}a_{22}a_{13} + a_{32}a_{23}a_{11} + a_{33}a_{21}a_{12})$$

***Example.***

Find the value of

$$\det. A = \begin{vmatrix} 1 & 2 & -3 \\ 2 & 0 & 4 \\ 3 & 2 & 1 \end{vmatrix}$$

by the rule of Sarrus.

***Solution :***

The rule of Sarrus is

$$\begin{array}{ccccc|l} 1 & 2 & -3 & 1 & 2 & - \\ 2 & 0 & 4 & 2 & 0 & \\ 3 & 2 & 1 & 3 & 2 & + \end{array}$$

$= (0 + 24 - 12) - (0 + 8 + 4)$

$= 0$

So the value of det. $A = 0$

**Note** that this rule does not work for determinants of order greater than 3.

## 9.15 Some Properties of a 3 × 3 Determinant

We have seen how to evaluate a determinant. Evaluation becomes easy with the help of the following properties. We shall simply verify these properties.

*Property 1*

The value of the determinant is unaltered by interchanging its rows and columns.

Let

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$= a_1 (b_2c_3 - c_2b_3) - b_1 (a_2c_3 - c_2a_3) + c_1 (a_2b_3 - a_3b_2)$

$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1$ ...(1)

If $|A'|$ is the determinant obtained from $|A|$ by changing its rows into columns, we get

$$|A'| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Then expanding it by the first column we get

$$|A'| = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}$$

$= a_1 (b_2c_3 - b_3 c_2) - b_1 (a_2c_3 - a_3c_2) + c_1 (a_2b_3 - b_2a_3)$

$= a_1b_2c_3 - a_1b_3c_2 - b_1a_2c_3 + b_1a_3c_2 + c_1 a_2b_3 - c_1b_2 a_3$ ...... (2)

$\therefore$ from (1) and (2) $|A| = |A'|$

***Property 2***

Interchanging any two adjacent rows (or columns) changes the sign of the determinant.

Let

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

And $|A^*|$ be the determinant obtained by interchanging the first and second rows of $|A|$ then

$$|A^*| = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Then $|A| = a_1 (b_2c_3 - c_2b_3) - b_1 (a_2c_3 - c_2a_3) + c_1 (a_2b_3 - b_2a_3)$

$= a_1 b_2c_3 - a_1b_3 c_2 - b_1 a_2c_3 + b_1a_3 c_2 + c_1 a_2b_3 - c_1b_2 a_3$

$|A^*| = a_2(b_1c_3 - c_1b_3) - b_2(a_1c_3 - c_1a_3) + c_2 (a_1b_3 - b_1a_3)$

$= a_2b_1c_3 - a_2c_1b_3 - b_2a_1c_3 + b_2 c_1a_3 + c_2a_1b_3 - c_2b_1a_3$

$= - (a_1 b_2c_3 - a_1b_3 c_2 - b_1 a_2c_3 + b_1a_3 c_2 + c_1 a_2b_3 - c_1b_2 a_3)$

$= - |A|$

*Property 3*

If any two rows (or columns) of a determinant are identical, then the value of the determinant is zero.

$$\text{Let} \quad |A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

In this determinant first two rows are identical.

Let $|A_1|$ be the determinant obtained by interchanging first two rows in $|A|$. Then the determinant itself remains unchanged, but by Prop. 2. its value is $-|A|$ i.e. $|A_1| = -|A|$.

Hence $\quad |A| = -|A|$

or $\quad 2|A| = 0$

$\therefore \quad |A| = 0.$

*Property 4.*

If all the elements of any row (or column) are multiplied by a constant $k$, then the value of the determinant is multiplied by $k$.

Let

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

And $|A_1|$ be the determinant obtained from $|A|$ by multiplying the elements of the first column by a constant $k$ (say). Then

$$|A_1| = \begin{vmatrix} ka_1 & b_1 & c_1 \\ ka_2 & b_2 & c_2 \\ ka_3 & b_3 & c_3 \end{vmatrix}$$

We have to show that $\quad |A_1| = k\,|A|$.

As before $|A| = a_1 (b_2c_3 - c_2b_3) - b_1 (a_2c_3 - c_2a_3) + c_1 (a_2b_3 - b_2a_3)$

$$|A_1| = ka_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} ka_2 & c_2 \\ ka_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} ka_2 & b_2 \\ ka_3 & b_3 \end{vmatrix}$$

$$= ka_1 (b_2c_3 - c_2b_3) - b_1 (ka_2c_3 - kc_2a_3) + c_1 (ka_2b_3 - ka_3b_2).$$

$$= k\{ a_1 (b_2c_3 - c_2b_3) - b_1 (a_2c_3 - c_2a_3) + c_1 (a_2b_3 - a_3b_2)\}$$

$$= k\,|A|.$$

***Property 5.***

If each element of any row (or column) of a determinant is expressed as the sum of two terms, then the determinant can be expressed as a sum of two determinants,

i.e.
$$\begin{vmatrix} a_1+\alpha & b_1 & c_1 \\ a_2+\beta & b_2 & c_2 \\ a_3+\gamma & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha & b_1 & c_1 \\ \beta & b_2 & c_2 \\ \gamma & b_3 & c_3 \end{vmatrix}$$

$$\text{L.H.S} = (a_1+\alpha)\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - (a_2+\beta)\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + (a_3+\gamma)\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$+ \alpha\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - \beta\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + \gamma\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha & b_1 & c_1 \\ \beta & b_2 & c_2 \\ \gamma & b_3 & c_3 \end{vmatrix}.$$

***Property 6.***

If to the elements of any row (or column) a multiple of any other row (or column) is added, the value of the determinant remains unaltered.

$$\text{Let } |A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Let us multiply the elements of second row by $k$ and add these products to the elements of the first row and obtain the determinant

$$|A^*| = \begin{vmatrix} a_1+ka_2 & b_1+kb_2 & c_1+kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

We have to show that $|A^*| = |A|$

Now $|A^*| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_3 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} ka_2 & kb_2 & kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ (by Prop. 5.)

$$= |A| + k \begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$= |A| + k \,.\, 0$ (by Prop. 3.) $= |A|$

***Example 1.***

Without expanding show that $\begin{vmatrix} 30 & 2 & 5 \\ 18 & 4 & 3 \\ 6 & 8 & 1 \end{vmatrix} = 0$.

***Solution.***

$$\begin{vmatrix} 30 & 2 & 5 \\ 18 & 4 & 3 \\ 6 & 8 & 1 \end{vmatrix} = \begin{vmatrix} 6\times5 & 2 & 5 \\ 6\times3 & 4 & 3 \\ 6\times1 & 8 & 1 \end{vmatrix}$$

$= 6 \begin{vmatrix} 5 & 2 & 5 \\ 3 & 4 & 3 \\ 1 & 8 & 1 \end{vmatrix}$ (Taking 6 common from $C_1$)

$= 6 \times 0$ (as $C_1 = C_3$)

$= 0.$

***Example 2.***

Find the value of

$\begin{vmatrix} 3 & 4 & 5 \\ 15 & 21 & 26 \\ 21 & 29 & 34 \end{vmatrix}$ (T.U. 2053)

***Solution.***

Let $|A|$ denote the given determinant. Multiplying the first row by 5 and subtracting it from the second row (i.e. applying $R_2 - 5R_1$), and multiplying the first row by 7 and subtracting it from the third row (i.e. applying $R_3 - 7R_1$). We get

$$|A| = \begin{vmatrix} 3 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{vmatrix} = 3\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} - 0\begin{vmatrix} 4 & 5 \\ 1 & -1 \end{vmatrix} + 0\begin{vmatrix} 4 & 5 \\ 1 & 1 \end{vmatrix}$$

$$= 3(-1-1) = -6.$$

***Example 3***

Without expanding the determinant show that

$$\begin{vmatrix} b-c & b+c & b \\ c-a & c+a & c \\ a-b & a+b & a \end{vmatrix} = 0$$

***Solution.***

$$\text{L.H.S} = \begin{vmatrix} 2b & b+c & b \\ 2c & c+a & c \\ 2a & a+b & a \end{vmatrix}$$ by adding the column 2 to the column 1 i.e. by applying $C_1 + C_2$

$$= 2\begin{vmatrix} b & b+c & b \\ c & c+a & c \\ a & a+b & a \end{vmatrix}$$ by taking 2 common from $C_1$

$= 2 \times 0$ as $C_1 = C_3$

$= 0 =$ R.H.S.

***Example 4.***

Show that $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (y-z)(z-x)(x-y).$ (T.U.2048)

***Solution.***

$$\text{L.H.S.} = \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix}$$ by applying $R_2 - R_1$, $R_3 - R_1$

$$= (y-x)(z-x)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{vmatrix}$$ by taking $(y-x)$ common from $R_2$ and $(z-x)$ from $R_3$

$$= (y - x)(z - x)\begin{vmatrix} 1 & y + x \\ 1 & z + x \end{vmatrix}$$
$$= (y - x)(z - x)(z + x - y - x)$$
$$= (y - x)(z - x)(z - y)$$
$$= (x - y)(y - z)(z - x) = \text{R.H.S}$$

**Example 5**

Prove that : $\begin{vmatrix} a+x & b & c \\ a & b+y & c \\ a & b & c+z \end{vmatrix} = xyz\left(1 + \frac{a}{x} + \frac{b}{y} + \frac{c}{z}\right)$

**Solution :**

$$\begin{vmatrix} a + x & b & c \\ a & b + y & c \\ a & b & c + z \end{vmatrix}$$
$$= \begin{vmatrix} x & -y & 0 \\ a & b+y & c \\ 0 & -y & z \end{vmatrix} \quad \begin{matrix}(R_1 \to R_1 - R_2 \\ R_3 \to R_3 - R_2)\end{matrix}$$
$$= x\begin{vmatrix} b+y & c \\ -y & z \end{vmatrix} + y\begin{vmatrix} a & c \\ 0 & z \end{vmatrix}$$
$$= x(bz + yz + cy) + y(az)$$
$$= bzx + xyz + cxy + ayz$$
$$= xyz\left(1 + \frac{a}{x} + \frac{b}{y} + \frac{c}{z}\right)$$

**Example 6**

Prove that $\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix} = 2(a + b)(b + c)(c + a)$

**Solution :**

$$\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix}$$

$$= \begin{vmatrix} a+b & -c & -(b+c) \\ a+b & a+b+c & b+c \\ -(a+b) & -a & b+c \end{vmatrix} \quad \begin{pmatrix} C_1 \to C_1 + C_2, \\ C_3 \to C_3 + C_2 \end{pmatrix}$$

$$= (a+b)(b+c) \begin{vmatrix} 1 & -c & -1 \\ 1 & a+b+c & 1 \\ -1 & -a & 1 \end{vmatrix} \quad \text{(Taking } a+b \text{ common from } C_1 \text{ and } b+c \text{ from } C_3\text{)}$$

$$= (a+b)(b+c) \begin{vmatrix} 0 & -c & -1 \\ 2 & a+b+c & 1 \\ 0 & -a & 1 \end{vmatrix} \quad (C_1 \to C_1 + C_3)$$

$$= -2(a+b)(b+c) \begin{vmatrix} -c & -1 \\ -a & 1 \end{vmatrix}$$

$$= -2(a+b)(b+c)(-c-a)$$

$$= 2(a+b)(b+c)(c+a)$$

***Example 7***

Prove that

$$\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix} = (a^2+b^2+c^2)(a+b+c)(a-b)(b-c)(c-a)$$

***Solution :***

$$\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix}$$

$$= \begin{vmatrix} (b+c)^2+a^2-2bc & a^2 & bc \\ (c+a)^2+b^2-2ca & b^2 & ca \\ (a+b)^2+c^2-2ab & c^2 & ab \end{vmatrix} \quad (C_1 \to C_1 + C_2 - 2C_3)$$

$$= \begin{vmatrix} a^2+b^2+c^2 & a^2 & bc \\ a^2+b^2+c^2 & b^2 & ca \\ a^2+b^2+c^2 & c^2 & ab \end{vmatrix}$$

$$= (a^2+b^2+c^2)\begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ca \\ 1 & c^2 & ab \end{vmatrix} \quad \text{(Taking } a^2+b^2+c^2 \text{ common from } C_1\text{)}$$

$$= (a^2+b^2+c^2)\begin{vmatrix} 0 & a^2-b^2 & bc-ca \\ 1 & b^2 & ca \\ 0 & c^2-b^2 & ab-ca \end{vmatrix} \quad (R_1 \to R_1 - R_2,\ R_3 \to R_3 - R_2)$$

$$= (a^2+b^2+c^2)(a-b)(b-c)\begin{vmatrix} 0 & a+b & -c \\ 1 & b^2 & ca \\ 0 & -(b+c) & a \end{vmatrix} \quad \text{(Taking } a-b \text{ common from } R_1 \text{ and } b-c \text{ from } R_3\text{)}$$

$$= -(a^2+b^2+c^2)(a-b)(b-c)\{a^2+ab-bc-c^2\}$$
$$= -(a^2+b^2+c^2)(a-b)(b-c)\{(a+c)(a-c)+b(a-c)\}$$
$$= -(a^2+b^2+c^2)(a+b+c)(a-b)(b-c)(a-c)$$
$$= (a^2+b^2+c^2)(a+b+c)(a-b)(b-c)(c-a)$$

**Example 8**

Show that : $\begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2$

**Solution :**

$$\begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -2c^2 & -2b^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix} \quad R_1 \to R_1 - R_2 - R_3$$

$$= -2\begin{vmatrix} 0 & c^2 & b^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix} \quad \text{(Taking } -2 \text{ common from } R_1\text{)}$$

$$= -2\begin{vmatrix} 0 & c^2 & b^2 \\ b^2 & a^2 & 0 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix} \quad (R_2 \to R_2 - R_1)$$

$$= -2 \begin{vmatrix} 0 & c^2 & b^2 \\ b^2 & a^2 & 0 \\ c^2 & 0 & a^2 \end{vmatrix} \quad (R_3 \to R_3 - R_1)$$

$$= -2\left\{-c^2 \begin{vmatrix} b^2 & 0 \\ c^2 & a^2 \end{vmatrix} + b^2 \begin{vmatrix} b^2 & a^2 \\ c^2 & 0 \end{vmatrix}\right\}$$

$$= -2\{-a^2b^2c^2 - a^2b^2c^2\}$$

$$= 4a^2b^2c^2$$

## EXERCISE 9.2

1. Evaluate the determinants of the following matrices.

a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, b) $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$, c) $\begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \\ 1 & 2 & 3 \end{pmatrix}$

d) $\begin{pmatrix} -1 & 0 & 3 \\ 2 & 1 & 4 \\ -2 & -3 & -1 \end{pmatrix}$ e) $\begin{pmatrix} 3 & -2 & 2 \\ 4 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix}$ f) $\begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & 4 \\ -4 & 5 & 6 \end{pmatrix}$

2. Solve for $x$.

a) $\begin{vmatrix} x & 2 & 3 \\ -1 & 0 & 1 \\ 2 & -2 & 0 \end{vmatrix} = 0$, b) $\begin{vmatrix} x & 3 & 3 \\ 2 & 3 & x \\ 2 & 3 & 3 \end{vmatrix} = 0$

3. Without expanding the determinants, show that the value of each of the following determinants is zero.

i) $\begin{vmatrix} 6 & 1 & 9 \\ 2 & 4 & 7 \\ 18 & 3 & 27 \end{vmatrix}$ (T.U. 2048H) ii) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

iii) $\begin{vmatrix} (a-b) & (b-c) & (c-a) \\ (b-c) & (c-a) & (a-b) \\ (c-a) & (a-b) & (b-c) \end{vmatrix}$ iv) $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$

v) $\begin{vmatrix} 1 & bc & bc(b+c) \\ 1 & ca & ca(c+a) \\ 1 & ab & ab(a+b) \end{vmatrix}$

vi) $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$, where $\omega$ is an imaginary cube root of unity.

4. Evaluate.

i) $\begin{vmatrix} 51 & 61 & 71 \\ 5 & 6 & 7 \\ 1 & 1 & 1 \end{vmatrix}$

ii) $\begin{vmatrix} 16 & 19 & 23 \\ 15 & 18 & 22 \\ 13 & 17 & 20 \end{vmatrix}$

5. Without expanding the determinants, prove that

i) $\begin{vmatrix} 1 & 7 & 8 \\ 4 & -3 & 4 \\ -5 & 10 & 2 \end{vmatrix} = \begin{vmatrix} 8 & 4 & 2 \\ 1 & 4 & -5 \\ 7 & -3 & 10 \end{vmatrix}$

ii) $\begin{vmatrix} 3 & 5 & 7 \\ 9 & 2 & 1 \\ 7 & 8 & 5 \end{vmatrix} + \begin{vmatrix} 9 & 3 & 7 \\ 2 & 5 & 8 \\ 1 & 7 & 5 \end{vmatrix} = 0$

iii) $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$ (T.U. 2052H)

iv) $\begin{vmatrix} 1 & bc & b+c \\ 1 & ca & c+a \\ 1 & ab & a+b \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ (HSEB 2052)

v) $\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$ (HSEB 2054)

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6. Show that

i) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ca & ab \end{vmatrix} = (a-b)(b-c)(c-a)$

ii) $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ b+c & c+a & a+b \end{vmatrix} = (b-c)(c-a)(a-b)(a+b+c)$

iii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (b-c)(c-a)(a-b)(a+b+c)$

(T.U. 2050, 056S, HSEB 2055)

iv) $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = (y-z)(z-x)(x-y)(yz+zx+xy)$

v) $\begin{vmatrix} b+c & a & b \\ c+a & c & a \\ a+b & b & c \end{vmatrix} = (a+b+c)(a-c)^2$

vi) $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

(T.U. 2052)

vii) $\begin{vmatrix} 1+a_1 & a_2 & a_3 \\ a_1 & 1+a_2 & a_3 \\ a_1 & a_2 & 1+a_3 \end{vmatrix} = 1 + a_1 + a_2 + a_3$

(T.U. 2051H)

viii) $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = xyz(1/x + 1/y + 1/z + 1)$

(T.U. 2055S, HSEB 2053, 2056)

ix) $\begin{vmatrix} a^2 & bc & c^2+ac \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$

x) $\begin{vmatrix} x^2+1 & xy & xz \\ xy & y^2+1 & yz \\ xz & yz & z^2+1 \end{vmatrix} = 1 + x^2 + y^2 + z^2$

**Answers**

1\. (a) −2 (b) 0 (c) 0 (d) −23 (e) 31 (f) 4

2\. (a) −5 (b) 2 or 3 4. (i) 0 (ii) −7

## 9.16 Transpose of a Matrix

***Definition.*** The new matrix obtained from a given matrix A by interchanging its rows and columns is called *the transpose* of A. It is denoted by A' or $A^T$.

For example

i) If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, then $A' = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

Here A is of order 2×3 and A' is of order 3×2.

ii) If $A = \begin{pmatrix} 3 & 1 & 2 \end{pmatrix}$ then $A' = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$

Here, A is of order 1×3 and A' is of order 3×1.

iii) If $A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \\ 2 & 0 & 3 \end{pmatrix}$, then $A' = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 2 & 4 & 3 \end{pmatrix}$

Here A is a square matrix of order 3 and A' is also a square matrix of order 3.

Note :

i) If A is an $m\times n$ matrix, then A' is an $n\times m$ matrix.

ii) If A is a square matrix of order $n$, then A' is also a square matrix of order $n$.

iii) A + A' and A − A' are symmetric and skew-symmetric matrices respectively.

## 9.17 Properties of Transpose of Matrices

The transpose of matrices satisfy the following properties:

(i) $(A')' = A$.

(ii) $(A + B)' = A' + B'$.

(iii) $(cA)' = cA'$, $c$ is constant.

(iv) $(AB)' = B'A'$

where A and B are any two matrices. (*The proofs are omitted*)

The following examples verify the above properties:

Let $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \\ 3 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 3 \\ -1 & 2 \\ 0 & -3 \end{pmatrix}$ and $c = 3$.

Then, $A' = \begin{pmatrix} 1 & 0 & 3 \\ 2 & -1 & 1 \end{pmatrix}$ and $B' = \begin{pmatrix} 0 & -1 & 0 \\ 3 & 2 & -3 \end{pmatrix}$

$$(A')' = \begin{pmatrix} 1 & 2 \\ 0 & -1 \\ 3 & 1 \end{pmatrix} = A$$

which verifies property no. (i).

Further,

$$A + B = \begin{pmatrix} 1 & 2 \\ 0 & -1 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 3 \\ -1 & 2 \\ 0 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 1+0 & 2+3 \\ 0-1 & -1+2 \\ 3+0 & 1-3 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ -1 & 1 \\ 3 & -2 \end{pmatrix}$$

Then, $(A + B)' = \begin{pmatrix} 1 & -1 & 3 \\ 5 & 1 & -2 \end{pmatrix}$

and $A' + B' = \begin{pmatrix} 1 & 0 & 3 \\ 2 & -1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ 3 & 2 & -3 \end{pmatrix}$

$$= \begin{pmatrix} 1+0 & 0-1 & 3+0 \\ 2+3 & -1+2 & 1-3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 & 3 \\ 5 & 1 & -2 \end{pmatrix}$$

Thus, $(A + B)' = A' + B'$

which verifies property no. (ii).

Also $cA = 3\begin{pmatrix} 1 & 2 \\ 0 & -1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 0 & -3 \\ 9 & 3 \end{pmatrix}$

Then $(cA)' = (3A)' = \begin{pmatrix} 3 & 0 & 9 \\ 6 & -3 & 3 \end{pmatrix}$

and $cA' = 3\begin{pmatrix} 1 & 0 & 3 \\ 2 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 9 \\ 6 & -3 & 3 \end{pmatrix}$

Thus $(cA)' = cA'$

which verifies property no. (iii).

Now for the verification of property no. (iv), we proceed as follows:

Let $A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 2 \end{pmatrix}$

so that the product AB is defined.

Then, $A' = \begin{pmatrix} 1 & 2 \\ 0 & 0 \\ -1 & 3 \end{pmatrix}$ and $B' = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 2 \end{pmatrix}$

Now, $AB = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 2 \end{pmatrix}$

$$= \begin{pmatrix} 1+0-0 & 0+0-2 \\ 2+0+0 & 0+0+6 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & 6 \end{pmatrix}$$

so that $(AB)' = \begin{pmatrix} 1 & 2 \\ -2 & 6 \end{pmatrix}$

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and also, $B'A' = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 0 \\ -1 & 3 \end{pmatrix}$

$= \begin{pmatrix} 1+0-0 & 2+0+0 \\ 0+0-2 & 0+0+6 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 \\ -2 & 6 \end{pmatrix}$

Thus, $(AB)' = B'A'$ which verifies property no. (iv).

**Singular Matrix**

A square matrix is said to be a singular matrix if its determinant is zero. Otherwise the squre matrix is said to be non-singular. For example

$A = \begin{pmatrix} 4 & 3 \\ 8 & 6 \end{pmatrix}$

$|A| = \begin{vmatrix} 4 & 3 \\ 8 & 6 \end{vmatrix} = 4 \times 6 - 3 \times 8 = 0$

$\therefore$ A is a singular matrix.

Again if $A = \begin{pmatrix} 1 & 5 \\ 3 & 12 \end{pmatrix}$, then

$|A| = \begin{vmatrix} 1 & 5 \\ 3 & 12 \end{vmatrix} = 12 - 15 = -3 \neq 0$

$\therefore$ A is non-singular matrix.

## 9.18 Inverse of a Matrix

Consider the matrices

$A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$

Then $AB = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

And $BA = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

Then $AB = BA = I$

*Definition*. If A and B are square matrices such that

$$AB = BA = I,$$

then B is called the *inverse* of A and is written as $A^{-1}$ (= B). Similarly, A is called the *inverse* of B and written as $B^{-1}$ (= A).

## 9.19 Adjoint of a Matrix

*Definition*

Let $$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

be a 3 × 3 matrix, and $A_{ij}$ be the cofactor of $a_{ij}$, then ***adjoint*** or ***adjugate*** of A denoted by adj A is defined by

$$\text{adj } A = \text{transpose of} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

$$= \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$$

***Examples***

i) For the matrix $\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$

We have

$a_{11} = 2$, $a_{12} = 4$, $a_{21} = 1$ and $a_{22} = 3$

Their cofactors are $A_{11} = 3$, $A_{12} = -1$, $A_{21} = -4$, $A_{22} = 2$ .

Thus by definition,

$$\text{adj } A = \text{transpose of} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$= \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ -1 & 2 \end{pmatrix}.$$

ii) Also for the matrix $\begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$ (T.U. 2049)

We have

$a_{11} = 1, \quad a_{12} = 2, \quad a_{13} = -2$
$a_{21} = -1, \quad a_{22} = 3, \quad a_{23} = 0$
$a_{31} = 0, \quad a_{32} = -2, \quad a_{33} = 1$

so that their cofactors are

$A_{11} = \text{cofactor of } a_{11} = \begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix} = 3$

$A_{12} = \text{cofactor of } a_{12} = -\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

$A_{13} = \text{cofactor of } a_{13} = \begin{vmatrix} -1 & 3 \\ 0 & -2 \end{vmatrix} = 2$

$A_{21} = \text{cofactor of } a_{21} = -\begin{vmatrix} 2 & -2 \\ -2 & 1 \end{vmatrix} = 2$

$A_{22} = \text{cofactor of } a_{22} = \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = 1$

$A_{23} = \text{cofactor of } a_{23} = -\begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} = 2$

$A_{31} = \text{cofactor of } a_{31} = \begin{vmatrix} 2 & -2 \\ 3 & 0 \end{vmatrix} = 6$

$A_{32} = \text{cofactor of } a_{32} = -\begin{vmatrix} 1 & -2 \\ -1 & 0 \end{vmatrix} = 2$

$A_{33} = \text{cofactor of } a_{33} = \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = 5$

Thus by definition

$$\text{adj } A = \text{transpose of} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

$$= \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}$$

## 9.20 Inverse of a 3 × 3 Matrix by the Adjoint Method

The inverse of a $3 \times 3$ matrix follows from the relation

$$A.(\text{adj } A) = (\text{adj } A) \,.\, A = |A|\, I \;\ldots\ldots (1)$$

This relation is true for any square matrix A whose determinant $|A| \neq 0$.

It can be proved ; but the proof is beyond the scope of the present text. So we shall just verify it with some examples

(i) Let $A = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 2 & -2 \\ -2 & 1 \end{vmatrix} + 0\begin{vmatrix} 2 & -2 \\ 3 & 0 \end{vmatrix}$$

$$= 1\,.\,3 + 1\,.\,(-2) = 1$$

As calculated above, the adjoint of matrix of A is

$$\text{adj } A = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}$$

So

$$A.\text{adj } A = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= I = |A|.I \,. \qquad (\text{because } |A| = 1)$$

$$\text{and} \quad (\text{adj } A).A = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= I = |A|.I, \qquad (\text{because } |A| = 1)$$

Thus we see that

$$A.(\text{adj } A) = (\text{adj } A).A = |A|.I$$

(ii) Let $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ 0 & 3 & -1 \end{pmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ 0 & 3 & -1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 0 & 1 \\ 3 & -1 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} - 1\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix}$$

$$= -3 + 4 - 6 = -5$$

$A_{11}$ = cofactor of $a_{11} = \begin{vmatrix} 0 & 1 \\ 3 & -1 \end{vmatrix} = -3$

$A_{12}$ = cofactor of $a_{12} = -\begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} = 2$

$A_{13}$ = cofactor of $a_{13} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$

$A_{21}$ = cofactor of $a_{21} = -\begin{vmatrix} 2 & -1 \\ 3 & -1 \end{vmatrix} = -(-2 + 3) = -1$

$A_{22}$ = cofactor of $a_{22} = \begin{vmatrix} 1 & -1 \\ 0 & -1 \end{vmatrix} = -1$

$A_{23}$ = cofactor of $a_{23} = -\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = -3$

$A_{31}$ = cofactor of $a_{31} = \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = 2$

$$A_{32} = \text{cofactor of } a_{32} = -\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = -(1+2) = -3$$

$$A_{33} = \text{cofactor of } a_{33} = \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} = -4$$

The matrix of cofactors of A is

$$C = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} -3 & 2 & 6 \\ -1 & -1 & -3 \\ 2 & -3 & -4 \end{pmatrix}$$

adj A = transpose of C

$$= \begin{pmatrix} -3 & -1 & 2 \\ 2 & -1 & -3 \\ 6 & -3 & -4 \end{pmatrix}$$

$$\text{Now, } A.(\text{adj } A) = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ 0 & 3 & -1 \end{pmatrix}\begin{pmatrix} -3 & -1 & 2 \\ 2 & -1 & -3 \\ 6 & -3 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{pmatrix}$$

$$= -5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= |A|.I \qquad \text{(because } |A| = -5\text{)}$$

Thus we get $\quad A.(\text{adj } A) = (\text{adj } A).A = |A|.I$

In this way, we can verify the relation (1) for any $3 \times 3$ matrix A, when $|A| \neq 0$.

Now from this relation we can get

$$A \frac{(\text{adj } A)}{|A|} = \frac{(\text{adj. } A)}{|A|} A = I,$$

which shows that $\dfrac{(\text{adj } A)}{|A|} = A^{-1}$, the inverse matrix of A.

## Worked Out Examples

***Example 1***

If $A = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$, prove that $AA' = A'A = 5I$ where I is a unit matrix of order 2.

***Solution :***

$$A = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$$

Then, $A' = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$

$$AA' = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 4+1 & 2-2 \\ 2-2 & 1+4 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= 5I$$

Again, $A'A = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$

$$= \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= 5I$$

$\therefore AA' = A'A = 5I$

***Example 2***

If $A = \begin{pmatrix} 2 & 0 & -6 \\ 5 & 1 & 2 \\ 7 & -3 & 0 \end{pmatrix}$, find $A^T$.

i) Show that sum of the given matrix and its transpose is the symmetric matrix.

ii) Show that the difference of the given matrix and its transpose is the skew-symmetric matrix.

***Solution :***

We have

$$A = \begin{pmatrix} 2 & 0 & -6 \\ 5 & 1 & 2 \\ 7 & -3 & 0 \end{pmatrix} \Rightarrow A' = \begin{pmatrix} 2 & 5 & 7 \\ 0 & 1 & -3 \\ -6 & 2 & 0 \end{pmatrix}$$

Now $$A + A' = \begin{pmatrix} 2 & 0 & -6 \\ 5 & 1 & 2 \\ 7 & -3 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 5 & 7 \\ 0 & 1 & -3 \\ -6 & 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2+2 & 0+5 & -6+7 \\ 5+0 & 1+1 & 2-3 \\ 7-6 & -3+2 & 0+0 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 5 & 1 \\ 5 & 2 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

which is a symmetric matrix.

Also, $$A - A' = \begin{pmatrix} 2 & 0 & -6 \\ 5 & 1 & 2 \\ 7 & -3 & 0 \end{pmatrix} - \begin{pmatrix} 2 & 5 & 7 \\ 0 & 1 & -3 \\ -6 & 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2-2 & 0-5 & -6-7 \\ 5-0 & 1-1 & 2+3 \\ 7+6 & -3-2 & 0-0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -5 & -13 \\ 5 & 0 & 5 \\ 13 & -5 & 0 \end{pmatrix}$$

which is a skew-symmetric matrix.

***Example 3***

Find the inverse of $\begin{pmatrix} 3 & 2 \\ -1 & 6 \end{pmatrix}$.

***Solution***

Let $A = \begin{pmatrix} 3 & 2 \\ -1 & 6 \end{pmatrix}$.

then $|A| = \begin{vmatrix} 3 & 2 \\ -1 & 6 \end{vmatrix} = 3 \,.\, 6 - 2\,(-1) = 18 + 2 = 20$

$A_{11}$ = cofactor of $a_{11} = 6$
$A_{12}$ = cofactor of $a_{12} = -(-1) = 1$
$A_{21}$ = cofactor of $a_{21} = -2$
$A_{22}$ = cofactor of $a_{22} = 3$

Matrix of cofactors $= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ -2 & 3 \end{pmatrix}$

adj A = transpose of $\begin{pmatrix} 6 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 6 & -2 \\ 1 & 3 \end{pmatrix}$

and $A^{-1} = \dfrac{\text{adj } A}{|A|} = \dfrac{1}{20}\begin{pmatrix} 6 & -2 \\ 1 & 3 \end{pmatrix}$.

***Example 4***

Find the inverse of $\begin{pmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ 0 & 3 & -1 \end{pmatrix}$ (HSEB 2052)

***Solution***

Let $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ 0 & 3 & -1 \end{pmatrix}$

The matrix of the cofactors of A is C and

$C = \begin{pmatrix} -3 & 2 & 6 \\ -1 & -1 & -3 \\ 2 & -3 & -4 \end{pmatrix}$ (from Art. 10.20 (ii))

adj A = transpose of $C = \begin{pmatrix} -3 & -1 & 2 \\ 2 & -1 & -3 \\ 6 & -3 & -4 \end{pmatrix}$

and $|A| = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ 0 & 3 & -1 \end{vmatrix}$

$$= 1\begin{vmatrix} 0 & 1 \\ 3 & -1 \end{vmatrix} - 2\begin{vmatrix} 2 & -1 \\ 3 & -1 \end{vmatrix} + 0\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix}$$

$$= 1(-3) - 2\,(1) = -3 - 2 = -5$$

$$A^{-1} = \frac{\text{adj } A}{|\,A\,|} = \frac{-1}{5}\begin{pmatrix} -3 & -1 & 2 \\ 2 & -1 & -3 \\ 6 & -3 & -4 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{1}{5} & \frac{-2}{5} \\ \frac{-2}{5} & \frac{1}{5} & \frac{3}{5} \\ \frac{-6}{5} & \frac{3}{5} & \frac{4}{5} \end{pmatrix}$$

***Example 5***

Prove that the following two matrices are inverse of each other :

$$A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$$

***Solution :***

$$AB = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 6-5 & -2+2 \\ 15-15 & -5+6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\text{Again, } BA = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 6-5 & 3-3 \\ -10+10 & -5+6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$AB = BA = I$$

Hence the two matrices are inverse of each other.

## EXERCISE 9.3

1. a) If $A = \begin{pmatrix} 4 & -5 \\ 3 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$ find $A^T$, $B^T$, $(AB)^T$ and show that $(AB)^T = B^TA^T$.

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b) If $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 2 & 3 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 2 & 1 \\ 3 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}$ find A', B', (AB)' and B'A'.

2. a) If $A = \begin{pmatrix} 1 & 2 \\ -3 & 6 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 3 \\ 5 & 7 \\ 1 & -4 \end{pmatrix}$ and k = 3; compute A', B' and verify that

(i) $(A')' = A$ (ii) $(A + B)' = A' + B'$ (iii) $(kA)' = kA'$

b) If $A = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{pmatrix}$, verify that $AA' = A'A = I$ where I is a unit matrix of order 3.

3. If $A = \begin{pmatrix} 2 & 4 & 3 \\ 2 & 3 & 4 \\ 5 & 2 & 6 \end{pmatrix}$ find $A^T$.

a) Show that the sum of the given matrix and its transpose is a symmetric matrix.

b) Show that the difference of the given matrix and its tranpose is a skew-symmetric matrix.

4. Find the adjoints and inverses of the following matrices, if possible.

i) $\begin{pmatrix} 2 & 1 \\ 4 & 6 \end{pmatrix}$. ii) $\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$. iii) $\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$.

iv) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. v) $\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 6 \\ 1 & 3 & 2 \end{pmatrix}$. vi) $\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{pmatrix}$

5. If $A = \begin{pmatrix} 7 & -3 \\ 6 & 2 \end{pmatrix}$, prove that $A^{-1} = \frac{1}{32}\begin{pmatrix} 2 & 3 \\ -6 & 7 \end{pmatrix}$.

6. Prove that the two matrices $\begin{pmatrix} -3 & -2 \\ 5 & 3 \end{pmatrix}$ and $\begin{pmatrix} 3 & 2 \\ -5 & -3 \end{pmatrix}$ are the inverses of each other.

7. If $A = \begin{pmatrix} 5 & 3 \\ 4 & 2 \end{pmatrix}$ verify that $A.(\text{adj } A) = (\text{adj } A).A = |A| I$.

8. If $A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 6 & 0 \\ -5 & 2 \end{pmatrix}$ verify that $(AB)^{-1} = B^{-1}A^{-1}$.

9. If $A = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$ find $A^{-1}$ and verify that $AA^{-1} = I$.

## *Answers*

1. a) $\begin{pmatrix} 4 & 3 \\ -5 & 6 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$ and $\begin{pmatrix} 13 & 0 \\ 22 & -3 \end{pmatrix}$

   b) $\begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 3 & 4 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 3 & 2 \\ 2 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 11 & 11 & 7 \\ 5 & 4 & 4 \\ -1 & -1 & -1 \end{pmatrix}$

2. a) $\begin{pmatrix} 1 & -3 & 0 \\ 2 & 6 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 5 & 1 \\ 3 & 7 & -4 \end{pmatrix}$

4. (i) $\begin{pmatrix} 6 & -1 \\ -4 & 2 \end{pmatrix}, \begin{pmatrix} 3/4 & -1/8 \\ -1/2 & 1/4 \end{pmatrix}$, (ii) $\begin{pmatrix} 3 & -4 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 3/2 & -2 \\ -1/2 & 1 \end{pmatrix}$,

   (iii) $\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$.

   (iv) $\begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, -\frac{1}{2}\begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$

   (v) $\begin{pmatrix} -16 & 8 & 8 \\ 6 & -2 & -6 \\ -1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -1 & -1 \\ -3/4 & 1/4 & 3/4 \\ 1/8 & 1/8 & -1/8 \end{pmatrix}$

   (vi) $\begin{pmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{pmatrix}, \frac{1}{10}\begin{pmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{pmatrix}$ 9. $\begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}$

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## ADDITIONAL QUESTIONS
### (Matrix)

1. When will two matrices be inverse of each other? Show that the matrices $\begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix}$ and $\begin{pmatrix} 2 & -1 \\ 5 & -3 \end{pmatrix}$ are inverse of each other and verify that their transposes are also inverse of each other. (T.U. 2050)

2. Let A = $\begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}$ be a $2 \times 2$ matrix. Find Adj. A and $A^{-1}$. Show that $AA^{-1} = A^{-1}A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. (HSEB 2050)

3. Given a matrix $\begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix}$, find a matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ such that they are inverse of each other. (T.U 2051.)

4. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, show that $A^{-1} = \frac{1}{19} A$.

5. Find the adjoint of the matrix $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$. Verify that this matrix and its adjoint matrix are inverses of each other. (T.U. 2052).

6. When does a matrix have its inverse? If $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ 0 & 3 & -1 \end{bmatrix}$, find its inverse. (HSEB 2052)

7. Prove that $\begin{pmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{pmatrix}$ and $\begin{pmatrix} -9 & 8 & -5 \\ -8 & 7 & -4 \\ -2 & 2 & 1 \end{pmatrix}$ are inverse of each other.

8. If $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$.

9. If $A = \begin{bmatrix} 2 & -3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, verify that $A.\ (\text{Adj. } A) = (\text{Adj. } A)\ .\ A = |A|\, I_3$.

10. State the conditions under which two matrices can be added and multiplied.

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ are two matrices, verify that $(AB)' = B'A'$. (T.U. 2055S)

### *Answers*

2. $\begin{pmatrix} 8 & -2 \\ -3 & 1 \end{pmatrix}, \begin{pmatrix} 4 & -1 \\ -3/2 & 1/2 \end{pmatrix}$ 3. $\begin{pmatrix} 2 & -1 \\ 5 & -3 \end{pmatrix}$

5. $\begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}$ 6. $\begin{pmatrix} 3/5 & 1/5 & -2/5 \\ -2/5 & 1/5 & 3/5 \\ -6/5 & 3/5 & 4/5 \end{pmatrix}$

## ADDITIONAL QUESTIONS
## (Determinants)

Prove that (1-5)

1. $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc.$

2. $\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$ is a perfect square.

3. $\begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ ax+by & bx+cy & 0 \end{vmatrix} = (b^2 - ac)(ax^2 + 2bxy + cy^2).$

4. Show that $x = 2$ is a root of the equation

$$\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0$$ and solve it completely.

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5. Prove that $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

6. Show that (i) $\begin{vmatrix} 1/a & bc & 1 \\ 1/b & ca & 1 \\ 1/c & ab & 1 \end{vmatrix} = 0.$

(ii) $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix} = 0.$

7. If $xyz + 1 = 0$, then show that

$$\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0.$$

8. Prove that if to the elements of any row (or column) a multiple of any other row (or column) is added or subtracted, the value of the determinant remains unchanged.

Without expanding show that

$$\begin{vmatrix} 3 & 4 & 5 \\ 15 & 21 & 26 \\ 21 & 29 & 36 \end{vmatrix} = 0 \text{ (T.U. 2053).}$$

9. Prove the following

(a) $\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} = 3abc - a^3 - b^3 - c^3$

(b) $\begin{vmatrix} 1 & x+y & x^2+y^2 \\ 1 & y+z & y^2+z^2 \\ 1 & z+x & z^2+x^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

(c) $\begin{vmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{vmatrix} = 0$

(d) $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = x^2(x+a+b+c)$

10. Show that $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc\,(a+b+c)^3.$

(T.U. 2054S)

## Answers

4. $x = 1, 2, -3$.

# CHAPTER 10
# System of Linear Equations

## 10.1 Introduction

We come across equations, when we try to solve some problems in mathematics. These equations may be of one or more variables. The solutions of the equations give the solutions of the problems. So it is quite natural that we should have knowledge of different methods of solving equations. The methods, we consider here, are row equivalent method, Cramer's rule and matrix method.

## 10.2 Systems of Linear Equations

A simple example of *a linear equation* in two variables $x$ and $y$ is

$$3x - y = 5$$

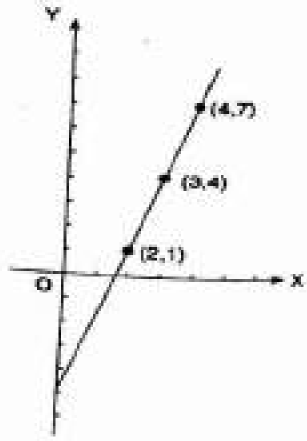
It is evident that solutions of this equation are ordered pairs such as (2, 1), (3, 4), (4, 7), etc. That is, each of the ordered pairs (2, 1), (3, 4), (4, 7), etc. satisfies the equation. For instance

$$3.[3] - [4] = 5. \text{ (true)}.$$

Moreover, there are infinite number of such pairs. If we plot each of these pairs, they will lie in a straight line. That is, the graph of the equation $3x - y = 5$ is a straight line. If we are interested in finding various pairs satisfying the given equation, we may write it in the form

$$y = 3x - 5,$$

which gives $y$ in terms of $x$.

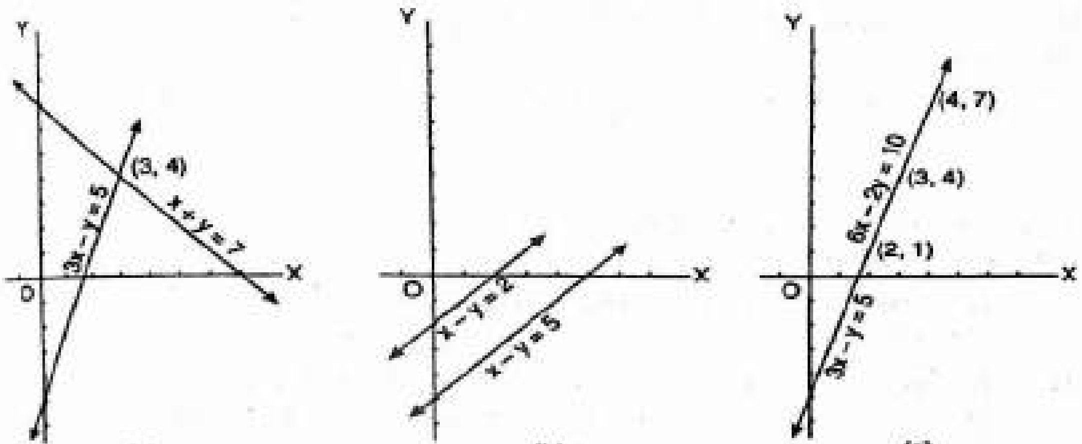



Given a value to $x$ in this equation, a corresponding value of $y$ can be found and each pair of values furnishes a solution of the equation.

The conjunction (pair) of two such linear equations in two variables is referred to as *a system*. Three typical examples of systems are:

(a) $3x - y = 5$, $x + y = 7$ (b) $x - y = 2$, $x - y = 5$ (c) $3x - y = 5$, $6x - 2y = 10$

The graphs of these equations are shown in (Fig. a, b, c.)




(a) (b) (c)

We shall now discuss the three cases somewhat elaborately.

## Case I. Intersecting Lines

The graphs of the first pair $3x - y = 5$ and $x + y = 7$ intersect at the point (3, 4). This means that both equations have the common solution $x = 3, y = 4$. Although there is an infinite number of solution sets which separately satisfy these equations, the solution $x = 3$, $y = 4$ is the only solution which satisfies both of them at the same time. This is why such a system of equations is called *simultaneous equations*.

Two linear equations in two variables which have at least one solution in common are said to be *consistent*. If consistent linear equations have just one solution in common, they are said to be *independent*. The system of equations $x + y = 7$ and $3x - y = 5$ are not only consistent but also independent.

## Case II. Parallel Lines

The graphs of the second pair $x - y = 2$ and $x - y = 5$ are *parallel lines*, that is they never meet. No set of values of $x$ and $y$ which satisfies one equation will satisfy the other equation. In other words, they have no common solution. These equations are *inconsistent* and *independent*. We can see why they are so called. For one equation states that the difference of two numbers $x$ and $y$ is 2 while the other states that their difference is 5, and two such numbers whose difference is both 2 and 5 at the same time do not exist.

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### Case III. Coincident Lines

The graphs of the pair of equations $3x - y = 5$ and $6x - 2y = 10$ *coincide* (i.e. one fits exactly on the other). Obviously, these equations are consistent, that is, a set of values of $x$ and $y$ which satisfies the first equation satisfies the second also. Moreover, these equations are *dependent*, because one of them can be obtained from the other just by multiplying both sides by a constant. In the above case, the second can be obtained from the first by multiplying both sides of the first by 2.

In the light of the discussion we had so far, a system of linear equations

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

where $x$ and $y$ are two variables and the rest are all constants, may be classified in the following way:

I. The system is consistent and independent if the equations have exactly one solution, i.e. if their graphs intersect in exactly one point.

II. The system is inconsistent and independent if the equations have no solution common to both, i.e. if the graphs of the equations do not intersect.

III. The system is consistent and dependent if every solution of one of the equations is also a solution of the other i.e. if the graphs of the equations coincide.

## EXERCISE 10.1

1. By drawing graphs or otherwise, classify each of the following systems:

(a) $4x - 3y = -6$, $-4x + 2y = 16$

(b) $9x - 2y = -4$, $3x + 4y = 1$

(c) $-6x + 4y = 10$, $3x - 2y = -5$

(d) $3x - 4y = 1$, $6x - 8y = 7$

(e) $25x - 15y = 45$, $-5x - 3y = 24$

***Answers***

1. (a) Consistent and Independent (b) Consistent and Independent
(c) Consistent and Dependent (d) Inconsistent and Independent
(e) Consistent and Independent

The system of equations considered so far consists of two variables only. Similar discussion holds good if the number of variable increased to three or more. An example of a linear equation in the three variables $x$, $y$ and $z$ is

$$3x + y - z = 14.$$

Solutions of this equation consist of the ordered triples such as (5, 6, 7) which satisfies the equation. Here (5, 6, 7) is just one solution such that

$$3.[5] + [6] - [7] = 14 \text{ (True)}.$$

One set of such triples is {(5, 0, 1),(5, 1, 2),(5, 3, 4) ......}. It is easy to verify that each of these triples satisfies the linear equation

$$x + y - z = 4$$

also. We thus observe that a system of two equations in three variables have an infinite number of solutions. But, if we further consider a third linear equation such as

$$x + y + z = 6,$$

then we notice that the triple which satisfies the third also is (5, 0, 1).

Actually, there is no other triple of numbers which satisfies the system of three equations:

$$\begin{aligned} 3x + y - z &= 14 \\ x + y - z &= 4 \\ x + y + z &= 6 \end{aligned}$$

We shall, as before, refer to such a system of equations to be consistent and independent.

## 10.3 Solutions of System of Linear Equations

One of the most fundamental techniques of solving a system like

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

is the technique of elimination. We can illustrate this technique on the linear system

$$\begin{aligned} x - 2y &= 3 \\ 2x + y &= 1 \end{aligned}$$

If we add 2 times the second equation to the first equation, we obtain

$$5x = 5 \qquad \text{or} \qquad x = 1$$

If we add $-2$ times the first equation to the second equation, we obtain

$$5y = -5 \qquad \text{or} \qquad y = -1$$

So we conclude that the ordered pair (1, −1) is the solution. It is not difficult to notice that in solving the above linear system, we actually work with constants (coefficients) and not with the variables. In the other words, there is no need to continue writing the variables $x$ and $y$, since one actually computes with the coefficients and the numbers in the right side of the equations. For this type of computation, it is helpful to list just the coefficient only in the form of a rectangular array, and enclose them within square brackets or parentheses. For instance,

$$\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \text{ or } \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

is such an arrangement and is called the *matrix of coefficients* or *coefficient matrix* of the system.

$$x - 2y = 3$$
$$2x + y = 1$$

For a system

$$5x - 2y = -2$$
$$2x + 5y = 24$$

the square matrix

$$\begin{bmatrix} 5 & -2 \\ 2 & 5 \end{bmatrix}$$

is the *coefficient matrix*. If, in this matrix, we include a third column consisting of the numbers on the right sides of the equations, we get another matrix of the form

$$\begin{bmatrix} 5 & -2 & -2 \\ 2 & 5 & 24 \end{bmatrix} \text{ or } \begin{bmatrix} 5 & -2 & : & -2 \\ 2 & 5 & : & 24 \end{bmatrix}$$

This matrix is called the *augmented matrix* of the system.

Let us now see how the idea of the matrices can be used in solving system of the linear equations. We shall present a comparative sketch of the various operations that we perform while solving the system by usual method (Addition method) and those which we perform with the entries (or elements) of rows of the augmented matrix. For instance, consider the following example.

***Example 1.***

Solve the system

$$-x + y = -9$$
$$x - 3y = 5$$

| Addition method | Row equivalent matrix method |
|---|---|
| $-x + y = -9$ <br> $x - 3y = 5$ | $\begin{bmatrix} -1 & 1 & : & -9 \\ 1 & -3 & : & 5 \end{bmatrix}$ |

| | |
|---|---|
| **Interchange first and second equations** $x - 3y = 5$ <br> $-x + y = -9$ | **Interchange first and second rows** $\begin{bmatrix} 1 & -3 & : & 5 \\ -1 & 1 & : & -9 \end{bmatrix}$ |
| **Add the first equation to the second** $x - 3y = 5$ <br> $-2y = -4$ | **Add the first row to the second** $\begin{bmatrix} 1 & -3 & : & 5 \\ 0 & -2 & : & -4 \end{bmatrix}$ |
| **Multiply the second equation by** $-\frac{1}{2}$ <br> $x - 3y = 5$ <br> $y = 2$ | **Multiply the second row by** $-\frac{1}{2}$ $\begin{bmatrix} 1 & -3 & : & 5 \\ 0 & 1 & : & 2 \end{bmatrix}$ |
| **Multiply the second equation by 3 and add to the first** $x = 11$ <br> $y = 2$ <br> The solution is (11,2) | **Multiply the second row by 3 and add it to the first** $\begin{bmatrix} 1 & 0 & : & 11 \\ 0 & 1 & : & 2 \end{bmatrix}$ <br> The solution is (11, 2) |

**Note:** In the *row equivalent matrix method*, the most important point to be remembered is that we always try to get a matrix in the form

$$\begin{bmatrix} 1 & 0 & : & p \\ 0 & 1 & : & q \end{bmatrix}$$

called the *reduced form* in which the coefficient matrix becomes a unit matrix, by which we mean a square matrix in which every entry along the leading diagonal (or principal diagonal) is unity and all other entries are zero. For instance, the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is a $2 \times 2$ unit matrix (or identity matrix).

The matrices obtained in the above procedure are known as *row equivalent matrices*.

## Elementary row operations:

Row-equivalent matrices are obtained by performing one or more of the following elementary row operations :

(i) Interchange of any two rows

(ii) Addition of one row to another

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(iii) Multiplication of any row by a non-zero number

(iv) Multiplication of any row by a non-zero number and addition of the resulting row to another.

Let us now illustrate this procedure with more examples.

## Worked Out Examples

***Example 1***

Solve the system $3x + 2y = -9$

$2x - 3y = -6$

by using row-equivalent matrices.

***Solution.***

Here the augmented matrix is

$$\begin{bmatrix} 3 & 2 & : & -9 \\ 2 & -3 & : & -6 \end{bmatrix}$$

Multiply the first row by $\frac{1}{3}$ to obtain

$$\begin{bmatrix} 1 & 2/3 & : & -3 \\ 2 & -3 & : & -6 \end{bmatrix}$$

Multiply the first row by $-2$ and add to the second (or perform $R_2 - 2R_1$)

$$\begin{bmatrix} 1 & 2/3 & : & -3 \\ 0 & -13/3 & : & 0 \end{bmatrix}$$

Multiply the second row by $-\frac{3}{13}$

$$\begin{bmatrix} 1 & 2/3 & : & -3 \\ 0 & 1 & : & 0 \end{bmatrix}$$

Multiply the second row by $-\frac{2}{3}$ and add it to the first row (or perform $R_1 - \frac{2}{3} R_2$)

$$\begin{bmatrix} 1 & 0 & : & -3 \\ 0 & 1 & : & 0 \end{bmatrix}$$

This gives us the solution

$x = -3,\ y = 0$ i.e. $(-3, 0)$.

Let us condense this procedure as follows:

$$\left[\begin{array}{cccc} 3 & 2 & : & -9 \\ 2 & -3 & : & -6 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc} 1 & 2/3 & : & -3 \\ 2 & -3 & : & -6 \end{array}\right] \quad R_1 \to \tfrac{1}{3} R_1$$

$$\sim \left[\begin{array}{cccc} 1 & 2/3 & : & -3 \\ 0 & -13/3 & : & 0 \end{array}\right] \quad R_2 \to R_2 - 2R_1$$

$$\sim \left[\begin{array}{cccc} 1 & 2/3 & : & -3 \\ 0 & 1 & : & 0 \end{array}\right] \quad R_2 \to -\tfrac{3}{13} R_2$$

$$\sim \left[\begin{array}{cccc} 1 & 0 & : & -3 \\ 0 & 1 & : & 0 \end{array}\right] \quad R_1 \to R_1 - \tfrac{2}{3} R_2$$

This gives us the solution

$$x = -3, \ y = 0 \quad \text{i.e. } (-3, 0).$$

Note: The symbol $\sim$ means "is equivalent to" and is used as we proceed by an elementary row operation from one step to the next.

For a system of three equations in three variables we try to get augmented matrix in the following reduced form:

$$\left(\begin{array}{ccccc} 1 & 0 & 0 & : & p \\ 0 & 1 & 0 & : & q \\ 0 & 0 & 1 & : & r \end{array}\right)$$

from which we obtain the solution $(p, q, r)$.

***Example 2***

Solve the system

$$2x - y + z = -1$$
$$x - 2y + 3z = 4$$
$$4x + y + 2z = 4 \qquad \text{(T.U. 2050)}$$

***Solution***

| Addition method | Row – equivalent matrix method |
|---|---|
| $2x - y + z = -1$<br>$x - 2y + 3z = 4$<br>$4x + y + 2z = 4$ | $\left[\begin{array}{ccccc} 2 & -1 & 1 & : & -1 \\ 1 & -2 & 3 & : & 4 \\ 4 & 1 & 2 & : & 4 \end{array}\right]$ |

| | |
|---|---|
| Interchange the first two equations<br>$x - 2y + 3z = 4$<br>$2x - y + z = -1$<br>$4x + y + 2z = 4$ | Interchange the first two rows<br>$\left[\begin{array}{cccc} 1 & -2 & 3 & : & 4 \\ 2 & -1 & 1 & : & -1 \\ 4 & 1 & 2 & : & 4 \end{array}\right]$ |
| Multiply the first equation by –2 and add to second<br>$x - 2y + 3z = 4$<br>$3y - 5z = -9$<br>$4x + y + 2z = 4$ | Multiply the first row by –2 and add to the second<br>$\left[\begin{array}{ccccc} 1 & -2 & 3 & : & 4 \\ 0 & 3 & -5 & : & -9 \\ 4 & 1 & 2 & : & 4 \end{array}\right]$ |
| Multiply the first equation by –4 and add to the third<br>$x - 2y + 3z = 4$<br>$3y - 5z = -9$<br>$9y - 10z = -12$ | Multiply the first row by –4 and add to the third<br>$\left[\begin{array}{ccccc} 1 & -2 & 3 & : & 4 \\ 0 & 3 & -5 & : & -9 \\ 0 & 9 & -10 & : & -12 \end{array}\right]$ |
| Multiply the second equation by $\frac{1}{3}$<br>$x - 2y + 3z = 4$<br>$y - \frac{5z}{3} = -3$<br>$9y - 10z = -12$ | Multiply the second row by $\frac{1}{3}$<br>$\left[\begin{array}{ccccc} 1 & -2 & 3 & : & 4 \\ 0 & 1 & -5/3 & : & -3 \\ 0 & 9 & -10 & : & -12 \end{array}\right]$ |
| Multiply the second equation by –9 and add to the third<br>$x - 2y + 3z = 4$<br>$y - \frac{5z}{3} = -3$<br>$5z = 15$ | Multiply the second row by –9 and add to the third<br>$\left[\begin{array}{ccccc} 1 & -2 & 3 & : & 4 \\ 0 & 1 & -5/3 & : & -3 \\ 0 & 0 & 5 & : & 15 \end{array}\right]$ |
| Multiply the third equation by $\frac{1}{5}$<br>$x - 2y + 3z = 4$<br>$y - \frac{5z}{3} = -3$<br>$z = 3$ | Multiply the third row by $\frac{1}{5}$<br>$\left[\begin{array}{ccccc} 1 & -2 & 3 & : & 4 \\ 0 & 1 & -5/3 & : & -3 \\ 0 & 0 & 1 & : & 3 \end{array}\right]$ |

| | |
|---|---|
| Multiply the third equation by –3 and $+\frac{5}{3}$, and then add to the first and second equations respectively<br>$x - 2y = -5$<br>$y = 2$<br>$z = 3$ | Multiply the third row by –3 and $\frac{5}{3}$ and then add to the first and second row respectively<br>$\begin{bmatrix} 1 & -2 & 0 & : & -5 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$ |
| Multiply the second equation by 2 and add to the first<br>$x = -1$<br>$y = 2$<br>$z = 3$<br>Hence the solution is (–1,2,3 ). | Multiply the second row by 2 and add to the first<br>$\begin{bmatrix} 1 & 0 & 0 & : & -1 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$<br>Hence the solution is (–1,2,3) |

***Example 3***

Solve the system

$$\begin{aligned} 2x - y + 4z &= -3 \\ x \quad\;\; - 4z &= 5 \\ 6x - y + 2z &= 10 \end{aligned}$$

by using row–equivalent matrices. (T.U. 2056S, HSEB 2053)

***Solution.***

Here the second equation does not contain y and so we find some difficulty in writing the augmented matrix. To remove this difficulty, we may write the system as

$$\begin{aligned} 2.x - 1.y + 4.z &= -3 \\ 1.x + 0.y - 4.z &= 5 \\ 6.x - 1.y + 2.z &= 10 \end{aligned}$$

The augmented matrix is

$$\begin{bmatrix} 2 & -1 & 4 & : & -3 \\ 1 & 0 & -4 & : & 5 \\ 6 & -1 & 2 & : & 10 \end{bmatrix}$$

Interchange the first two rows

$$\begin{bmatrix} 1 & 0 & -4 & : & 5 \\ 2 & -1 & 4 & : & -3 \\ 6 & -1 & 2 & : & 10 \end{bmatrix}$$

Perform $R_2 - 2R_1$ and $R_3 - 6R_1$

$$\begin{bmatrix} 1 & 0 & -4 & : & 5 \\ 0 & -1 & 12 & : & -13 \\ 0 & -1 & 26 & : & -20 \end{bmatrix}$$

Perform $R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & -4 & : & 5 \\ 0 & -1 & 12 & : & -13 \\ 0 & 0 & 14 & : & -7 \end{bmatrix}$$

Perform $(-1)R_2$ and $\frac{1}{14} R_3$

$$\begin{bmatrix} 1 & 0 & -4 & : & 5 \\ 0 & 1 & -12 & : & 13 \\ 0 & 0 & 1 & : & -1/2 \end{bmatrix}$$

Perform $R_1 + 4R_3$ and $R_2 + 12R_3$

$$\begin{bmatrix} 1 & 0 & 0 & : & 3 \\ 0 & 1 & 0 & : & 7 \\ 0 & 0 & 1 & : & -\frac{1}{2} \end{bmatrix}$$

This gives us the solution $x = 3, y = 7, z = -\frac{1}{2}$, i.e. $(3, 7, -\frac{1}{2})$.

Let us condense this procedure as follows.

$$\begin{bmatrix} 2 & -1 & 4 & : & -3 \\ 1 & 0 & -4 & : & 5 \\ 6 & -1 & 2 & : & 10 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -4 & : & 5 \\ 2 & -1 & 4 & : & -3 \\ 6 & -1 & 2 & : & 10 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -4 & : & 5 \\ 0 & -1 & 12 & : & -13 \\ 0 & -1 & 26 & : & -20 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 6R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -4 & : & 5 \\ 0 & -1 & 12 & : & -13 \\ 0 & 0 & 14 & : & -7 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -4 & : & 5 \\ 0 & 1 & -12 & : & 13 \\ 0 & 0 & 1 & : & -1/2 \end{bmatrix} \quad \begin{matrix} R_2 \to (-1)R_2 \\ R_3 \to \frac{1}{14} R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 3 \\ 0 & 1 & 0 & : & 7 \\ 0 & 0 & 1 & : & -\frac{1}{2} \end{bmatrix} \quad \begin{matrix} R_1 \to R_1 + 4R_3 \\ R_2 \to R_2 + 12R_3 \end{matrix}$$

This gives us the solution $x = 3, y = 7, z = -\frac{1}{2}$, i.e. $(3, 7, -\frac{1}{2})$.

A condensed flow chart of this process

Start

↓

Given three equations in three unknown

↓

Form the augmented matrix

↓

Is the matrix in the form $\begin{bmatrix} 1 & 0 & 0 & : & p \\ 0 & 1 & 0 & : & q \\ 0 & 0 & 1 & : & r \end{bmatrix}$ — No → Use the elementary row operations

↓ Yes

The solution is $x = p, y = q, z = r$.

↓

Stop

## EXERCISE 10.2

1. Solve, by using both addition method and row–equivalent matrices, the following system of linear equations:

(a) $5x - 2y = -2$, $2x + 5y = -24$
(b) $x - y = 2$, $2x + 3y = 9$
(c) $x + y - 2z = 7$, $2x - 3y - 2z = 0$, $x - 2y - 3z = 3$

2. Solve the following systems by using row–equivalent matrices:

(a) $8x - 3y = -31$, $2x + 6y = 26$
(b) $3x - 3y = 11$, $9x - 2y = 5$
(c) $x = 2y + 3$, $3x - 5y = 8$
(d) $3x - 2y = 8$, $5x + 3y = 7$
(e) $5x - 3y = -2$, $4x + 2y = 5$
(f) $6x - 4y = 0$, $x + y - 5 = 0$
(g) $x + y = 5$, $2x + 3y = 12$
(h) $5x - 3y = 20$, $2x + 3y = 8$
(i) $2x + 12y = 16$, $3x + 10y = 8$
(j) $3x - 2y = -8$, $-x + 4y = 6$

3. Use the row–equivalent matrices to solve the system:

(a) $x - y + 2z = 0$, $x - 2y + 3z = -1$, $2x - 2y + z = -3$
(b) $x + 2y - 3z = 9$, $2x - y + 2z = -8$, $3x - y - 4z = 3$ (T.U. 2049)
(c) $x + y + z = 1$, $x + 2y + 3z = 4$, $x + 3y + 7z = 13$ (T.U. 2051)
(d) $3x - 5z = -7$, $3x + 5y = 3$, $3z - 3y = 2$
(e) $9y - 5x = 3$, $x + z = 1$, $z + 2y = 2$
(f) $x + 4y + 3z = 6$, $3x + 9y = 18$, $-5x - 6y + 2z = -5$
(g) $x - y + z = -3$, $x + y + z = 1$, $3x - 4y - z = 1$
(h) $x - y - z = -2$, $5x + 10z = 20$, $10y - 20z = 10$
(i) $2y + 6z = 2$, $3y + 6z = 6$, $2x + 4y + 6z = 22$
(j) $x + 4y + z = 18$, $3x + 3y - 2z = 2$, $-4y + z = -7$

### Answers

1. (a) (–2, –4) (b) (3,1) (c) (1,2, –2)
2. (a) (–2,5) (b) (–1/3,–4) (c) (1, –1) (d) (2, –1)

(e) (1/2, 3/2) (f) (2,3) (g) (3,2) (h) (4,0)
(i) (–4,2) (j) (–2,1)

3. (a) (0,2,1) (b) (–1,2, –2) (c) (1, –3,3) (d) (–1/9,2/3,4/3)
(e) (3,2, –2) (f) (–3,3, –1) (g) (2,2, –3) (h) (2,3,1)
(i) (6,4, –1) (j) (1,3,5)

## 10.4 Application of Matrices and Determinants

a) A system of linear equations in two or more variables appear in various branches of pure, applied and applicable mathematics. Representation of such a system in the form of *a matrix equation* and use of *matrix methods* have proved to be of great help in solving systems of linear equations in two or more variables. To illustrate this beautiful technique, we consider a system of linear equations in two variables.

Let the system of the linear equations in two variables $x$ and $y$ be

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

Collecting the coefficients of the variables and the constants separately in the order in which they occur in the equations and enclosing them within parentheses, we have the matrices

$$A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Also, writing the two variables $x$ and $y$ as a column matrix, we have

$$X = \begin{pmatrix} x \\ y \end{pmatrix}$$

Then, by virtue of the definition of the product and equality of the two matrices, the system of the linear equations may be written in the compact and elegant matrix form:

$$AX = C,$$

$$\text{or,} \quad \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

$$\text{or} \quad \begin{pmatrix} a_1x + b_1y \\ a_2x + b_2y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

which implies

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

Extension of the above process to a system of linear equations in three or more variables is obvious.

Let us now see how such a matrix equation may be solved by the methods of matrix algebra.

We know that the inverse $A^{-1}$ of the matrix A exists, if A is non-singular, i.e. $|A| \neq 0$. Multiplying both sides of the matrix equation

$$AX = C$$

by $A^{-1}$ from the left, we get

$$A^{-1}(AX) = A^{-1}C$$

or, $(A^{-1}A)X = A^{-1}C$

or $IX = D$, (say) with $A^{-1}C = D = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$

or $X = D$

or $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$

Hence, $x = d_1$ and $y = d_2$.

## Worked Out Examples

***Example 1***

Solve $4x + 5y = 2$
$2x + 3y = 0$

***Solution :***

Writing the system of linear equations in the matrix form, we have,

$$AX = C,$$

where, $A = \begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $C = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

Here $|A| = \begin{vmatrix} 4 & 5 \\ 2 & 3 \end{vmatrix} = 12 - 10 = 2 \neq 0$

$A_{11}$ = cofactor of $a_{11} = 3$ $A_{12}$ = cofactor of $a_{12} = -2$

$A_{21}$ = cofactor of $a_{21} = -5$ $A_{22}$ = cofactor of $a_{22} = 4$

Matrix of cofactors $= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -5 & 4 \end{pmatrix}$

$$\text{adj } A = \begin{pmatrix} 3 & -5 \\ -2 & 4 \end{pmatrix}$$

Now, $A^{-1} = \frac{1}{|A|}$ adj A $= \frac{1}{2}\begin{pmatrix} 3 & -5 \\ -2 & 4 \end{pmatrix}$

Since $X = A^{-1}C$, $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3 & -5 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$$= \frac{1}{2}\begin{pmatrix} 6 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

Hence $x = 3$ and $y = -2$

***Example 2.***

Solve $\quad x + y - z = 1$
$\quad y + z = 2$
$\quad x - y = 0$

***Solution :***

Writing this system of linear equations in the matrix form, we have

$$AX = C$$

where

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } C = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + (-1)\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= 1 + 1 + 1 = 3 \neq 0$$

$$A_{11} = \text{cofactor of } a_{11} = \begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} = 1$$

$$A_{12} = \text{cofactor of } a_{12} = -\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 1$$

$$A_{13} = \text{cofactor of } a_{13} = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1$$

$$A_{21} = \text{cofactor of } a_{21} = -\begin{vmatrix} 1 & -1 \\ -1 & 0 \end{vmatrix} = 1$$

$$A_{22} = \text{cofactor of } a_{22} = \begin{vmatrix} 1 & -1 \\ -1 & 0 \end{vmatrix} = 1$$

$$A_{23} = \text{cofactor of } a_{23} = -\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 2$$

$$A_{31} = \text{cofactor of } a_{31} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$$

$$A_{32} = \text{cofactor of } a_{32} = -\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = -1$$

$$A_{33} = \text{cofactor of } a_{33} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

$$\text{Matrix of cofactors} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 2 \\ 2 & -1 & 1 \end{pmatrix}$$

$$\text{adj } A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & -1 \\ -1 & 2 & 1 \end{pmatrix}$$

Since, $X = A^{-1}C = \dfrac{1}{|A|} \text{ adj } (A).C$,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & -1 \\ -1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 1+2 \\ 1+2 \\ -1+4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and hence $x = 1$, $y = 1$ and $z = 1$.

b) Another important technique of solving a system of linear equations is the application of *Cramer's rule*. This technique is also a consequence of the technique of elimination which we have discussed in the previous pages. But here we use the notion of a determinant associated with a square matrix instead of the matrix itself.

Let us now see how the idea of a determinant can be used in solving system of linear equations. For instance consider the system of two linear equations

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

where $x$ and $y$ are variables and the rest are all constants.

In order to facilitate computation and to save space, we present computations involving $x$ and $y$ side by side:

| | |
|---|---|
| Multiply the first equation by $b_2$, and the second by $-b_1$ to find | Multiply the first equation by $-a_2$ and the second by $a_1$ to find |
| $a_1b_2x + b_1b_2y = c_1b_2$ | $-a_1a_2x - a_2b_1y = -a_2c_1$ |
| $-a_2b_1x - b_1b_2y = -c_2b_1$ | $a_1a_2x + a_1b_2y = a_1c_2$ |
| Add the two equations to get | Add the two equations to get |
| $(a_1b_2 - a_2b_1)\, x = (c_1b_2 - c_2b_1)$. | $(a_1b_2 - a_2b_1)\, y = (a_1c_2 - a_2c_1)$. |

Using the determinant notations, we have

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

***Alternative method***

The system of linear equations is

$$\left.\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}\right\} \quad \ldots(1)$$

We have

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} x = \begin{vmatrix} a_1x & b_1 \\ a_2x & b_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1x + b_1y & b_1 \\ a_2x + b_2y & b_2 \end{vmatrix} \quad \text{(by changing } C_1 \to C_1 + C_2y)$$

$$= \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \quad \text{because of (1)}$$

Also we have $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} y = \begin{vmatrix} a_1 & b_1 y \\ a_2 & b_2 y \end{vmatrix}$.

$$= \begin{vmatrix} a_1 & a_1 x + b_1 y \\ a_2 & a_2 x + b_2 y \end{vmatrix} \quad \text{(by applying } C_2 \to C_2 + C_1 x)$$

$$= \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \quad \text{because of equation (1)]}$$

Let us now see how the determinants determine the nature of the solution of the system, i.e. what values can be assigned to $x$ and $y$.

*Case I.*

Suppose $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$, then we can divide in each case by this quantity (number) to obtain

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \quad \text{and} \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

This pair of values gives unique solution. In this case the system is *consistent* and *independent*

*Case II.*

Suppose $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$ and $\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = 0.$

Then we have

$$0.x = 0 \quad \text{and} \quad 0.y = 0$$

Clearly, these two are true whatever may be the values of $x$ and $y$. In other words, given a value to $x$, we may find the corresponding value of $y$ from one of the equations such that the pair of values satisfies both equations. Since we can assign an infinite number of values to $x$, we can find an infinite number of values of $y$. Thus we have an infinite number of solutions of the system. In such a case, we say that system is *consistent* and *dependent*.

*Case III.*

Suppose

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0 \text{ and either } \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \neq 0 \quad \text{or} \quad \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \neq 0.$$

Then we have

$$0.x \neq 0 \quad \text{or} \quad 0.y \neq 0, \quad \text{which is obviously false.}$$

In such a case, we say that the system is *inconsistent* and *independent*. In other words, the system has no solution.

In conclusion, we say that the system is consistent and has exactly one solution when determinant of the coefficients, i.e.

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0.$$

In what follows, we always assume that determinant of the coefficient matrix is NEVER zero, and in this case we have the following rules for the determinations in the values of x and y:

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad \text{and} \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

where the numerator determinants in the values of $x$ and $y$ may be obtained by replacing the first and second columns of the denominator determinant by

$$\begin{matrix} c_1 \\ c_2 \end{matrix}$$

in turn.

The above rule of finding the solution of a system of linear equations is known as Cramer's Rule.

Cramer's rule for a system of linear equations in two variables can be extended to a system of linear equations in three or more variables. The procedure of obtaining formulae for the variables is the same, i.e. gradual elimination of variables till we get an equation involving a single variable.

We can use the alternative method also.

Consider the system of linear equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{aligned}\right\} \quad \ldots\ldots\ldots\text{(ii)}$$

We have

$$x\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1x + b_1y + c_1z & b_1 & c_1 \\ a_2x + b_2y + c_2z & b_2 & c_2 \\ a_3x + b_3y + c_3z & b_3 & c_3 \end{vmatrix}$$

(by changing $C_1 \to C_{1+} C_2y + C_3z$)

$$= \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \quad \text{because of (ii)}$$

Also we have

$$y\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1y & c_1 \\ a_2 & b_2y & c_2 \\ a_3 & b_3y & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_1x + b_1y + c_1z & c_1 \\ a_2 & a_2x + b_2y + c_2z & c_2 \\ a_3 & a_3x + b_3y + c_3z & c_3 \end{vmatrix}$$

(by applying $C_2 \to C_1x + C_2 + C_3z$)

$$= \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \quad \text{because of (ii)}$$

Similarly

$$z\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Hence, the formulae for $x$, $y$ and $z$ are

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{D_1}{D} \qquad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{D_2}{D}$$

$$z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{D_3}{D}$$

where the quantities in the numerator and denominator of each formula is a determinant of third order (or three-by-three determinant). It is defined by the formula of the form

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_1 - b_2c_1)$$

Note: the numerator determinants $D_1$, $D_2$, $D_3$ of the above formulae for $x$, $y$ and $z$ can be obtained by replacing the first, second and third columns of the denominator determinant D by the column

$$\begin{matrix} d_1 \\ d_2 \\ d_3 \end{matrix}$$

in turn.

## Worked Out Examples

***Example 1***

To illustrate the use of Cramer's rule, consider the system

$$x - 2y = -7$$
$$3x + 7y = 5.$$

Using Cramer's Rule, we have

$$x = \frac{\begin{vmatrix} -7 & -2 \\ 5 & 7 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 3 & 7 \end{vmatrix}} = \frac{(-7).7 - 5.(-2)}{1.7 - 3.(-2)} = \frac{-39}{13} = -3$$

$$y = \frac{\begin{vmatrix} 1 & -7 \\ 3 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 3 & 7 \end{vmatrix}} = \frac{1.5 - 3.(-7)}{1.7 - 3.(-2)} = \frac{26}{13} = 2$$

Hence the solution is (–3, 2).

***Example 2***

Solve the following equations using Cramer's rule.

$$\frac{3}{x} + \frac{2}{y} = \frac{19}{20}$$

and $$\frac{4}{x} + \frac{10}{y} = 2$$

***Solution :***

$$\left.\begin{aligned} \frac{3}{x} + \frac{2}{y} &= \frac{19}{20} \\ \frac{4}{x} + \frac{10}{y} &= 2 \end{aligned}\right\}$$

$$D = \begin{vmatrix} 3 & 2 \\ 4 & 10 \end{vmatrix} = 30 - 8 = 22$$

$$D_1 = \begin{vmatrix} 19/20 & 2 \\ 2 & 10 \end{vmatrix} = \frac{19}{2} - 4 = \frac{11}{2}$$

$$D_2 = \begin{vmatrix} 3 & 19/20 \\ 4 & 2 \end{vmatrix} = 6 - \frac{19}{5} = \frac{11}{5}$$

Using Cramer's rule,

$$\frac{1}{x} = \frac{D_1}{D} = \frac{11/2}{22} = \frac{1}{4} \qquad \text{and} \qquad \frac{1}{y} = \frac{D_2}{D} = \frac{11/5}{22} = \frac{1}{10}$$

$\therefore \quad x = 4 \qquad\qquad \therefore \quad y = 10$

$\therefore$ the solution is (4, 10)

***Example 3***

Solve by using Cramer's rule, the system

$$x - 2y - z = -7$$
$$2x + y + z = 0$$
$$3x - 5y + 8z = 13$$

(T.U. 2055S)

***Solution***

Rule of Sarrus used to find the value of D :

$$\begin{array}{ccccc} 1 & -2 & -1 & 1 & -2 \\ 2 & 1 & 1 & 2 & 1 \\ 3 & -5 & 8 & 3 & -5 \end{array}$$

$$D = 8 + (-6) + 10 - (-3) - (-5) - (-32)$$
$$= 52$$

Hence

$$D = \begin{vmatrix} 1 & -2 & -1 \\ 2 & 1 & 1 \\ 3 & -5 & 8 \end{vmatrix} = 52$$

Similarly,

$$D_1 = \begin{vmatrix} -7 & -2 & -1 \\ 0 & 1 & 1 \\ 13 & -5 & 8 \end{vmatrix} = -104\,, \qquad D_2 = \begin{vmatrix} 1 & -7 & -1 \\ 2 & 0 & 1 \\ 3 & 13 & 8 \end{vmatrix} = 52$$

and $D_3 = \begin{vmatrix} 1 & -2 & -7 \\ 2 & 1 & 0 \\ 3 & -5 & 13 \end{vmatrix} = 156$

Hence $\quad x = \dfrac{D_1}{D} = -\dfrac{104}{52} = -2$

$$y = \frac{D_2}{D} = \frac{52}{52} = 1 \qquad\qquad z = \frac{D_3}{D} = \frac{154}{52} = 3.$$

Hence the solution is (–2,1,3).

Note: In practice, it is actually not necessary to evaluate $D_3$, since if we know the values of $x$ and $y$, we can use them in one of the equations to find $z$. We know $x = -2$ and $y = 1$ in the above case, so by using the equation $-2 - 2.1 - z = -7$, we find $z = 3$.

## EXERCISE 10.3

1. Apply matrix method or Cramer's rule to solve the following systems:

(a) $2x - y = 5$
$x - 2y = 1$

(b) $2x + 5y = 17$
$5x - 2y = -1$

(c) $3x + 4y = -2$
$5x - 7y = 24$

(d) $-2x + 4y = 3$
$3x - 7y = 1$

(e) $5x - 4y = -3$
$7x + 2y = 49$

(f) $2x + 5y = 24$
$2x + 3y = 12$

(g) $\frac{2x}{3} + y = 16$
$x + \frac{y}{4} = 14$

(h) $3x + \frac{4}{y} = 10$
$-2x + \frac{3}{y} = -1$

2. Solve the following system by using matrix method or Cramer's Rule:

(a) $x - 3y - 7z = 6$
$2x + 3y + z = 9$
$4x + y \quad = 7$

(b) $2x - 3y - z = 4$
$x - 2y - z = 1$
$x - y + 2z = 9$ (T.U. 2053)

(c) $3x - y - 2z = 1$
$x - y + 2z = 3$
$-2x + 3y + z = 8$

(d) $3x + 5y = 2$
$2x - 3z = -7$
$4y + 2z = 2$

(e) $6y + 6z = -1$
$8x + 6z = -1$
$4x + 9y = 8$

(f) $x + 2y - z = -5$
$2x - y + z = 6$
$x - y - 3z = -3$

### Answers

1. (a) (3,1) (b) (1, 3) (c) (2, –2) (d) (–25/2, –11/2)
(e) (5, 7) (f) (–3, 6) (g) (12, 8) (h) (2, 1)

2. (a) (1,3–2) (b) (2, –1,3) (c) (3, 4, 2)
(d) (4, –2, 5) (e) (1/2,2/3,–5/6) (f) ( 1, –2, 2)

# CHAPTER 11
# Complex Numbers

## 11.1 Introduction

Earlier in the text we discussed about different types of numbers, namely, the natual numbers (the counting numbers), the integers, the rational numbers and the irrational numbers. All these numbers taken together form the set of real numbers. We have also seen how real numbers can be represented by points in a straight line, called the real line or the number line. The set of real numbers is everywhere dense—between every two real numbers there is a real number. Also we have noted that the system of real numbers is complete – every real number corresponds to a point in the real line and conversely. One important fact about real number is

*"The square of a real number is never negative."*

Though the system of real numbers is complete in itself we find it inadequate to find a solution for equation of the type $x^2 + 4 = 0$ which demands a value of $x$ whose square is negative. We cannot find a solution for the equation in the system of real numbers. If we were to make equations like $x^2 + 4 = 0$ sensible naturally we have to extend the system of numbers and we shall be dealing with this extended system of numbers in this chapter. This is done by introducing new set of numbers called **complex numbers**. It is believed that Cardon used complex numbers in 1546.

## 11.2 Complex Numbers

An ordered pair of real numbers is defined to be a **Complex Number**.

Thus, if $a$ and $b$ are real numbers, the ordered pair written as $(a, b)$ is a **complex number**. The first number, $a$, is called the **real part** and the second number, $b$, the **imaginary part** of the complex number. The imaginary part (the second number) is as good a real number as the first part. We shall agree to call it the imaginary part since we have called the first number, the real part, and there is nothing imaginary about it. In line with this interesting terminology, the $x$-axis is frequently referred to as the real axis, and the $y$-axis as the imaginary axis.

A complex number is usually denoted by a single letter such as $z$, $w$ etc. If $z = (a, b)$ is a complex number, the real and the imaginary parts of $z$ are respectively denoted by Re($z$) and Im($z$). Thus Re($z$) = $a$ and Im($z$) = $b$.

Since a complex number $(a, b)$ is an ordered pair of real numbers $a$ and $b$, two complex numbers $(a, b)$ and $(c, d)$ are said to be equal if and only if $a = c$ and $b = d$.

## a) Three Definitions

i) The sum of two complex numbers $z = (a, b)$ and $w = (c, d)$ is defined to be a complex number $z + w$ such that

$$z + w = (a + c, b + d).$$

For example, $(2, 1) + (3, 4) = (5, 5)$

ii) The product of two complex numbers $z = (a, b)$ and $w = (c, d)$ is defined to be a complex number $zw$ such that

$$zw = (ac - bd, ad + bc)$$

For example, $(2, 1) \cdot (3, 4) = (2\cdot3 - 1\cdot4, 2\cdot4 + 1\cdot3)$
$= (2, 11)$

iii) If $z = (a, b) \neq (0, 0)$ be a complex number then the reciprocal of $z$, denoted by $\frac{1}{z}$ or $z^{-1}$, is defined as

$$z^{-1} = \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right)$$

This is the **multiplicative inverse** of $z$, as $zz^{-1} = (1, 0)$

For example $\frac{1}{(3, 1)} = \left(\frac{3}{10}, \frac{-1}{10}\right)$

We have $\frac{w}{z} = wz^{-1}$ where $w$ and $z$ are complex numbers.

Also if $z = (a, b)$ and $w = (c, d)$

then $$\frac{w}{z} = (c, d)\left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right)$$
$$= \left(\frac{ac + bd}{a^2 + b^2}, \frac{ad - bc}{a^2 + b^2}\right)$$

## b) Some Theorems

*Theorem I.*

The operations of addition and multiplication satisfy the commutative, associative and distributive laws.

*Proof.*

Let $z_1 = (a, b)$, $z_2 = (c, d)$ and $z_3 = (e, f)$, then

i) $$z_1 + z_2 = (a, b) + (c, d) = (a + c, b + d) = (c + a, d + b) = z_2 + z_1$$

$\therefore$ addition of complex numbers is commutative.

ii) $$(z_1 + z_2) + z_3 = (a + c, b + d) + (e, f) = (a + c + e, b + d + f) = (a, b) + (c + e, d + f) = z_1 + (z_2 + z_3)$$

$\therefore$ addition of complex numbers is associative.

iii) $$z_1z_2 = (a, b)(c, d) = (ac - bd, ad + bc) = (ca - db, da + cb) = (c, d)(a, b) = z_2z_1$$

Hence, multiplication of complex numbers is commutative.

iv) $$(z_1z_2)z_3 = (ac - bd, ad + bc)(e, f) = (ace - bde - adf - bcf, acf - bdf + ade + bce)$$

Also, $$z_1(z_2z_3) = (a, b)(ce - df, cf + de) = (ace - adf - bcf - bde, acf + ade + bce - bdf)$$

$\therefore$ $(z_1z_2)z_3 = z_1(z_2z_3)$

Hence, multiplication of complex numbers is associative.

v) $$z_1(z_2 + z_3) = (a, b)[(c, d) + (e, f)] = (a, b)(c + e, d + f) = (ac + ae - bd - bf, ad + af + bc + be)$$

$$z_1z_2 + z_1z_3 = (a, b)(c, d) + (a, b)(e, f) = (ac - bd, ad + bc) + (ae - bf, af + be) = (ac + ae - bd - bf, ad + af + bc + be)$$

$\therefore$ $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$

Thus, addition and multiplication of complex numbers are distributive.

*Theorem II.*

If $z = (a, b)$ be a complex number

i) $(a, b) + (0, 0) = (a, b)$

ii) $(a, b)\ (0, 0) = (0, 0)$

iii) $(a, b)\ (1, 0) = (a, b)$

iv) $\dfrac{(a, b)}{(1, 0)} = (a, b)$

v) $(a, b) + (-a, -b) = (0, 0)$

Proofs are immediate from definitions, for example

i) $(a, b) + (0, 0) = (a + 0, b + 0) = (a, b)$

iv) $\dfrac{(a, b)}{(1, 0)} = (a, b)\dfrac{1}{(1, 0)} = (a, b)\ (1, 0) = (a, b)$

We observe from the above theorems that the complex numbers $(0, 0)$ and $(1, 0)$ have properties similar to those of the real numbers 0 and 1, and so we shall take them as the real numbers 0 and 1; and the complex number $(-a, -b)$ acts as the additive inverse of $z = (a, b)$. These complex numbers are therefore denoted by the symbols 0, 1 and $-z$.

Thus by writing $(0, 0) = 0$, $(1, 0) = 1$ and $(-a, -b) = -z$ the theorems will be restated as

i) $z + 0 = z$

ii) $z \cdot 0 = 0$

iii) $z \cdot 1 = z$

iv) $\dfrac{z}{1} = z$

v) $z + (-z) = z - z = 0$

Here, $-z$ is said to be the **additive inverse of** $z$.

It can be easily seen that for any non-zero real numbers $a, b$

i) $(a, 0) + (b, 0) = (a + b, 0)$

ii) $(a, 0)\ (b, 0) = (ab, 0)$

iii) $\dfrac{(a, 0)}{(b, 0)} = \left(\dfrac{a}{b}, 0\right)$

Hence a complex number of the form $(a, 0)$ has properties similar to the real number $a$. Hence we shall henceforth identify the complex number $(a, 0)$ with the real number $a$.

In other words, a complex number $(a, 0)$ with the imaginary part 0 is the real number '$a$' itself. Hence the real number system is a special case of the complex number system.

## 11.3 The Imaginary Unit

We have seen that the complex number $(a, 0)$ plays the same role as the real number $a$. Hence the real number $a$ can be treated as the complex number $(a, 0)$ with the imaginary part 0. Thus the complex number $(1, 0)$ is same as the real number 1. Naturally, the question arises – "*What if the real part is zero ?*" The complex number $(0, 1)$ with the real part zero and the imaginary part 1 is denoted by the letter '$i$' (the *iota* in the Greek alphabet and the first letter of the word imaginary in the English alphabet) and it is called the **imaginary unit**. The symbol $i$ was introduced by Euler in 1777.

*The complex number (0, 1) is denoted by i and is called the imaginary unit.*

### a) Theorems

***Theorem I.***

$i^2 = -1$

***Proof***

$$\begin{aligned} i^2 = i \cdot i &= (0, 1)\,(0, 1) = (0 - 1, 0 + 0) \\ &= (-1, 0) = -(1, 0) \\ &= -1 \end{aligned}$$

***Theorem II.***

If $a$ and $b$ are real numbers then the complex number $(a, b)$ can be written as $a + ib$ where $i = (0, 1)$, and $i^2 = -1$.

***Proof***

$$\begin{aligned} a + ib &= (a, 0) + (0, 1)\,(b, 0) \\ &= (a, 0) + (0, b) \\ &= (a, b) \end{aligned}$$

So, if $a$ and $b$ are real numbers then $a + ib$ is said to be a complex number where $i^2 = -1$.

Now we are in a position to say that there is a solution to an equation like $x^2 + 4 = 0$

$$\begin{aligned} & x^2 + 4 = 0 \\ \Rightarrow\quad & x^2 = -4 \\ \Rightarrow\quad & x^2 = 4i^2 \\ \therefore\quad & x = \pm 2i \end{aligned}$$

Henceforth the complex number $(a, 0)$ with the imaginary part zero will be written as the real number $a$, and it is said to be a *purely real* number (or simply a real number); the complex number (0, $b$) with the real part zero will be written as $bi$ and it will be said to be a *purely imaginary* number. The complex number ($a$, $b$) will be written as $a + ib$.

Taking $i$ as the imaginary unit such that $i^2 = -1$ and the complex number $z = a + ib$ as a complex number, it is not necessary to memorize the definitions of the sum and the product of two complex numbers. We can use the simple rules of algebra for computing the sum, difference or products of complex numbers.

For example,

i) $(2 + 3i) + (-1 + 4i) = 2 + 3i - 1 + 4i$
$= 2 - 1 + 3i + 4i$
$= 1 + 7i$

ii) $(2 + 3i) - (-1 + 4i) = 2 + 3i + 1 - 4i$
$= 3 - i$

iii) $(2 + 3i) \times (-1 + 4i) = -2 + 8i - 3i + 12i^2$
$= -2 + 5i - 12 = -14 + 5i$

The set of complex numbers, denoted by the letter C, obviously contains the set of real numbers R, i.e. R $\subset$ C.

**b) Powers of $i$**

We have defined $i$ as that number whose square is $-1$, and it is the imaginary unit. Any real multiple of $i$ is an imaginary number.

Since $i^2 = -1$, we have

$i^3 = i^2 \cdot i = (-1) \cdot i = -i$

$i^4 = (i^2)^2 = (-1)^2 = 1$

$i^5 = (i^4)i = (1)i = i$

$i^6 = (i^2)^3 = (-1)^3 = -1$

$i^{10} = (i^2)^5 = (-1)^5 = -1$

$(i^{-1}) = \frac{1}{i} = \frac{i}{i^2} = -i$

$i^{-10} = \frac{1}{i^{10}} = \frac{1}{-1} = -1$ etc.

Thus any integral power of $i$ is one of the four numbers $1, -1, i$ or $-i$.

**c) Some Examples to Illustrate the Use of $i$**

i) $\sqrt{-3} = \sqrt{i^2\ 3} = i\sqrt{3}$

ii) $\sqrt{-5} \times \sqrt{-9} = \sqrt{5\ i^2} \times \sqrt{9\ i^2}$

$= i\sqrt{5} \times i\sqrt{9} = i^2 \sqrt{5 \times 9} = -\sqrt{45} = -3\sqrt{5}$

Note : It is wrong to write

$$\sqrt{-5} \times \sqrt{-9} = \sqrt{-5 \times -9} = \sqrt{45} = 3\sqrt{5}$$

as the square root of a negative real number is undefined.

iii) $a^2 + b^2 = a^2 - (-1)b^2$

$= a^2 - i^2b^2 = (a + ib)(a - ib)$

## d) Properties of Complex Numbers

i) A real number cannot be equal to an imaginary number unless each is zero.

Let $a$ and $b$ be real numbers such that $a = ib$.

We have $a = ib$

$\Rightarrow a^2 = i^2b^2$

$\Rightarrow a^2 = -b^2$

$\Rightarrow a^2 + b^2 = 0$

$\Rightarrow a = 0, b = 0$

Hence the statement.

ii) If $a + ib = 0$, then $a = 0$ and $b = 0$.

Here, $a + ib = 0$

$\Rightarrow a = -ib$

$\Rightarrow a^2 = i^2b^2$

$\Rightarrow a^2 = -b^2$

$\Rightarrow a^2 + b^2 = 0$

$\Rightarrow a = 0$ and $b = 0$

iii) $a + ib = c + id$ iff $a = c, b = d$.

Here, $a + ib = c + id$

$\Leftrightarrow a - c = i(d - b)$

$\Leftrightarrow (a - c)^2 = i^2(d - b)^2$

$\Leftrightarrow (a - c)^2 = -(d - b)^2$

$\Leftrightarrow (a - c)^2 + (d - b)^2 = 0$

$\Leftrightarrow a - c = 0, d - b = 0$

$\Leftrightarrow a = c, b = d$

## Worked Out Examples

***Example 1***

Evaluate :

a) $(1, 0)^6$ b) $(0, 1)^7$

***Solution:***

a) $(1, 0)^6 = (1)^6 = 1$

b) $(0, 1)^7 = (i)^7 = i^4.i^2.i = 1.(-1).i = -i$

***Example 2***

1. Find the values of $x$ and $y$ if

a) $(x, y) = (1, 2) + (2, 3)$

b) $(x, y) = (3, 1) . (2, 3)$

c) $(3, 1) = (x, y) + (5, -1)$

d) $(x, y) = \dfrac{(2, -1)}{(-1, 3)}$

***Solutions:***

(a) $(x, y) = (1, 2) + (2, 3) = (3, 5)$

$\therefore x = 3, y = 5$

(b) $(x, y) = (3, 1) . (2, 3) = (6 - 3, 9 + 2) = (3, 11)$

$\therefore x = 3, y = 11$

(c) $(3, 1) = (x, y) + (5, -1) = (x + 5, y - 1)$

Hence $x + 5 = 3, y - 1 = 1$

$\therefore x = -2, y = 2$

(d) $(x, y) = \dfrac{(2, -1)}{(-1, 3)} = \left(\dfrac{-2 - 3}{1 + 9}, \dfrac{-6 + 1}{1 + 9}\right) = \left(-\dfrac{1}{2}, -\dfrac{1}{2}\right)$

$\therefore x = -\dfrac{1}{2}, y = -\dfrac{1}{2}$

***Example 3***

Simplify : $3\sqrt{-4} + 5\sqrt{-9} - 4\sqrt{-25}$

***Solution :***

$3\sqrt{-4} + 5\sqrt{-9} - 4\sqrt{-25}$

$= 3\sqrt{4i^2} + 5\sqrt{9i^2} - 4\sqrt{25i^2}$

$= 6i + 15i - 20i$

$= i$

***Example 4***

Find the values of the real numbers $x$ and $y$ if

$$(x + 2) + yi = (3 + i)(1 - 2i)$$

***Solution :***

$x + 2 + yi = (3 + i)(1 - 2i)$

or, $x + 2 + yi = 3 - 6i + i - 2i^2$

or, $x + 2 + yi = 5 - 5i$

Equating real and imaginary parts,

$x + 2 = 5$ and $y = -5$

$\therefore$ $x = 3$ and $y = -5$

## EXERCISE 11.1

1. **Evaluate :**
   a) $(1, 0)^2$ b) $(1, 0)^5$
   c) $(0, 1)^5$ d) $(0, 1)^{11}$
2. Find the values of $x$ and $y$ in each of the following
   a) $(x, y) = (2, 3) + (3, 2)$ b) $(x, y) = (2, 1) + (-2, -1)$
   c) $(x, y) = (2, 3) - (3, 2)$ d) $(2, 3) = (1, 1) + (x, y)$
   e) $(x, y) = (1, 1).(2, 3)$ f) $(x, y) = \dfrac{(1, 1)}{(3, 4)}$
3. **Simplify :**
   a) $\sqrt{-9} + \sqrt{-25} - \sqrt{-36}$ b) $\left(3 - \sqrt{-4}\right)\left(2 + \sqrt{-9}\right)$
   c) $\sqrt{-16}.\sqrt{-1}$ d) $3i^2 + i^3 + 9i^4 - i^7$
   e) $\frac{1}{i} - \frac{1}{i^2} + \frac{1}{i^3} - \frac{1}{i^4}$ f) $\frac{1}{i} + \frac{1}{i^2} + \frac{1}{i^3} + \frac{1}{i^4}$
4. Prove that : $(1 + i)^4.\left(1 + \frac{1}{i}\right)^4 = 16$
5. Find the real numbers $x$ and $y$ if
   a) $x + iy = (2 - 3i)(3 - 2i)$
   b) $(x - 1)i + (y + 1) = (1 + i)(4 - 3i)$
6. Show that $\dfrac{3 + 2i}{2 - 5i} + \dfrac{3 - 2i}{2 + 5i}$ is purely a real number

***Answers***

1. a) 1 b) 1 c) $i$ d) $-i$
2. a) $x = 5, y = 5$ b) $x = 0, y = 0$ c) $x = -1, y = 1$ d) $x = 1, y = 2$
   e) $x = -1, y = 5$ f) $x = \frac{7}{25}, y = -\frac{1}{25}$
3. a) $2i$ b) $12 + 5i$ c) $-4$ d) 6 e) 0 f) 0
5. a) $x = 0, y = -13$ b) $x = 2, y = 6$

## 11.4 Conjugate of a Complex Number

Let $a$ and $b$ be real numbers and let $z = a + ib$ be a complex number. The complex number $a - ib$ is called the **conjugate of** $z$, and it is denoted by $\bar{z}$ (read 'bar z'). We note here that two complex numbers are said to be conjugate of one another if their real parts are same and the imaginary parts differ in sign only.

Thus $3 + 4i$ and $3 - 4i$ are conjugate complex numbers. Each is the conjugate of the other.

### a) Properties of Conjugates

Let $z = a + ib$ and $w = c + id$ be two complex numbers, then we have the following properties

i) $\text{Re}(z) = \frac{1}{2}(z + \bar{z})$ ii) $\text{Im}(z) = \frac{1}{2i}(z - \bar{z})$

iii) $\overline{z + w} = \bar{z} + \bar{w}$ iv) $\overline{zw} = \bar{z}\,\bar{w}$

v) $\overline{z^2} = (\bar{z})^2$ vi) $\bar{\bar{z}} = z$

vii) $z\bar{z}$ = a real number

***Proofs :***

i) $z + \bar{z} = a + ib + a - ib = 2a$

$\therefore \quad \frac{1}{2}(z + \bar{z}) = a = \text{Re}(z)$

ii) $z - \bar{z} = a + ib - a + ib = 2ib$

$\therefore \quad \frac{1}{2i}(z - \bar{z}) = b = \text{Im}(z)$

iii) $\overline{z + w} = \overline{a + ib + c + id}$

$= \overline{a + c + i(b + d)}$

$= a + c - i(b + d)$

$= a - ib + c - id = \bar{z} + \bar{w}$

iv) $\overline{zw} = \overline{(a + ib)\,(c + id)}$

$= \overline{(ac - bd) + i(ad + bc)}$

$= ac - bd - i(ad + bc)$

$= (a - ib)\,(c - id) = \bar{z} \cdot \bar{w}$

v) In the result of (iv) if $z = w$, we get

$\overline{z^2} = (\bar{z})^2$

vi) $\bar{\bar{z}} = \overline{\overline{a + ib}} = \overline{a - ib} = a + ib = z$

vii) $z\,\bar{z} = (a + ib)\,(a - ib) = a^2 - i^2b^2$

$= a^2 + b^2$, a real number.

## 11.5 Absolute Value of a Complex Number

The **absolute value** of a complex number $z = a + ib$ is defined as the non-negative real number $\sqrt{a^2 + b^2}$. It is denoted by $|z|$ or $|a + ib|$.

The absolute value of a complex number is also known as the **modulus** of the complex number.

Examples: $|3 + 4i| = \sqrt{3^2 + 4^2} = 5$

$|4 - 3i| = \sqrt{4^2 + (-3)^2} = 5$

### a) Properties of Moduli of Complex Numbers

i) $|z| = |\bar{z}|$

ii) $|z| = 0$ iff $z = 0$

iii) $|zw| = |z| . |w|$ for two complex numbers

iv) $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$ if $|w| \neq 0$

v) $z\,\bar{z} = |z|^2$

vi) $\text{Re}\,(z) \leq |z|$ and $\text{Im}\,(z) \leq |z|$

***Proofs***

i) Let $z = a + ib$, then $\overline{z} = a - ib$

By definition $|z| = \sqrt{a^2 + b^2}$

and $|\overline{z}| = \sqrt{a^2 + (-b)^2}$

$= \sqrt{a^2 + b^2}$

$\therefore \quad |z| = |\overline{z}|$

ii) Let $z = a + ib$

So, $|z| = \sqrt{a^2 + b^2}$

$|z| = 0 \quad \Rightarrow \quad \sqrt{a^2 + b^2} = 0$

or, $a^2 + b^2 = 0$

or, $a = 0$ and $b = 0$

$\therefore \quad z = 0$

Also, $z = 0 \Rightarrow |z| = 0$.

iii) Let $z = a + ib$ and $w = c + id$

so that $|z| = \sqrt{a^2 + b^2}$ and $|w| = \sqrt{c^2 + d^2}$

and $zw = ac - bd + i(ad + bc)$

hence $|zw| = \sqrt{(ac - bd)^2 + (ad + bc)^2}$

$= \sqrt{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2}$

$= \sqrt{(a^2 + b^2)(c^2 + d^2)}$

$= \sqrt{a^2 + b^2}\sqrt{c^2 + d^2}$

$= |z||w|$

iv) $\frac{z}{w} = \frac{a + ib}{c + id} = \frac{a + ib}{c + id} \times \frac{c - id}{c - id}$

$= \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2}$

$= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i$

Now, $\left|\frac{z}{w}\right| = \sqrt{\left(\frac{ac + bd}{c^2 + d^2}\right)^2 + \left(\frac{bc - ad}{c^2 + d^2}\right)^2}$

$= \sqrt{\frac{a^2c^2 + b^2d^2 + b^2c^2 + a^2d^2}{(c^2 + d^2)^2}}$

$$= \sqrt{\frac{(a^2 + b^2)(c^2 + d^2)}{(c^2 + d^2)^2}}$$

$$= \sqrt{\frac{a^2 + b^2}{c^2 + d^2}} = \frac{\sqrt{a^2 + b^2}}{\sqrt{c^2 + d^2}} = \frac{|z|}{|w|}$$

v) $z\bar{z} = (a + ib)(a - ib)$

$= a^2 - i^2b^2$

$= a^2 + b^2$

$= |z|^2$

vi) Re $(z) \le |z|$ and Im$(z) \le |z|$

Let $z = a + ib$

Then Re$(z) = a$, Im$(z) = b$ and $|z| = \sqrt{a^2 + b^2}$

Since, $a^2 \le a^2 + b^2$

$\Rightarrow \quad a \le \sqrt{a^2 + b^2}$

$\Rightarrow$ Re$(z) \le |z|$

Similarly, $b \le \sqrt{a^2 + b^2}$

$\Rightarrow$ Im$(z) \le |z|$

**b) The Triangle Inequality**

If $z$ and $w$ are complex numbers

$|z| + |w| \ge |z + w|$

***Proof.***

Let $z = a + ib$ and $w = c + id$ so that $z + w = a + c + i(b + d)$, then

$|z| = \sqrt{a^2 + b^2}$, $|w| = \sqrt{c^2 + d^2}$

Also $|z + w| = \sqrt{(a + c)^2 + (b + d)^2}$

Now, $|z| + |w| \ge |z + w|$ will be true

if $\sqrt{a^2 + b^2} + \sqrt{c^2 + d^2} \ge \sqrt{(a + c)^2 + (b + d)^2}$

i.e. $a^2 + b^2 + c^2 + d^2 + 2\sqrt{(a^2 + b^2)(c^2 + d^2)} \ge (a + c)^2 + (b + d)^2$

i.e. $\sqrt{(a^2 + b^2)(c^2 + d^2)} \ge ac + bd$

i.e. $(a^2 + b^2)(c^2 + d^2) \ge a^2c^2 + b^2d^2 + 2abcd$

i.e. $a^2d^2 + b^2c^2 \ge 2abcd$

i.e. $a^2d^2 + b^2c^2 - 2abcd \ge 0$

i.e. $(ad - bc)^2 \geq 0$

which is true for all real numbers $a, b, c, d$.

Hence $|z| + |w| \geq |z + w|$.

***Second Method***

$$|z + w|^2 = (z + w)^2$$

$$= (z + w)\overline{(z + w)} \qquad (\because |z|^2 = z\overline{z})$$

$$= (z + w)(\overline{z} + \overline{w}) \qquad (\because \overline{z + w} = \overline{z} + \overline{w})$$

$$= z\overline{z} + z\overline{w} + \overline{z}w + w\overline{w}$$

$$= |z|^2 + |w|^2 + \left(z\overline{w} + \overline{z}w\right)$$

$$= |z|^2 + |w|^2 + \left(z\overline{w} + \overline{z\overline{\overline{w}}}\right) \qquad (\because \overline{\overline{w}} = w)$$

$$= |z|^2 + |w|^2 + 2\text{Re}(z\overline{w}) \qquad (\because z + \overline{z} = 2\,\text{Re}(z))$$

$$\leq |z|^2 + |w|^2 + 2|z\overline{w}| \qquad (\because \text{Re}(z) \leq |z|)$$

$$= |z|^2 + |w|^2 + 2|z||\overline{w}| \qquad (\because |zw| = |z||w|)$$

$$= |z|^2 + |w|^2 + 2|z||w| \qquad (\because |w| = |\overline{w}|)$$

$$= (|z| + |w|)^2$$

$\therefore$ $|z + w| \leq |z| + |w|$

i.e. $|z| + |w| \geq |z + w|$

## 11.6 Square Roots of Complex Numbers

Let $z = a + ib$ be a complex number whose square roots we want to compute.

Let a square root of $z = a + ib$ be $x + iy$, so that

$(x + iy)^2 = a + ib$

or, $x^2 - y^2 + 2xyi = a + ib$

$\therefore$ $x^2 - y^2 = a$ and $2xy = b$

Solving these two equations we get two sets of values of $x$ and $y$, thus giving us two square roots of $a + ib$.

## Worked Out Examples

***Example 1***

Express the following complex number into $a + ib$ form : $\dfrac{(1-i)^2}{1+2i}$

***Solution :***

$$\frac{(1-i)^2}{1+2i} = \frac{1-2i+i^2}{1+2i}$$

$$= \frac{1-2i-1}{1+2i} = \frac{-2i}{1+2i}$$

$$= \frac{-2i}{1+2i} \times \frac{1-2i}{1-2i} = \frac{-2i+4i^2}{1-4i^2}$$

$$= \frac{-2i-4}{1+4}$$

$$= -\frac{4}{5} - \frac{2}{5}i$$

which is in the form of $a + ib$ where $a = -\dfrac{4}{5}$, $b = -\dfrac{2}{5}$

***Example 2***

Find the absolute value of $\dfrac{1-2i}{2+i}$

***Solution :***

$$\left|\frac{1-2i}{2+i}\right| = \frac{|1-2i|}{|2+i|}$$

$$= \frac{\sqrt{(1)^2+(-2)^2}}{\sqrt{(2)^2+(1)^2}} = 1$$

***Example 3***

If $x - iy = \dfrac{3-2i}{3+2i}$, prove that $x^2 + y^2 = 1$.

***Solution :***

$$x - iy = \frac{3-2i}{3+2i}$$

or, $$x - iy = \frac{3-2i}{3+2i} \times \frac{3-2i}{3-2i}$$

$$= \frac{5-12i}{9+4}$$

or, $x - iy = \frac{5}{13} - \frac{12}{13}i$

Equating real and imaginary parts,

$$x = \frac{5}{13}, \quad y = \frac{12}{13}$$

Now, $x^2 + y^2 = \left(\frac{5}{13}\right)^2 + \left(\frac{12}{13}\right)^2$

$$= \frac{25 + 144}{169} = 1$$

***Example 4***

If $(x + iy)(3 + 2i) = 1 + i$, show that $x^2 + y^2 = \frac{2}{13}$

***Solution :***

$$(x + iy)(3 + 2i) = 1 + i$$

Taking modulus on both sides,

$$|(x + iy)(3 + 2i)| = |1 + i|$$

or, $|x + iy||3 + 2i| = |1 + i|$

or, $\sqrt{x^2 + y^2}\sqrt{9 + 4} = \sqrt{1 + 1}$

or, $x^2 + y^2 = \frac{2}{13}$

***Example 5***

Find the multiplicative inverse of $\frac{2 + 3i}{3 - i}$

***Solution :***

Let $z$ be the multiplicative inverse of $\frac{2 + 3i}{3 - i}$. Then,

$$z \cdot \frac{2 + 3i}{3 - i} = 1$$

$$z = \frac{3 - i}{2 + 3i} = \frac{3 - i}{2 + 3i} \times \frac{2 - 3i}{2 - 3i}$$

$$= \frac{6 - 9i - 2i - 3}{4 + 9} = \frac{3}{13} - \frac{11}{13}i$$

***Example 6***

Find the square roots of $3 - 4i$.

**Solution.**

Let $x + iy$ be the square roots of $3 - 4i$ so that

$$(x + iy)^2 = 3 - 4i$$

or, $x^2 - y^2 + 2xyi = 3 - 4i$

so $x^2 - y^2 = 3$ and $2xy = -4$

We have $(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2$

$$= 9 + 16 = 25$$

$\therefore$ $x^2 + y^2 = 5$ , since $x^2 + y^2$ cannot be negative.

So we have $x^2 - y^2 = 3$

and $x^2 + y^2 = 5$

Solving the two equations, we have

$$x = \pm 2, \quad y = \pm 1$$

Since $xy = -2$, $x$ and $y$ are of opposite signs

$\therefore$ $x = 2, y = -1$

or, $x = -2, y = 1$

Hence the square roots are

$2 - i$ and $-2 + i$

i.e. $\pm (2 - i)$

**Note 1.** If we want the square roots of $3 + 4i$, i.e. $x$ and $y$ have the same sign, proceeding as above we see that $xy = 2$, so $x = 2, y = 1$ and $x = -2, y = -1$ are the permissible sets of values of $x$ and $y$ and so the square roots will be

$$\pm (2 + i)$$

**Note 2.** Sometimes the square roots of a complex number can be found by inspection. For example,

a) $3 - 4i = 4 - 1 - 4i = 2^2 + i^2 - 2.2.i = (2 - i)^2$

$\therefore$ the square roots of $3 - 4i$ are $\pm(2 - i)$

b) $5 + 12i = 9 - 4 + 12i = 3^2 + (2i)^2 + 2.3.2i = (3 + 2i)^2$

$\therefore$ the square roots of $5 + 12i$ are $\pm(3 + 2i)$

***Example 7***

Find the square roots of the complex number $\dfrac{(8, -15)}{(0, 1)}$

**Solution :**

$$\frac{(8, -15)}{(0, 1)} = \frac{8 - 15i}{0 + i} = \frac{8 - 15i}{i} = -15 - 8i$$

Let $x + iy$ be the square roots of $-15 - 8i$. Then,

$(x + iy)^2 = -15 - 8i$

or, $(x^2 - y^2) + i.2xy = -15 - 8i$

Equating real and imaginary parts,

$x^2 - y^2 = -15$ ...... (i)

$2xy = -8(-ve)$ ...... (ii)

Again, $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$

$= (-15)^2 + (-8)^2$

$= (17)^2$

$\therefore$ $x^2 + y^2 = 17$ ...... (iii)

Adding (i) and (iii)

$2x^2 = 2$ $\therefore x = \pm 1$

Substituting the value of $x$ in (i)

$y^2 = 16$ $\therefore y = \pm 4$

Since $xy$ is negative (from (ii)), so the possible values of $x$ and $y$ are as follows:

$x = 1, y = -4$ and $x = -1, y = 4$

$\therefore$ the required square roots of $\frac{(8, -15)}{(0, 1)}$ i.e. of $-15 - 8i$ are $\pm(1 - 4i)$

## EXERCISE 11.2

1. Express each of the following complex numbers in the form of $a + ib$
   a) $(2 + 5i) + (1 - i)$ b) $(2 + 5i) - (4 - i)$
   c) $(2 + 3i)(3 - 2i)$ d) $\frac{3 + 4i}{4 - 3i}$
   e) $\frac{i}{2 + i}$ f) $\frac{1 - i}{(1 + i)^2}$
   g) $\frac{2 - \sqrt{-25}}{1 - \sqrt{-16}}$ h) $\sqrt{\frac{1 + i}{1 - i}}$
2. If $z = 2 + 3i$ and $w = 3 - 2i$, find $\overline{z}^2 + \overline{w}^2$.
3. Compute the absolute values of the following
   a) $1 + 2i$ b) $1 + \sqrt{3}i$ c) $(1 + 2i)(2 + i)$
   d) $(3 + 4i)(3 - 4i)$ e) $(1 + i)^{-1}$ f) $\frac{1 + i}{1 - i}$

4. If $z = 1 + 2i$ and $w = 2 - i$, verify that

a) $\overline{zw} = \overline{z}\ \overline{w}$ b) $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$

c) $|zw| = |z||w|$ d) $|z + w| \le |z| + |w|$

5. Prove that $\frac{\overline{z}}{|z|^2}$ is the multiplicative inverse of $z$.

6. a) If $(3 - 4i)(x + iy) = 3\sqrt{5}$, show that $5x^2 + 5y^2 = 9$.

b) If $x + iy = \frac{a - ib}{a + ib}$, show that $x^2 + y^2 = 1$.

c) If $\frac{1 - ix}{1 + ix} = a - ib$, prove that $a^2 + b^2 = 1$.

d) If $x - iy = \sqrt{\frac{1 - i}{1 + i}}$, prove that $x^2 + y^2 = 1$.

7. If $z$ and $w$ are two complex numbers, prove that

$$|z + w|^2 = |z|^2 + |w|^2 + 2\text{Re}(z\overline{w})$$

8. Find the multiplicative inverse of the following complex numbers

a) $(3 + i)^2$ b) $\frac{2 - 5i}{6 + i}$

9. If $z$ and $w$ are two complex numbers, prove that

$|z - w| \ge |z| - |w|$

10. Determine the square roots of the following complex numbers

a) $5 + 12i$ (T.U. 2051) H b) $-5 + 12i$

c) $8 + 6i$ d) $-8 + 6i$

e) $7 - 24i$ f) $-7 + 24i$

g) $i$ h) $12 - 5i$

i) $\frac{2 - 36i}{2 + 3i}$ j) $\frac{(5, 12)}{(3, -4)}$

## Answers

1. a) $3 + 4i$ b) $-2 + 6i$ c) $12 + 5i$ d) $0 + i$

e) $\frac{1}{5} + \frac{2}{5}i$ f) $-\frac{1}{2} - \frac{1}{2}i$ g) $\frac{22}{17} + \frac{3}{17}i$ h) $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

2. 0

3. a) $\sqrt{5}$ b) 2 c) 5 d) 25 e) $\frac{1}{\sqrt{2}}$ f) 1

8. a) $\frac{2}{25} - \frac{3}{50}i$ b) $\frac{7}{29} + \frac{32}{29}i$

10. a) $\pm(3 + 2i)$ b) $\pm(2 + 3i)$ c) $\pm(3 + i)$ d) $\pm(1 + 3i)$ e) $\pm(4 - 3i)$ f) $\pm(3 + 4i)$ g) $\pm\frac{1}{\sqrt{2}}(1 + i)$ h) $\pm\frac{1}{\sqrt{2}}(5 - i)$ i) $\pm(1 - 3i)$ j) $\pm\frac{1}{5}(4 + 7i)$

## 11.7 The Cube Roots of Unity

An important type of complex number arises when we consider the cube roots of unity. We shall find a number $z$ such that its cube is unity, i.e. to find $z$ such that $z^3 = 1$.

We have $z^3 = 1$

or, $z^3 - 1 = 0$

or, $(z - 1)(z^2 + z + 1) = 0$

Either $z - 1 = 0$

or, $z^2 + z + 1 = 0$

$\therefore$ $z = 1$ or $\frac{-1 \pm \sqrt{1 - 4}}{2}$

i.e. $z = 1$ or $\frac{-1 \pm \sqrt{-3}}{2}$

i.e. $z = 1$, or $\frac{-1 \pm \sqrt{3}i}{2}$

So the three cube roots of unity are 1, $\frac{-1 + \sqrt{3}i}{2}$ and $\frac{-1 - \sqrt{3}i}{2}$.

The first one is a real number and the other two are imaginary or complex numbers, and these are often known as the **imaginary cube roots** of unity, any one of which is denoted by the greek letter $\omega$ (omega).

### Properties of the Cube Roots of Unity

i) Each imaginary cube root of unity is the square of the other.

For, $\left(\frac{-1 + \sqrt{3}i}{2}\right)^2 = \frac{1 - 2\sqrt{3}i + 3i^2}{4}$

$$= \frac{1 - 2\sqrt{3}i - 3}{4}$$

$$= \frac{-1 - \sqrt{3}i}{2}$$

and $$\left(\frac{-1 - \sqrt{3}i}{2}\right)^2 = \frac{1 + 2\sqrt{3}i + 3i^2}{4}$$

$$= \frac{-2 + 2\sqrt{3}i}{4}$$

$$= \frac{-1 + \sqrt{3}i}{2}$$

Thus if we write $\omega$ for any one of the imaginary cube roots, the other will be $\omega^2$. Hence the three cube roots of unity are $1, \omega, \omega^2$.

ii) The product of the two imaginary cube roots of unity is equal to 1.

For $\omega \cdot \omega^2 = \frac{-1 + \sqrt{3}i}{2} \times \frac{-1 - \sqrt{3}i}{2}$

$$= \frac{1 - 3i^2}{4} = \frac{4}{4} = 1$$

As a direct consequence, we have

$\omega^3 = \omega \cdot \omega^2 = 1, \omega^{3n} = 1$ for any integral value of $n$.

Also $\omega^2 = \frac{1}{\omega}$ and $\omega = \frac{1}{\omega^2}$

Thus, one imaginary cube root of unity is the reciprocal of the other.

iii) The sum of the three cube roots of unity is zero.

For $1 + \omega + \omega^2 = 1 + \frac{-1 + \sqrt{3}i}{2} + \frac{-1 - \sqrt{3}i}{2}$

$$= \frac{2 - 1 + \sqrt{3}i - 1 - \sqrt{3}i}{2} = 0$$ $1 + \omega^2 + \omega = 0$

Thus, we have two important relations

$1 + \omega + \omega^2 = 0$ and $\omega^3 = 1$

It may be noted here that any integral power of $\omega$ will reduce to $1, \omega$ or $\omega^2$.

For examples,

$\omega^4 = \omega^3 \cdot \omega = \omega$, $\omega^5 = \omega^3 \cdot \omega^2 = \omega^2$

$\omega^6 = 1$, $\omega^{20} = \omega^{18} \cdot \omega^2 = (\omega^3)^6 \omega^2 = \omega^2$

$\omega^{-1} = \frac{1}{\omega} = \frac{\omega^3}{\omega} = \omega^2$

$\omega^{-10} = \frac{1}{\omega^{10}} = \frac{1}{\omega^9 \cdot \omega} = \frac{1}{\omega} = \omega^2$, etc.

GK

## Worked Out Example

***Example 1***

Show that $(1-\omega+\omega^2)^4+(1+\omega-\omega^2)^4=-16$

***Solution :***

Since $1+\omega+\omega^2=0$,

we have $1+\omega=-\omega^2$ and $1+\omega^2=-\omega$

$\therefore$ L.H.S. $=(1-\omega+\omega^2)^4+(1+\omega-\omega^2)^4$

$=(-2\omega)^4+(-2\omega^2)^4$

$=16\omega^4+16\omega^8$

$=16(\omega^4+\omega^8)$

$=16(\omega+\omega^2)$

$=16(-1)=-16$

***Example 2***

Prove that : $\dfrac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}=\omega$

***Solution :***

$\dfrac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}$

$=\dfrac{a\omega^3+b\omega+c\omega^2}{b+c\omega+a\omega^2}$ $(\because \omega^3=1)$

$=\dfrac{\omega(a\omega^2+b+c\omega)}{b+c\omega+a\omega^2}$

$=\omega$

## EXERCISE 11.3

1. If $\omega$ be a complex cube root of unity, show that
   a) $(1+\omega-\omega^2)^3-(1-\omega+\omega^2)^3=0$
   b) $(2+\omega+\omega^2)^3+(1+\omega-\omega^2)^8-(1-3\omega+\omega^2)^4=1$
   c) $(1-\omega+\omega^2)^4\,(1+\omega-\omega^2)^4=256$
   d) $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8)=9.$

e) $\dfrac{a + b\omega + c\omega^2}{a\omega + b\omega^2 + c} + \dfrac{a + b\omega + c\omega^2}{a\omega^2 + b + c\omega} = -1$

2. a) If $\alpha = \frac{1}{2}(-1 + \sqrt{-3})$ and $\beta = \frac{1}{2}(-1 - \sqrt{-3})$, show that

$$\alpha^4 + \alpha^2\beta^2 + \beta^4 = 0$$

b) If $\alpha$ and $\beta$ are the complex cube roots of unity, prove that

$$\alpha^4 + \beta^4 + \alpha^{-1}\beta^{-1} = 0$$

3. Prove that

a) $\left(\dfrac{-1 + \sqrt{-3}}{2}\right)^6 + \left(\dfrac{-1 - \sqrt{-3}}{2}\right)^6 = 2$

b) $\left(\dfrac{-1 + \sqrt{-3}}{2}\right)^4 + \left(\dfrac{-1 - \sqrt{-3}}{2}\right)^4 = -1$

4. If $x = a + b$, $y = a\omega + b\omega^2$, $z = a\omega^2 + b\omega$ show that

i) $x + y + z = 0$

ii) $xyz = a^3 + b^3$

iii) $x^3 + y^3 + z^3 = 3(a^3 + b^3)$

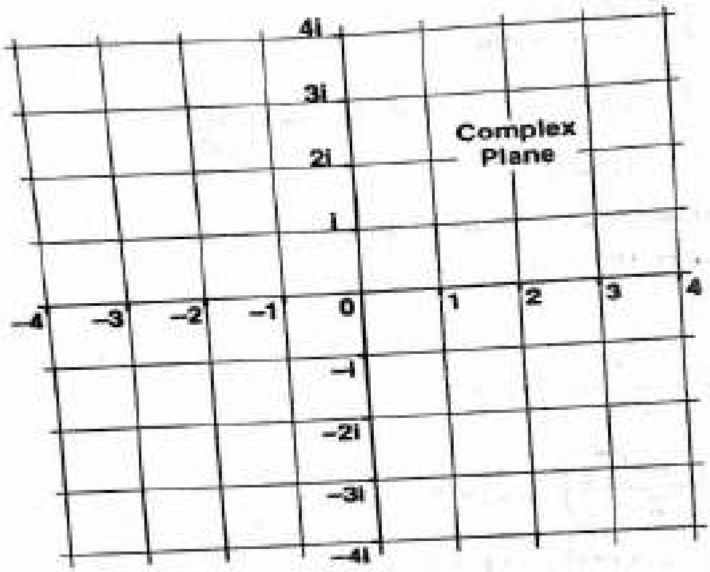
## 11.8 Geometrical Representation of Complex Numbers

A complex number, denoted by $z$, is defined as an ordered pair $(x, y)$ of real numbers. Since every ordered pair of real numbers can be represented by a point in the Cartesian plane, the complex number $z = (x, y)$ can be identified with a point P in the Cartesian plane with coordinates $(x, y)$; and conversely every point in the plane represents a complex number.

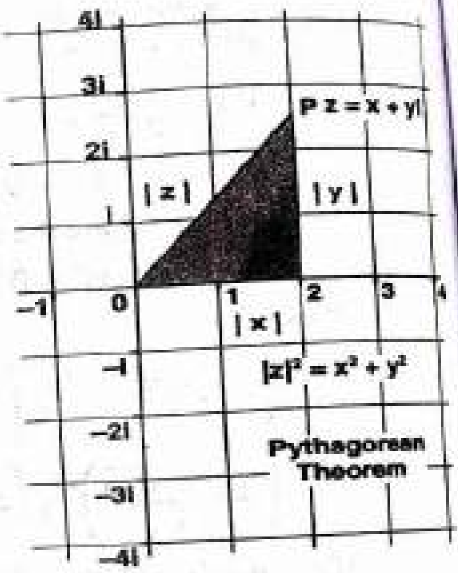
### a) The Complex Plane

When complex numbers are represented by points in the Cartesian plane, the plane is called the **complex plane**; and the set of such points form an **Argand diagram** Figure (a). The idea of expressing complex numbers geometrically was formulated by Argand (French) and Gauss (German). The name complex number is due to Gauss.

In figure (b), we have $z = (x, y) = x + iy$ and $|z| = OP = \sqrt{x^2 + y^2}$ is the modulus of the complex number $z$. In other words, the modulus of a complex number is the distance of the point representing the complex number from the origin in the complex plane.



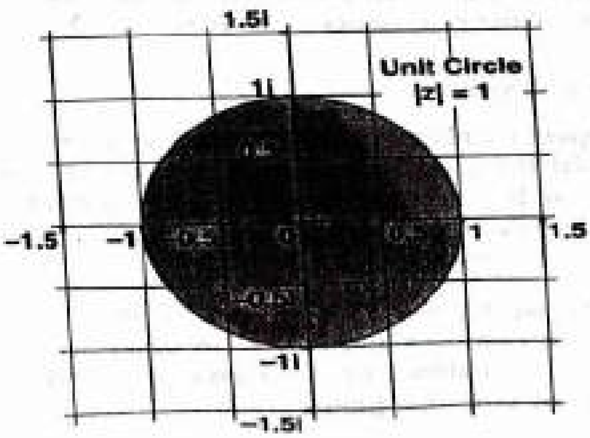

(a)




(b)

## b) The Unit Circle

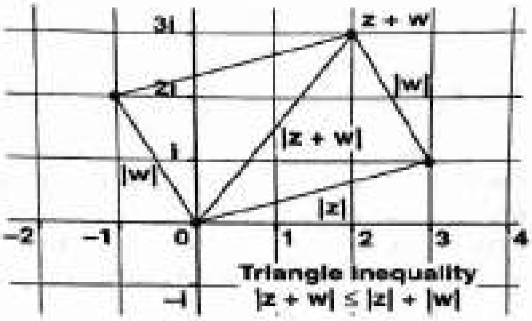
Of particular importance is the **unit circle**. This is the set of all points in the complex numbers with absolute value 1. Of course, 1 is the absolute value of both 1 and –1, but it's also the absolute value of both *i* and –*i* since they're both one unit away from 0 on the imaginary axis. The *unit circle* is the circle of radius 1 centered at 0. It includes all complex numbers of absolute value 1, so it has the equation $|z| = 1$.



### c) The Triangle Inequality

An important relation relating complex numbers with addition of absolute values is the **triangle inequality**. If $z$ and $w$ are any two complex numbers, then

$$|z + w| \leq |z| + |w|$$

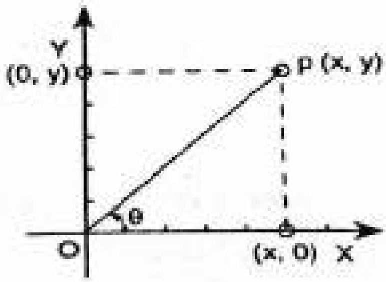


Triangle Inequality
$|z + w| \leq |z| + |w|$

One can see this from the parallelogram rule for addition. Consider the triangle whose vertices are 0, $z$ and $z + w$. One side of the triangle, the one from 0 to $z + w$ has length $|z + w|$. A second side of the triangle, the one from 0 to $z$, has length $|z|$. And the third side of the triangle, the one from $z$ to $z + w$, is parallel and equal to the line from 0 to $w$, and therefore has length $|w|$. Now, in any triangle, any one side is less than or equal to the sum of the other two sides, and, therefore, we have the triangle inequality displayed aside.

## 11.9 Polar Form of a Complex Number

Let $z = (x, y) = x + iy$ be a complex number. It can be represented by a point P in the complex plane with cartesian coordinates $(x, y)$. Let $\theta$ be the angle in the standard position with OP as its terminal arm, and $r$ the length of the line segment OP.




So, we have

$$x = r\cos\theta,$$
$$y = r\sin\theta$$

Thus, the complex number, $z = x + iy$, may be written in the following trigonometric form (polar form)

$$z = r(\cos\theta + i\sin\theta)$$

where $r = \sqrt{x^2 + y^2}$ and $\tan\theta = \frac{y}{x}$, $x \neq 0$.

Actually $r = |z|$ is the modulus of $z$ and the angle $\theta$ is called the **amplitude** or the **argument of** $z$ and is written as amp $(z)$ or arg $(z)$.

Since $\sin\theta$ and $\cos\theta$ are both periodic with a period $2\pi$ or $360°$, the complex number

$$z = x + iy$$
$$= r(\cos\theta + i\sin\theta)$$

may be written in the general form as

$$z = r[\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi)], \quad \text{if } \theta \text{ is in radians,}$$
$$= r[\cos(\theta + n\,360°) + i\sin(\theta + n\,360°)],$$
if $\theta$ is in degrees, $n$ is an integer

## 11.10 Products and Quotients in Polar Form

One of the important uses of the polar (or trigonometric) form of complex numbers is in the computation of products of complex numbers. It provides us a quick and efficient method for calculating the product and hence that of the quotient. Other important uses include the computation of powers and roots of complex numbers.

### Theorem

The product and quotient of two complex numbers

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

are given by

$$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

and
$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

***Proof:***

$$z_1z_2 = r_1(\cos\theta_1 + i\sin\theta_1).\ r_2(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1r_2\{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\}$$
$$= r_1r_2\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}, \text{ thus proving the first part.}$$

To prove the second part, we have

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$
$$= \frac{r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}$$
$$= \frac{r_1(\cos\theta_1 + i\sin\theta_1)[\cos(-\theta_2) + i\sin(-\theta_2)]}{r_2(\cos^2\theta_2 + \sin^2\theta_2)}$$
$$= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)].$$

If we now introduce a third factor in the product considered, we have

$$r_1(\cos\theta_1 + i\sin\theta_1).\ r_2(\cos\theta_2 + i\sin\theta_2).\ r_3(\cos\theta_3 + i\sin\theta_3)$$
$$= r_1r_2r_3[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)].(\cos\theta_3 + i\sin\theta_3)$$
$$= r_1r_2r_3[\cos(\theta_1 + \theta_2 + \theta_3) + i\sin(\theta_1 + \theta_2 + \theta_3)],$$

and we obtain, in a similar manner, the product of four or more complex numbers. In case of $n$ such numbers, we obtain the formula:

$$r_1(\cos\theta_1 + i\sin\theta_1).\ r_2(\cos\theta_2 + i\sin\theta_2).....\ r_n(\cos\theta_n + i\sin\theta_n)$$
$$= r_1r_2r_3....\ r_n\ [\cos(\theta_1 + \theta_2 + ....+ \theta_n) + i\sin(\theta_1 + \theta_2 + ... + \theta_n)].$$

This formula for the product of $n$ complex numbers gives us the value of the $n^{th}$ power of a complex number which can be stated as a theorem known as **De Moivre's Theorem.**

From the above results, we have the following relations.

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

give $\quad |z_1| = r_1, \quad |z_2| = r_2$

and $\quad \text{amp}(z_1) = \theta_1, \quad \text{amp}(z_2) = \theta_2$

Now, $z_1z_2 = r_1r_2\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$

gives $\quad |z_1z_2| = r_1r_2 = |z_1||z_2|$

and $\quad \text{amp}(z_1z_2) = \theta_1 + \theta_2 = \text{amp}(z_1) + \text{amp}(z_2)$

Again, $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\}$

gives, $\quad \left|\dfrac{z_1}{z_2}\right| = \dfrac{r_1}{r_2} = \dfrac{|z_1|}{|z_2|}$

and $\quad \text{amp}\left(\dfrac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \text{amp}(z_1) - \text{amp}(z_2)$

The product and the quotient of two complex numbers can be represented as follows:

If the complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ with magnitudes $r_1$, $r_2$ and the amplitudes $\theta_1$, $\theta_2$ be represented in the complex plane by the points $P(r_1, \theta_1)$ and $Q(r_2, \theta_2)$ then the complex number $z_1z_2$ with magnitude $r_1r_2$ and the amplitude $\theta_1 + \theta_2$ will be represented by the point R such that $\angle XOR = \theta_1 + \theta_2$ and $OR = r_1r_2$.

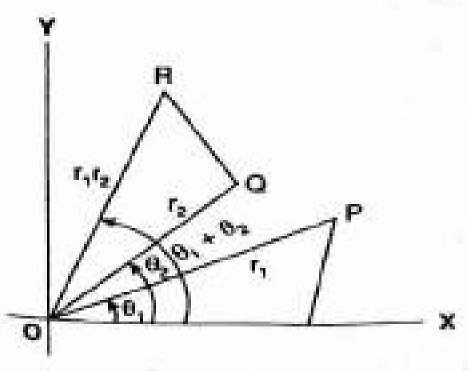



Fig. (i)

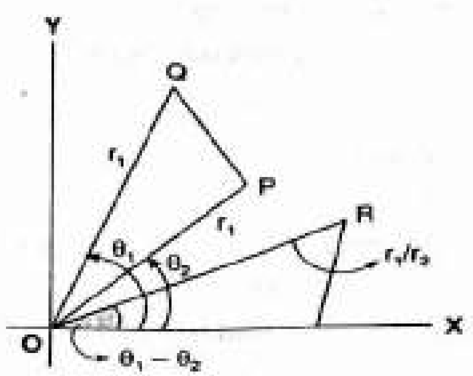



Fig. (ii)

In the same way, the complex number $\frac{z_1}{z_2}$ with magnitude $\frac{r_1}{r_2}$ and amplitude $\theta_1 - \theta_2$ will be represented by the point R such that

$$\angle XOR = \theta_1 - \theta_2 \text{ and } OR = \frac{r_1}{r_2}.$$

## 11.11 Integral Powers and Roots of Complex Numbers

The product of a complex number $z$ by itself, i.e., $z.\ z$ is denoted by $z^2$ and is called the square or second power of $z$. The cube or the third power of $z$, denoted by $z^3$, is defined by $z^3 = z^2.\ z$. In general, for any positive integer $n$, the $n$th power of a complex number $z$ is defined by

$$z^n = z^{n-1}.\ z \qquad \text{and} \qquad z^0 = 1.$$

Obviously, the $n$th power of a complex number is also a complex number. Let us denote it by $w$, so that

$$z^n = w.$$

Conversely, if $w \neq 0$, and if $n$ is a positive integer, then any complex number $z$ whose $n^{th}$ power is $w$, is known as the $n^{th}$ root of $w$. In other words, any complex number $z$ such that

$$z^n = w, \quad w \neq 0, \quad n = 1, 2, 3, \ldots\ldots$$

is known as the $n^{th}$ root of $w$. The $n^{th}$ root of $w$ is usually denoted by

$$w^{1/n} \text{ or } \sqrt[n]{w}.$$

The computation of the $n^{th}$ power and the $n^{th}$ root of a complex number may be carried out with comparative case with the help of a theorem known as De Moivre's theorem, which uses the polar (or trigonometric) form of a complex number.

The method of finding the $n^{th}$ roots of a complex number will be given after De-Moivre's Theorem.

### De Moivre's Theorem

If $n$ is any positive integer,

$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$.

***Proof.***

Obviously, when $n = 1$,

$[r(\cos\theta + i\sin\theta)]^1 = r(\cos\theta + i\sin\theta)$.

For $n = 2$, we have

$[r(\cos\theta + i\sin\theta)]^2$

$$= r^2(\cos^2\theta + 2i\cos\theta\sin\theta + i^2\sin^2\theta)$$

$= r^2 (\cos^2 \theta - \sin^2 \theta + i\, 2 \sin \theta \cos \theta)$
$= r^2 (\cos 2\theta + i \sin 2\theta)$.

Thus the theorem is true for $n = 1$ and $n = 2$. We prove the theorem by induction.

Let us assume that the theorem is true for some positive integer $k$.

By assumption,

$[r (\cos \theta + i \sin \theta)]^k = r^k (\cos k\theta + i \sin k\theta)$.

Multiplying both sides by $r (\cos \theta + i \sin \theta)$, we get

$[r (\cos \theta + i \sin \theta)]^{k+1}$

$$= r^{k+1} (\cos k\theta + i \sin k\theta)(\cos \theta + i \sin\theta)$$
$$= r^{k+1} [\cos (k\theta + \theta) + i \sin ( k\theta + \theta)]$$
$$= r^{k+1} [\cos (k + 1) \theta + \sin (k + 1) \theta],$$

which shows that the theorem is true for $n = k + 1$ whenever it is true for $n = k$. But we know that it is true for $n = 1$ and $n = 2$. When it is true for $n = 2$, the above proof shows that it is true for $n = 3$. Continuing this way, we come to the conclusion that the theorem is true for every positive integer $n$. This completes the proof.

In fact, the theorem is true not only for a positive integer but also for any real number $n$. We shall assume this without proof.

Let us now apply this theorem to compute the integral powers of complex numbers.

## $n^{th}$ root of a given complex number

Let $z = r (\cos \theta + i \sin \theta)$ be the given complex number whose $n^{th}$ root is required.

Let $w = R(\cos \phi + i \sin \phi)$ be the $n^{th}$ root of the complex number $z$.

Then, $\quad w^n = z$

$\Rightarrow \quad \{R(\cos \phi + i \sin \phi)\}^n = r (\cos \theta + i \sin \theta)$

$\Rightarrow \quad R^n(\cos n\phi + i \sin n\phi) = r (\cos \theta + i\sin \theta)$

Since the two complex numbers are equal, so

$$R^n = r \Rightarrow R = r^{1/n} = \sqrt[n]{r}$$

and $\quad \cos n\phi + i \sin n\phi = \cos \theta + i \sin \theta$

$\Rightarrow \quad \cos n\phi = \cos \theta$ and $\sin n\phi = \sin \theta$

$\Rightarrow \quad n\phi = 2k.\pi + \theta \quad$ or, $\quad k.360° + \theta$

$$\therefore \quad \phi = \frac{k.360° + \theta}{n}$$

$$\therefore \quad w = R(\cos \phi + i \sin \phi)$$

$$= \sqrt[n]{r}\left\{\cos \frac{k.360^\circ + \theta}{n} + i \sin \frac{k.360^\circ + \theta}{n}\right\}$$

For k = 0, 1, 2, 3, ......, $n - 1$, we get $n$ different roots of $z$. For other integral value of k will give the root obtained earlier i.e. roots will be repeated.

Thus the k$^{th}$ root of $z$ denoted by $z_k$ is given by

$$z_k = \sqrt[k]{r}\left\{\cos \frac{k.360^\circ + \theta}{n} + i \sin \frac{k.360^\circ + \theta}{n}\right\}$$

$$k = 0, 1, 1, 2, 3, \ldots\ldots, n-1$$

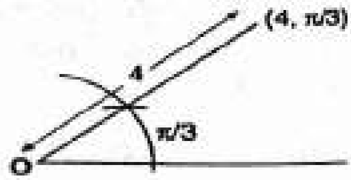
## Worked Out Examples

***Example 1.***

Express $2 + 2\sqrt{3}\, i$ in the polar form. Represent the complex number in the complex plane.

***Solution.***

Let $2 + 2\sqrt{3}\, i = r(\cos \theta + i \sin \theta)$.
Then by the definition of the equality of two complex numbers




$$r \cos \theta = 2$$

and $\quad r \sin \theta = 2\sqrt{3}$.

This gives $r^2 = 4 + 12 = 16$.

or $\quad r = 4 \qquad$ (positive value only).

Also, $\cos \theta = \dfrac{1}{2}$

and $\sin \theta = \dfrac{\sqrt{3}}{2}$,

giving $\theta = 60^\circ$

Hence $2 + 2\sqrt{3}i = 4(\cos 60^\circ + i \sin 60^\circ)$.

***Example 2***

Express $6(\cos 30^\circ + i \sin 30^\circ)$ in the form $x + iy$.

***Solution***

Let $6(\cos 30^\circ + i \sin 30^\circ) = x + iy$.

Equating real and imaginary parts, we have

$x = 6\cos 30°$ and $y = 6\sin 30°$

$= 6 . \frac{\sqrt{3}}{2}$ $\qquad = 6 . \frac{1}{2}$

$= 3\sqrt{3}$ $\qquad = 3.$

Hence $6(\cos 30° + i\sin 30°) = 3\sqrt{3} + 3i.$

***Example 3***

Find the value of $\frac{2(\cos 70° + i\sin 70°)}{\cos 10° + i\sin 10°}$

***Solution :***

$$\frac{2(\cos 70° + i\sin 70°)}{\cos 10° + i\sin 10°}$$

$$= 2\{\cos(70° - 10°) + i\sin(70° - 10)\}$$

$$= 2\{\cos 60° + i\sin 60°\}$$

$$= 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 1 + i\sqrt{3}$$

***Example 4***

Prove that $\frac{\cos 8\theta + i\sin 8\theta}{(\cos\theta + i\sin\theta)^6} = \cos 2\theta + i\sin 2\theta$

***Solution :***

$$\frac{\cos 8\theta + i\sin 8\theta}{(\cos\theta + i\sin\theta)^6}$$

$$= \frac{(\cos\theta + i\sin\theta)^8}{(\cos\theta + i\sin\theta)^6} \quad \text{(using De-Moivre's theorem)}$$

$$= (\cos\theta + i\sin\theta)^2$$

$$= \cos 2\theta + i\sin 2\theta$$

***Example 5***

Using De-Moivre's theorem, evaluate $(1 - \sqrt{3}i)^6$.

***Solution :***

Let $1 - \sqrt{3}i = r(\cos\theta + i\sin\theta)$

Equating real and imaginary parts,

$$r\cos\theta\ (x) = 1, \quad r\sin\theta\ (y) = -\sqrt{3}$$

$$r = \sqrt{(1)^2 + (-\sqrt{3})^2} = 2$$

$$\tan\theta = \frac{-\sqrt{3}}{1} = -\sqrt{3}$$

$\therefore \quad \theta = 300°$

$\therefore \quad 1 - \sqrt{3}i = 2(\cos 300° + i\sin 300°)$

Now, $(1 - \sqrt{3}i)^6 = \{2\cos(300° + i\sin 300°)\}^6$

$= 2^6\{\cos 1800° + i\sin 1800°\}$

$= 2^6\{\cos 0° + i\sin 0°\} = 2^6 = 64$

***Example 6***

Show that $z^3 = 1$, if $z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$.

***Solution.***

Let $z = x + iy = r(\cos\theta + i\sin\theta) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$.

Here, $x = -\frac{1}{2}$ and $y = \frac{\sqrt{3}}{2}$.

Then, $r = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$.

$\tan\theta = \dfrac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}$, or $\theta = 120°$ $\left[\begin{array}{l}\text{in the second quadrant}\\ \text{as } x \text{ is negative and}\\ y \text{ is positive}\end{array}\right]$

Thus, $\quad z = \cos 120° + i\sin 120°$.

Again, $z^3 = (\cos 120° + i\sin 120°)^3$

$= \cos 3(120°) + i\sin 3(120°)$

$= \cos 360° + i\sin 360° = 1$.

Another important use of De Moivre's theorem is in the computation of the roots of complex numbers. We shall illustrate the method with some examples.

***Example 7***

Find the square roots of $2 + 2\sqrt{3}i$

***Solution :***

Let $z = 2 + 2\sqrt{3}i$

Here, $x = 2$ and $y = 2\sqrt{3}$

To write $z$ in polar form we note that

$$r = \sqrt{2^2 + (2\sqrt{3})^2} = 4$$

and $\tan\theta = \dfrac{2\sqrt{3}}{2} = \sqrt{3}$

or, $\theta = 60°$

$\therefore$ in polar form

$$z = 4(\cos 60° + i\sin 60°)$$

$= 4[\cos(60° + n\,360°) + i\sin(60° + n\,360°)]$ in general polar form

$\therefore \sqrt{z} = z^{1/2} = \sqrt{4}\,[\cos(60° + n\,360°) + i\sin(60° + n\,360°)]^{1/2}$

$= 2[\cos(30° + n\,180°) + i\sin(30° + n\,180°)]$ where $n = 0, 1$

(Using De Moivre's theorem)

When $n = 0$, $z^{1/2} = 2(\cos 30° + i\sin 30°) = \sqrt{3} + i$

When $n = 1$, $z^{1/2} = 2(\cos 210° + i\sin 210°) = -\sqrt{3} - i$

For $n = 2, 3, \ldots$ the values obtained above will repeat, as the angles differ by multiples of 360°.

Note: The square roots can be obtained directly using the formula given in Art. 11.11

***Example 8***

Find the cube roots of 1 using De Moivre's theorem.

***Solution :***

Let $z^3 = 1$ so that we need to find $z$.

Writing in polar form

$z^3 = 1 = 1 + 0i$

$= \cos 0 + i\sin 0 \quad [r = 1, \theta = 0]$

$= \cos 2n\pi + i\sin 2n\pi$, in general polar form.

$\therefore z = (\cos 2n\pi + i\sin 2n\pi)^{1/3}$

$= \cos\dfrac{2n\pi}{3} + i\sin\dfrac{2n\pi}{3}$, where $n = 0, 1, 2.$

When $n = 0$, $z = \cos 0 + i\sin 0 = 1$

When $n = 1$, $z = \cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3} = -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$

When $n = 2$, $z = \cos\dfrac{4\pi}{3} + i\sin\dfrac{4\pi}{3} = -\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i$

Hence $1, \dfrac{-1 + \sqrt{3}i}{2}, \dfrac{-1 - \sqrt{3}i}{2}$ are the three cube roots of 1.

[Remember that we have computed these cube roots earlier without using De-Moivre's Theorem]

Note: The cube roots can be obtained directly using the formula given in Art. 11.11

***Example 9***

If $z$ be a complex number, prove that

$$\left|\frac{1}{z}\right| = \frac{1}{|z|} \quad \text{and} \quad \text{amp}\left(\frac{1}{z}\right) = -\text{amp}(z)$$

***Solution :***

Let $z = r(\cos\theta + i\sin\theta)$

Then, $|z| = r$ and $\text{amp}(z) = \theta$

$$\frac{1}{z} = \frac{1}{r(\cos\theta + i\sin\theta)}$$

$$= \frac{1}{r(\cos\theta + i\sin\theta)} \times \frac{\cos\theta - i\sin\theta}{(\cos\theta - i\sin\theta)}$$

$$= \frac{\cos\theta - i\sin\theta}{r(\cos^2\theta + \sin^2\theta)}$$

$$= \frac{1}{r}\{\cos(-\theta) + i\sin(-\theta)\}$$

Now, $\left|\frac{1}{z}\right| = \frac{1}{r} = \frac{1}{|z|}$

Again, $\text{amp}\left(\frac{1}{z}\right) = -\theta = -\text{amp}(z)$

***Example 10***

If $z = \cos\theta + i\sin\theta$, prove that $z^n + \frac{1}{z^n} = 2\cos n\theta$

***Solution :***

$$z = \cos\theta + i\sin\theta$$

Then, $z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

and $\frac{1}{z^n} = z^{-n} = \cos n\theta - i\sin n\theta$

Now, $z^n + \frac{1}{z^n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$$= 2\cos n\theta$$

## EXERCISE 11.4

1. Express the following complex numbers in the polar form:
   (a) $2 + 2i$ (b) $\sqrt{3} + i$ (c) $2i$ (d) $i - \sqrt{3}$
   (e) $1 - i$ (f) $\dfrac{(1 + i)}{(1 - i)}$ (g) $\dfrac{i}{(1 + i)}$ (h) $\sqrt{\dfrac{1 - i}{1 + i}}$
2. Express the following in the form $x + iy$:
   (a) $3 (\cos 60° + i \sin 60°)$
   (b) $3 (\cos 120° + i \sin 120°)$
   (c) $2 (\cos 150° + i \sin 150°)$
   (d) $2 \cos (-45°) + i\, 2 \sin (-45°)$.
3. Simplify :
   a) $(\cos 32° + i \sin 32°) (\cos 13° + i \sin 13°)$
   b) $(\sin 40° + i \cos 40°) (\cos 40° + i \sin 40°)$
   c) $\dfrac{\cos 80° + i \sin 80°}{\cos 20° + i \sin 20°}$
   d) $\dfrac{(\cos 3\theta + i \sin 3\theta) (\cos \theta - i \sin \theta)}{(\cos \theta + i \sin \theta)^2}$
4. Apply De Moivre's Theorem to compute
   a) $[2(\cos 15° + i \sin 15°)]^6$
   b) $[3(\cos 120° + i \sin 120°)]^3$
   c) $(\cos 18° + i \sin 18°)^5$
   d) $[\cos 9° + i \sin 9°]^{40}$
   e) $(1 + i)^{20}$
   f) $(-1 + i)^{14}$
   g) $\left(\frac{1}{2} + i \frac{\sqrt{3}}{2}\right)^7$
   i) $i^2$
5. Using De–Moivre's Theorem, find the square roots of
   a) $4 + 4\sqrt{3}i$ b) $-1 + \sqrt{3}i$ c) $-2 - 2\sqrt{3}i$
   d) $2i$ e) $-i$
6. Determine the cube roots of $-1$.
7. Solve the following equations
   a) $z^4 = 1$ b) $z^6 = 1$ c) $z^4 + 1 = 0$ d) $z^3 = 8i$
8. Find the fourth roots of $-\frac{1}{2} + i \frac{\sqrt{3}}{2}$.

9. If $\overline{z}$ be the conjugate of the complex number z, prove that

$$\text{Arg}(\overline{z}) = 2\pi - \text{Arg}(z).$$

10. If $z = \cos\theta + i\sin\theta$, prove that $z^n - \frac{1}{z^n} = 2\sin n\theta i$.

### *Answers*

1. (a) $2\sqrt{2}(\cos 45° + i\sin 45°)$ (b) $2(\cos 30° + i\sin 30°)$
(c) $2(\cos 90° + i\sin 90°)$ (d) $2(\cos 150° + i\sin 150°)$
(e) $\sqrt{2}(\cos 315° + i\sin 315°)$ (f) $(\cos 90° + i\sin 90°)$
(g) $\frac{1}{\sqrt{2}}(\cos 45° + i\sin 45°)$ (h) $\cos 315° + i\sin 315°$

2. (a) $\frac{3}{2} + i\frac{3\sqrt{3}}{2}$ (b) $-\frac{3}{2} + i\frac{3\sqrt{3}}{2}$
(c) $-\sqrt{3} + i$ (d) $\sqrt{2} - i\sqrt{2}$

3. a) $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ b) $i$ c) $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ d) 1

4. a) $64i$ b) 27 c) $i$ d) 1 e) $-2^{10}$ f) $2^7 i$ g) $\frac{1}{2} + i\frac{\sqrt{3}}{2}$
h) $-1$

5. a) $\pm(\sqrt{6} + i\sqrt{2})$ b) $\pm\frac{1}{\sqrt{2}}(1 + i\sqrt{3})$ c) $\pm(-1 + i\sqrt{3})$
d) $\pm(1 + i)$ e) $\pm\frac{1-i}{\sqrt{2}}$

6. $-1, \frac{1}{2}(1 + i\sqrt{3}), \frac{1}{2}(1 - i\sqrt{3})$

7. a) $\pm 1, \pm i$ b) $\pm 1, \frac{1}{2}(1 \pm i\sqrt{3}), \frac{1}{2}(-1 \pm i\sqrt{3})$
c) $\pm\frac{1+i}{\sqrt{2}}, \pm\frac{1-i}{\sqrt{2}}$ d) $\sqrt{3} + i, -\sqrt{3} + i, -2i$

8. $\pm\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right), \pm\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

## ADDITIONAL QUESTIONS

1. Define a complex number. If $z = \frac{2+i}{3-i}$, find the real and the imaginary parts of (i) $\frac{1+z}{1-z}$ (ii) $z + \frac{1}{z}$.

2. If $z_1 = 3 - 2i$ and $z_2 = 4 + 3i$, find $z_1^{-1}$, $z_2^{-1}$, $\frac{z_1}{z_2}$ and $\frac{z_2}{z_1}$.

3. Find the multiplicative inverse of :

   (i) $(2+i)^2$ (ii) $\frac{3+4i}{4-5i}$

4. If $a + ib = \sqrt{\frac{1+i}{1-i}}$, prove that $a^2 + b^2 = 1$.

5. If $\sqrt{x+iy} = a + ib$, prove that $\sqrt{x - iy} = a - ib$.

6. If $x + iy = \frac{u+iv}{u-iv}$, prove that $x^2 + y^2 = 1$.

   *Solution:*

$$x + iy = \frac{u+iv}{u-iv}$$

or, $$x + iy = \frac{u+iv}{u-iv} \times \frac{u+iv}{u+iv}$$

or, $$x + iy = \frac{(u+iv)^2}{u^2+v^2}$$

$$x + iy = \frac{u^2-v^2}{u^2+v^2} + i\frac{2uv}{u^2+v^2}$$

Equating real and imaginary parts,

$$x = \frac{u^2-v^2}{u^2+v^2} \quad \text{and} \quad y = \frac{2uv}{u^2+v^2}$$

Now, $$x^2 + y^2 = \left(\frac{u^2-v^2}{u^2+v^2}\right)^2 + \left(\frac{2uv}{u^2+v^2}\right)^2$$

$$= \frac{(u^2-v^2)^2 + 4u^2v^2}{(u^2+v^2)^2} = \frac{(u^2+v^2)^2}{(u^2+v^2)^2} = 1$$

7. State De' Moivre's Theorem. Use it to find the cube roots of unity. (HSEB 2056, 2058)

8. State De-Moivre's theorem and use it to find the cube roots of unity. Verify that the sum of the three cube roots of unity is zero. (T.U. 2052)

9. Simplify (Use De-Moivre's theorem)

i) $(\cos 40° + i \sin 40°)(\cos 50° + i \sin 50°)$

*Solution :*

$$\begin{aligned}(\cos 40° + i \sin 40°)(\cos 50° + i \sin 50°) &= \cos(40° + 50°) + i \sin(40° + 50°)\\ &= \cos 90° + i \sin 90°\\ &= 0 + i.1 = i\end{aligned}$$

ii) $(\cos 75° + i \sin 75°)(\cos 15° - i \sin 15°)$

iii) $\dfrac{(\cos\theta + i \sin\theta)^4}{(\cos\theta - i \sin\theta)^5}$

10. If $\omega$ be one of the complex cube roots of unity, prove that

(i) $\omega^3 = 1$, (ii) $1 + \omega + \omega^2 = 0$

11. Prove that each complex cube root of unity is the reciprocal of the other.

12. Express a complex number in a polar form. State De-Moivre's theorem. Using De-Moivre's theorem find the cube roots of unity. If $\omega = \frac{-1+\sqrt{3}i}{2}$ be a complex cube root of unity, prove that

$$\omega^2 = \frac{-1 - \sqrt{3}\,i}{2}$$

13. What do you mean by a complex number ? Express it in the polar form. Find the square root of

$$\frac{2 - 36i}{2 + 3i}$$ (T.U. 2053)

14. Prove that :

i) $\left(\dfrac{-1 + \sqrt{-3}}{2}\right)^6 + \left(\dfrac{-1 - \sqrt{-3}}{2}\right)^6 = 2$

ii) $\left(\dfrac{-1 + \sqrt{-3}}{2}\right)^5 + \left(\dfrac{-1 - \sqrt{-3}}{2}\right)^5 = -1$

15. Show that $\dfrac{a + ib}{c + id}$ will be real if $ad = bc$.

*Solution:*

$$\frac{a + ib}{c + id} = \frac{a + ib}{c + id} \times \frac{c - id}{c - id}$$

$$= \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2}$$

$$= \frac{ac + bd}{c^2 + d^2} + i\,\frac{bc - ad}{c^2 + d^2}$$

The given complex number wil be real if

$$\frac{bc - ad}{c^2 + d^2} = 0$$

i.e. $\quad ad = bc$

16. If $z$ and $w$ are two complex numbers, prove that

a) $|z - w| \le |z| + |w|$

b) $|z + w| \ge |z| - |w|$

17. Prove that

$$\left(\frac{-1 + \sqrt{-3}}{2}\right)^n + \left(\frac{-1 - \sqrt{-3}}{2}\right)^n = 2, \text{ if n is a multiple of 3}$$

$$= -1, \text{ if n is any other integer}$$

### Answers

1. i) 1, 2 $\quad$ ii) $\frac{3}{2}, -\frac{1}{2}$

2. $\frac{3}{13} + \frac{2}{13}i,\ \frac{4}{25} - \frac{3}{25}i,\ \frac{6}{25} - \frac{17}{25}i,\ \frac{6}{13} + \frac{17}{13}i$

3. i) $\frac{3}{25} - \frac{4}{25}i$ $\quad$ ii) $-\frac{8}{25} - \frac{31}{25}i$

9. ii) $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ $\quad$ iii) $\cos 9\theta + i \sin 9\theta$

13. i) $\frac{1 + i}{\sqrt{2}}, \frac{-1 - i}{\sqrt{2}}$ $\quad$ ii) $\frac{-1 + i}{\sqrt{2}}, \frac{1 - i}{\sqrt{2}}$

13. $\pm(3i - 1)$

# CHAPTER 12
# Polynomial Equations

## 12.1 Introduction

A function $f$ defined by

$$f(x) = a_0x^n + a_1x^{n-1} + \ldots\ldots + a_{n-1}x + a_n, \quad (a_0 \neq 0) \qquad \ldots(1)$$

where $n$ is non-negative integer, $a_0, a_1, \ldots\ldots\ldots, a_{n-1}, a_n$ are all constants is called a **rational integral function** or **polynomial of degree $n$ in $x$**. The constants $a_0, a_1, \ldots\ldots, a_n$ are respectively called the **coefficients** of $x^n, x^{n-1}, \ldots\ldots\ldots, x^0 = 1$, and each of $a_0x^n, a_1x^{n-1}, \ldots\ldots, a_n$ is called a **term** of the polynomial. The term $a_0x^n$ is called the **leading term** of the polynomial. Particular examples of polynomials written as functions are

$$g(x) = 2x - 4, \qquad G(x) = x^2 + 2x - 3,$$
$$h(x) = 5x^3 - 3x^2 + x - 1, \text{ etc.}$$

A polynomial is sometimes called a **quantic**. The quantics of various successive degrees have special names. For instances, the polynomials of degrees one, two, three and four are respectively called **linear (first degree), quadratic (or quadric), cubic** and **biquadratic (or quartic)**. Thus the polynomials defined by

$$g(x) = ax + b, \qquad G(x) = ax^2 + bx + c,$$
$$h(x) = ax^3 + bx^2 + cx + d \text{ and}$$
$$H(x) = ax^4 + bx^3 + cx^2 + dx + e$$

are linear, quadratic, cubic and biquadratic respectively.

If for a certain value $a$ of $x$, $f(a) = 0$, the value $x = a$ is called a **zero** of the polynomial defined by (1). For instance, the value $x = 2$ is a zero of the linear function (polynomial) defined by $g(x) = 2x - 4$, since

$$g(2) = 2.2 - 4 = 0.$$

Note that there is no other zero (i.e. no other value of $x$ which makes it zero). Similarly, it is easy to verify that $x = 1$ and $x = -3$ are the zeros of the quadratic defined by $G(x) = x^2 + 2x - 3$. Here the number of zeros is exactly two. Thus we notice that a polynomial may have one or more zeros depending upon the degree of the polynomial.

## 12.2 Polynomial Equations

A polynomial of degree $n$ in $x$ defined by

$$f(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n \qquad \dots(1)$$

where $n$ is a non-negative integer and $a_0, a_1, \dots, a_n$ are all constants may vanish for one or more values of $x$.

Suppose $f(x) = 0$ for some values of $x$, then

$$f(x) = 0 \qquad \dots(2)$$

is called a general equation of degree $n$ in $x$ and the values of $x$ for which (2) is true are known as the solutions of the equation. This equation is sometimes known as a polynomial equation of degree $n$ in $x$. Some special types of polynomial equations are :

(a) Linear equation: $ax + b = 0$

(b) Quadratic equation: $ax^2 + bx + c = 0$

(c) Cubic equation: $ax^3 + bx^2 + cx + d = 0$

and (d) Biquadratic equation: $ax^4 + bx^3 + cx^2 + dx + e = 0$

Regarding polynomial equations, we assume without proof, the following fundamental theorem of algebra :

'EVERY EQUATION HAS AT LEAST A ROOT'

The proof of this theorem is beyond the level of the present text. On the basis of this theorem and the 'Factor Theorem', we shall now prove a proposition.

### Theorem 1

Every equation of degree $n$ in $x$ has $n$ roots, and no more.

***Proof.***

Let the given equation be denoted by $f(x) = 0$,

where $\quad f(x) = a_0x^n + a_1x^{n-1} + \dots \dots + a_{n-1}x + a_n$.

From the 'Fundamental Theorem of Algebra', the equation $f(x) = 0$ has a root, real or imaginary. Let this be denoted by $\alpha_1$; then, by the 'Factor Theorem', $x - \alpha_1$ is a factor of $f(x)$, so that

$$f(x) = (x - \alpha_1)\,\phi_1(x),\ \phi_1(x) = a_0x^{n-1} + \dots\dots$$

where $\phi_1(x)$ is a polynomial of degree $n - 1$. Again, the equation $\phi_1(x) = 0$ has a root, real or imaginary; let it be denoted by $\alpha_2$ then $x - \alpha_2$ is factor of $\phi_1(x)$, so that

$$\phi_1(x) = (x - \alpha_2)\,\phi_2(x),\ \phi_2(x) = a_0x^{n-2} + \dots\dots$$

where $\phi_2(x)$ is polynomial of degree $n - 2$.

Thus $\quad f(x) = (x - \alpha_1)(x - \alpha_2)\,\phi_2(x)$.

Proceeding in this way, we obtain

$$f(x) = a_0(x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \ldots\ldots (x - \alpha_n)$$

Hence the equation $f(x) = 0$ has $n$ solutions (i.e. roots), since $f(x)$ vanishes when $x$ has any of the values

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots, \alpha_n.$$

Also the equation cannot have more than $n$ roots; for if $x$ has any value different from any of the quantities $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_n$, all the factors on the right are different from zero, and therefore $f(x)$ cannot vanish for that value of $x$.

This theorem may be used to deduce that

i) A biquadratic equation in $x$ has exactly four roots and no more.

ii) A cubic equation has three and only three roots.

iii) A quadratic equation has two and only two roots.

It may also be used in the investigation of the relations between the roots and the coefficients of any equation.

## 12.3 Quadratic Equation

Various properties of quadratic equation may be derived directly from those of the general equation of degree $n$. But, in many occasions, we do not need the general theory of equation of degree $n$. It is generally felt that a systematic study of the theory of quadratic equation throws sufficient lights on the general theory. We shall therefore consider the quadratic equation in a greater detail in the present and the following sections:

**Theorem 1.**

The two roots of a quadratic equation,

$$ax^2 + bx + c = 0 \qquad (a \neq 0),$$

are $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

***Proof.***

Let us divide both sides of

$$ax^2 + bx + c = 0$$

by the coefficient of $x^2$ and transpose the constant to the right side of the equation, so we have

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Adding $\dfrac{b^2}{4a^2}$ to both sides of the equation,

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

or $$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Taking square roots we get

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a},$$

or $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The use of the plus and the minus signs gives the two roots of the quadratic equation. This completes the proof.

**Theorem 2**

The quadratic equation $ax^2 + bx + c = 0$ cannot have more than two roots.

***Proof.***

For, if possible, let $\alpha$, $\beta$, $\gamma$ be three different roots of the quadratic equation

$$ax^2 + bx + c = 0 \qquad (a \neq 0).$$

Then, since each of these values must satisfy the equation, we have

$$a\alpha^2 + b\alpha + c = 0 \qquad \ldots\ldots (1)$$
$$a\beta^2 + b\beta + c = 0 \qquad \ldots\ldots (2)$$
$$a\gamma^2 + b\gamma + c = 0 \qquad \ldots\ldots (3)$$

From (1) and (2), by subtraction,

$$a(\alpha^2 - \beta^2) + b(\alpha - \beta) = 0$$

Since $\alpha \neq \beta$, divide out by $\alpha - \beta$; then

$$a(\alpha + \beta) + b = 0$$

Similarly, from (2) and (3)

$$a(\beta + \gamma) + b = 0$$

Hence by subtraction,

$$a(\alpha - \gamma) = 0;$$

which is impossible, since, by, hypothesis $a \neq 0$, and $\alpha$ is not equal to $\gamma$. Hence there cannot be more than two different roots.

*Cor. A quadratic equation has two and only two roots.*

Combining theorems 1 and 2, we get the required proof.

(Note : An argument similar to the one used in the case of the general equation of degree $n$ can also be used.)

## 12.4 Nature of the Roots of a Quadratic Equation

Let the two roots of the quadratic equation

$$ax^2 + bx + c = 0 \qquad (a \neq 0)$$

be denoted by $\alpha$ and $\beta$, so that

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and $$\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

(Remember that the constants $a$, $b$, $c$ are rational numbers).

The expression $b^2 - 4ac$, under the radical sign, occurs in both the roots. Its value depends on the coefficients $a$, $b$ and $c$. There are three possibilities:

I. If $b^2 - 4ac > 0$, (the quantity under the radical sign is positive), then the roots are real and unequal.

In particular, if $b^2 - 4ac$ is positive as well as a perfect square, the roots are rational and unequal, provided $a$, $b$, $c$ are rational.

II. If $b^2 - 4ac = 0$, then the roots are real and equal, each being $-\frac{b}{2a}$.

III. If $b^2 - 4ac < 0$, then the roots are imaginary and unequal.

Thus computing the value of $b^2 - 4ac$, it is possible to determine the nature of the quadratic equation without actually solving the equation. This quantity is therefore known as the **discriminant** of the quadratic equation.

Two conclusions that can be derived from the above discussion are as follows:

(a) **In a quadratic equation with rational coefficients, irrational roots always occur in conjugate pair.** If $b^2 - 4ac$ is positive but not a perfect square, then $\sqrt{b^2 - 4ac}$ will be irrational. Since $\sqrt{b^2 - 4ac}$ occurs in both roots $\alpha$ and $\beta$, so if we put

$$-\frac{b}{2a} = p \qquad \text{and} \qquad \frac{\sqrt{b^2 - 4ac}}{2a} = \sqrt{q}\text{, then}$$

$$\alpha = p + \sqrt{q} \quad \text{and} \quad \beta = p - \sqrt{q}$$

are the two roots, both of which are irrational, each being the conjugate of the other. Hence in a quadratic equation with rational coefficients, the irrational roots occur in pair.

(b) In a quadratic equation with real coefficients, imaginary (complex) roots always occur in pair. If $b^2 - 4ac$ is negative, then $\sqrt{b^2 - 4ac}$ will be imaginary. Since $\sqrt{b^2 - 4ac}$ occurs in both roots $\alpha$ and $\beta$, so if we put

$$-\frac{b}{2a} = p \qquad \text{and} \qquad \frac{\sqrt{b^2 - 4ac}}{2a} = iq$$

then $\qquad \alpha = p + iq \quad$ and $\quad \beta = p - iq$

are the two roots, both of which are imaginary (complex), each being the conjugate of the other. Hence in a quadratic equation with real coefficients, the imaginary (complex) roots occur in pair.

## Worked Out Examples

***Example 1***

Determine the nature of the roots of

$$2x^2 - 3x - 2 = 0.$$

***Solution.***

Since $a = 2$, $b = -3$ and $c = -2$,

$b^2 - 4ac = 25 > 0$, $\quad$ the roots are real, rational and unequal.

***Example 2***

Prove that the roots of $2x^2 - 6x + 7 = 0$ are imaginary.

***Solution.***

Here $a = 2$, $b = -6$ and $c = 7$, so that

$b^2 - 4ac = -20$. $\qquad$ Hence the result.

***Example 3***

If the equation $x^2 + (k + 2)x + 2k = 0$ has equal roots, find $k$.

***Solution.***

Here $a = 1$, $b = k + 2$ and $c = 2k$. The condition for equal roots gives

$$(k + 2)^2 - 4.1.2k = 0$$

or $\quad k^2 - 4k + 4 = 0$

$\therefore \qquad k = 2$

***Example 4***

If the equation $(1 + m^2)x^2 + 2mcx + c^2 - a^2 = 0$ has equal roots, show that $c^2 = a^2(1 + m^2)$

***Solution :***

Comparing the given equation with $Ax^2 + Bx + C = 0$

we have $A = 1 + m^2$, $B = 2mc$ and $C = c^2 - a^2$

$B^2 - 4AC = 4m^2c^2 - 4.(1 + m^2)(c^2 - a^2)$

$$= 4m^2c^2 - 4c^2 + 4a^2 - 4m^2c^2 + 4m^2a^2$$
$$= -4c^2 + 4a^2 + 4m^2a^2$$

For equal roots, $B^2 - 4AC = 0$

or, $-4c^2 + 4a^2 + 4m^2a^2 = 0$

$$c^2 = a^2(1 + m^2)$$

***Example 5***

If the roots of the equation $x^2 - 2cx + ab = 0$ be real and unequal, prove that the roots of $x^2 - 2(a + b)x + a^2 + b^2 + 2c^2 = 0$ will be imaginary.

***Solution :***

The given equations are

$$x^2 - 2cx + ab = 0 \quad \ldots\ldots \text{(i)}$$

and $$x^2 - 2(a + b)x + a^2 + b^2 + 2c^2 = 0 \quad \ldots\ldots \text{(ii)}$$

Discriminant of (i) $= (-2c)^2 - 4.1.ab$

$= 4c^2 - 4ab$

$= 4(c^2 - ab) > 0 \quad \ldots\ldots$ (iii)

as the roots are real and unequal.

Again, the discriminant of (ii)

$$= \{-2(a + b)\}^2 - 4.1.(a^2 + b^2 + 2c^2)$$
$$= 4(a + b)^2 - 4(a^2 + b^2 + 2c^2)$$
$$= 4\{a^2 + 2ab + b^2 - a^2 - b^2 - 2c^2\}$$
$$= -8(c^2 - ab) < 0 \quad \text{(from (iii))}$$

Hence the roots of (ii) are imaginary.

## EXERCISE 12.1

1. Determine the nature of the roots of each of the following equations:
   a) $x^2 - 6x + 5 = 0$
   b) $x^2 - 4x - 3 = 0$
   c) $x^2 - 6x + 9 = 0$
   d) $4x^2 - 4x + 1 = 0$
   e) $2x^2 - 9x + 35 = 0$
   f) $4x^2 + 8x - 5 = 0$
2. For what values of $p$ will the equation $5x^2 - px + 45 = 0$ have equal roots ? (HSEB 2056)
3. If the equation $x^2 + 2(k + 2)x + 9k = 0$ has equal roots, find $k$.
4. For what value of a will the equation $x^2 - (3a - 1)x + 2(a^2 - 1) = 0$ have equal roots ?

5. If the roots of the equation $(a^2 + b^2)x^2 - 2(ac + bd)x + (c^2 + d^2) = 0$ are equal, then $\frac{a}{b} = \frac{c}{d}$

6. Show that the roots of the equation
$$(a^2 - bc)x^2 + 2(b^2 - ca)x + (c^2 - ab) = 0$$
will be equal, if either $b = 0$, or $a^3 + b^3 + c^3 - 3abc = 0$.

7. If $a, b, c$ are rational and $a + b + c = 0$, show that the roots of $(b + c - a)x^2 + (c + a - b)x + (a + b - c) = 0$ are rational.

8. Prove that the roots of the equation $(x - a)(x - b) = k^2$ are real for all values of $k$.

9. Show that the roots of the equation $x^2 - 4abx + (a^2 + 2b^2)^2 = 0$ are imaginary.

10. If the roots of the quadratic equation $qx^2 + 2px + 2q = 0$ are real and unequal, prove that the roots of the equation $(p + q)x^2 + 2qx + (p - q) = 0$ are imaginary.

**Answers**

1. a) real, rational and unequal, b) real, irrational and unequal, c) real, rational and equal d) real, rational and equal. e) imaginary and unequal. f) real, rational and unequal

2. $p = \pm 30$ 3. $k = 1$ or 4 4. $a = 3$

## 12.5 Relations between Roots and Coefficients

Let $\alpha$ and $\beta$ be the roots of the quadratic equation $ax^2 + bx + c = 0$, then

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

We have, by addition

$$\alpha + \beta = \frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a}$$

$$= -\frac{2b}{2a} = -\frac{b}{a}$$

$$= -\frac{\text{coefficient of } x}{\text{coefficient of } x^2}$$

Again, by multiplication, we have

$$\alpha\beta = \frac{(-b + \sqrt{b^2 - 4ac})(-b - \sqrt{b^2 - 4ac})}{4a^2}$$

$$= \frac{(-b)^2 - (b^2 - 4ac)}{4a^2} = \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

$\therefore$ the sum of the roots $= -\frac{b}{a}$, the product of the roots $= \frac{c}{a}$.

Note: In a quadratic equation where the coefficient of $x^2$ is unity,

(i) the sum of the roots is equal to the coefficient of $x$ with its sign changed;

(ii) the product of the roots is equal to the constant term.

In any equation the constant term (i.e. the term without $x$) is frequently called the absolute term.

## 12.6 Special Roots

Under the following conditions the given quadratic equation will have the special roots :

### i) Roots equal in magnitude but opposite in sign :

Two roots will be equal in magnitude but opposite in magnitude if there sum is zero

i.e. $\alpha + \beta = -\frac{b}{a} = 0$

$\therefore \quad b = 0$

### ii) Reciprocal roots :

The two roots will be reciprocal to each other if their product is 1

i.e. $\alpha\beta = \frac{c}{a} = 1$

$\therefore \quad c = a$

### iii) One root zero

If one root is zero, then product of the roots is zero.

i.e. $\frac{c}{a} = 0 \quad \Rightarrow \quad c = 0$

### iv) Both roots zero

If both roots are zero, then

$\alpha + \beta = 0 \quad \Rightarrow \quad -\frac{b}{a} = 0$

and $\alpha\beta = 0 \quad \Rightarrow \quad \frac{c}{a} = 0$

$\therefore b = 0$ and $c = 0$

## 12.7 Formation of a Quadratic Equation

Suppose $ax^2 + bx + c = 0$ is the required equation, and $\alpha$ and $\beta$ are the given roots.

The required equation may be written as

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0,$$

or, $$x^2 - (\alpha + \beta)x + \alpha\beta = 0 \quad (\because \alpha + \beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a})$$

Hence any quadratic equation may be expressed in the form

$x^2$ – (sum of the roots) $x$ + product of the roots = 0.

## 12.8 Symmetric Functions of Roots

Symmetric functions of the roots of a quadratic equation are those functions in which the two roots are so involved that the function is unaltered when the two roots are interchanged. For instance, the sum $\alpha + \beta$ and the product $\alpha\beta$ of the roots, we have considered in the last two sections, are symmetric functions of $\alpha$ and $\beta$. Further examples of symmetric functions are

$$\frac{1}{\alpha} + \frac{1}{\beta}, \qquad \alpha^2 + \beta^2, \qquad \frac{\alpha + \beta}{\alpha\beta}, \qquad \alpha^2\beta^2, \text{ etc.}$$

One of the interesting characteristics of symmetric functions in $\alpha$ and $\beta$ is that they can be expressed in terms of $\alpha + \beta$ and $\alpha\beta$. Hence a symmetric function in $\alpha$ and $\beta$ can be expressed in terms of the coefficients of the equation.

For example, if $\alpha$ and $\beta$ are the roots of $x^2 - px + q = 0$, then

$$\alpha + \beta = p$$

and $$\alpha\beta = q.$$

From these values, it is easy to show that

(i) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= p^2 - 2q, \ \alpha^2\beta^2 = q^2$

(ii) $\alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 + \beta^2 - \alpha\beta)$

$= p\{(\alpha + \beta)^2 - 3\alpha\beta)$

$= p(p^2 - 3q)$

and $\alpha^3\beta^3 = q^3$

***Example 1***

Form the equation whose roots are 2, – 3.

***Solution.***

We have sum of roots = –1, product of roots = –6.

Hence the required equation is

$$x^2 + x - 6 = 0$$

***Example 2***

Form the equation whose one root is $2 + \sqrt{3}$.

***Solution.***

Since the irrational roots occur in pair, so if one root = $2 + \sqrt{3}$, then other root will be $2 - \sqrt{3}$.

Sum of the roots = $2 + \sqrt{3} + 2 - \sqrt{3} = 4$

Product of the roots = $(2 + \sqrt{3})\ (2 - \sqrt{3})$

$$= 4 - 3 = 1$$

Now the required quadratic equation is $x^2 - 4x + 1 = 0$

***Example 3***

Form a quadratic equation whose roots are the squares of the roots of

$$4x^2 + 8x - 5 = 0.$$

***Solution.***

Let $\alpha$ and $\beta$ be the roots of this equation, then

$$\alpha + \beta = -2 \quad \text{and} \quad \alpha\beta = -\frac{5}{4}$$

Since the roots of the required equation are the squares of $\alpha$ and $\beta$, we have :

Sum of the roots = $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$$= 4 + \frac{5}{2} = \frac{13}{2}$$

Product of the roots = $\alpha^2\beta^2 = \frac{25}{16}$

Hence the required equation is

$$x^2 - \frac{13}{2}x + \frac{25}{16} = 0$$

or $$16x^2 - 104x + 25 = 0$$

**Example 4**

Find the value of $k$ so that the equation $(3k + 1)x^2 + 2(k + 1)x + k = 0$ may have reciprocal roots.

**Solution :**

Comparing the given equation with $ax^2 + bx + c = 0$, we have

$a = 3k + 1, \quad b = 2(k + 1), \quad c = k$

For reciprocal roots,

$c = a$

$\Rightarrow \quad k = 3k + 1$

$\Rightarrow \quad -2k = 1$

$\therefore \quad k = -\frac{1}{2}$

**Example 5**

Show that 1 is a root of the equation $(b - c)x^2 + (c - a)x + (a - b) = 0$. Also, find the other root.

**Solution :**

Putting $x = 1$ in the given equation, we have $b - c + c - a + a - b = 0$ which is true, so 1 is the root of the given equation.

If $\alpha$ be the other root, then

$$\text{Product of the roots} = \frac{a - b}{b - c}$$

$$\Rightarrow \quad 1.\alpha = \frac{a - b}{b - c}$$

$$\therefore \quad \alpha = \frac{a - b}{b - c}$$

**Example 6**

If $\alpha$ and $\beta$ are the roots of the equation $2x^2 - 3x - 5 = 0$, form a quadratic equation whose roots are $2\alpha + \frac{1}{\beta}$ and $2\beta + \frac{1}{\alpha}$.

**Solution :**

From the given quadratic equation,

$$\alpha + \beta = -\frac{(-3)}{2} = \frac{3}{2}$$

$$\alpha.\beta = -\frac{5}{2}$$

For the required quadratic equation:

sum of the roots $= 2\alpha + \frac{1}{\beta} + 2\beta + \frac{1}{\alpha}$

$= 2(\alpha + \beta) + \left(\frac{\alpha + \beta}{\alpha\beta}\right)$

$= 2\left(\frac{3}{2}\right) + \left(\frac{3/2}{-5/2}\right) = \frac{12}{5}$

Product of the roots $= \left(2\alpha + \frac{1}{\beta}\right)\left(2\beta + \frac{1}{\alpha}\right)$

$= 4\alpha\beta + 4 + \frac{1}{\alpha\beta}$

$= 4\left(-\frac{5}{2}\right) + 4 - \frac{2}{5} = -\frac{32}{5}$

The required quadratic eqution is

$$x^2 - \frac{12}{5}x - \frac{32}{5} = 0$$

or, $5x^2 - 12x - 32 = 0$

***Example 7***

If one root of the equation $x^2 - px + q = 0$ be twice the other, show that $2p^2 = 9q$.

**Solution.**

If $\alpha$ is a root of the equation $x^2 - px + q = 0$ then the other root is $2\alpha$; and

$\alpha + 2\alpha = p$ and $\alpha.2\alpha = q$

or $3\alpha = p$, and $2\alpha^2 = q$

$2\alpha^2 = q \Rightarrow 2.\left(\frac{p}{3}\right)^2 = q$

Hence $2p^2 = 9q$.

***Example 8***

Find the condition that the roots of the quadratic equation

$$ax^2 + cx + c = 0$$

may be in the ratio $m : n$

**Solution.**

If $\alpha$ and $\beta$ are the roots of the equation $ax^2 + cx + c = 0$, then

$\alpha + \beta = \frac{-c}{a}$ and $\alpha\beta = \frac{c}{a}$

By given $\alpha : \beta = m : n$.

We know $\dfrac{\alpha + \beta}{\sqrt{\alpha\beta}} = \dfrac{-c/a}{\sqrt{\frac{c}{a}}}$

or $\sqrt{\dfrac{\alpha}{\beta}} + \sqrt{\dfrac{\beta}{\alpha}} = -\sqrt{\dfrac{c}{a}}$

Hence $\sqrt{\dfrac{m}{n}} + \sqrt{\dfrac{n}{m}} + \sqrt{\dfrac{c}{a}} = 0$

is the required condition.

***Example 9***

If $\alpha$ and $\beta$ are the roots of the equation $x^2 - px + q = 0$, find the equation whose roots are $\alpha^2\beta^{-1}$ and $\beta^2\alpha^{-1}$.

***Solution :***

From the given quadratic equation,

$$\alpha + \beta = -(-p) = p \quad , \quad \alpha.\beta = q$$

For the required quadratic equation:

sum of the roots $= \alpha^2\beta^{-1} + \beta^2\alpha^{-1}$

$$= \frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$$

$$= \frac{\alpha^3 + \beta^3}{\alpha\beta}$$

$$= \frac{(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)}{\alpha\beta}$$

$$= \frac{p^3 - 3qp}{q}$$

product of the roots $= \alpha^2\beta^{-1}.\beta^2\alpha^{-1}$

$$= \alpha\beta = q$$

Now, the required quadratic equation is

$$x^2 - \frac{p^3 - 3pq}{q}x + q = 0$$

or, $qx^2 - p(p^2 - 3q)x + q^2 = 0$

## EXERCISE 12.2

1. Form the equation whose roots are
   a) $3, -2$ b) $-5, 4$
   c) $\sqrt{3}, -\sqrt{3}$ d) $\frac{1}{2}(-1+\sqrt{5}), \frac{1}{2}(-1-\sqrt{5})$
   e) $-3+5i, -3-5i$ f) $a+ib, a-ib$
2. a) Find a quadratic equation whose roots are twice the roots of
   $$4x^2 + 8x - 5 = 0$$
   b) Find a quadratic equation whose roots are the reciprocals of the roots of
   $$3x^2 - 5x - 2 = 0.$$ (HSEB 2051)
   c) Find a quadratic equation whose roots are greater by $h$ than the roots of
   $$x^2 - px + q = 0$$
   d) Form a quadratic equation whose roots are the squares of the roots of
   $$3x^2 - 5x - 2 = 0$$
3. Find the quadratic equation with rational coefficient one of whose roots is
   a) $4+3i$ b) $\frac{1}{5+3i}$ c) $2+\sqrt{3}$
4. Find the value of $k$ so that the equation
   a) $2x^2 + kx - 15 = 0$ has one root $= 3$
   b) $3x^2 + kx - 2 = 0$ has roots whose sum is equal to 6
   c) $2x^2 + (4-k)x - 17 = 0$ has roots equal but opposite in sign
   d) $3x^2 - (5+k)x + 8 = 0$ has roots numerically equal but opposite in sign.
   e) $3x^2 + 7x + 6 - k = 0$ has one root equal to zero
   f) $4x^2 - 17x + k = 0$ has the reciprocal roots.
5. Show that $-1$ is a root of the equation
   $$(a+b-2c)x^2 + (2a-b-c)x + (c+a-2b) = 0.$$
   Find the other root.
6. Find the value of $m$ for which the equation
   $$(m+1)x^2 + 2(m+3)x + (2m+3) = 0$$
   will have (a) reciprocal roots (b) one root zero.
7. If the roots of the equation $x^2 + ax + c = 0$ differ by 1, prove that
   $$a^2 = 4c + 1.$$

8. If $\alpha$ and $\beta$ be the roots of the equation $x^2 - x - 6 = 0$, find the equation whose roots are
   a) $\alpha^2\beta^{-1}$ and $\beta^2\alpha^{-1}$  b) $\alpha + \frac{1}{\beta}$ and $\beta + \frac{1}{\alpha}$
9. If $\alpha$ and $\beta$ are the roots of the equation $ax^2 + bx + c = 0$, find the equation whose roots are
   a) $\alpha\beta^{-1}$ and $\beta\alpha^{-1}$
   b) $\alpha^3$ and $\beta^3$
   c) $(\alpha - \beta)^2$ and $(\alpha + \beta)^2$
   d) the reciprocal of the roots of the given equation.
10. a) If the roots of the equation $ax^2 + bx + c = 0$ be in the ratio of $3 : 4$, prove that $12b^2 = 49ac$. (T.U. 2049 H, 2058 S)
    b) If one root of the equation $ax^2 + bx + c = 0$ be four times the other root, show that $4b^2 = 25ac$.
    c) For what value of $m$, the equation $x^2 - mx + m + 1 = 0$ may have its roots in the ratio 2:3.
11. a) If $\alpha$ and $\beta$ are the roots of $px^2 + qx + q = 0$, prove that

$$\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{q}{p}} = 0$$

$x^2 - 3x + 1 = 0$

    b) If the roots of the equation $lx^2 + nx + n = 0$ be in the ratio $p : q$, prove that

$$\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{n}{l}} = 0$$ [HSEB 2055]

12. If one root of the equation $ax^2 + bx + c = 0$ be the square of the other, prove that $b^3 + a^2c + ac^2 = 3abc$.

## Answers

1. a) $x^2 - x - 6 = 0$  b) $x^2 + x - 20 = 0$  c) $x^2 - 3 = 0$
   d) $x^2 + x - 1 = 0$  e) $x^2 + 6x + 34 = 0$  f) $x^2 - 2ax + (a^2 + b^2) = 0$
2. a) $x^2 + 4x - 5 = 0$  b) $2x^2 + 5x - 3 = 0$
   c) $x^2 - (p + 2h)x + (q + ph + h^2) = 0$  d) $9x^2 - 37x + 4 = 0$
3. a) $x^2 - 8x + 25 = 0$  b) $34x^2 - 10x + 1 = 0$  c) $x^2 - 4x + 1 = 0$
4. a) $k = -1$  b) $k = -18$  c) $k = 4$
   d) $k = -5$  e) $k = 6$  f) $k = 4$
5. $\frac{2b - c - a}{a + b - 2c}$  6. a) $-2$  b) $-\frac{3}{2}$

8. a) $6x^2 + 19x - 36 = 0$ b) $6x^2 - 5x - 25 = 0$
9. a) $acx^2 - (b^2 - 2ac)x + ac = 0$ b) $a^3x^2 + b(b^2 - 3ca)x + c^3 = 0$
   c) $a^4x^2 - 2a^2(b^2 - 2ca)x + b^2(b^2 - 4ac) = 0$
   d) $cx^2 + bx + a = 0$
10. c) $5$ or $-\frac{5}{6}$

## 12.9 Miscellaneous Results

In this section, we shall obtain conditions under which two given quadratic equations may have one root common or both roots common.

### a) One Root Common

Let $ax^2 + bx + c = 0$ and $a'x^2 + b'x + c' = 0$ be two equations. Suppose that $\alpha$ is a root common to both the equations. Then

$$a\,\alpha^2 + b\,\alpha + c = 0$$
$$a'\,\alpha^2 + b'\,\alpha + c' = 0$$

By the rule of cross-multiplication

$$\frac{\alpha^2}{bc' - b'c} = \frac{\alpha}{ca' - c'a} = \frac{1}{ab' - a'b}$$

This gives $\alpha = \dfrac{bc' - b'c}{ca' - c'a}$ and also $\alpha = \dfrac{ca' - c'a}{ab' - a'b}$

Combining the two, we have the required condition

$$(bc' - b'c)\,(ab' - a'b) = (ca' - c'a)^2$$

and the common root is

$$\frac{bc' - b'c}{ca' - c'a} \quad \text{or} \quad \frac{ca' - c'a}{ab' - a'b}$$

The condition of one root common of the two given quadratic equations can be remembered in the following way:

(List the coeffs. and repeat the first)

$$\begin{array}{ccccccc}
a & & b & & c & & a \\
 & \times & & \times & & \times & \\
a' & & b' & & c' & & a' \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & ab' - a'b & & bc' - b'c & & a'c - ac' &
\end{array}$$

The left hand expression of the condition = $(ab' - a'b)\,(bc' - b'c)$
and the right hand expression of the condition = $(a'c - ac')^2$

### b) Two Roots Common

If the quadratic equations have both roots common and if $\alpha$ and $\beta$ be the common roots, then

$$\alpha + \beta = -\frac{b}{a} = -\frac{b'}{a'} \qquad \text{or} \qquad \frac{a}{a'} = \frac{b}{b'}$$

$$\text{and} \quad \alpha\beta = \frac{c}{a} = \frac{c'}{a'} \qquad \text{or} \qquad \frac{a}{a'} = \frac{c}{c'}$$

$$\text{Hence } \frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'},$$

which is the required condition for the equations to have both roots common.

## Worked Out Examples

***Example 1***

Show that the following two quadratic equations $2x^2 + x - 3 = 0$ and $3x^2 - 4x + 1 = 0$ have one root common.

***Solution :***

Writing the coefficients in order and repeating the first one, we have

$$\begin{array}{cccc} 2 & 1 & -3 & 2 \\ 3 & -4 & 1 & 3 \end{array}$$

The left hand expression of the condition

$$= (2 \times -4 - 3 \times 1)\,(1 \times 1 - (-4) \times (-3))$$
$$= (-11)\,(-11) = 121$$

The right hand expression of the condition

$$= (-3 \times 3 - 1 \times 2)^2 = 121$$

Since the two results are equal, so the two equations have one root common.

Note: we can prove the result by solving the equations as well.

***Example 2***

Find the value of $k$ so that $x^2 - kx - 21 = 0$ and $x^2 - 3kx + 35 = 0$ have a common root.

***Solution :***

Writing the coefficients in order and repeating the first one, we have

$$\begin{array}{cccc} 1 & -k & -21 & 1 \\ 1 & -3k & 35 & 1 \end{array}$$

Now, using the condition of one root common, we have

$(-3k + k).(-35k - 63k) = (-21 - 35)^2$

$\Rightarrow$ $(-2k).(-98k) = (-56)^2$

$\Rightarrow$ $196k^2 = 56 \times 56$

$\Rightarrow$ $k^2 = 16$

$\therefore$ $k = \pm 4$

***Example 3***

If the quadratic equations $x^2 + qx + pr = 0$ and $x^2 + rx + pq = 0$ have one root common, prove that $p + q + r = 0$

***Solution :***

If $\alpha$ be a common root, then

$\alpha^2 + q\alpha + pr = 0$ ...... (i)

$\alpha^2 + r\alpha + pq = 0$ ...... (ii)

$-\quad-\quad-\quad-$

Subtracting, $(q - r)\alpha - p(q - r) = 0$

$(q - r)(\alpha - p) = 0$

$\therefore$ $\alpha = p$

Substituting the value of $\alpha$ in (i) we have, $p + q + r = 0$

## EXERCISE 12.3

1. Show that each pair of the following equations has a common root
   a) $x^2 - 8x + 15 = 0$ and $2x^2 - x - 15 = 0$
   b) $3x^2 - 8x + 4 = 0$ and $4x^2 - 7x - 2 = 0$
2. Find the value of $p$ so that each pair of the equations may have one root common.
   a) $4x^2 + px - 12 = 0$ and $4x^2 + 3px - 4 = 0$
   b) $2x^2 + px - 1 = 0$ and $3x^2 - 2x - 5 = 0$
3. If the quadratic equations $x^2 + px + q = 0$ and $x^2 + p'x + q' = 0$ have a common root, show that it must be either

$$\frac{pq' - p'q}{q - q'} \text{ or } \frac{q - q'}{p' - p}.$$

4. If the equations $x^2 + px + q = 0$ and $x^2 + qx + p = 0$ have a common root, prove that either $p = q$, or $p + q + 1 = 0$. (HSEB 2050)

5. If the quadratic equations $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a common root, then either $a + b + c = 0$ or $a = b = c$.

6. Prove that if the equations

$x^2 + bx + ca = 0$ and $x^2 + cx + ab = 0$

have a common root, their other roots will satisfy

$$x^2 + ax + bc = 0.$$

### *Answers*

2. (i) $p = \pm 2$ (ii) $p = 1$ or $-\dfrac{41}{15}$

( a b ) c

a

## ADDITIONAL QUESTIONS

1. What do you mean by the roots of a quadratic equation ? Can there be a quadratic equation with real coefficients which has one root real and another root imaginary ? Justify your answer. (T.U. 2053)

2. Consider the quadratic equation $ax^2 + bx + c = 0$. Under what conditions are the roots

   i) real and unequal; ii) equal; iii) imaginary

   Find a quadratic equation whose roots are the reciprocals of the roots of $3x^2 - 5x - 3 = 0$. (HSEB 2051)

3. Distinguish between an identity and an equation. What are the roots of the quadratic equation $ax^2 + bx + c = 0$ ? Prove that a quadratic equation cannot have more than two roots ? (T.U. 2049)

4. Find the condition that $ax^2 + bx + c = 0$ may have

   i) one root reciprocal of the other
   ii) roots equal in magnitude but opposite in sign
   iii) both roots positive
   iv) both roots negative

5. Prove that the roots of the equation $x^2 + (2k - 1)x + k^2 = 0$ are real if $k \le \frac{1}{4}$.

6. Show that the roots of the equation

$$2(a^2 + b^2)x^2 + 2(a + b)x + 1 = 0$$

can never be real unless $a = b$.

7. Show that the roots of the quadratic equation
$$(b-c)x^2 + 2(c-a)x + (a-b) = 0$$
are always real.

8. For all $a, b, c \in R$, prove that the roots of the quadratic equation
$$(a^2+b^2)x^2 + 2(bc+ad)x + (c^2+d^2) = 0$$
are imaginary. Also, prove that the roots will be real only when $\frac{a}{b} = \frac{d}{c}$.

9. Determine the value of $m$ for which
$$3x^2 + 4mx + 2 = 0 \quad \text{and} \quad 2x^2 + 3x - 2 = 0$$
may have a common root.

10. If the ratio of the roots of $ax^2 + bx + c = 0$ be equal to that of the roots of $a'x^2 + b'x^2 + c' = 0$ prove that $\frac{b^2}{b'^2} = \frac{ac}{a'c'}$

11. If $-4$ is a root of the equation $x^2 + px - 4 = 0$ and the equation $x^2 + px + q = 0$ has equal roots, find the value of $q$.

12. For what value of $k$ is one root of the equation
$$x^2 + 3x - 6 = k(x-1)^2$$
double the other ?

13. For what values of $k$ are the roots of the equation $3x^2 - 2kx + k = 0$ in the ratio of 3 : 1 ?

14. The sum of the roots of the equation
$$\frac{1}{x+a} + \frac{1}{x+b} = \frac{1}{c}$$
is zero. Prove that the product of the roots is $-\frac{1}{2}(a^2+b^2)$

15. For what values of $m$ will the quadratic equation
$$x^2 - 2(5+2m)x + 3(7+10m) = 0$$
have i) equal roots; ii) reciprocal roots ?

16. The sum of the roots of a quadratic equation is 1 and the sum of their squares is 13. Find the equation.

17. If $\alpha, \beta$ be the roots of $ax^2 + bx + c = 0$, obtain an equation whose roots are
i) $(a\alpha + b)^{-1}$ and $(a\beta + b)^{-1}$
ii) $\sqrt{\frac{\alpha}{\beta}}$ and $\sqrt{\frac{\beta}{\alpha}}$

18. If the roots of the equation $lx^2 + nx + n = 0$ be in the ratio $p : q$, find the value of $\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}}$. (HSEB 2056)

19. If the difference of the roots of $x^2 + 2px + q = 0$ be equal to the difference of the roots of $x^2 + 2qx + p = 0$ prove that $p + q + 1 = 0$

**Answers**

2. $3x^2 + 5x - 3 = 0$
4. i) $c = a$ ii) $b = 0$
   iii) $a$ and $c$ should be of same sign and $b$ of opposite sign and discriminant positive.
   iv) $a, b, c$ should be of the same sign
9. $\frac{7}{4}$ or $-\frac{11}{8}$ 11. $\frac{9}{4}$ 12. $-24$ or $3$ 13. 4
15. i) $\frac{1}{2}$ or 2 ii) $-\frac{2}{3}$ 16. $x^2 - x - 6 = 0$
17. i) $acx^2 - bx + 1 = 0$ ii) $\sqrt{ac}\, x^2 + bx + \sqrt{ac} = 0$ 18. $-\sqrt{\frac{n}{l}}$

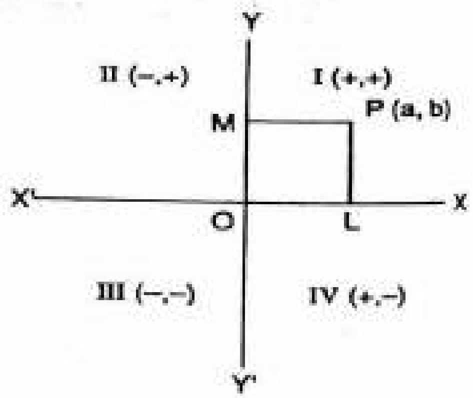
CHAPTER 13

# Straight Lines

## 13.1 Review of equation of a straight line in different forms

### Coordinates of a point on the plane

Let XOX' and YOY', the two mutually perpendicular lines to each other meeting at the point O be taken as x-axis and y-axis respectively. The point O is known as the origin. Let P be a point on the plane. From P, draw PM perpendicular to XOX' meeting at the point M. If OM = $x$ and MP = $y$, then the point P is associated with the numbers $x$ and $y$. We say that P has the coordinates $(x, y)$, $x$ being the x-coordinate (abscissa) and $y$ being the y-cordinate (ordinate).




XOX' and YOY' divide the whole plane into four parts known as the quadrants. The sign of the coordinates of the point on I, II, III and IV quadrants are (+, +), (−, +), (−, −) and (+, −).

### Some Fundamental Formulae

### Distance formula

If $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points on the plane then the distance between them denoted by $d$ is given by

$$d = AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

## Slope formula

If $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points then

rise = $y_2 - y_1$ and run = $x_2 - x_1$.

The slope of the line AB is defined as the ratio of rise to the run.

The slope of a line is also defined as the tangent of the angle made by the line AB with the positive x-axis. Thus if $\theta$ is the angle made by the line AB with positive x-axis, then

$$\text{slope of the line} = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$$

Slope of the line is denoted by $m$.

## Section formula

**Internal Division:** If $C(x, y)$ be a point dividing the line joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the ratio of $m_1{:}m_2$, then

$$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}$$

$$\text{and} \quad y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}$$

If $C(x, y)$ is the middle point of the line AB, then

$$x = \frac{x_1 + x_2}{2}$$

$$\text{and} \quad y = \frac{y_1 + y_2}{2}$$

**External division:** If $C(x, y)$ be a point dividing the line joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ externally in the ratio of $m_1{:}m_2$, then

$$x = \frac{m_1x_2 - m_2x_1}{m_1 - m_2}$$

$$\text{and} \quad y = \frac{m_1y_2 - m_2y_1}{m_1 - m_2}$$

## Centroid formula

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a triangle. If $G(x, y)$ be the centroid of the triangle, then

$$x = \frac{x_1 + x_2 + x_3}{3}$$

$$\text{and} \quad y = \frac{y_1 + y_2 + y_3}{3}$$

### Area of a triangle

If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a triangle ABC, then the area of the triangle ABC

$$= \frac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3)$$

$$= \frac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}$$

The expression for the area of the triangle with vertices $(x_1, y_1)$, $(x_2, y_2)$ and $(x_3, y_3)$ can be obtained from the following multiplication operation

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{array}$$

The expression for the area of the triangle may also be expressed in the following determinant form

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

If the three points A, B and C are collinear, then

area of the $\Delta ABC = 0$

i.e. $x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$

### Relation between cartesian coordinates and polar coordinates

If $(x, y)$ be the cartesian coordinates of a point on the plane and $(r, \theta)$ be the polar coordinates of the same point, then

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

where $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\frac{y}{x}$

## 13.2 Equations of Straight Lines Parallel to the Axes

We have seen that equation of x-axis is $y = 0$. The equation of y-axis can similarly be written as $x = 0$, since any point on the y-axis will have its abscissa equal to zero.

Now consider a straight line parallel to x-axis at a distances '$a$' units from it. Any point on the straight line has its ordinate equal to $a$. Hence $y = a$ is the equation of the straight line parallel to x-axis.

Similarly the equation of a straight line parallel to the y-axis at a distance '$b$' units from it is $x = b$.

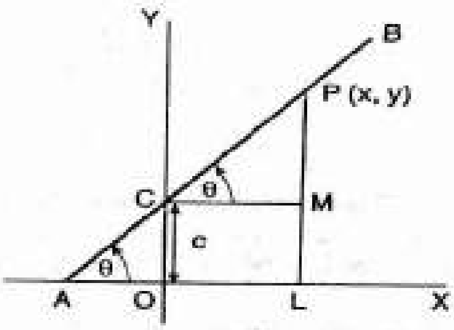
## 13.3 Three Standard Forms of Equation of Straight Lines

### (a) Slope Intercept Form

Let AB be a straight line meeting the x- and y-axes at A and C respectively.

Let OC = $c$, $\angle OAC = \theta$. So slope of the line = $\tan\theta$ = m, say, and the intercept on the y axis (the y intercept) = $c$.

Let P $(x, y)$ be any point on the line. Draw PL perpendicular to x-axis and CM parallel to x-axis meeting PL at M.




Then $\angle PCM = \theta$, CM = OL = $x$, LP = $y$.

Also $\tan\theta = \dfrac{MP}{CM}$

i.e. $m = \dfrac{y - c}{x}$

or, $y - c = mx$

or, $y = mx + c$.

Since this equation is satisfied by the coordinates $(x, y)$ of any point P on the line, it is the equation of the line.

Note 1. This equation is referred to as the slope-intercept form or simply the slope form of an equation of a straight line.

Note 2. If the line passes through the origin, $c = 0$, the equation will be $y = mx$.

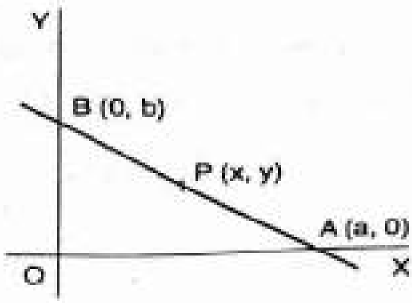
If the line be parallel to the x-axis, i.e. its slope equals zero ($m = 0$), the equation becomes $y = c$.

If the line be parallel to the y-axis, the equation cannot be written in this form.

### (b) Double Intercept Form

Let the straight line AB cut the axes at A and B.

Let OA = $a$, OB = $b$. These are the intercepts on the x- and y-axes respectively. Obviously the coordinates of A and B are respectively $(a, 0)$ and $(0, b)$. Let P $(x, y)$ be a point on AB.




Then slope of AP $= \dfrac{y - 0}{x - a}$

and slope of PB $= \dfrac{b - y}{0 - x}$ (Art. 9.4)

But AP and PB are the segments of the same straight line

So $\dfrac{y-0}{x-a} = \dfrac{b-y}{0-x}$

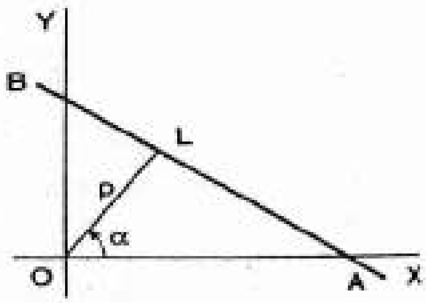
i.e. $-xy = bx - ab - xy + ay$

or, $bx + ay = ab$

or, $\dfrac{x}{a} + \dfrac{y}{b} = 1.$

This is the equation of the line whose intercepts are $a$ and $b$. This form of the equation of a straight line cannot be used if the line passes through the origin or if it be parallel to any one of the axes.

### (c) Normal Form or Perpendicular Form




Let AB be a straight line on which the length of the perpendicular, OL, from the origin is of length $p$. Let the angle AOL made by the perpendicular OL with the positive x-axis be $\alpha$.

We have $\angle OBL = 90° - \angle BOL$

$= 90° - (90° - \alpha) = \alpha.$

From $\Delta$ OAL, $\cos\alpha = \dfrac{OL}{OA} = \dfrac{p}{OA}$

or, $OA = \dfrac{p}{\cos\alpha}$

From $\Delta$ OBL, $\sin\alpha = \dfrac{OL}{OB} = \dfrac{p}{OB}$

or, $OB = \dfrac{p}{\sin\alpha}$

Thus the intercepts on the axes are $OA\left(= \dfrac{p}{\cos\alpha}\right)$ and $OB\left(= \dfrac{p}{\sin\alpha}\right)$.

Using the double intercept form, the equation of the line is

$$\frac{x}{OA} + \frac{y}{OB} = 1$$

or, $$\frac{x}{\frac{p}{\cos\alpha}} + \frac{y}{\frac{p}{\sin\alpha}} = 1$$

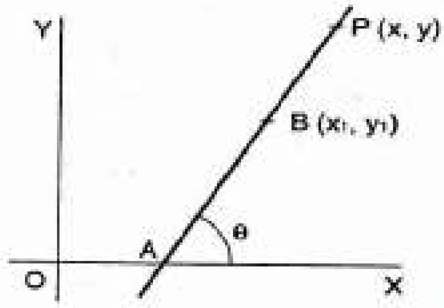
or, $$x\cos\alpha + y\sin\alpha = p.$$

## 13.4 Equation of Lines in Special Cases

### (a) Point Slope Form:

To find the equation of a line whose slope is given and which passes through a given point.

Let a line passing through a given point B $(x_1, y_1)$ have its slope equal to $m$.




Let P $(x, y)$ be any point on the line.

Then slope of BP $= \dfrac{y - y_1}{x - x_1}$ but slope of the line $= m$.

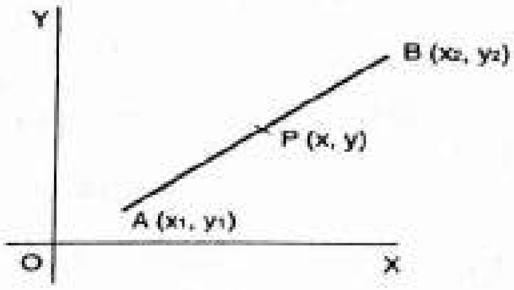
$$\therefore \quad \frac{y - y_1}{x - x_1} = m$$

$$\therefore \quad y - y_1 = m\ (x - x_1)$$

is the equation of the line.

### (b) Two Points Form:

To determine the equation of a line passing through two given points.




Let a straight line pass through $A(x_1, y_1)$ and B $(x_2, y_2)$.

Take any point $P(x, y)$ on the line. Then

slope of AP $= \dfrac{y - y_1}{x - x_1}$

slope of AB $= \dfrac{y_2 - y_1}{x_2 - x_1}$.

But AP and AB are segments of the same straight line,

$$\therefore \quad \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

or, $\quad y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}\ (x - x_1)$ is the equation of the straight line.

## 13.5 Linear Equation $Ax + By + C = 0$.

The general equation of first degree in $x$ & $y$ is $Ax + By + C = 0$, where A, B, C are constants, and A and B are not simultaneously zero. This is also called the linear equation in $x$ and $y$ as we shall subsequently see that it always represents a straight line.

To prove that $Ax + By + C = 0$ always represents a straight line. Let $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ be any three points on the locus represented by the equation $Ax + By + C = 0$. The coordinates of the points must satisfy the equation.

Hence $\quad Ax_1 + By_1 + C = 0$

$$Ax_2 + By_2 + C = 0$$
$$Ax_3 + By_3 + C = 0.$$

From the first two equations, by the rule of cross multiplication, we have

$$\frac{A}{y_1 - y_2} = \frac{B}{x_2 - x_1} = \frac{C}{x_1y_2 - x_2y_1} = k, \text{ say}$$

$\therefore \quad A = (y_1 - y_2)\,k, \qquad B = (x_2 - x_1)\,k, \qquad C = (x_1y_2 - x_2y_1)\,k$

Substituting the values of A, B and C in the third equation, we have

$$k\,(y_1 - y_2)\,x_3 + k(x_2 - x_1)\,y_3 + k(x_1y_2 - x_2y_1) = 0$$

or, $\quad x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 = 0$

The L.H.S. is twice the area of the triangle PQR. Hence the area of the triangle is zero, which means that P, Q, R lie in a straight line.

Thus the linear equation represents a straight line.

The converse is easily seen to be true. Because equation of any line not parallel to y-axis can be written as $y = mx + c$, and that of any line parallel to y-axis is $x = a$.

Both of these equations are linear.

## 13.6 Reduction of the Linear Equation to Three Standard Forms

The linear equation $Ax + By + C = 0$ can be reduced to the three standard forms : slope-intercept form, double intercept form and the normal form.

### i) Reduction to the Slope Intercept Form

$Ax + By + C = 0$ can be written as

$$By = -Ax - C$$

i.e. $\quad y = -\frac{A}{B}x - \frac{C}{B}$ which is of the form $y = mx + c$

where slope $(m) = -\frac{A}{B}$ and y – intercept $(c) = -\frac{C}{B}$.

### ii) Reduction to the Double Intercept Form

$Ax + By + C = 0$ can be written as

$$Ax + By = -C$$

or $\dfrac{A}{-C}x + \dfrac{B}{-C}y = 1$

or $\dfrac{x}{-C/A} + \dfrac{y}{-C/B} = 1$, which is of the form $\dfrac{x}{a} + \dfrac{y}{b} = 1$, where

x intercept $= a = -\dfrac{C}{A}$, y intercept $= b = -\dfrac{C}{B}$.

**iii) Reduction to the Normal Form**

The equations

$$Ax + By + C = 0$$

and $x \cos\alpha + y \sin\alpha - p = 0$

will represent one and the same straight line if their corresponding coefficients are proportional.

i.e. $\dfrac{\cos\alpha}{A} = \dfrac{\sin\alpha}{B} = \dfrac{-p}{C} = k$, say

so that $\cos\alpha = Ak$, $\sin\alpha = Bk$, $-p = Ck$.

$\therefore A^2k^2 + B^2k^2 = 1$, i.e. $k^2 = \dfrac{1}{A^2 + B^2}$

or $k = \pm\dfrac{1}{\sqrt{A^2 + B^2}}$

If $C > 0$, then $k = -\dfrac{1}{\sqrt{A^2 + B^2}}$

So, $p = \dfrac{C}{\sqrt{A^2 + B^2}}$, $\cos\alpha = \dfrac{-A}{\sqrt{A^2 + B^2}}$ and $\sin\alpha = \dfrac{-B}{\sqrt{A^2 + B^2}}$

Hence the equation of the line in the normal form is

$$-\frac{A}{\sqrt{A^2 + B^2}}x - \frac{B}{\sqrt{A^2 + B^2}}y = \frac{c}{\sqrt{A^2 + B^2}}$$

Again, if $C < 0$, then $k = \dfrac{1}{\sqrt{A^2 + B^2}}$

So, $p = -\dfrac{C}{\sqrt{A^2 + B^2}}$, $\cos\alpha = \dfrac{A}{\sqrt{A^2 + B^2}}$ and $\sin\alpha = \dfrac{B}{\sqrt{A^2 + B^2}}$

Hence, the equation of the line in the normal form is

$$\frac{A}{\sqrt{A^2 + B^2}}x + \frac{B}{\sqrt{A^2 + B^2}}y = -\frac{C}{\sqrt{A^2 + B^2}}$$

## 13.7 The Point of Intersection of Two Straight Lines

Let $A_1 x + B_1 y + C_1 = 0$ and $A_2 x + B_2 y + C_2 = 0$ be any two given lines.

The coordinates of the point of intersection of two given lines is

$$\left(\frac{B_1C_2 - C_1B_2}{A_1B_2 - B_1A_2}, \frac{C_1A_2 - A_1C_2}{A_1B_2 - B_1A_2}\right)$$

provided that $A_1B_2 - B_1A_2 \neq 0$.

## 13.8 Condition for Concurrency of Three Straight Lines

Let the three straight lines be

$$(L_1) : A_1x + B_1y + C_1 = 0$$
$$(L_2) : A_2x + B_2y + C_2 = 0$$
$$(L_3) : A_3x + B_3y + C_3 = 0$$

The condition for three lines to be concurrent is

$$A_1 (B_2C_3 - B_3C_2) + B_1 (C_2A_3 - C_3A_2) + C_1 (A_2B_3 - A_3B_2) = 0.$$

The condition for the three lines to be concurrent expressed in determinant form is as follows:

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = 0.$$

In practice we may follow the following rule. If three quantities $l, m, n$ can be determined such that

$$l(A_1x + B_1y + C_1) + m(A_2x + B_2y + C_2) + n (A_3x + B_3y + C_3) = 0$$

then the three lines are concurrent.

## 13.9 Any Line Through the Intersection of Two Given Lines

Let $\quad A_1x + B_1y + C_1 = 0 \quad \text{.........} \quad (i)$

and $\quad A_2x + B_2y + C_2 = 0 \quad \text{.........} \quad (ii)$

be two given lines.

Then the equation of the line through the intersection of the lines (i) and (ii) is

$$(A_1x + B_1y + C_1) + k(A_2x + B_2y + C_2) = 0 \text{ .........} \quad (iii)$$

where k is any constant.

## Worked Out Examples

***Example 1***

Prove that the line whose intercepts on the axes of x and y are respectively $-2$ and 3, passes through the point (2, 6).

***Solution***

Here, $\quad x - \text{intercept} = a = -2$

$\quad y - \text{intercept} = b = 3$

Then the equation of the line is

$$\frac{x}{-2} + \frac{y}{3} = 1$$

or, $\quad -3x + 2y = 6 \quad \ldots\ldots\ldots\ldots\ldots (i)$

Put $x = 2$ and $y = 6$ in equation (i)

Then, $\quad -3 \times 2 + 2 \times 6 = 6$

or, $\quad 6 = 6$ which is true.

Hence the line (i) passes through the point (2, 6).

***Example 2***

Determine the equation of the straight line passing through the point $(-1, 3)$ and $x$-intercept is thrice the $y$-intercept.

***Solution :***

If $y$-intercept be $b$, then $x$-intercept will be $3b$. Then the equation of the line is

$$\frac{x}{3b} + \frac{y}{b} = 1$$

or, $\quad x + 3y = 3b \quad \ldots\ldots (i)$

The line (i) passes through the point $(-1, 3)$, so

$$-1 + 3 \times 3 = 3b$$

$\therefore \quad 3b = 8$

Substituting the value of $3b$ in (i),

$$x + 3y = 8$$

which is the required equation of the line.

***Example 3***

Obtain the equations of the straight lines through the point $\left(\frac{1}{\sqrt{3}}, 1\right)$ whose perpendicular distance from the origin is unity.

***Solution.***

Let the equation of the line be

$$x \cos \alpha + y \sin \alpha = p.$$

Since it passes through the point $\left(\frac{1}{\sqrt{3}}, 1\right)$ and its perpendicular distance from the origin i.e. $p = 1$, so by substitution, we have,

$$\frac{1}{\sqrt{3}} \cdot \cos \alpha + 1 \cdot \sin \alpha = 1$$

or, $\frac{1}{2} \cos \alpha + \frac{\sqrt{3}}{2} \cdot \sin \alpha = \frac{\sqrt{3}}{2}$

or, $\cos 60° \cos \alpha + \sin 60° \sin \alpha = \frac{\sqrt{3}}{2}$

or, $\cos (\alpha - 60°) = \cos (\pm 30°)$

$$\alpha - 60° = \pm 30°$$

$\therefore$ $\alpha = 60° \pm 30°$

$\therefore$ $\alpha = 90°, 30°$

when $\alpha = 90°$, the equation of the line is

$$x \cos 90° + y \sin 90° = 1$$

$\therefore$ $y = 1$

when $\alpha = 30°$, the equation of the line is

$$x \cos 30° + y \sin 30° = 1$$

or, $x \cdot \frac{\sqrt{3}}{2} + y \cdot \frac{1}{2} = 1$

$\therefore$ $\sqrt{3}x + y = 2$

***Example 4***

If a line with equation $5x + 6y = 2k$ together with the coordinate axes form a triangle of area 135 sq. units, find the value of $k$.

***Solution :***

The given equation of the line is

$$5x + 6y = 2k$$

Reducing this equation to double intercept form, we have

$$\frac{x}{2k/5} + \frac{y}{2k/6} = 1$$

where x-intercept $= \frac{2k}{5}$ and y-intercept $= \frac{2k}{6}$

The area of the triangle $= \frac{1}{2} \cdot \frac{2k}{5} \cdot \frac{2k}{6}$

or, $135 = \frac{k^2}{15}$

$k^2 = 15 \times 135 = 2025$

$\therefore \quad k = \pm 45$

***Example 5***

Find the equation of the line passing through the point of intersection of the lines $x + 2y = 9$ and $3x - 2y + 5 = 0$ and making equal intercepts on the axes.

***Solution :***

The equations of the given lines are

$x + 2y = 9$ ...... (i)

and $3x - 2y = -5$ ...... (ii)

Solving equations (i) and (ii), we have

$x = 1, \quad y = 4.$

$\therefore$ the point of intersection of the lines (i) and (ii) is (1, 4).

If $a$, $a$ be the intercepts made by the line on the axes, then the equation of the line is

$$\frac{x}{a} + \frac{y}{a} = 1$$

or, $x + y = a$ ...... (iii)

The line (iii) passes through the point (1, 4), so

$1 + 4 = a$

$\therefore a = 5$

Substituting the value of a in (iii), we have

$x + y = 5$ which is the required equation of the line.

***Example 6***

Find the equation of the straight line passing through the intersection of the lines $3x - 4y - 10 = 0$ and $5x + 3y - 7 = 0$ and making angle 135° with the positive x-axis.

***Solution :***

The given equations of the lines are

$3x - 4y - 10 = 0$ ...... (i)

$5x + 3y - 7 = 0$ ...... (ii)

Solving equations (i) and (ii), we have

$$x = 2, \qquad y = -1$$

$\therefore$ the point of intersection of the lines (i) and (ii) is (2, –1).

slope of the line = $m = \tan 135° = -1$

The equation of the line through a given point $(x_1, y_1)$ is

$$y - y_1 = m(x - x_1)$$

The equation of the line through (2, –1) and slope –1 is

$$y - (-1) = -1(x - 2)$$

$$x + y = 1$$

**Alternative Method**

The equation of the line through the intersection of given lines is

$$(3x - 4y - 10) + k(5x + 3y - 7) = 0 \qquad \text{...... (i)}$$

where $k$ is a constant to be determined.

or, $\quad (3 + 5k)x + (-4 + 3k)y + (-10 - 7k) = 0$

Slope of this line $= -\dfrac{3 + 5k}{-4 + 3k}$

Also, slope of the line = $\tan 135° = -1$

$\therefore \quad -\dfrac{3 + 5k}{-4 + 3k} = -1$

or, $\quad 3 + 5k = -4 + 3k$

or, $\quad 2k = -7$

$\therefore \quad k = -\dfrac{7}{2}$

Substituting the value of $k$ in (iii), we have

$$3x - 4y - 10 - \frac{7}{2}(5x + 3y - 7) = 0$$

$$x + y = 1$$

***Example 7***

Show that the following set of three lines are concurrent.

$x - y + 1 = 0$, $2x - y - 1 = 0$ and $x + 3y - 11 = 0$

***Solution :***

Taking the first two equations,

we have $\quad x - y + 1 = 0 \quad$ ...... (i)

$\qquad\qquad 2x - y - 1 = 0 \quad$ ...... (ii)

Solving equations (i) and (ii)

$x = 2, y = 3$

$\therefore$ the point of intersection of the lines (i) and (ii) is (2, 3)

Put $x = 2, y = 3$ in third equation

$x + 3y - 11 = 0$ i.e. $2 + 3 \times 3 - 11 = 0$

which is true

Hence the three lines are concurrent.

## EXERCISE 13.1

1. Find the equations of the lines
   i) making an angle of 60° with the positive x-axis and cutting an intercept 3 from the y-axis.
   ii) making an angle of 135° with the positive x-axis and cutting an intercept 3 from the negative y-axis.
   iii) bisecting the angle between the axes.
2. i) Obtain the equation of the straight line passing through the point (3, 4) cutting off equal intercepts on the axes.
   ii) Find the equation of the straight line which passes through the point (7, 11) and has intercepts on the axes equal in magnitude but opposite in sign.
3. Find the equation of the straight line through the point (2, 3) whose intercept on x-axis is twice that on y-axis.
4. Find the equation of the straight line which passes through the point (3, 4) and makes intercepts on the axes, the sum of whose lengths is 14.
5. Determine the equation of the line the portion of which, intercepted by the axes, is divided by the point (–5, 4) in the ratio 1:2.
6. Determine the equations of the sides and medians of the triangle whose vertices are (–2, 0), (2, 4), (4, 1).
7. Find the value of k so that the line whose equation is $2x + 3y + k = 0$ will form a triangle with the coordinate axes whose area is 27 sq. units.
8. In what ratio is the line joining (1, 3) and (2, 7) divided by the line $3x + y = 9$ ?
9. For what value of $k$ will make the three points (1, 4), (–3, 16) and ($k$, –2) collinear ?

10. Prove that the following triplets of lines are concurrent
    (i) $x + 2y = 0,\ 3x - 4y - 10 = 0,\ 5x + 3y - 7 = 0$
    (ii) $2x - 7y + 10 = 0\ \ 3x - 2y = 1,\ x - 12y + 21 = 0$
    (iii) $9x - 13y - 90 = 0,\ 2x + 11y - 20 = 0,\ 7x + y - 70 = 0$

11. Find the value of '$a$' for which the lines $3x + y - 2 = 0$, $ax + 2y - 3 = 0$ and $2x - y - 3 = 0$ may be concurrent.
    Also find the point of concurrence.

12. Find the equations of the lines joining the point of intersection of the lines $x + 3y + 2 = 0$, $2x - y - 3 = 0$ to the (i) origin; (ii) point (3, 1).

13. Obtain the equation of the straight line which makes equal intercepts on the axes and passes through the point of intersection of the lines $2x - 3y + 1 = 0$ and $x + 2y = 2$.

14. P and Q are two points on the line $x - y + 1 = 0$ and are at distance 5 from the origin. Find the area of the triangle OPQ.

15. Obtain the equations of two lines passing through the intersection of the lines $4x - 3y - 1 = 0$ and $2x - 5y + 3 = 0$ and equally inclined to the axes.

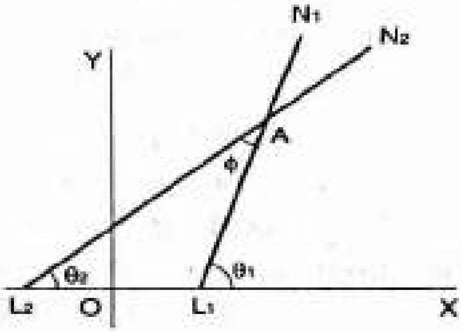
16. Find the equation of the straight line passing through the intersection of
    i) $x + 3y + 2 = 0,\ 2x - y - 3 = 0$ and slope $= \frac{1}{2}$
    ii) $3x - 4y - 13 = 0,\ 8x - 11y - 33 = 0$ and slope $= -\frac{2}{3}$

## *Answers*

1. i) $y = \sqrt{3}\,x + 3$ ii) $y + x + 3 = 0$ iii) $y = \pm x$
2. i) $x + y = 7$ ii) $x - y + 4 = 0$
3. $x + 2y = 8$
4. $4x + 3y = 24,\ x + y = 7$
5. $5y = 8x + 60$
6. $x - y + 2 = 0,\ 3x + 2y - 14 = 0,\ x - 6y + 2 = 0;$
   $x + 4y - 8 = 0,\ x - 2y + 2 = 0,\ 7x - 2y = 6.$
7. $k = \pm 18$
8. 3 : 4
9. $k = 3$
11. $a = 5;\ (1, -1)$
12. i) $x + y = 0$; ii) $x - y = 2$
13. $7x + 7y = 9$ 14. 3.5 sq. units
15. $x - y = 0,\ x + y = 2.$ 16. i) $x - 2y = 3$ ii) $2x + 3y = 37$

## 13.10 Angle between Two Lines

(i) Let $y = m_1 x + c_1$ and $y = m_2 x + c_2$ be the equations of two lines in slope intercept form. Let the lines be $N_1AL_1$ and $N_2AL_2$ meeting the x-axis at $L_1$ and $L_2$. Let $\angle AL_1X = \theta_1$ and $\angle AL_2X = \theta_2$




$\therefore \quad \tan\theta_1 = m_1$

$\tan\theta_2 = m_2$

If the angle between the lines be $ø$ $(=\angle L_1AL_2)$

$$ø = \theta_1 - \theta_2.$$

$$\therefore \quad \tan ø = \tan(\theta_1 - \theta_2)$$

$$= \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$$

$$= \left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$$

Hence $$ø = \tan^{-1}\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$$

If we consider the angle $L_1AN_2$ as the angle between the lines,

$$\angle L_1AN_2 = \pi - ø.$$

$$\therefore \quad \tan(L_1AN_2) = -\tan ø = -\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$$

$$\therefore \text{ Angle between the lines} = \tan^{-1}\left(\pm\frac{m_1 - m_2}{1 + m_1 m_2}\right)$$

Note : The positive value of $\tan ø$ gives the acute angle between the two lines and the negative value gives the obtuse angle.

*Condition of Perpendicularity.*

The two lines will be perpendicular to each other if $ø = 90°$.

$$\text{i.e. } \frac{1}{\tan ø} = 0$$

$$\text{or} \quad \frac{1 + m_1 m_2}{m_1 - m_2} = 0$$

$$\text{or} \quad 1 + m_1 m_2 = 0$$

$$\text{or} \quad m_1 . m_2 = -1$$

Thus if the product of the slopes $= -1$, the two lines will be at right angles.

*Condition of Parallelism*

(i) The two lines will be parallel to one another if $\phi = 0°$.

$$\text{i.e. } \tan\phi = 0$$

$$\text{or } \frac{m_1 - m_2}{1 + m_1 m_2} = 0$$

$$\text{or } m_1 = m_2$$

Thus if the slopes are equal, the two lines will be parallel to each other.

(ii) Let the equations of the lines be given in the general form

$$A_1x + B_1y + C_1 = 0 \text{ and } A_2x + B_2y + C_2 = 0 \qquad \text{(T.U. 2049)}$$

Their slopes are respectively $-\frac{A_1}{B_1}$ and $-\frac{A_2}{B_2}$ (Art. 10.5)

Hence the angle between them is given by

$$\phi = \tan^{-1}\left(\pm\frac{-\frac{A_1}{B_1} + \frac{A_2}{B_2}}{1 + \frac{A_1A_2}{B_1B_2}}\right)$$

$$= \tan^{-1}\left(\pm\frac{A_1B_2 - A_2B_1}{A_1A_2 + B_1B_2}\right).$$

**Perpendicular lines**

The two lines will be perpendicular to each other if $\phi = 90°$.

$$\text{or, } \frac{1}{\tan 90°} = 0 \quad \Rightarrow \quad \frac{A_1A_2 + B_1B_2}{A_1B_2 - A_2B_1} = 0$$

which gives $A_1A_2 + B_1B_2 = 0$

**Parallel lines**

The two lines will be parallel to one another if $\phi = 0°$.

$$\text{i.e. } \tan\phi = 0 \Rightarrow \frac{A_1B_2 - A_2B_1}{A_1A_2 + B_1B_2} = 0$$

$$\text{or, } A_1B_2 - A_2B_1 = 0$$

$$\text{i.e. } \frac{A_1}{A_2} = \frac{B_1}{B_2}$$

**Equation of any line parallel to the line $ax + by + c = 0$**

The given equation of the line is

$$ax + by + c = 0 \qquad \ldots\ldots \text{(i)}$$

Slope of the line (i) $= -\frac{a}{b}$

Let the equation of the line parallel to the line (i) be

$$y = mx + c \qquad \ldots\ldots \text{(ii)}$$

Slope of the line (ii) $= m$

Since (i) and (ii) are parallel, so

$$m = -\frac{a}{b}$$

Substituting the value of $m$ in (ii), we have

$$y = -\frac{a}{b}x + c$$

$$\Rightarrow \quad ax + by - bc = 0$$

which is in the form of $ax + by + k = 0$ where $k = -bc$

Thus, to get an equation of a line parallel to the given line, we simply change the constant term of the given equation.

## Equation of any line perpendicular to the line $ax + by + c = 0$

The given equation of the line is

$$ax + by + c = 0 \qquad \ldots\ldots \text{(i)}$$

and its slope $= -\frac{a}{b}$

Let the equation of the line perpendicular to the line (i) be

$$y = mx + c \qquad \ldots\ldots \text{(ii)}$$

whose slope $= m$

Since the lines (i) and (ii) are perpendicular to each other, so

$$m\left(-\frac{a}{b}\right) = -1$$

$$\therefore \qquad m = \frac{b}{a}$$

Substituting the value of $m$ in (ii), we have

$$y = \frac{b}{a}x + c$$

or, $\quad bx - ay + ac = 0$

which is in the form of $bx - ay + k = 0$ where $k = ac$.

Thus to get an equation of a line perpendicular to the given line, we interchange the coefficient of $x$ and $y$, change the sign of one of them and also the constant term.

## 13.11 Important Points of Concurrencies

The following are the four important definitions of point of concurrencies of three straight lines.

i) ***Orthocentre*** : The perpendiculars drawn from the vertices on their opposite sides meet at a point. This point is known as 'Orthocentre'.

ii) ***Circumcentre*** : The perpendicular bisectors of the sides of a triangle meet at a point. This point is known as 'circumcentre'.

iii) ***Incentre*** : The bisectors of the internal angles of a triangle meet at a point. This point is known as 'Incentre'.

iv) ***Centroid*** : The medians of a triangle meet at a point. This point is known as 'centroid'.

### Worked out examples

***Example 1.***

Find the acute angle between the lines

$$x - 3y - 6 = 0 \text{ and } y = 2x + 5.$$

***Solution.***

The slopes of the two lines are respectively $+\frac{1}{3}$ and 2.

If $\phi$ be the angle between them,

$$\tan\phi = \pm\frac{+\frac{1}{3} - 2}{1 + (+\frac{1}{3})(2)} = \pm\frac{-\frac{5}{3}}{\frac{5}{3}} = \pm 1$$

$$\phi = 45^\circ \text{ or } 135^\circ.$$

$\therefore$ the acute angle between them is $45^\circ$.

***Example 2***

Find the equation of the line through (4, 1) which is

(i) perpendicular (ii) parallel

to the line $x - 2y - 4 = 0$.

***Solution.***

i) Any line perpendicular to $x - 2y - 4 = 0$ is $2x + y + k = 0$

If it passes through (4, 1), then

$$2.4 + 1 + k = 0, \quad \text{i.e. } k = -9.$$

$\therefore$ the required line is $2x + y - 9 = 0$.

ii) Any line parallel to $x - 2y - 4 = 0$ is $x - 2y + k = 0$

If it passes through (4, 1),

$$4 - 2.1 + k = 0 \qquad \text{or, } k = -2.$$

$\therefore$ the required line is $x - 2y - 2 = 0$.

***Example 3***

Find the equation of a straight line through the intersection of the lines $2x - 3y + 1 = 0$ and $x + y - 2 = 0$ and parallel to the line joining the points (1, 2) and (–3, 5).

***Solution :***

Solving the two given equations

$$2x - 3y + 1 = 0 \qquad \text{and} \quad x + y - 2 = 0$$

we have $x = 1, y = 1$. So, the point of intersection of the two given lines is (1, 1).

The equation of the line through (1, 1) is

$$y - 1 = m(x - 1) \qquad \text{...... (i)}$$

where $m$ = slope of the line (i).

Also, $m_1$ = slope of the line through (1, 2) and (–3, 5)

$$= \frac{5 - 2}{-3 - 1} = -\frac{3}{4}$$

As the two lines are parallel, so

$$m = m_1$$

or, $m = -\frac{3}{4}$

Substituting the value of $m$ in (i), we have

$$y - 1 = -\frac{3}{4}(x - 1)$$

or, $3x + 4y = 7$

***Example 4***

Find the equation of a straight line drawn at right angles to the line $\frac{x}{a} + \frac{y}{b} = 1$ through the point where it meets the x-axis.

***Solution :***

The given line meets the x-axis at the point where $y = 0$

$\therefore \quad \frac{x}{a} = 1 \qquad$ i.e. $x = a$

$\therefore$ the line meets the x-axis at $(a, 0)$

The equation of a line perpendicular to $\frac{x}{a} + \frac{y}{b} = 1$ is

$$\frac{x}{b} - \frac{y}{a} = k \quad \ldots\ldots \text{(i)}$$

If the line (i) passes through $(a, 0)$, then

$$\frac{a}{b} = k$$

Subsituting the value of k in (i), we have

$$\frac{x}{b} - \frac{y}{a} = \frac{a}{b}$$

***Example 5***

Determine the equation of the straight lines through (1, –4) that make an angle of 45° with the straight line $2x + 3y + 7 = 0$. (HSEB 2053)

***Solution.***

Let m be the slope of the required line, then its equation through (1, –4) is

$$y + 4 = m(x - 1).$$

Slope of the given line $2x + 3y + 7 = 0$ is $-\frac{2}{3}$.

Hence $\tan 45° = \pm \dfrac{m + \frac{2}{3}}{1 - m \cdot \frac{2}{3}}$

or, $\dfrac{3m + 2}{3 - 2m} = \pm 1$

or, $3m + 2 = \pm (3 - 2m)$

or, $m = -5, \frac{1}{5}$.

Substituting these values of m, the required equations of the lines are

$$y + 4 = -5(x - 1),\ y + 4 = \frac{1}{5}(x - 1)$$

or, $y + 5x = 1$ and $5y - x + 21 = 0$.

***Example 6***

Find the ratio in which the perpendicular from (4, 1) to the line segment joining (2, –1) and (6, 1) divides the segment.

***Solution.***

The slope of the line segment joining (2, –1) and (6, 1) is

$$\frac{1 - (-1)}{6 - 2} = \frac{1}{2}.$$

Hence slope of any line perpendicular to the segment is $-2$. If $k : 1$ be the ratio in which the segment is divided by the line drawn perpendicular from the point (4, 1) to the line segment and using the section formula, the coordinates of the foot of the perpendicular are

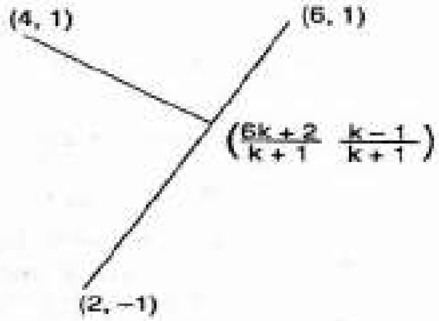
$$\frac{k.6 + 1.2}{k + 1} \quad \text{and} \quad \frac{k.1 + 1.(-1)}{k + 1}$$

i.e. $$\left(\frac{6k + 2}{k + 1}, \frac{k - 1}{k + 1}\right).$$

Hence, the slope of the line perpendicular to the line segment is also

$$\frac{\frac{k - 1}{k + 1} - 1}{\frac{6k + 2}{k + 1} - 4} = \frac{k - 1 - k - 1}{6k + 2 - 4k - 4}$$

$$= -\frac{1}{k - 1}$$




$\therefore \quad \frac{-1}{k - 1} = -2$

or, $\quad 2k - 2 = 1$

or, $\quad 2k = 3$

or $\quad k = \frac{3}{2}$

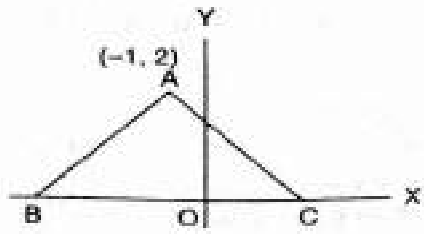
Hence the ratio is 3:2.

***Example 7***

Find the equation of the sides of an equilateral triangle whose vertex is (–1, 2) and base is $y = 0$. (T.U. 2056 S, 2058 S)

***Solution.***

Let ABC be an equilateral triangle whose base BC is on the x-axis i.e. $y = 0$ and the vertex A is at the point (–1, 2)




Here,

$\angle CBA = 60°$ and $\angle XCA = 120°$

$\therefore$ Slopes of the lines BA and CA are $\sqrt{3}$ and $-\sqrt{3}$ respectively.

Now, the equation of the side BA is

$y - 2 = \sqrt{3}(x + 1)$

or, $\quad y - 2 = \sqrt{3}x + \sqrt{3}$

$\therefore \quad \sqrt{3}x - y + 2 + \sqrt{3} = 0$

Again, the equation of the side CA is

$$y - 2 = -\sqrt{3}(x + 1)$$

or, $y - 2 = -\sqrt{3}x - \sqrt{3}$

$\therefore$ $\sqrt{3}x + y + \sqrt{3} - 2 = 0.$

***Example 8***

Prove that the equation of the straight line passing through the point $(a \cos^3\theta, a \sin^3\theta)$ and is parallel to the straight line $x \operatorname{cosec}\theta - y \sec\theta = a$ is $x \cos\theta - y \sin\theta = a \cos 2\theta$.

***Solution :***

The given equation of the line is

$$x \operatorname{cosec}\theta - y \sec\theta = a \qquad \text{...... (i)}$$

The equation of the line parallel to the line (i) is

$$x \operatorname{cosec}\theta - y \sec\theta + k = 0 \qquad \text{...... (ii)}$$

The line (ii) passes through the point $(a \cos^3\theta, a \sin^3\theta)$, so

$$a \cos^3\theta \frac{1}{\sin\theta} - a \sin^3\theta \frac{1}{\cos\theta} + k = 0$$

or, $\dfrac{a(\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta)}{\sin\theta \cos\theta} + k = 0$

or, $\dfrac{a \cos 2\theta}{\sin\theta \cos\theta} + k = 0$

$\therefore$ $k = -\dfrac{a \cos 2\theta}{\sin\theta \cos\theta}$

From (ii)

$$x \frac{1}{\sin\theta} - y \frac{1}{\cos\theta} - \frac{a \cos 2\theta}{\sin\theta \cos\theta} = 0$$

or, $x \cos\theta - y \sin\theta = a \cos 2\theta$

***Example 9***

The three sides of a triangle have the equations $7x - y + 11 = 0$, $x + y - 15 = 0$ and $7x + 17y + 65 = 0$. Find the equation of the straight line through a vertex and parallel to the opposite side with equation $x + y - 15 = 0$

***Solution :***

The given equations of the sides of the triangle are

$$7x - y + 11 = 0 \qquad \text{...... (i)}$$

$$x + y - 15 = 0 \qquad \text{...... (ii)}$$

$$7x + 17y + 65 = 0 \qquad \text{...... (iii)}$$

The equation of the line through the vertex which is the point of intersection of the lines (i) and (iii) is

$$7x - y + 11 + k(7x + 17y + 65) = 0 \qquad \text{...... (iv)}$$

where $k$ is a constant to be determined.

or, $(7 + 7k)x + (-1 + 17k)y + 11 + 65k = 0$

Slope of this line (iv) $= -\dfrac{7 + 7k}{-1 + 17k}$

Slope of the line (ii) (i.e. opposite side) $= -1$

Since the line (iv) is parallel to (ii), so

$$-\frac{7 + 7k}{-1 + 17k} = -1$$

or, $7 + 7k = -1 + 17k$

$\therefore \quad k = \dfrac{4}{5}$

From (iv), $7x - y + 11 + \dfrac{4}{5}(7x + 17y + 65) = 0$

or, $35x - 5y + 55 + 28x + 68y + 260 = 0$

$$x + y + 5 = 0$$

***Example 10***

Find the orthocentre of the triangle whose sides are $x + ay - a^2 = 0$, $x + by - b^2 = 0$, $x + cy - c^2 = 0$.

***Solution:***

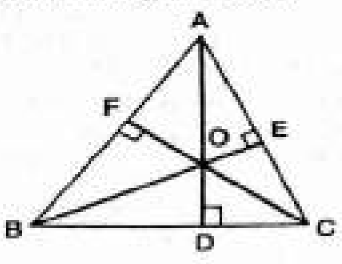
Let the equations of the sides BC, CA and AB of the triangle ABC be

$x + ay - a^2 = 0$ ...... (i)

$x + by - b^2 = 0$ ...... (ii)

$x + cy - c^2 = 0$ ...... (iii)

respectively.




To get the co-ordinates of A, we solve equations (ii) and (iii).

Solving equations (ii) and (iii) we have

$$x = -bc,\ y = b + c$$

$\therefore$ the co-ordinates of A $= (-bc, b + c)$

Similarly, the co-ordinates of B $= (-ca, c + a)$

and the co-ordinates of C $= (-ab, a + b)$

Let AD, BE and CF be the perpendiculars drawn from the vertices A, B and C on their opposite sides. These perpendiculars meet at O. Then O is the orthocentre.

The equation of AD perpendicular to BC is

$$ax - y + k = 0$$

This line passes through the vertex A, so

$$a(-bc) - (b + c) + k = 0$$

$\therefore \quad k = abc + b + c$

$\therefore$ the equation of AD is

$$ax - y + abc + b + c = 0 \qquad \text{...... (I)}$$

Similarly, the equations of BE and CF are

$$bx - y + abc + a + c = 0 \qquad \text{...... (II)}$$

$$cx - y + abc + a + b = 0 \qquad \text{...... (III)}$$

respectively.

Now for the orthocentre, we solve any two of the equations (I), (II) and (III) of the perpendiculars AD, BE and CF.

Solving equations (I) and (II), we have

$$x = 1,\ y = abc + a + b + c$$

$\therefore$ the orthocentre is $(1, abc + a + b + c)$

The equation (III) is also satisfied by the co-ordinates

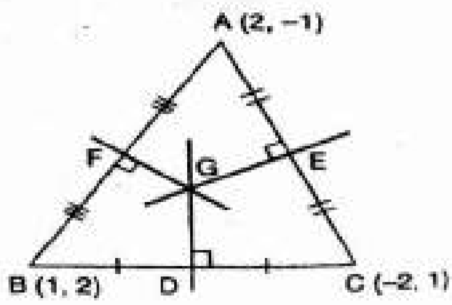
$$(1, abc + a + b + c).$$

***Example 11***

Find the circumcentre of the triangle where vertices are (2, –1), (1, 2) and (–2, 1)

***Solution:***

Let A (2, –1), B (1, 2) and C (–2, 1) be the vertices of the triangle ABC. Let GD, GE and GF be the perpendicular bisector of the sides BC, CA and AB of the triangle ABC, where G $(h, k)$ is the circumcentre of the triangle




$\therefore \quad$ D = mid-point of BC

$$= \left(\frac{1-2}{2}, \frac{2+1}{2}\right) = \left(-\frac{1}{2}, \frac{3}{2}\right)$$

and $\quad$ E = mid-point of CA

$$= \left(\frac{-2+2}{2}, \frac{1-1}{2}\right) = (0, 0)$$

Since GD $\perp$ BC,

slope of GD $\times$ slope of BC = –1

or $\dfrac{k - \frac{3}{2}}{h + \frac{1}{2}} \times \dfrac{1 - 2}{-2 - 1} = -1$

or $\dfrac{2k - 3}{2h + 1} \times \dfrac{-1}{-3} = -1$ i.e. $3h + k = 0$ ...... (1)

Also GE $\perp$ CA

or slope of GE $\times$ slope of CA $= -1$

or $\dfrac{k - 0}{h - 0} \times \dfrac{-1 - 1}{2 + 2} = -1$

or $\dfrac{k}{h} \times \dfrac{-2}{4} = -1$

i.e. $2h - k = 0$ ...... (2)

Solving (1) and (2) we get $h = 0$, $k = 0$

$\therefore$ the circumcentre is (0, 0)

## EXERCISE 13.2

1. **Find the angle between each pair of lines**
   a) $x - \sqrt{3}y = a$, $\sqrt{3}x - y = b$
   b) $3x + 2y = 7$, $2x - 3y = 5$
   c) $2x - 3y + 7 = 0$, $7x + 4y - 9 = 0$
2. Without finding the angles determine whether the following pair of lines are (i) perpendicular (ii) parallel or (iii) neither
   a) $x + y + 7 = 0$, $2x + 2y - 7 = 0$
   b) $x + y + 7 = 0$, $x - y + 7 = 0$
   c) $x + y + 7 = 0$, $2x + 3y + 7 = 0$
3. Find the equation of the
   a) line through the point (2, 3) and parallel to the straight line $4x - 3y = 10$
   b) line through (5, 4) and perpendicular to the line $4x - 3y = 10$
   c) line through the mid-point of the segment connecting (2, –4) and (2, 4) and parallel to the line $3x - 2y = 4$
   d) line parallel to the line $5x + 4y = 9$ and making an intercept $-5$ on the x-axis
   e) lines through the centroid of the triangle with vertices at (3, –4), (–2, 1), (5, 0) and
      i) parallel to the line $x - 3y = 4$ (T.U. 2049)
      ii) perpendicular to the line $x - 3y = 4$ (HSEB 2051)

f) line through the point (–4, –5) and perpendicular to the line joining the points (1, 2) and (5, 6)

g) line through the point that divides the join of (–3, –4) and (7, 1) in the ratio 3 : 2 and perpendicular to the join.

h) line through the point of intersection of the straight lines $2x - 3y + 4 = 0$ and $3x + 4y - 45 = 0$ and parallel to the straight line $2x + 3y = 5$.

i) line through the intersection of the lines $2x - 3y + 5 = 0$ and $7x + 4y = -3$ and perpendicular to the line $6x - 7y + 8 = 0$

j) lines through the point (3, 2) and making angle 45° with the line $x - 2y = 3$.

k) line through the vertex A of the triangle ABC and parallel to the opposite side, where the vertices are A(–5, 6), B(–1, –4) and C(3, 2). Also find the equation of the perpendicular bisector of BC.

4. Determine k so that the line $3x - ky - 8 = 0$ shall make an angle of 45° with the line $3x + 5y - 17 = 0$

5. Find the coordinates of the foot of the perpendicular from (6, 8) to the line through (1, 5) and (9, 3).

6. Find the coordinates of the feet of the perpendiculars from the origin to the lines $x + y - 4 = 0$ and $x + 5y - 26 = 0$. Find the equation of the straight line joining them.

Ex. 8 7. Prove that the equation of the straight line which passes through the point $(a \cos^3\theta, a \sin^3\theta)$ and is perpendicular to the straight line $x \sec\theta + y \operatorname{cosec}\theta = a$ is $x \cos\theta - y \sin\theta = a \cos 2\theta$. (T.U. 2053)

8. Find the equation of the sides of the right angled isosceles triangle whose vertex is (–2, –3) and whose base is $x = 0$.

9. Find the coordinates of the orthocentre and the circumcentre of the triangle whose vertices are (0, 1), (1, –2) and (2, –3).

10. The opposite corners A and C of a square have the coordinates (–2, 7) and (4, –3). Find the equation of the diagonal BD.

11. The equation of one diagonal of a square is $2x + 3y = 5$ and the coordinates of one vertex is (1, –3). Find the equations of two sides of the square which pass through the given vertex.

### *Answers*

1. a) 30° or 150° b) 90° c) $\tan^{-1}\left(\pm \frac{29}{2}\right)$
2. a) parallel b) perpendicular c) neither
3. a) $4x - 3y + 1 = 0$ b) $3x + 4y = 31$ c) $3x - 2y = 6$
   d) $5x + 4y + 25 = 0$ e) $x - 3y = 5$, $3x + y = 5$

f) $x + y + 9 = 0$  g) $2x + y = 5$  h) $2x + 3y = 33$
i) $7x + 6y + 1 = 0$  j) $3x - y = 7,\ x + 3y = 9$
k) $3x - 2y + 27 = 0,\ 2x + 3y + 1 = 0$

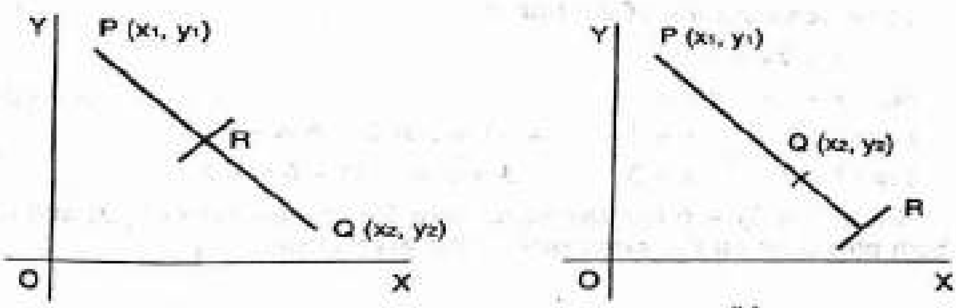
4. $k = 12$ or $-\frac{3}{4}$  5. (5, 4)  6. (2, 2), (1, 5); $3x + y - 8 = 0$

8. $x = y + 1,\ x + y + 5 = 0$  9. (−7, −6), (5, 1)

10. $3x - 5y + 7 = 0$  11. $x - 5y = 16,\ 5x + y = 2$

## 13.12 The Two Sides of a Line

Let P $(x_1, y_1)$ and Q$(x_2, y_2)$ be any two points and $Ax + By + C = 0$ a line.




(a) (b)

Join PQ. Let PQ (produced if necessary) meet the given line at R. Let PR:RQ = $m{:}n$, $m{:}n$ is positive if R divides PQ internally i.e. P and Q are on opposite sides of the line (Fig (a)); and $m{:}n$ is negative if R divides PQ externally, i.e. P and Q are on the same sides of the line (Fig. (b)).

From the section formula coordinates of R are given by

$$\left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}\right)$$

But R lies on the line $Ax + By + C = 0$, hence the coordinates of R must satisfy the equation of the line.

i.e. $$A\left(\frac{mx_2 + nx_1}{m + n}\right) + B\left(\frac{my_2 + ny_1}{m + n}\right) + C = 0$$

or $$m(Ax_2 + By_2 + C) + n\,(Ax_1 + By_1 + C) = 0$$

or $$\frac{Ax_1 + By_1 + C}{Ax_2 + By_2 + C} = -\frac{m}{n}$$

Case (i) If the points P and Q are on opposite sides of the line, $\frac{m}{n}$ is positive, hence $Ax_1 + By_1 + C$ and $Ax_2 + By_2 + C$ are of opposite signs.

Case (ii) If the point P and Q are on the same side of the line, $\frac{m}{n}$ is negative, hence $Ax_1 + By_1 + C$ and $Ax_2 + By_2 + C$ are of same sign.

Note. Any given point $(x', y')$ and the origin are on the same side of the line $Ax + By + C = 0$ if $Ax' + By' + C$ and C have the same sign. They are on opposite sides if they are of opposite signs.

***Example***

Are the points $(-1, 2)$ and $(3, -2)$ lie on the same side of the line with equation $x + 3y = 6$

***Solution :***

The given equation of the line is

$$x + 3y = 6$$

or, $x + 3y - 6 = 0$

For $(-1, 2)$ : $x + 3y - 6 = -1 + 3 \times 2 - 6 = -1$

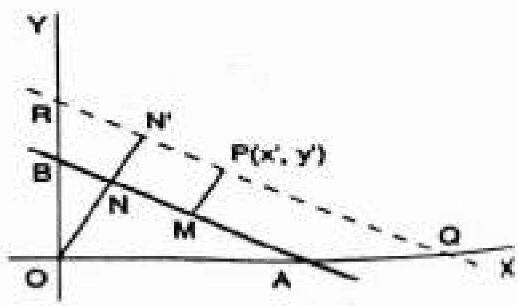
For $(3, -2)$ : $x + 3y - 6 = 3 + 3 \times (-2) - 6 = -9$

Since, $x + 3y - 6$ has the same sign for the points $(-1, 2)$ and $(3, -2)$, so both points lie on the same side of the given line.

## 13.13 Length of the Perpendicular from a Point on a Straight Line $x \cos \alpha + y \sin \alpha = p$

I) Let the equation of any line AB be $x \cos \alpha + y \sin \alpha = p$ so that the length of the perpendicular from the origin on the line is $p$, i.e. ON $= p$ and $\angle AON = \alpha$.

Let $P(x', y')$ be the point from which PM is drawn perpendicular to AB.




Draw a line QR through $P(x', y')$ parallel to the given line AB. Let the perpendicular from O to this line be ON' $(=p')$. Then PM $= p' - p$ or $p - p'$ according as $p' > p$ or $p' < p$, i.e. PM $= \pm (p' - p)$, the proper sign is taken so as to make PM positive. Now equation of QR is

$$x \cos \alpha + y \sin \alpha = p'.$$

Since P lies on this line, so

$$x' \cos \alpha + y' \sin \alpha = p'$$

$$\therefore \quad PM = \pm (p' - p)$$

$$= \pm (x' \cos \alpha + y' \sin \alpha - p).$$

ii) Length of the perpendicular drawn from the point $(x', y')$ on the line whose equation is $Ax + By + C = 0$

Firstly, we express the equation $Ax + By + C = 0$ into normal form

If $C > 0$,

$$p = \frac{C}{\sqrt{A^2 + B^2}}, \cos \alpha = \frac{-A}{\sqrt{A^2 + B^2}} \text{ and } \sin \alpha = \frac{-B}{\sqrt{A^2 + B^2}}$$

If $C < 0$,

$$p = -\frac{C}{\sqrt{A^2 + B^2}}, \cos \alpha = \frac{A}{\sqrt{A^2 + B^2}} \text{ and } \sin \alpha = \frac{B}{\sqrt{A^2 + B^2}}$$

(By Art. 13.6 (iii))

For both cases, the length of the perpendicular drawn from the point $(x_1, y_1)$ on the line

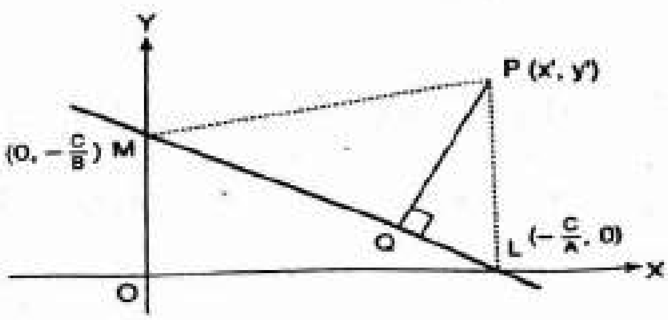
$x \cos \alpha + y \sin \alpha = p$ is

$$= |x' \cos \alpha + y' \sin \alpha - p| = \left| \frac{Ax' + By' + C}{\sqrt{A^2 + B^2}} \right|$$

*Alternatively,*

The given line $Ax + By + C = 0$ can be reduced to the double intercept form

$$\frac{x}{\left(-\frac{C}{A}\right)} + \frac{y}{\left(-\frac{C}{B}\right)} = 1.$$



Let the given line meet the axes in L and M so that the coordinates of L and M are respectively $\left(-\frac{C}{A}, 0\right)$ and $\left(0, -\frac{C}{B}\right)$.

Let PQ be the perpendicular from $P(x', y')$ on LM. Join LP and MP.

$\therefore$ area of $\Delta$LMP

$$= \frac{1}{2}\left|x'\left(0 + \frac{C}{B}\right) + \frac{-C}{A}\left(-\frac{C}{B} - y'\right) + 0\,(y' - 0)\right|$$

$$= \frac{1}{2}\left|\frac{Cx'}{B} + \frac{C^2}{AB} + \frac{Cy'}{A}\right|$$

$$= \frac{1}{2}\left|\frac{C}{AB}(Ax' + By' + C)\right|$$

$$= \frac{1}{2}\left|\frac{C}{AB}\right| |Ax' + By' + C|$$

Also area of $\Delta LMP = \frac{1}{2}$ LM.PQ

$$= \frac{1}{2}\sqrt{\left(0 + \frac{C}{A}\right)^2 + \left(\frac{-C}{B} - 0\right)^2}\ .\ PQ$$

$$= \frac{1}{2}\sqrt{\frac{C^2(A^2 + B^2)}{A^2B^2}}\ .\ PQ$$

$$= \frac{1}{2}\left|\frac{C}{AB}\right|\sqrt{A^2 + B^2}\ .\ PQ$$

$$\therefore\ \frac{1}{2}\left|\frac{C}{AB}\right|\sqrt{A^2 + B^2}\ .\ PQ = \frac{1}{2}\left|\frac{C}{AB}\right| |Ax' + By' + C|$$

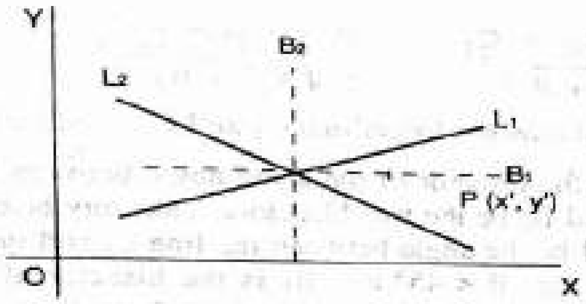
$$\text{or}\quad PQ = \frac{|Ax' + By' + C|}{\sqrt{A^2 + B^2}} = \left|\frac{Ax' + By' + C}{\sqrt{A^2 + B^2}}\right|$$

$\therefore$ the length of the perpendicular from $(x'\ y')$ on the line $Ax + By + C = 0$ is equal to $\left|\frac{Ax' + By' + C}{\sqrt{A^2 + B^2}}\right|$.

## 13.14 Bisectors of the Angles between Two Lines

Let $L_1 : A_1x + B_1y + C_1 = 0$ and $L_2: A_2x + B_2y + C_2 = 0$ be two given lines meeting at M. Let $B_1$ and $B_2$ be the lines bisecting the angles between the lines $L_1$ and $L_2$.

Now we shall proceed to determine the equation of the two bisectors $B_1$ and $B_2$ of the angles between $L_1$ and $L_2$.




We know that any point on a bisector is equidistant from the two lines. Let $P(x', y')$ be any point in any of the bisectors.

Distance of $L_1$ from $P = \pm\dfrac{A_1x' + B_1y' + C_1}{\sqrt{A_1^2 + B_1^2}}$

Distance of $L_2$ from $P = \pm\dfrac{A_2x' + B_2y' + C_2}{\sqrt{A_2^2 + B_2^2}}$

$$\therefore \quad \pm\frac{A_1x' + B_1y' + C_1}{\sqrt{A_1^2 + B_1^2}} = \pm\frac{A_2x' + B_2y' + C_2}{\sqrt{A_2^2 + B_2^2}}$$

or, $$\frac{A_1x' + B_1y' + C_1}{\sqrt{A_1^2 + B_1^2}} = \pm\frac{A_2x' + B_2y' + C_2}{\sqrt{A_2^2 + B_2^2}}$$

Since these relations are true for any point $(x', y')$ on the bisectors, so the equations of the bisectors are

$$\frac{A_1x + B_1y + C_1}{\sqrt{A_1^2 + B_1^2}} = \pm\frac{A_2x + B_2y + C_2}{\sqrt{A_2^2 + B_2^2}}$$

Note. 1 To identify the bisector of the angle in which the origin lies:

In $L_1 : A_1x + B_1y + C_1 = 0$ and $L_2 : A_2x + B_2y + C_2 = 0$,

Let the equations be taken so that $C_1$ and $C_2$ are both positive. Hence the coordinates of the origin when substituted in the equations will make both $A_1x + B_1y + C_1$ and $A_2x + B_2y + C_2$ positive. If Q be any point on the bisector containing the origin then the coordinates of Q, make $A_1x + B_1y + C_1$ positive because both the points O and Q lie on the same side of the line $L_1$. Similarly, $A_2x + B_2y + C_2$ must also be positive. Hence the equation of the bisector of an angle which contains the origin is

$$\frac{A_1x + B_1y + C_1}{\sqrt{A_1^2 + B_1^2}} = +\frac{A_2x + B_2y + C_2}{\sqrt{A_2^2 + B_2^2}}$$

The equation of the bisector of the angle in which the origin does not lie is, therefore,

$$\frac{A_1x + B_1y + C_1}{\sqrt{A_1^2 + B_1^2}} = -\frac{A_2x + B_2y + C_2}{\sqrt{A_2^2 + B_2^2}}$$

(To apply this let us not forget that $C_1$ and $C_2$ must both be positve).

Note. 2 To identify the bisector of the acute angle between the lines $L_1$ and $L_2$: Let $B_1$ and $B_2$ be the two bisectors. Take any bisector $B_1$ and any line $L_1$. Let $\theta$ be the angle between the line $L_1$ and the bisector $B_1$. If $|\tan\theta| < 1$, then $\theta < 45°$ and $B_1$ is the bisector of an acute angle between the lines $L_1$ and $L_2$. On the other hand if $|\tan\theta| > 1$, then $\theta > 45°$ and $B_1$ bisects the obtuse angle between the lines $L_1$ and $L_2$.

## Worked out Examples

***Example 1.***

Prove that two of the vertices of the triangle formed by the lines $y - x = 0$, $2y - x = 0$ and $y = 1$ lie on one side and the third vertex lies on the other side of the line $2x + 3y - 4 = 0$.

***Solution.***

The equations of the sides of the given triangle are $y - x = 0$, $2y - x = 0$, $y = 1$.

Hence the vertices of the triangle are A(0, 0), B(1, 1) and C(2, 1).

(Obtained by solving the equations taken two at a time).

Substituting the coordinates of these points in the expression

$$2x + 3y - 4 = 0$$

We have

for A(0, 0), $0 + 0 - 4 = -4$, –ve

for B(1, 1), $2.1 + 3.1 - 4 = 1$, +ve

for C (2, 1), $2.2 + 3.1 - 4 = 3$, + ve

Hence B and C lie on one side and A lies on the other side of the line

$$2x + 3y - 4 = 0.$$

***Example 2.***

Find the length of the perpendicular from the point (1, 1) to the line joining (0, 4) to the point of intersection of $3x + 4y - 9 = 0$ with the x-axis.

***Solution.***

The equation of x-axis is $y = 0$.

The line $3x + 4y - 9 = 0$ cuts the x-axis ($y = 0$) at (3, 0).

The equation of the line joining (0, 4) and (3, 0) is

$$y - 4 = \frac{0-4}{3-0}(x-0)$$

or, $4x + 3y - 12 = 0$

Length of perpendicular from (1, 1) to this line

$$= -\frac{4.1 + 3.1 - 12}{\sqrt{4^2 + 3^2}} = \frac{5}{5} = 1.$$

***Example 3.***

Find the distance between the parallel lines $y = 2x + 4$ and $6x - 3y = 5$.

***Solution.***

Any point on the line $y = 2x + 4$ is (0, 4) which is obtained by putting $x = 0$. Length of perpendicular from (0, 4) on the line $6x - 3y = 5$ is

$$= -\frac{6.0 - 3.4 - 5}{\sqrt{6^2 + 3^2}} = \frac{17}{\sqrt{45}}.$$

***Example 4.***

Determine the equations of the bisectors of the angles between the lines $3x - 2y + 1 = 0$ and $18x + y - 5 = 0$. Identify the bisector of the acute angle.

***Solution.***

The equations of the lines are $3x - 2y + 1 = 0$ and $-18x - y + 5 = 0$.

The equations of the bisectors are

$$\frac{3x - 2y + 1}{\sqrt{3^2 + 2^2}} = \pm\frac{-18x - y + 5}{\sqrt{18^2 + 1^2}}$$

or, $$\frac{3x - 2y + 1}{\sqrt{13}} = \pm\frac{-18x - y + 5}{\sqrt{325}}$$

or, $$3x - 2y + 1 = \pm\frac{-18x - y + 5}{5}$$

or, $15x - 10y + 5 = \pm(-18x - y + 5)$.

Taking positive sign, the equation of the bisector is

$15x - 10y + 5 = -18x - y + 5$

or, $33x - 9y = 0$

$11x - 3y = 0$

Again taking negative sign, the equation of the other bisector is

$15x - 10y + 5 = 18x + y - 5$

or, $3x + 11y - 10 = 0$

Now, to see which is the bisector of an acute angle, we find the angle between one of the bisectors and any one of the given lines.

Slope of the bisector $11x - 3y = 0$ is $\frac{11}{3}$

Slope of the line $3x - 2y + 1 = 0$ is $\frac{3}{2}$

Let $\theta$ be the angle between the bisector $11x - 3y = 0$ and the line $3x - 2y + 1 = 0$, then

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{\frac{11}{3} - \frac{3}{2}}{1 + \frac{11}{3}\cdot\frac{3}{2}}$$

or, $\tan\theta = \frac{1}{3} < 1$

$\therefore \quad \theta < 45°$

$\therefore$ $11x - 3y = 0$ is the bisector of an acute angle.

***Example 5.***

Find the equations to the bisectors of the angles between the straight lines $x = y$ and $x + y = 1$. Identify the bisector of the angle which includes the point $(\frac{1}{2}, 0)$.

***Solution.***

The equations of the bisectors are

$$\frac{x - y}{\sqrt{2}} = \pm\frac{-x - y + 1}{\sqrt{2}}$$

Now taking positive sign,

$$x - y = -x - y + 1$$

$\therefore \quad 2x - 1 = 0.$

Again taking negative sign,

$$x - y = x + y - 1$$

$\therefore \quad 2y - 1 = 0$

Since $2x - 1 = 0$ is satisfied by $x = \frac{1}{2}$ and $y = 0$, so $2x - 1 = 0$ is the equation of the bisector of an angle which includes the point $(\frac{1}{2}, 0)$.

***Example 6***

If $p$ is the length of the perpendicular dropped from the point $(a, b)$ on the line $\frac{x}{a} + \frac{y}{b} = 1$, prove that

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2}$$

***Solution :***

The given equation of the line is

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad \text{...... (i)}$$

$p$ = length of the perpendicular dropped from the point $(a, b)$ on the line (i)

$$= \frac{1+1-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}} = \frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}$$

or, $p^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right) = 1$

$\therefore \quad \frac{1}{a^2}+\frac{1}{b^2} = \frac{1}{p^2}$

**Example 7**

Find the points on the y-axis each of whose perpendicular distance from the line $ax + by + c = 0$ is $b$.

**Solution :**

The given equation of the line is

$$ax + by + c = 0 \quad \text{...... (i)}$$

Let $(0, y')$ be a point on y-axis such that its perpendicualr distance from the line (i) is $b$.

Now, $b = \pm\frac{a.0 + by' + c}{\sqrt{a^2+b^2}}$

or, $by' + c = \pm b\sqrt{a^2+b^2}$

$by' = -c \pm b\sqrt{a^2+b^2}$

$\therefore \quad y' = \frac{1}{b}\left\{-c \pm b\sqrt{a^2+b^2}\right\}$

$\therefore$ the required points on y-axis = $\left(0, \frac{1}{b}\left\{-c \pm b\sqrt{a^2+b^2}\right\}\right)$

**Example 8**

If $p_1$ and $p_2$ are the lengths of the perpendicular drawn from the points $(\cos\theta, \sin\theta)$ and $(-\sec\theta, \operatorname{cosec}\theta)$ on the line $x\sec\theta + y\operatorname{cosec}\theta = 0$ respectively, prove that

$$\frac{4}{{p_1}^2} - {p_2}^2 = 4.$$

**Solution :**

The given equation of the line is

$$x\sec\theta + y\operatorname{cosec}\theta = 0 \quad \text{...... (i)}$$

$p_1$ = length of the perpendicular from $(\cos\theta, \sin\theta)$ on the line (i)

$$= \pm\frac{\cos\theta\sec\theta + \sin\theta\operatorname{cosec}\theta}{\sqrt{\sec^2\theta + \operatorname{cosec}^2\theta}}$$

$$= \pm 2\sin\theta\cos\theta = \pm\sin 2\theta$$

Again, $p_2$ = length of the perpendicular from $(-\sec\theta, \operatorname{cosec}\theta)$ on the line (i)

$$= \pm \frac{-\sec^2\theta + \operatorname{cosec}^2\theta}{\sqrt{\sec^2\theta + \operatorname{cosec}^2\theta}} = \pm \frac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta}$$

$$= \pm \frac{2\cos 2\theta}{\sin 2\theta} = \pm 2\cot 2\theta$$

Now, $\frac{4}{p_1^2} - p_2^2 = \frac{4}{\sin^2 2\theta} - 4\cot^2 2\theta$

$$= 4(\operatorname{cosec}^2 2\theta - \cot^2 2\theta)$$

$$= 4$$

## EXERCISE 13.3

1. Find the lengths of perpendiculars drawn from
   a) (0, 0) to the line $3x + y + 1 = 0$
   b) (–3, 0) to the line $3x + 4y + 7 = 0$
   c) (2, 3) to the line $8x + 15y + 24 = 0$
2. If $p$ is the length of the perpendicular dropped from the origin on the line $\frac{x}{a} + \frac{y}{b} = 1$, prove that $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2}$

   (T.U. 2048 H), (HSEB 2055)
3. Find the distance between the parallel lines
   a) $3x + 5y = 11$ and $3x + 5y = -23$.
   b) $2x - 5y = 6$ and $6x - 15y + 11 = 0$
4. Find the equations of the bisectors of the angles between the lines
   a) $3x - 4y + 2 = 0$ and $5x + 12y + 5 = 0$
   b) $3x - 4y + 6 = 0$ and $5x + 12y + 10 = 0$
   c) $x - 2y = 0$ and $2y - 11x = 6$
5. a) Find the equations of the bisectors of the angles between the lines $y = x$ and $y = 7x + 4$. Identify the bisector of the acute angle.

   b) Find the equations of the bisectors of the angles between the lines $4x - 3y + 1 = 0$ and $12x - 5y + 7 = 0$ and prove that the bisectors are at right angles to each other. Also identify the bisector of the angle between the lines containing the origin.

6. a) The length of the perpendicular drawn from the point $(a, 3)$ on the line $3x + 4y + 5 = 0$ is 4. Find the value of a.

   b) What are the points on the axis of x whose perpendicular distance from the straight line $\frac{x}{a} + \frac{y}{b} = 1$ is $a$ ?

7. Show that two of the three points (0, 0), (2, 3) and (3, 4) lie on one side and the remaining on the other side of the line $x - 3y + 3 = 0$.

8. a) Determine the equation and length of the altitude drawn from the vertex A to the opposite side of the triangle whose vertices are A(0, 1), B(1, 3), C(4, –2).

   b) Find the equations of two straight lines each of which is parallel to and at a distance of $\sqrt{5}$ from the line $x + 2y - 7 = 0$.

   c) Find the equations of the two straight lines drawn through the point (0, a) on which the perpendiculars drawn from the point $(2a, 2a)$ are each of length a.

   d) Find the equation of the line which is at right angles to $3x + 4y = 12$, such that its perpendicular distance from the origin is equal to the length of the perpendicular from (3, 2) on the given line.

   e) The equation of the diagonal of a parallelogram is $3y = 5x + k$. The two opposite vertices of a parallelogram are the points (1, –2) and (–2, 1). Find the value of k.

9. If p and p' be the length of the perpendiculars from the origin upon the straight line whose equations are

$$x \sec\theta + y \operatorname{cosec}\theta = a \quad \text{and} \quad x\cos\theta - y\sin\theta = a\cos 2\theta,$$

prove that $4p^2 + p'^2 = a^2$. (T.U. 2051, 2055 S)

10. Show that the product of the perpendiculars drawn from the two points $(\pm\sqrt{a^2 - b^2}, 0)$ upon the line $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ is $b^2$.

11. The origin is a corner of a square and two of its sides are $y + 2x = 0$ and $y + 2x = 3$. Find the equation of the other two sides.

### *Answers*

1. a) $\frac{1}{\sqrt{10}}$ b) $\frac{2}{5}$ c) 5 3. a) $\sqrt{34}$ b) $\frac{29}{\sqrt{261}}$

4. a) $14x - 112y + 1 = 0$; $64x + 8y + 51 = 0$
   b) $x - 8y + 2 = 0$; $8x + y + 16 = 0$
   c) $8x - 6y + 3 = 0$, $3x + 4y + 3 = 0$

5. a) $6x - 3y + 2 = 0$ (acute angle bisector), $x + 2y + 2 = 0$
   b) $4x + 7y + 11 = 0$ (containing the origin), $7x - 4y + 3 = 0$
6. a) 1 or $\frac{-37}{3}$ b) $\left(\frac{a}{b}\left(b \pm \sqrt{a^2 + b^2}\right), 0\right)$
8. a) $3x - 5y + 5 = 0, \frac{11}{\sqrt{5}}$ b) $x + 2y - 2 = 0, x + 2y - 12 = 0$
   c) $y = a, 4x - 3y + 3a = 0$ d) $4x - 3y = \pm 5$ e) 1
11. $x - 2y = 0, 2y - x = \pm 3$

# CHAPTER 14
# Pair of Straight Lines

## 14.1 Introduction

Let $\quad A_1x + B_1y + C_1 = 0$ ......(1)

and $\quad A_2x + B_2y + C_2 = 0$ ......(2)

be the equations of two straight lines.

Consider the equation

$$(A_1x + B_1y + C_1)(A_2x + B_2y + C_2) = 0 \quad ......(3)$$

The coordinates of any point lying in straight line (1) will satisfy (3) also. Hence all points on the straight line (1) lie on (3). Similarly all points lying on the straight line (2) lie on (3).

Conversely, the coordinates of any point lying on (3) will satisfy (1) or (2) or both.

Hence equation (3) represents a pair of the straight lines given by (1) and (2). If the left hand side of equation (3) be expanded we get an equation of second degree in $x$ and $y$, i.e. an equation of the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

Thus we see that the equation of a pair of lines is a second degree equation. But the converse is not always true.As we shall see later, equations of second degree will represent a pair of straight lines only if the left hand side can be resolved into two linear factors.

### Homogeneous Equation

An equation with two variables $x$ and $y$ in which each term has the same degree is known as a homogeneous equation. If the degree of each term is 2, then the equation is known as the homogeneous equation of degree two or a second degree homogeneous equation. The most general form of a homogeneous equation of degree two is

$$ax^2 + 2hxy + by^2 = 0.$$

## 14.2 The Homogeneous Equation of Second Degree $ax^2 + 2hxy + by^2 = 0$, always Represents a Pair of Straight Lines through the Origin.

The homogeneous equation of second degree in x and y is

$$ax^2 + 2hxy + by^2 = 0.$$

The equation can be written as

$$y^2 + \frac{2h}{b}\,xy + \frac{a}{b}\,x^2 = 0, \qquad \text{if } b \neq 0$$

or, $\left(\frac{y}{x}\right)^2 + \frac{2h}{b}\left(\frac{y}{x}\right) + \frac{a}{b} = 0.$

This is quadratic in $\frac{y}{x}$.

Hence $\frac{y}{x}$ will have two values, $m_1$ and $m_2$, say

$\therefore \quad \frac{y}{x} = m_1 \quad$ and $\quad \frac{y}{x} = m_2$

or $\quad y = m_1x \quad$ and $\quad y = m_2x$

But these are straight lines through the origin.

If $b = 0$, the given equation becomes

$$ax^2 + 2hxy = 0$$

$$\therefore \quad x(ax + 2hy) = 0.$$

which represents two straight lines through the origin *viz*, $x = 0$ and $ax + 2hy = 0$.

Hence in all cases the homogeneous equation will represent two straight lines through the origin.

**Note:** Since $m_1$ and $m_2$ are two values of $\frac{y}{x}$ of $\left(\frac{y}{x}\right)^2 + \frac{2h}{b}\left(\frac{y}{x}\right) + \frac{a}{b} = 0$ we have, $m_1 + m_2 = -\frac{2h}{b}$ and $m_1m_2 = \frac{a}{b}$.

## 14.3 Angle between the line pair represented by $ax^2 + 2hxy + by^2 = 0$

Let $\quad y = m_1x \quad \ldots\ldots$ (i)

and $\quad y = m_2x \quad \ldots\ldots$ (ii)

be the equations of two lines represented by the equation

$$ax^2 + 2hxy + by^2 = 0$$

where $m_1 + m_2 = -\frac{2h}{b}$ and $m_1m_2 = \frac{a}{b}$.

Slopes of the lines (i) and (ii) are $m_1$ and $m_2$ respectively.

If $\theta$ be the angle between the lines (i) and (ii), then

$$\tan\theta = \pm\frac{m_1 - m_2}{1 + m_1m_2}$$

$$= \pm\frac{\sqrt{(m_1+m_2)^2 - 4m_1m_2}}{1+m_1m_2}$$

$$= \pm\frac{\sqrt{\left(-\frac{2h}{b}\right)^2 - 4.\frac{a}{b}}}{1+\frac{a}{b}}$$

$$= \pm\frac{2\sqrt{h^2 - ab}}{a+b}$$

$$\therefore \quad \theta = \tan^{-1}\left(\pm\frac{2\sqrt{h^2-ab}}{a+b}\right)$$

## Condition of perpendicularity

Two lines will be perpendicular to each other if $\theta = 90°$.

So, $\cot 90° = \dfrac{a+b}{2\sqrt{a^2\ ab}}$

or, $0 = \dfrac{a+b}{2\sqrt{h^2-ab}}$

$\therefore \quad a + b = 0$

i.e. coeff. of $x^2$ + coeff. of $y^2 = 0$

## Condition for coincident lines

Two lines will be coincident if $\theta = 0°$

So, $\tan 0° = \dfrac{2\sqrt{h^2-ab}}{a+b}$

or, $0 = h^2 - ab$

$\therefore \quad h^2 = ab$

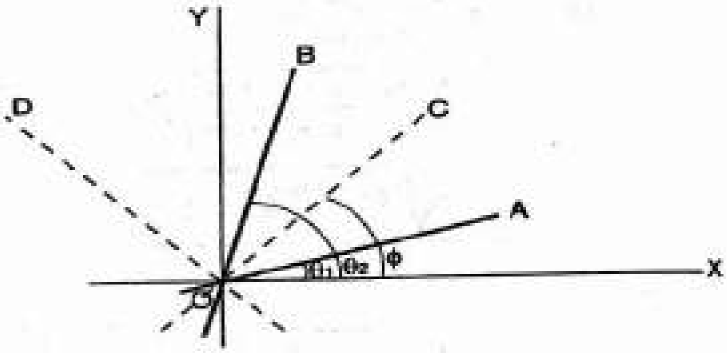
**Note:** The homogeneous equation of degree two i.e. $ax^2 + 2hxy + by^2 = 0$ represents

i) two real and distinct lines if $h^2 - ab > 0$

ii) two real and coincident lines if $h^2 - ab = 0$

iii) two imaginary lines if $h^2 - ab < 0$.

## 14.4 Bisectors of the Angles between the Pair of Lines represented by $ax^2 + 2bxy + by^2 = 0$. (T.U. 2058 S)




Let $y = m_1 x$ and $y = m_2 x$ be the two lines through origin represented by $ax^2 + 2hxy + by^2 = 0$, so that $m_1 + m_2 = -\frac{2h}{b}$ and $m_1 m_2 = \frac{a}{b}$. Let OA and OB be the two lines as in the figure and let OC and OD be the bisectors of the angles between them. If OA and OB make angles $\theta_1$ and $\theta_2$ with OX, tan $\theta_1 = m_1$ and $\tan \theta_2 = m_2$.

Let OC make an angle $\phi$ with OX.

From the figure, we have

$$\angle XOC = \angle XOA + \angle AOC$$

$$\phi = \theta_1 + \tfrac{1}{2}(\theta_2 - \theta_1) = \tfrac{1}{2}(\theta_1 + \theta_2)$$

or, $\quad 2\phi = \theta_1 + \theta_2$

or, $\quad \tan 2\phi = \tan(\theta_1 + \theta_2)$

i.e. $\tan(2\angle XOC) = \tan 2\phi = \tan(\theta_1 + \theta_2)$

Also $\quad \angle XOD = \angle XOC + 90° = \phi + 90°$ and

$\tan(2\angle XOD) = \tan(2\phi + 180°) = \tan 2\phi$.

Hence if $(x, y)$ be any point in OC, or OD, we have tan $\angle XOC$ or tan $\angle XOD = \frac{y}{x}$, and $\tan 2\angle XOD = \tan 2\angle XOC = \tan 2\phi$.

But $\quad \tan 2\phi = \tan(\theta_1 + \theta_2)$

$$\therefore \quad \frac{2\tan\phi}{1 - \tan^2\phi} = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2}$$

$$\text{or,} \quad \frac{2\frac{y}{x}}{1 - \left(\frac{y}{x}\right)^2} = \frac{m_1 + m_2}{1 - m_1 m_2}$$

$$\text{or,} \quad \frac{2\frac{y}{x}}{1 - \frac{y^2}{x^2}} = \frac{-\frac{2h}{b}}{1 - \frac{a}{b}}$$

or, $\quad h(x^2 - y^2) = (a - b)xy$.

Since this is true for any point $(x, y)$ in OC or OD, it represents the two bisectors.

*Alternative Method:*

Let the two straight lines represented by the equation

$$ax^2 + 2hxy + by^2 = 0$$

be $y = m_1x$ and $y = m_2x$

so that $m_1 + m_2 = -\dfrac{2h}{b}$ and $m_1m_2 = \dfrac{a}{b}$.

The equations of the bisectors of the angles between the lines

$$y - m_1x = 0 \quad \text{and} \quad y - m_2x = 0$$

are
$$\frac{y - m_1x}{\sqrt{1 + m_1^2}} = \pm\frac{y - m_2x}{\sqrt{1 + m_2^2}}.$$

Their combined equation is

$$\left(\frac{y - m_1x}{\sqrt{1 + m_1^2}} + \frac{y - m_2x}{\sqrt{1 + m_2^2}}\right)\left(\frac{y - m_1x}{\sqrt{1 + m_1^2}} - \frac{y - m_2x}{\sqrt{1 + m_2^2}}\right) = 0$$

or $$\left(\frac{y - m_1x}{\sqrt{1 + m_1^2}}\right)^2 - \left(\frac{y - m_2x}{\sqrt{1 + m_2^2}}\right)^2 = 0$$

or $$(1 + m_2^2)(y^2 - 2m_1xy + m_1^2x^2) - (1 + m_1)(y^2 - 2m_2xy + m_2^2x^2) = 0$$

or $$x^2 (m_1^2 - m_2^2) - 2xy (m_1 - m_2) (1 - m_1m_2) - y^2 (m_1^2 - m_2^2) = 0$$

or $$(x^2 - y^2) (m_1^2 - m_2^2) = 2xy (m_1 - m_2) (1 - m_1m_2)$$

or $$x^2 - y^2 = 2xy\frac{1 - m_1m_2}{m_1 + m_2}$$

or $$x^2 - y^2 = 2xy\frac{1 - \frac{a}{b}}{-\frac{2h}{b}}$$

or $$x^2 - y^2 = 2xy\frac{(a - b)}{2h}$$

or $$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

or $$h(x^2 - y^2) = (a - b)xy$$

which is the required equation.

## 14.5 Condition that the General Equation of Second Degree may Represent a Line Pair

The general equation of second degree in $x$ and $y$ is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad (a \neq 0).$$

The equation may be written as a quadratic in $x$, viz.

$$ax^2 + 2x\,(hy + g) + by^2 + 2fy + c = 0$$

Solving for x,

$$x = \frac{1}{a}\left\{-(hy + g) \pm \sqrt{(hy + g)^2 - a(by^2 + 2fy + c)}\right\}.$$

These equations will be linear only if $(hy + g)^2 - a(by^2 + 2fy + c)$ be a perfect square

i.e. $(h^2 - ab)\,y^2 + 2(gh - af)\,y + (g^2 - ac)$ be a perfect square.

i.e. $(gh - af)^2 - (h^2 - ab)\,(g^2 - ac) = 0.$

i.e. $a(abc + 2fgh - af^2 - bg^2 - ch^2) = 0$

Since $a \neq 0$, the required conditions is

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$

If $a = 0$ but $b \neq 0$, then the given equation can be expressed as a quadratic in $y$ and we can find the condition in the same way.

**Note 1:** This condition may also be expressed in the following determinant form

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

**Note 2:** If $a = b = 0$ and $h \neq 0$, then the given equation will reduce to

$$2hxy + 2gx + 2fy + c = 0$$

This can be written as

$$2x(hy + g) + \frac{2f}{h}(hy + g) - \frac{2fg}{h} + c = 0$$

or, $2(hy + g)\left(x + \frac{f}{h}\right) - \frac{2fg}{h} + c = 0$

This represents two straight lines if $-\frac{2fg}{h} + c = 0$

or, $2fg - ch = 0$

## 14.6 If the Equations $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represent a Pair of Lines, then $ax^2 + 2hxy + by^2 = 0$ represent a Pair of Lines through the Origin Parallel to the above Pair. (T.U. 2056 S)

Let $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represent a pair of straight lines.

So the left hand side can be resolved into two linear factors and the equation may then be written as

$$(l_1x + m_1y + n_1)\,(l_2x + m_2y + n_2) = 0$$

where $l_1l_2 = a$, $m_1m_2 = b$, $n_1n_2 = c$, $l_1m_2 + m_1l_2 = 2h$,

$$l_1n_2 + n_1l_2 = 2g, \quad m_1n_2 + m_2n_1 = 2f.$$

The separate equations of the lines are

$$l_1x + m_1y + n_1 = 0 \text{ and } l_2x + m_2y + n_2 = 0$$

Hence the equations of the lines through the origin parallel to the above lines are $l_1x + m_1y = 0$ and $l_2x + m_2y = 0$.

The combined equation is

$$(l_1x + m_1y)\,(l_2x + m_2y) = 0$$

or, $l_1l_2x^2 + (l_1m_2 + l_2m_1)\,xy + m_1m_2y^2 = 0$

or, $ax^2 + 2hxy + by^2 = 0$.

**Note 1 :** Angle between two lines represented by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \text{...... (i)}$$

is same as the angle between the two lines represented by

$$ax^2 + 2hxy + by^2 = 0 \qquad \text{...... (ii)}$$

Since the two lines represented by the equation (ii) are parallel to the two lines represented by the equation (i). So, the angle between the two lines given by (i) is same as the angle between the lines given by (ii). But the angle between the lines represented by (ii) is given by

$$\tan\theta = \pm\frac{2\sqrt{h^2 - ab}}{a + b}$$

So, the angle between the two lines represented by equation (i) is also given by the same formula

$$\tan\theta = \pm\frac{2\sqrt{h^2 - ab}}{a + b}$$

The two lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ will be perpendicular to each other if $a + b = 0$ and they will represent parallel lines if $h^2 = ab$.

**Note 2:** To find the equation of the bisectors of the angles between the lines represented by the general equation of the second degree we proceed as follows. Find the separate equations of the two lines, and find the equations of the bisectors. Combine the two equations to get a single equation of the bisectors.

**Note 3:** The point of intersection of the two lines represented by the general equation of second degree can be obtained by solving the separate equations

$$l_1x + m_1y + n_1 = 0, \text{ and } l_2x + m_2y + n_2 = 0$$

for $x$ and $y$, and writing down the result in terms of the coefficients of the given equation by using the relations

$l_1l_2 = a,\quad m_1m_2 = b,\quad n_1n_2 = c,\quad l_1m_2 + m_1l_2 = 2h.$

$l_1n_2 + n_1l_2 = 2g,\quad m_1n_2 + m_2n_1 = 2f.$

The coordinates of the point of intersection will be

$$\left(\sqrt{\frac{f^2 - bc}{h^2 - ab}}, \sqrt{\frac{g^2 - ca}{h^2 - ab}}\right)$$

## 14.7 Lines Joining the Origin to the Intersection of a Line and a Curve

**Find the equation to the pair of lines joining the origin to the points of intersection of the line $lx + my = n$ and the curve**

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

***Solution.***

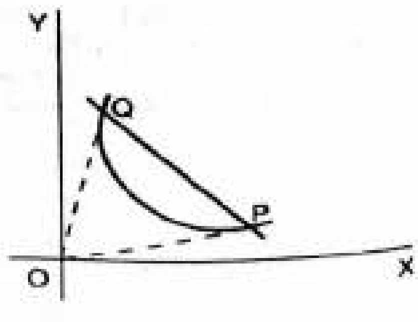
The general equation of second degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \text{ ......... (i)}$$

in general, represents a curve, except in the case when it represent a pair of straight lines.

Let the straight line $lx + my = n$ meet the curve (or pair of straight lines) in two points P and Q. We have to find the equation representing the line pair OP, OQ. The equation of given straight line may be written as

$$\frac{lx + my}{n} = 1 \text{ ......... (ii)}$$




Now consider the homogeneous equation of second degree in x and y,

$$ax^2 + 2hxy + by^2 + 2(gx+fy)\left(\frac{lx+my}{n}\right) + c\left(\frac{lx+my}{n}\right)^2 = 0 \text{ ...... (iii)}$$

Equation (iii), being homogeneous in $x$ and $y$, represents a pair of straight lines through the origin.

Also the coordinates of the point of intersection of the straight line (ii) and the curve (i) satisfy both the equations (i) and (ii), and hence they satisfy the equation (iii). The equation (iii) therefore represents a pair of lines which passes through the origin and the points of intersection of the line (ii) and the curve (i).

## Worked out Examples

***Example 1.***

Find the single equation representing the pair of lines

$$y = x \quad \text{and} \quad y = -x.$$

***Solution.***

Writing the given equations with R.H.S. zero, we have

$$y - x = 0 \qquad \text{and} \qquad y + x = 0$$

Hence the combined equation is

$$(y - x)(y + x) = 0$$

i.e. $$y^2 - x^2 = 0$$

***Example 2.***

Find the separate equations of the lines represented by

i) $x^2 - 5xy + 4y^2 = 0$

ii) $x^2 + 2xy \sec\theta + y^2 = 0.$

***Solution.***

i) We have $x^2 - 5xy + 4y^2 = 0$

i.e. $(x - 4y)(x - y) = 0$

$\therefore$ $x - 4y = 0$ and $x - y = 0$ are the two lines represented by the given equation.

ii) Here $x^2 + 2xy \sec\theta + y^2 = 0$

i.e. $$\left(\frac{y}{x}\right)^2 + 2\left(\frac{y}{x}\right)\sec\theta + 1 = 0.$$

Since this is quadratic in $\frac{y}{x}$, it will have two values $m_1$ and $m_2$, say, so that

$$m_1 + m_2 = -2\sec\theta \text{ and } m_1m_2 = 1$$

Also $$(m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1m_2$$
$$= 4\sec^2\theta - 4 = 4\tan^2\theta$$

$$m_1 - m_2 = \pm\ 2\tan\theta$$

$\therefore \quad m_1 = -\sec\theta \pm \tan\theta.$

and $\quad m_2 = \sec\theta \pm \tan\theta.$

The two values may be written as

$$m_1 = -(\sec\theta + \tan\theta),$$

and $\quad m_2 = -(\sec\theta - \tan\theta).$

Hence the two lines are

$$y = -(\sec\theta + \tan\theta)\,x \text{ and}$$
$$y = -(\sec\theta - \tan\theta)x.$$

***Example 3.***

Find the angle between the line-pair $2x^2 + 7xy + 3y^2 = 0$.

***Solution.***

Comparing the given equation with the equation

$$ax^2 + 2hxy + by^2 = 0$$

$$a = 2, \qquad h = \frac{7}{2}, \qquad b = 3.$$

If $\alpha$ be the angle between the lines,

then, $\tan\alpha = \pm\dfrac{2\sqrt{h^2 - ab}}{a + b} = \pm\dfrac{2\sqrt{\frac{49}{4} - 6}}{2 + 3} = \pm 1$

$\therefore \quad \alpha = 45^\circ$ or $135^\circ$

***Example 4.***

Find the equation of the bisectors of the angles between the lines represented by $2x^2 - 6xy - y^2 = 0$. (HSEB 2050)

***Solution.***

Here $a = 2, h = -3, b = -1$.

The equations of the bisectors are given by

$$h(x^2 - y^2) = (a - b)\,xy$$

i.e. $\quad -3(x^2 - y^2) = (2 + 1)\,xy$

or $\quad -x^2 + y^2 = xy$

or $\quad x^2 + xy - y^2 = 0.$

***Example 5.***

Find the single equation of the lines through the origin and perpendicular to the lines represented by

$$ax^2 + 2hxy + by^2 = 0. \qquad \text{(HSEB 2051)}$$

***Solution.***

Let $y = m_1x$ and $y = m_2x$ be the separate equations of the given line pair $ax^2 + 2hxy + by^2 = 0$. So

$$m_1 + m_2 = -\frac{2h}{b} \qquad \text{and} \qquad m_1m_2 = \frac{a}{b}.$$

Equations of the lines through the origin and perpendicular to

$y = m_1 x$ and $y = m_2x$ are respectively

$y = -\frac{1}{m_1}x$ and $y = -\frac{1}{m_2}x$

i.e. $m_1y + x = 0$ and $m_2y + x = 0.$

The combined equation is

$$(m_1y + x)\,(m_2y + x) = 0$$

i.e. $m_1m_2y^2 + (m_1 + m_2)\,xy + x^2 = 0$

i.e. $\frac{a}{b}y^2 - \frac{2h}{b}xy + x^2 = 0$

i.e. $ay^2 - 2hxy + bx^2 = 0.$

***Example 6.***

If the pair of lines $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ be such that each pair bisects the angles between the other pair, prove that $pq = -1$.

(HSEB 2057)

***Solution.***

The pair of lines bisecting the angles between the pair

$x^2 - 2pxy - y^2 = 0$ is given by

$-p(x^2 - y^2) = [1 - (-1)]\,xy$

or $-px^2 + py^2 = 2xy$

or $px^2 + 2xy - py^2 = 0.$

By the question, the pair of bisectors is given by

$$x^2 - 2qxy - y^2 = 0$$

Hence $px^2 + 2xy - py^2 = 0$

and $x^2 - 2qxy - y^2 = 0$

must represent the same pair of lines

$\therefore \quad \frac{p}{1} = \frac{2}{-2q} = \frac{-p}{-1}$

or $pq = -1.$

***Example 7.***

Prove that $2x^2 + 7xy + 3y^2 - 4x - 7y + 2 = 0$ represents two straight lines.

***Solution.***

Comparing the given equation with the general equation of second degree,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

We have $a = 2,\ h = \frac{7}{2},\ b = 3,\ g = -2,\ f = -\frac{7}{2},\ c = 2.$

The condition for the general equation of second degree to represent two straight lines is

$$abc + 2ghf - af^2 - bg^2 - ch^2 = 0.$$

Substituting the values of the coefficients in the L.H.S.

We have

$$\text{L.H.S.} = 2.3.2 + 2\,(-2)\left(\frac{7}{2}\right)\left(-\frac{7}{2}\right) - 2\left(-\frac{7}{2}\right)^2 - 3(-2)^2 - 2\left(\frac{7}{2}\right)^2$$

$$= 12 + 49 - \frac{49}{2} - 12 - \frac{49}{2} = 0.$$

$\therefore$ the given equation represents two straight lines.

***Example 8.***

Determine the two straight lines represented by

$$6x^2 - xy - 12y^2 - 8x + 29y - 14 = 0.$$

Hence find the point of intersection of the lines.

***Solution.***

The given equation may be written as

$$-12y^2 - y\,(x - 29) + 6x^2 - 8x - 14 = 0.$$

Solving for y, we have

$$y = \frac{+(x - 29) \pm \sqrt{(x - 29)^2 - 4\,(-12)\,(6x^2 - 8x - 14)}}{2.\,(-12)}$$

or $\quad y = \dfrac{x - 29 \pm \sqrt{x^2 - 58x + 841 + 288x^2 - 384x - 672}}{-24}$

or $\quad y = \dfrac{x - 29 \pm \sqrt{289x^2 - 442x + 169}}{-24}$

or $\quad y = \dfrac{18x - 42}{-24}, \qquad$ and $\quad y = \dfrac{-16x - 16}{-24}$

or $\quad -4y = 3x - 7, \qquad$ and $\quad 3y = 2x + 2.$

or $\quad 3x + 4y - 7 = 0 \qquad$ and $\quad 2x - 3y + 2 = 0.$

These are the required equations of the lines. Also solving these equations simultaneously,

we have $x = \frac{13}{17}$ and $y = \frac{20}{17}$

Hence the point of intersection is $\left(\frac{13}{17}, \frac{20}{17}\right)$

***Example 9.***

Find the equation of the straight lines which pass through the point (1, 2) and are perpendicular to the straight lines

$$12x^2 - 7xy - 12y^2 + 19x - 17y + 5 = 0. \qquad \text{(T.U. 2052)}$$

***Solution.***

Let $m_1$ and $m_2$ be the slopes of lines represented by the given equation so that

$$m_1 + m_2 = -\frac{7}{12} \qquad \text{...... (i)}$$

$$m_1 m_2 = -1 \qquad \text{...... (ii)}$$

Then the slopes of the lines perpendicular to the lines represented by given equation are $-\frac{1}{m_1}$ and $-\frac{1}{m_2}$.

The equations of the lines through (1, 2) and perpendicular to given lines, are

$$y - 2 = -\frac{1}{m_1}(x - 1) \qquad \text{...... (iii)}$$

and $$y - 2 = -\frac{1}{m_2}(x - 1) \qquad \text{...... (iv)}$$

Solving (i) and (ii)

We have $m_1 = \frac{3}{4}$, $m_2 = -\frac{4}{3}$; $m_1 = -\frac{4}{3}$, $m_2 = \frac{3}{4}$

Substituting the values of $m_1$ and $m_2$ in (iii) and (iv) respectively, we have

$$y - 2 = -\frac{4}{3}(x - 1)$$

and $$y - 2 = \frac{3}{4}(x - 1)$$

or, $4x + 3y - 10 = 0$

and $3x - 4y + 5 = 0.$

$\therefore$ The lines through (1, 2) and perpendicular to given lines are

$$(4x + 3y - 10)(3x - 4y + 5) = 0$$

$\therefore$ $12x^2 - 7xy - 12y^2 - 10x + 55y - 50 = 0$

The same equation will be obtained if we take $m_1 = -\frac{4}{3}, m_2 = \frac{3}{4}$.

*Alternative Method:*

Consider, $12x^2 - 7xy - 12y^2$

$$= 12x^2 - 16xy + 9xy - 12y^2$$
$$= 4x(3x - 4y) + 3y(3x - 4y)$$
$$= (3x - 4y)(4x + 3y)$$

Now, the equation of the lines perpendicular to the lines represented by given equation $12x^2 - 7xy - 12y^2 + 19x - 17y + 5 = 0$ is

$$(4x + 3y + c_1)(3x - 4y + c_2) = 0 \text{ ...... (i)}$$

If the lines (i) pass through the point (1, 2), then

$$(4 \times 1 + 3 \times 2 + c_1)(3 \times 1 - 4 \times 2 + c_2) = 0$$

or, $(10 + c_1)(-5 + c_2) = 0$

$\therefore c_1 = -10, \ c_2 = 5$

Substituting the values of $c_1$ and $c_2$ in (i), we have

$$(4x + 3y - 10)(3x - 4y + 5) = 0$$
$$12x^2 - 7xy - 12y^2 - 10x + 55y - 50 = 0$$

which is the equation of the lines through (1, 2) and perpendicular to the lines represented by given equation.

**Example 10.**

If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represent a pair of parallel lines, prove that (T.U. 2057 S)

i) $\frac{a}{h} = \frac{h}{b} = \frac{g}{f}$, and

ii) the distance between them is $2\sqrt{\frac{g^2 - ac}{a^2 + ab}}$

**Solution.**

Let the given equation represent two parallel lines whose separate equations are

$$lx + my + n = 0 \text{ and } lx + my + n' = 0,$$

then $(lx + my + n)(lx + my + n')$

$$= ax^2 + 2hxy + by^2 + 2gx + 2fy + c.$$

Equating coefficients of like terms on the two sides, we have

$$l^2 = a, \quad m^2 = b, \ nn' = c.$$
$$m(n + n') = 2f, \quad l(n + n') = 2g, \quad 2lm = 2h.$$

i) we have $ab = l^2m^2 = h^2$

or $\frac{a}{h} = \frac{h}{b}$

Also $\frac{g}{f} = \frac{l}{m} = \frac{lm}{m^2} = \frac{h}{b}$

Hence $\frac{a}{h} = \frac{h}{b} = \frac{g}{f}$

ii) the perpendicular distance between the parallel lines $lx + my + n = 0$ and $lx + my + n' = 0$

$$= \frac{n - n'}{\sqrt{l^2 + m^2}} = \frac{\sqrt{(n + n')^2 - 4nn'}}{\sqrt{l^2 + m^2}}$$

$$= \frac{\sqrt{\frac{4g^2}{l^2} - 4c}}{\sqrt{a + b}} = \frac{\sqrt{\frac{4g^2 - 4ac}{a}}}{\sqrt{a + b}}$$

$$= 2\sqrt{\frac{g^2 - ac}{a(a + b)}}$$

**Example 11.**

Prove that the straight lines joining the origin to the point of intersection of the line $\frac{x}{a} + \frac{y}{b} = 1$ and the curve $x^2 + y^2 = c^2$ are at right angles if $\frac{1}{a^2} + \frac{1}{b^2} = \frac{2}{c^2}$. (T.U. 2053, 055 S, 057 S) (HSEB 2055,2056)

**Solution.**

The equation of the given curve is

$$x^2 + y^2 = c^2 \quad \text{...... (i)}$$

The equation of the given line is

$$\frac{x}{a} + \frac{y}{b} = 1$$

or, $$\frac{bx + ay}{ab} = 1 \quad \text{...... (ii)}$$

Making equation (i) homogeneous with the help of equation (ii), we have

$$x^2 + y^2 = c^2\left(\frac{bx + ay}{ab}\right)^2$$

or, $a^2b^2x^2 + a^2b^2y^2 = c^2(b^2x^2 + 2abxy + a^2y^2)$

or, $(a^2b^2 - b^2c^2)x^2 - 2abc^2xy + (a^2b^2 - c^2a^2)y^2 = 0$

This is the equation of the lines joining the origin and the points of intersection of the given curve and the line.These straight lines are at right angles to each other

if coeff. of $x^2$ + coeff. of $y^2 = 0$

$$a^2b^2 - b^2c^2 + a^2b^2 - c^2a^2 = 0$$

$$b^2c^2 + c^2a^2 = 2a^2b^2$$

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{2}{c^2}$$

***Example 12***

If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two straight lines, prove that the product of the perpendiculars from the origin on these lines is

$$\frac{c}{\sqrt{(a-b)^2 + 4h^2}}$$

***Solution :***

If $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ be two lines represented by the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

then

$$(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$$

Equating the coefficients of like terms, we have,

$$a_1a_2 = a,\ b_1b_2 = b,\ c_1c_2 = c \text{ and } a_1b_2 + a_2b_1 = 2h$$

Length of the perpendicular from (0, 0) on the line $a_1x + b_1y + c_1 = 0$ is

$$\frac{c_1}{\sqrt{a_1^2 + b_1^2}}$$

Again, the length of the perpendicular from (0, 0) on the line

$a_2x + b_2y + c_2 = 0$ is $\dfrac{c_2}{\sqrt{a_2^2 + b_2^2}}$

Product of the lengths of the perpendiculars

$$= \frac{c_1}{\sqrt{a_1^2 + b_1^2}} \cdot \frac{c_2}{\sqrt{a_2^2 + b_2^2}}$$

$$= \frac{c_1c_2}{\sqrt{a_1^2a_2^2 + (a_1^2b_2^2 + a_2^2b_1^2) + b_1^2b_2^2}}$$

$$= \frac{c_1c_2}{\sqrt{(a_1a_2)^2 + (b_1b_2)^2 + (a_1b_2 + a_2b_1)^2 - 2a_1a_2b_1b_2}}$$

$$= \frac{c}{\sqrt{a^2 + b^2 + 4h^2 - 2ab}}$$

$$= \frac{c}{\sqrt{(a-b)^2 + 4h^2}}$$

***Example 13***

Find the condition that one of the lines given by $ax^2 + 2hxy + by^2 = 0$ may be perpendicular to one of the lines given by $a'x^2 + 2h'xy + b'y^2 = 0$.

(T.U. 2054)

***Solution :***

Let $y = mx$ be an equation of a line given by $ax^2 + 2hxy + by^2 = 0$

Then, $\quad ax^2 + 2hx.mx + b.m^2x^2 = 0$

$\therefore \quad bm^2 + 2hm + a = 0 \quad \ldots\ldots \text{(i)}$

The equation of the line perpendicular to $y = mx$ and through origin is

$$y = -\frac{1}{m}x$$

which is one of the lines given by

$$a'x^2 + 2h'xy + b'y^2 = 0$$

So, $\quad a'x^2 + 2h'x.\left(-\frac{1}{m}x\right) + b'.\frac{1}{m^2}x^2 = 0$

$\therefore \quad a'm^2 - 2h'm + b' = 0 \quad \ldots\ldots \text{(ii)}$

Using the rule of cross-multiplication (from (i) and (ii))

$$\frac{m^2}{2(hb' + h'a)} = \frac{m}{aa' - bb'} = \frac{1}{-2(bh' + a'h)}$$

$$m = \frac{aa' - bb'}{-2(bh' + a'h)}$$

and $\quad m = \frac{2(b'h + ah')}{aa' - bb'}$

So, $\quad \frac{aa' - bb'}{-2(bh' + a'h)} = \frac{2(b'h + ah')}{aa' - bb'}$

$\therefore \quad (aa' - bb')^2 + 4(b'h + ah')(bh' + a'h) = 0$

which is the required condition.

6 Mark ques. will asked

## EXERCISE 14.1

1. Find a single equation representing the line pair
   a) $x + y = 0, x + 2y = 0$
   b) $ax - by = 0, bx + ay = 0$
   c) $x + y + 2 = 0, x + 2y + 1 = 0$
2. Determine the lines represented by each of the following equations:
   a) $x^2 - 2xy = 0$
   b) $x^2 - 5xy + 4y^2 = 0$
   c) $xy - 3x + 2y - 6 = 0$
   d) $x^2 + 2xy + y^2 - 2x - 2y - 15 = 0.$
3. Find the angle between the following pairs of lines
   a) $x^2 + 9xy + 14y^2 = 0$
   b) $x^2 - 2xy \cot\theta - y^2 = 0.$
   c) $x^2 - 5xy + 4y^2 + x + 2y - 2 = 0$
4. Find the equations of the bisectors of the angles between the pair of lines
   a) $3x^2 - 15xy + 2y^2 = 0$
   b) $x^2 + 2xy \operatorname{cosec}\theta + y^2 = 0$
   c) $ax^2 + 2hxy + by^2 + k(x^2 + y^2) = 0.$
5. Prove that each of the following equations represents a pair of lines:
   a) $2x^2 + 7xy + 3y^2 - 4x - 7y + 2 = 0$ (HSEB 2053)
   b) $6x^2 - xy - 12y^2 - 8x + 29y - 14 = 0$
6. Find the value of $k$ so that the following equations may represent a line pair
   a) $x^2 + kxy + 2y^2 + 3x + 5y + 2 = 0$
   b) $2x^2 + 7xy + 3y^2 - 4x - 7y + k = 0$
7. a) Find the equation of the two lines represented by $x^2 + 6xy + 9y^2 + 4x + 12y - 5 = 0$. Prove that the two lines are parallel. Also, find the distance between them. (T.U. 2049)

   b) Find the equations of the two lines represented by the equation $2x^2 + 3xy + y^2 + 5x + 2y - 3 = 0$. Find their points of intersection and also the angle between them. (T.U. 2048, 2058 S)
8. If the line pairs $ax^2 + 2hxy + by^2 = 0$ and $a'x^2 + 2h'xy + b'y^2 = 0$ have the same bisectors, prove that $h(a' - b') = h'(a - b)$

9. a) Find the equation of the straight lines through the origin and at right angles to the lines $x^2 - 5xy + 4y^2 = 0$. (HSEB 2052)

   b) Find the equation to the straight lines passing through (1, 1) and parallel to the lines represented by $x^2 - 5xy + 4y^2 + x + 2y - 2 = 0$

   c) Find the equation of the straight lines perpendicular to the lines given by $x^2 + xy - 6y^2 + 7x + 31y - 18 = 0$ and passing through the origin.

10. a) Show that the lines joining the origin to the point of intersection of the line $x + y = 1$ with the curve $4x^2 + 4y^2 + 4x - 2y - 5 = 0$ are at right angles to each other. (T.U. 2050)

   V.I

   b) Show that the lines joining the origin to the points common to $x^2 + hxy - y^2 + gx + fy = 0$ and $fx - gy = \lambda$ are at right angles for all values of $\lambda \neq 0$.

   c) Find the equation of the lines joining the origin to the points of intersection of $x + 2y = 3$ and $4x^2 + 16xy - 12y^2 - 8x + 12y - 3 = 0$. Also find the angle between the two lines.

   d) Find the equation to the pair of straight lines joining the origin to the intersection of the straight line $y = mx + c$ and the curve $x^2 + y^2 = a^2$. Prove that they are at right angles if

   $$2c^2 = a^2(1 + m^2)$$ (T.U. 2052, HSEB 2053, 2058)

11. Prove that the two straight lines

    $(x^2 + y^2) \sin^2 \alpha = (x \cos \theta - y \sin \theta)^2$ include an angle $2\alpha$.

12. Find the condition that one of the lines $ax^2 + 2hxy + by^2 = 0$ may coincide with one of the lines $a_1x^2 + 2h_1xy + b_1y^2 = 0$.

13. Prove that the product of the perpendiculars drawn from the point $(x_1, y_1)$ on the lines represented by $ax^2 + 2hxy + by^2 = 0$ is

$$\frac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{(a-b)^2 + 4h^2}}$$

## Answers

1. a) $x^2 + 3xy + 2y^2 = 0$
   b) $abx^2 + (a^2 - b^2)xy - aby^2 = 0$
   c) $x^2 + 2y^2 + 3xy + 3x + 5y + 2 = 0$
2. a) $x = 0; x - 2y = 0$ b) $x - 4y = 0; x - y = 0$
   c) $x + 2 = 0; y - 3 = 0$ d) $x + y - 5 = 0; x + y + 3 = 0$
3. a) $\tan^{-1}\left(\pm\frac{1}{3}\right)$ b) $\frac{\pi}{2}$ c) $\tan^{-1}\left(\pm \frac{3}{5}\right)$

4. a) $15x^2 + 2xy - 15y^2 = 0$   b) $x^2 - y^2 = 0$
   c) $h(x^2 - y^2) = (a - b)\, xy$
6. a) $k = \frac{9}{2}$ or 3   b) $k = 2$
7. a) $x + 3y + 5 = 0,\ x + 3y - 1 = 0,\ \frac{3\sqrt{10}}{5}$
   b) $x + y + 3 = 0,\ 2x + y - 1 = 0,\ (4, -7),\ \tan^{-1}\left(\pm \frac{1}{3}\right)$
9. a) $4x^2 + 5xy + y^2 = 0$   b) $x^2 - 5xy + 4y^2 + 3x - 3y = 0$
   c) $6x^2 + xy - y^2 = 0$
10. c) $3x^2 + 40xy - 16y^2 = 0,\ \tan^{-1}\left(\pm \frac{16\sqrt{7}}{13}\right)$
   d) $(c^2 - a^2m^2)x^2 + 2a^2mxy + (c^2 - a^2)y^2 = 0$
13. $(ab_1 - a_1b)^2 = 4(bh_1 - b_1h)\,(a_1h - ah_1)$

## ADDITIONAL QUESTIONS

1. What is the homogeneous equation of degree two ? Prove that the equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines passing through the origin. Are they always real ? (T.U. 2048, 2055 S)

2. Prove that the homogeneous equation of second degree $ax^2 + 2hxy + by^2 = 0$ always represents a pair of straight lines through the origin and hence find the angle between them. (HSEB 2055)

3. Find the angles between two lines represented by $ax^2 + 2hxy + by^2 = 0$. When these lines will be perpendicular ? What happens when $h^2 = ab$ ? (HSEB 2054)

4. Show that every pair of straight lines can be represented by a second degree equation in $x$ and $y$. Obtain the condition for the second degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ to represent a pair of straight lines. (HSEB 2051)

5. Find the condition that the general equation of second degree may represent a pair of straight lines. Use your result to show that the equation $2x^2 + 7xy + 3y^2 = 0$ represents two straight lines. (HSEB 2052)

6. Obtain the condition that the general equation of second degree in $x$ and $y$ may represent a pair of straight lines. Does the equation $2x^2 + 7xy + 3y^2 - 4x - 7y + 2 = 0$ represent a pair of straight lines ? (HSEB 2053)

7. Find the equation of the bisectors of the angle between the lines given by $ax^2 + 2hxy + by^2 = 0$. Use it to find the bisectors of the angles between $2x^2 - 6xy - y^2 = 0$. (HSEB 2050)

8. For what value of $\lambda$ does the equation $12x^2 - 10xy + 2y^2 + 11x - 5y + \lambda = 0$ represent two straight lines ? Show that the angle between them is $\tan^{-1}\left(\frac{1}{7}\right)$.

9. If the two pair of lines represented by $x^2 - 2axy - 3y^2 = 0$ and $x^2 - 2bxy - y^2 = 0$ be such that each pair bisects the angles between the other pair, prove that $ab + 2 = 0$

10. Show that the lines joining the origin to the point of intersection of the curve $3x^2 + 5xy - 3y^2 + 2x + 3y = 0$ and the line $3x - 2y = 1$ are at right angles. (T.U. 2051 H)

11. If the lines joining the origin to the point of intersection $y = mx + 1$ with $x^2 + y^2 = 1$ are perpendicular, find the value of m.

12. Find the value of k so that the lines which join the origin to the point of intersection of the lines $y - x = k$ and the curve $x^2 + y^2 + 4x - 6y - 36 = 0$ may be at right angles.

13. Prove that the equation of the lines bisecting the angles between th bisectors of the pair of lines $ax^2 + 2hxy + by^2 = 0$ is
$$(a - b)(x^2 - y^2) + 4hxy = 0$$

14. Prove that one of the lines $ax^2 + 2hxy + by^2 = 0$ will bisect the angle between the co-ordinate axes if $(a + b)^2 = 4h^2$.

15. Show that the pair of lines $a^2x^2 + 2h(a + b)xy + b^2y^2 = 0$ is equally inclined to the pair of the lines $ax^2 + 2hxy + by^2 = 0$.

16. Prove that the straight line joining the origin to the points of intersection of the straight line $lx + my = n$ with the curve $x^2 + y^2 = a^2$ are at right angles if $a^2(l^2 + m^2) = 2n^2$.

17. The equations of the two pairs of opposite sides of a rectangle are $x^2 - 7x + 6 = 0$ and $y^2 - 14y + 40 = 0$. Find the equation of its diagonals.

18. Prove that the two pairs of lines given by $y^2 + xy - 12x^2 = 0$ and $4y^2 - 13xy + 3x^2 = 0$ have one line in common and the other two liens are perpendicular to each other.

19. Show that the two straight lines $x^2(\tan^2\theta + \cos^2\theta) - 2xy\tan\theta + y^2\sin^2\theta = 0$ make with x-axis angles such that the difference of their tangents is 2.

***Answers***

6. yes 6. $x^2 + xy - y^2 = 0$
8. 2 11. $\pm 1$
12. 9 or $-4$
17. $6x - 5y + 14 = 0$, $6x + 5y - 56 = 0$

# CHAPTER 15
# The Circle

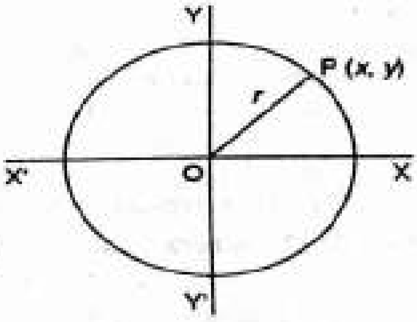
## 15.1 Circle

A *circle* is a closed curve such that every point on the curve is at a constant distance from a fixed point. In terms of locus, a circle may be defined as the locus of a point which moves so that its distance from a fixed point is constant. The fixed point is called the *centre* and the constant distance is called the *radius* of the circle.

## 15.2 Equation of a Circle

### (i) Centre at the origin
**(Standard form)**

Let O(0, 0) be the centre and $r$ the radius of the circle. Let P($x$, $y$) be any point on the circle.




Then $\quad OP = r$

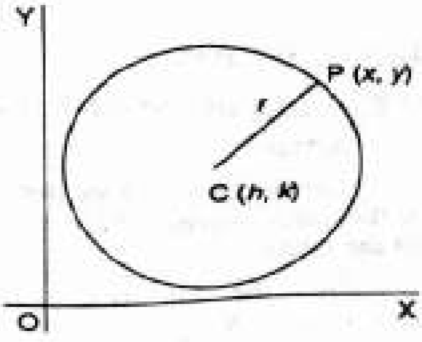
or, $\quad OP^2 = r^2$

$\therefore \quad x^2 + y^2 = r^2.$

This relation is true for any point P($x$, $y$) on the circle. Hence it is the equation of the circle.

### (ii) Centre at any point
**(Central form)**

Let C($h$, $k$) be the centre and $r$ the radius of the circle. Let P($x$, $y$) be any point on the circle so that




$$CP = r:$$

or, $\quad CP^2 = r^2$

or, $(x - h)^2 + (y - k)^2 = r^2.$

This is the equation of the circle.

### (iii) General equation of the circle

Consider the equation

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

in which the coefficients of $x^2$ and $y^2$ are equal, each being unity and there is no term containing $xy$.

The given equation may be written as

$$x^2 + 2gx + y^2 + 2fy = -c$$

i.e. $$x^2 + 2gx + g^2 + y^2 + 2fy + f^2 = g^2 + f^2 - c$$

$$(x + g)^2 + (y + f)^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2 \quad \text{...... (i)}$$

which is in the form of $(x - h)^2 + (y - k)^2 = r^2$, hence, represents a circle.

Comparing equation (i) with the equation of a circle

$$(x - h)^2 + (y - k)^2 = r^2,$$

we have $h = -g, \quad k = -f, \quad r = \sqrt{g^2 + f^2 - c}$

Hence $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle whose centre is at $(-g, -f)$ and radius equal to $\sqrt{g^2 + f^2 - c}$ .

The equation of a circle in this form is called the *general equation of a circle.*

**Note 1:** If $g^2 + f^2 - c > 0$, the radius is *real*, hence the equation gives a real geometrical locus.

If $g^2 + f^2 - c = 0$, the radius is *zero*. The circle in this case is called a point-circle.

If $g^2 + f^2 - c < 0$, the radius becomes *imaginary*. We say that the equation represents a circle with a real centre and an *imaginary radius.*

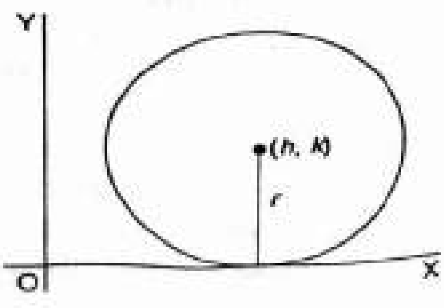
**Note 2:** The general equation of second degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents a circle if $a = b$, (coefficients of $x^2$ and $y^2$ are equal) and $h = 0$, (coefficient of $xy$ is zero).

#### Particular Cases

**i) Equating of the circle touching the x-axis**




Let $(h, k)$ be the centre of the circle. If the circle touches the x-axis, then radius of the circle $(r) = k$.

So, the equation of the circle touching the x-axis is

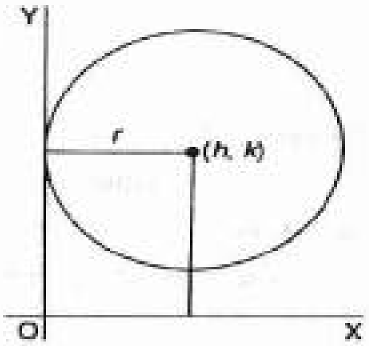
$$(x - h)^2 + (y - k)^2 = k^2$$

**ii) Equation of the circle touching the y-axis**

Let $(h, k)$ be the centre of the circle. If the circle touches the y-axis, then the radius of the circle $(r) = h$.

So, the equation of the circle touching the y-axis is

$$(x - h)^2 + (y - k)^2 = h^2$$

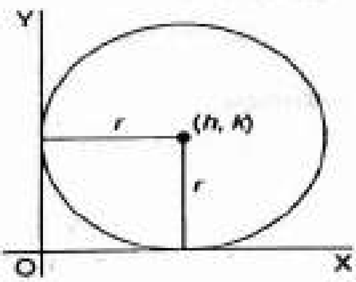



**iii) Equation of the circle touching both axes:**

Let $(h, k)$ be the centre of the circle. If the circle touches both axes, then radius of the circle $(r) = h = k$.

So, the equation of the circle touching both axes is

$$(x - h)^2 + (y - h)^2 = h^2$$

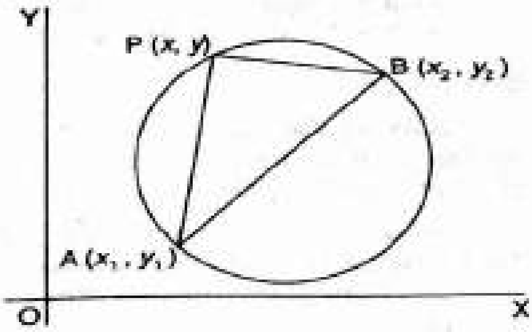



**(iv) Circle with a given diameter (Diameter form)**

Let A $(x_1, y_1)$ and B$(x_2, y_2)$ be the ends of a diameter of a circle. Let P$(x, y)$ be any point on the circle.

Join AP, BP and AB

Since AB is a diameter of the circle, $\angle APB$ is a right angle.




Now slope of AP $= \dfrac{y - y_1}{x - x_1}$

and slope of BP $= \dfrac{y - y_2}{x - x_2}$

Since AP is perpendicular to BP, the product of their slopes should be $-1$.

Hence $\qquad \dfrac{y - y_1}{x - x_1} \cdot \dfrac{y - y_2}{x - x_2} = -1$

or, $\quad (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

This equation is satisfied by any point on the circle, hence it is the equation of the circle.

## Worked Out Examples

***Example 1.***

Find the equation of the circle with centre at (2, 3) and radius 5.

***Solution:***

The required equation is

$$(x-2)^2 + (y-3)^2 = 5^2$$

i.e. $x^2 + y^2 - 4x - 6y - 12 = 0.$

***Example 2.***

Find the centre and radius of the circle whose equation is

$$x^2 + y^2 + 4x - 6y + 4 = 0.$$

***Solution:***

The given equation may be written as

$$x^2 + 4x + y^2 - 6y = -4$$

or, $x^2 + 4x + 4 + y^2 - 6y + 9 = 4 + 9 - 4$

or, $(x+2)^2 + (y-3)^2 = 9$

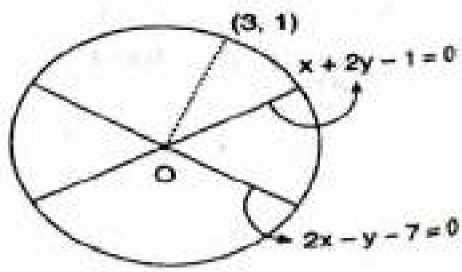
Hence the centre of the circle is at (–2, 3) and the radius 3.

***Example 3.***

Find the equation of the circle whose centre is the point of intersection of $x + 2y - 1 = 0$ and $2x - y - 7 = 0$ and which passes through (3, 1).

***Solution:***

Solving the equations of lines, we get the point of intersection (3, –1). Hence the centre of the circle is at (3, –1).




Since the circle passes through (3, 1), the radius of the circle is equal to the distance of this point from the centre.

Hence,

$$\text{radius} = \sqrt{(3-3)^2 + (-1-1)^2} = 2$$

Therefore the equation of the circle is

$$(x-3)^2 + (y+1)^2 = 4$$

or, $x^2 + y^2 - 6x + 2y + 6 = 0.$

*Example 4*

Find the equation of the circle which touches the positive $x$- and $y$- axes and whose radius is 5 units.

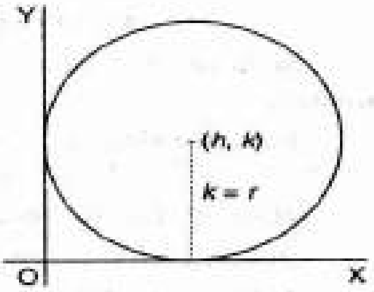
**Solution**

Let $(h, k)$ be the centre of the circle. We have, $h = k = 5$, as the circle touches the axes.

Hence the equation of the circle is

$$(x - 5)^2 + (y - 5)^2 = 5^2$$

or, $x^2 + y^2 - 10x - 10y + 25 = 0.$




*Example 5*

Determine the equation of the circle if the ends of a diameter be at (3, 0) and (7, –1).

**Solution:**

The equation of a circle in diameter form is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

or, $(x - 3)(x - 7) + (y - 0)(y + 1) = 0$

or, $x^2 + y^2 - 10x + y + 21 = 0.$

*Example 6*

Find the equation of the circle whose centre is at (–4, –5) and of which the tangent is the line $3x - 4y + 2 = 0$.

**Solution.**

The length of perpendicular from the centre of a circle to any tangent to the circle is equal to the radius of the circle.

Hence, radius $= \dfrac{3(-4) - 4(-5) + 2}{\sqrt{(3^2 + 4^2)}} = 2$

∴ the equation of the circle is

$$(x + 4)^2 + (y + 5)^2 = 2^2$$

$$x^2 + y^2 + 8x + 10y + 37 = 0.$$

*Example 7*

Derive the equation of the circle with centre on the line $2x + y - 1 = 0$, radius 5 and through (4, 3).

***Solution***

Let the centre of the circle be at $(h, k)$, so that the equation of the circle is

$$(x-h)^2 + (y-k)^2 = 5^2 \quad \text{(radius = 5)}$$

Since the centre lies on $2x + y - 1 = 0$,
we have $2h + k - 1 = 0$ ... ... (i).

Since the circle passes through (4, 3),

$$(4-h)^2 + (3-k)^2 = 25 \quad \text{... ... (ii).}$$

Now we have to solve (i) and (ii) to get the values of $h$ and $k$.

From (i) $k = 1 - 2h$

Substituting this value of k in (ii) we get

$$(4-h)^2 + (3-1+2h)^2 = 25$$

or, $5h^2 = 5$

$\therefore$ $h = 1, -1$

a) When $h = 1$, $k = -1$

$\therefore$ the equation of the circle is

$$(x-1)^2 + (y+1)^2 = 25$$

or, $x^2 + y^2 - 2x + 2y - 23 = 0$

b) When $h = -1$, $k = 3$

The equation of the circle is

$$(x+1)^2 + (y-3)^2 = 25$$

or, $x^2 + y^2 + 2x - 6y - 15 = 0$

***Example 8***

Find the equation of the circle passing through the points (1, 0), (2, –2) and (3, 1).

***Solution:***

Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Since it passes through (1, 0), (2, –2) and (3, 1), the coordinates will satisfy the equation. Hence substituting the coordinates in the equation, we have

$$2g + c = -1$$
$$-4f + 4g + c = -8$$
$$2f + 6g + c = -10$$

Solving, we get $\quad g = -\frac{5}{2}, \; f = \frac{1}{2}, \quad c = 4.$

Hence the equation of the circle is

$$x^2 + y^2 - 5x + y + 4 = 0.$$

***Example 9***

Find the equation of the circle which touches the $x$-axis at the point (3, 0) and passes through the point (1, 2).

**Solution:**

Let O' $(h, k)$ be the centre of the circle. Since the circle touches the $x$-axis, so radius of the circle = $k$ and also $h = 3$.

Now the equation of the circle touching the $x$-axis at the point (3, 0) is

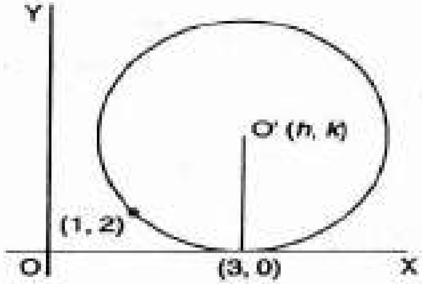
$$(x - 3)^2 + (y - k)^2 = k^2 \quad \ldots \ldots (i)$$

Since this circle (i) passes through the point (1, 2), so

$$4 + (2 - k)^2 = k^2$$

or, $\quad 4k = 8$

$\therefore \quad k = 2$




Now the reqd. equation of the circle is

$$(x - 3)^2 + (y - 2)^2 = (2)^2$$

$\therefore \quad x^2 + y^2 - 6x - 4y + 9 = 0.$

***Example 10***

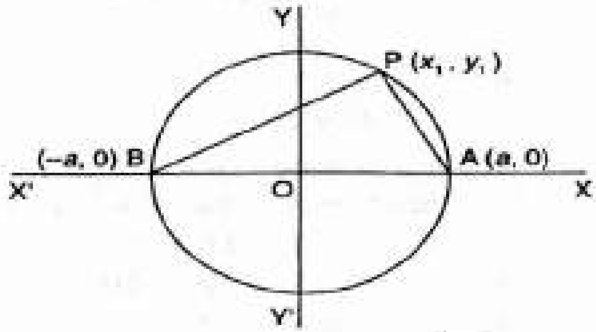
Prove analytically that the angle in a semi-circle is a right angle.

**Solution:**

Let O(0, 0) be the centre and $a$ the radius of a circle, so that its equation is

$$x^2 + y^2 = a^2.$$

Let the circle cut the x-axis at A and B. Clearly, AB is a diameter of the circle and the coordinates of A and B are $(a, 0)$ and $(-a, 0)$ respectively. Let $P(x_1, y_1)$ be any point on the circle. It is required to show that $\angle APB$ is a right angle.




Since $P(x_1, y_1)$ is a point on the circle $x^2 + y^2 = a^2$, we have

$$x_1^2 + y_1^2 = a^2 \quad \ldots \ldots \ldots (i).$$

Slope of AP = $\dfrac{y_1 - 0}{x_1 - a} = \dfrac{y_1}{x_1 - a}$

Slope of BP = $\dfrac{y_1 - 0}{x_1 + a} = \dfrac{y_1}{x_1 + a}$

Product of the slopes of AP and BP

$$= \frac{y_1}{x_1 - a} \cdot \frac{y_1}{x_1 + a} = \frac{y_1^2}{x_1^2 - a^2}$$

$$= \frac{y_1^2}{-y_1^2} \qquad \text{(by virtue of (i))}$$

$$= -1.$$

$\therefore$ AP is perpendicular to BP i.e. $\angle APB$ is a right angle.

But P is any point on the circle, hence the result.

***Example 11***

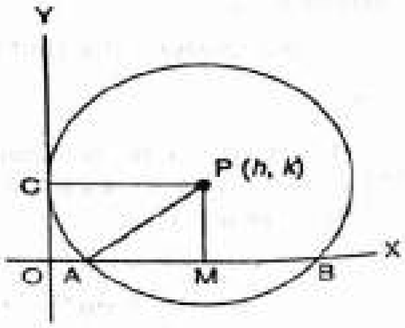
Find the equation to the circle which touches the positive y axis at a distance of 4 from the origin and cuts off an intercept 6 from the axis of x.

***Solution :***

Let $P(h, k)$ be the centre of the circle touching the y-axis at C such that OC = 4. From P draw PM $\perp$ AB.

Then, $MP = OC = k = 4$

$AM = MB = 3$




From the right angled triangle AMP,

$AP^2 = AM^2 + MP^2$

$= 9 + 16 = 25$

or, $CP^2 = 25$

$h^2 = 25$

$\therefore$ $h = \pm 5$

Now the equation of the circle touching the y-axis is

$(x - h)^2 + (y - k)^2 = h^2$

or, $(x \pm 5)^2 + (y - 4)^2 = (5)^2$

or, $x^2 + y^2 \pm 10x - 8y + 16 = 0$

## EXERCISE 15.1

1. Find the equations of the circles with
   a) Centre at (4, 5) and radius 3
   b) Centre at (0, 0) and diameter 8
   c) Centre at $(p, q)$ and radius $\sqrt{p^2 + q^2}$
   d) Centre at (4, –1) and through the origin
   e) two of the diameters are $x + y = 6$ and $x + 2y = 8$ and radius 10.
   f) centre at (+1, 5) and through the point of intersection of the lines $2x - y = 5$ and $3x + y = 10$.
   g) with (0, 0) and (4, 7) as the ends of a diameter
   h) concentric with the circle $x^2 + y^2 + 8x - 6y + 1 = 0$ and radius 3
   i) concentric with the circle $x^2 + y^2 - 8x + 12y + 15 = 0$ and passing through (5, 4)
   j) passing through the origin and making intercepts equal to 3 and 4 from the positive $x$ and $y$-axis respectively.
   k) centre on $x + 3y + 1 = 0$ and through (5, 3) and (–2, 2)
2. Find the equation of the circle
   a) with centre at (3, 4) and touching the x-axis
   b) with centre at $(a, b)$ and touching the y-axis
   c) with centre at (4, 5) and touching the line $3x - 4y + 18 = 0$
   d) with centre on the line $3x - 5y = 4$ and touching both axes of coordinates
   e) which touches each axis and passes through the point (2, 1)
   f) which touches the coordinate axes at $(a, 0)$ and $(0, a)$
   g) with radius 4, centre on the line $13x + 4y = 32$ and touching the line $3x + 4y = 12$
3. Find the centre and the radius of the following circles
   a) $x^2 + y^2 - 12x - 4y = 9$
   b) $x^2 + y^2 - 3x + 2y - \frac{3}{4} = 0$
   c) $4(x^2 + y^2) + 12ax - 6ay - a^2 = 0$

   (3, 4)
4. Find the equation of the circle
   a) passing through the points (0, 0), $(a, 0)$ and $(0, b)$
   b) through the points (1, 2), (3, 1) and (–3, –1)
   c) circumscribing the triangle whose sides are $x + y = 6$, $2x + y = 4$ and $x + 2y = 5$

5. Show that the points (3, 3), (6, 4), (7, 1) and (4, 0) are concyclic. Find the centre and the radius of the circle.

6. If $y - x = 2$ is the equation of a chord of the circle $x^2 + y^2 + 2x = 0$, find the equation of the circle of which this chord is a diameter.

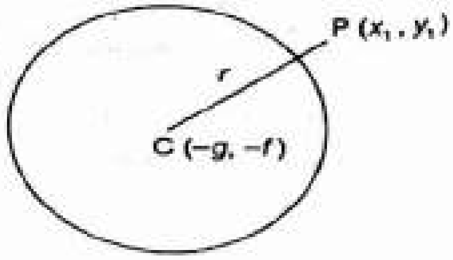
***Answer***

1. a) $x^2 + y^2 - 8x - 10y + 32 = 0$ b) $x^2 + y^2 = 16$
   c) $x^2 + y^2 - 2px - 2qy = 0$ d) $x^2 + y^2 - 8x + 2y = 0$
   e) $x^2 + y^2 - 16x + 4y - 32 = 0$ f) $x^2 + y^2 + 2x - 10y - 6 = 0$
   g) $x^2 + y^2 - 4x - 7y = 0$ h) $x^2 + y^2 + 8x - 6y + 16 = 0$
   i) $x^2 + y^2 - 8x + 12y - 49 = 0$ j) $x^2 + y^2 - 3x - 4y = 0$
   k) $x^2 + y^2 - 4x + 2y - 20 = 0$
2. a) $x^2 + y^2 - 6x - 8y + 9 = 0$ b) $x^2 + y^2 - 2ax - 2by + b^2 = 0$
   c) $x^2 + y^2 - 8x - 10y + 37 = 0$ d) $x^2 + y^2 + 4x + 4y + 4 = 0$
   e) $x^2 + y^2 - 2x - 2y + 1 = 0$, $x^2 + y^2 - 10x - 10y + 25 = 0$
   f) $x^2 + y^2 - 2ax - 2ay + a^2 = 0$
   g) $x^2 + y^2 - 16y + 48 = 0$, $x^2 + y^2 - 8x + 10y + 25 = 0$
3. a) (6, 2), 7 b) $\left(\frac{3}{2}, -1\right)$, 2 c) $\left(-\frac{3a}{2}, \frac{3a}{4}\right)$, $\frac{7a}{4}$
4. a) $x^2 + y^2 - ax - bx = 0$ b) $x^2 + y^2 - x + 3y - 10 = 0$
   c) $x^2 + y^2 - 17x - 19y + 50 = 0$
5. (5, 2), $\sqrt{5}$ 6. $x^2 + y^2 + 3x - y + 2 = 0$

## 15.3 A Point and a Circle

Let $P(x_1, y_1)$ be any point and $x^2 + y^2 + 2gx + 2fy + c = 0$ be a circle with centre $C(-g, -f)$ and radius

$$r = \sqrt{g^2 + f^2 - c}$$




Denoting

$$x^2 + y^2 + 2gx + 2fy + c$$

by $F(x, y)$,

$$F(x_1, y_1) = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$
$$= (x_1 + g)^2 + (y_1 + f)^2 - (g^2 + f^2 - c)$$
$$= CP^2 - r^2.$$

If P lies on the circle, $CP^2 = r^2$, so $F(x_1, y_1) = 0$;
If P lies inside the circle, $CP^2 < r^2$, so $F(x_1, y_1) < 0$;
If P lies outside the circle, $CP^2 > r^2$, so $F(x_1, y_1) > 0$.

As a particular case, any point $(x_1, y_1)$ will lie inside, on or outside the circle $x^2 + y^2 = a^2$ according as $x_1^2 + y_1^2$ is $<\, =$ or $> a^2$.

## 15.4 A Line and a Circle

Let $y = mx + c$ and $x^2 + y^2 = a^2$ be the equations of a line and a circle respectively. If the line intersects the circle, the points of intersection can be obtained by solving the two equations simultaneously.

Thus, using $\quad y = mx + c$ in $x^2 + y^2 = a^2$, we get

$$x^2 + (mx + c)^2 = a^2$$

$$\therefore \quad (1 + m^2)x^2 + 2mcx + c^2 - a^2 = 0 \quad \ldots\ \ldots\ldots\ \ldots\ldots\ \ldots(1)$$

This is quadratic in $x$, hence, in general, $x$ has two values. Corresponding to the two values of $x$, we can obtain the two values of $y$. Hence the coordinates of the points of intersection of the line and the circle are obtained. The following three cases correspond to the three values of $x$.

i) The roots of (1), i.e. the values of $x$ are real and distinct. In this case the line cuts the circle at *two distinct points*.

ii) The roots of (1) are real and equal. In this case the line meets the circle at *two coincident points*. We say that the line is *tangent* to the circle.

iii) The roots of (1) are *imaginary*. In this case the line *does not intersect* the circle. It is customary to say that the line cuts the circle at the *imaginary points*.

Since the discriminant of (1) is

$$4m^2c^2 - 4(1 + m^2)(c^2 - a^2)$$

i.e. $\quad 4(m^2c^2 - c^2 + a^2 - m^2c^2 + m^2a^2)$

i.e. $\quad 4[a^2(1 + m^2) - c^2]$

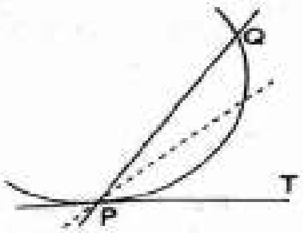
Hence the line $y = mx + c$ will

(i) intersect the circle $x^2 + y^2 = a^2$ at two distinct points,

(ii) meet the circle at two coincident points

or, (iii) not cut the circle, according as $a^2(1 + m^2) >\, =$ or $< c^2$.

## 15.5 Tangents and Normals to any Curve




Let P be any point on a curve. Take another point Q close to P on the curve. Join the secant line PQ. Now, keeping P fixed, let Q move along the curve towards the point P. In the limiting case when Q tends to P, the secant line PQ will tend to a definite line PT through P. This position PT (when $Q \to P$) of PQ is called the *tangent to the curve at P*; and the point P is called the *point of contact*. In other words, the tangent to the curve at any point meets the curve at *two coincident points*.

A straight line drawn perpendicular to the tangent at the point of contact is called the *normal to* the curve at that point.

In case of a circle, a tangent is also a line which is perpendicular to the radius at the point of contact; and the normal to a circle always passes through the centre.

## 15.6 Equation of the Tangent to the Circle $x^2 + y^2 = a^2$ at a Point $(x_1, y_1)$ on the Circle

### Method I (General method)

Let $P(x_1, y_1)$ be the point on the circle

$$x^2 + y^2 = a^2.$$

Take another point $Q(x_2, y_2)$ on the circle, near to P.

then, $\quad x_1^2 + y_1^2 = a^2 \quad \text{... ... (1)}$

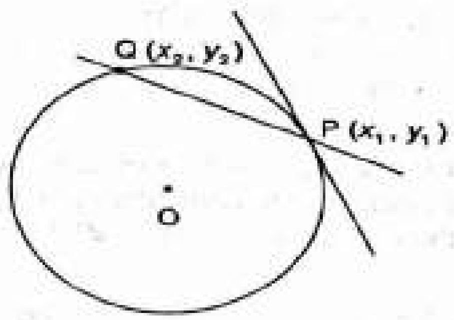
and $\quad x_2^2 + y_2^2 = a^2. \quad \text{... ... (2)}$

Subtracting, we have

$$y_2^2 - y_1^2 = x_1^2 - x_2^2$$

or, $\quad (y_2 - y_1)(y_2 + y_1) = (x_1 + x_2)(x_1 - x_2)$

or, $\quad \dfrac{y_2 - y_1}{x_2 - x_1} = -\dfrac{x_1 + x_2}{y_2 + y_1} \quad \text{... ... (3)}$




Also equation of PQ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1). \qquad \text{(two points formula)}$$

i.e. $\quad y - y_1 = -\dfrac{x_1 + x_2}{y_1 + y_2}(x - x_1) \qquad \text{(using 3)}$

By definition, the secant PQ becomes the tangent at P when Q approaches P, i.e. when $x_2 \to x_1$, and $y_2 \to y_1$.

Hence the equation of the tangent at P is

$$y - y_1 = -\frac{x_1}{y_1}(x - x_1)$$

i.e. $\quad yy_1 - y_1^2 = -xx_1 + x_1^2$

or, $\quad xx_1 + yy_1 = x_1^2 + y_1^2$

or $\quad xx_1 + yy_1 = a^2.$

**Note :** The method of finding the equation of a tangent by the above method is perfectly general. This method is used to determine the equation of tangent to any curve, namely parabola, etc. as we shall see in subsequent chapter.

But in case of a circle we can find the equation of a tangent at any point of it using the geometrical property of the tangent to a circle as shown in the special method.

### Method II (Special method)

Let $P(x_1, y_1)$ be a point on the circle $x^2 + y^2 = a^2$.

so that $\quad x_1^2 + y_1^2 = a^2 \quad \ldots\ldots(1)$

The tangent at P is perpendicular to the radius through P, i.e. perpendicular to OP.

The slope of $OP = \dfrac{y_1}{x_1}$.

Hence slope of the tangent at $P = -\dfrac{x_1}{y_1}$

$\therefore$ the equation of the tangent at P is

$$y - y_1 = -\frac{x_1}{y_1}(x - x_1)$$

i.e. $\quad xx_1 + yy_1 = x_1^2 + y_1^2$

or, $\quad xx_1 + yy_1 = a^2 \qquad$ using (1)

## 15.7 Tangent to the General Circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at a given point on it

Let $P(x_1, y_1)$ be a point on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$. Take another point $Q(x_2, y_2)$ on the circle close to P. Since P and Q lie on the circle, we have

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \qquad \ldots\ldots\ldots(1)$$

and

$$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0 \qquad \ldots\ldots\ldots(2)$$

Subtracting, we get

$$x_2^2 - x_1^2 + y_2^2 - y_1^2 + 2g(x_2 - x_1) + 2f(y_2 - y_1) = 0$$

or $(x_2 - x_1)(x_2 + x_1 + 2g) + (y_2 - y_1)(y_2 + y_1 + 2f) = 0$

or $$\frac{y_2 - y_1}{x_2 - x_1} = -\frac{x_1 + x_2 + 2g}{y_1 + y_2 + 2f} \quad \dots\dots\dots (3)$$

Now the equation of the secant PQ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

i.e. $$y - y_1 = -\frac{x_1 + x_2 + 2g}{y_1 + y_2 + 2f}(x - x_1) \quad \dots\dots (4) \text{ (using 3)}$$

As Q approaches and coincides with P, the secant PQ becomes the tangent at P.

Hence, writing $x_1$ for $x_2$ and $y_1$ for $y_2$ in (4), we get the equation of the tangent at P as

$$y - y_1 = -\frac{2x_1 + 2g}{2y_1 + 2f}(x - x_1)$$

or, $(y - y_1)(y_1 + f) + (x - x_1)(x_1 + g) = 0$

or, $xx_1 + yy_1 + gx + fy = x_1^2 + y_1^2 + x_1g + y_1f$

Adding $x_1g + y_1f + c$ to both sides, we get

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

Using (1) the equation of the tangent is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

**Note 1 :** From the equation of the tangent obtained above, it can be seen that the equation of the tangent at any point $(x_1, y_1)$ on the circle can be written down by replacing $x^2$ by $xx_1$, $y^2$ by $yy_1$, $2x$ by $x + x_1$ and $2y$ by $y + y_1$ in the equation of the circle.

**Note 2 :** The slope of the tangent to the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ at the point $(x_1, y_1)$ is $-\dfrac{x_1 + g}{y_1 + f}$.

## 15.8 Condition of Tangency of a Straight Line to a Circle

Let $y = mx + c$ be a straight line and $x^2 + y^2 = a^2$ be a circle.

Consider the points of intersection of the line and the circle.

From $y = mx + c$ and $x^2 + y^2 = a^2$,

we have $$x^2 + (mx + c)^2 = a^2$$

i.e. $$(1 + m^2)x^2 + 2mcx + (c^2 - a^2) = 0$$

The straight line will touch the circle if the two values of $x$ obtained from the above quadratic equation are equal. This will be so if

$$m^2c^2 - (1 + m^2)(c^2 - a^2) = 0$$

i.e. $$m^2a^2 - c^2 + a^2 = 0$$

or, $$c = \pm a\sqrt{1 + m^2}$$

Thus the line $y = mx + c$ will be a tangent to the circle

$$x^2 + y^2 = a^2 \quad \text{if} \quad c = \pm a\sqrt{1 + m^2}.$$

Hence the straight lines $y = mx \pm a\sqrt{1 + m^2}$ are always tangent to the circle $x^2 + y^2 = a^2$.

*Alternative Method*

The condition of tangency may also be deduced by using the geometrical property of a circle that a tangent line is always perpendicular to the radius passing through the point of contact. In other words, the length of the perpendicular from the centre of the circle to the line must be equal to the radius of the circle, if the line be a tangent to the circle. In the circle $x^2 + y^2 = a^2$, the centre is at the origin (0, 0) and the radius = $a$.

The length of the perpendicular from the centre on the line

$$y = mx + c \quad \text{is} \quad \pm \frac{c}{\sqrt{1 + m^2}}$$

$\therefore$ the condition of tangency is

$$\pm \frac{c}{\sqrt{1 + m^2}} = a$$

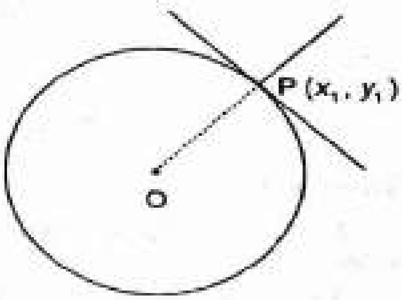
or $$c = \pm a\sqrt{1 + m^2}$$

## 15.9 Equation of the Normal to the Circle $x^2 + y^2 = a^2$ at the point $P(x_1, y_1)$ on the Circle

The equation of the tangent to the circle at $P(x_1, y_1)$ is

$$xx_1 + yy_1 = a^2.$$

So, the slope of the tangent $= -\dfrac{x_1}{y_1}$




Since the normal at P is perpendi-cular to the tangent at P, the slope of the normal is $\dfrac{y_1}{x_1}$.

Also the normal passes through $P(x_1, y_1)$

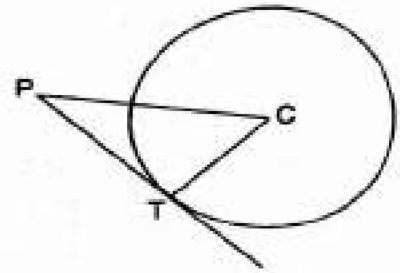
Hence its equation is

$$y - y_1 = \frac{y_1}{x_1}(x - x_1)$$

or, $x_1y = y_1x$.

Equation of the normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at any point on it can be similarly deduced.

## 15.20 Length of the Tangent from an External Point to a Circle




Let a tangent PT be drawn from an external point $P(x_1, y_1)$ to the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$

The centre of the circle is $C(-g, -f)$ and its radius $= \sqrt{g^2 + f^2 - c}$

Hence $PC^2 = (x_1 + g)^2 + (y_1 + f)^2$

and, $CT^2 = g^2 + f^2 - c$

Also $CT \perp PT$,

$$\therefore \quad PT^2 = PC^2 - CT^2$$
$$= (x_1 + g)^2 + (y_1 + f)^2 - (g^2 + f^2 - c)$$
$$= x_1^2 + 2gx_1 + y_1^2 + 2fy_1 + c$$
$$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c.$$

Hence the length of the tangent PT

$$= \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$

**Note.** If the equation of the circle be $x^2 + y^2 = a^2$, the length of the tangent from an external point $(x_1, y_1)$ can be similarly shown to be equal to

$$\sqrt{x_1^2 + y_1^2 - a^2}.$$

## Worked Out Examples

***Example 1***

Find the equations of tangents and normals to each of the following circles :

(a) $x^2 + y^2 = 25$ at $(3, 4)$

(b) $x^2 + y^2 - 2x - 4y + 3 = 0$ at $(2, 3)$.

***Solution:***

**(a)** Equation of the tangent at (3, 4) is

$$x.3 + y.4 = 25$$

or, $3x + 4y = 25.$

Hence, the slope of the tangent $= -\frac{3}{4}$.

Slope of the normal $= \frac{4}{3}$.

Equation of the normal at (2, 3) is

$$y - 4 = \frac{4}{3}(x - 3)$$

or, $3y - 12 = 4x - 12$

or, $3y - 4x = 0.$

(b) Equation of the tangent at (2, 3) of the circle

$$x^2 + y^2 - 2x - 4y + 3 = 0$$

is $x.2 + y.3 - (x + 2) - 2(y + 3) + 3 = 0$

or, $x + y - 5 = 0.$

Hence, the slope of the tangent $= -1$.

Slope of the normal $= 1$.

Equation of the normal at (2, 3) is

$$y - 3 = 1(x - 2)$$

or $x - y + 1 = 0.$

***Example 2***

Find the equations of the tangents to the circle

$$x^2 + y^2 - 4x - 6y - 12 = 0$$

which are parallel to the line $3x + 4y + 1 = 0$.

***Solution:***

The centre of the circle is at (2, 3) and its radius = 5. Equation of any line parallel to $3x + 4y + 1 = 0$ is $3x + 4y + k = 0$. If this line be a tangent to the given circle, the perpendicular distance from the centre on the line should be equal to the radius.

i.e. $\pm \frac{3.2 + 4.3 + k}{\sqrt{3^2 + 4^2}} = 5$

or, $(k + 18) = \pm 25$

or, $k = -43, 7.$

Hence the required equations of the tangents are

$$3x + 4y - 43 = 0$$

and $3x + 4y + 7 = 0.$

***Example 3***

Show that the line $\frac{x}{a} + \frac{y}{b} = 1$ will be tangent to the circle

$$(x-a)^2 + (y-b)^2 = r^2 \qquad \text{if} \qquad \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{r^2}$$

***Solution :***

The given equation of the circle is

$$(x-a)^2 + (y-b)^2 = r^2 \qquad \text{...... (i)}$$

Centre of the circle = $(a, b)$ and radius = $r$

Length of the perpendicular from the centre $(a, b)$ on the line $\frac{x}{a} + \frac{y}{b} = 1$

is $$\frac{1+1-1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} = \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}$$

The line will be tangent to the circle (i) if

$$\frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} = r$$

or, $$r^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = 1$$

i.e. $$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{r^2}$$

***Example 4***

Find the equations of the tangents to the circle $x^2 + y^2 = 25$ drawn through the point (13, 0)

***Solution :***

The given equation of the circle is

$$x^2 + y^2 = 25 \qquad \text{...... (i)}$$

Centre of the circle = (0, 0) and radius = 5

The equation of the line through the point (13, 0) is

$$y = m(x - 113)$$

$$y - mx + 13m = 0 \qquad \text{...... (ii)}$$

where $m$ is the slope of the line (ii)

Length of the perpendicular drawn from the centre (0, 0) on the line (ii)

$$= \pm\frac{0 - 0 + 13m}{\sqrt{1 + m^2}} = \pm\frac{13m}{\sqrt{1+m^2}}$$

The line (ii) will be tangent to the circle (i) if

$$\pm\frac{13m}{\sqrt{1+m^2}} = 5$$

or, $169m^2 = 25(1 + m^2)$

or, $144m^2 = 25$

$\therefore \quad m = \pm\dfrac{5}{12}$

Substituting the value of $m$ in (ii), the equations of the tangents are

$$12y = \pm 5(x - 13)$$

***Example 5***

Find the value of k so that the length of the tangent from (5, 4) to the circle $x^2 + y^2 + 2ky = 0$ is 1.

***Solution:***

The length of the tangent from (5, 4) to the circle

$= \sqrt{5^2 + 4^2 + 2.k.4}$

$\therefore \quad \sqrt{25 + 16 + 8k} = 1$

$\Rightarrow \quad 41 + 8k = 1$

$\Rightarrow \quad 8k = -40$

$\therefore \quad k = -5$

## EXERCISE 15.2

1. Find the equations of tangents and normals to the following circles at the points mentioned against each.
   (a) $x^2 + y^2 = 8$ at (2, 2)
   (b) $x^2 + y^2 = 36$ at (– 6, 0)
   (c) $x^2 + y^2 - 3x + 10y - 15 = 0$ at (4, – 11).
2. a) Show that the line $3x - 4y = 25$ and the circle $x^2 + y^2 = 25$ intersect in two coincident points.

   b) Prove that the line $5x + 12y + 78 = 0$ is tangent to the circle $x^2 + y^2 = 36$.

   c) Prove that the tangent to the circle $x^2 + y^2 = 5$ at the point (1, –2) also touches the circle $x^2 + y^2 - 8x + 6y + 20 = 0$ and find the point of contact.

3. Find the equation of the tangents to the circle
   (a) $x^2 + y^2 = 4$, which are parallel to $3x + 4y - 5 = 0$.
   (b) $x^2 + y^2 = 5$, which are perpendicular to $x + 2y = 0$.
   (c) $x^2 + y^2 - 6x + 4y = 12$, which are parallel to the line $4x + 3y + 5 = 0$.
   (d) $x^2 + y^2 - 2x - 4y - 4 = 0$, which are perpendicular to the line $3x - 4y = 1$.

4. Find the value of $k$ so that
   a) the line $4x + 3y + k = 0$ may touch the circle $x^2 + y^2 - 4x + 10y + 4 = 0$
   b) the line $2x - y + 4k = 0$ touches the circle $x^2 + y^2 - 2x - 2y - 3 = 0$

5. a) Find the condition that the line $px + qy = r$ is tangent to the circle $x^2 + y^2 = a^2$
   b) Deduce the condition that $lx + my + n = 0$ may be a tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$
   c) Find the condition that the line $lx + my + n = 0$ should be a normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.

6. a) Show that the tangents to the circle $x^2 + y^2 = 100$ at the points (6, 8) and (8, –6) are perpendicular to each other.
   b) Prove that the tangents to the circle $x^2 + y^2 + 4x + 8y + 2 = 0$ at the points (1, –1) and (–5, –7) are parallel.

7. Find the equation of the circle whose centre is at the point ( $h$, $k$) and which passes through the origin and prove that the equation of the tangent at the origin is $hx + ky = 0$

8. If the line $lx + my = 1$ touches the circle $x^2 + y^2 = a^2$, prove that the point $(l, m)$ lies on a circle whose radius is $\frac{1}{a}$

9. a) Find the conditions for the two circles $x^2 + y^2 = a^2$ and $(x - c)^2 + y^2 = b^2$ to touch (i) externally (ii) internally.
   b) Prove that the two circles $x^2 + y^2 + 2ax + c^2 = 0$ and $x^2 + y^2 + 2by + c^2 = 0$ touch if $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$.

10. Find the equations of the tangents drawn from the point (11, 3) to the circle $x^2 + y^2 = 65$.

11. Find the points of intersection of the line $x + y - 3 = 0$ and the circle $x^2 + y^2 - 2x - 3 = 0$. Also find the length of the intercept made by the line with the circle.

$h=3$
$k=-2$
$d=\left|\frac{c_2-c_1}{\sqrt{A^2+B^2}}\right|$

12. A circle touches the parallel lines $3x - 4y = 7$ and $3x - 4y + 43 = 0$ and has its centre on the line $2x - 3y + 13 = 0$. Find its equation.

13. Prove that the straight line $y = x + a\sqrt{2}$ touches the circle $x^2 + y^2 = a^2$ and find its point of contact.

14. Tangents are drawn from the origin to the circle $x^2 + y^2 + 10x + 10y + 40 = 0$. Find their equations.

15. Determine the length of the tangent to the circle
    a) $x^2 + y^2 = 25$ from (3, 5)
    b) $x^2 + y^2 + 4x + 6y - 19 = 0$ from (6, 4)

16. Determine the value of k so that the length of the tangent from (5, 4) to the circle $x^2 + y^2 + 2ky = 0$ is 5.

17. Show that the length of the tangent drawn from any point on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ to the circle $x^2 + y^2 + 2gx + 2fy + c_1 = 0$ is $\sqrt{c_1 - c}$.

18. Find the eq$^n$ of the line through the point (1, -1) which cuts of a chord of length $4\sqrt{3}$ m from the circle $x^2+y^2-6x+4y-3=0$

## Answer

Ans: $\boxed{3x-4y-7=0}$

1. (a) $x + y = 4, x - y = 0$ (b) $x = -6, y = 0$
   (c) $5x - 12y - 152 = 0$, $12x + 5y + 7 = 0$
2. (c) (3, −1)
3. (a) $3x + 4y \pm 10 = 0$ (b) $2x - y \pm 5 = 0$
   (c) $4x + 3y + 19 = 0$, $4x + 3y - 31 = 0$
   (d) $4x + 3y + 5 = 0$, $4x + 3y - 25 = 0$
4. (a) $k = 32$ or $-18$ b) 1 or $-\frac{3}{2}$
5. (a) $a^2(p^2 + q^2) = r^2$ b) $(g^2 + f^2 - c)(l^2 + m^2) = (ln - lg - mf)^2$
   c) $n = gl + mf$
7. $x^2 + y^2 - 2hx - 2ky = 0$
9. a) $a + b = c$ b) $a - b = c$
10. $7x - 4y - 65 = 0$, $4x + 7y - 65 = 0$
11. (1, 2), (3, 0); $2\sqrt{2}$
12. $x^2 + y^2 + 4x - 6y - 12 = 0$
13. $(-a/\sqrt{2}, a/\sqrt{2})$
14. $3x - y = 0$, $x - 3y = 0$
15. a) 3 b) 9
16. −2

36+16

# CHAPTER 16
# Limits and Continuity

## 16.1 Introduction

We have discussed a function and its graph in chapter I. As a review, we give the definition of a function, its domain, range and its graph once again before defining the limit of a function.

We deal with limits and continuity which are quite fundamental for the development of calculus. These two concepts are closely linked together with the involvement of the concept of limit in the definition of continuity. So in the sequence, limit comes first and it is proper to begin with some discussion about it. The discussion is initiated with some examples so as to give some intuitive idea about it. Then follows the precise definition of the limit. The same line of approach is being followed in the case of continuity as well. We shall also mention some limit theorems and properties of continuous functions without proof.

### Function

Let X and Y be two non-empty sets. Then a function $f$ from X to Y is a rule which assigns a unique element of Y to each element of X. The unique element of Y which $f$ assigns corresponding to an element $x \in X$ is denoted by $f(x)$. So, we also write $y = f(x)$. The symbol $f : X \to Y$ usually means '$f$ is a function from X to Y'. The element $f(x)$ of Y is called the image of $x$ under the function $f$.

### Value of the Function

If $f$ is a function from X to Y and $x = a$ is an element in the domain of $f$, then the image $f(a)$ corresponding to $x = a$ is said to be the value of the function at $x = a$.

If the value of the function $f(x)$ at $x = a$ denoted by $f(a)$ is a finite number, then $f(x)$ exists or is defined at $x = a$ otherwise, $f(x)$ does not exist or is not defined at $x = a$. For example :

i) $y = f(x) = 3x + 5$ exists or is defined at $x = 2$ as

$f(2) = 3 \times 2 + 5 = 11$ is a finite number.

ii) $y = f(x) = \dfrac{1}{x-1}$ is not defined at $x = 1$ as

$$f(1) = \frac{1}{0} \text{ is not a finite number.}$$

Hence $f(x)$ does not exist or is undefined at $x = 1$.

Consider the function $y = f(x) = \dfrac{x^2 - 1}{x - 1}$ and $x = 1$.

When $x = 1$, $y = f(0) = \dfrac{0}{0}$. This does not give any number. It simply indicates that the number in the numerator and denominator are each zero. So, there are some functions that take the form $\dfrac{0}{0}$ for some value of $x$. Such form is said to be an **indeterminate form.** Other indeterminate forms which a function may take for some values of $x$ are $\dfrac{\infty}{\infty}$, $\infty - \infty$, $1^{\infty}$ and $0^{\infty}$.

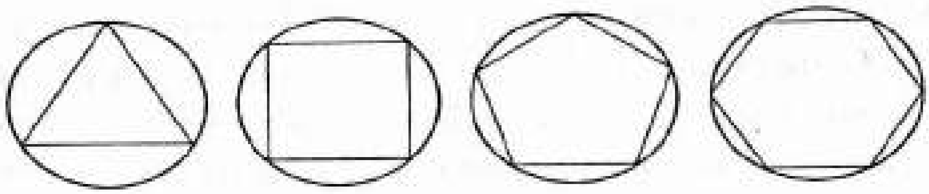
## 16.2 Meaning of x → a

Before giving the meaning of $x \to a$, we consider an example to illustrate the meaning of $x \to 2$. Let $x$ be a variable and let us make the variable $x$ to take the values 1.9, 1.99, 1.999, 1.9999, .... As the number of 9's increases, the value of $x$ will be nearer and nearer to 2 but will never be 2. In such a situation, the numerical difference between $x$ and 2 will be ve[illegible] small. Again we let the variable $x$ take the values 2.1, 2.01, 2.001, 2.000[illegible] .... As number of zeros increases, but due to the presence of 1 at the end, the value of $x$ will be nearer and nearer to 2 but will never be equal to 2. In such a situation also, the numerical difference between $x$ and 2 will be sufficiently small. Thus if $x$ takes the value greater than 2 or less than 2 but the numerical difference between $x$ and 2 is sufficiently small, then we say that $x$ approaches 2 or $x$ tends to 2 and is written as $x \to 2$.

Let $x$ be a variable and '$a$' a constant number. If $x$ takes a value such that the numerical difference between $x$ and $a$ is sufficiently small, then we say that $x$ tends to $a$ and is written as $x \to a$.

## 16.3 Intuitive Idea of Limit

In this section we try to give a clear concept of limit by means of some examples. Consider a regular polygon inscribed in a circle. Keeping the circle fixed, as we increase the number of sides of the polygon, the area (or the perimeter) of the polygon also increases. But, no matter however large may be the number of sides, the area (or the perimeter) of the polygon can never be greater than that of the circle. But by taking the number of the sides of the polygon sufficiently large, we can make the difference between the area (or the perimeter) of the polygon and the area (or the perimeter) of the circle sufficiently small (or as small as we please). If we put this in the mathematical language, we say that given a positive small number $\epsilon$, no matter however small it may be, it is always possible to obtain a number $n$ such that the difference between the area (or the perimeter) of the polygon with $n$ sides inscribed in the circle and the area (or the circumference) of the circle is less than $\epsilon$. In such a case we say that the area (or the circumference) of the circle is the limit of a series of areas (or perimeters) of the polygons obtained by giving the sequence of integral values to $n$.



Now, let us put the above example in a functional notation. For this, we shall denote the area of a regular polygon with $n$ sides inscribed in a fixed circle by $A_n$, $n > 2$. Then the set of ordered pairs $(n, A_n)$ gives the increasing sequence of areas of polygons. This functional relation can also be written as

$$f(n) = A_n$$

When $n$ tends to infinity, $f(n)$ or $A_n$ gets close to (or approaches) the area of the circle, that is to say, the limiting value of $f(n)$ or $A_n$ is the area of the circle and is denoted by

$$\lim_{n \to \infty} f(n) \quad \text{or} \quad \lim_{n \to \infty} A_n$$

The next example we take is the sequence of numbers

0.9, 0.99, 0.999, ......

In this sequence the terms are gradually increasing but remain always less than 1. But by making a proper choice of the term, we can make the term sufficiently close to 1 or make the difference between 1 and the term *sufficiently small* (or *as small as we please*). In such a case we say that the sequence tends to the limiting value 1.

The above sequence can also be put in a functional notaton by defining the function $f$ by

| | |
|---|---|
| f(1) = 0.9, | the 1st term |
| f(2) = 0.99, | the 2nd term |
| f(3) = 0.999, | the 3rd term |
| ... ...... | ............... |
| $f(n) = 0.99 \ldots 9$(n 9's), | the $n^{th}$ term. |

When $n$ tends to infinity, $f(n)$ becomes almost equal to 1. So the limiting value of $f(n)$ is 1 and it is denoted by

$$\lim_{n \to \infty} f(n) = 1$$

Now we conclude this section with one more example.This example is the series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \ldots$$

The series is such that no matter how many terms we may take, the sum of the series can never be equal to 2. But by taking sufficiently large number of terms, we can make the *sum approach* sufficiently near to 2. This we can see more clearly by using functional notation.

Let $S_n$ be the sum of $n$ terms of the series. So,

$$S_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\ldots \text{ to n terms}$$

$$= 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots \text{ to n terms}$$

$$= \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} = 2 - \frac{1}{2^{n-1}}$$

$$S_n = 2 - \frac{1}{2^{n-1}}$$

Now it is obvious that when $n$ is sufficiently large, $\frac{1}{2^{n-1}}$ is sufficiently small. So when $n$ tends to infinity, $S_n$ tends to 2; i.e., the *limiting value* of $S_n$ is 2 and it is denoted by

$$\lim_{n \to \infty} S_n = 2$$

## 16.4 Limit of a Function

We use the concept of the limit of a sequence to understand the meaning of the limit of a function.

First, we consider the function $y = f(x) = 2x + 3$.

Considering the sequence of values of $x$ to be 0.5, 0.75, 0.9, 0.99, 0.999, 0.9999, ... whose limit is 1, we see the corresponding values of $f(x)$ are 4, 4.5, 4.8, 4.98, 4.998, 4.9998, ... which go nearer and nearer to 5 when $x$ is very near to 1. So, when $x$ is sufficiently close to 1, $f(x)$ is very close to 5.

Again if we consider the sequence of values of $x$ to be 2, 1.5, 1.25, 1.1, 1.01, 1.001, 1.0001, ... whose limit is 1, we shall find the corresponding values of $f(x)$ to be 7, 6, 5.5, 5.2, 5.02, 5.002, 5.0002, ... which go nearer and nearer to 5 when $x$ is very close to 1. So, when $x$ is sufficiently close to 1, $f(x)$ is very close to 5. That is, when $x \to 1, f(x) \to 5$. In symbol, we write

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} (2x + 3) = 5$$

Hence, we have the following definition of limit of a function.

The number '$l$' to which the value of a function $f(x)$ approaches when $x$ approaches a certain number '$a$' is said to be the limiting value of $f(x)$.

We can define limit of a function in the following way also.

A function $f(x)$ is said to tend to a limit '$l$' when $x \to a$ if the numerical difference between $f(x)$ and $l$ can be made as small as we please by making $x$ sufficiently close to $a$ and we write

$$\lim_{x \to a} f(x) = l$$

### Meaning of Infinity ($\infty$)

Let us consider the function $y = f(x) = \frac{1}{x}$

If we consider the sequence of values of $x$ to be 1, 0.5, 0.1, 0.01, 0.001, 0.0001, ... whose limit is 0, we see that the corresponding values of $f(x)$ are 1, 2, 10, 100, 1000, 10000, ... which go on increasing. If we take $x$ small enough, the corresponding value of $f(x)$ will be large enough. Taking the value of $x$ to be sufficiently close to 0, the value of $f(x)$ will be greater than any positive number, however large. In such a case, we say that as $x$ tends to zero, $f(x)$ tends to infinity and is indicated by the symbol, $f(x) \to \infty$ as $x \to 0$

or, $$\lim_{x \to 0} \frac{1}{x} = \infty$$

### Infinity as a Limit of a Function

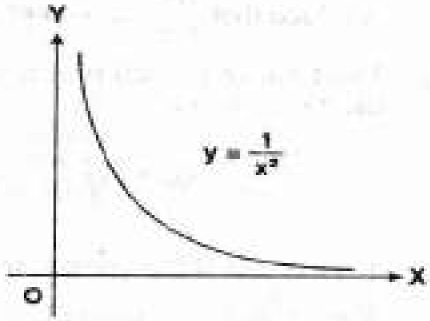
Let $f(x)$ be a function of $x$. Making $x$ sufficiently close to $a$, if the value of $f(x)$ obtained **is greater** than any **pre-assigned number**, however large, we say that the limit of $f(x)$ is infinity as $x$ tends to $a$. Symbolically, we write

$$\lim_{x \to a} f(x) = \infty$$

### Limit at Infinity

Let us consider the function $f(x) = \frac{1}{x^2}$ and we see its nature when the value of $x$ goes on increasing. The following table shows the values of $x$ and the corresponding values of $y$.

| $x$ | 1 | 10 | 100 | 1000 | ... |
|---|---|---|---|---|---|
| $f(x)$ | 1 | 0.01 | 0.0001 | 0.000001 | ... |




From the above table, we see that when the value of $x$ increases, the corresponding value of $f(x)$ decreases. When the value of $x$ becomes large enough, the corresponding value of $f(x)$ becomes small enough. That is, taking the value of $x$ to be sufficiently large i.e. the value greater than any positive number, however large, the value of $f(x)$ can be made sufficiently close to 0. In such a situation, we say that as $x$ tends to infinity, $f(x)$ tends to zero and is indicated by the symbol, $f(x)\to 0$ when $x\to\infty$,

$$\text{or,} \qquad \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{x^2} = 0$$

A function $f(x)$ is said to tend to '$l$' when $x \to \infty$ if $f(x)$ can be made close to '$l$' when $x$ is greater than any pre-assigned number, however large. Symbolically, we write,

$$\lim_{x\to\infty} f(x) = l$$

## 16.5 Limit Theorems

Let $f(x)$ and $g(x)$ be two functions of $x$ such that $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$, then we have the following theorems on limits :

i) The limit of the sum (or difference) of the functions $f(x)$ and $g(x)$ is the sum (or difference) of the limits of the functions. i.e.

$$\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = l \pm m$$

ii) The limit of the product of the functions $f(x)$ and $g(x)$ is the product of the limits of the functions. i.e.

$$\lim_{x\to a} [f(x) \,.\, g(x)] = \left(\lim_{x\to a} f(x)\right) . \left(\lim_{x\to a} g(x)\right) = l.m$$

iii) The limit of the quotient of the function $f(x)$ and $g(x)$ is the quotient of the limits of the functions, provided that the limit of the denominator is not zero i.e.

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}=\frac{l}{m}$$

provided that $\lim_{x\to a} g(x) = m \neq 0$.

iv) The limit of the $n$th root of a function $f(x)$ is the $n$th root of the limit of the function. i.e.

$$\lim_{x\to a}\sqrt[n]{f(x)}=\sqrt[n]{\lim_{x\to a}f(x)}=\sqrt[n]{l}$$

## A. Important Theorem on Limit

1. For all rational values of $n$,

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$$

The proof of this theorem consists of the following three cases.

**Case I :**

When $n$ is a positive integer :

By actual division,

$$\frac{x^n-a^n}{x-a}=x^{n-1}+x^{n-2}.a+x^{n-3}.a^2+\dots+a^{n-1}$$

Now, $\lim_{x\to a}\frac{x^n-a^n}{x-a}=\lim_{x\to a}[x^{n-1}+x^{n-2}.a+x^{n-3}.a^2+\dots a^{n-1}]$

$$=a^{n-1}+a^{n-1}+a^{n-1}+\dots+a^{n-1}$$

$$=n\,a^{n-1}$$

**Case II :**

When $n$ is a negative integer :

Let $n=-m$ where $m$ is a positive integer

Then,

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=\lim_{x\to a}\frac{x^{-m}-a^{-m}}{x-a}$$

$$=\lim_{x\to a}\frac{\frac{1}{x^m}-\frac{1}{a^m}}{x-a}$$

$$=\lim_{x\to a}\frac{a^m-x^m}{x^m a^m(x-a)}$$

$$= \lim_{x \to a} \left[ -\frac{x^m - a^m}{x - a} \times \frac{1}{x^m a^m} \right]$$

$$= -\left( \lim_{x \to a} \frac{x^m - a^m}{x - a} \right) \left( \lim_{x \to a} \frac{1}{x^m a^m} \right)$$

$$= -m.\, a^{m-1} \frac{1}{a^m . a^m} \quad \text{(using case I)}$$

$$= (-m)\, a^{(-m)-1} = n\, a^{n-1}$$

**Case III :**

**When $n$ is a rational fraction:**

Let $n = \frac{p}{q}$ where $p$ and $q$ are integers and $q \neq 0$.

Then, $\lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{x \to a} \frac{x^{p/q} - a^{p/q}}{x - a}$

$$= \lim_{x \to a} \frac{(x^{1/q})^p - (a^{1/q})^p}{x - a}$$

Put $x^{1/q} = y$ and $a^{1/q} = b$ so that $x = y^q$ and $a = b^q$

when $x \to a$, $y \to b$

Now, $\lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{y \to b} \frac{y^p - b^p}{y^q - b^q} = \lim_{y \to b} \frac{\frac{y^p - b^p}{y - b}}{\frac{y^q - b^q}{y - b}}$

$$= \frac{\lim_{y \to b} \frac{y^p - b^p}{y - b}}{\lim_{y \to b} \frac{y^q - b^q}{y - b}} = \frac{p\, b^{p-1}}{q\, b^{q-1}} = \frac{p}{q} . b^{p-q}$$

$$= \frac{p}{q} b^{q(p/q - 1)} = \frac{p}{q} . (b^q)^{p/q - 1} = n\, a^{n-1}.$$

$\therefore$ for all rational values of $n$,

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = n\, a^{n-1}$$

## A. Limits of Algebraic Functions

***Example 1.***

Find the limiting value of $f(x) = 3x - 2$, when $x$ approaches 3 or evaluate $\lim_{x \to 3} f(x)$.

***Solution:***

When $x$ approaches 3, $3x$ approaches $3 \times 3 = 9$. So $3x - 2$ approaches $9 - 2 = 7$. Therefore,

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} (3x - 2) = 7$$

***Example 2.***

Find the limiting value of $f(x) = 3x^2 - 5x + 6$, when $x$ approaches 2.

***Solution:***

When $x$ approaches 2, $3x^2$ approaches $3 \times 2^2 = 12$, and $5x$ approaches $5 \times 2 = 10$

Therefore, $$\lim_{x \to 2} (3x^2 - 5x + 6) = 12 - 10 + 6 = 8$$

***Example 3.***

Evaluate $\lim_{x \to 3} \dfrac{4x - 5}{2x + 3}$

***Solution:***

When $x$ approaches 3, $4x - 5$ approaches $4 \times 3 - 5 = 7$ and $2x + 3$ approaches $2 \times 3 + 3 = 9$.

Therefore, $$\lim_{x \to 3} \frac{4x - 5}{2x + 3} = \frac{7}{9}$$

***Example 4.***

Evaluate $\lim_{x \to 0} \dfrac{5x^2 + 3x}{x}$

***Solution:***

When $x = 0$, the function $\dfrac{5x^2 + 3x}{x}$ takes the form $\frac{0}{0}$, which is indeterminate. Therefore,

$$\lim_{x \to 0} \frac{5x^2 + 3x}{x} = \lim_{x \to 0} \frac{x(5x + 3)}{x}$$
$$= \lim_{x \to 0} (5x + 3)$$
$$= 0 + 3 = 3$$

***Example 5.***

Evaluate $\lim_{x \to a} \dfrac{x^5 - a^5}{x^4 - a^4}$

**Solution:**

Following the argument used in the solution of Ex. 4.

We have,

$$\lim_{x\to a} \frac{x^5 - a^5}{x^4 - a^4}$$

$$= \lim_{x\to a} \frac{(x-a)(x^4 + x^3a + x^2a^2 + xa^3 + a^4)}{(x-a)(x^3 + x^2a + xa^2 + a^3)}$$

$$= \lim_{x\to a} \frac{x^4 + x^3a + x^2a^2 + xa^3 + a^4}{x^3 + x^2a + xa^2 + a^3}$$

$$= \frac{5a^4}{4a^3} = \frac{5a}{4}$$

**Example 6.**

Evaluate $\lim_{x\to a} \frac{x^{1/3} - a^{1/3}}{x^{1/2} - a^{1/2}}$

**Solution:**

The given function takes the indeterminate form $\frac{0}{0}$, when $x = a$. But

$$\lim_{x\to a} \frac{x^{1/3} - a^{1/3}}{x^{1/2} - a^{1/2}} = \lim_{x\to a} \frac{(x^{1/6})^2 - (a^{1/6})^2}{(x^{1/6})^3 - (a^{1/6})^3}$$

$$= \lim_{x\to a} \frac{(x^{1/6} - a^{1/6})(x^{1/6} + a^{1/6})}{(x^{1/6} - a^{1/6})(x^{2/6} + x^{1/6}a^{1/6} + a^{2/6})}$$

$$= \lim_{x\to a} \frac{x^{1/6} + a^{1/6}}{x^{1/3} + x^{1/6}a^{1/6} + a^{1/6}}$$

$$= \frac{2a^{1/6}}{3a^{1/3}} = \frac{2}{3a^{1/6}}$$

**Example 7**

Evaluate $\lim_{x\to a} \frac{\sqrt{x+a} - \sqrt{3x-a}}{x-a}$

**Solution:**

The given function takes the indeterminate form $\frac{0}{0}$, when $x = a$. But

$$\lim_{x\to a} \frac{\sqrt{x+a} - \sqrt{3x-a}}{x-a}$$

$$= \lim_{x\to a} \frac{\left(\sqrt{x+a} - \sqrt{3x-a}\right)\left(\sqrt{x+a} + \sqrt{3x-a}\right)}{(x-a)\left(\sqrt{x+a} + \sqrt{3x-a}\right)}$$

$$= \lim_{x \to a} \frac{x + a - 3x + a}{(x - a)\left(\sqrt{x + a} + \sqrt{3x - a}\right)}$$

$$= \lim_{x \to a} \frac{-2(x - a)}{(x - a)\left(\sqrt{x + a} + \sqrt{3x - a}\right)}$$

$$= \lim_{x \to a} \frac{-2}{\sqrt{x + a} + \sqrt{3x - a}}$$

$$= \frac{-2}{\sqrt{2a} + \sqrt{2a}} = \frac{-1}{\sqrt{2a}}$$

***Example 8***

Evaluate $\lim_{x \to \infty} \frac{3x^2 + 2x + 1}{4x^2 + x + 5}$

***Solution:***

The given function takes the indeterminate form $\frac{\infty}{\infty}$, when $x = \infty$.

We get $\lim_{x \to \infty} \frac{3 + \frac{2}{x} + \frac{1}{x^2}}{4 + \frac{1}{x} + \frac{5}{x^2}} = \frac{3 + 0 + 0}{4 + 0 + 0} = \frac{3}{4}$

***Example 9***

Evaluate $\lim_{x \to \infty} \left(\sqrt{x + a} - \sqrt{x}\right)$ (HSEB 2058)

***Solution:***

The given function takes the indeterminate form $\infty - \infty$, when $x = \infty$. So by multiplying the numerator and the denominator by $\sqrt{x + a} + \sqrt{x}$, we have

$$\lim_{x \to \infty} \frac{\left(\sqrt{x + a} - \sqrt{x}\right)\left(\sqrt{x + a} + \sqrt{x}\right)}{\left(\sqrt{x + a} + \sqrt{x}\right)}$$

$$= \lim_{x \to \infty} \frac{x + a - x}{\sqrt{x + a} + \sqrt{x}} = \lim_{x \to \infty} \frac{a}{\sqrt{x + a} + \sqrt{x}}$$

$$= \frac{a}{\infty + \infty} = \frac{a}{\infty}$$

$$= 0$$

## EXERCISE 16.1

1. Find the following limits:

(a) $\lim_{x\to 2} (2x^2 + 3x - 14)$ (b) $\lim_{x\to 5} (x^2 + 2x - 9)$

(c) $\lim_{x\to 1} \dfrac{3x^2 + 2x - 4}{x^2 + 5x - 4}$ (d) $\lim_{x\to 3} \dfrac{6x^2 + 3x - 12}{2x^2 + x + 1}$

2. Compute the following limits :

(a) $\lim_{x\to 0} \dfrac{4x^3 - x^2 + 2x}{3x^2 + 4x}$ (b) $\lim_{x\to 4} \dfrac{x^3 - 64}{x^2 - 16}$

(c) $\lim_{x\to a} \dfrac{x^{2/3} - a^{2/3}}{x - a}$ (d) $\lim_{x\to 1} \dfrac{x^2 + 3x - 4}{x - 1}$

(e) $\lim_{x\to 2} \dfrac{x^2 - 5x + 6}{x^2 - x - 2}$ (f) $\lim_{x\to 2} \dfrac{x^2 - 4x + 4}{x^2 - 7x + 10}$

(g) $\lim_{x\to a} \dfrac{\sqrt{3x} - \sqrt{2x + a}}{2(x - a)}$ (h) $\lim_{x\to a} \dfrac{\sqrt{2x} - \sqrt{3x - a}}{\sqrt{x} - \sqrt{a}}$

(T.U. 2053 H)

(i) $\lim_{x\to 1} \dfrac{\sqrt{2x} - \sqrt{3 - x^2}}{x - 1}$ (j) $\lim_{x\to 2} \dfrac{\sqrt{x} - \sqrt{6 - x^2}}{x - 2}$

(k) $\lim_{x\to 64} \dfrac{\sqrt[6]{x} - 2}{\sqrt[3]{x} - 4}$ (l) $\lim_{x\to a} \dfrac{\sqrt{3a - x} - \sqrt{x + a}}{4(x - a)}$

3. Calculate the following limits:

(a) $\lim_{x\to\infty} \dfrac{2x^2}{3x^2 + 2}$ (b) $\lim_{x\to\infty} \dfrac{3x^2 - 4}{4x^2}$

(c) $\lim_{x\to\infty} \dfrac{4x^2 + 3x + 2}{5x^2 + 4x - 3}$ (d) $\lim_{x\to\infty} \dfrac{5x^2 + 2x - 7}{3x^2 + 5x + 2}$

4. Calculate the following limits:

(a) $\lim_{x\to\infty} \left(\sqrt{x} - \sqrt{x - 3}\right)$ (b) $\lim_{x\to\infty} \left(\sqrt{x - a} - \sqrt{x - b}\right)$

(T.U. 2048 H)

(c) $\lim_{x\to\infty} \left(\sqrt{3x} - \sqrt{x - 5}\right)$ (d) $\lim_{x\to\infty} \sqrt{x}\left(\sqrt{x} - \sqrt{x - a}\right)$

(T.U. 2052)

(e) $\lim\limits_{x \to \infty} \left(\sqrt{x - a} - \sqrt{bx}\right)$

5. (a) $\lim\limits_{x \to 2} \dfrac{x - \sqrt{8 - x^2}}{\sqrt{x^2 + 12} - 4}$ (b) $\lim\limits_{x \to 1} \dfrac{x - \sqrt{2 - x^2}}{2x - \sqrt{2 + 2x^2}}$

***Answers***

| | | | | |
|---|---|---|---|---|
| 1. | (a) 0. | (b) 26. | (c) $\frac{1}{2}$ | (d) $\frac{51}{22}$ |
| 2. | (a) $\frac{1}{2}$ | (b) 6 | (c) $\frac{2}{3}$ $a^{-1/3}$ | (d) 5 |
| | (e) $-\frac{1}{3}$ | (f) 0 | (g) $\frac{1}{4\sqrt{3a}}$ | (h) $-\frac{1}{\sqrt{2}}$ |
| | (i) $\sqrt{2}$ | (j) $\frac{5}{2\sqrt{2}}$ | (k) $\frac{1}{4}$ | (l) $-\frac{1}{4\sqrt{2a}}$ |
| 3. | (a) $\frac{2}{3}$ | (b) $\frac{3}{4}$ | (c) $\frac{4}{5}$ | (d) $\frac{5}{3}$ |
| 4. | (a) 0 | (b) 0 | (c) $\infty$ | (d) a/2 |

(e) $\infty$ if $b \neq 1$ and 0 if $b = 1$
(If the limiting value of a function is $\infty$, we say that the limit of the function does not exist)

5 (a) 4 (b) 2

## B. Limits of Trigonometric Functions

### Standard Results

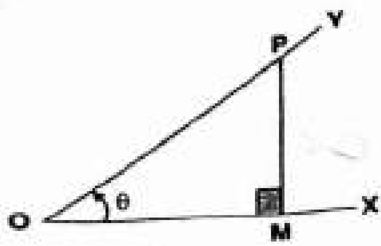
(i) $\lim\limits_{\theta \to 0} \sin \theta = 0$

(ii) $\lim\limits_{\theta \to 0} \cos \theta = 1$

Let OX be the initial line and $\angle XOY = \theta$. Take any point P on the line OY. From P draw PM perpendicular to OX. Then,

$$\sin \theta = \frac{MP}{OP}$$

and $$\cos \theta = \frac{OM}{OP}$$




When $\theta$ is small, MP will be small and P will be near to M. When $\theta$ is small enough, MP will be small enough and P will be very close to M. This implies that as $\theta \to 0$, $MP \to 0$ and $OP \to OM$.

Therefore, (i) $\lim_{\theta\to 0} \sin\theta = \lim_{\theta\to 0} \frac{MP}{OP} = 0$

and (ii) $\lim_{\theta\to 0} \cos\theta = \lim_{\theta\to 0} \frac{OM}{OP} = 1$

(iii) $\lim_{\theta\to\alpha} \sin\theta = \sin\alpha$

Put $\theta = \alpha + h$ so that when $\theta\to\alpha$, $h\to 0$.

Now, $\lim_{\theta\to\alpha} \sin\theta = \lim_{h\to 0} \sin(\alpha + h)$

$$= \lim_{h\to 0} \{\sin\alpha\cos h + \cos\alpha\sin h\}$$

$$= \sin\alpha \lim_{h\to 0}\cos h + \cos\alpha \lim_{h\to 0}\sin h$$

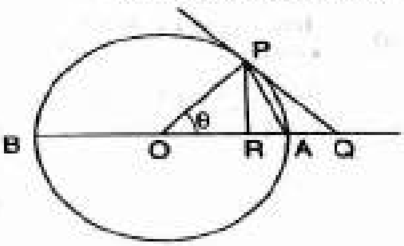
$$= \sin\alpha \,.\, 1 + \cos\alpha \,.\, 0$$

$$= \sin\alpha$$

$\therefore \quad \lim_{\theta\to\alpha} \sin\theta = \sin\alpha$

**Theorem.** $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ where $\theta$ is measured in radian.

**(HSEB 2056, 2058)**




Let ABC be a circle of radius $r$ and AP be an arc which subtends an angle $\theta$ at the centre O. Let PQ be the tangent at the point P of the circle which meets BA produced at Q. Join PA and draw PR perpendicular to BA. Then

Area of $\Delta OPA \leq$ Area of sector OAP

$\leq$ Area of $\Delta OPQ$

Now, Area of $\Delta OPA = \frac{1}{2}$ OA. PR

$= \frac{1}{2} r^2 \sin\theta \qquad (\because OA = r, PR = r\sin\theta)$

Area of sector OAP $= \frac{1}{2} r^2 \theta$

Area of $\Delta OPQ = \frac{1}{2}$ OP.PQ $= \frac{1}{2} r^2 \tan\theta$

$\therefore \quad \frac{1}{2} r^2 \sin\theta \leq \frac{1}{2} r^2 \theta \leq \frac{1}{2} r^2 \tan\theta$

$$1 \leq \frac{\theta}{\sin\theta} \leq \frac{1}{\cos\theta}$$

or, $1 \geq \frac{\sin\theta}{\theta} \geq \cos\theta$

or, $\lim\limits_{\theta\to 0} 1 \geq \lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta} \geq \lim\limits_{\theta\to 0} \cos\theta$

or, $1 \geq \lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta} \geq 1$

$\therefore \lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

## C. Limits of Logarithmic and Exponential Functions

For the limits of logarithmic and exponential functions, we recall the following definition of $e$.

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

If we put $n = \frac{1}{h}$ so that when $n\to\infty$, $h\to 0$

then $\lim\limits_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = \lim\limits_{h\to 0}(1 + h)^{1/h} = e$

### Some Standard Results

a) $\lim\limits_{x\to 0}\frac{\log(1 + x)}{x} = 1$

$$\lim_{x\to 0}\frac{\log(1+x)}{x} = \lim_{x\to 0}\frac{1}{x}\log(1+x)$$
$$= \lim_{x\to 0}\log(1+x)^{1/x}$$
$$= \log\left\{\lim_{x\to 0}(1+x)^{1/x}\right\}$$
$$= \log e = 1$$

b) $\lim\limits_{x\to 0}\frac{e^x - 1}{x} = 1$

Put $e^x - 1 = y$ then $e^x = 1 + y$ and $x = \log(1 + y)$ so that when $x\to 0$, $y\to 0$.

Now, $\lim\limits_{x\to 0}\frac{e^x - 1}{x} = \lim\limits_{y\to 0}\frac{y}{\log(1+y)}$

$$= \lim_{y\to 0}\frac{1}{\frac{1}{y}\log(1+y)} = \frac{1}{1} = 1$$

c) $\lim\limits_{x\to 0} \dfrac{a^x - 1}{x} = \log a$

Put $a^x - 1 = y$ then $a^x = 1 + y$ which implies $x \log a = \log (1 + y)$

and $\quad x = \dfrac{\log (1 + y)}{\log a} \quad$ so that when $x \to 0$, $y \to 0$.

Now, $\lim\limits_{x\to 0} \dfrac{a^x - 1}{x} = \lim\limits_{y\to 0} \dfrac{y}{\dfrac{\log (1 + y)}{\log a}}$

$$= \log a \lim_{y\to 0} \frac{1}{\frac{1}{y} \log (1 + y)}$$

$$= \log a \,.\, 1 = \log a$$

## Worked Out Examples

***Example 1.***

Show that $\quad \lim\limits_{x\to 0} \dfrac{\tan x}{x} = 1$

**Solution:**

$\dfrac{\tan x}{x}$ takes the indeterminate form $\dfrac{0}{0}$ at $x = 0$. So we write,

$$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \left(\frac{\sin x}{x} \cdot \frac{1}{\cos x}\right) = 1 \cdot \frac{1}{\cos 0} = 1$$

***Example 2.***

Show that $\lim\limits_{x\to 0} \dfrac{1 - \cos 3x}{3x^2} = \dfrac{3}{2}$

**Solution:**

$\dfrac{1 - \cos 3x}{3x^2}$ takes the indeterminate form $\dfrac{0}{0}$ at $x = 0$. So we write

$$\lim_{x\to 0} \frac{1 - \cos 3x}{3x^2} = \lim_{x\to 0} \frac{2 \sin^2 \frac{3x}{2}}{3x^2}$$

$$= \frac{2}{3} \lim_{x\to 0} \left(\frac{\sin \frac{3x}{2}}{x}\right)^2 = \frac{2}{3} \lim_{x\to 0} \left(\frac{\sin \frac{3x}{2}}{\frac{3x}{2}} \cdot \frac{3}{2}\right)^2$$

$$= \frac{2}{3} \cdot \left(1 \cdot \frac{3}{2}\right)^2 = \frac{3}{2}.$$

*Example 3.*

Evaluate $\lim_{x\to 0} \dfrac{\text{cosec } x - \cot x}{x}$

*Solution:*

$\dfrac{\text{cosec } x - \cot x}{x}$ takes the indeterminate form $\dfrac{\infty - \infty}{0}$ when $x = 0$.

So we write,

$$\lim_{x\to 0} \frac{\text{cosec } x - \cot x}{x} = \lim_{x\to 0} \frac{1}{x}\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right)$$

$$= \lim_{x\to 0} \frac{1 - \cos x}{x \sin x}$$

$$= \lim_{x\to 0} \frac{2 \sin^2 \frac{x}{2}}{x.2 \sin \frac{x}{2} \cos \frac{x}{2}}$$

$$= \lim_{x\to 0} \frac{\tan \frac{x}{2}}{2 \cdot \frac{x}{2}} = \frac{1}{2}.$$

*Example 4*

Evaluate : $\lim_{x\to a} \dfrac{\sqrt{x} - \sqrt{a}}{\tan (x - a)}$

*Solution :*

The given function takes the form $\frac{0}{0}$ when $x = a$.

Now, $\lim_{x\to a} \dfrac{\sqrt{x} - \sqrt{a}}{\tan (x - a)}$

$$= \lim_{x\to a} \left\{ \frac{\sqrt{x} - \sqrt{a}}{\tan (x - a)} \cdot \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}} \right\}$$

$$= \lim_{x\to a} \left\{ \frac{(x - a) \cos (x - a)}{\sin (x - a)} \cdot \frac{1}{\sqrt{x} + \sqrt{a}} \right\}$$

$$= \lim_{(x - a)\to 0} \frac{1}{\frac{\sin (x - a)}{x - a}} \lim_{(x-a)\to 0} \cos (x - a)$$

$$\times \lim_{x\to a} \frac{1}{\sqrt{x} + \sqrt{a}}$$

$$= \frac{1}{1} \cdot 1 \cdot \frac{1}{\sqrt{a} + \sqrt{a}}$$

$$= \frac{1}{2\sqrt{a}}$$

**Example 5**

Evaluate $\lim_{x \to \theta} \frac{x \sin \theta - \theta \sin x}{x - \theta}$ (T.U. 2050H)

**Solution :**

$\frac{x \sin \theta - \theta \sin x}{x - \theta}$ takes the indeterminate form $\frac{0}{0}$ at $x = \theta$.

Put $x - \theta = h$ then $x = \theta + h$

so that when $x \to \theta$, $h \to 0$.

Now,

$$\lim_{x \to \theta} \frac{x \sin \theta - \theta \sin x}{x - \theta}$$

$$= \lim_{h \to 0} \frac{(\theta + h) \sin \theta - \theta \sin (\theta + h)}{h}$$

$$= \lim_{h \to 0} \frac{h \sin \theta + \theta\{\sin \theta - \sin (\theta + h)\}}{h}$$

$$= \lim_{h \to 0} \left[\sin \theta + \frac{\theta}{h} 2 \cos \left(\frac{2\theta + h}{2}\right) \sin \left(\frac{-h}{2}\right)\right]$$

$$= \lim_{h \to 0} \sin \theta - \lim_{h \to 0} \theta \cos \left(\theta + \frac{h}{2}\right) \frac{\sin h/2}{h/2}$$

$$= \sin \theta - \theta \cos \theta$$

**Example 6**

Evaluate : $\lim_{x \to 0} \frac{e^{3x} - 1}{x.5^x}$

**Solution:**

$$\lim_{x \to 0} \frac{e^{3x} - 1}{x.5^x} \qquad [\text{form} \frac{0}{0}]$$

$$= \lim_{x \to 0} \frac{e^{3x} - 1}{3x} \cdot 3 \cdot \frac{1}{5^x}$$

$$= \left(\lim_{x \to 0} \frac{e^{3x} - 1}{3x} \cdot 3\right) \left(\lim_{x \to 0} \frac{1}{5^x}\right)$$

$$= 1.3.1 = 3$$

***Example 7***

Evaluate : $\lim\limits_{x\to 0} \dfrac{a^x - b^x}{x}$

***Solution:***

$$\lim_{x\to 0} \frac{a^x - b^x}{x} \qquad [\text{form } \tfrac{0}{0}]$$

$$= \lim_{x\to 0} \frac{(a^x - 1) - (b^x - 1)}{x}$$

$$= \lim_{x\to 0} \frac{a^x - 1}{x} - \lim_{x\to 0} \frac{b^x - 1}{x}$$

$$= \log a - \log b$$

$$= \log \left(\frac{a}{b}\right)$$

***Example 8***

Evaluate : $\lim\limits_{x\to 1} \dfrac{\log x}{x - 1}$

***Solution:***

$$\lim_{x\to 1} \frac{\log x}{x - 1} \qquad [\text{form } \tfrac{0}{0}]$$

Put $x - 1 = y$ then $x = 1 + y$ so that when $x\to 1$, $y\to 0$.

Now, $\lim\limits_{x\to 1} \dfrac{\log x}{x - 1}$

$$= \lim_{y\to 0} \frac{\log (1 + y)}{y} = 1$$

## EXERCISE 16.2

**Evaluate the following:**

1. $\lim\limits_{x\to 0} \dfrac{\sin ax}{x}$
2. $\lim\limits_{x\to 0} \dfrac{\tan bx}{x}$
3. $\lim\limits_{x\to 0} \dfrac{\sin mx}{\sin nx}$
4. $\lim\limits_{x\to 0} \dfrac{\tan ax}{\tan bx}$
5. $\lim\limits_{x\to 0} \dfrac{\sin px}{\tan qx}$
6. $\lim\limits_{x\to a} \dfrac{\sin (x - a)}{x^2 - a^2}$

7. $\lim_{x \to p} \frac{x^2 - p^2}{\tan(x - p)}$

8. $\lim_{x \to 0} \frac{\sin ax . \cos bx}{\sin cx}$

9. $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$

10. $\lim_{x \to 0} \frac{1 - \cos 6x}{x^2}$

11. $\lim_{x \to 0} \frac{1 - \cos 9x}{x^2}$

12. $\lim_{x \to 0} \frac{\cos ax - \cos bx}{x^2}$

13. $\lim_{x \to 0} \frac{\sin ax - \sin bx}{x}$

14. $\lim_{x \to 0} \frac{1 - \cos px}{1 - \cos qx}$

15. $\lim_{x \to 0} \frac{\tan x - \sin x}{x^3}$

16. $\lim_{x \to 0} \frac{\tan 2x - \sin 2x}{x^3}$

17. $\lim_{x \to \frac{\pi}{2}} (\sec x - \tan x)$

18. $\lim_{x \to \frac{\pi}{4}} \frac{\sec^2 x - 2}{\tan x - 1}$

19. $\lim_{x \to \frac{\pi}{4}} \frac{2 - \operatorname{cosec}^2 x}{1 - \cot x}$

20. $\lim_{x \to y} \frac{\tan x - \tan y}{x - y}$

21. $\lim_{x \to y} \frac{\sin x - \sin y}{x - y}$ (HSEB 2053, 2057)

22. $\lim_{x \to y} \frac{\cos x - \cos y}{x - y}$ (T.U. 2049 H)

23. $\lim_{x \to \theta} \frac{x \cot \theta - \theta \cot x}{x - \theta}$ (T.U. 2049, HSEB 2055)

24. $\lim_{x \to \theta} \frac{x \cos \theta - \theta \cos x}{x - \theta}$ (T.U. 2052 H, 2057 S)

25. $\lim_{x \to 1} \frac{1 + \cos \pi x}{\tan^2 \pi x}$

26. $\lim_{x \to \theta} \frac{x \tan \theta - \theta \tan x}{x - \theta}$ (T.U. 2050 H)

27. $\lim_{\theta \to \frac{\pi}{4}} \frac{\cos \theta - \sin \theta}{\theta - \frac{\pi}{4}}$

28. $\lim_{x \to c} \frac{\sqrt{x} - \sqrt{c}}{\sin x - \sin c}$

29. Find the limits of

a) $\lim_{x \to 0} \frac{e^{6x} - 1}{x}$

b) $\lim_{x \to 0} \frac{e^{2x} - 1}{x . 2^{x+1}}$

c) $\lim_{x \to 0} \frac{e^{ax} - e^{bx}}{x}$

d) $\lim_{x \to 0} \frac{a^x + b^x - 2}{x}$

30. Evaluate the limits of

a) $\lim_{x \to 2} \frac{x - 2}{\log(x - 1)}$

b) $\lim_{x \to \frac{\pi}{2}} \frac{\cos x}{\log\left(x - \frac{\pi}{2} + 1\right)}$

2

**Answers**

| | | | | |
|---|---|---|---|---|
| 1. $a$ | 2. $b$ | 3. $\frac{m}{n}$ | 4. $\frac{a}{b}$ | 5. $\frac{p}{q}$ |
| 6. $\frac{1}{2a}$ | 7. $2p$ | 8. $\frac{a}{c}$ | 9. $\frac{1}{2}$ | 10. 18 |
| 11. $\frac{81}{2}$ | 12. $\frac{1}{2}(b^2 - a^2)$ | 13. $a - b$ | 14. $\frac{p^2}{q^2}$ | 15. $\frac{1}{2}$ |
| 16. 4 | 17. 0 | 18. 2 | 19. 2 | 20. $\sec^2 y$ |
| 21. $\cos y$ | 22. $-\sin y$ | 23. $\cot\theta + \frac{\theta}{\sin^2\theta}$ | | |
| 24. $\cos\theta + \theta\sin\theta$ | 25. $\frac{1}{2}$ | 26. $\tan\theta - \theta\sec^2\theta$ | 27. $-\sqrt{2}$ | |
| 28. $\frac{\sec c}{2\sqrt{c}}$ | 29. a) 6 b) 1 | c) $a - b$ | d) $\log(ab)$ | |
| 30. a) 1 | b) $-1$ | | | |

## 16.6 Limits

A neighbourhood of a point '$a$' is an open interval containing the point '$a$'. It is generally denoted by $(a - \delta, a + \delta)$. Also,

$$x \in (a - \delta, a + \delta) \Rightarrow |x - a| < \delta$$

As we have seen above that $\lim\limits_{x \to a} f(x) = l$ means $f(x)$ approaches $l$ as $x$ approaches $a$ or the difference between $f(x)$ and $l$ i.e. $|f(x) - l|$ is very small when the difference between $x$ and $a$ i.e. $|x - a|$ is very small.

### Definition:

Let $f(x)$ be defined in a neighbourhood of $a$ ($f(x)$ may or may not be defined at $x = a$). Then $f(x)$ is said to tend to the limit $l$, as $x$ approaches $a$. Symbolically,

$$\lim_{x \to 0} f(x) = l$$

if to every positive number $\varepsilon$, however small, there corresponds a positive number $\delta$, such that $|f(x) - l| < \varepsilon$, whenever $|x - a| < \delta$.

### Right hand limit and Left hand limit

A function $f(x)$ is said to have the right hand limit $l_1$ at $x = a$ as $x$ approaches $a$ through value greater than $a$ (i.e. $x$ approaches $a$ from the right) and symbolically it is written as $\lim\limits_{x \to a^+} f(x) = l_1$. The right hand limit of $f(x)$ 5. $= a$ is also written as $\lim\limits_{x \to a+0} f(x)$ or $f(a + 0)$.

A function $f(x)$ is said to have the left hand limit $l_2$ at $x = a$ as $x$ approaches $a$ through value less than $a$ (i.e. $x$ approaches $a$ from left) and symbolically it is written as $\lim_{x \to a^-} f(x) = l_2$. The left hand limit of $f(x)$ at $x = a$ is also written as $\lim_{x \to a-0} f(x)$ or $f(a - 0)$.

The necessary and sufficient condition for $f(x)$ to have a limit at $x = a$ is $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ should exist and coincide. That is, $\lim_{x \to a} f(x)$ exists if and only if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$.

***Example 1***

**Calculate the limit of the function f at the point specified below:**

$$f(x) \begin{array}{ll} = 3x^2 - 1, & \text{when } x \le 2 \\ = 4x + 3, & \text{when } x > 2 \end{array} \Bigg\} \text{ at } x = 2$$

***Solution:***

Left-hand limit at $x = 2$ is

$$\lim_{x \to 2-0} f(x) = \lim_{x \to 2-0} (3x^2 - 1) = 12 - 1 = 11$$

Right-hand limit at $x = 2$ is

$$\lim_{x \to 2+0} f(x) = \lim_{x \to 2+0} (4x + 3) = 8 + 3 = 11.$$

As the left-hand limit is equal to the right-hand limit, the limit of the function $f(x)$ at the point $x = 2$ exists and is equal to 11; i.e.,

$$\lim_{x \to 2} f(x) = 11.$$

***Example 2***

Find the limit if it exists $\lim_{x \to 2} \frac{x - 2}{|x - 2|}$

***Solution :***

Let $f(x) = \frac{x - 2}{|x - 2|}$

By the definition,

$$|x - 2| = \begin{cases} x - 2 & \text{if } x > 2 \\ -(x - 2) & \text{if } x < 2 \end{cases}$$

Then, $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x-2}{x-2}$

$= 1$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x-2}{-(x-2)}$

$= -1$

Since, $\lim_{x\to 2^+} f(x) \neq \lim_{x\to 2^-} f(x)$, so $\lim_{x\to 2} f(x)$ does not exist.

## EXERCISE 16.3

1. **Find the limits at the points specified :**

   (a) $f(x) \begin{array}{l} = x + 2 \text{ for } x \geq 0 \\ = 4x + 2 \text{ for } x < 0 \end{array} \Big\}$ at $x = 0$

   (b) $f(x) \begin{array}{l} = 3x + 2 \text{ for } x \geq 1 \\ = 2x \text{ for } x < 1 \end{array} \Big\}$ at $x = 1$

   (c) $f(x) \begin{array}{l} = 5x + 2 \text{ for } x \geq 2 \\ = 7x - 2 \text{ for } x < 2 \end{array} \Big\}$ at $x = 2$

   (d) $f(x) \begin{array}{l} = 3x + 1 \text{ for } x \geq 2 \\ = 2x^2 - 1 \text{ for } x < 2 \end{array} \Big\}$ at $x = 2$ (T.U. 2058 S)

   (e) $f(x) \begin{array}{l} = 2x + 1 \text{ for } x \geq 1 \\ = 4x^2 - 1 \text{ for } x < 1 \end{array} \Big\}$ at $x = 1$

   (f) $f(x) \begin{array}{l} = 3x - 2 \text{ for } x \geq 2 \\ = 2x^2 + 1 \text{ for } x < 2 \end{array} \Big\}$ at $x = 2$.

2. **Evaluate the following limits :**

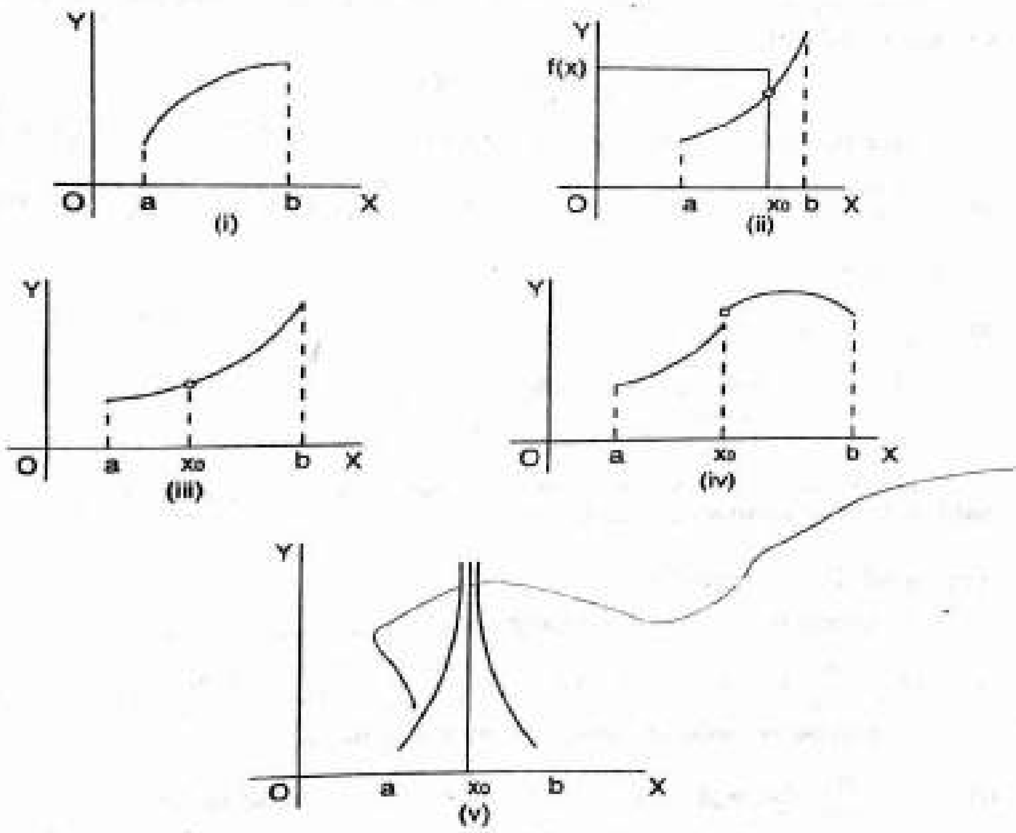
   (a) $\lim_{x\to 2} |x - 2|$  (b) $\lim_{x\to 0} \frac{|x|}{x}$

### *Answers*

1. (a) 2 (b) The limit does not exist. (c) 12
   (d) 7. (e) 3 (f) The limit does not exist.
2. a) 0 b) The limit does not exist.

## 16.7 Continuity

The intuitive idea of a continuous functions $f$ in the interval $[a, b]$ gives the impression that the graph of the function $f$ in this interval is a smooth curve without any break in it. Actually this curve is such that it can be drawn by the continuous motion of pencil without lifting it in a sheet of paper. Similarly, a discontinuous function gives the picture consisting of disconnected curves. Let us actually look at their graphs and discuss their nature.




Now it is obvious that leaving aside the graph of the function $f$ in fig. (i) all other graphs in the remaining four figures are discontinuous. In fig (ii) $\lim_{x \to x_0} f(x)$ exists and $f(x_0)$ is also defined, but $\lim_{x \to x_0} f(x) \neq f(x_0)$. In fig. (iii) $\lim_{x \to x_0} f(x)$ exists but $f(x)$ is not defined at $x = x_0$. In fig. (iv) $\lim_{x \to x_0} f(x)$ does not exist, while $f(x)$ is defined at $x = x_0$. But in fig. (v) neither $\lim_{x \to x_0} f(x)$

exists nor the function $f(x)$ is defined at $x = x_0$. Only in fig (i), the curve is continuous and $\lim_{x \to x_0} f(x)$ exists, $f(x_0)$ is defined and $\lim_{x \to x_0} f(x) = f(x_0)$.

So the relation $\lim_{x \to x_0} f(x) = f(x_0)$ can be considered to be the necessary and sufficient condition for a curve or the function $f$ to be continuous at $x = x_0$.

*Definition.* The function $f(x)$ is said to be continuous at the point $x = x_0$, if and only if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

This definition of continuity of the function $f(x)$ at $x = x_0$ implies that

a) $\lim_{x \to x_0} f(x)$ exists i.e. $\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^+} f(x)$ are finite and equal.

b) $f(x_0)$ exists.

c) $\lim_{x \to x_0} f(x) = f(x_0)$

Hence $f(x)$ will be continuous at $x = x_0$ if

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = f(x_0)$$

If any of the above conditions is not satisfied then the function $f(x)$ is said to be discontinuous at that point.

## Types of discontinuities

A discontinuous function may be of the following types :

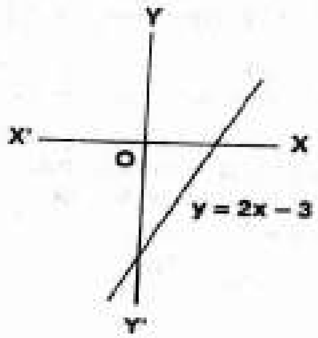
i) If $\lim_{x \to x_0} f(x)$ does not exist i.e. $\lim_{x \to x_0^-} f(x) \neq \lim_{x \to x_0^+} f(x)$ then $f(x)$ is said to be an ordinary discontinuity or a jump.

ii) If $\lim_{x \to x_0} f(x) \neq f(x_0)$ then the function $f(x)$ is said to have a **removable discontinuity** at $x = x_0$. This type of discontinuity can be removed by redefining the function.

iii) If $\lim_{x \to x_0} f(x) \to \infty$ or $-\infty$, then $f(x)$ is said to have infinite discontinuity at $x = x_0$.

Let us see the following examples to have the idea about the continuity and the different types of discontinuities :

i) The graph of $y = 2x - 3$ is given aside. It is continuous at every point.

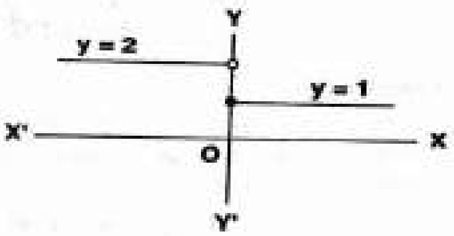



ii) A function $f(x)$ is defined below

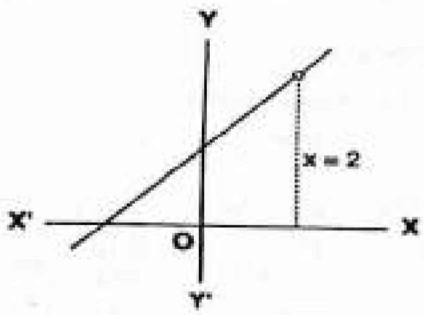
$$f(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 2 & \text{for } x < 0 \end{cases}$$

Its graph is given aside.

$\therefore$ the function $f(x)$ is discontinuous at $x = 0$. There is a jump at $x = 0$.



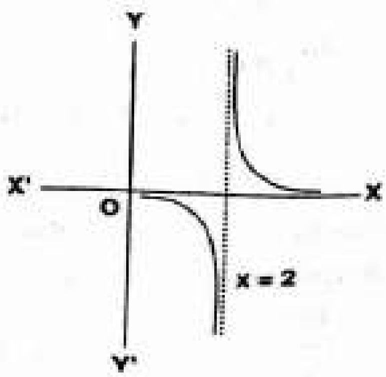

iii) The graph of $y = f(x) = \frac{x^2 - 4}{x - 2}$ is given aside. But $f(2) = \frac{0}{0}$ which is an indeterminate form. So, $f(2)$ does not exist i.e. $f(2)$ is not defined at $x = 2$. $f(x)$ is discontinuous at $x = 2$.




iv) The graph of $y = f(x) = \frac{1}{x - 2}$ is given aside. Here

$$\lim_{x \to 2^-} \frac{1}{x - 2} \to -\infty$$

and $$\lim_{x \to 2^+} \frac{1}{x - 2} \to \infty.$$



## 16.8 Continuity in an Interval

A function $f(x)$ is said to be continuous in an **open interval** $(a, b)$, if it is continuous at evey point in $(a, b)$.

A function $f(x)$ is said to be continuous in the **closed interval** $[a, b]$, if it is continuous at every point of the open interval $(a, b)$ and if it is continuous at the point a from the right and continuous at the point $b$ from the left.

i.e. $\lim\limits_{x \to a^+} f(x) = f(a)$ and $\lim\limits_{x \to b^-} f(x) = f(b)$

### Worked Out Example

***Example 1.***

Test the continuity or discontinuity of the following functions by calculating the left hand limit, the right hand limit and the value of the function at the points mentioned :

i) $f(x) = 2x^2 - 3x + 10$ at $x = 1$

ii) $f(x) = \dfrac{1}{x - 2}$ at $x = 2$

***Solution :***

i) Left hand limit at $x = 1$ is

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2x^2 - 3x + 10) = 2 - 3 + 10 = 9$$

Right hand limit at $x = 1$ is

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2x^2 - 3x + 10) = 2 - 3 + 10 = 9$$

$\lim\limits_{x \to 1^-} f(x)$ and $\lim\limits_{x \to 1^+} f(x)$ are finite and equal so $\lim\limits_{x \to 1} f(x)$ exists.

$\therefore \lim\limits_{x \to 1} f(x) = 9$

Also, $f(1) = 2 \times 1 - 3 \times 1 + 10 = 9$

$\lim\limits_{x \to 1} f(x) = f(1)$

Hence $f(x)$ is continuous at $x = 1$.

ii) Left hand limit at $x = 2$ is

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{1}{x - 2} = -\infty \text{ which does not exist.}$$

Hence $f(x)$ is discontinuous at $x = 2$.

**Example 2.**

A function $f(x)$ is defined as follows :

$$f(x) = \begin{cases} 2x + 3 & \text{for } x < 1 \\ 4 & \text{for } x = 1 \\ 6x - 1 & \text{for } x > 1 \end{cases}$$

Is the function continuous at $x = 1$ ? If not, how can you make it continuous ?

**Solution :**

Left hand limit at $x = 1$ is

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2x + 3) = 2 \times 1 + 3 = 5$$

Right hand limit at $x = 1$ is

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (6x - 1) = 6 \times 1 - 1 = 5$$

$\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$ are finite and equal.

So, $\lim_{x \to 1} f(x)$ exist and $\lim_{x \to 1} f(x) = 5$

But $f(1) = 4$

$\therefore \quad \lim_{x \to 1} f(x) \neq f(1)$

Hence $f(x)$ is not continuous at $x = 1$.

This is a case of **removable discontinuity.**

The given function will be continuous if $f(1)$ is also equal to 5. Thus the given function can be made continuous by defining the function in the following way :

$$f(x) = \begin{cases} 2x + 3 & \text{for } x < 1 \\ 5 & \text{for } x = 1 \\ 6x - 1 & \text{for } x > 1 \end{cases}$$

**Example 3**

A function $f(x)$ is defined as follows:

$$f(x) = \begin{cases} x^2 - 1 & \text{for} \quad x < 3 \\ 2kx & \text{for} \quad x \geq 3 \end{cases}$$

Find the value of k so that $f(x)$ is continuous at $x = 3$.

**Solution :**

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x^2 - 1) = 9 - 1 = 8$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} 2kx = 6k$$

and $f(3) = 6k$

Since $f(x)$ is continuous at $x = 3$,

so $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x) \Rightarrow 6k = 8$

$\therefore \quad k = \frac{4}{3}$

## EXERCISE 16.4

1. Test the continuity or discontinuity of the following functions by calculating the left-hand limits, the right-hand limits and the values of the functions at points specified:

(i) $f(x) = x^2$ at $x = 4$ (ii) $f(x) = 2 - 3x^2$ at $x = 0$

(iii) $f(x) = 3x^2 - 2x + 4$ at $x = 1$ (iv) $f(x) = \frac{1}{2x}$ at $x = 0$

(v) $f(x) = \frac{1}{x-2}$ at $x \neq 2$ (vi) $f(x) = \frac{1}{3x}$ at $x \neq 0$

(vii) $f(x) = \frac{1}{1-x}$ at $x = 1$ (T.U. 2048)

(viii) $f(x) = \frac{1}{x-3}$ at $x = 3$ (ix) $f(x) = \frac{x^2-9}{x-3}$ at $x = 3$

(x) $f(x) = \frac{x^2-16}{x-4}$ at $x = 4$ (xi) $f(x) = \frac{|x-2|}{x-2}$ at $x = 2$

2. Discuss the continuity of functions at the points specified:

(i) $f(x) \begin{array}{ll} = 2 - x^2 & \text{for } x \leq 2 \\ = x - 4 & \text{for } x > 2 \end{array} \Bigg\}$ at $x = 2$ (T.U. 2053 H)

(ii) $f(x) \begin{array}{ll} = 2x^2 + 1 & \text{for } x \leq 2 \\ = 4x + 1 & \text{for } x > 2 \end{array} \Bigg\}$ at $x = 2$

(iii) $f(x) \begin{array}{ll} = 2x & \text{for } x \leq 3 \\ = 3x - 3 & \text{for } x > 3 \end{array} \Bigg\}$ at $x = 3$

(iv) $f(x) \begin{array}{ll} = 2x + 1 & \text{for } x < 1 \\ = 2 & \text{for } x = 1 \\ = 3x & \text{for } x > 1 \end{array} \Bigg\}$ at $x = 1$. (T.U.051 H, 056 S)

(HSEB 2052, 2057, 2058)

3\. i) A function f(x) is defined as follows:

$$f(x) = \begin{cases} x^2 + 2 & \text{for } x < 5 \\ 20 & \text{for } x = 5 \\ 3x + 12 & \text{for } x > 5 \end{cases}$$

Show that $f(x)$ has removable discontinuity at $x = 5$.

ii) A function $f(x)$ is defined as follows :

$$f(x) = \begin{cases} 2x - 3 & \text{for } x < 2 \\ 2 & \text{for } x = 2 \\ 3x - 5 & \text{for } x > 2 \end{cases}$$

Is the function $f(x)$ continuous at $x = 2$ ? If not, how can the function $f(x)$ be made continuous at $x = 2$ ?

4\. (i) A function $f(x)$ is defined below

$$f(x) = \begin{cases} kx + 3 & \text{for } x \geq 2 \\ 3x - 1 & \text{for } x < 2 \end{cases}$$

Find the value of k so that $f(x)$ is continuous at $x = 2$.

(ii) A function $f(x)$ is defined as follows:

$$f(x) = \begin{cases} \dfrac{2x^2 - 18}{x - 3} & \text{for } x \neq 3 \\ k & \text{for } x = 3 \end{cases}$$

Find the value of k so that $f(x)$ is continuous at $x = 3$.

***Answers***

1 (i) continuous (ii) continuous (iii) continuous
(iv) discontinuous (v) continuous (vi) continuous
(vii) discontinuous (viii) discontinuous (ix) discontinuous
(x) discontinuous (xi) discontinuous

2\. (i) continuous (ii) continuous (iii) continuous
(iv) discontinuous

3\. ii) No, $f(x) = \begin{cases} 2x - 3 & \text{for } x < 2 \\ 1 & \text{for } x = 2 \\ 3x - 5 & \text{for } x > 2 \end{cases}$

4\. (i) 1 (ii) 12

## ADDITIONAL QUESTIONS
## (Limits)

1\. Prove that $\lim\limits_{x \to -2/3} \dfrac{2}{2 + 3x}$ does not exist.

2. Do the following function define for the value $x = 1$ ?

   (i) $f(x) = \frac{x - 1}{x + 2}$ (ii) $f(x) = \frac{x^3 + 1}{x - 1}$

3. What do you mean by the left hand limit and right hand limit of a function ? What is the condition for the limit of a function to exist at a point ?

   Prove that $\lim\limits_{x \to 0} |x| = 0$ but $\lim\limits_{x \to 0} \frac{|x|}{x}$ does not exist.

4. Distinguish between the limit and value of the function at a point.

   It is given that $f(x) = \frac{ax + b}{x + 1}$, $\lim\limits_{x \to 0} f(x) = 2$ and $\lim\limits_{x \to \infty} f(x) = 1$. Prove that $f(-2) = 0$.

5. Define limit of a function at a point. It is given that $f(x) = \frac{x + 6}{cx - d}$, $\lim\limits_{x \to 0} f(x) = -6$ and $\lim\limits_{x \to \infty} f(x) = \frac{1}{3}$, prove that $f(13) = \frac{1}{2}$.

6. What do you mean by an indeterminate form ? State their different forms. Evaluate the following limit

   $$\lim_{x \to \infty} \sqrt{x}\left(\sqrt{x} - \sqrt{x - a}\right)$$ (T.U. 2052)

7. Let $f: R \to R$ be defined by

   $$f(x) = \begin{cases} x \text{ if } x \text{ is an integer} \\ 0 \text{ if } x \text{ is not an integer} \end{cases}$$

   Find $\lim\limits_{x \to 1} f(x)$. Is it same as $f(1)$ ?

8. Prove that :

   (i) $\lim\limits_{x \to 3} \left(\frac{1}{x - 3} - \frac{9}{x^3 - 3x^2}\right) = \frac{2}{3}$

   (ii) $\lim\limits_{x \to 3} \left(\frac{x^2 + 9}{x^2 - 9} - \frac{3}{x - 3}\right) = \frac{1}{2}$

9. Evaluate :

   (i) $\lim\limits_{x \to 2} \frac{x^{-3} - 2^{-3}}{x - 2}$ (ii) $\lim\limits_{x \to \infty} \frac{(2x - 1)^6 (3x - 1)^4}{(2x + 1)^{10}}$

   (iii) $\lim\limits_{x \to 0} \frac{(1 + x)^6 - 1}{(1 + x)^2 - 1}$ (iv) $\lim\limits_{x \to a} \frac{(x + 2)^{5/2} - (a + 2)^{5/2}}{x - a}$

10. If $\lim\limits_{x \to a} \frac{x^3 - a^3}{x - a} = 27$ find all possible values of $a$.

11. Find the limiting values of

(i) $\lim_{x \to 0} \frac{\sin x^0}{x}$ (ii) $\lim_{x \to 0} \frac{1 - \cos 4\theta}{1 - \cos 6\theta}$

(iii) $\lim_{x \to \pi/2} \frac{\cos x}{\pi/2 - x}$ (iv) $\lim_{x \to 0} \frac{\tan 2x - x}{3x - \sin x}$

(v) $\lim_{x \to \pi} \frac{1 - \sin x/2}{(\pi - x)^2}$ (vi) $\lim_{x \to \pi/2} \frac{1 + \cos 2x}{(\pi - 2x)^2}$

(vii) $\lim_{x \to 0} \sin \frac{1}{x}$ (viii) $\lim_{x \to 0} x \sin \frac{1}{x}$

(ix) $\lim_{x \to a} \frac{\sin x - \sin a}{\sqrt{x} - \sqrt{a}}$ (x) $\lim_{x \to a} (a - x) \tan \frac{\pi x}{2a}$

(xi) $\lim_{y \to 0} \frac{(x + y) \sec (x + y) - x \sec x}{y}$

12. (i) A function is defined as

$$f(x) = \begin{cases} 3x^2 + 2 & \text{if } x < 1 \\ 2x + 3 & \text{if } x \geq 1 \end{cases}$$

find $\lim_{x \to 1} f(x)$.

(ii) A function $f(x)$ is defined by

$$f(x) = \begin{cases} 3 + 2x & \text{for } -3/2 \leq x < 0 \\ 3 - 2x & \text{for } 0 \leq x < 3/2 \\ -3 - 2x & \text{for } x \geq 3/2 \end{cases}$$

Find $\lim_{x \to 0} f(x)$ and $\lim_{x \to 3/2} f(x)$ if they exist.

13. Evaluate the limits of

a) $\lim_{x \to 0} \frac{e^{px} - 1}{e^{qx} - 1}$ b) $\lim_{x \to 0} \frac{e^x - e^{-x} + x}{x}$

c) $\lim_{x \to 0} \frac{a^x - 1}{b^x - 1}$

14. a) $\lim_{x \to 0} \frac{2^x - 1}{\sin x}$ b) $\lim_{x \to 0} \frac{e^{\sin x} - \sin x - 1}{x}$

c) $\lim_{x \to \frac{\pi}{2}} \frac{e^{\cos x} - 1}{\frac{\pi}{2} - x}$ d) $\lim_{x \to e} \frac{\log x - 1}{x - e}$

***Answers***

2. (i) Defined (ii) Not defined
6. $a/2$ 7. 0, No
9. (i) $-\frac{3}{16}$ (ii) $\frac{81}{16}$ (iii) 3 (iv) $\frac{5}{2}(a+2)^{3/2}$ 10. $\pm 3$
11. (i) $\frac{\pi}{180}$ (ii) $\frac{4}{9}$ (iii) 1 (iv) $\frac{1}{2}$ (v) $\frac{1}{8}$
(vi) $\frac{1}{2}$ (vii) Does not exist (viii) 0 (ix) $2\sqrt{a}\cos a$
(x) $\frac{2a}{\pi}$ (xi) $\sec x + x\tan x\sec x$
12. (i) 5 (ii) 3, does not exist.
13. a) $\frac{p}{q}$ b) 3 c) $\frac{\log a}{\log b}$
14. a) log 2 b) 0 c) 1 d) $\frac{1}{e}$

## ADDITIONAL QUESTIONS
## (Continuity)

1. **Define the continuity of a function at a point. Give with reason, an example of a continuous function at a point. Is the function $f(x) = \frac{1}{1-x}$ continuous at the point $x = 1$?** (T.U. 2048)

2. **When a function $f(x)$ is said to be continuous at a given point $x = a$? Discuss the continuity of**
$$f(x) = \begin{cases} x^2 + 2 & \text{for } x \le 5 \\ 3x + 12 & \text{for } x > 5 \end{cases} \quad \text{at } x = 5.$$ (HSEB 2054)

3. **At what points is the function**
$$f(x) = \frac{x+1}{(x-2)(x-3)}$$
(i) discontinuous (ii) continuous ?

4. Discuss the continuity of the function $f(x)$ at the point $x = 0$.
$$f(x) = \begin{cases} x & \text{if } x > 0 \\ 1 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

5. A function $f(x)$ is defined as follows :

$$f(x) = \begin{cases} 2x + 1 & \text{for } x < 1 \\ 2 & \text{for } x = 1 \\ 3x & \text{for } x > 1 \end{cases}$$

Calculate the left hand limit and the right hand limit of $f(x)$ at $x = 1$. Is the function continuous at $x = 1$ ? (HSEB 2051)

6. What do you understand by the limit of a function ? Let a function $f(x)$ be defined by

$$f(x) = \begin{cases} 2 - x^2 & \text{for } x < 2 \\ 3 & \text{for } x = 2 \\ x - 4 & \text{for } x > 2 \end{cases}$$

Verify that the limit of the function $f(x)$ exists at $x = 2$. Is the function $f(x)$ continuous at $x = 2$ ? If not why ? State how can you make it continuous. (T.U. 2050)

7. A function $f(x)$ is defined as under

$$f(x) = \begin{cases} \dfrac{x^2 - x - 6}{x^2 - 2x - 3} & x \neq 3 \\ \dfrac{5}{3} & x = 3 \end{cases}$$

Prove that $f(x)$ is discontinuous at $x = 3$. Can the definition of $f(x)$ for $x = 3$ be modified so as to make it continuous there ?

8. (i) A function $f(x)$ is defined as follows :

$$f(x) = \begin{cases} \frac{1}{2} + x & \text{when } 0 < x < \frac{1}{2} \\ \frac{1}{2} & \text{when } x = \frac{1}{2} \\ \frac{3}{2} - x & \text{when } \frac{1}{2} < x < 1 \end{cases}$$

Show that $f(x)$ has removable discontinuity at $x = \frac{1}{2}$.

(ii) A function $f(x)$ is defined in (0, 3) in the following way :

$$f(x) = \begin{cases} x^2 & \text{when } 0 < x < 1 \\ x & \text{when } 1 \le x < 2 \\ 1/4\, x^3 & \text{when } 2 \le x < 3 \end{cases}$$

Show that $f(x)$ is continuous at $x = 1$ and $x = 2$.

(iii) A function $f(x)$ is defined as follows :

$$f(x) = \begin{cases} 3 + 2x & \text{for } -3/2 \le x < 0 \\ 3 - 2x & \text{for } 0 \le x < 3/2 \\ -3 - 2x & \text{for } x \ge 3/2 \end{cases}$$

Show that $f(x)$ is continuous at $x = 0$ & discontinuous at $x = \frac{3}{2}$. (HSEB 2056)

(iv) A function $f(x)$ is defined as follows :

$$f(x) = \begin{cases} 1 & \text{when } x > 0 \\ 0 & \text{when } x = 0 \\ -1 & \text{when } x < 0 \end{cases}$$

Show that it is discontinuous at $x = 0$.

9. In the following, determine the value of the constant so that the given function is continuous at the point mentioned.

(i) $f(x) = \begin{cases} kx^2 & \text{if } x \le 2 \\ 3 & \text{if } x > 2 \end{cases}$ at $x = 2$

(ii) $g(x) = \begin{cases} ax + 5 & \text{if } x \le 2 \\ x - 1 & \text{if } x > 2 \end{cases}$ at $x = 2$

(iii) $f(x) = \begin{cases} 2px + 3 & \text{if } x < 1 \\ 1 - px^2 & \text{if } x \ge 1 \end{cases}$ at $x = 1$

(iv) $f(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3} & \text{if } x \ne 3 \\ k & \text{if } x = 3 \end{cases}$ at $x = 3$

10. Let $f(x) = \begin{cases} 2x & \text{if } x < 2 \\ 2 & \text{if } x = 2 \\ x^2 & \text{if } x > 2 \end{cases}$

Show that $f(x)$ has removable discontinuity at $x = 2$.

[Hint : $f(x)$ has removable discontinuity if

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) \ne f(2)\ ]$$

11. What condition is necessary for a function $f(x)$ to be continuous at the point $x = a$ ? In what conditions will $f(x)$ be discontinuous at $x = a$ ?

12. A function $f(x)$ is defined as

$$f(x) = \begin{cases} 1 & \text{when } x \ne 0 \\ 2 & \text{when } x = 0 \end{cases}$$

Find $\lim_{x\to 0} f(x)$ if it exists. Is the function continuous at $x = 0$ ?

*Solution :*

$$\lim_{x\to 0^-} f(x) = 1$$

$$\lim_{x\to 0^+} f(x) = 1$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0+} f(x)$$

So, limit $\lim_{x\to 0} f(x)$ exists.

$$\therefore \lim_{x \to 0} f(x) = 1$$

Again, f(0) = 2 $\quad \lim_{x \to 0} f(x) \neq f(0)$

$\therefore$ $f(x)$ is not continuous at $x = 0$. This is the case of removable discontinuity.

13. Find the point of discontinuities of the following functions :

i) $f(x) = \dfrac{x + 1}{x - 1}$ $\qquad$ ii) $f(x) = \dfrac{3x - 1}{x^3 - 5x^2 + 6x}$

### *Answers*

1. No $\quad$ 2. continuous

3. (i) 2 and 3 $\quad$ (ii) continuous at all points except at $x = 2$ and $x = 3$

4. discontinuous $\quad$ 5. 3, 3, No

6. No, because $\lim_{x \to 2} f(x) \neq f(2)$. The given function can be made continuous at $x = 2$ by redefining $f(x)$ in the following way

$$f(x) = \begin{cases} 2 - x^2 & \text{for } x < 2 \\ -2 & \text{for } x = 2 \\ x - 4 & \text{for } x > 2 \end{cases}$$

7. Yes, by defining $f(x)$ in the following way

$$f(x) = \begin{cases} \dfrac{x^2 - x - 6}{x^2 - 2x - 3}, & x \neq 3 \\ \dfrac{5}{4}, & x = 3 \end{cases}$$

9. (i) $\frac{3}{4}$ $\quad$ (ii) $-2$ $\quad$ (iii) $-\frac{2}{3}$ $\quad$ (iv) 6 $\qquad$ 12. 1

13. (i) $x = 1$ $\quad$ (ii) $x = 0, 2, 3$

CHAPTER 17

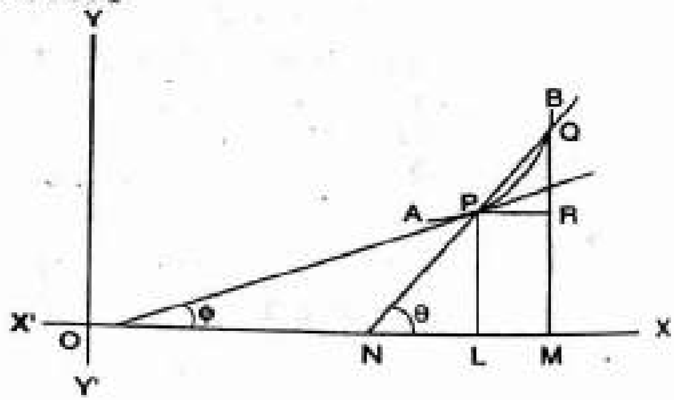
# The Derivatives

## 17.1 Introduction

Differential calculus is a theory which has its origin in the solution of two old problems – one of drawing a tangent line to a curve and the other of calculating the velocity of non-uniform motion of a particle. In both the problems, the curves involved are continuous curves and the process used is the limiting process. So the objects of study in the differential calculus are continuous functions. These problems were solved in a certain sense by Isaac Newton (English, 1642-1727) and Gottfried Willhelm Leibnitz (German, 1646-1716) and in the process differential calculus is discovered.

## 17.2 Tangent Line to a Curve

Let AB be a continuous curve given by $y = f(x)$ and P, Q be any two points in it. Let the coordinates of P and Q be $(x, y)$ and $(x', y')$. When a point moves along the curve from the point P to the point Q, it moves horizontaly through the distance PR and vertically through the distance RQ.




$$PR = LM = OM - OL$$
$$= x' - x$$

$$RQ = QM - RM = y' - PL = y' - y$$

These quantities $x' - x$ and $y' - y$ are called the increments in $x$ and $y$ respectively and are denoted by $\Delta x$ and $\Delta y$, i.e.

$$\Delta x = x' - x \qquad \text{and} \qquad \Delta y = y' - y.$$

Also $\qquad \Delta y = f(x') - f(x) = f(x + \Delta x) - f(x)$

If we join the points P and Q, we get secant PQ which makes an angle $\theta$ with the $x$-axis, i.e. $\angle QNM = \theta$. So $\angle QPR = \angle QNM = \theta$ and

$$\tan\theta = \frac{QR}{PR} = \frac{\Delta y}{\Delta x},$$

which is the slope of the secant PQ. As Q moves along the curve and approaches P, the secant rotates about P. The limiting position of the secant, when Q ultimately coincides with P, is the tangent at P, making the angle $\phi$ with the $x$-axis. In that situation, $\Delta x$, $\Delta y$ tend to zero. So

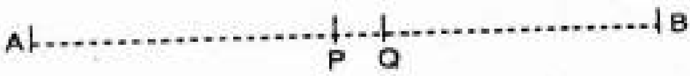
$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\tan\theta = \tan\phi$$

or $$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \tan\phi$$

Thus $\displaystyle\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$ or $\displaystyle\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$ gives the slope of a tangent to the curve given by the function $f$.

## 17.3 Instantaneous Velocity

Suppose a particle is moving in a straight line AB. Then the distance described increases with time. So the distance $s$ can be considered to be a function $f$ of the time $t$, and $s = f(t)$




At times $t$ and $t + \Delta t$, suppose the particle is at the points P and Q respectively such that $AP = s$ and $AQ = s + \Delta s$.

Then, $\quad PQ = AQ - AP = s + \Delta s - s = \Delta s$

Also, $\quad \Delta s = s + \Delta s - s = f(t+\Delta t) - f(t)$.

So the average velocity, $v_{av}$, during the time interval $(t, t+\Delta t)$ is

$$v_{av} = \frac{\Delta s}{\Delta t} = \frac{f(t+\Delta t)-f(t)}{\Delta t}$$

Now as $\Delta t \to 0$, Q tends to P. So the instantaneous velocity $v$ of the particle at P or in time $t$ is the limit to which $v_{av}$ tends as $\Delta t\to 0$, and

$$v = \lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t} = \lim_{\Delta t\to 0}\frac{f(t+\Delta t)-f(t)}{\Delta t}$$

## 17.4 Derivative

Both of the above problems lead us to the calculation of the limit of the ratio of the increment of a function to the corresponding increment of the independent variable as the latter tends to zero. This method of calculating a limit is an operation which is called differentiation of the function. The operation is denoted by $\frac{d}{dx}$.

The result of operation is called the *derivative* or the *differential coefficient of the function.*

*Definition.* Let the function $f$ be defined in the interval $(a, b)$. Then the *derivative* or the *differential coefficient of the function* $f$ at a point $x$ of the interval is defined to be the limiting value of

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Again if a is a fixed point then, the derivative of $f(x)$ at $x = a$ denoted by $f'(a)$ is defined by

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that the limit exists.

The symbols used to denote the derivative of $f$ with respect to $x$ are

$$f'(x),\ \frac{d\,f(x)}{dx},\ y',\ \frac{dy}{dx}.$$

With this definition, we can state the above problems as follows:

1. The slope of the tangent at a point $(x, y)$ of a curve given by a function $y = f(x)$ is equal to the derivative of the function with respect to $x$, i.e., $f'(x)$ or $\frac{df(x)}{dx}$.

2. The velocity of a particle describing a path given by $s = f(t)$ at a time $t$ is equal to the derivative of the function with respect to $t$, i.e., $s'$ or $\frac{df(t)}{dt}$

Now let us calculate the derivatives of the functions $x$, $x^2$ and $x^3$ and deduce the derivative of $x^n$.

(i) Let $y = f(x) = x$.

Let $\Delta x$ be a small increment in $x$ and $\Delta y$ be the corresponding small increment in $y$. Then

$$y + \Delta y = x + \Delta x$$

$$\Delta y = x + \Delta x - y = x + \Delta x - x = \Delta x$$

or, $$\frac{\Delta y}{\Delta x} = 1$$

$$\therefore \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 1$$

(ii) Let $y = f(x) = x^2$.

Let $\Delta x$ be a small increment in $x$ and $\Delta y$ be the corresponding small increment in $y$. Then

$$\begin{aligned}
y + \Delta y &= (x + \Delta x)^2 \\
\Delta y &= (x + \Delta x)^2 - y = (x + \Delta x)^2 - x^2 \\
&= x^2 + 2x.\Delta x + (\Delta x)^2 - x^2 \\
&= 2x.\Delta x + (\Delta x)^2 \\
\frac{\Delta y}{\Delta x} &= 2x + \Delta x \\
\therefore \quad \frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x) \\
&= 2x.
\end{aligned}$$

(iii) Let $y = f(x) = x^3$.

Let $\Delta x$ be a small increment in $x$ and $\Delta y$ be the corresponding small increment in $y$. Then

$$\begin{aligned}
y + \Delta y &= (x + \Delta x)^3 \\
\Delta y &= (x + \Delta x)^3 - y = (x + \Delta x)^3 - x^3 \\
&= x^3 + 3x^2.\Delta x + 3x.(\Delta x)^2 + (\Delta x)^3 - x^3 \\
\Delta y &= 3x^2.\Delta x + 3x.(\Delta x)^2 + (\Delta x)^3 \\
\frac{\Delta y}{\Delta x} &= 3x^2 + 3x.\Delta x + (\Delta x)^2 \\
\therefore \quad \frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\
&= \lim_{\Delta x \to 0} [3x^2 + 3x.\Delta x + (\Delta x)^2] = 3x^2
\end{aligned}$$

Thus we see that

the derivative of $x = \dfrac{dx}{dx} = x^{1-1} = 1$

the derivative of $x^2 = \dfrac{dx^2}{dx} = 2.x^{2-1} = 2x$

the derivative of $x^3 = \dfrac{dx^3}{dx} = 3.x^{3-1} = 3x^2$

From these results, we can conclude that

The derivative of $x^n = \dfrac{dx^n}{dx} = nx^{n-1}$

(This formula holds not only for natural numbers, it holds for any rational number).

### Derivative of $x^n$ from First Principles (Using Limit Theorem)

Let $y = x^n$ ...... (i)

Let $\Delta x$ be a small increment in $x$ and $\Delta y$, a corresponding increment in $y$. Then,

$$y + \Delta y = (x + \Delta x)^n \text{ ...... (ii)}$$

Subtracting (i) from (ii), we get

$$\Delta y = (x + \Delta x)^n - x^n$$

or, $$\frac{\Delta y}{\Delta x} = \frac{(x + \Delta x)^n - x^n}{\Delta x}$$

Now, $$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^n - x^n}{(x + \Delta x) - x}$$

$$= \lim_{(x + \Delta x) \to x} \frac{(x + \Delta x)^n - x^n}{(x + \Delta x) - x}$$

$$= nx^{n-1} \qquad \left(\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = n\,a^{n-1}\right)$$

Now we shall show that the derivative of a constant is zero.

(iv) Let $y = f(x) = c$, a constant

Then $y + \Delta y = c$

$$\Delta y = c - y = c - c = 0$$

$$\frac{\Delta y}{\Delta x} = 0$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 0$$

***Example 1***

Find from first principles the derivative of $(ax + b)^n$.

***Solution :***

Let $y = (ax + b)^n$

Let $\Delta x$ and $\Delta y$ be the small increments in $x$ and $y$ respectively. Then,

$$y + \Delta y = \{a(x + \Delta x) + b\}^n$$

$$\Delta y = (ax + a\Delta x + b)^n - (ax + b)^n$$

or $$\frac{\Delta y}{\Delta x} = \frac{(ax + a\Delta x + b)^n - (ax + b)^n}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(ax + a\Delta x + b)^n - (ax + b)^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(ax + a\Delta x + b)^n - (ax + b)^n}{(ax + a\Delta x + b) - (ax + b)} \cdot a$$

$$= \lim_{(ax+a\Delta x+b) \to (ax+b)} \frac{(ax + a\Delta x + b)^n - (ax + b)^n}{(ax + a\Delta x + b) - (ax + b)} \cdot a$$

$$= n.(ax + b)^{n-1} \cdot a \quad \left(\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right)$$

$$= n.a\ (ax + b)^{n-1}$$

***Example 2***

Find from first principles, the derivative of $\dfrac{1}{\sqrt{x + 2}}$

***Solution :***

Let $\quad y = \dfrac{1}{\sqrt{x + 2}}$.

Let $\Delta x$ be a small increment in $x$ and $\Delta y$ be the corresponding small increment in $y$. Then

$$y + \Delta y = \frac{1}{\sqrt{(x + \Delta x + 2)}}$$

$$\Delta y = \frac{1}{\sqrt{(x + \Delta x + 2)}} - \frac{1}{\sqrt{(x + 2)}}$$

$$= \frac{\sqrt{(x + 2)} - \sqrt{(x + \Delta x + 2)}}{\sqrt{(x + \Delta x + 2)}\ \sqrt{(x + 2)}}$$

$$= \frac{[\sqrt{x + 2} - \sqrt{x + \Delta x + 2}]\ [\sqrt{x + 2} + \sqrt{x + \Delta x + 2}]}{\sqrt{x + \Delta x + 2}\ \sqrt{x + 2}\ [\sqrt{x + 2} + \sqrt{x + \Delta x + 2}]}$$

$$= \frac{x + 2 - x - \Delta x - 2}{\sqrt{x + \Delta x + 2}\ \sqrt{x + 2}\ [\sqrt{x + 2} + \sqrt{x + \Delta x + 2}]}$$

$$= \frac{-\Delta x}{\sqrt{x + \Delta x + 2}\ \sqrt{x + 2}\ [\sqrt{x + 2} + \sqrt{x + \Delta x + 2}]}$$

$$\frac{\Delta y}{\Delta x} = \frac{-1}{\sqrt{x + \Delta x + 2}\ \sqrt{x + 2}\ [\sqrt{x + 2} + \sqrt{x + \Delta x + 2}]}$$

$$\therefore \quad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-1}{\sqrt{x + \Delta x + 2}\ \sqrt{x + 2}\ [\sqrt{x + 2} + \sqrt{x + \Delta x + 2}]}$$

$$= \frac{-1}{\sqrt{x + 2}\ \sqrt{x + 2}\ [\sqrt{x + 2} + \sqrt{x + 2}]}$$

$$= \frac{-1}{2(x + 2)^{3/2}}$$

## 17.5 Techniques of Differentiation

Here we shall deduce some fundamental formulae of differentiation.

### I. The Sum Rule

Let $f(x)$ and $g(x)$ be any two differentiable functions of $x$.

Let $\quad h(x) = f(x) \pm g(x)$

Then $\quad h'(x) = f'(x) \pm g'(x)$

or, $\quad \frac{d}{dx} h(x) = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x).$

*Proof.* Let $a$ be a fixed point. Then, by definition,

$$h'(a) = \lim_{x \to a} \frac{h(x) - h(a)}{x - a}$$

$$= \lim_{x \to a} \frac{[f(x) \pm g(x)] - [f(a) \pm g(a)]}{x - a}$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \pm \lim_{x \to a} \frac{g(x) - g(a)}{x - a}$$

$$= f'(a) \pm g'(a).$$

But $a$ is an arbitrary fixed number so

$$h'(x) = f'(x) \pm g'(x)$$

or, $\quad \frac{d}{dx} \{f(x) \pm g(x)\} = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x).$

Put $f(x) = u$ and $g(x) = v$. Then we have

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

From this sum Rule, we can deduce a formula for the differentiation of 'a constant times a function'.

(i) $$\frac{d(2u)}{dx} = \frac{d(u+u)}{dx} = \frac{du}{dx} + \frac{du}{dx} = 2 \cdot \frac{du}{dx}$$

(ii) $$\frac{d(3u)}{dx} = \frac{d(2u+u)}{dx} = \frac{d(2u)}{dx} + \frac{du}{dx}$$

$$= 2 \cdot \frac{du}{dx} + \frac{du}{dx} = 3\frac{du}{dx}.$$

(iii) $$\frac{d(4u)}{dx} = \frac{d(3u+u)}{dx} = \frac{d(3u)}{dx} + \frac{du}{dx}$$

$$= 3 \cdot \frac{du}{dx} + \frac{du}{dx} = 4\frac{du}{dx}$$

So in general, we have

$$\frac{d(nu)}{dx} = n\frac{du}{dx} \text{ for any integer } n.$$

(This rule holds not only for an integer, but also for an rational number).

***Example 3***

Find the derivative of $5x^3 + 4x^2 - 2x + 7$

**Solution:**

Let $y = 5x^3 + 4x^2 - 2x + 7$

Differentiating both sides w.r.t. 'x'.

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(5x^3 + 4x^2 - 2x + 7)$$

$$= \frac{d(5x^3)}{dx} + \frac{d(4x^2)}{dx} - \frac{d(2x)}{dx} + \frac{d(7)}{dx}$$

$$= 5 \cdot \frac{dx^3}{dx} + 4 \cdot \frac{d(x^2)}{dx} - 2 \cdot \frac{dx}{dx}$$

or, $$\frac{dy}{dx} = 15x^2 + 8x - 2.$$

## II. The Product Rule

Let $f(x)$ and $g(x)$ be any two differentiable functions of $x$. Let $h(x) = f(x) \cdot g(x)$. Then

$$h'(x) = f(x).g'(x) + g(x).f'(x)$$

or, $$\frac{d}{dx}h(x) = f(x) \cdot \frac{d}{dx}g(x) + g(x) \cdot \frac{d}{dx}f(x).$$

*Proof:* Let $a$ be a fixed point. Then, by definition

$$h'(a) = \lim_{x\to a} \frac{h(x) - h(a)}{x - a}$$

$$= \lim_{x\to a} \frac{f(x)\,g(x) - f(a)\,g(a)}{x - a}$$

$$= \lim_{x\to a} \frac{f(x)\,g(x) - f(x)\,g(a) + f(x)\,g(a) - f(a)\,g(a)}{x - a}$$

$$= \lim_{x\to a} f(x) \lim_{x\to a} \frac{g(x) - g(a)}{x - a} + g(a) \cdot \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

$$= f(a)\,g'(a) + g(a)\,f'(a).$$

But $a$ is an arbitrary fixed number, so

$$h'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x).$$

$$\frac{d}{dx}\{f(x) \cdot g(x)\} = f(x) \cdot \frac{d}{dx} g(x) + g(x) \cdot \frac{d}{dx} f(x).$$

Put $f(x) = u$ and $g(x) = v$. Then we get

$$\frac{d}{dx}(u.v) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

***Example 4***

Find the derivative of $(3x^2 - 5x)(2x + 3)$

***Solution:***

Let $u = 3x^2 - 5x$ and $v = 2x + 3$. Then we have

$$\frac{d}{dx}(u.v) = u.\frac{dv}{dx} + v\frac{du}{dx}$$

$$= (3x^2 - 5x)\frac{d}{dx}(2x + 3) + (2x + 3)\frac{d}{dx}(3x^2 - 5x)$$

$$= (3x^2 - 5x)\,2 + (2x + 3)(6x - 5)$$

$$= 18x^2 - 2x - 15$$

## III. The Power Rule

If $u$ is the function of $x$, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}.$$

*Proof:* By the Product Rule, we have

$$\frac{d}{dx}(u^2) = \frac{d}{dx}(u.u) = u\frac{du}{dx} + u\frac{du}{dx} = 2u\frac{du}{dx}.$$

Also, $\frac{d}{dx}(u^3) = \frac{d(u.u^2)}{dx} = u\frac{du^2}{dx} + u^2\frac{du}{dx}$

$= u.2u\frac{du}{dx} + u^2\frac{du}{dx}$

$= 3u^2\frac{du}{dx}$.

Similarly, for any natural number n, we have

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

(But in fact, it is true for any rational number $n$).

*Example 5*

Find the derivative of $(4x^3 + 5)^{3/2}$.

**Solution:**

Let $u = 4x^3 + 5$

$\because \quad \frac{du}{dx} = \frac{d}{dx}(4x^3 + 5) = 12x^2.$

$\therefore \quad \frac{d}{dx}(4x^3 + 5)^{3/2} = \frac{d(u^{3/2})}{dx}$

$= \frac{3}{2}u^{1/2}\frac{du}{dx} = \frac{3}{2}(4x^3 + 5)^{1/2} . 12x^2$

$= 18x^2\sqrt{4x^3 + 5}$

## IV. The Quotient Rule

Let $f(x)$ and $g(x)$ be any two differentiable functions of $x$. Let $h(x) = \frac{f(x)}{g(x)}$. Then

$$h'(x) = \frac{g(x)\, f'(x) - f(x)\, g'(x)}{\{g(x)\}^2}$$

*Proof:* We have, by definition,

$$h'(a) = \lim_{x \to a} \frac{h(x) - h(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}}{x - a} \qquad g(a) \neq 0$$

$$= \lim_{x \to a} \frac{g(a)\, f(x) - f(a)\, g(x)}{(x - a)\, g(x)\, g(a)}$$

$$= \lim_{x \to a} \frac{g(a) f(x) - g(a) f(a) + g(a) f(a) - f(a) g(x)}{(x - a) g(x) g(a)}$$

$$= \lim_{x \to a} \left[ \frac{1}{g(a) g(x)} \left\{ g(a) \frac{f(x) - f(a)}{x - a} - f(a) \frac{g(x) - g(a)}{x - a} \right\} \right]$$

$$= \frac{1}{g(a) g(a)} \{ g(a) f'(a) - f(a) g'(a) \}$$

$$= \frac{g(a) f'(a) - f(a) g'(a)}{\{g(a)\}^2}$$

$\therefore$ In general, we have

$$h'(x) = \frac{g(x) f'(x) - f(x) g'(x)}{\{g(x)\}^2}$$

or, $$\frac{d}{dx} \left\{ \frac{f(x)}{g(x)} \right\} = \frac{g(x) \frac{d}{dx} f(x) - f(x) \frac{d}{dx} g(x)}{\{g(x)\}^2}$$

Put $u = f(x)$ and $v = g(x)$. Then we have

$$\frac{d}{dx} \left( \frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

***Example 6***

Find the derivative of $\dfrac{4x^2 + 3}{3x^2 - 2}$.

***Solution:***

Let $u = 4x^2 + 3$ and $v = 3x^2 - 2$. Then

$$\frac{du}{dx} = 8x \text{ and } \frac{dv}{dx} = 6x$$

$$\therefore \frac{d}{dx} \left( \frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

$$= \frac{(3x^2 - 2) \cdot 8x - (4x^2 + 3) \cdot 6x}{(3x^2 - 2)^2} = \frac{-34x}{(3x^2 - 2)^2}$$

## V. The Chain Rule

If $y = f(u)$ and $u = g(x)$, where $f$ and $g$ are differentiable functions, then $\dfrac{dy}{dx}$ exists and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = f'(u) \frac{du}{dx}$$

*Proof:* We have $u = g(x)$. Let $\Delta x$ be a small increment in $x$ and $\Delta u$ be the corresponding small increment in $u$. Then

$$u + \Delta u = g(x + \Delta x)$$

or, $\Delta u = g(x + \Delta x) - u = g(x + \Delta x) - g(x)$

since $g(x)$ is differentiable, it is continuous. So,

$$\lim_{\Delta x \to 0} \Delta u = \lim_{\Delta x \to 0} [g(x + \Delta x) - g(x)]$$

$$= g(x) - g(x) = 0$$

Thus $\Delta u \to 0$, as $\Delta x \to 0$. Then

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \left[ \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \right]$$

$$= \left( \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \right) \left( \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \right)$$

$$= \frac{dy}{du} \cdot \frac{du}{dx}$$

**Example 7**

Find $\frac{dy}{dx}$, if $y = 4u^2 - 3u + 5$ and $u = 2x^2 - 3$.

**Solution:**

We have $\frac{dy}{du} = 8u - 3$ and $\frac{du}{dx} = 4x$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (8u - 3)\, 4x$$

$$= \{8(2x^2 - 3) - 3\}\, 4x$$

$$= 64x^3 - 108x.$$

## 17.6 Second and Higher Derivatives

Let $y = f(x)$ be a differentiable function. Then $\frac{dy}{dx} = f'(x)$ is called the *first derivative* or the first differential coefficient of $f(x)$ with respect to $x$. If we differentiate $\frac{dy}{dx}$ again, we get $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ written as $\frac{d^2y}{dx^2}$ or $f''(x)$, which is called the *second derivative* or the second differential coefficient of $f(x)$ with respect to $x$ and so on.

***Example 8***

Find the second and higher derivatives of

$$y = 5x^4 - 3x^2 + 11$$

***Solution:***

We have $\quad y = 5x^4 - 3x^2 + 11$

$\therefore \quad \frac{dy}{dx} = 20x^3 - 6x,$ the first derivative

$\frac{d^2y}{dx^2} = 60x^2 - 6,$ the second derivative

$\frac{d^3y}{dx^3} = 120x,$ the third derivative

$\frac{d^4y}{dx^4} = 120,$ the fourth derivative

$\frac{d^5y}{dx^5} = 0,$ the fifth derivative

and all other higher derivatives are also zero.

## 17.7 Implicit Function and Implicit Differentiation

Let $f(x, y)$ be an arbitrary function of two variables $x$ and $y$ and let

$$f(x, y) = 0 \;\ldots\ldots\; \text{(i)}$$

This equation may or may not be solvable for $y$. But we can differentiate (i) term by term with respect to $x$ and solve for $\frac{dy}{dx}$. This process of finding the value of $\frac{dy}{dx}$ without solving the equation for $y$ is called *implicit differentiation.*

***Example 9***

Use implicit differentiation to find $\frac{dy}{dx}$ in $2x^2 - 3y^2 = 16$.

***Solution:***

The given equation is $\quad 2x^2 - 3y^2 = 16$

Differentiating both sides with respect to $x$, we get

$$\frac{2dx^2}{dx} - \frac{3dy^2}{dy} \cdot \frac{dy}{dx} = \frac{d(16)}{dx}$$

or, $\quad 4x - 6y \cdot \frac{dy}{dx} = 0$

$\therefore \quad \frac{dy}{dx} = \frac{4x}{6y} = \frac{2x}{3y}$.

## Worked Out Examples

***Example 1***

Find from first principles, the derivative of $2x^2 + 3x - 6$.

***Solution :***

Let $y = 2x^2 + 3x - 6$

Let $\Delta x$ and $\Delta y$ be the small increments in $x$ and $y$ respectively. Then,

$$y + \Delta y = 2(x + \Delta x)^2 + 3(x + \Delta x) - 6$$

or, $$\Delta y = 2(x + \Delta x)^2 + 3(x + \Delta x) - 6 - 2x^2 - 3x + 6$$
$$= 4x\Delta x + 2(\Delta x)^2 + 3\Delta x$$

or, $$\frac{\Delta y}{\Delta x} = 4x + 2\Delta x + 3$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$
$$= \lim_{\Delta x \to 0} (4x + 2\Delta x + 3)$$
$$= 4x + 3$$

$\therefore$ $$\frac{dy}{dx} = 4x + 3$$

***Example 2***

Find from first principles, the derivative of $(2 - 3x)^{1/2}$.

***Solution :***

Let $y = \sqrt{2 - 3x}$

Let $\Delta x$ and $\Delta y$ be the small increments in $x$ and $y$ respectively. Then,

$$y + \Delta y = \sqrt{2 - 3(x + \Delta x)}$$
$$\Delta y = \sqrt{2 - 3(x + \Delta x)} - \sqrt{2 - 3x}$$

or, $$\frac{\Delta y}{\Delta x} = \frac{\sqrt{2 - 3(x + \Delta x)} - \sqrt{2 - 3x}}{\Delta x}$$
$$= \frac{2 - 3(x + \Delta x) - 2 + 3x}{\Delta x\left(\sqrt{2 - 3(x + \Delta x)} + \sqrt{2 - 3x}\right)}$$
$$= \frac{-3}{\left(\sqrt{2 - 3(x + \Delta x)} + \sqrt{2 - 3x}\right)}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-3}{\sqrt{2 - 3(x + \Delta x)} + \sqrt{2 - 3x}}$$

$$= -\frac{3}{2\sqrt{2 - 3x}}$$

**Alternative Method**

Let $y = (2 - 3x)^{1/2}$

$y + \Delta y = \{2 - 3(x + \Delta x)\}^{1/2}$

where $\Delta x$ and $\Delta y$ are the small increments in $x$ and $y$ respectively.

$$\Delta y = \{2 - 3(x + \Delta x)\}^{1/2} - (2 - 3x)^{1/2}$$

or, $$\frac{\Delta y}{\Delta x} = \frac{(2 - 3x - 3\Delta x)^{1/2} - (2 - 3x)^{1/2}}{\Delta x}$$

$$= \frac{(2 - 3x - 3\Delta x)^{1/2} - (2 - 3x)^{1/2}}{(2 - 3x - 3\Delta x) - (2 - 3x)} \times -3$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{(2-3x-3\Delta x) \to (2-3x)} \frac{(2 - 3x - 3\Delta x)^{1/2} - (2 - 3x)^{1/2}}{(2 - 3x - 3\Delta x) - (2 - 3x)} \times -3$$

$$= \frac{1}{2} \cdot (2 - 3x)^{-1/2} \cdot (-3)$$

$$= -\frac{3}{2\sqrt{2 - 3x}} \qquad \left(\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right)$$

***Example 3***

Find the derivative of $\dfrac{\sqrt{x} + a}{\sqrt{x} + b}$.

***Solution :***

Let $$y = \frac{\sqrt{x} + a}{\sqrt{x} + b}$$

Then, $$\frac{dy}{dx} = \frac{(\sqrt{x} + b)\frac{d}{dx}(\sqrt{x} + a) - (\sqrt{x} + a)\frac{d}{dx}(\sqrt{x} + b)}{(\sqrt{x} + b)^2}$$

$$= \frac{(\sqrt{x} + b) \cdot \frac{1}{2\sqrt{x}} - (\sqrt{x} + a)\frac{1}{2\sqrt{x}}}{(\sqrt{x} + b)^2}$$

$$= \frac{b - a}{2\sqrt{x}(\sqrt{x} + b)^2}$$

**Example 4**

Find the derivative of $\frac{1}{x + \sqrt{x^2 - a^2}}$

**Solution :**

Let $\quad y = \frac{1}{x + \sqrt{x^2 - a^2}}$

Then, $\quad y = \frac{1}{x + \sqrt{x^2 - a^2}} \times \frac{x - \sqrt{x^2 - a^2}}{x - \sqrt{x^2 - a^2}}$

$$= \frac{1}{a^2}\left(x - \sqrt{x^2 - a^2}\right)$$

$$\frac{dy}{dx} = \frac{1}{a^2}\frac{d}{dx}\left\{x - \sqrt{x^2 - a^2}\right\}$$

$$= \frac{1}{a^2}\left\{\frac{d}{dx}(x) - \frac{d(x^2 - a^2)^{1/2}}{d(x^2 - a^2)} \cdot \frac{d(x^2 - a^2)}{dx}\right\}$$

$$= \frac{1}{a^2}\left\{1 - \frac{1}{2\sqrt{x^2 - a^2}} \times 2x\right\}$$

$$= \frac{\sqrt{x^2 - a^2} - x}{a^2\sqrt{x^2 - a^2}}$$

**Example 5**

Find the derivative of $\sqrt{\frac{1 - x}{1 + x}}$.

**Solution :**

Let $\quad y = \sqrt{\frac{1 - x}{1 + x}} = \left(\frac{1 - x}{1 + x}\right)^{1/2}$

Then, $\quad \frac{dy}{dx} = \frac{d\left(\frac{1 - x}{1 + x}\right)^{1/2}}{d\left(\frac{1 - x}{1 + x}\right)} \cdot \frac{d\left(\frac{1 - x}{1 + x}\right)}{dx}$

$$= \frac{1}{2}\left(\frac{1 - x}{1 + x}\right)^{-1/2} \frac{(1 + x)\frac{d}{dx}(1 - x) - (1 - x)\frac{d}{dx}(1 + x)}{(1 + x)^2}$$

$$= \frac{1}{2} \cdot \frac{(1 - x)^{-1/2}}{(1 + x)^{-1/2}} \cdot \frac{-2}{(1 + x)^2}$$

$$= -\frac{1}{\sqrt{1 - x} \cdot (1 + x)^{3/2}}$$

***Example 6***

Find $\frac{dy}{dx}$ if $x = t + \frac{1}{t}$ and $y = t - \frac{1}{t}$.

***Solution :***

$$x = t + \frac{1}{t}$$

$$\frac{dx}{dt} = \frac{d}{dt}\left(t + \frac{1}{t}\right) = 1 - \frac{1}{t^2}$$

Again, $y = t - \frac{1}{t}$

$$\frac{dy}{dt} = \frac{d}{dt}\left(t - \frac{1}{t}\right) = 1 + \frac{1}{t^2}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 + \frac{1}{t^2}}{1 - \frac{1}{t^2}} = \frac{t^2 + 1}{t^2 - 1}$$

***Example 7***

Find $\frac{dy}{dx}$ for $x^3 + y^3 - 3axy = 0$

***Solution :***

$$x^3 + y^3 - 3axy = 0$$

$$\frac{d}{dx}(x^3 + y^3 - 3axy) = 0$$

or, $\frac{dx^3}{dx} + \frac{dy^3}{dy}\frac{dy}{dx} - 3a\left\{x\frac{dy}{dx} + y\frac{dx}{dx}\right\} = 0$

or, $3x^2 + 3y^2\frac{dy}{dx} - 3ax\frac{dy}{dx} - 3ay = 0$

or, $3(y^2 - ax)\frac{dy}{dx} = 3(ay - x^2)$

$\therefore \quad \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$

***Example 8***

Find $\frac{dy}{dx}$ for $xy^2 = (x + 2y)^3$

***Solution :***

We have, $xy^2 = (x + 2y)^3$

Then, $\frac{d}{dx}(xy^2) = \frac{d}{dx}(x + 2y)^3$

$$x\frac{dy^2}{dy}\cdot\frac{dy}{dx} + y^2\frac{d}{dx}(x) = \frac{d(x+2y)^3}{d(x+2y)}\cdot\frac{d(x+2y)}{dx}$$

$$x.2y\frac{dy}{dx} + y^2.1 = 3(x+2y)^2.\left(1 + 2\frac{dy}{dx}\right)$$

or, $2xy\frac{dy}{dx} + y^2 = 3(x+2y)^2 + 6(x+2y)^2\frac{dy}{dx}$

or, $\{2xy - 6(x+2y)^2\}\frac{dy}{dx} = 3(x+2y)^2 - y^2$

$$\frac{dy}{dx} = \frac{3(x+2y)^2 - y^2}{2xy - 6(x+2y)^2}$$

$$= \frac{\frac{3(x+2y)^3}{x+2y} - y^2}{2xy - \frac{6(x+2y)^3}{x+2y}}$$

$$= \frac{3xy^2 - y^2(x+2y)}{2xy(x+2y) - 6xy^2} \quad (\because xy^2 = (x+2y)^3)$$

$$= \frac{2y^2(x-y)}{2xy(x-y)} = \frac{y}{x}$$

***Example 9***

Differentiate $(3x - 1)^2$ w.r.t. $2x + 1$.

***Solution :***

Let $y = (3x-1)^2$

Then, $\frac{dy}{dx} = \frac{d}{dx}(3x-1)^2$

$$= \frac{d(3x-1)^2}{d(3x-1)}\cdot\frac{d(3x-1)}{dx}$$

$$= 2.(3x-1).3.1$$

$$= 6(3x-1)$$

Again let $z = 2x + 1$

Then, $\frac{dz}{dx} = \frac{d}{dx}(2x+1) = 2$

Now, $\frac{d(3x-1)^2}{d(2x+1)} = \frac{dy}{dz} = \frac{dy/dx}{dz/dx}$

$$= \frac{6(3x-1)}{2} = 3(3x-1)$$

## EXERCISE 17.1

1. Find, from definition, the derivatives of the following:
   (i) $3x^2$ (ii) $x^2 - 2$ (iii) $x^2 + 5x - 3$
   (iv) $3x^2 - 2x + 1$ (v) $\frac{1}{x}$ (vi) $\frac{3}{2x^2}$
   (vii) $\frac{1}{x-1}$ (viii) $\frac{1}{5-x}$ (ix) $\frac{1}{2x+3}$
   (x) $\frac{ax+b}{x}$ (xi) $x^{1/2}$ (T.U. 2049) (xii) $x + \sqrt{x}$ (T.U. 2051)
   (xiii) $(1+x)^{1/2}$ (xiv) $(2x+3)^{1/2}$ (T.U. 2048) (xv) $(1+x^2)^{1/2}$
   (xvi) $\frac{1}{x^{1/2}}$ (T.U. 2050) (xvii) $\frac{1}{(1-x)^{1/2}}$ (xviii) $\frac{1}{(3x+4)^{1/2}}$
   (xix) $\frac{ax+b}{\sqrt{x}}$

2. Find the derivatives of the following:
   (i) $x^5$ (ii) $5x^7$
   (iii) $3x^2 - 5x + 7$ (iv) $\frac{3x^3 + 2x - 1}{2x^2}$
   (v) $2x^{3/4} - 3x^{1/2} - 5x^{1/4}$ (vi) $\frac{2x + 3x^{3/4} + x^{1/2} + 1}{x^{1/4}}$
   (vii) $x^{3/4}(x^{2/3} + x^{1/3} + 1)$ (viii) $(x^{1/2} + x^{-1/2})^2$

3. Use the product rule to calculate the derivatives of :
   (i) $3x^2(2x-1)$ (ii) $(2x^2+1)(3x^2-2)$
   (iii) $(3x^4+5)(4x^5-3)$ (iv) $(3x^2+5x-1)(x^2+3)$
   (v) $(a+\sqrt{x})(a-\sqrt{x})$ (vi) $(a+x^{3/4})(a-x^{1/4})$

4. Use the quotient rule to find the derivatives of :
   (i) $\frac{x}{1+x}$ (ii) $\frac{x^2}{1-x^2}$ (iii) $\frac{x^2-a^2}{x^2+a^2}$
   (iv) $\frac{3}{x^2}$ (v) $\frac{x^2-2x}{x+1}$ (vi) $\frac{x^2+2x-1}{x^2-2x+1}$
   (vii) $\frac{\sqrt{x}}{\sqrt{x}+1}$

5. Use the general power rule to calculate the derivatives of :

(i) $(2x+3)^2$ (ii) $(3-2x)^3$

(iii) $(3x^2+2x-1)^4$ (iv) $(2x^2+3x-3)^{-6}$

(v) $\sqrt{(8-5x)}$ (vi) $(2x^2-3x+1)^{3/4}$

(vii) $\dfrac{1}{\sqrt{(ax^2+bx+c)}}$

(viii) $\dfrac{1}{\sqrt[3]{3x^2-4x-1}}$ (ix) $\dfrac{1}{\sqrt{a^n-x^n}}$

(x) $\dfrac{1}{\sqrt{(x+a)}-\sqrt{x}}$ (xi) $\dfrac{1}{x-\sqrt{a^2+x^2}}$ (HSEB 2051)

(xii) $\dfrac{1}{\sqrt{x+a}+\sqrt{x-a}}$ (xiii) $\dfrac{1}{\sqrt{x^2+a^2}-\sqrt{x^2-a^2}}$ (HSEB 2050)

(xiv) $\sqrt{\dfrac{x^2+a^2}{x^2-a^2}}$ (T.U. 2055 S)

6. Use the chain rule to calculate $\dfrac{dy}{dx}$, if

(i) $y=2u^2-3u+1$ and $u=2x^2$.

(ii) $y=2u^2+3$ and $u=3x^2-1$ (T.U. 2048)

(iii) $y=5t^2+6t-7$ and $t=x^3-2$ (T.U. 2049)

(iv) $y=\dfrac{t-1}{2t}$ and $t=\sqrt{x+1}$

(v) $y=(2u^2+3)^{1/3}$ and $u=\sqrt{(2x+1)}$

(vi) $y=\dfrac{t}{t^2-1}$ and $t=3x^2+1$.

7. Find the second derivatives of the following:

(i) $y=7x^2+6x-5$ (ii) $y=3x^4-x^2+1$

(iii) $y=\dfrac{3}{x^2}$ (iv) $y=\dfrac{1}{2x+1}$

8. Use implicit differentiation to obtain $\dfrac{dy}{dx}$ in the following.

(i) $x^2+y^2=16$ (ii) $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$

(iii) $y^2 = 4ax$ (iv) $2x^2 + 3xy + 2y^2 = 0$ (T.U. 2051 H)

(v) $x^2 + 2x^2y = y^3$ (vi) $x^3y^6 = (x + y)^9$ (T.U. 058S)

(vii) $x^2y + xy^2 = a^3$ (T.U. 052H)

(viii) $x^3 + y^3 = 3xy^2$ (T.U. 2049 H, 057 S) (ix) $x^2y^2 = x^2 + y^2$ (T.U. 2049)

9. i) Differentiate $x^6$ w.r.t. $x^2$.
   ii) Differentiate $(3x + 1)^4$ w.r.t. $(3x + 1)$
   iii) Differentiate $2x^6 - 3x^4 + x^2$ w.r.t. $x^3$
   iv) Differentiate $(5x - 1)^6$ w.r.t. $(x + 1)^2$

### Answers

1. (i) $6x$ (ii) $2x$ (iii) $2x + 5$
   (iv) $6x - 2$ (v) $-\frac{1}{x^2}$ (vi) $-\frac{3}{x^3}$
   (vii) $-\frac{1}{(x-1)^2}$ (viii) $\frac{1}{(5-x)^2}$ (ix) $-\frac{2}{(2x+3)^2}$
   (x) $-\frac{b}{x^2}$ (xi) $\frac{1}{2}x^{-1/2}$ (xii) $1 + \frac{1}{2}x^{-1/2}$
   (xiii) $\frac{1}{2}(1 + x)^{-1/2}$ (xiv) $(2x + 3)^{-1/2}$ (xv) $\frac{x}{\sqrt{1+x^2}}$
   (xvi) $-\frac{1}{2x^{3/2}}$ (xvii) $\frac{1}{2(1-x)^{3/2}}$ (xviii) $\frac{-3}{2(3x+4)^{3/2}}$
   (xix) $\frac{ax - b}{2x^{3/2}}$
2. (i) $5x^4$ (ii) $35x^6$ (iii) $6x - 5$
   (iv) $\frac{3x^3 - 2x + 2}{2x^3}$ (v) $\frac{6x^{1/2} - 6x^{1/4} - 5}{4x^{3/4}}$
   (vi) $\frac{3}{2x^{1/4}} + \frac{3}{2x^{1/2}} + \frac{1}{4x^{3/4}} - \frac{1}{4x^{5/4}}$
   (vii) $\frac{17}{12}x^{5/12} + \frac{13}{12}x^{1/12} + \frac{3}{4}x^{-1/4}$ (viii) $1 - \frac{1}{x^2}$
3. (i) $6x(3x - 1)$ (ii) $2x(12x^2 - 1)$
   (iii) $4x^3(27x^5 + 25x - 9)$ (iv) $12x^3 + 15x^2 + 16x + 15$
   (v) $-1$ (vi) $\frac{3a}{4}x^{-1/4} - \frac{a}{4}x^{-3/4} - 1$
4. (i) $\frac{1}{(1+x)^2}$ (ii) $\frac{2x}{(1-x^2)^2}$ (iii) $\frac{4a^2x}{(x^2+a^2)^2}$

(iv) $-\frac{6}{x^3}$ (v) $\frac{x^2+2x-2}{(x+1)^2}$ (vi) $\frac{4x}{(1-x)^3}$

(vii) $\frac{1}{2\sqrt{x}\left(\sqrt{x}+1\right)^2}$

5. (i) $4(2x+3)$ (ii) $-6(3-2x)^2$

(iii) $8(3x+1)(3x^2+2x-1)^3$

(iv) $-6(4x+3)(2x^2+3x-3)^{-7}$ (v) $\frac{-5}{2\sqrt{8-5x}}$

(vi) $\frac{3}{4}(4x-3)(2x^2-3x+1)^{-1/4}$

(vii) $-\frac{1}{2}(2ax+b)(ax^2+bx+c)^{-3/2}$

(viii) $-\frac{2}{3}(3x-2)(3x^2-4x-1)^{-4/3}$

(ix) $\frac{1}{2}nx^{n-1}(a^n-x^n)^{-3/2}$

(x) $\frac{1}{2a}\left(\frac{1}{\sqrt{x+a}}+\frac{1}{\sqrt{x}}\right)$ (xi) $-\frac{1}{a^2}\left(1+\frac{x}{\sqrt{a^2+x^2}}\right)$

(xii) $\frac{1}{4a}\left(\frac{1}{\sqrt{x+a}}-\frac{1}{\sqrt{x-a}}\right)$ (xiii) $\frac{x}{2a^2}\left(\frac{1}{\sqrt{x^2+a^2}}+\frac{1}{\sqrt{x^2-a^2}}\right)$

(xiv) $\frac{-2a^2x}{\sqrt{x^2+a^2}\,(x^2-a^2)^{3/2}}$

6. (i) $4x(8x^2-3)$ (ii) $24x(3x^2-1)$ (iii) $6x^2(5x^3-7)$

(iv) $\frac{1}{4(x+1)^{3/2}}$ (v) $\frac{4}{3}(4x+5)^{-2/3}$ (vi) $-\frac{2(9x^4+6x^2+2)}{3x^3(3x^2+2)^2}$

7. (i) 14 (ii) $36x^2-2$ (iii) $\frac{18}{x^4}$

(iv) $\frac{8}{(2x+1)^3}$

8. (i) $-\frac{x}{y}$ (ii) $\frac{b^2x}{a^2y}$ (iii) $\frac{2a}{y}$

(iv) $-\frac{(4x+3y)}{(3x+4y)}$ (v) $\frac{2x(1+2y)}{3y^2-2x^2}$ (vi) $\frac{y}{x}$

(vii) $-\frac{y(2x+y)}{x(x+2y)}$ (viii) $\frac{y^2-x^2}{y(y-2x)}$ (ix) $\frac{x(1-y^2)}{y(x^2-1)}$

9. (i) $3x^4$ (ii) $4(3x+1)^3$ (iii) $4x^3-4x+\frac{2}{3x}$

(iv) $\frac{15(5x-1)^5}{x+1}$

$\sqrt{a}-\sqrt{b} \times \sqrt{a}+\sqrt{b}$
$\sqrt{a}+\sqrt{b}$

## 17.8 Derivatives of the Trigonometrical Functions

**(i) Derivative of sin x** (T.U. 2052)

Let $y = \sin x$ ......... (i)

Let $\Delta x$ be a small increment in $x$ and $\Delta y$ be the corresponding increment in $y$. Then

$y + \Delta y = \sin(x + \Delta x)$ ......... (ii)

Now subtracting (i) from (ii) we get

$$\Delta y = \sin(x + \Delta x) - \sin x$$

$$= 2\sin\frac{\Delta x}{2} \cdot \cos\frac{2x + \Delta x}{2}$$

$$\therefore \quad \frac{\Delta y}{\Delta x} = \frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cdot \cos\frac{2x + \Delta x}{2}$$

$$\therefore \quad \frac{dy}{dx} = \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}\left(\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cdot \cos\frac{2x + \Delta x}{2}\right) = \cos x$$

Hence $\dfrac{d(\sin x)}{dx} = \cos x$

**(ii) Derivative of cos x**

With the same process that we followed to derive the derivative of $\sin x$, we get the derivative of $\cos x$, as

$$\frac{d(\cos x)}{dx} = -\sin x$$

**(iii) Derivative of tan x** (T.U. 2055 S)

Let $y = \tan x$ ......... (i)

Let $\Delta x$ be a small increment in $x$ and $\Delta y$ be the corresponding increment in $y$. Then

$y + \Delta y = \tan(x + \Delta x)$ ...... (ii)

Now subtracting (i) from (ii), we get

$$\Delta y = \tan(x + \Delta x) - \tan x$$

$$= \frac{\sin(x + \Delta x)}{\cos(x + \Delta x)} - \frac{\sin x}{\cos x}$$

$$= \frac{\sin(x + \Delta x)\cos x - \cos(x + \Delta x)\sin x}{\cos(x + \Delta x)\cos x}$$

$$= \frac{\sin (x + \Delta x - x)}{\cos (x + \Delta x) \cos x}$$

$$= \frac{\sin \Delta x}{\cos (x + \Delta x) \cos x}$$

$$\therefore \quad \frac{\Delta y}{\Delta x} = \frac{\sin \Delta x}{\Delta x} \cdot \frac{1}{\cos (x + \Delta x) \cos x}$$

$$\therefore \quad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \left( \frac{\sin \Delta x}{\Delta x} \cdot \frac{1}{\cos (x + \Delta x) \cos x} \right)$$

$$= \frac{1}{\cos x \cdot \cos x} = \sec^2 x$$

Hence, $\dfrac{d (\tan x)}{dx} = \sec^2 x$

### (iv) Derivative of cot x

With the same process that we followed to derive the derivative of tan $x$, we can get the derivative of cot $x$, as

$$\frac{d (\cot x)}{dx} = - \operatorname{cosec}^2 x$$

### (v) Derivative of sec x

Let $\quad y = \sec x = \dfrac{1}{\cos x}$ ...... (i)

Let $\Delta x$ be a small increment in $x$ and $\Delta y$ be the corresponding increment in $y$.

Then $\quad y + \Delta y = \dfrac{1}{\cos (x + \Delta x)}$ ...... (ii)

Now subtracting (i) from (ii), we get

$$\Delta y = \frac{1}{\cos (x + \Delta x)} - \frac{1}{\cos x}$$

$$= \frac{\cos x - \cos (x + \Delta x)}{\cos (x + \Delta x) \cos x}$$

$$= \frac{2 \sin \frac{2x + \Delta x}{2} \sin \frac{\Delta x}{2}}{\cos x . \cos (x + \Delta x)}$$

or, $$\frac{\Delta y}{\Delta x} = \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cdot \frac{\sin \frac{2x + \Delta x}{2}}{\cos x . \cos (x + \Delta x)}$$

$$\therefore \quad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left( \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cdot \frac{\sin \frac{2x + \Delta x}{2}}{\cos x . \cos (x + \Delta x)} \right)$$

$$= \frac{\sin x}{\cos x . \cos x} = \tan x \sec x.$$

Hence, $\frac{d (\sec x)}{dx} = \sec x . \tan x$

### (vi) Derivative of cosec x

(T.U. 2049 H)

With the same process that we followed to derive the derivative of sec $x$, we can get the derivative of cosec $x$, as

$$\frac{d (\text{cosec } x)}{dx} = - \text{cosec } x . \cot x$$

## Worked Out Examples

***Example 1***

Find, from first principles, the differential coefficient of $\sin (ax + b)$.

***Solution:***

Let $\quad y = \sin (ax + b)$

If $\Delta x$ is a small increment in $x$ and $\Delta y$ the corresponding increment in $y$, then

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin [a(x + \Delta x) + b] - \sin (ax + b)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2 \sin \left( \frac{ax + a\Delta x + b - ax - b}{2} \right) \cos \left( \frac{ax + a\Delta x + b + ax + b}{2} \right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left( \frac{\sin \frac{a\Delta x}{2}}{\frac{a \Delta x}{2}} \right) . a . \cos \left( ax + b + \frac{a \Delta x}{2} \right)$$

$$= 1 . a \cos (ax + b)$$

$$= a \cos (ax + b).$$

**Example 2**

Find the derivative of $\sin(ax^2 - b)$

**Solution:**

Let $y = \sin(ax^2 - b)$. Put $u = ax^2 - b$.

$\therefore \quad \dfrac{du}{dx} = 2ax$

Also $\quad y = \sin u \quad \therefore \dfrac{dy}{du} = \cos u$

Now, $\quad \dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = \cos u.\ 2ax = 2ax \cos(ax^2 - b)$

This can be worked out directly also

Let $\quad y = \sin(ax^2 - b)$. Then

$$\frac{dy}{dx} = \frac{d \sin(ax^2 - b)}{dx} = \frac{d \sin(ax^2 - b)}{d(ax^2 - b)} \cdot \frac{d(ax^2 - b)}{dx}$$

$$= \cos(ax^2 - b) \cdot 2ax = 2ax \cos(ax^2 - b)$$

**Example 3**

Find the derivative of $\sqrt{\tan x^2}$

**Solution:**

Let $y = \sqrt{\tan x^2}$. Put $u = \tan x^2$ and $v = x^2$

Then $\quad y = \sqrt{u} = u^{1/2}$. So

$$\frac{dy}{du} = \frac{1}{2} u^{-1/2} = \frac{1}{2} (\tan x^2)^{-1/2}$$

$$\frac{du}{dv} = \frac{d \tan v}{dv} = \sec^2 v = \sec^2 x^2.$$

$$\frac{dv}{dx} = \frac{dx^2}{dx} = 2x$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

$$= \frac{1}{2} (\tan x^2)^{-1/2} \sec^2 x^2 \cdot 2x$$

$$= \frac{x \sec^2 x^2}{\sqrt{\tan x^2}}$$

Alt. Method.

$$\frac{dy}{dx} = \frac{d (\tan x^2)^{1/2}}{dx}$$

$$= \frac{d(\tan x^2)^{1/2}}{d(\tan x^2)} \cdot \frac{d(\tan x^2)}{dx^2} \cdot \frac{dx^2}{dx}$$

$$= \frac{1}{2} (\tan x^2)^{-1/2} \sec^2 x^2 \cdot 2x$$

$$= \frac{x \sec^2 x^2}{\sqrt{\tan x^2}}$$

**Example 4**

Find the derivative of $x^2 \sec (ax - b)$

**Solution:**

Let $y = x^2 \sec (ax - b)$. Then

$$\frac{dy}{dx} = x^2 \frac{d \sec (ax - b)}{dx} + \sec (ax - b) \cdot \frac{dx^2}{dx}$$

$$= x^2 \cdot \frac{d \sec (ax - b)}{d(ax - b)} \cdot \frac{d(ax - b)}{dx} + \sec (ax - b) \cdot 2x$$

$$= x^2 \sec (ax - b) \tan (ax - b) \cdot a + 2x \sec (ax - b)$$

$$= x \sec (ax - b) \{ax \tan (ax - b) + 2\}$$

**Example 5**

Find the derivative of $\frac{1 + \sin x}{1 - \sin x}$

**Solution:**

Let $y = \frac{1 + \sin x}{1 - \sin x}$. Then

$$\frac{dy}{dx} = \frac{(1 - \sin x) \cdot \frac{d(1 + \sin x)}{dx} - (1 + \sin x) \frac{d(1 - \sin x)}{dx}}{(1 - \sin x)^2}$$

$$= \frac{(1 - \sin x) \cos x + (1 + \sin x) \cos x}{(1 - \sin x)^2}$$

$$= \frac{2 \cos x}{(1 - \sin x)^2}$$

**Example 6**

Find the derivative of $\frac{1 - \tan x}{\sec x}$.

**Solution:**

Let $y = \frac{1 - \tan x}{\sec x} = \cos x (1 - \tan x)$

$= \cos x - \sin x$

$$\therefore \quad \frac{dy}{dx} = \frac{d\cos x}{dx} - \frac{d\sin x}{dx}$$
$$= -\sin x - \cos x.$$

***Example 7***

Find $\frac{dy}{dx}$, when $x - y = \sin xy$.

***Solution:***

We have $x - y = \sin xy$.

Differentiating both sides with respect to $x$, we get

$$\frac{d(x-y)}{dx} = \frac{d(\sin xy)}{dx}$$

$$\text{or, } 1 - \frac{dy}{dx} = \frac{d(\sin xy)}{d(xy)} \cdot \frac{d(xy)}{dx}$$

$$= \cos xy \left(y + x\frac{dy}{dx}\right)$$

$$= y\cos xy + x\cos xy \cdot \frac{dy}{dx}$$

$$\text{or, } \quad (1 + x\cos xy)\frac{dy}{dx} = 1 - y\cos xy$$

$$\therefore \quad \frac{dy}{dx} = \frac{1 - y\cos xy}{1 + x\cos xy}$$

***Example 8***

Find $\frac{dy}{dx}$, when $y = 2\theta - \tan\theta$ and $x = \tan\theta$

***Solution:***

We have $y = 2\theta - \tan\theta$ and $x = \tan\theta$

$$\frac{dy}{d\theta} = 2 - \sec^2\theta \text{ and } \frac{dx}{d\theta} = \sec^2\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$= \frac{2 - \sec^2\theta}{\sec^2\theta}$$

$$= 2\cos^2\theta - 1$$

$$= \cos 2\theta$$

## 17.9 Derivatives of Inverse Circular Functions

### (i) Derivative of $\sin^{-1} x$

Let $y = \sin^{-1}x$. So $\sin y = x$

Differentiating both sides with respect to $y$, we get

$$\cos y = \frac{dx}{dy}$$

or, $\frac{dx}{dy} = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

### (ii) Derivative of $\cos^{-1}x$

Let $y = \cos^{-1}x$ $\quad \therefore x = \cos y$

Differentiating both sides with respect to $y$, we get

$$\frac{dx}{dy} = -\sin y = -\sqrt{1 - \cos^2 y} = -\sqrt{1 - x^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{-1}{\sqrt{1 - x^2}}$$

### (iii) Derivative of $\tan^{-1}x$

Let $y = \tan^{-1}x$ $\quad \therefore x = \tan y$.

Now differentiating both sides with respect to $y$, we get

$$\frac{dx}{dy} = \sec^2 y = 1 + \tan^2 y = 1 + x^2$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{1 + x^2}$$

Similarly, we get

(iv) $\frac{d(\text{cosec}^{-1}x)}{dx} = \frac{-1}{x\sqrt{x^2 - 1}}$

(v) $\frac{d(\sec^{-1}x)}{dx} = \frac{1}{x\sqrt{x^2 - 1}}$

(vi) $\frac{d(\cot^{-1}x)}{dx} = \frac{-1}{1 + x^2}$

## Worked Out Examples

***Example 1.***

Find $\frac{dy}{dx}$, when $y = x^3.\cot^{-1}x$.

***Solution:***

We have $y = x^3 \cot^{-1}x$

Now differentiating both sides with respect to $x$, we get

$$\frac{dy}{dx} = \frac{dx^3}{dx}.\cot^{-1}x + x^3\frac{d\cot^{-1}x}{dx}$$

$$= 3x^2\cot^{-1}x - \frac{x^3}{1+x^2}$$

***Example 2.***

Find $\frac{dy}{dx}$, when $y = \tan^{-1}\frac{2x}{1-x^2}$

***Solution:***

We have $y = \tan^{-1}\frac{2x}{1-x^2}$. Put $x = \tan\theta$

$$y = \tan^{-1}\frac{2\tan\theta}{1-\tan^2\theta} = \tan^{-1}(\tan 2\theta)$$

$$= 2\theta = 2\tan^{-1}x$$

$$\frac{dy}{dx} = \frac{2}{1+x^2}$$

## EXERCISE 17.2

1. Find, from the first principles, the differential coefficients of
   (i) $\sin 4x$ (ii) $\cos(ax - b)$ (iii) $\tan(3x - 4)$
   (iv) $\sin\frac{3x}{2}$ (v) $\tan\frac{5x}{3}$ (vi) $\cos^2 x$ (T.U. 057S, 058S)
   (vii) $\sin^2 3x$ (viii) $\sqrt{\sin 2x}$ (ix) $\sqrt{\sec x}$
2. Find the derivatives of
   (i) $\sin(4x - 5)$ (ii) $\cos(ax + b)$ (iii) $\tan(5x^2 + 6)$

(iv) $\cot\sqrt{x}$ (v) $\sec\frac{1}{x}$ (vi) $\operatorname{cosec}\frac{ax^2}{b}$

(vii) $\sin^5(ax^2 - c)$ (viii) $\cos^3(2ax - 3b)$ (ix) $a\sqrt{\tan(5x - 7)}$

(x) $\sec^5\left(\frac{ax + b}{c}\right)$ (xi) $\operatorname{cosec}^n\left(\frac{px^2 - q}{r}\right)$ (T.U. 2056 S)

3. Find the differential coefficients of

(i) $\tan(\cos 5x)$ (ii) $\cos(\sin(3x^2 + 2))$

(iii) $\sin(\tan(ax + b))$ (iv) $\sec^2(\tan\sqrt{x})$ (T.U. 2052)

(v) $\tan^5(\sin(px - q))$ (T.U. 2050)

(vi) $\operatorname{cosec}^3(\cot 4x)$

(vii) $\cot(\sqrt{\tan 3x})$ (T.U. 2049 H)

(viii) $\sin^2(\cos 6x)$

4. Find the derivatives of

(i) $(x^2 + 3x)\sin 5x$ (ii) $x^3\tan(2x^3 + 3x)$

(iii) $a\sqrt{x}\cos(ax^2 - b)$ (iv) $(x + \sin 2x)\sec 3x^2$

(v) $ax^3\operatorname{cosec}(p - qx)$ (vi) $\frac{1}{\sqrt{x}}\cdot\sin\sqrt{x}$ (HSEB 2051)

(vii) $\frac{x^2 - 1}{\cos 4x}$ (viii) $\frac{\sec nx}{ax - b}$

5. Find the differential coefficients of

(i) $\frac{1 - 2\sin^2\frac{x}{2}}{\cos^2 x}$ (ii) $\frac{\sin 2nx}{\cos^2 nx}$

(iii) $\frac{\sin ax - \sin bx}{\cos ax + \cos bx}$ (iv) $\frac{1 - \cos x}{1 + \cos x}$

(v) $\sqrt{\frac{1 - \sin x}{1 + \sin x}}$ (HSEB 2051) (vi) $\frac{\cos 2x + 1}{\sin 2x}$

(vii) $\frac{\cos 2x}{1 - \sin 2x}$ (viii) $\frac{1}{\sec x - \tan x}$

(ix) $\frac{\sec x + \tan x}{\sec x - \tan x}$ (x) $\frac{1 + \tan x}{1 - \tan x}$

6. Find the derivatives of

(i) $\sin 6x\cos 4x$ (ii) $\sin 2mx\sin 2nx$

(iii) $\cos 7x\cdot\cos 5x$ (iv) $\sin 3x\cdot\cos 5x$

7. Find the derivatives of

(i) $\sin^{-1}(3x-4)$

(ii) $\cos^{-1}\frac{3x^2-2}{2}$

(iii) $\tan^{-1}\frac{1}{5x-3}$

(iv) $\sec^{-1}(\tan x)$

(v) $\sin^{-1}(1-2x^2)$ (T.U. 2050 H)

(vi) $\cos^{-1}(4x^3-3x)$

(vii) $\cos^{-1}\frac{1-x^2}{1+x^2}$ (T.U. 2057 S)

(viii) $\tan^{-1}\frac{1}{1-x^2}$

(ix) $\tan^{-1}\frac{\sin 2x}{1+\cos 2x}$

(x) $\sec^{-1}\frac{1}{\sqrt{(1-x^2)}}$

(xi) $\tan^{-1}\frac{2x}{1-x^2}$ (T.U. 2056 S)

8. Find $\frac{dy}{dx}$, when

(i) $x+y=\sin y$

(ii) $x+y=\cos(x-y)$

(iii) $x-y=\tan xy$

(iv) $x^2y=\sec xy^2$ (HSEB 2054)

(v) $x^2y^2=\tan(ax+by)$ (T.U. 2051)

(vi) $x^2+y^2=\sin xy$ (T.U. 2052) (HSEB 2052)

(vii) $x^2y^2=\tan xy$ (T.U. 2048)

(viii) $xy=\sec(x-y)$

(ix) $xy=\tan(x^2+y^2)$

9. Find $\frac{dy}{dx}$, when

(i) $x=a\cos^2\theta$, $y=b\sin^2\theta$ (T.U. 2055 S)

(ii) $x=2a\sin t\cos t$, $y=b\cos 2t$

(iii) $x=2a\tan\theta$, $y=a\sec^2\theta$ (T.U. 2052 H)

(iv) $x=\tan t$, $y=\sin t\cos t$

(v) $x=a(\cos t+t\sin t)$, $y=a(\sin t-t\cos t)$ (T.U. 2050)

(vi) $x=a(\tan t-t\sec^2 t)$, $y=a\sec^2 t$ (T.U. 2049)

(vii) $x=a(t+\sin t)$, $y=a(1-\cos t)$ (T.U. 051H, 058 S)

10. Differentiate

(i) $\sin x$ with respect to $\cos x$. (ii) $\tan x$ with respect to $\sec x$

(iii) $\sec^2 x$ with respect to $\tan x$ (iv) $\operatorname{cosec} x$ with respect to $\cot x$

11. Find the derivatives of

a) $\cos x^\circ$ b) $x \sin x^\circ$

### Answers

1. (i) $4 \cos 4x$ (ii) $-a \sin (ax - b)$ (iii) $3 \sec^2(3x - 4)$

(iv) $\frac{3}{2} \cos \frac{3x}{2}$ (v) $\frac{5}{3} \sec^2 \frac{5x}{3}$ (vi) $-\sin 2x$

(vii) $3 \sin 6x$ (viii) $\dfrac{\cos 2x}{\sqrt{\sin 2x}}$ (ix) $\frac{1}{2} \sqrt{\sec x} \cdot \tan x$

2. (i) $4 \cos (4x - 5)$ (ii) $-a \sin (ax + b)$

(iii) $10x \sec^2 (5x^2 + 6)$ (iv) $-\dfrac{1}{2\sqrt{x}} \operatorname{cosec}^2 \sqrt{x}$

(v) $-\dfrac{1}{x^2} \sec \dfrac{1}{x} \tan \dfrac{1}{x}$ (vi) $-\dfrac{2ax}{b} \operatorname{cosec} \dfrac{ax^2}{b} \cot \dfrac{ax^2}{b}$

(vii) $10ax \sin^4 (ax^2 - c) \cos (ax^2 - c)$

(viii) $-6a \cos^2 (2ax - 3b) \sin (2ax - 3b)$

(ix) $\dfrac{5a \sec^2(5x - 7)}{2\sqrt{\tan (5x - 7)}}$

(x) $\dfrac{5a}{c} \sec^5 \dfrac{ax + b}{c} \tan \dfrac{ax + b}{c}$

(xi) $\dfrac{-2npx}{r} \operatorname{cosec}^n \left(\dfrac{px^2 - q}{r}\right) \cot \left(\dfrac{px^2 - q}{r}\right)$

3. (i) $-5 \sec^2 (\cos 5x) \sin 5x$

(ii) $-6x \sin (\sin (3x^2 + 2)) \cos (3x^2 + 2)$

(iii) $a \cos (\tan (ax + b)) \sec^2(ax + b)$

(iv) $\dfrac{1}{\sqrt{x}} \sec^2(\tan \sqrt{x}) \tan (\tan \sqrt{x}) \sec^2\sqrt{x}$

(v) $5p \tan^4(\sin (px - q)) \sec^2(\sin (px - q)) \cos (px - q)$

(vi) $12 \operatorname{cosec}^3(\cot 4x) \cot (\cot 4x) \operatorname{cosec}^2 4x$

(vii) $\dfrac{-3}{2\sqrt{\tan 3x}} \operatorname{cosec}^2 \left(\sqrt{\tan 3x}\right) \sec^2 3x$

(viii) $-6 \sin (2 \cos 6x) \sin 6x$

4. (i) $(2x + 3) \sin 5x + 5(x^2 + 3x) \cos 5x$

(ii) $3x^2 \tan (2x^3 + 3x) + 3x^3(2x^2 + 1) \sec^2 (2x^3 + 3x)$

(iii) $\frac{a}{2\sqrt{x}}\cos(ax^2-b)-2a^2x^{3/2}\sin(ax^2-b)$

(iv) $(1+2\cos 2x)\sec 3x^2+6x(x+\sin 2x)\sec 3x^2 . \tan 3x^2$

(v) $3ax^2 \operatorname{cosec}(p-qx)+aqx^3\operatorname{cosec}(p-qx)\cot(p-qx)$

(vi) $\frac{1}{2x}(\cos\sqrt{x}-\frac{1}{\sqrt{x}}\sin\sqrt{x})$

(vii) $\frac{2x\cos 4x+4(x^2-1)\sin 4x}{\cos^2 4x}$

(viii) $\frac{n(ax-b)\sec nx\tan nx-a\sec nx}{(ax-b)^2}$

5. (i) $\sec x\tan x$ (ii) $2n\sec^2 nx$

(iii) $\frac{1}{2}(a-b)\sec^2\frac{1}{2}(a-b)x$ (iv) $\tan\frac{1}{2}x\sec^2\frac{1}{2}x$

(v) $-\frac{1}{2}\sec^2\left(\frac{\pi}{4}-\frac{x}{2}\right)$ (vi) $-\operatorname{cosec}^2 x$

(vii) $\sec^2\left(\frac{\pi}{4}+x\right)$ (viii) $\sec x(\tan x+\sec x)$

(ix) $2\sec x(\sec x+\tan x)^2$ (x) $\sec^2\left(\frac{\pi}{4}+x\right)$

6. (i) $5\cos 10x+\cos 2x$

(ii) $(m+n)\sin 2(m+n)x-(m-n)\sin 2(m-n)x$

(iii) $-6\sin 12x-\sin 2x$

(iv) $4\cos 8x-\cos 2x$

7. (i) $\frac{3}{\sqrt{\{1-(3x-4)^2\}}}$ (ii) $\frac{-2\sqrt{3}}{\sqrt{(4-3x^2)}}$

(iii) $-\frac{5}{1+(5x-3)^2}$ (iv) $\frac{\sec^2 x}{\tan x\sqrt{(\tan^2 x-1)}}$

(v) $\frac{-2}{\sqrt{1-x^2}}$ (vi) $\frac{-3}{\sqrt{(1-x^2)}}$

(vii) $\frac{2}{1+x^2}$ (viii) $\frac{2x}{1+(1-x^2)^2}$

(ix) 1 (x) $\frac{1}{\sqrt{(1-x^2)}}$

(xi) $\frac{2}{1+x^2}$

8. (i) $\frac{1}{\cos y-1}$ (ii) $\frac{1+\sin(x-y)}{\sin(x-y)-1}$

(iii) $\dfrac{1 - y\sec^2 xy}{1 + x\sec^2 xy}$

(iv) $\dfrac{2xy - y^2\sec xy^2 \tan xy^2}{2xy\sec xy^2 \tan xy^2 - x^2}$

(v) $\dfrac{2xy^2 - a\sec^2(ax + by)}{b\sec^2(ax + by) - 2x^2y}$

(vi) $\dfrac{2x - y\cos xy}{x\cos xy - 2y}$

(vii) $-\dfrac{y}{x}$

(viii) $\dfrac{\sec(x - y)\tan(x - y) - y}{x + \sec(x - y)\tan(x - y)}$

(ix) $\dfrac{y - 2x\sec^2(x^2 + y^2)}{2y\sec^2(x^2 + y^2) - x}$

9. (i) $\dfrac{-b}{a}$ (ii) $\dfrac{-b}{a}\tan 2t$ (iii) $\tan\theta$

(iv) $\cos^2 t(\cos^2 t - \sin^2 t)$ (v) $\tan t$ (vi) $-\dfrac{1}{t}$

(vii) $\tan\frac{1}{2}t$

10. (i) $-\cot x$ (ii) $\operatorname{cosec} x$ (iii) $2\tan x$ (iv) $\cos x$

11. a) $-\dfrac{\pi}{180}\sin x^\circ$ b) $\dfrac{\pi x}{180}\cos x^\circ + \sin x^\circ$

## 17.10 Derivatives of Logarithmic and Exponential Functions

Any function of the form

$$f(x) = a^x$$

is called an exponential function, in which the base $a$ is a constant and the index $x$ is a variable. The inverse of an exponential function is called a logarithmic function which is denoted by $\log_a x$.

So, if $y = a^x$, we have $\log_a y = x$

There is a special type of exponential function $e^x$, where $e$ is the limiting value $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$. The value of $e$ lies between 2 and 3 and is approximately 2.718. The corresponding logarithmic function is called the natural logarithmic function and is denoted by $\log x$, base $e$ being understood.

Thus $$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

Further put $n = \frac{1}{h}$ so that $h = \frac{1}{n}$

When $n \to \infty$, $h \to 0$.

So, $$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$
$$= \lim_{h\to 0}(1 + h)^{1/h}$$

The following properties of the logarithmic functions can be easily deduced.

1. $\log_a(x.y) = \log_a x + \log_a y$
2. $\log_a \frac{x}{y} = \log_a x - \log_a y$
3. $\log_a x^n = n \log_a x$
4. $\log_a a = 1$
5. $\log_a m = \log_a b . \log_b m$
6. $\log_a 1 = 0$

Now, we find the derivatives of the exponential and the logarithmic functions by first principles using the following limit theorems

1. $\lim_{x\to 0} \frac{\log(1 + x)}{x} = 1$
2. $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$
3. $\lim_{x\to 0} \frac{a^x - 1}{x} = \log a$

## I. Derivative of Natural Logarithmic Function

Let $y = \log x$

Let $\Delta x$ and $\Delta y$ be the small increments in $x$ and $y$ respectively.

Then,
$$y + \Delta y = \log(x + \Delta x)$$

or, $$\Delta y = \log(x + \Delta x) - \log x$$

or, $$\frac{\Delta y}{\Delta x} = \frac{\log(x + \Delta x) - \log x}{\Delta x}$$

$$= \frac{\log\left(\frac{x + \Delta x}{x}\right)}{\Delta x}$$

$$= \frac{1}{\Delta x}\log\left(1 + \frac{\Delta x}{x}\right)$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\log\left(1 + \frac{\Delta x}{x}\right)$$

$$= \lim_{\Delta x \to 0} \frac{\log\left(1 + \frac{\Delta x}{x}\right)}{\frac{\Delta x}{x}\cdot x}$$

$$= \frac{1}{x}\lim_{\Delta x \to 0} \frac{\log\left(1 + \frac{\Delta x}{x}\right)}{\frac{\Delta x}{x}}$$

$$= \frac{1}{x}\cdot 1 = \frac{1}{x}$$

## II. Derivative of $e^x$ (from First Principles)

Let $\quad y = e^x \quad$ ......... (1)

Let $\Delta x$ be a small increment in $x$ and $\Delta y$, a corresponding increment in $y$. Then,

$$y + \Delta y = e^{x + \Delta x} \quad \text{......... (2)}$$

Subtracting (1) from (2), we get

$$\Delta y = e^{x + \Delta x} - e^x = e^x.e^{\Delta x} - e^x$$

$$= e^x(e^{\Delta x} - 1)$$

or, $\quad \dfrac{\Delta y}{\Delta x} = \dfrac{e^x(e^{\Delta x} - 1)}{\Delta x}$

Now, $\quad \dfrac{dy}{dx} = \lim_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{e^x(e^{\Delta x} - 1)}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{e^x(e^{\Delta x} - 1)}{\Delta x}$$

$$= e^x \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x}$$

$$= e^x.1$$

$$= e^x \qquad \left(\because \lim_{x \to 0} \frac{e^x - 1}{x} = 1\right)$$

## III. Derivative of Logarithmic Functions $\log_a x$

Let $y = \log_a x$

$= \log_a e.\log x$, by changing the base from $a$ to $e$.

$$\therefore \quad \frac{dy}{dx} = \log_a e \, \frac{d(\log x)}{dx}$$

$$= (\log_a e) \cdot \frac{1}{x}$$

## IV. Derivative of $a^x$ (from First Principle)

Let $y = a^x$

Let $\Delta x$ and $\Delta y$ be the small increments in $x$ and $y$ respectively. Then,

$$y + \Delta y = a^{x + \Delta x}$$

or, $$\Delta y = a^{x + \Delta x} - a^x$$

$$= a^x (a^{\Delta x} - 1)$$

$$\frac{\Delta y}{\Delta x} = \frac{a^x (a^{\Delta x} - 1)}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} a^x \frac{(a^{\Delta x} - 1)}{\Delta x}$$

$$= a^x . \log a \qquad \left(\because \lim_{\Delta x \to 0} \frac{a^x - 1}{x} = \log a\right)$$

## Worked Out Examples

***Example 1.***

Find the derivative of $\log (5x^2 + 6)$

***Solution:***

Let $y = \log (5x^2 + 6)$. Put $u = 5x^2 + 6$

Then $y = \log u$ and $u = 5x^2 + 6$.

$$\because \quad \frac{dy}{du} = \frac{d \log u}{du} = \frac{1}{u}, \qquad \frac{du}{dx} = 10x.$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot 10x = \frac{10x}{5x^2 + 6}$$

**Alt. Method**

Let $y = \log (5x^2 + 6)$

$$\frac{dy}{dx} = \frac{d \log (5x^2 + 6)}{dx}$$

$$= \frac{d \log (5x^2 + 6)}{d(5x^2 + 6)} \cdot \frac{d(5x^2 + 6)}{dx}$$

$$= \frac{1}{5x^2 + 6} \cdot 10x$$

$$= \frac{10x}{5x^2 + 6}$$

***Example 2.***

Find the derivative of $\sin 3x . \log (ax + b)$

***Solution:***

Let $y = \sin 3x . \log (ax + b)$

$$\because \quad \frac{dy}{dx} = \frac{d (\sin 3x)}{d (3x)} \cdot \frac{d(3x)}{dx} . \log (ax + b)$$

$$+ \sin 3x . \frac{d \log (ax + b)}{d(ax + b)} \cdot \frac{d(ax + b)}{dx}$$

$$= \cos 3x . 3 . \log(ax + b) + \sin 3x . \frac{1}{ax + b} . a$$

$$= 3 \cos 3x . \log (ax + b) + \frac{a \sin 3x}{ax + b}$$

***Example 3.***

Find the derivative of $\dfrac{\sin x}{\log(3x + 4)}$

**Solution:**

Let $y = \dfrac{\sin x}{\log(3x+4)}$

$$\therefore \quad \frac{dy}{dx} = \frac{\log(3x+4)\,.\,\dfrac{d \sin x}{dx} - \sin x\,.\,\dfrac{d \log(3x+4)}{dx}}{\{\log(3x+4)\}^2}$$

$$= \frac{\log(3x+4)\,.\,\cos x - \sin x \dfrac{3}{3x+4}}{\{\log(3x+4)\}^2}$$

$$= \frac{(3x+4)\cos x \log(3x+4) - 3\sin x}{(3x+4)\{\log(3x+4)\}^2}$$

**Example 4**

Find the derivative of $e^{ax+b}$

**Solution:**

Let $y = e^{ax+b}$. Put $u = ax + b$

$\because \quad y = e^u$ and $u = ax + b$

$\dfrac{dy}{du} = \dfrac{de^u}{du}$ and $\dfrac{du}{dx} = \dfrac{d(ax+b)}{dx}$

$\dfrac{dy}{du} = e^u$ and $\dfrac{du}{dx} = a.$

$$\therefore \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= e^u \,.\, a$$

$$= ae^{ax+b}.$$

**Example 5.**

Find the derivative of $e^{5x}$ . sin $6x$

**Solution:**

Let $y = e^{5x} \sin 6x$. Then

$$\frac{dy}{dx} = \frac{d\,e^{5x}}{dx}\,.\,\sin 6x + e^{5x}\,.\,\frac{d \sin 6x}{dx}$$

$$= 5.e^{5x}\sin 6x + 6e^{5x}\cos 6x$$

$$= e^{5x}(5\sin 6x + 6\cos 6x)$$

## EXERCISE 17.3

1. Find, from the first principle, the derivatives of:

   (i) $\log(ax + b)$ (ii) $\log_5 x$ (iii) $\log \frac{x}{10}$

   (iv) $e^{ax+b}$ (v) $e^{x/3}$

2. Find the derivatives of

   (i) $\log(\sin x)$ (ii) $\log(x + \tan x)$

   (iii) $\log(1 + e^{5x})$ (iv) $\log(\log x)$

   (v) $\log(\sec x)$ (vi) $\log(1 + \sin^2 x)$

   (vii) $ln(e^{ax} + e^{-ax})$ (viii) $\log\left(\sqrt{a^2 + x^2} + b\right)$

   (ix) $\log\left(\sqrt{a + x} + \sqrt{a - x}\right)$ x) $ln\,|x - 4|$

3. Find the differential coefficient of

   (i) $e^{\sin x}$ (ii) $e^{\sqrt{\cos x}}$ (iii) $e^{(1 + \log x)}$

   (iv) $e^{\sin(\log x)}$ (v) $\tan(\log x)$ (vi) $\sin(1 + e^{ax})$

   (vii) $\cos(\log \sec x)$ (viii) $\sec(\log \tan x)$ (ix) $\sin \log \sin e^{x^2}$

4. Differentiate the following with respect to $x$:

   (i) $x^2 \log(1 + x)$ (ii) $x^5 e^{ax}$

   (iii) $\sin ax.\log x$ (iv) $e^{ax} \cos bx$

   (v) $(\tan x + x^2) \log x$ (vi) $(\sin x + \cos x)e^{ax}$

5. Calculate the derivatives of

   (i) $\frac{\log x}{\sin x}$ (ii) $\frac{\log(ax + b)}{e^{px}}$ (iii) $\frac{e^{ax}}{\cos bx}$

   (iv) $\frac{\sin ax}{1 + \log x}$ (v) $\frac{\log x}{a^2 + x^2}$ (vi) $\frac{x^n}{e^{ax+b}}$

6. Find $\frac{dy}{dx}$, when

   (i) $xy = \log(x^2 + y^2)$ (T.U. 2052 H)

   (ii) $x^2 + y^2 = \log(x + y)$

   (iii) $e^{xy} = xy$ (T.U. 2053 H)

   (iv) $x = e^{\cos 2t}$, $y = e^{\sin 2t}$

   (v) $x = \cos(\log t)$, $y = \log(\cos t)$ (T.U. 2049 H)

   (vi) $x = \log t + \sin t$, $y = e^t + \cos t$.

## Answers

1. (i) $\frac{a}{ax+b}$ (ii) $\frac{\log_5 e}{x}$ (iii) $\frac{1}{x}$

(iv) $ae^{ax+b}$ (v) $\frac{1}{3}e^{x/3}$

2. (i) $\cot x$ (ii) $\frac{1+\sec^2 x}{x+\tan x}$ (iii) $\frac{5e^{5x}}{1+e^{5x}}$

(iv) $\frac{1}{x\log x}$ (v) $\tan x$ (vi) $\frac{2\sin x\cos x}{1+\sin^2 x}$

(vii) $\frac{a(e^{ax}-e^{-ax})}{e^{ax}+e^{-ax}}$ (viii) $\frac{x}{\sqrt{a^2+x^2}\,(\sqrt{a^2+x^2}+b)}$

(ix) $\frac{1}{2(\sqrt{a+x}+\sqrt{a-x})}\left(\frac{1}{\sqrt{a+x}}-\frac{1}{\sqrt{a-x}}\right)$ (x) $\frac{1}{x-4}$

3. (i) $e^{\sin x}\cos x$ (ii) $\frac{-e^{\sqrt{\cos x}}\sin x}{2\sqrt{\cos x}}$ (iii) $\frac{e^{(1+\log x)}}{x}$

(iv) $\frac{1}{x}e^{\sin(\log x)}.\cos(\log x)$ (v) $\frac{\sec^2(\log x)}{x}$ (vi) $ae^{ax}\cos(1+e^{ax})$

(vii) $-\tan x.\sin(\log\sec x)$

(viii) $\frac{\sec(\log\tan x)\tan(\log\tan x)\sec^2 x}{\tan x}$

(ix) $2x\,e^{x^2}\cot(e^{x^2})\cos\log\sin(e^{x^2})$

4. (i) $2x\log(1+x)+\frac{x^2}{1+x}$ (ii) $5x^4e^{ax}+ax^5e^{ax}$

(iii) $a\cos ax.\log x+\frac{\sin ax}{x}$ (iv) $(a\cos bx-b\sin bx)e^{ax}$

(v) $(\sec^2 x+2x)\log x+\frac{\tan x+x^2}{x}$

(vi) $(\cos x-\sin x)e^{ax}+a(\sin x+\cos x)e^{ax}$

5. (i) $\frac{\sin x-x\cos x.\log x}{x\sin^2 x}$ (ii) $\frac{a-p(ax+b)\log(ax+b)}{(ax+b)e^{px}}$

(iii) $\frac{(a\cos bx+b\sin bx)e^{ax}}{\cos^2 bx}$ (iv) $\frac{ax(1+\log x)\cos ax-\sin ax}{x(1+\log x)^2}$

(v) $\frac{a^2+x^2-2x^2\log x}{x(a^2+x^2)^2}$ (vi) $\frac{(n-ax)x^{n-1}}{e^{ax+b}}$

6. (i) $\frac{2x-x^2y-y^3}{x^3+xy^2-2y}$ (ii) $\frac{1-2x^2-2xy}{2xy+2y^2-1}$

(iii) $-\frac{y}{x}$ (iv) $-e^{\sin 2t-\cos 2t}.\cot 2t$

(v) $\frac{t\tan t}{\sin(\log t)}$ (vi) $\frac{t(e^t-\sin t)}{1+t\cos t}$

## ADDITIONAL QUESTIONS
## (Derivative)

1. Define differential coefficient of a function at a given point. Find from first principles, the derivative of $x + \sqrt{x}$ with respect to $x$. (T.U. 2051)

2. Define the differential coefficient of a function at a given point. What are its physical meanings ? Find from definition, the differential coefficient of $\sin x$ with respect to $x$. (T.U. 2052)

3. Define the differential coefficient of a function $f(x)$ at a point and interprete it geometrically. Find from first principles, the derivative of $\sqrt{2x+3}$ with respect to $x$. (T.U. 2053)

4. Find from first principles, the derivatives of

   (i) $x + \sqrt{x+1}$ (T.U. 2051 H)

   (ii) $\sqrt{ax+b}$ . Define $\frac{dy}{dx}$ as a slope. (T.U. 2052 H)

5. Define the derivative of a function at any point. Prove that

   (i) $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$

   (ii) $\frac{d}{dx}[f(x) \cdot g(x)] = f(x)\frac{d}{dx}(g(x)) + g(x)\frac{d}{dx}(f(x))$ (T.U. 2054)

6. Find from first principles, the derivatives of

   (i) $(2x+3)^5$ (ii) $x^{3/4}$ (iii) $\cos(ax+b)$

   (iv) $5^x$ (v) $\log(3x+2)$ (vi) $(3x-4)^{1/3}$

   (vii) $\frac{1}{\sqrt[3]{4x+1}}$

7. Find $\frac{dy}{dx}$ of

   (i) $y = \sqrt{\frac{a}{a-x}}$ (ii) $y = \sqrt{\frac{a-x}{a+x}}$

   (iii) $x^{2/3} + y^{2/3} = a^{2/3}$ (iv) $\sqrt{x} + \sqrt{y} = \sqrt{a}$

   (v) $x^2y^3 = (x+2y)^5$ (vi) $y = x^2 \sin\frac{1}{x}$

   (vii) $y = \sin^{-1}(3x - 4x^3)$ (T.U 2048)

   (viii) $x = \frac{3t}{1+t^3}, y = \frac{3t^2}{1+t^3}$ (T.U. 2048 H)

(ix) $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$

(x) $y = \dfrac{x^2\sqrt{2x+3}}{\sqrt[3]{4x+1}}$ (xi) $y = \sqrt{\dfrac{(x+1)(x+2)}{(x-3)(x-4)}}$

8. If $f(x) = mx + c$ and $f(0) = f'(0) = 1$, what is $f(2)$ ?

9. If $f(x) = 4x^3 - 8x + 3$, find $f'(0)$, $f''(1)$.

10. (a) If $\sin y = x\sin(a+y)$, prove that

$$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

(b) If $\sin y = x\cos(a+y)$, show that

$$\frac{dy}{dx} = \frac{\cos^2(a+y)}{\cos a}$$

11. If $y = \sqrt{\dfrac{1+x}{1-x}}$, prove that $\dfrac{dy}{dx} = \dfrac{y}{1-x^2}$.

12. Find the derivatives of

(i) $x^{\sin x}$ (ii) $(\sin x)^x$ (iii) $(\sin x)^{\log x}$

(iv) $e^{x^x}$ (v) $x^{e^x}$ (HSEB 2050, 052) (vi) $(\log x)^{\tan x}$ (T.U. 2050 H)

(vii) $(\tan x)^{\log x}$ (T.U. 2048 H) (viii) $x^{\sec x}$ (ix) $(\sin x)^{\cos x}$ (T.U. 2053)

13. Find $\dfrac{dy}{dx}$, when

(i) $y = x^y$

(ii) $x^y . y^x = 1$ (T.U. 2051 H) (iii) $x^m y^n = (x+y)^{m+n}$

(iv) $e^{\sin x} + e^{\sin y} = 1$ (v) $x^y = y^x$ (T.U. 2048)

(vi) $x^{\sin x} = y^{\sin y}$ (T.U. 2049)

## Answers

1. $1 + \dfrac{1}{2\sqrt{x}}$ 2. $\cos x$ 3. $\dfrac{1}{\sqrt{2x+3}}$

4. (i) $1 + \dfrac{1}{2\sqrt{x+1}}$ (ii) $\dfrac{a}{2\sqrt{ax+b}}$

6. (i) $10(2x+3)^4$ (ii) $\dfrac{3}{4}\dfrac{1}{x^{1/4}}$ (iii) $-a\sin(ax+b)$

(iv) $5^x \log_e 5$ (v) $\dfrac{3}{3x+2}$ (vi) $(3x-4)^{-2/3}$

(vii) $-\dfrac{4}{3}(4x+1)^{-4/3}$

7. (i) $\dfrac{1}{2}\sqrt{\dfrac{a}{(a-x)^3}}$ (ii) $-\dfrac{a}{(a-x)^{1/2}\,.\,(a+x)^{3/2}}$

(iii) $-\dfrac{y^{1/3}}{x^{1/3}}$ (iv) $-\dfrac{y^{1/2}}{x^{1/2}}$ (v) $\dfrac{y}{x}$

(vi) $-\cos\dfrac{1}{x}+2x\sin\dfrac{1}{x}$ (vii) $\dfrac{3}{\sqrt{1-x^2}}$ (viii) $\dfrac{2t-t^4}{1-2t^3}$

(ix) $\cot\dfrac{\theta}{2}$ (x) $y\left[\dfrac{2}{x}+\dfrac{1}{2x+3}-\dfrac{4}{3(4x+1)}\right]$

(xi) $\dfrac{1}{2}y\left[\dfrac{1}{x+1}+\dfrac{1}{x+2}-\dfrac{1}{x-3}-\dfrac{1}{x-4}\right]$

8. 3

9. –8, 24

12. (i) $x^{\sin x}\left(\dfrac{\sin x}{x}+\cos x\,.\,\log x\right)$

(ii) $(\sin x)^x\,(x\cot x+\log\sin x)$

(iii) $(\sin x)^{\log x}\left(\dfrac{1}{x}\log\sin x+\cot x\log x\right)$

(iv) $e^{x^x}\,.\,x^x\,(1+\log x)$

(v) $x^{e^x}\,.\,e^x\left(\log x+\dfrac{1}{x}\right)$

(vi) $(\log x)^{\tan x}\left\{\sec^2x.\log(\log x)+\dfrac{\tan x}{x\log x}\right\}$

(vii) $(\tan x)^{\log x}\left\{\dfrac{1}{x}\log\tan x+\dfrac{\sec^2x}{\tan x}\log x\right\}$

(viii) $x^{\sec x}\,.\,\sec x\left\{\tan x\log x+\dfrac{1}{x}\right\}$

(ix) $(\sin x)^{\cos x}\,\{\cos x.\cot x-\sin x.\log(\sin x)\}$

13. (i) $\dfrac{y^2}{x(1-y\log x)}$ (ii) $-\dfrac{y(y+x\log y)}{x(y\log x+x)}$ (iii) $\dfrac{y}{x}$

(iv) $-\dfrac{e^{\sin x}\cos x}{e^{\sin y}\cos y}$ (v) $\dfrac{y(x\log y-y)}{x(y\log x-x)}$

(vi) $\dfrac{y(x\cos x\log x+\sin x)}{x(y\cos y\log y+\sin y)}$

●

CHAPTER 18

# Application of Derivatives

## 18.1 Introduction

The application of derivatives plays a very important role in the field of Science, engineering, economics, commerce and so on. The problems of the type "the increase in the length of an iron rod due to the increase in temperature"; "the increasing and decreasing tendency of the cost functions"; "the maximum profit to be made with the help of minimum investment" and so on can betterly be solved with the help of derivatives.

## 18.2 Geometrical Interpretation of Derivative of a function

If $y = f(x)$ be the given function which represents a continuous curve, then the derivative of $f(x)$ with respect to $x$ denoted by $\frac{dy}{dx}$ or $f'(x)$ is the slope of the tangent at any point of the curve represented by the given function $y = f(x)$. So, the derivative of the given function at $x = x_0$ means the slope of the tangent to the curve at the point $x = x_0$.

The proof of this result is given in Art 18.2

## 18.3 Increasing and Decreasing Functions

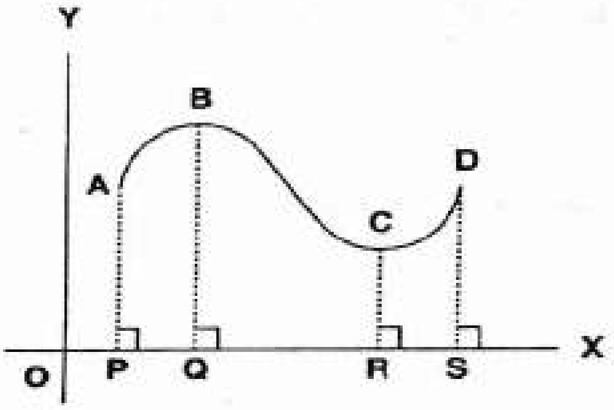
A function can be presented in various forms. One of the most important forms of the presentation is the graph. To know the nature of the graph of the function, we must have the idea of increasing and decreasing tendency of the function on an interval.

Later on, the increasing and decreasing nature of the function are also used in finding the maximum and the minimum values of the function.

Here, in this section as an application, we use derivatives to examine the increasing and decreasing tendency of the function in an interval.

The figure given below is the continuous curve represented by the given function $y = f(x)$. Consider the points A, B, C and D on the curve where OP =

$x_1$, OQ = $x_2$, OR = $x_3$, OS = $x_4$ and PA = $f(x_1)$, QB = $f(x_2)$, RC = $f(x_3)$ and SD = $f(x_4)$




From the figure, we see that the curve (i.e. the function) increases from A to B then decreases from B to C and again increases from C to D.

That is, in the part AB, $x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$

in the part BC, $x_3 > x_2 \Rightarrow f(x_3) < f(x_2)$

and in the part CD, $x_4 > x_3 \Rightarrow f(x_4) > f(x_3)$

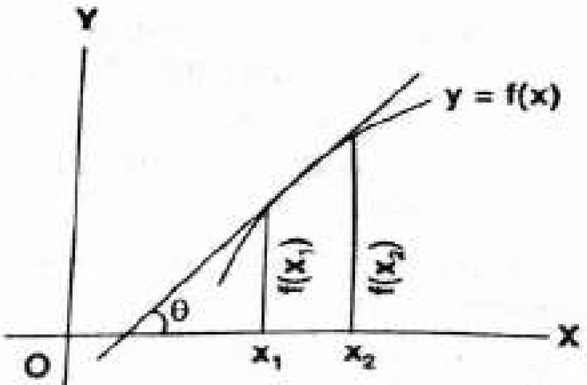
Now, we have the following definitions of increasing and decreasing functions.

## Increasing Function

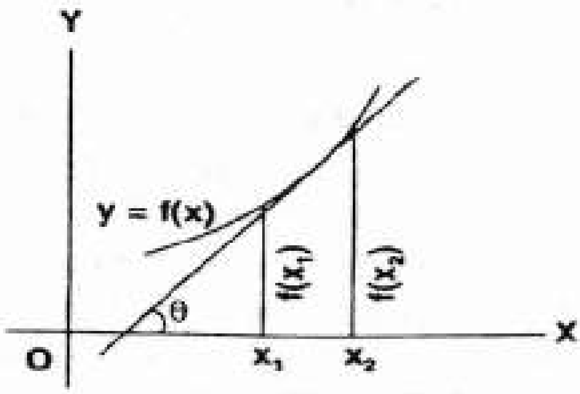
A function $y = f(x)$ is said to be increasing in the interval $(a, b)$ if for every $x_1, x_2 \in (a, b)$

$$x_2 > x_1 \quad \Rightarrow \quad f(x_2) > f(x_1)$$

This result shows that as $x$ increases, $y$ i.e. $f(x)$ also increases. So, the slope of the tangent at any point of such a curve is positive. Thus, $y = f(x)$ is an increasing function if $\frac{dy}{dx} = f'(x) > 0$.




$x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$

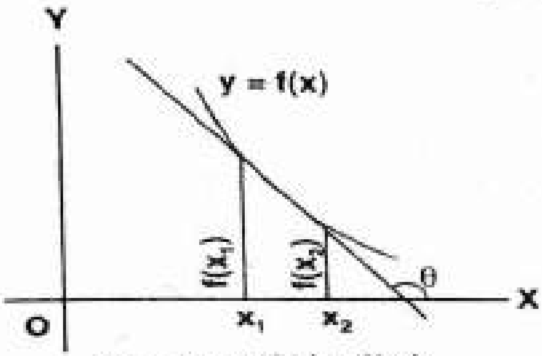



$x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$

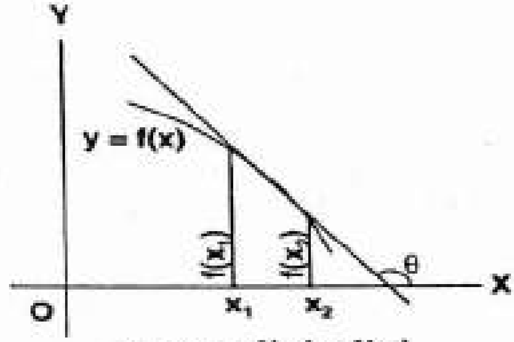
## Decreasing Function

A function $y = f(x)$ is said to be decreasing in the interval $(a, b)$ if for every $x_1, x_2 \in (a, b)$

$$x_2 > x_1 \Rightarrow f(x_2) < f(x_1)$$




$x_2 > x_1 \Rightarrow f(x_2) < f(x_1)$




$x_2 > x_1 \Rightarrow f(x_2) < f(x_1)$

This result shows that as $x$ increases, $y$ i.e. $f(x)$ decreases. So, the slope of the tangent at any point of such a curve is negative. Thus $y = f(x)$ is a decreasing function if $\frac{dy}{dx} = f'(x) < 0$.

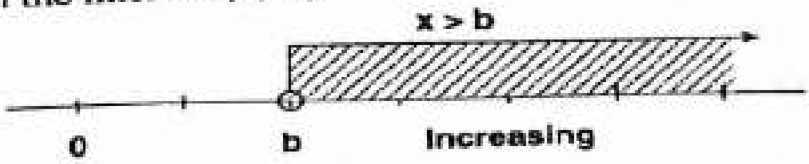
A function is increasing (or decreasing) at the point $x = a$ means the function is increasing (or decreasing) at $x = a$ which is an interior point of an interval.

The following table shows the sign of the derivative of the given function $y = f(x)$ defined in an interval $(a, b)$ and the nature of the curve represented by the given function.

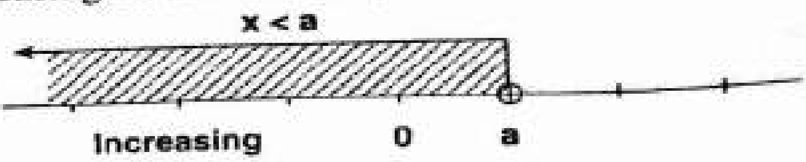
| Sign of the derivative | Nature of $f(x)$ |
|---|---|
| $f'(x) > 0$ | Function is increasing. |
| $f'(x) < 0$ | Function is decreasing. |

## Intervals of Increasing and Decreasing Functions

If the function $y = f(x)$ is increasing for $x > b$ $(b > 0)$, then the function is increasing on the interval $(b, \infty)$

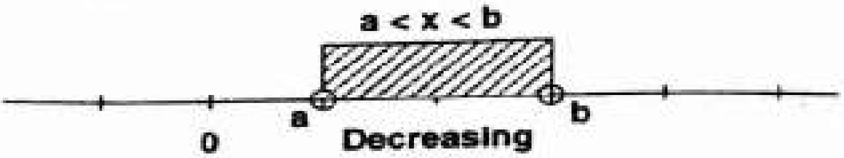



Again if the function $y = f(x)$ is increasing for $x < a$ $(a > 0)$, then the function is increasing on the interval $(-\infty, a)$.

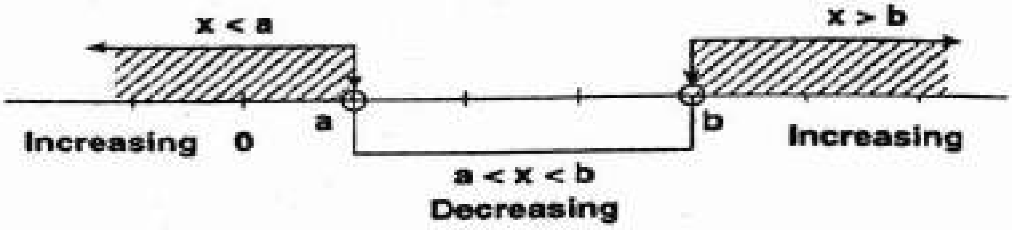


Thus the function $y = f(x)$ is increasing for $x < a$ and $x > b$ means increasing for $x \in (-\infty, a) \cup (b, \infty)$

But if the function $y = f(x)$ is decreasing for $a < x < b$, then the function is dencreasing on $(a, b)$.




Thus the intervals in which the function $y = f(x)$ are increasing and decreasing are shown below:




***Example***

Show that the function $f(x) = \frac{1}{2}x^2 - 3x$ increases in the interval $(3, \infty)$ and decreases in the interval $(-\infty, 3)$.

***Solution :***

$$\text{We have} \quad f(x) = \frac{1}{2}x^2 - 3x$$

$$\Rightarrow \quad f'(x) = x - 3$$

$$\text{For } x > 3, \quad f'(x) > 0$$

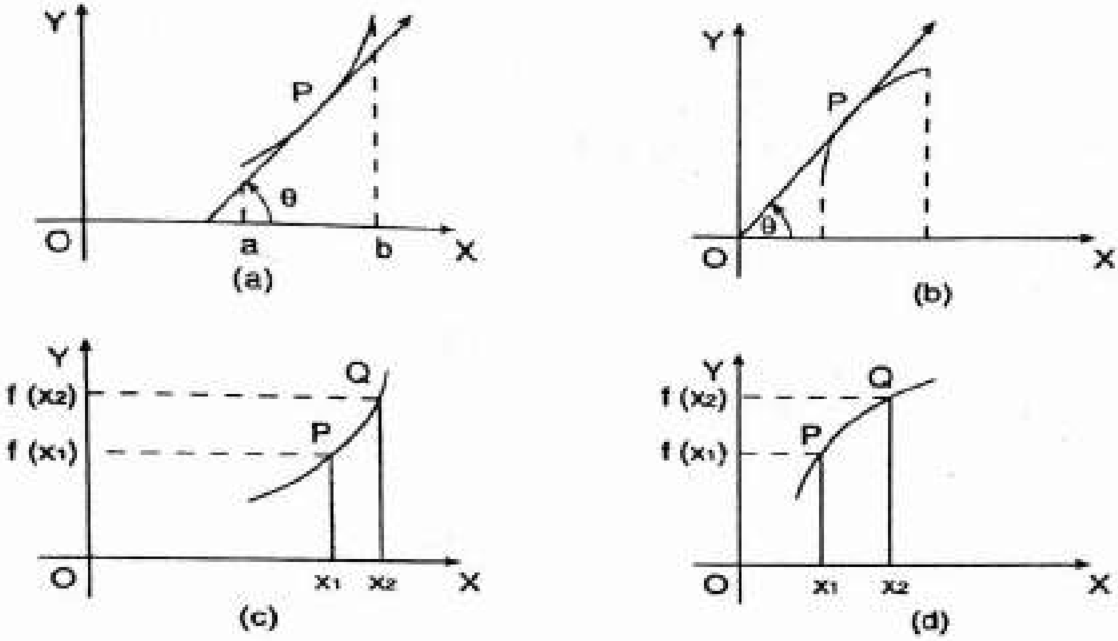
$$\text{and for } x < 3, \quad f'(x) < 0$$

$\therefore f(x)$ is increasing function for $x > 3$ i.e. on the interval $(3, \infty)$ and $f(x)$ decreases for $x < 3$ i.e. on the interval $(-\infty, 3)$.

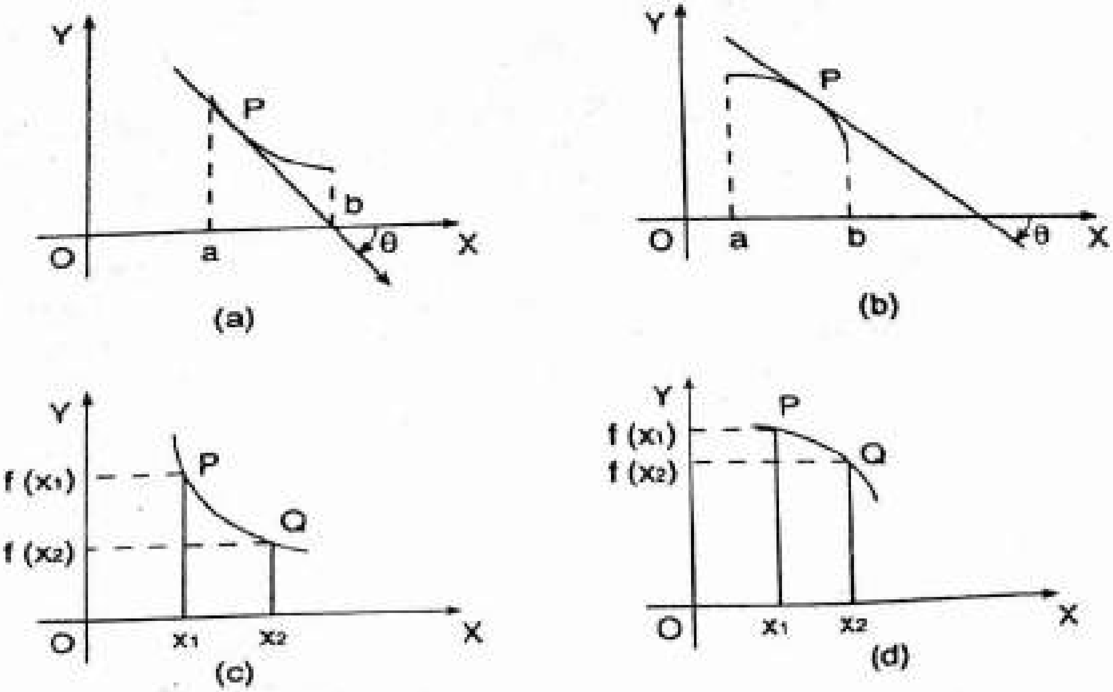
Note: At $x = 4$, $f'(x) = 4 - 3 = 1 > 0$, so $f(x)$ is increasing at $x = 4$ which is a point on $(3, \infty)$.

## 18.4 Maxima and Minima

The basic problem that we are going to solve in this section is to find a maximum or minimum value of a function $f$ in an interval $(a, b)$. To understand the problem properly, it is necessary to know the nature of the curve in the neighbourhood of the point where the maximum or the minimum value of the function occurs. The nature of the curve at a particular point is known by studying the slopes of the tangents at the points in its neighbourhood.



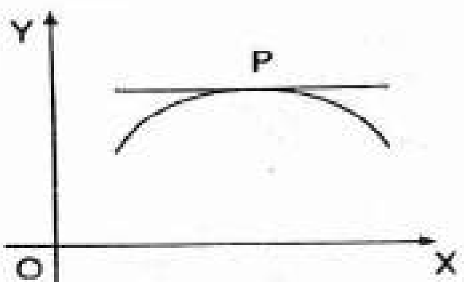

The slope or the gradient of a tangent line at a point of the graph of a function $f$ may be positive, negative or zero. If the gradient of $f(x)$ is positive the inclination $\theta$ of the tangent line is positive and the tangent line slopes upwards (or goes up-hill) from left to right as shown in the figures. In the figures, within the interval $(a, b)$, where the slope of the tangent line at any point is positive, $x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$, i.e., the function is increasing. So we can say that if $f'(x)$ is positive at a certain point, the function $f$ is increasing in its neighbourhood.



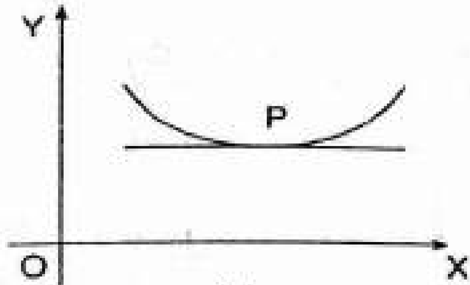
Similarly, if $f'(x)$ is negative, the inclination $\theta$ of the tangent line is negative and the tangent line slopes downwards (or goes down-hill) from left to right as shown in the figures. In the figures, within the interval $(a, b)$, where the slope of the tangent line at any point is negative,

$$x_2 > x_1 \Rightarrow f(x_2) < f(x_1),$$

i.e., the function $f$ is decreasing in its neighbourhood. So we can say that if $f'(x)$ is negative at a certain point, the function $f$ is decreasing in its neighbourhood.



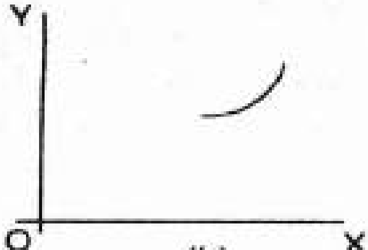
(a)



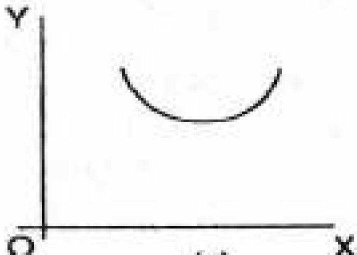
(b)

When $f'(x) = 0$, the tangent line neither goes 'uphill' nor goes 'downhill'. It remains horizontal or stationary. The points at which $f'(x) = 0$ or $f'(x)$ is not defined are called critical (or stationary) points. We shall come back to these points while dealing with maxima and minima of a function.

(a)

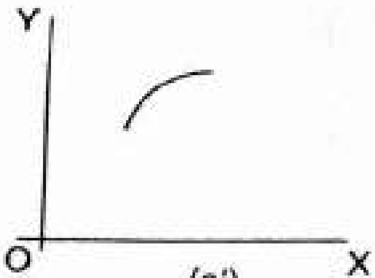


(b)



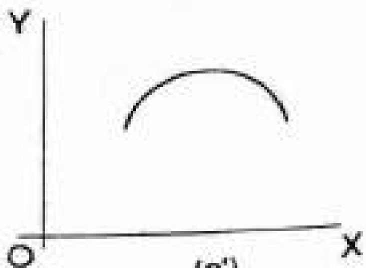
(c)

$f''(x)$ is the derivative of $f'(x)$. So if $f''(x) > 0$, $f'(x)$ is increasing (analogous to the above case where $f'(x)$ is the derivative of $f(x)$. $f'(x) > 0$ implies that $f(x)$ is an increasing function). But $f'(x)$ is the slope of a tangent to the graph $y = f(x)$. So if $f''(x) > 0$, the gradient or the slope of smaller positive value to a larger positive value and in the figure (b) the slope increases from a larger negative value to a smaller negative value. The curve in the figure (c) is just a combination of the curve in figures (a) and (b). These curves are said to be concave upwards. So the condition for a graph to be concave upwards is that the second derivative of the function must be positive.



(a')

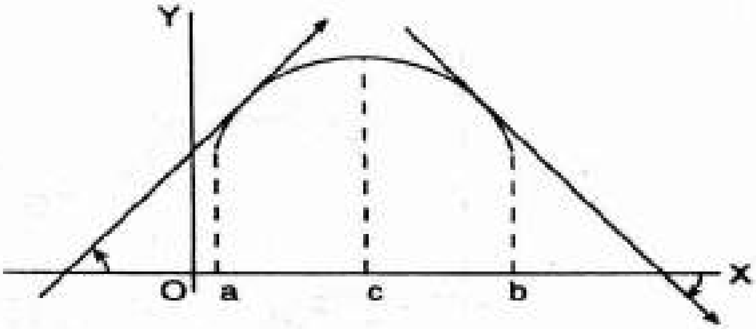
(b')



(c')

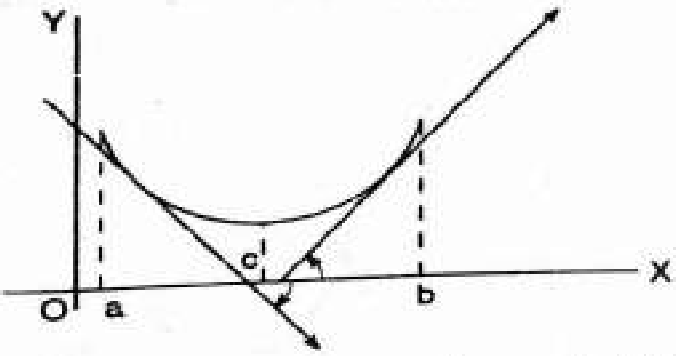
Similarly, if $f''(x) < 0$, then $f'(x)$ is decreasing, i.e. the slope of the tangent line is decreasing. If the slope is decreasing, the curve must be one of the following types. In the figure (a'), the slope $f'(x)$ is decreasing from the greater positive value to a smaller positive value, while in the figure (b') the slope $f'(x)$ decreases from a smaller negative value to a larger negative value. The curve in the figure (c') is just a combination of the curves in the figures (a') and (b'). These curves are said to be concave downwards. So the condition for a curve to be concave downwards is that the second derivative $f''(x)$ of the function $f$ must be negative.

Thus we see that if $f''(x) < 0$ in an interval $(a, b)$ the graph of $f$ is concave downwards in the interval and if $f''(x) > 0$ in the interval, the graph of $f$ is concave upwards in the interval.




Further, consider the function $f$ with a portion of the curve as given in the figure in the interval $(a, b)$ which is concave downwards. So $f''(x) < 0$ and $f'(x)$ decreases throughout the portion of the curve in the interval $(a, b)$. But in the interval $(a, c)$, $f'(x) > 0$ and in the interval $(c, b)$, $f'(x) < 0$. As the curve is continuous there must be a point $x_0$ such that $f'(x_0) = 0$. At that point $x_0$, the function $f$ has a local maximum. So the conditions for a function $f$ to have a local maximum at a point $x_0$ are

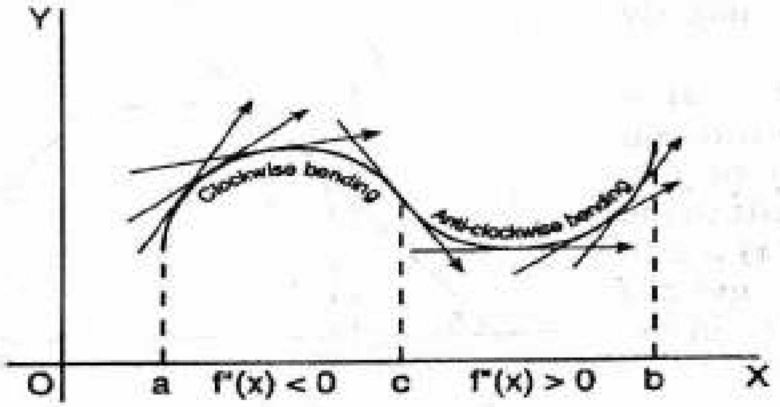
(i) $f'(x_0) = 0$
(ii) $f''(x_0) < 0$




Similarly, let us consider a function $f$ with a portion of the curve concave upwards in the interval $(a, b)$. As the curve is concave upwards, $f''(x) > 0$ and so $f'(x)$ is increasing in the interval $(a, b)$. But $f'(x) < 0$ in the interval $(a, c)$ and $f'(x) > 0$ in the interval $(c, b)$, i.e. $f'(x)$ changes sign in the interval $(a, b)$. As $f'(x)$ is also continuous in that interval, there must exist a point $x_0$ in the interval at which $f'(x)$ takes a value zero, i.e., $f'(x_0) = 0$. So, $x_0 \in (a, b)$ is the point at which the function $f$ has a local minimum. Hence the conditions for a function $f$ to have a local minimum at $x_0$ are

(i) $f'(x_0) = 0$
(ii) $f''(x_0) > 0$.

Again consider the graph of a function $f$ in the following figure. In the interval $(a, c)$, the portion of the graph is concave downwards and so $f''(x) < 0$, while the portion of the graph in the interval $(c, b)$ is concave upwards and so $f''(x) > 0$. So naturally there must exist a point in the interval at which $f''(x) = 0$. This point is of considerable importance. It separates the portion of the graph which is concave downwards from the portion which is concave upwards. Such a point of the graph is called a **point of inflection.**

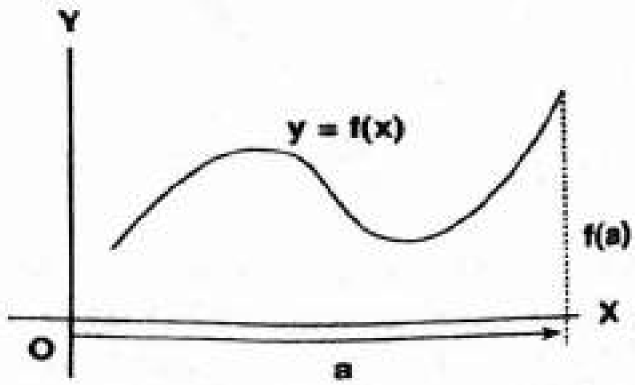



The maximum and the minimum values of the function need not be the greatest and the least values of the function but simply the maximum and the minimum values on the neighbourhood of the point $x = a$ and hence termed as the local maxima and local minima.

A function may have several maximum and minimum values but they occur alternatively. But besides these maximum and the minimum values, there may be the values greater than the maximum value and less than the minimum value i.e. these may be the greatest and the least values of the function which are termed as absolute maximum value and the absolute minimum value of the function.

### Absolute maxima

A function $y = f(x)$ is said to have the absolute maximum value or absolute maxima at $x = a$ if $f(a)$ is the greatest of all its values for all $x$ belonging to the domain of the function. The absolute maximum value is also known as the global maximum value.

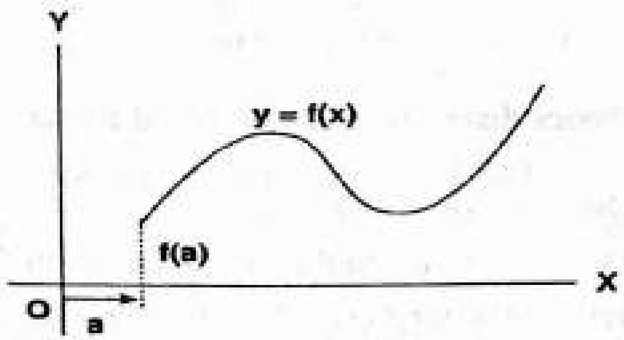



In other words, $f(a)$ is the absolute maximum value of $f(x)$ if $f(a) \geq f(x)$ for all $x \in D(f)$.

In the adjoining figure, $f(a)$ is the absolute maximum value of $f(x)$ at $x = a$.

### Absolute minima

A function $y = f(x)$ is said to have the absolute minimum value or absolute minima at $x = a$ if $f(a)$ is the smallest of all of its values for all $x$ belonging to the domain of the function. The absolute minimum value is also known as the global minimum value.




In other words, $f(a)$ is the absolute minimum value of $f(x)$ if $f(a) \leq f(x)$ for all $x \in D(f)$.

In the adjoining figure, $f(a)$ is the absolute minimum value of $f(x)$ at $x = a$.

### Stationary point

A point on the graph of the function $y = f(x)$ where the tangent is parallel to the $x$-axis is known as the stationary point or critical point. At the stationary point, $\frac{dy}{dx} = f'(x) = 0$

### Local maxima

A function $y = f(x)$ is said to have the local maximum value or local maxima at $x = a$ if $f(a) > f(a \pm h)$ for sufficiently small positive value of $h$. The local maxima is also known as the relative maxima of the function.

### Local minima

A function $y = f(x)$ is said to have the local minimum value or local minima at $x = a$ if $f(a) < f(a \pm h)$ for sufficiently small positive value of $h$. The local minima is also known as the relative minima of the function.

### Procedure to find the absolute maxima and minima

Let $y = f(x)$ be the given function defined in an interval $[a, b]$.

i) Find the first derivative of $f(x)$ i.e. find $f'(x)$.

ii) Making $f'(x) = 0$, solve for $x$ to get the stationary points. Let the values of $x$ be $c$ and $d$.

iii) Find the values of $f(x)$ at $x = a, b, c$ and $d$.

The least of the value gives the absolute minimum value and the greatest value gives the absolute maximum value.

Thus if a function $y = f(x)$ is defined in an interval $[a, b]$, then $f(x)$ may have the absolute maximum value or absolute minimum value at $x = a$, $x = b$ or an interior point $c \in [a, b]$ where $f'(c) = 0$

## Procedure to find the local maxima and local minima

The following steps are to be used in finding the maxima and minima of the function $f(x)$ at a point.

i) Find $f'(x)$ and $f''(x)$ of the given function $y = f(x)$.

ii) Making $f'(x) = 0$, solve for $x$ to get the stationary points. Let one of the stationary point be $a$ i.e. $x = a$.

iii) Find $f''(a)$. If $f''(a) < 0$, then $f(x)$ has maximum value at $x = a$ and the maximum value $= f(a)$
If $f''(a) > 0$, then $f(x)$ has minimum value at $x = a$ and the minimum value $= f(a)$.
If $f''(a) = 0$ and $f'''(a) \neq 0$, then $f(x)$ has no maximum and no minimum value at $x = a$.

The condition for the function $y = f(x)$ to have the maximum, minimum or no maxima no minima at the point $x = a$ are given below.

| | For the function $y = f(x)$ | | | |
|---|---|---|---|---|
| Conditions | Maxima | Minima | No max. or no min. | |
| First order derivative | $\frac{dy}{dx} = f'(x) = 0$ | $\frac{dy}{dx} = f'(x) = 0$ | $\frac{dy}{dx} = f'(x) = 0$ | $\frac{dy}{dx} = f'(x) \neq 0$ |
| Second order derivative | $\frac{d^2y}{dx^2} = f''(x) < 0$ | $\frac{d^2y}{dx^2} = f''(x) > 0$ | $\frac{d^2y}{dx^2} = f''(x) = 0$ | |
| Third order derivative | | | $\frac{d^3y}{dx^3} = f'''(x) \neq 0$ | |

## Alternative method to find the local maxima and local minima with first derivative

For the maximum or minimum value of the function $y = f(x)$, we use the following steps:

i) Find $f'(x)$ or $\frac{dy}{dx}$.

ii) Making $f'(x) = 0$, solve for $x$. Let $x = a$ be one of the stationary points.

iii) Note the sign of $\frac{dy}{dx}$ when $x$ changes its value from $a - h$ to $a + h$.

iv) If $\frac{dy}{dx}$ changes its sign from +ve to −ve, $y$ or $f(x)$ has the maximum value at $x = a$. If $\frac{dy}{dx}$ changes its sign from −ve to +ve, $f(x)$ has minimum value at $x = a$.

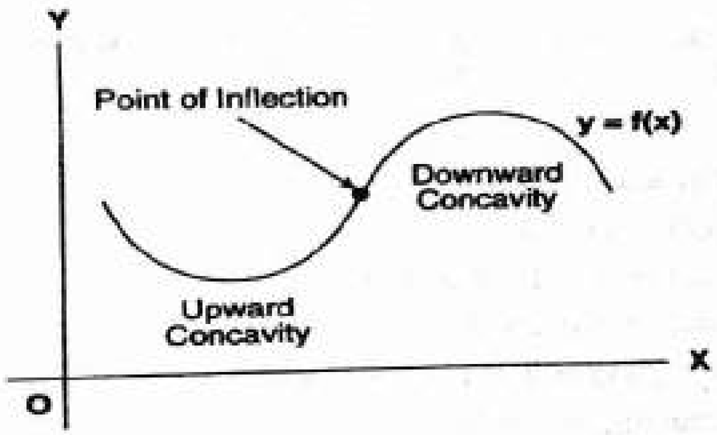
The maximum or minimum value of $f(x)$ at $x = a$ is $f(a)$.

If $\frac{dy}{dx}$ does not change its sign, then $f(x)$ has no maximum or minimum value.

## Concavity and Convexity of Curves

The graph of the function $y = f(x)$ defined in an interval $(a, b)$ is **concave upward** (or convex downward) if each point (except the point of contact) on the graph (i.e. curve) lies above any tangent to it in that interval.

But if each point on the graph (i.e. curves) lies below any tangent to the curve in the interval, then the graph of the function is said to be **concave downward** (or convex upward).




The graph of the function $y = f(x)$ will be concave upward or concave downward according as $\frac{d^2y}{dx^2} = f''(x) > 0$ or $\frac{d^2y}{dx^2} = f''(x) < 0$.

## Procedure to find the concave upward and concave downward of the graph of the function

Let $y = f(x)$ be the given function. For the concave upward or concave downward of the graph of the function in an interval we use the following steps:

i) Find the second derivative of $f(x)$ i.e. find $f''(x)$.

ii) Find the interval within which $f''(x) > 0$. Then we conclude that the graph of $y = f(x)$ will be concave upward in the interval.

iii) Again find the interval within which $f''(x) < 0$. Then we conclude that the graph of $y = f(x)$ will be concave downward in that interval.

### Point of inflection

The point which divides the graph of the function from the shape of upward concavity (or downward concavity) to the downward concavity (or upward concavity) is known as the point of inflection. If $y = f(x)$ be the given function which represent a continuous curve, then the point where $\frac{d^2y}{dx^2} = f''(x) = 0$ and $\frac{d^3y}{dx^3} = f'''(x) \neq 0$ is said to be the point of inflection.

## Worked out examples

***Example 1***

Show that the function $f(x) = x^2 - 3x + 4$ is increasing at the point $x = 2$ and is decreasing at the point $x = 1$.

***Solution :***

$$f(x) = x^2 - 3x + 4$$

Then, $f'(x) = 2x - 3$

At $x = 2$, $f'(x) = 2 \times 2 - 3 = 1 > 0$

$\therefore f(x)$ is increasing at $x = 2$.

Again at $x = 1$, $f'(x) = 2 \times 1 - 3 = -1 < 0$

$\therefore f(x)$ is decreasing at $x = 1$.

***Example 2***

Show that the function $f(x) = x - \frac{1}{x}$ is increasing for all $x \in R$ $(x \neq 0)$.

***Solution :***

$$f(x) = x - \frac{1}{x}$$

Then, $f'(x) = 1 + \frac{1}{x^2}$ which is positive for $x \in R$ except $x = 0$

$\therefore f(x)$ is increasing for all $x \in R$ $(x \neq 0)$, that is for

$$x \in (-\infty, 0) \cup (0, \infty)$$

***Example 3***

Find the interval in which the function $f(x) = 2x^3 - 15x^2 + 36x + 1$ is increasing or decreasing.

***Solution :***

$$f(x) = 2x^3 - 15x^2 + 36x + 1$$

Then, $f'(x) = 6x^2 - 30x + 36$

$= 6(x^2 - 5x + 6)$

$= 6(x - 2)(x - 3)$

$f'(x) = 0$ gives $x = 2$ and $3$ which are the stationary points.

For $x > 3$, $f'(x) > 0$

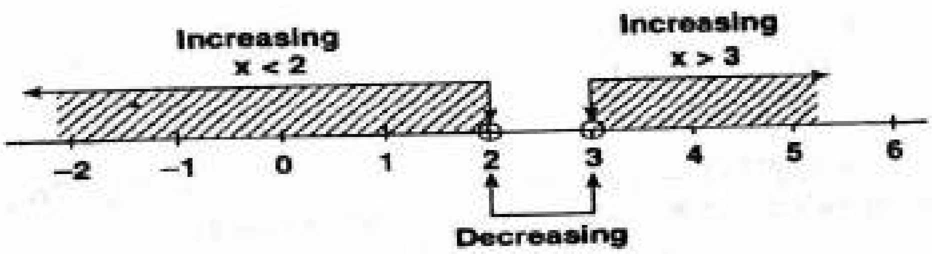
$\therefore f(x)$ is increasing for $x > 3$ i.e. on the interval $(3, \infty)$

Again for $x < 2$, $f'(x) > 0$

$\therefore f(x)$ is increasing for $x < 2$ i.e. on the interval $(-\infty, 2)$

For $2 < x < 3$, $f'(x) < 0$

$\therefore f(x)$ is decreasing in $2 < x < 3$.




$\therefore f(x)$ is increasing on $(-\infty, 2) \cup (3, \infty)$ and decreasing on $x \in (2, 3)$.

***Example 4***

Find the absolute maximum (greatest value) and the absolute minimum value (least value) of the function $f(x) = 2x^3 - 9x^2 + 12x + 20$ defined on an interval $[-1, 5]$

***Solution :***

$$f(x) = 2x^3 - 9x^2 + 12x + 20$$

Then, $f'(x) = 6x^2 - 18x + 12$

$f'(x) = 0$ gives

$6x^2 - 18x + 12 = 0$

$\Rightarrow x^2 - 3x + 2 = 0$

$\Rightarrow (x - 1)(x - 2) = 0$

$\therefore x = 1, 2$

When $x = -1$, $f(x) = 2(-1)^3 - 9(-1)^2 + 12(-1) + 20 = -3$

$x = 5$, $f(x) = 2(5)^3 - 9(5)^2 + 12(5) + 20 = 105$

$x = 1$, $f(x) = 2(1)^3 - 9(1)^2 + 12(1) + 20 = 24$

$x = 2$, $f(x) = 2(2)^3 - 9(2)^2 + 12(2) + 20 = 72$

$\therefore$ absolute max. value = 105 and absolute min. value = –3

***Example 5***

Calculate the maximum and minimum values of

$$f(x) = 2x^3 - 3x^2 - 36x$$

***Solution:***

We have $f(x) = 2x^3 - 3x^2 - 36x$

$\therefore$ $f'(x) = 6x^2 - 6x - 36$

$f''(x) = 12x - 6$

For the maximum or minimum value of $f(x)$,

$$f'(x) = 0$$

$\therefore$ $6x^2 - 6x - 36 = 0$

$x^2 - x - 6 = 0$

or, $(x - 3)(x + 2) = 0$

$\therefore$ $x = 3, -2$

(i) When $x = 3$, $f''(x) = 36 - 6 = 30$, which is positive. So $f(x)$ has a minimum value at $x = 3$ and the minimum value is

$$= 54 - 27 - 108 = -81$$

(ii) When $x = -2$, $f''(x) = -24 - 6 = -30$, which is negative. So $f(x)$ has a maximum value at $x = -2$ and the maximum value is

$$= -16 - 12 + 72 = 44$$

***Example 6***

Show that $f(x) = x^3 - 3x^2 + 6x + 4$ has neither a maximum nor a minimum value.

***Solution :***

Here, $f(x) = x^3 - 3x^2 + 6x + 4$

$f'(x) = 3x^2 - 6x + 6$

$= 3(x^2 - 2x + 2)$

$= 3\{(x - 1)^2 + 1\}$

which is always positive for all real values of $x$ and can never be zero.

$\therefore$ $f(x)$ has neither a maximum nor a minimum value.

**Example 7**

Let $f(x) = 2x^3 - 6x^2 + 5$. Find where the graph is concave downward and where it is concave upward.

**Solution:**

We have
$$f(x) = 2x^3 - 6x^2 + 5$$
$$f'(x) = 6x^2 - 12x$$
$$f''(x) = 12x - 12 = 12(x - 1)$$

$f''(x) = 0$ gives $x = 1$ which is the point of inflection.

For $x > 1, f''(x) > 0$ and for $x < 1, f''(x) < 0$.

Hence the graph is concave downwards if $x < 1$ and is concave upwards if $x > 1$.

**Example 8**

Find the maximum area of a rectangular plot of land which can be enclosed by a rope of length 60 metres.

**Solution:**

Let the sides of the rectangular plot of land be $x$ and $y$. So
$$2x + 2y = 60$$
or
$$x + y = 30$$
The area of the land is
$$A = xy = x(30 - x) = 30x - x^2$$
$$\therefore \quad \frac{dA}{dx} = 30 - 2x$$
and
$$\frac{d^2A}{dx^2} = -2$$

For the maximum or minimum value of A, $\frac{dA}{dx} = 0$

$$30 - 2x = 0$$
or
$$x = 15$$
and
$$y = 30 - x = 30 - 15 = 15$$

Since $\frac{d^2A}{dx^2} = -2 < 0$, so A has maximum value when $x = 15$.

$\therefore$ the maximum area $= 15 \times 15 = 225$ sq.m.

## EXERCISE 18.1

1. i) Examine whether the function $f(x) = 15x^2 - 14x + 1$ is increasing or decreasing at $x = \frac{2}{5}$ and $x = \frac{5}{2}$.

   ii) Show that the function $f(x) = 2x^3 - 24x + 15$ is increasing at $x = 3$ and decreasing at $x = \frac{3}{2}$.

2. i) Test whether the function $f(x) = 2x^2 - 4x + 3$ is increasing or decreasing on the interval $(1, 4]$.

   ii) Examine whether the function $f(x) = 16x - \frac{4}{3}x^3$ is increasing or decreasing on the interval $(-\infty, -2)$.

   iii) Show that the function $f(x) = -x^3 + 6x^2 - 13x + 20$ is decreasing for all $x \in R$.

   iv) Show that the function $f(x) = 4x - \frac{9}{x} + 6$ is increasing for all $x \in R$ except at $x = 0$.

3. Find the intervals in which the following functions are increasing or decreasing.

   i) $f(x) = 3x^2 - 6x + 5$

   ii) $f(x) = x^4 - \frac{1}{3}x^3$

   iii) $f(x) = 5x^3 - 135x + 22$

   iv) $f(x) = 6 + 12x + 3x^2 - 2x^3$

   v) $f(x) = x^3 - 12x$ defined on $[-3, 5]$

4. Find the absolute maximum and the absolute minimum values of the following function on the given intervals:

   i) $f(x) = 3x^2 - 6x + 4$ on $[-1, 2]$

   ii) $f(x) = 2x^3 - 9x^2$ on $[-2, 4]$

   iii) $f(x) = x^3 - 6x^2 + 9x$ on $[0, 5]$

   iv) $f(x) = 2x^3 - 15x^2 + 36x + 10$ on $[1, 4]$

5. Find the local maxima and minima and points of inflection:

   (i) $f(x) = 3x^2 - 6x + 3$

   (ii) $f(x) = x^3 - 12x + 8$

   (iii) $f(x) = x^3 - 6x^2 + 3$

(iv) $f(x) = 2x^3 - 15x^2 + 36x + 5$ (HSEB 2050, 052)
(v) $f(x) = 2x^3 - 9x^2 - 24x + 3$
(vi) $f(x) = 4x^3 - 15x^2 + 12x + 7$ (HSEB 2051)
(vii) $f(x) = 4x^3 - 6x^2 - 9x + 1$ on the inverval $(-1, 2)$
(viii) $f(x) = x + \frac{100}{x} - 5$

6. Show that the following functions have neither maximum nor minimum value.
   i) $f(x) = x^3 - 6x^2 + 24x + 4$
   ii) $f(x) = x^3 - 6x^2 + 12x - 3$

7. Determine where the graph is concave upwards and where it is concave downwards of the following functions:
   (i) $f(x) = x^4 - 2x^3 + 5$
   (ii) $f(x) = x^4 - 8x^3 + 18x^2 - 24$ (HSEB 2056)
   (iii) $y = 3x^5 + 10x^3 + 15x$
   (iv) $f(x) = x^3 - 9x^2$ defined on $[-2, 5]$

8. A man who has 144 metres of fencing material wishes to enclose a rectangular garden. Find the maximum area he can enclose. (HSEB 2054)

9. Show that the rectangle of largest possible area, for a given perimeter, is a square. (HSEB 2058)

10. A window is in the form of a rectangle surmounted by a semi-circle. If the total perimeter is 9 metres, find the radius of the semi-circle for the greatest window area. (HSEB 2055)

11. A closed cylindrical can is to be made so that its volume is 52 $cm^3$. Find its dimensions if the surface is to be a minimum.

12. A gardener having 120 m. of fencing wishes to enclose a rectangular plot of land and also to erect a fence across the land parallel to two of the sides. Find the maximum area he can enclose.

13. Find two numbers whose sum is 10 and the sum of whose squares is minimum.

### *Answers*

1. (i) Decreasing at $x = \frac{2}{5}$ and increasing at $x = \frac{5}{2}$
2. (i) Increasing (ii) Decreasing
3. (i) Increasing on $(1, \infty)$ and decreasing on $(-\infty, 1)$

(ii) Increasing on $\left(\frac{1}{4}, \infty\right)$ and decreasing on $\left(-\infty, \frac{1}{4}\right)$
(iii) Increasing on $(-\infty, -3) \cup (3, \infty)$ and decreasing on $(-3, 3)$
(iv) Decreasing on $(-\infty, -1) \cup (2, \infty)$ and increasing on $(-1, 2)$
(v) Increasing on $[-3, -2) \cup (2, 5]$ and decreasing on $(-2, 2)$

4. (i) Absolute max = 13, Absolute min. = 1
(ii) Absolute max = 0, Absolute min. = –52
(iii) Absolute max = 20, Absolute min. = 0
(iv) Absolute max = 42, Absolute min. = 33

5. (i) min. value = 0 at $x = 1$. No point of inflection.
(ii) max. value = 24 at $x = -2$; min. value = –8 at $x = 2$. The point of inflection is at $x = 0$
(iii) max. value = 3 at $x = 0$; min. value = –29 at $x = 4$. The point of inflection is at $x = 2$
(iv) max. value = 33 at $x = 2$; min. value = 32 at $x = 3$. The point of inflection is at $x = \frac{5}{2}$.
(v) max. value = 16 at $x = -1$; min. value = –109 at $x = 4$. The point of inflection is at $x = \frac{3}{2}$.
(vi) max. value = $\frac{39}{4}$ at $x = \frac{1}{2}$; min. value = 3 at $x = 2$. The point of inflection is at $x = \frac{5}{4}$.
(vii) Max. value = $\frac{7}{2}$ at $x = -\frac{1}{2}$, Min. value = $-\frac{25}{2}$ at $x = \frac{3}{2}$. The point of inflection is at $x = \frac{1}{2}$
(viii) min. value = 15 at $x = 10$; max. value = –25 at $x = -10$. No point of inflection.

7. (i) Concave upwards for $x > 1$, Concave upwards for $x < 0$ and Concave downwards for $0 < x < 1$.
(ii) Concave upwards for $x < 1$; Concave downwards for $1 < x < 3$; Concave upwards for $x > 3$.
(iii) Concave upwards for $x > 0$; Concave downwards for $x < 0$.
(iv) Concave upward on $(3, 5]$ and concave down on $[-2, 3)$.

8. 1296 sq. m

10. $\frac{9}{4 + \pi}$ metres

11. height = 2 radius where radius = $\left(\frac{26}{\pi}\right)^{1/3}$

12. 600 m$^2$

13. 5, 5

## 18.5 Derivative as the Rate Measure

Let $y = f(x)$ be the continuous function. By the definition of a function, $y$ changes while $x$ will change. If $\Delta x$ and $\Delta y$ be the small changes in $x$ and $y$ respectively, then

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

is the change in $y$ per unit change in $x$ and hence is the average rate of change of $y$ with respect to $x$ on the interval $[x, x + \Delta x]$. The average rate of change becomes the instantaneous rate of change when $\Delta x \to 0$ provided that the limit exists.

Thus, $$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}\Big|_{x = x_0} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

i.e. $$\left(\frac{dy}{dx}\right)_{x = x_0} = f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

is the instantaneous rate of change of $y$ with respect to $x$ at $x = x_0$.

The most appropriate examples of the average rate of change and the instantaneous rate of change occur in case of average velocity and the velocity of the particle. If $\Delta s$ be the change in distance made by the particle in time $\Delta t$, then $\frac{\Delta s}{\Delta t}$ gives the average rate of change of the distance $s$ and $\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$ is the instantaneous rate of change of $s$ which gives the velocity of the particle in time $t$.

In the same way, if $v$ be the velocity of the particle in time $t$, then $\frac{dv}{dt}$ is the rate of change of velocity known as the acceleration of the particle at time $t$.

## Worked Out Examples

***Example 1.***

A particle moves in a straight line so that its distance in metres from a given point in the line after $t$ seconds is given $s = 3 + 5t + t^3$. Find,

(i) the velocity at the end of $2\frac{1}{4}$ secs,

(ii) the acceleration at the end of $3\frac{2}{3}$ secs,

(iii) the average velocity during the 5th second.

**Solution:**

$$s = 3 + 5t + t^3$$

$$\frac{ds}{dt} = 5 + 3t^2$$

$$\frac{d^2s}{dt^2} = 6t$$

(i) When $t = 2\frac{1}{4}$ sec, $\frac{ds}{dt}$ = velocity $= 5 + 3 \times \left(\frac{9}{4}\right)^2$

$$= 20\frac{3}{16} \text{ m/s}$$

(ii) When $t = 3\frac{2}{3}$ sec, $\frac{d^2s}{dt^2}$ = acceleration $= 6 \times \frac{11}{3} = 22$ m/s$^2$

(iii) $s(t) = 3 + 5t + t^3$

$s(5) = 3 + 5 \times 5 + (5)^3 = 153$

$s(4) = 3 + 5 \times 4 + (4)^3 = 87$

Average velocity $= \frac{\Delta s}{\Delta t} = \frac{s(5) - s(4)}{5 - 4}$

$$= \frac{153 - 87}{1} = 66 \text{ m/s}$$

**Example 2.**

A circular plate of metal expands by heat so that its radius increases at the rate of 0.25 cm/sec. Find the rate at which the surface-area is increasing when the radius is 7 cms.

**Solution:**

Let $r$ and $s$ be the radius and the surface-area of the circular plate in time $t$ respectively. Then,

$$s = \pi r^2$$

$$\frac{ds}{dt} = \frac{d}{dt}(\pi r^2) = 2\pi r\frac{dr}{dt}$$

When $r = 7$ cms,

$$\frac{ds}{dt} = 2 \times \frac{22}{7} \times 7 \times 0.25 = 11 \text{ cm}^2\text{/sec.}$$

**Example 3**

An edge of a cube is increasing at the rate of 8 cm/sec. How fast is the volume of the cube increasing when the edge is of length 10 cm?

***Solution :***

Let $l$ = length of the edge and

$v$ = volume of the cube in time $t$

By given, $\frac{dl}{dt} = 8$ cm/sec

We know that $v = l^3$

$$\Rightarrow \quad \frac{dv}{dt} = \frac{dl^3}{dt} = 3\,l^2 \frac{dl}{dt}$$

When $l = 10$ cm, $\frac{dv}{dt} = 3 \times (10)^2 \times 8$

$= 2400$ cm$^3$/sec

***Example 4***

A spherical ball of salt dissolving in water decreases its volume at the rate of 0.75 cm$^3$/min. Find the rate at which the radius of the salt is decreasing when its radius is 6 cm.

***Solution :***

Let $r$ = radius and $v$ = volume of the spherical ball of salt in time $t$.

By given, $\frac{dv}{dt} = 0.75$ cm$^3$/min

We know that

$$v = \frac{4}{3}\pi r^3$$

$$\Rightarrow \quad \frac{dv}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$= \frac{4}{3}\pi\, 3r^2 \frac{dr}{dt}$$

$$= 4\pi r^2 \frac{dr}{dt}$$

When $r = 6.0$ cm

$$0.75 = 4\pi \times (6)^2 \times \frac{dr}{dt}$$

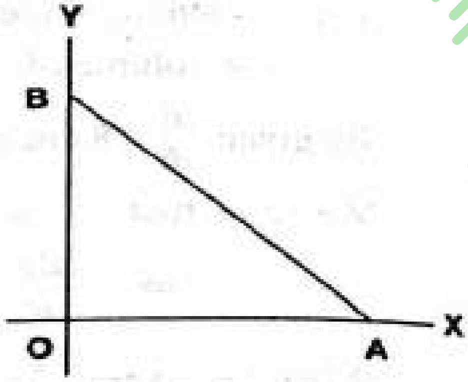
$$\frac{dr}{dt} = \frac{3}{4 \times 4\pi \times 36} = \frac{1}{192\pi} \text{ cm/min.}$$

***Example 5***

Two aeroplanes in flight cross above town at 1 PM one plane travels east at 300 km/hr, the other north at 400 km/hr. At what rate does the distance between the planes change at 3:00 PM?

**Solution :**

Let A and B be the positions of the planes in $t$ hours.




Let AB = $x$ be the distance between the positions of two planes in $t$ hours.

OA = $300t$ and OB = $400t$

Since, $AB^2 = OA^2 + OB^2$

we have, $x^2 = (300t)^2 + (400t)^2$

$\Rightarrow \quad x^2 = 90000t^2 + 160000t^2$

$\Rightarrow \quad x^2 = 250000t^2$

$\Rightarrow \quad x = 500t$

$\Rightarrow \quad \frac{dx}{dt} = 500$

At $t = 2$ hours, $\frac{dx}{dt} = 500$

$\therefore$ the two planes are changing their positions at the rate of 500 km/hr.

***Example 6***

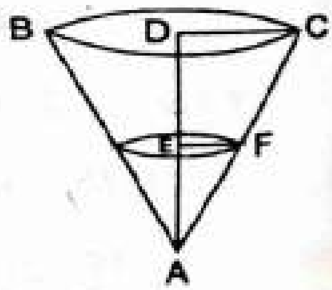
Water flows into an inverted conical tank at the rate of 24 cm³/min. When the depth of water is 9 cm., how fast is the level rising ? Assume that the height of the tank is 15 cm. and the radius at the top is 5 cm.

**Solution:**

Let ABC be the conical water tank into which water is flowing at the rate of 24 cm³/min. At a certain time $t$, let $h$ be the height AE of the water and $r$ the radius EF of the water surface. Now $\Delta^s$ ACD, AFE are similar. So

$$\frac{AE}{AD} = \frac{EF}{DC} \quad \text{or,} \quad \frac{h}{15} = \frac{r}{5} \qquad \therefore r = \frac{1}{3}h$$

Let V be the volume of water in the tank.




$$\therefore \quad V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \frac{1}{9} h^2 h = \frac{1}{27}\pi h^3.$$

$$\therefore \quad \frac{dV}{dt} = \frac{1}{27}\pi \,.\, 3h^2 \frac{dh}{dt}$$

$$\therefore \quad \frac{dh}{dt} = \frac{9}{\pi h^2} \,.\, \frac{dV}{dt} = \frac{9}{\pi h^2} \,.\, 24$$

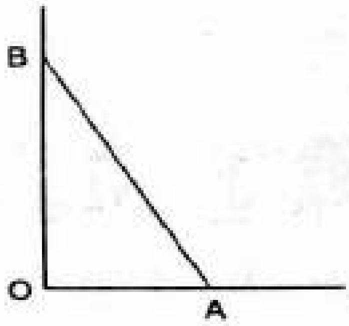
When $h = 9$ cm,

$$\frac{dh}{dt} = \frac{9}{\pi 81} \,.\, 24 = \frac{8}{3\pi} \text{ cm/min.}$$

***Example 7***

A 5 m. ladder leans against a vertical wall. If the top slides downwards at the rate of 12 m/min, find the speed of the lower end when it is 4 m. from the wall.

***Solution:***




At a certain time $t$, let AB be the position of the ladder such that

$$OA = x \quad \text{and} \quad OB = y$$

By given, $\frac{dy}{dt} = -12$

$\therefore \quad x^2 + y^2 = 5^2 = 25 \text{ ...... (i)}$

When $x = 4$ m,

$y^2 = 25 - 16 = 9$

$\therefore \quad y = 3$

Now differentiating both sides of (i) with respect to $t$, we get

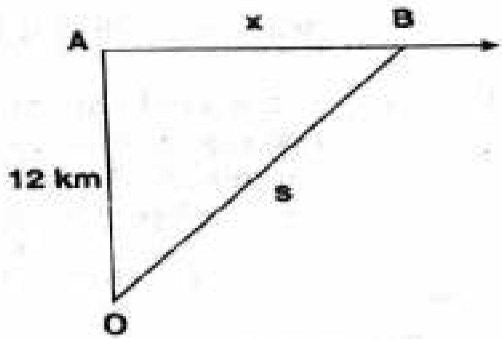
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

or, $\frac{dx}{dt} = -\frac{y}{x}\cdot\frac{dy}{dt} = -\frac{3}{4} \times -12 = 9$ m/min.

***Example 8***

An aeroplane is flying horizontally at a height of 12 km with a velocity of 270 km/hr. Find the rate at which it is receding from a fixed point on the ground which it passed over two minutes ago.

***Solution:***




Let the aeroplane flying horizontally at a height of 12 km from the ground with a velocity of 270 km/hr., be at position B in time $t$ hours.

Distance passed by the plane in

2 mins = $\left(270 \times \frac{2}{60}\right)$ mile = 9 km

Let $OB = s$ and $AB = x$

By given, $\frac{dx}{dt} = 270$

From the figure, $s^2 = (12)^2 + x^2 \text{ ..... (i)}$

when $x = 9$, $s^2 = (12)^2 + (9)^2 = (15)^2$

$\therefore \quad s = 15$ km

Differentiating (i) both sides w.r.t. '$t$'

When $x = 9$, $\quad 2s\dfrac{ds}{dt} = 2x\dfrac{dx}{dt}$

or, $\quad 15 \cdot \dfrac{ds}{dt} = 9 \cdot 270$

$\therefore \quad \dfrac{ds}{dt} = 162$ km/hr

## EXERCISE 18.2

1. (a) A particle moves in a straight line. The distance $s$ (measured in metre) covered by the particle in time $t$ (in second) is given by

$$s = 2t^2 + 5t - 4$$

Find the velocity and the acceleration of the particle in 6 secs.

(b) The displacement of the particle varies with time according to the relation $x = -15t^2 + 20t + 30$. Find the velocity and the acceleration of the particle in $\frac{1}{2}$ second. The distance is measured in metre and time in second.

2. (a) The side of a square sheet is increasing at the rate of 5 cm/min. At what rate is the area increasing when the side is 12 cm long?

(b) A stone thrown into a pond produces a circular ripples which expands from the point of impact. If the radius of the ripple increases at the rate of 3.5 cm/sec, how fast is the area growing when the radius is 15 cm? ($\pi = \frac{22}{7}$)

3. (a) From a cylindrical drum containing oil and kept vertical, the oil is leaking so that the level of the oil is decreasing at the rate of 2 cm/min. If the radius and the height of the drum are 10.5 cm and 40 cm respectively, find the rate at which the volume of the oil is decreasing. ($\pi = \frac{22}{7}$)

(b) Water is poured into a right circular cylinder of radius 8 cm at the rate of 18 cu.cm/min. Find the rate at which the level of water is rising in the cylinder.

(c) Gasoline is pumped into a vertical cylindrical tank at the rate of 24 cu.cm/min. The radius of the tank is 9 cm. How fast is the surface rising?

4. (a) A spherical balloon is inflated at the rate of 18 cu.cm/sec. At what rate is the radius increasing when the radius is 8 cm ?

   (b) A spherical ball of salt is dissolving in water in such a way that the rate of decrease in volume at any instant is proportional to the surface. Prove that the radius is decreasing at the constant rate.

5. (a) The radius of the conical tank is $\frac{1}{3}$ of the height. Water flows into an inverted conical tank at the rate of 4.4 cu.cm/sec. How fast is the level rising when the height of the water is 3.5 cm? ($\pi = \frac{22}{7}$)

   (b) Water flows into an inverted conical tank at the rate of 42 $cm^3$/sec. When the depth of the water is 8 cm, how fast is the level rising? Assume that the height of the tank is 12 cm and the radius of the top is 6 cm.

6. If the volume of the expanding cube is increasing at the rate of 24 $cm^3$/min, how fast is its surface area increasing when the surface area is 216 $cm^2$?

7. Two concentric circles are expanding in such a way that the radius of the inner circle is increasing at the rate of 10 cm/sec and that of the outer circle at the rate of 7 cm/sec. At a certain time, the radii of the inner and the outer circles are respectively 24 cm and 30 cm. At what time, is the area between the circles increasing or decreasing? How fast?

8. A man of height 1.5 m walks away from a lamp post of height 4.5 m at the rate of 20 cm/sec. How fast is the shadow lengthening when the man is 42 cm from the post?

9. A point is moving along the curve $y = 2x^3 - 3x^2$ in such a way that its x-coordinate is increasing at the rate of 2 cm/sec. Find the rate at which the distance of the point from the origin is increasing when the point is at (2, 4).

10. (a) A kite is 24 m high and there are 25 metres of cord out. If the kite moves horizontally at the rate of 36 km/hr directly away from the person who is flying it, how fast is the cord out?

    (b) A 2.5 m ladder leans against a vertical wall. If the top slides downwards at the rate of 12 cm/sec, find the speed of the lower end when it is 2 m from the wall.

### *Answers*

1. a) 29 m/s, 4 $m/s^2$ (b) 5 m/s, $-30$ $m/s^2$
2. (a) 120 $cm^2$/min (b) 330 $cm^2$/sec

3. (a) 693 cm$^3$/min (b) $\frac{9}{32\pi}$ cm/min (c) $\frac{8}{27\pi}$ cm/min

4. (a) $\frac{9}{128\pi}$ cm/sec 5. (a) $\frac{36}{35}$ cm/sec (b) $\frac{21}{8\pi}$ cm/sec

6. 16 cm$^2$/min 7. $60\pi$ cm$^2$/sec (decreasing) 8. 10 cm/sec

9. $10\sqrt{5}$ cm/sec 10. (a) 10.08 km/hr (b) 9 cm/sec

## ADDITIONAL QUESTIONS
## (Maxima and Minima and Rate Measure)

1. What are the maximum and minimum values of a function at a point ? What are the criteria to be satisfied in order to have maximum and minimum values of a function ?

2. Find the maximum and minimum values of the following functions:
   (i) $y = (x - 3)^4$ (ii) $y = (x - 1)(x + 2)^2$
   (iii) $y = \dfrac{x^2 - 7x + 6}{x - 10}$ (iv) $y = ax^2 + bx + c$
   (v) $f(x) = x^3 - 3x$ (vi) $f(x) = 2x^3 + 9x^2 + 12x + 2$
   (vii) $f(x) = x^4 - 14x^2 - 24x + 1$

3. Show that $x^5 - 5x^4 + 5x^3 - 1$ is maximum when $x = 1$, minimum when $x = 3$, neither when $x = 0$.

4. (i) Find two positive numbers whose product is 100 and whose sum is as small as possible. What is the minimum sum ?
   (ii) Divide 60 into two parts such that the sum of the squares of the two parts is minimum.

5. It is given that at $x = 1$, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value, on the interval [0, 2]. Find the value of $a$.

6. Determine where the graph is concave upwards and where it is concave downwards of the following
   (i) $f(x) = x^4 + 4x^3 - 1$ (ii) $y = (x^2 - 1)(x^2 - 5)$

7. (i) Find the minimum value of $x^2y$ when $x + y = 7$.
   (ii) If $x + y = a$ ($a > 0$) find the maximum value of $x^2y$.
   (iii) Find the maximum and minimum values of $u$ where
   $$u = \frac{4}{x} + \frac{36}{y} \text{ and } x + y = 2.$$

8. If the area of a circle increases at a uniform rate, prove that the rate of increase of the perimeter varies inversely as the radius.

9. Two cars start at the same time from the junction of two roads one on each road with uniform speed $v$ m.p.s. If the roads are inclined at 120°, show that the distance between them increases at the rate of $\sqrt{3}v$ m.p.s.

10. A ship leaves a port at noon and travels east at the rate of 9 km/hr. Another ship leaves the same port after one hour and sails south at the rate of 12 km/hr. How fast are they separating at 2 PM?

11. A man of height 1.5 m walks away from a lamp post of height 4.5 m at the rate of 20 cm/sec. How fast does the end of his shadow move?

## Answers

2. (i) min. value = 0 at $x = 3$,
   (ii) min. value = $-4$ at $x = 0$; max. value = 0 at $x = -2$
   (iii) max. value = 1 at $x = 4$, min. value = 25 at $x = 16$
   (iv) min. value = $\dfrac{4ac-b^2}{4a}$ at $x = -\dfrac{b}{2a}$ when $a > 0$
   max. value = $\dfrac{4ac-b^2}{4a}$ at $x = -\dfrac{b}{2a}$ when $a < 0$
   (v) max. value = 2 at $x = -1$; min. value = $-2$ at $x = 1$.
   (vi) max. value = $-2$ at $x = -2$; min. value = $-3$ at $x = -1$.
   (vii) max. value = 12 at $x = -1$; min. value = 9 at $x = -2$; min. value = $-116$ at $x = 3$.

4. (i) 10, 10, 20 (ii) 30, 30. 5. 120

6. (i) Concave upwards for $x < -2$; Concave downwards for $-2 < x < 0$; Concave upwards for $x > 0$.
   (ii) Concave upwards for $x < -1$; Concave downwards for $-1 < x < 1$; Concave upwards for $x > 1$.

7. (i) Min. value = 0 (ii) Max. value = $\dfrac{4a^3}{27}$
   (iii) Max. value = 8 at $x = -1$, Min. value = 32 at $x = \frac{1}{2}$

10. $\dfrac{51}{\sqrt{13}}$ km/hr 11. 30 cm/sec

CHAPTER 19

# Antiderivatives & Its Application

## 19.1 Antiderivatives (Indefinite Integrals)

Let $f$ be a continuous function defined in an open interval $(a, b)$. Then a function F is said to be an *antiderivative* of $f$ on the interval, if the derivative of F is equal to $f$ on the interval, i.e., if

$$\frac{dF(x)}{dx} = f(x), \quad x \in (a, b).$$

As the derivative of a constant $c$ is zero, $F(x) + c$ is also an antiderivative of f, whenever the function F is so. Actually, when

$$\frac{dF(x)}{dx} = f(x),$$

We have

$$\frac{d\,[F(x) + c]}{dx} = \frac{dF(x)}{dx} + \frac{dc}{dx}$$
$$= f(x) + 0$$
$$= f(x).$$

The converse is also true: any two antiderivatives of a function differ by a constant. Let F and G be antiderivatives of a function $f$. Then

$$\frac{d[F(x) - G(x)]}{dx} = \frac{dF(x)}{dx} - \frac{dG(x)}{dx}$$
$$= f(x) - f(x) = 0$$

From this it follows that there exists a constant $c$ such that

$$F(x) - G(x) = c.$$

All these go to establish the fact that if F is an antiderivative of $f$, $F(x) + c$ gives all the possible antiderivatives of $f$, when $c$ runs through all real numbers.

Now it is desirable to have a general form of all antiderivatives of $f$. This general form, which we call indefinite integral of $f$, is denoted by

$$\int f\,dx \qquad \text{or} \qquad \int f(x)\,dx$$

If F is an antiderivative of $f$, we have

$$\int f(x)\, dx = F(x) + c$$

One basic property of the indefinite integral is

$$\int \{c_1 f(x) + c_2\, g(x)\}\, dx = c_1 \int f(x)\, dx + c_2 \int g(x)\, dx + c.$$

where $f$ and $g$ are continuous functions in an interval $(a, b)$ and $c_1$ and $c_2$ are some constants.

We shall take following examples as formulae.

## 19.2 Techniques of Integration

### A. Formulae:

F is an antiderivative of $f$.

| $f(x)$ | $f(x) = \dfrac{dF(x)}{dx}$ |
|---|---|
| $x^n/n$ | $x^{n-1}$ |
| $\dfrac{\sin ax}{a}$ | $\cos ax$ |
| $-\dfrac{\cos ax}{a}$ | $\sin ax$ |
| $\dfrac{\sec ax}{a}$ | $\sec ax \cdot \tan ax$ |
| $\dfrac{\tan ax}{a}$ | $\sec^2 ax$ |
| $-\dfrac{\operatorname{cosec} ax}{a}$ | $\operatorname{cosec} ax \cdot \cot ax$ |
| $-\dfrac{\cot ax}{a}$ | $\operatorname{cosec}^2 ax$ |
| $\log x$ | $\dfrac{1}{x}$ |
| $\dfrac{e^{ax}}{a}$ | $e^{ax}$ |

So we have the formulae:

$$\int x^{n-1}\, dx = \frac{x^n}{n} + c \qquad (\because \int f(x)\, dx = F(x) + c)$$

$$\int \cos ax\, dx = \frac{\sin ax}{a} + c$$

$$\int \sin ax\, dx = -\frac{\cos ax}{a} + c$$

$$\int \sec ax \tan ax\, dx = \frac{\sec ax}{a} + c$$

$$\int \sec^2 ax\, dx = \frac{\tan ax}{a} + c$$

$$\int \operatorname{cosec} ax \,.\, \cot ax\, dx = -\frac{\operatorname{cosec} ax}{a} + c$$

$$\int \operatorname{cosec}^2 ax\, dx = -\frac{\cot ax}{a} + c$$

$$\int \frac{1}{x}\, dx = \log x + c$$

$$\int e^{ax}\, dx = \frac{e^{ax}}{a} + c$$

$$\int (ax + b)^n\, dx = \frac{(ax + b)^{n+1}}{(n+1)a} + c$$

$$\int \frac{1}{ax + b}\, dx = \frac{\log (ax + b)}{a} + c$$

## Worked out examples

***Example 1.***

Calculate $\int (4x^{1/3} + 5x^{2/3} + \frac{1}{x^2})\, dx$

***Solution:***

We have $\int (4x^{1/3} + 5x^{2/3} + \frac{1}{x^2})\, dx$

$$= \int 4x^{1/3} + \int 5x^{2/3} + \int x^{-2}\, dx$$

$$= 4 . \frac{x^{4/3}}{\frac{4}{3}} + 5 . \frac{x^{5/3}}{\frac{5}{3}} + \frac{x^{-1}}{-1} + c$$

$$= 3x^{4/3} + 3x^{5/3} - \frac{1}{x} + c$$

***Example 2.***

Calculate $\int (2x + 3)\,(4x + 5)^4\, dx$ (T.U. 2053 H)

***Solution:***

We have $\int (2x + 3)\,(4x + 5)^4\, dx$

$$= \frac{1}{2} \int (4x + 6)\,(4x + 5)^4\, dx$$

$$= \frac{1}{2} \int (4x + 5 + 1)\,(4x + 5)^4\, dx$$

$$= \frac{1}{2}\int [(4x + 5)^5 + (4x + 5)^4]\, dx$$

$$= \frac{1}{2}[\int (4x + 5)^5\, dx + \int (4x + 5)^4\, dx]$$

$$= \frac{1}{2}\left[\frac{(4x + 5)^6}{6.4} + \frac{(4x + 5)^5}{5.4}\right] + c$$

$$= \frac{1}{8}\left[\frac{(4x + 5)^6}{6} + \frac{(4x + 5)^5}{5}\right] + c$$

***Example 3.***

Calculate $\int \frac{x^2 + 3x + 3}{x + 2}\, dx$

***Solution :***

We have $\int \frac{x^2 + 3x + 3}{x + 2}\, dx$

$$= \int \left(x + 1 + \frac{1}{x + 2}\right) dx$$

$$= \frac{x^2}{2} + x + \log(x + 2) + c$$

***Example 4.***

Calculate $\int (3x + 2)(2x + 5)^3\, dx$

***Solution:***

We have $\int (3x + 2)(2x + 5)^3\, dx$

$$= \frac{3}{2}\int (2x + \frac{4}{3})(2x + 5)^3\, dx$$

$$= \frac{3}{2}\int (2x + 5 - \frac{11}{3})(2x + 5)^3\, dx$$

$$= \frac{3}{2}\int [(2x + 5)^4 - \frac{11}{3}(2x + 5)^3]\, dx$$

$$= \frac{3}{2}\left[\frac{(2x + 5)^5}{2.5} - \frac{11}{3}\cdot\frac{(2x + 5)^4}{2.4}\right] + c$$

$$= \frac{3}{4}\left[\frac{1}{5}(2x + 5)^5 - \frac{11}{12}(2x + 5)^4\right] + c$$

***Example 5***

Calculate $\int \frac{3x + 2}{(5x + 1)^2}\, dx$

***Solution:***

We have $\int \frac{3x + 2}{(5x + 1)^2}\, dx$

$$= \frac{3}{5}\int \frac{5x + \frac{10}{3}}{(5x+1)^2}\, dx$$

$$= \frac{3}{5}\int \frac{(5x + 1 + \frac{7}{3})\, dx}{(5x+1)^2}$$

$$= \frac{3}{5}\int \left[ \frac{1}{5x+1} + \frac{7}{3} \quad \frac{1}{(5x+1)^2} \right] dx$$

$$= \frac{3}{5}\left[ \int \frac{dx}{5x+1} + \frac{7}{3}\int (5x+1)^{-2}\, dx \right]$$

$$= \frac{3}{5}\left[ \frac{\log (5x+1)}{5} + \frac{7}{3}\cdot\frac{(5x+1)^{-1}}{-1.5} \right] + c$$

$$= \frac{3}{25}\left[ \log (5x+1) - \frac{7}{3(5x+1)} \right] + c$$

***Example 6.***

Calculate $\int \frac{1}{\sqrt{x+3} - \sqrt{x-1}}\, dx$

***Solution:***

We have $\int \frac{1}{\sqrt{x+3} - \sqrt{x-1}}\, dx$

$$= \int \frac{\sqrt{x+3} + \sqrt{x-1}}{(\sqrt{x+3} - \sqrt{x-1})(\sqrt{x+3} + \sqrt{x-1})}\, dx$$

$$= \int \frac{\sqrt{x+3} + \sqrt{x-1}}{x + 3 - x + 1}\, dx$$

$$= \frac{1}{4}\int [(x+3)^{1/2}\, dx + \int (x-1)^{1/2}\, dx]$$

$$= \frac{1}{4}\left[ \frac{(x+3)^{3/2}}{\frac{3}{2}} + \frac{(x-1)^{3/2}}{\frac{3}{2}} \right] + c$$

$$= \frac{1}{6}\,[(x+3)^{3/2} + (x-1)^{3/2}] + c$$

***Example 7***

Calculate $\int \sin^2 x\, dx$

***Solution:***

We have $\int \sin^2 x\, dx = \frac{1}{2}\int (1 - \cos 2x)\, dx$

$$= \frac{1}{2} [\int 1.dx - \int \cos 2x \, dx]$$

$$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + c$$

$$= \frac{1}{4}[2x - \sin 2x] + c$$

***Example 8.***

Calculate $\int \tan^2 ax \, dx$

***Solution:***

We have $\int \tan^2 ax \, dx = \int (\sec^2 ax - 1) \, dx$

$$= \int \sec^2 ax \, dx - \int 1.dx$$

$$= \frac{\tan ax}{a} - x + c$$

***Example 9.***

Evaluate $\int \sqrt{1 - \sin 2x} \; dx$

***Solution:***

We have $\int \sqrt{1 - \sin 2x} \; dx$

$$= \int \sqrt{(\sin^2 x + \cos^2 x - 2 \sin x \cos x)} \, dx$$

$$= \int \sqrt{(\sin x - \cos x)^2} \, dx$$

$$= \int (\sin x - \cos x) \, dx$$

$$= - \cos x - \sin x + c$$

***Example 10.***

Evaluate $\int \frac{dx}{1 - \cos x}$

***Solution:***

We have $\int \frac{dx}{1 - \cos x}$

$$= \int \frac{dx}{2 \sin^2 \frac{x}{2}} = \frac{1}{2} \int \operatorname{cosec}^2 \frac{x}{2} \, dx$$

$$= \frac{1}{2}\left(- \frac{\cot \frac{x}{2}}{\frac{1}{2}}\right) + c = - \cot \frac{x}{2} + c.$$

**Example 11**

Evaluate $\int \frac{dx}{1 - \sin x}$

**Solution:**

We have $\int \frac{dx}{1 - \sin x}$

$= \int \frac{(1 + \sin x)\, dx}{(1 - \sin x)(1 + \sin x)}$

$= \int \frac{1 + \sin x}{1 - \sin^2 x}\, dx$

$= \int \frac{1 + \sin x}{\cos^2 x}\, dx$

$= \int \sec^2 x\, dx + \int \tan x . \sec x\, dx$

$= \tan x + \sec x + c$

**Example 12.**

Calculate $\int \sin 6x . \cos 3x\, dx$

**Solution:**

We have $\int \sin 6x . \cos 3x\, dx$

$= \frac{1}{2}\int 2 \sin 6x \cos 3x\, dx$

$= \frac{1}{2}\int (\sin 9x + \sin 3x)\, dx$

$= \frac{1}{2}\left[- \frac{\cos 9x}{9} - \frac{\cos 3x}{3}\right] + c$

$= -\frac{1}{18}[\cos 9x + 3 \cos 3x] + c$

**Example 13**

Calculate $\int \frac{1 - e^{3x}}{e^{5x}}\, dx$

**Solution:**

We have $\int \frac{1 - e^{3x}}{e^{5x}}\, dx = \int (e^{-5x} - e^{-2x})\, dx$

$= \frac{e^{-5x}}{-5} - \frac{e^{-2x}}{-2} + c$

$= -\frac{1}{5e^{5x}} + \frac{1}{2e^{2x}} + c$

## EXERCISE 19.1

Find the indefinite integrals:

6 Mar

1. (i) $\int 5x^3\,dx$ (ii) $\int 7x^{5/2}\,dx$

   (iii) $\int 2x^{-7/2}\,dx$ (iv) $\int 4x^{-5}\,dx$

2. (i) $\int (x^2 + 2)\,dx$ (ii) $\int (3x^2 + 2x + 1)\,dx$

   (iii) $\int (x^{3/4} + x^{1/2} + 4x^{1/3})\,dx$ (iv) $\int (2x + 1)(3x + 2)\,dx$

   (v) $\int (x^2 - \frac{1}{x^2})\,dx$ (vi) $\int (\sqrt{x} - \frac{1}{\sqrt{x}})\,dx$

   (vii) $\int \sqrt{x}(x^2 - 5)\,dx$ (viii) $\int (x - 3)^2\,dx$

   (ix) $\int \frac{3x^2 - 5x + 2}{x}\,dx$ (x) $\int \frac{ax^2 + bx + c}{x^2}\,dx$

   (xi) $\int (x^2 + 3x + 5)\,x^{-1/3}\,dx$

3. (i) $\int (3x + 5)^4\,dx$ (ii) $\int (a - bx)^5\,dx$

   (iii) $\int (c + dx)^{-3/2}\,dx$ (iv) $\int \frac{dx}{\sqrt{2x + 7}}$

   (v) $\int [x + \frac{1}{(x + 3)^2}]\,dx$ (vi) $\int [4 + \frac{1}{(5x + 1)^2}]\,dx$

   (vii) $\int \frac{x + 3}{x - 3}\,dx$ (T.U. 2049 H) (viii) $\int \frac{3x - 1}{x - 2}\,dx$

   (ix) $\int \frac{x^2 + 3x + 3}{x + 1}\,dx$ (x) $\int \frac{x^2 + 5}{x + 2}\,dx$ (T.U. 2056S)

4. (i) $\int x\sqrt{x + 1}\,dx$ (ii) $\int x\sqrt{ax + b}\,dx$

   (iii) $\int 2x\sqrt{2x + 3}\,dx$ (iv) $\int 5x\sqrt{5x + 2}\,dx$ (HSEB 2051)

   (v) $\int (x + 2)\sqrt{3x + 2}\,dx$ (vi) $\int (2x + 3)\sqrt{3x + 1}\,dx$ (T.U. 2054)

   (vii) $\int (5x + 3)\sqrt{4x + 1}\,dx$ (viii) $\int \frac{x + 2}{\sqrt{x + 1}}\,dx$

$5x - x$

$4x + x + x + 2$

(ix) $\int \frac{3x + 4}{\sqrt{x + 1}} dx$

(x) $\int \frac{(2x + 1)}{\sqrt{3x + 2}} dx$

(xi) $\int \frac{2x + 3}{(3x + 1)^{3/2}} dx$

(xii) $\int \frac{3x + 2}{\sqrt{5x + 3}} dx$ (T.U. 2055S)

5. (i) $\int \frac{1}{\sqrt{x + 1} - \sqrt{x}} dx$

(ii) $\int \frac{dx}{\sqrt{x + a} - \sqrt{x - a}}$

(iii) $\int \frac{1}{\sqrt{2x + 1} - \sqrt{2x - 3}} dx$ (T.U. 2051 H)

(iv) $\int \frac{dx}{\sqrt{x + a} - \sqrt{x - b}}$

6. (i) $\int \sin 5x \, dx$

(ii) $\int \sin (ax + b) \, dx$

(iii) $\int \cos (a^2x + b) \, dx$

(iv) $\int \sec^2 (2x + 3) \, dx$

(v) $\int \sin^2 ax \, dx$

(vi) $\int \cos^2 bx \, dx$

(vii) $\int \tan^2 ax \, dx$

(viii) $\int \sin^4 x \, dx$

(ix) $\int \cos^4 nx \, dx$

(x) $\int \frac{1}{\cos^2 x \sin^2 x} dx$

Cot$^2$nn (xi) $\int \frac{1}{\sec^2 x \tan^2 x} dx$

7. (i) $\int \sqrt{1 + \cos nx} \, dx$

(ii) $\int \sqrt{1 - \cos px} \, dx$

(iii) $\int \sqrt{1 + \sin 2ax} \, dx$

(iv) $\int \frac{dx}{1 + \cos mx}$

(v) $\int \frac{dx}{1 - \cos nx}$

(vi) $\int \frac{dx}{1 - \sin ax}$ × 1 + sin ax

8. (i) $\int \sin 6x \cos 2x \, dx$

(ii) $\int \sin 7x \, . \sin 5x \, dx$

(iii) $\int \sin 6x \cos 8x \, dx$

(iv) $\int \cos px . \cos qx \, dx$

9. (i) $\int (e^{px} + e^{-qx}) \, dx$

(ii) $\int (e^{px} + e^{-px})^2 \, dx$

(iii) $\int \frac{e^{2x} + e^x + 1}{e^x} dx$

(iv) $\int e^x (e^{2x} + 1) \, dx$

## Answers

1. (i) $\frac{5}{4}x^4 + c$ (ii) $2x^{7/2} + c$
   (iii) $-\frac{4}{5}x^{-5/2} + c$ (iv) $-x^{-4} + c$

2. (i) $\frac{1}{3}x^3 + 2x + c$ (ii) $x^3 + x^2 + x + c$
   (iii) $\frac{4}{7}x^{7/4} + \frac{2}{3}x^{3/2} + 3x^{4/3} + c$ (iv) $2x^3 + \frac{7}{2}x^2 + 2x + c$
   (v) $\frac{1}{3}x^3 + \frac{1}{x} + c$ (vi) $\frac{2}{3}x^{3/2} - 2x^{1/2} + c$
   (vii) $\frac{2}{7}x^{7/2} - \frac{10}{3}x^{3/2} + c$ (viii) $\frac{1}{3}x^3 - 3x^2 + 9x + c$
   (ix) $\frac{3}{2}x^2 - 5x + 2\log x + c$ (x) $ax + b\log x - \frac{c}{x} + k$
   (xi) $\frac{3}{8}x^{8/3} + \frac{9}{5}x^{5/3} + \frac{15}{2}x^{2/3} + c$

3. (i) $\frac{(3x+5)^5}{15} + c$ (ii) $-\frac{(a-bx)^6}{6b} + c$
   (iii) $\frac{-2(c+dx)^{-1/2}}{d} + k$ (iv) $\sqrt{2x+7} + c$
   (v) $\frac{1}{2}x^2 - \frac{1}{x+3} + c$ (vi) $4x - \frac{1}{5(5x+1)} + c$
   (vii) $x + 6\log(x-3) + c$ (viii) $3x + 5\log(x-2) + c$
   (ix) $\frac{1}{2}x^2 + 2x + \log(x+1) + c$ (x) $\frac{1}{2}x^2 - 2x + 9\log(x+2) + c$

4. (i) $\frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + c$
   (ii) $\frac{1}{a^2}\left[\frac{2}{5}(ax+b)^{5/2} - \frac{2}{3}b(ax+b)^{3/2}\right] + c$
   (iii) $\frac{1}{5}(2x+3)^{5/2} - (2x+3)^{3/2} + c$
   (iv) $\frac{2}{25}(5x+2)^{5/2} - \frac{4}{15}(5x+2)^{3/2} + c$
   (v) $\frac{2}{9}\left[\frac{1}{5}(3x+2)^{5/2} + \frac{4}{3}(3x+2)^{3/2}\right] + c$
   (vi) $\frac{2}{9}\left[\frac{2}{5}(3x+1)^{5/2} + \frac{7}{3}(3x+1)^{3/2}\right] + c$
   (vii) $\frac{1}{8}[(4x+1)^{5/2} + \frac{7}{3}(4x+1)^{3/2}] + c$
   (viii) $\frac{2}{3}(x+1)^{3/2} + 2(x+1)^{1/2} + c$
   (ix) $2(x+1)^{3/2} + 2(x+1)^{1/2} + c$
   (x) $\frac{2}{9}\left[\frac{2}{3}(3x+2)^{3/2} - (3x+2)^{1/2}\right] + c$
   (xi) $\frac{2}{9}[2(3x+1)^{1/2} - 7(3x+1)^{-1/2}] + c$
   (xii) $\frac{2}{25}[(5x+3)^{3/2} + (5x+3)^{1/2}] + c$

5. (i) $\frac{2}{3}[(x+1)^{3/2}+x^{3/2}]+c$
(ii) $\frac{1}{3a}[(x+a)^{3/2}+(x-a)^{3/2}]+c$
(iii) $\frac{1}{12}[(2x+1)^{3/2}+(2x-3)^{3/2}]+c$
(iv) $\frac{2}{3(a+b)}[(x+a)^{3/2}+(x-b)^{3/2}]+c$

6. (i) $-\frac{1}{5}\cos 5x+c$ (ii) $-\frac{1}{a}\cos(ax+b)+c$
(iii) $\frac{\sin(a^2x+b)}{a^2}+c$ (iv) $\frac{1}{2}\tan(2x+3)+c$
(v) $\frac{1}{2}\left[x-\frac{\sin 2ax}{2a}\right]+c$ (vi) $\frac{1}{2}\left[x+\frac{\sin 2bx}{2b}\right]+c$
(vii) $\frac{1}{a}\tan ax-x+c$ (viii) $\frac{1}{32}[12x-8\sin 2x+\sin 4x]+c$
(ix) $\frac{1}{32n}[12nx+8\sin 2nx+\sin 4nx]+c$
(x) $\tan x-\cot x+c$ (xi) $-\cot x-\frac{3}{2}x-\frac{1}{4}\sin 2x+c$

7. (i) $\frac{2\sqrt{2}}{n}\sin\frac{1}{2}nx+c$ (ii) $-\frac{2\sqrt{2}}{p}\cos\frac{1}{2}px+c$
(iii) $\frac{1}{a}(\sin ax-\cos ax)+c$ (iv) $\frac{1}{m}\tan\frac{1}{2}mx+c$
(v) $-\frac{1}{n}\cot\frac{1}{2}nx+c$ (vi) $\frac{1}{a}(\tan ax+\sec ax)+c$

8. (i) $-\frac{\cos 8x}{16}-\frac{\cos 4x}{8}+c$ (ii) $\frac{\sin 2x}{4}-\frac{\sin 12x}{24}+c$
(iii) $\frac{\cos 2x}{4}-\frac{\cos 14x}{28}+c$
(iv) $\frac{1}{2}\left[\frac{\sin(p+q)x}{p+q}+\frac{\sin(p-q)x}{p-q}\right]+c$

9. (i) $\frac{e^{px}}{p}-\frac{e^{-qx}}{q}+c$ (ii) $\frac{e^{2px}}{2p}+2x-\frac{e^{-2px}}{2p}+c$
(iii) $e^x+x-e^{-x}+c$ (iv) $\frac{1}{3}e^{3x}+e^x+c$

## B. Antiderivatives by Substitution Method

In the preceding section we have seen that integrals can be integrated easily when they are in standard forms. A given integral may not always be in a standard form. So to reduce it to a standard form we have to change the variable into a new one by a suitable substitution. To make a proper choice one must have a lot of practice and skill. At the time of substitution, we express the given function x in term of the new variable y (say) and replace $dx$ by $\frac{dy}{dx} \cdot dx$.

**Form I.**

Integrals of the forms $\int \{f(x)\}^n f'(x)\, dx$ and $\int \frac{f'(x)}{\{f(x)\}^n} dx$

Here we substitute, $f(x) = y$ so that $f'(x)\, dx = dy$

Then the given integrals are $\int y^n\, dy$ and $\int y^{-n}\, dy$ which can easily be integrated.

**Form II**

Integrals of the forms $\int f(x)\, g(x)\, dx$ and $\int \frac{f(x)}{g(x)} dx$

Sometimes one function is not the derivative of the other, even then we integrate by substituting $g(x)$ (say) $= y$. This type of integral is given below in example 1.

## Worked Out Examples

***Example 1.***

Calculate $\int x\sqrt{x-4}\, dx$

***Solution:***

Put $y = x - 4$. $\therefore dx = dy$ and $x = y + 4$

So we have

$$\int x\sqrt{x-4}\, dx = \int (y+4)\, y^{1/2}\, dy$$
$$= \int y^{3/2}\, dy + 4\int y^{1/2}\, dy$$
$$= \frac{2}{5} y^{5/2} + \frac{8}{3} y^{3/2} + c$$
$$= \frac{2}{5}(x-4)^{5/2} + \frac{8}{3}(x-4)^{3/2} + c$$

***Example 2.***

Calculate $\int \frac{(2ax+b)}{(ax^2+bx+c)^{1/2}} dx$

***Solution:***

Put $y = ax^2 + bx + c$. Then

$dy = (2ax + b)\, dx$

So we have

$$\int \frac{(2ax+b)}{(ax^2+bx+c)^{1/2}}\,dx = \int \frac{dy}{y^{1/2}}$$
$$= \int y^{-1/2}\,dy = 2y^{1/2} + k$$
$$= 2\sqrt{ax^2+bx+c} + k$$

***Example 3.***

Integrate $\int \frac{xdx}{(2x^2+3)}$

***Solution:***

Put $y = 2x^2 + 3$. Then $dy = 4x\,dx$

or, $x\,dx = \frac{1}{4}\,dy$. So we have

$$\int \frac{x\,dx}{2x^2+3} = \frac{1}{4}\int \frac{dy}{y} = \frac{1}{4}\log y + c$$
$$= \frac{1}{4}\log(2x^2+3) + c$$

***Example 4.***

Integrate $\int x\cos(ax^2+b)\,dx$

***Solution:***

Put $y = ax^2 + b$. Then $dy = 2ax\,dx$

or $x\,dx = \frac{1}{2a}\,dy$. So we have

$$\int x\cos(ax^2+b)\,dx = \frac{1}{2a}\int \cos y\,dy$$
$$= \frac{1}{2a}\sin y + c$$
$$= \frac{1}{2a}\sin(ax^2+b) + c$$

***Example 5.***

Integrate $\int \sin^3x\cos^3x\,dx$

***Solution:***

Put $y = \sin x$. Then $dy = \cos x\,dx$. So we have

$$\int \sin^3x\cos^3x\,dx$$
$$= \int \sin^3x\,(1-\sin^2x)\cos x\,dx$$
$$= \int y^3(1-y^2)\,dy$$
$$= \int y^3\,dy - \int y^5\,dy$$

$= \frac{1}{4} y^4 - \frac{1}{6} y^6 + c$

$= \frac{1}{4} \sin^4 x - \frac{1}{6} \sin^6 x + c$

***Example 6.***

Calculate $\int \tan^n x \sec^2 x \, dx$

***Solution:***

Put $y = \tan x$. Then $dy = \sec^2 x \, dx$. So we have

$$\int \tan^n x \sec^2 x \, dx = \int y^n \, dy$$

$$= \frac{y^{n+1}}{n+1} + c$$

$$= \frac{1}{n+1} \tan^{n+1} x + c$$

***Example 7.***

Integrate $\int \sqrt{e^x - 1} \, e^x \, dx.$

***Solution:***

Put $y = e^x - 1$. They $dy = e^x \, dx$. So we have

$$\int \sqrt{e^x - 1} \, e^x \, dx = \int y^{1/2} \, dy$$

$$= \frac{2}{3} y^{3/2} + c = \frac{2}{3} (e^x - 1)^{3/2} + c.$$

***Example 8.***

Integrate $\int \frac{\log (ax + b)}{ax + b} dx$

***Solution:***

Put $y = \log (ax + b)$. Then $dy = \frac{1}{ax + b} . a \, dx$. So we have

$$\int \frac{\log (ax + b)}{(ax + b)} dx = \frac{1}{a} \int y \, dy$$

$$= \frac{1}{2a} y^2 + c = \frac{1}{2a} [\log (ax + b)]^2 + c$$

***Example 9***

Evaluate : $\int e^{\cos^2 x} \sin x \cos x \, dx$

***Solution :***

Let $y = \cos^2 x$. Then, $dy = -2 \cos x \sin x \, dx$

$\Rightarrow \quad -\frac{1}{2} dy = \sin x \cos x \, dx$

$\therefore \quad \int e^{\cos^2 x} \sin x \cos x \, dx$

$$= -\frac{1}{2}\int e^y \, dy$$

$$= -\frac{1}{2} e^y + c$$

$$= -\frac{1}{2} e^{\cos^2 x} + c$$

***Example 10***

Find $\int \frac{1}{1+e^x} dx$

***Solution :***

$$\int \frac{1}{1+e^x} dx = \int \frac{e^{-x}}{1+e^{-x}} dx$$

Let $y = 1 + e^{-x}$. Then, $dy = -e^{-x} dx$

$\Rightarrow \quad e^{-x} dx = -dy$

$\therefore \quad \int \frac{1}{1+e^x} dx = -\int \frac{dy}{y}$

$$= -\log y + c$$

$$= -\log (1 + e^{-x}) + c$$

## EXERCISE 19.2

Integrate the following:

1. (i) $\int 3x^2 (x^3 + 1)^3 \, dx$ (T.U. 052 H) (ii) $\int x (a^2 + x^2)^{3/2} \, dx$

(iii) $\int \frac{x \, dx}{3x^2 - 4}$ (iv) $\int \frac{x^{n-1}}{\sqrt{a^n + x^n}} dx$

(v) $\int \frac{2x + 3}{(3x^2 + 9x + 5)^3} dx$ (vi) $\int \frac{(3x + 2) \, dx}{(3x^2 + 4x + 1)^3}$

(vii) $\int 3x(x + 2)(x^3 + 3x^2 + 1)^2 \, dx$

(viii) $\int \frac{(x + 2)}{\sqrt{x^2 + 4x + 3}} dx$ (ix) $\int \frac{(x^2 + 1) \, dx}{\sqrt{x^3 + 3x + 4}}$

2. (i) $\int x \sin(ax^2 + b)\,dx$

(ii) $\int x^2 \cos(x^3 + 1)\,dx$

(iii) $\int \frac{1}{x} \sin(\log x)\,dx$ (T.U. 2051 H)

(iv) $\int \sin^3 x \cos x\,dx$

(v) $\int \sin^2 x \cos^3 x\,dx$ (T.U. 2049 H)

(vi) $\int \cos^5 x \sin^3 x\,dx$

(vii) $\int (a \sin x - b)^3 \cos x\,dx$

(viii) $\int \frac{\sin x\,dx}{(1 - \cos x)^m}$ (HSEB 2054)

(ix) $\int \cot x (\log \sin x)^3\,dx$ (T.U. 2052)

(x) $\int \tan^2\theta \sec^4\theta\,d\theta$ (T.U. 2050 H)

(xi) $\int \tan^3 x \sec^4 x\,dx$

(xii) $\int \tan^3 x \sec^{3/2} x\,dx$

(xiii) $\int \cot^{3/2} x \operatorname{cosec}^4 x\,dx$

(xiv) $\int \frac{\sin ax + \cos ax}{\sin ax - \cos ax}\,dx$

(xv) $\int \cot x\,dx$

(xvi) $\int \tan x\,dx$

(xvii) $\int \sec x\,dx$ $\sec x \frac{(\sec x + \tan x)}{\sec x + \tan x}$

(xviii) $\int \operatorname{cosec} x\,dx$

(xix) $\int \tan^3 x\,dx$

(xx) $\int \cot^3 x\,dx$

(xxi) $\int \sec^4 x\,dx$

(xxii) $\int \tan^4 x\,dx$ (T.U. 2053)

(xxiii) $\int \tan^5 x\,dx$

(xxiv) $\int (\tan^2 x + \tan^4 x)\,dx$

(xxv) $\int e^{\sin x \cos x} \cos 2x\,dx$

(xxvi) $\int e^{\sin^2 x} \sin 2x\,dx$ Let, $y = \sin^2 x$ (T.U. 2048 H)

(xxvii) $\int \left(1 - \frac{1}{x^2}\right) e^{x + 1/x}\,dx$ $y = x + \frac{1}{x}$ (T.U. 2048, 2055 S, 2058 S)

(xxviii) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$ Let, $y = \sqrt{x}$ (T.U. 2050)

(xxix) $\int \frac{e^{2x}}{1 + e^x}\,dx$ (T.U. 2052 H)

(xxx) $\int \frac{e^x - 1}{e^x + 1}\,dx$

(xxxi) $\int \frac{xe^x\,dx}{\cos^2(xe^x - e^x)}$

### *Answers*

1. (i) $\frac{1}{4}(x^3 + 1)^4 + c$ (ii) $\frac{1}{5}(a^2 + x^2)^{5/2} + c$
(iii) $\frac{1}{6}\log(3x^2 - 4) + c$ (iv) $\frac{2}{n}\sqrt{a^n + x^n} + c$
(v) $-\frac{1}{6(3x^2 + 9x + 5)^2} + c$ (vi) $-\frac{1}{4(3x^2 + 4x + 1)^2} + c$
(vii) $\frac{1}{3}(x^3 + 3x^2 + 1)^3 + c$ (viii) $\sqrt{x^2 + 4x + 3} + c$
(ix) $\frac{2}{3}\sqrt{x^3 + 3x + 4} + c$

2. (i) $-\frac{1}{2a}\cos(ax^2 + b) + c$ (ii) $\frac{1}{3}\sin(x^3 + 1) + c$
(iii) $-\cos(\log x) + c$ (iv) $\frac{1}{4}\sin^4 x + c$
(v) $\frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + c$ (vi) $\frac{1}{8}\cos^8 x - \frac{1}{6}\cos^6 x + c$
(vii) $\frac{1}{4a}(a\sin x - b)^4 + c$ (viii) $\frac{1}{1-m}(1 - \cos x)^{1-m} + c$
(ix) $\frac{1}{4}(\log \sin x)^4 + c$ (x) $\frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta + c$
(xi) $\frac{1}{4}\tan^4 x + \frac{1}{6}\tan^6 x + c$ (xii) $\frac{2}{7}\sec^{7/2} x - \frac{2}{3}\sec^{3/2} x + c$
(xiii) $-\frac{2}{5}\cot^{5/2} x - \frac{2}{9}\cot^{9/2} x + c$ (xiv) $\frac{1}{a}\log(\sin ax - \cos ax) + c$
(xv) $\log(\sin x) + c$ (xvi) $\log \sec x + c$
(xvii) $\log(\sec x + \tan x) + c$ (xviii) $\log(\operatorname{cosec} x - \cot x) + c$
(xix) $\frac{1}{2}\tan^2 x + \log(\cos x) + c$ (xx) $-\frac{1}{2}\cot^2 x - \log(\sin x) + c$
(xxi) $\tan x + \frac{1}{3}\tan^3 x + c$ (xxii) $\frac{1}{3}\tan^3 x - \tan x + x + c$
(xxiii) $\frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \log(\sec x) + c$
(xxiv) $\frac{1}{3}\tan^3 x + c$ (xxv) $e^{\sin x \cos x} + c$
(xxvi) $e^{\sin^2 x} + c$ (xxvii) $e^{x + 1/x} + c$
(xxviii) $-2\cos\sqrt{x} + c$ (xxix) $e^x - \log(1 + e^x) + c$
(xxx) $2\log(e^{x/2} + e^{-x/2}) + c$ (xxxi) $\tan(xe^x - e^x) + c$

## C. Trigonometrical Substitution

Here we shall consider integrals which involve $a^2 - x^2$, $a^2 + x^2$ or $x^2 - a^2$. It is quite obvious that the substitution $x = a\sin\theta$ turns $a^2 - x^2$ into $a^2\cos^2\theta$; the substitution $x = a\tan\theta$ turns $a^2 + x^2$ into $a^2\sec^2\theta$ and the substitution $x = a\sec\theta$ turns $x^2 - a^2$ into $a^2\tan^2\theta$. These substitutions will make the resulting function easily integrable. This can be better understood with some illustrations.

But if the integral invlves $\frac{1}{\sqrt{x^2+a^2}}$ or $\frac{1}{\sqrt{x^2-a^2}}$ only, besides the trigonometrial substitution, we can follow the following method as well.

Put $\qquad x + \sqrt{x^2+a^2} = y$

so that $\qquad \left(1 + \frac{1}{2\sqrt{x^2+a^2}}2x\right)dx = dy$

$$\Rightarrow \qquad \frac{x+\sqrt{x^2+a^2}}{\sqrt{x^2+a^2}}dx = dy$$

$$\Rightarrow \qquad \frac{dx}{\sqrt{x^2+a^2}} = \frac{dy}{y}$$

Integrating both sides will give the required result.

Similarly for second, we put $x + \sqrt{x^2-a^2} = y$ and the result will be obtained in the same way.

## Worked Out Examples

***Example 1***

Evaluate $\int \frac{1}{x^2-a^2}dx$

***Solution :***

Let $x = a \sec\theta$. Then, $dx = a\sec\theta\tan\theta\, d\theta$

$$\therefore \int \frac{1}{x^2-a^2}dx = \int \frac{a\sec\theta\tan\theta\, d\theta}{a^2\sec^2\theta - a^2}$$

$$= \int \frac{a\sec\theta\tan\theta\, d\theta}{a^2\tan^2\theta}$$

$$= \frac{1}{a}\int \frac{\sec\theta}{\tan\theta}d\theta = \frac{1}{a}\int \operatorname{cosec}\theta\, d\theta$$

$$= \frac{1}{a}\log|\operatorname{cosec}\theta - \cot\theta| + c$$

$$= \frac{1}{a}\log\left|\frac{x}{\sqrt{x^2-a^2}} - \frac{a}{\sqrt{x^2-a^2}}\right| + c$$

$$= \frac{1}{a}\log\left|\frac{x-a}{\sqrt{x^2-a^2}}\right| + c$$

$$= \frac{1}{a} \log \left| \sqrt{\frac{x-a}{x+a}} \right| + c$$

$$= \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c$$

***Example 2***

Find the indefinite integral $\int \sqrt{a^2 - x^2}\, dx$

***Solution :***

Let $x = a \sin \theta$. Then, $dx = a \cos \theta\, d\theta$

$$\therefore \int \sqrt{a^2 - x^2}\, dx = \int \sqrt{a^2 - a^2 \sin^2\theta}\; a \cos \theta\, d\theta$$

$$= \int a \cos \theta \,.\, a \cos \theta\, d\theta$$

$$= \frac{a^2}{2} \int 2 \cos^2\theta\, d\theta$$

$$= \frac{a^2}{2} \int (1 + \cos 2\theta)\, d\theta$$

$$= \frac{a^2}{2} \left[ \theta + \frac{\sin 2\theta}{2} \right] + c$$

$$= \frac{a^2}{2} [\theta + \sin \theta \cos \theta] + c$$

$$= \frac{a^2}{2} \left[ \sin^{-1} \frac{x}{a} + \frac{x}{a} \sqrt{1 - \frac{x^2}{a^2}} \right] + c$$

$$= \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{1}{2} x \sqrt{a^2 - x^2} + c$$

***Example 3***

Evaluate : $\int \frac{dx}{x\sqrt{x^2 - a^2}}$

***Solution :***

Let $x = a \sec \theta$. Then, $dx = a \sec \theta \tan \theta\, d\theta$

$$\therefore \int \frac{dx}{x\sqrt{x^2 - a^2}} = \int \frac{a \sec \theta \tan \theta\, d\theta}{a \sec \theta \sqrt{a^2 \sec^2\theta - a^2}}$$

$$= \int \frac{a \sec \theta \tan \theta\, d\theta}{a \sec \theta\; a \tan \theta}$$

$$= \frac{1}{a} \int d\theta$$

$$= \frac{1}{a}\theta + c$$

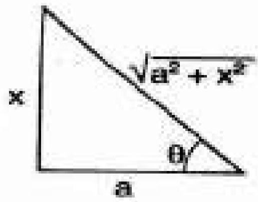
$$= \frac{1}{a}\sec^{-1}\frac{x}{a} + c$$

**Example 4.**

Evaluate $\int \frac{dx}{(a^2 + x^2)^2}$

**Solution:**

Let $x = a\tan\theta$. Then $dx = a\sec^2\theta\, d\theta$, and
$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2\sec^2\theta$




$$\therefore \int \frac{dx}{(a^2 + x^2)^2} = \int \frac{a\sec^2\theta\, d\theta}{a^4\sec^4\theta}$$

$$= \frac{1}{a^3}\int \cos^2\theta\, d\theta$$

$$= \frac{1}{2a^3}\int (1 + \cos 2\theta)\, d\theta$$

$$= \frac{1}{2a^3}[\theta + \tfrac{1}{2}\sin 2\theta] + c$$

$$= \frac{1}{2a^3}[\theta + \sin\theta . \cos\theta] + c$$

$$= \frac{1}{2a^3}\left[\tan^{-1}\frac{x}{a} + \frac{x}{\sqrt{a^2 + x^2}}\frac{a}{\sqrt{a^2 + x^2}}\right] + c$$

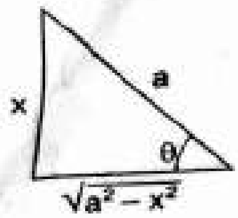
$$= \frac{1}{2a^3}\left[\tan^{-1}\frac{x}{a} + \frac{ax}{a^2 + x^2}\right] + c$$

**Example 5.**

Evaluate $\int \frac{a^2\, dx}{x^2(a^2 - x^2)^{1/2}}$

**Solution:**

Let $x = a\sin\theta$. Then $dx = a\cos\theta\, d\theta$ and
$a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2\cos^2\theta.$




$$\therefore \int \frac{a^2\, dx}{x^2(a^2 - x^2)^{1/2}} = \int \frac{a^2 . a\cos\theta\, d\theta}{a^2\sin^2\theta\, a\cos\theta}$$

$$= \int \operatorname{cosec}^2\theta\, d\theta = -\cot\theta + c$$

$$= -\frac{\sqrt{a^2 - x^2}}{x} + c$$

**Example 6**

Find the indefinite integral : $\int \sqrt{\frac{a-x}{x}}\, dx$

**Solution :**

Let $x = a \sin^2\theta$. Then, $dx = a.2 \sin\theta \cos\theta\, d\theta$

$$\therefore \int \sqrt{\frac{a-x}{x}}\, dx = \int \sqrt{\frac{a - a\sin^2\theta}{a \sin^2\theta}} . 2a \sin\theta \cos\theta\, d\theta$$

$$= \int \frac{\cos\theta}{\sin\theta} . 2a \sin\theta \cos\theta\, d\theta$$

$$= a \int 2\cos^2\theta\, d\theta$$

$$= a \int (1 + \cos 2\theta)\, d\theta$$

$$= a \left[\theta + \frac{\sin 2\theta}{2}\right] + c$$

$$= a\, [\theta + \sin\theta \cos\theta] + c$$

$$= a \left[ \sin^{-1}\sqrt{\frac{x}{a}} + \sqrt{\frac{x}{a}} . \sqrt{1 - \frac{x}{a}} \right] + c$$

$$= a \sin^{-1}\sqrt{\frac{x}{a}} + \sqrt{ax - x^2} + c$$

## EXERCISE 19.3

**Find the indefinite integrals:**

1. i) $\int \frac{dx}{\sqrt{1-x^2}}$ ii) $\int \frac{dx}{x^2\sqrt{9-x^2}}$

   iii) $\int \frac{dx}{(a^2-x^2)^{3/2}}$ $x = a\sin\theta$ iv) $\int \frac{x^2\, dx}{\sqrt{a^2-x^2}}$ $x = a\sin\theta$

2. i) $\int \frac{\sqrt{x^2-9}}{x}\, dx$ $x = 3\sec\theta$ ii) $\int \frac{dx}{x^2\sqrt{x^2-a^2}}$

   iii) $\int \frac{dx}{(x^2-a^2)^{3/2}}$ iv) $\int \frac{dx}{\sqrt{x^2-4}}$ Q and (i) are same.

   v) $\int \frac{x^3\, dx}{(x^2-a^2)^{3/2}}$

3. i) $\int \frac{dx}{x^2 + a^2}$  ii) $\int \frac{dx}{x^2\sqrt{x^2+1}}$

iii) $\int \frac{x^2\,dx}{(1+x^2)^2}$  iv) $\int \frac{dx}{\sqrt{a^2+x^2}}$ i.e. $y = x + \sqrt{x}$

v) $\int \frac{dx}{x\sqrt{x^2+1}}$  vi) $\int \frac{\tan^{-1}x}{1+x^2}\,dx$

4. i) $\int \sqrt{\frac{a+x}{a-x}}\,dx$  ii) $\int \sqrt{\frac{x}{a-x}}\,dx$  $x = a\sin^2\theta$

**Answers**

1. i) $\sin^{-1}x + c$  ii) $-\frac{\sqrt{9-x^2}}{9x} + c$

iii) $\frac{x}{a^2\sqrt{a^2-x^2}} + c$  iv) $\frac{a^2}{2}\sin^{-1}\frac{x}{a} - \frac{1}{2}x\sqrt{(a^2-x^2)} + c$

2. i) $\sqrt{x^2-9} - 3\sec^{-1}\frac{x}{3} + c$

ii) $\frac{\sqrt{x^2-a^2}}{a^2x} + c$  iii) $-\frac{x}{a^2\sqrt{x^2-a^2}} + c$

iv) $\log\left|x + \sqrt{x^2-4}\right| + c$  v) $\frac{x^2-2a^2}{\sqrt{x^2-a^2}}$

3. i) $\frac{1}{a}\tan^{-1}\frac{x}{a} + c$  ii) $-\frac{\sqrt{x^2+1}}{x} + c$

iii) $\frac{1}{2}\tan^{-1}x - \frac{1}{2}\frac{x}{x^2+1} + c$  iv) $\log\left|x + \sqrt{a^2+x^2}\right| + c$

v) $\log\left|\frac{\sqrt{x^2+1}-1}{x}\right|$  vi) $\frac{1}{2}(\tan^{-1}x)^2 + c$

4. i) $a\sin^{-1}\frac{x}{a} - \sqrt{a^2-x^2} + c$  ii) $a\sin^{-1}\sqrt{\frac{x}{a}} - \sqrt{x(a-x)} + c$

## 19.3 Integration of a Product Function "by Parts"

If a given function to be integrated is in the product form and it cannot be integrated either by reducing the integrand into the standard form or by substitution, we use the following rule known as the integration by parts.

Let $u$ and $w$ be two differential functions of $x$. Then using the product rule of differentiation, we have

$$\frac{d}{dx}(uw) = u\frac{dw}{dx} + w\frac{du}{dx}$$

or, $$u\frac{dw}{dx} = \frac{d}{dx}(uw) - w\frac{du}{dx}$$

Integrating both sides w.r.t. '$x$', we have

$$\int\left(u\frac{dw}{dx}\right)dx = \int\left(\frac{d}{dx}(uw)\right)dx - \int\left(w\frac{du}{dx}\right)dx$$

$$\Rightarrow \int\left(u\frac{dw}{dx}\right)dx = uw - \int\left(w\frac{du}{dx}\right)dx \quad \text{...... (i)}$$

Let $\frac{dw}{dx} = v$, then $w = \int v\,dx$

Now, the relation (i) takes the form,

$$\int uv\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx$$

This formula can be stated as follows :

The integral of the product of two functions

= First function × Integral of second

− Integral of (Derivative of first × Integral of second)

This is the formula of the integration of the product of two functions and is known as the "Integration by parts". The successfulness of the use of the above formula depends upon the proper choice of the first function. The first function must be chosen such that its derivative reduces to a simple form and second function should be easily integrable.

## Worked Out Examples

***Example 1.***

Calculate the integral $\int (2x-3)\,e^x\,dx$

***Solution :***

Take $2x-3$ as the first function and $e^x$ as the second. Let $u = 2x-3$, $v = e^x$, then

$$\int (2x-3)\, e^x\, dx = \int u\, v\, dx$$
$$= u \int v\, dx - \int \left(\frac{d}{dx}(u) \int v\, dx\right) dx$$
$$= (2x-3) \int e^x\, dx - \int \left(\frac{d}{dx}(2x-3) \int e^x\, dx\right) dx$$
$$= (2x-3)\, e^x - \int 2.1\, e^x\, dx$$
$$= (2x-3)\, e^x - 2e^x + c$$
$$= (2x-5)e^x + c$$

**Example 2.**

Calculate the integral $\int \log x\, dx$.

**Solution :**

Take $\log x$ as the first function and 1 as the second. Then,

$$\int \log x\, dx = \int 1 \,.\, \log x\, dx$$
$$= \log x \int 1.dx - \int \left(\frac{d}{dx}(\log x) \int 1.dx\right) dx$$
$$= \log x \,.\, x - \int \frac{1}{x} \,.\, x\, dx$$
$$= x \log x - \int 1 \,.\, dx$$
$$= x \log x - x + c$$

**Example 3.**

Calculate the integral $\int x \cos x\, dx$

**Solution :**

Take $x$ as the first function and $\cos x$ as the second. Then,

$$\int x \cos x\, dx = x \int \cos x\, dx - \int \left(\frac{d}{dx}(x) \int \cos x\, dx\right) dx$$
$$= x \sin x - \int 1 \,.\, \sin x\, dx$$
$$= x \sin x - (-\cos x) + c$$
$$= x \sin x + \cos x + c$$

**Example 4.**

Evaluate $\int x^2 \sin 2x\, dx$

**Solution :**

Take $x^2$ as the first function and $\sin 2x$ as the second. Then,

$$\int x^2 \sin 2x\, dx = x^2 \int \sin 2x\, dx - \int \left(\frac{d}{dx}(x^2) \int \sin 2x\, dx\right) dx$$

$$= x^2\left(\frac{-\cos 2x}{2}\right) - \int 2x\left(\frac{-\cos 2x}{2}\right) dx$$

$$= -\frac{1}{2}x^2 \cos 2x + \int x \cos 2x\, dx$$

$$= -\frac{1}{2}x^2 \cos 2x + x\int \cos 2x\, dx - \int \left(\frac{d}{dx}(x)\int \cos 2x\, dx\right) dx$$

$$= -\frac{1}{2}x^2 \cos 2x + \frac{1}{2}x \sin 2x - \int 1 . \frac{\sin 2x}{2} dx$$

$$= -\frac{1}{2}x^2 \cos 2x + \frac{1}{2}x \sin 2x - \frac{1}{2} . \frac{1}{2}(-\cos 2x) + c$$

$$= -\frac{1}{2}x^2 \cos 2x + \frac{1}{2}x \sin 2x + \frac{1}{4}\cos 2x + c$$

***Example 5***

Evaluate $\int x \cos^2 x\, dx$

***Solution :***

$$\int x \cos^2 x\, dx = \frac{1}{2}\int x\,(2\cos^2 x)\, dx$$

$$= \frac{1}{2}\int x(1 + \cos 2x)\, dx$$

$$= \frac{1}{2}\int x\, dx + \frac{1}{2}\int x \cos 2x\, dx$$

$$= \frac{1}{2} \cdot \frac{x^2}{2} + \frac{1}{2}\left\{x\int \cos 2x\, dx - \int \left(\frac{d}{dx}(x)\int \cos 2x\, dx\right) dx\right\}$$

$$= \frac{1}{4}x^2 + \frac{1}{2}\left\{\frac{1}{2} . x \sin 2x - \int 1 . \frac{1}{2}\sin 2x\, dx\right\}$$

$$= \frac{1}{4}x^2 + \frac{1}{4}x \sin 2x - \frac{1}{4} . \frac{1}{2}(-\cos 2x) + c$$

$$= \frac{1}{4}x^2 + \frac{1}{4}x \sin 2x + \frac{1}{8}\cos 2x + c$$

***Example 6***

Evaluate $\int e^x \cos x\, dx$

***Solution :***

Let $\quad I = \int e^x \cos x\, dx$

$$= e^x\int \cos x\, dx - \int \left\{\frac{d}{dx}(e^x)\int \cos x\, dx\right\} dx$$

$$= e^x \sin x - \int e^x \sin x\, dx$$

$$= e^x \sin x - \left\{ e^x \int \sin x\, dx - \int \left( \frac{d}{dx}(e^x) \int \sin x\, dx \right) dx \right\}$$

$$= e^x \sin x - \{e^x (-\cos x) - \int e^x (-\cos x)\, dx\}$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x\, dx$$

$$I = e^x (\sin x + \cos x) - I + 2c$$

or, $\quad 2I = e^x (\sin x + \cos x) + 2c$

$\therefore \quad I = \frac{1}{2} e^x (\sin x + \cos x) + c$

## EXERCISE 19.4

1. Calculate the integrals:

(a) $\int x \log x\, dx$

(b) $\int (5x - 2) \log x\, dx$

(c) $\int x^n \log x\, dx$ (T.U. 2051 H)

(d) $\int (\log x)^2\, dx$

(e) $\int x \log (x - 1)\, dx$ (T.U. 049 H)

(f) $\int x^2 e^{ax}\, dx$ (HSEB 2058)

(g) $\int xe^{5x}\, dx$

(h) $\int (2x - 1)\, e^{2x}\, dx$ (T.U. 048)

(i) $\int x \sin x\, dx$ (T.U. 2053 H, HSEB 2057)

(j) $\int x \sec^2 x\, dx$ (T.U. 048 H)

(k) $\int x \sec x \tan x\, dx$

(l) $\int x \cos nx\, dx$

(m) $\int x \sin ax\, dx$

(n) $\int x \cos (ax - b)\, dx$

(o) $\int x^2 \sin x\, dx$ (T.U. 2049, HSEB 2057)

(p) $\int x^2 \cos nx\, dx$

(q) $\int x^2 \sin (3x - 2)\, dx$

(r) $\int x \sin^2 x\, dx$ (T.U. 2050 H)

(s) $\int x \cos^2 2x\, dx$

(t) $\int x \sin^2 nx\, dx$

(u) $\int x \tan^2 nx\, dx$

(v) $\int \sec^3 x\, dx$ (T.U. 057)(HSEB 051)

(w) $\int \operatorname{cosec}^3 x\, dx$

(x) $\int e^x \sin x\, dx$ (T.U. 2050)

(y) $\int e^{ax} \sin bx\, dx$ (T.U. 051, 056)

(z) $\int e^{ax} \cos bx\, dx$ (T.U. 2053)

**Answers**

1. (a) $\frac{1}{4}x^2(2\log x - 1) + c$ (b) $\frac{1}{2}x(5x-4)\log x - \frac{5x^2}{4} + 2x + c$

(c) $\frac{x^{n+1}}{(n+1)^2}[(n+1)\log x - 1] + c$ (d) $x(\log x)^2 - 2x\log x + 2x + c$

(e) $\frac{1}{2}(x^2-1)\log(x-1) - \frac{x^2}{4} - \frac{x}{2} + c$ (f) $\frac{x^2e^{ax}}{a} - \frac{2xe^{ax}}{a^2} + \frac{2e^{ax}}{a^3} + c$

(g) $\frac{1}{25}(5x-1)e^{5x} + c$ (h) $(x-1)e^{2x} + c$

(i) $\sin x - x\cos x + c$ (j) $x\tan x + \log\cos x + c$

(k) $x\sec x - \log(\sec x + \tan x) + c$ (l) $\frac{1}{n}x\sin nx + \frac{1}{n^2}\cos nx + c$

(m) $\frac{1}{a^2}\sin ax - \frac{1}{a}x\cos ax + c$

(n) $\frac{1}{a}x\sin(ax-b) + \frac{1}{a^2}\cos(ax-b) + c$

(o) $-x^2\cos x + 2x\sin x + 2\cos x + c$

(p) $\frac{1}{n}x^2\sin nx + \frac{2}{n^2}x\cos nx - \frac{2}{n^3}\sin nx + c$

(q) $-\frac{1}{3}x^2\cos(3x-2) + \frac{2}{9}x\sin(3x-2) + \frac{2}{27}\cos(3x-2) + c$

(r) $\frac{1}{4}x^2 - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + c$

(s) $\frac{1}{4}x^2 + \frac{1}{8}x\sin 4x + \frac{1}{32}\cos 4x + c$

(t) $\frac{1}{4}x^2 - \frac{1}{4n}x\sin 2nx - \frac{1}{8n^2}\cos 2nx + c$

(u) $\frac{1}{n}x\tan nx + \frac{1}{n^2}\log\cos nx - \frac{1}{2}x^2 + c$

(v) $\frac{1}{2}\sec x\tan x + \frac{1}{2}\log(\sec x + \tan x) + c$

(w) $-\frac{1}{2}\operatorname{cosec} x\cot x + \frac{1}{2}\log(\operatorname{cosec} x - \cot x) + c$

(x) $\frac{1}{2}e^x(\sin x - \cos x) + c$

(y) $\frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$

(z) $\frac{e^{ax}}{a^2+b^2}(b\sin bx + a\cos bx) + c$

## 19.4 Definite Integral

The expression

$$\int_a^b f(x)\,dx$$

is called the definite integral of $f(x)$ from $a$ to $b$. Here, $a$ is said to be lower limit and $b$, the upper limit. The definite integral has a definite value. Now we derive the formula to evaluate this value with the help of the fundamental theorem of integral calculus. This theorem states that if $f$ is a continuous function and $F(x) = \int_a^x f(t)\,dt$, then

$$\frac{d}{dx}F(x) = f(x).$$

The Fundamental theorem of integral calculus establishes the relation between the two basic concepts of the calculus the derivative and the definite integral. Now with the help of this fundamental theorem, we can prove the following corollary.

**Corollary.** If $f$ is continuous on $[a, b]$ and $\phi$ is an antiderivative of $f$, then

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a).$$

**Proof:** Let $F(x) = \int_a^x f(t)\,dt.$

Obviously, we get $F(a) = 0$. Now, as F and $\phi$ are antiderivative of the same function $f$, they differ only by a constant. So

$F(x) = \phi(x) + c$, for some constant c.

$\therefore \quad F(a) = \phi(a) + c$

or, $\quad 0 = \phi(a) + c$

or $\quad \phi(a) = -c$

$\quad F(x) = \phi(x) - \phi(a)$

and $\quad F(b) = \phi(b) - \phi(a).$

But we have

$$F(b) = \int_a^b f(t)\,dt$$

$$\therefore \quad \int_a^b f(t)\,dt = \phi(b) - \phi(a).$$

This theorem is known as the **Fundamental Theorem of Integral Calculus.**

Thus in evaluating a definite integral $\int_a^b f(x)\,dx$, we use the following steps :

1. Evaluate the indefinite integral $\int f(x)\, dx$. Denote this evaluation by $\phi(x)$.
2. Substitute $x = b$ in $\phi(x)$ to get $\phi(b)$.
3. Substitute $x = a$ in $\phi(x)$ to get $\phi(a)$.
4. Then subtract $\phi(a)$ from $\phi(b)$ which will give the value of the definite integral $\int_a^b f(x)\, dx$.

## Change of Limits

In the evaluation of a definite integral, sometime we use substitution method. If we put $f(x) = y$, the variable $x$ has been changed into the variable $y$. But in a definite integral $\int_a^b f(x)\, dx$, the limits are of the variable $x$ but not of $y$. So, while evaluating a definite integral by substitution method, the limits of $x$ must be changed into the limits of $y$.

## Worked Out Examples

***Example 1.***

Evaluate $\int_0^3 x^5\, dx$

***Solution:***

We have $\int_0^3 x^5\, dx = \left[\frac{1}{6} x^6\right]_0^3 = \frac{1}{6}(3^6 - 0) = \frac{243}{2}$

***Example 2.***

Evaluate $\int_{-1}^2 (x^2 + x + 1)\, dx$

***Solution:***

$$\text{We have } \int_{-1}^2 (x^2 + x + 1)\, dx = \left[\frac{1}{3}x^3 + \frac{1}{2}x^2 + x\right]_{-1}^2$$

$$= \left(\frac{1}{3}\cdot 8 + \frac{1}{2}\cdot 4 + 2\right) - \left(-\frac{1}{3} + \frac{1}{2} - 1\right) = 7\frac{1}{2}.$$

***Example 3***

Evaluate $\int_0^2 \frac{x\, dx}{\sqrt{x^2 + 4}}$

***Solution:***

Put $y = x^2 + 4$. Then $dy = 2x\, dx$.

When $x = 0$, $y = 4$; and when $x = 2$, $y = 8$.

So we have

$$\int_0^2 \frac{x\,dx}{\sqrt{x^2+4}} = \int_4^8 \frac{1}{2} y^{-1/2}\,dy = \left[ y^{1/2} \right]_4^8$$

$$= (8)^{1/2} - (4)^{1/2} = 2\sqrt{2} - 2.$$

**Example 4.**

Evaluate $\int_0^{\sqrt{3}/2} \frac{dx}{\sqrt{(1-x^2)}}$ (T.U. 2048)

**Solution:**

Put $x = \sin\theta$. Then $dx = \cos\theta\, d\theta$.

When $x = 0$, $\theta = 0$ and when $x = \frac{\sqrt{3}}{2}$, $\theta = \frac{\pi}{3}$

So we have

$$\int_0^{\sqrt{3}/2} \frac{dx}{\sqrt{(1-x^2)}}$$

$$= \int_0^{\pi/3} d\theta = [\theta]_0^{\pi/3} = \frac{\pi}{3}$$

**Example 5.**

Evaluate $\int_0^{\pi/2} \frac{\cos\theta\, d\theta}{\sqrt{1-\sin\theta}}$ (T.U. 2051 H)

**Solution:**

Put $y = 1 - \sin\theta$. Then $dy = -\cos\theta\, d\theta$.

When $\theta = 0$, $y = 1$ and when $\theta = \frac{\pi}{2}$, $y = 0$

So we have

$$\int_0^{\pi/2} \frac{\cos\theta\, d\theta}{\sqrt{1-\sin\theta}}$$

$$= -\int_1^0 y^{-1/2}\,dy = \left[-2y^{1/2}\right]_1^0 = 2$$

**Example 6.**

Evaluate $\int_0^{\pi/6} 5\sqrt{3}\tan^4 x \sec^2 x\, dx$

**Solution:**

Put $y = \tan x$. Then $dy = \sec^2 x\, dx$

When $x = 0, y = 0$ and when $x = \frac{\pi}{6}$, $y = \frac{1}{\sqrt{3}}$

So we have

$$\int_0^{\pi/6} 5\sqrt{3} \tan^4 x \sec^2 x \, dx = \int_0^{1/\sqrt{3}} 5\sqrt{3}\, y^4 \, dy$$

$$= [\sqrt{3}\, y^5]_0^{1/\sqrt{3}} = \frac{1}{9}$$

***Example 7.***

Evaluate $\int_1^2 \frac{\sin(\log t)}{t} dt.$ (T.U. 2057 S)

***Solution:***

Put $y = \log t$. Then $dy = \frac{1}{t} dt$.

When $t = 1, y = 0$ and when $t = 2, y = \log 2$.

So we have

$$\int_1^2 \frac{\sin(\log t)\, dt}{t} = \int_0^{\log 2} \sin y \, dy$$

$$= [-\cos y]_0^{\log 2}$$

$$= 1 - \cos(\log 2).$$

***Example 8.***

Evaluate $\int_1^e \ln x \, dx$

***Solution :***

1nd. Int. $= \int \ln x \, dx$

$= \int \ln x \, . \, 1 \, dx$

$= \ln x \int 1. \, dx - \int \left\{ \frac{d}{dx} (\ln x) \int 1. dx \right\} dx$

$= x \ln x - \int \frac{1}{x} . \, x \, dx$

$= x \ln x - x$

Def. Int. $= \int_1^e \ln x \, dx$

$= [x \ln x - x]_1^e$

$= (e \ln e - e) - (0 - 1)$

$= 0 + 1 = 1$

**Example 9**

Evaluate $\int_0^{\pi/2} x \sin x \, dx$

**Solution :**

Here, we consider $x$ as the first function and $\sin x$ as the second. Now,

$\int_0^{\pi/2} x \sin x \, dx$

$$= [-x \cos x]_0^{\pi/2} - \int_0^{\pi/2} 1.(-\cos x) \, dx$$

$$= [-x \cos x]_0^{\pi/2} + \int_0^{\pi/2} \cos x \, dx$$

$$= [-x \cos x]_0^{\pi/2} + [\sin x]_0^{\pi/2}$$

$$= -\frac{\pi}{2} \cos \frac{\pi}{2} + 0 + \sin \frac{\pi}{2} - 0 = 1$$

## EXERCISE 19.5

**Evaluate:**

1. $\int_0^1 (x^2 + 5) \, dx$
2. $\int_1^2 (2x^2 + 3x + 4) \, dx$
3. $\int_{-1}^2 (x + 2)^2 \, dx$
4. $\int_0^{-1} \frac{dx}{x + 2}$
5. $\int_0^1 x^3 \sqrt{1 + 2x^4} \, dx$
6. $\int_1^2 e^{2x^2 - 1} \, 2x \, dx$
7. $\int_0^a \frac{x \, dx}{(a^2 + x^2)^{3/2}}$
8. $\int_0^1 \frac{2x \, dx}{x^2 + 3}$
9. $\int_0^a x \sqrt{a^2 + x^2} \, dx$
10. $\int_{-\pi/3}^{\pi/3} \cos t \, dt$ (HSEB 2057)
11. $\int_0^{\pi/4} \sec^2\theta \, d\theta$
12. $\int_0^1 \cos^2\pi x \, dx$
13. $\int_0^{\pi/4} \sin^2 x \, dx$
14. $\int_0^{\pi/6} \cos^3 x \, dx$
15. $\int_0^{\pi/2} \sin^3 x \, dx$
16. $\int_0^{\pi/3} \frac{dx}{1 + \cos x}$
17. $\int_{\pi/2}^{\pi/6} \frac{dx}{1 - \cos 2x}$
18. $\int_0^{\pi/4} \frac{dx}{1 - \sin x}$ (T.U. 2051)

19. $\int_0^{\pi/2} \frac{dx}{1 + \sin x}$ (multiplication of the numerator and the denominator by $1 - \sin x$ is not admissible as $1 - \sin x = 0$ at $x = \pi/2$. So put $x = \pi/2 - y$) (T.U. 2050)

20. $\int_0^{\pi} \sqrt{1 + \cos x}\, dx$ (HSEB 054)

21. $\int_0^{\pi/2} \sqrt{1 + \sin x}\, dx$

22. $\int_0^{\pi/3} \sin 2\theta \cos \theta\, d\theta$

23. $\int_0^{\pi/2} \cos 3x \cos 2x\, dx$

24. $\int_0^{\pi/4} \cos^3 x \,.\, \sin^2 x\, dx$

25. $\int_0^{\pi/2} (1 + \cos x)^2 \sin x\, dx$

26. $\int_0^{\pi/4} \tan x\, dx$

27. $\int_0^{\pi/4} \tan^3 x\, dx$ (T.U. 2052)

28. $\int_0^{\pi/4} \tan^2 x \sec^4 x\, dx$ (T.U. 049)

29. $\int_1^{e} x \log x\, dx$

30. $\int_0^{1} xe^x\, dx$

31. $\int_1^{e} x^2 \log x\, dx$

32. $\int_{\pi}^{\pi/2} x \cos x\, dx$

33. $\int_0^{\pi} x \sin^2 x\, dx$ (T.U. 2058 S)

34. $\int_0^{\pi} x \cos^2 x\, dx$ (HSEB 2053)

35. $\int_0^{a} \sqrt{a^2 - x^2}\, dx$

36. $\int_0^{1/2} \frac{dx}{\sqrt{1 - x^2}}$ (T.U. 2048 H, HSEB 2052)

37. $\int_0^{1} \frac{dx}{1 + x^2}$ (T.U. 2049 H)

38. $\int_0^{\pi/2} \frac{\cos x}{1 + \sin^2 x}\, dx$ (T.U. 2054)

39. $\int_0^{-1} \frac{dx}{\sqrt{4 - x^2}}$ (T.U. 2056 S)

40. $\int_0^{a} \frac{dx}{a^2 + x^2}$ (T.U. 2052 H)

### Answers

1. $5\frac{1}{3}$ 2. $13\frac{1}{6}$ 3. 21
4. $-\log 2$ 5. $\frac{1}{12}(3\sqrt{3} - 1)$ 6. $\frac{1}{2} e (e^6 - 1)$
7. $\frac{\sqrt{2} - 1}{\sqrt{2a}}$ 8. $\log 4 - \log 3$ 9. $\frac{2\sqrt{2} - 1}{3} a^3$
10. $\sqrt{3}$ 11. 1 12. $\frac{1}{2}$ 13. $\frac{\pi}{8} - \frac{1}{4}$ 14. $\frac{11}{24}$ 15. $\frac{2}{3}$
16. $\frac{1}{\sqrt{3}}$ 17. $-\frac{\sqrt{3}}{2}$ 18. $\sqrt{2}$ 19. 1 20. $2\sqrt{2}$ 21. 2
22. $\frac{7}{12}$ 23. $\frac{3}{5}$ 24. $\frac{7}{60\sqrt{2}}$ 25. $\frac{7}{3}$ 26. $\frac{1}{2}\log 2$
27. $\frac{1}{2}(1 - \log 2)$ 28. $\frac{8}{15}$ 29. $\frac{1}{4}(e^2 + 1)$ 30. 1

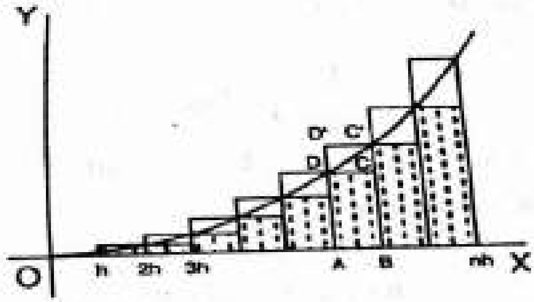
31. $\frac{1}{9}(2e^3 + 1)$ 32. $\frac{\pi}{2} + 1$ 33. $\frac{\pi^2}{4}$ 34. $\frac{\pi^2}{4}$ 35. $\frac{\pi a^3}{4}$

36. $\frac{\pi}{6}$ 37. $\frac{\pi}{4}$ 38. $\frac{\pi}{4}$ 39. $-\frac{\pi}{6}$ 40. $\frac{\pi}{4a}$

## 19.5 Area as a Definite Integral

Computation of the area under the curve of function was really a challenge for mathematicians in the early days. The great Greek mathematician Archimedes solved the problem for some special curves by using some special methods. So before Leibnitz and Newton the areas under the graphs of some simple curves such as linear, quadratic, cubic and of higher powers were only known. Lastly Leibnitz and some other mathematicians brought the problem into a right track by considering the area under the curve as 'an infinite sum of infinitely thin rectangles.' Leibnitz also introduced the integral sign in 1676 by elongating S which stands for 'sum'. The subject concerning this theory was decided by the agreement between Jacob Bernoulli and Leibnitz to be called 'Integral Calculus' in 1696, while the term 'Integral' itself was introduced by Bernoulli in 1690.

As an illustration let us consider an example. We shall find the area included between the curve $y = x^2$, the $x$-axis and the ordinates $x = 0$ and $x = a$.




Draw the parabola $y = x^2$, and divide the interval $(0, a)$ into $n$ sub-intervals of equal length $h = a/n$. Draw n rectangles as shown in the figure. All the shaded rectangles such as ABCD lie under the curve and constitute a set of rectangles inscribed in the parabolic region whose area we have to find out. Let us denote the sum of the areas of these inscribed rectangles by $s_n$. So (as $y = x^2$ gives $y = h^2$, when $x = h$, we have)

$$\begin{aligned} s_n &= 0 + h(h)^2 + h(2h)^2 + h(3h)^2 + \ldots + h(n-1)h^2 \\ &= h^3(1^2 + 2^2 + \ldots + (n-1)^2) \\ &= \frac{a^3}{n^3}\left(\frac{1}{6}(n-1).n.(2n-1)\right) \\ &= \frac{1}{6}a^3\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right) \end{aligned}$$

Therefore, $\lim\limits_{n\to\infty} s_n = \frac{1}{3}a^3$

Let us consider the rectangles of the type ABC'D'. They constitute a set of rectangles circumscribing the parabolic region. Let us denote the sum of the areas of these circumscribing rectangles by $S_n$. Therefore,

$$\begin{aligned} S_n &= h.h^2 + h(2h)^2 + h(3h)^2 + \ldots\ldots + h(nh)^2 \\ &= h^3\,(1^2 + 2^2 + \ldots + n^2) \\ &= \frac{a^3}{n^3}\frac{1}{6}n\,(n+1)\,(2n+1) \\ &= \frac{1}{6}a^3\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) \end{aligned}$$

therefore, $\lim\limits_{n\to\infty} S_n = \frac{1}{3}a^3$

If we denote the area of the parabolic region by A, obviously we have

$$\lim_{n\to\infty} s_n \le A \le \lim_{n\to\infty} S_n$$

$$\frac{1}{3}a^3 \le A \le \frac{1}{3}a^3$$

Therefore, $A = \frac{1}{3}a^3$.

With this method, we can easily find the areas of plane regions.

## Definite Integral as the Limit of a Sum

If $f(x)$ be a function continuous in an interval $[a, b]$ and if the interval $[a, b]$ be divided into $n$ equal parts with each of length $h$, so that $nh = b - a$, then

$$\lim_{h\to 0} h\,[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f(a+\overline{n-1}\,h)]$$

is known as the definite integral of $f(x)$ w.r.t. $x$, the range is from $a$ to $b$ and is written as $\int_a^b f(x)\,dx$.

Hence, $\int_a^b f(x)\,dx = \lim\limits_{h\to 0} h\,[f(a) + f(a+h) + f(a+2h) + \ldots\ldots$

$+ f(a+\overline{n-1}\,h)]$

where $nh = b - a$

The above relation can also be written as

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\,[f(a+h) + f(a+2h) + f(a+3h) + \ldots\ldots + f(a+nh)]$$

Here, $b$ is called the upper limit and $a$ the lower limit.

In particular if $a = 0$, then

$$\int_0^b f(x)\,dx = \lim_{h\to 0} h\,[f(h) + f(2h) + f(3h) + \ldots\ldots + f(nh)]$$

where $nh = b - 0 = b$

This relation gives the definition of an integral as the limit of a sum. The above example shows that the integral gives the area of the plane region under given conditions.

***Example 1***

Find the area bounded by the curve $y = 2x^2 - 3$, x-axis $x = 0$, $x = a$. (Use the limit of a sum)

***Solution :***

Here $f(x) = 2x^2 - 3$

Then using $\int_0^a f(x)\,dx = \lim_{h \to 0} h\,[f(h) + f(2h) + f(3h) + \ldots\ldots + f(nh)]$

We have, $\int_0^a (2x^2 - 3)\,dx = \lim_{h\to 0} [(2h^2 - 3) + (2.2^2h^2 - 3) + (2.3^2h^2 - 3) + \ldots\ldots (2.n^2h^2 - 3)]$

where $nh = a - 0 = a$

$$= \lim_{h\to 0} h\,[2h^2(1^2 + 2^2 + 3^2 + \ldots + n^2) - 3n]$$

$$= \lim_{h\to 0} \left\{ 2h^3 . \frac{n(n+1)(2n+1)}{6} - 3nh \right\}$$

$$= \lim_{h\to 0} \left\{ \frac{1}{3} . nh(nh + h)(2nh + h) - 3nh \right\}$$

$$= \frac{1}{3} . a(a + 0)(2a + 0) - 3a$$

$$= \frac{2}{3}a^3 - 3a$$

## EXERCISE 19.6

Find the area of the plane region using the limit of a sum: (x-axis is included in each of the following question)

1. $y = x;\ x = 0,\ x = a$
2. $y = 4x;\ x = 0,\ x = a$
3. $y = 1 - x,\ x = 0\ x = b$
4. $y = 4x^2;\ x = 0,\ x = c$
5. $y = 4ax^2;\ x = 0,\ x = a$
6. $y = e^x;\ x = 0,\ x = a$
7. $y = e^{3x};\ x = 0,\ x = b$

*Answers*

1. $\frac{a^2}{2}$ 2. $2a^2$ 3. $b - \frac{b^2}{2}$ 4. $\frac{4c^3}{3}$
5. $\frac{4a^4}{3}$ 6. $e^a - 1$ 7. $\frac{1}{3}(e^{3b} - 1)$

## 19.6 Reimann Sums and Integrals

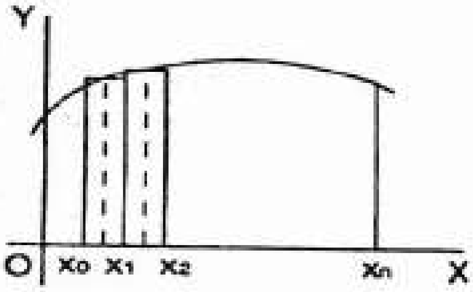
Now we shall generalise the idea developed in the last section. For this, the function, we consider, must be continuous. Let $f$ be a function defined and continuous in an interval $[a, b]$. Consider a finite set of points $x_0, x_1, \ldots, x_n$ such that

$$a = x_0 < x_1 < \ldots\ldots < x_{n-1} < x_n = b$$

This set of points is called a partition of the interval $[a, b]$. This set divides the interval $[a, b]$ into n subintervals whose lengths we denote by

$$\Delta x_i = x_i - x_{i-1}, \qquad i = 1, 2, 3, \ldots\ldots n.$$

The greatest length of these sub-intervals is called the norm or mesh of the partition that is, mesh = max $\Delta x_i$, $i = 1, 2, \ldots n$.




Take any point $t_i$ in the interval $[x_{i-1}, x_i]$. Then the sum

$$\sum_{i=1}^{n} f(t_i)\,\Delta x_i$$

is called a Riemann sum for the function in the interval $[a, b]$. Now as $n$ tends to infinity, $\Delta x_i \to 0$. In this case, as we have seen earlier, the Riemann sum tends to a limit which is equal to the area under the curve of the continuous function $f$.

If the Riemann sum $\sum_{i=1}^{n} f(t_i)\,\Delta x_i$ tends to a limit I, as the mesh approaches zero for every choice of the point $t_i$ in $[x_{i-1}, x_i]$, then I is called the definite integral of $f$ from $a$ to $b$. We denote this limit by

$$I = \int_a^b f(x)\,dx = \lim_{\text{mesh}\to 0} \Sigma f(t_i)\,\Delta x_i$$

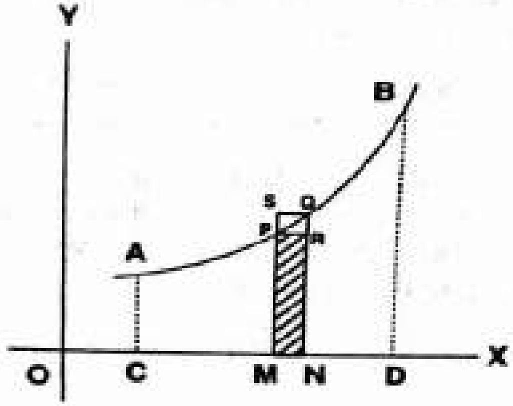
(Here $a$ and $b$ are respectively called the lower and upper limits of the integral).

The existance of definite integral for some functions is guaranted by the theorem "If f is continuous in the interval $(a, b)$ then $\int_a^b f(x)\,dx$ exists".

The proof of this theorem is not so easy. So we leave the proof.

**If $f(x)$ is continuous on $[a, b]$, then the area bounded by the curve $y = f(x)$, the $x$-axis and the two ordinates $x = a$, $x = b$ is equal to $\int_a^b y\,dx$.**

The curve represented by $y = f(x)$ is shown in the figure. Let A and B be two points on the curve with the ordinates AC and BD at $x = a$ and $x = b$ respectively.




Let $P(x, y)$ and $Q(x + \Delta x, y + \Delta y)$ be two neighbouring points on the curve. We complete the rectangles PMNR and SMNQ.

Let $A(x)$ = area of the figure ACMP

= area bounded by the curve $y = f(x)$, $x$-axis, the fixed ordinate AC and the variable ordinate PM

As $x$ increases, the corresponding area $A(x)$ will increase. So, if $x$ is increased to $x + \Delta x$, then $A(x + \Delta x)$ = area of the figure ACNQ.

Then $A(x + \Delta x) - A(x)$ = area of the figure PMNQ

From the figure,

Rect. PMNR < Area of the figure PMNQ < Rect. SMNQ

$$\Rightarrow \quad y.\Delta x < A(x + \Delta x) - A(x) < (y + \Delta y).\Delta x$$

$$\Rightarrow \quad y < \frac{A(x + \Delta x) - A(x)}{\Delta x} < y + \Delta y$$

$$\Rightarrow \quad \lim_{\Delta x \to 0} y < \lim_{\Delta x \to 0} \frac{A(x + \Delta x) - A(x)}{\Delta x} < \lim_{\Delta x \to 0} (y + \Delta y)$$

$$\Rightarrow \quad y < A'(x) < \lim_{\Delta x \to 0} (y + \Delta y)$$

Since, $\lim_{\Delta x \to 0} (y + \Delta y) = \lim_{\Delta y \to 0} (y + \Delta y) = y$

$$\therefore \quad A'(x) = y = f(x)$$

That is, $A(x)$ is the antiderivative of $f(x)$.

Now, using the fundamental theorem of Integral Calculus, we have

$$\int_a^b f(x)\,dx = A(b) - A(a)$$

$$\Rightarrow \quad \int_a^b f(x)\,dx = (\text{value of } A(x) \text{ when } x = b) - (\text{value of } A(x) \text{ when } x = a)$$

$$= \text{Area of the figure ACDB} - 0$$

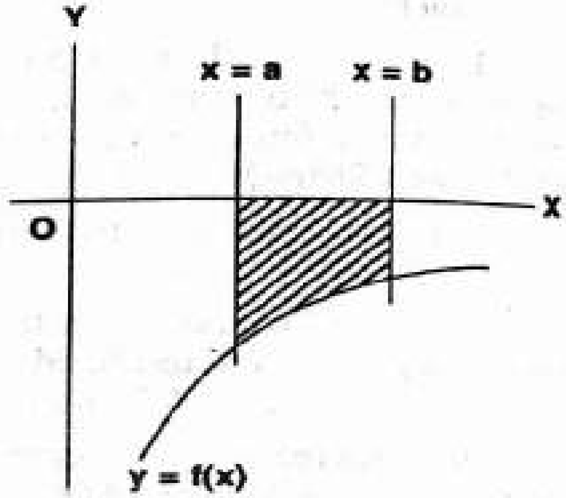
$$\int_a^b y\,dx = \text{Area of the figure ACDB}$$

which is the area bounded by the curve $y = f(x)$, $x$-axis and the two ordinates $x = a$ and $x = b$.

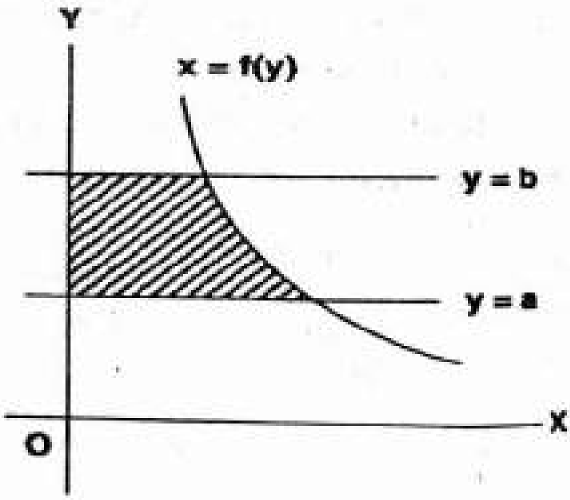
**Note :** In the proof of the above result, we have assumed that $f(x) \geq 0$ (i.e. the curve lies above the $x$-axis.)

**Cor. 1 :** If the curve $y = f(x)$ lies below the $x$-axis (i.e. $f(x) \leq 0$) then the area bounded by the curve $y = f(x)$, $x$-axis and the two ordinates is given by

$$\int_a^b (-y)\,dx = -\int_a^b f(x)\,dx$$



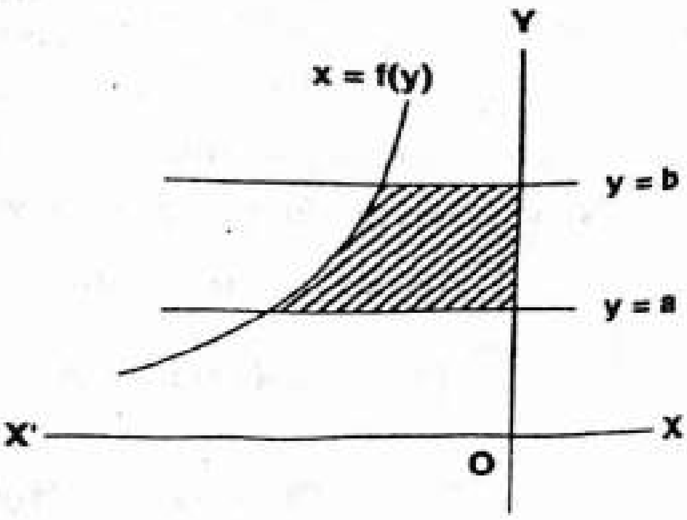

**Cor. 2 :** The area bounded by the curve $x = f(y)$ ($f(y) \geq 0$), the $y$-axis and the two abscissae $y = a$ and $y = b$ is equal to $\int_a^b x\,dy$. (Here the curve lies on the right of $y$-axis)




**Cor. 3 :** If the curve $x = f(y)$ lies on the left of $y$-axis, (i.e. $f(y) \leq 0$) then the area bounded by the curve $x = f(y)$, $y$-axis and the abscissae $y = a$ and $y = b$ is equal to

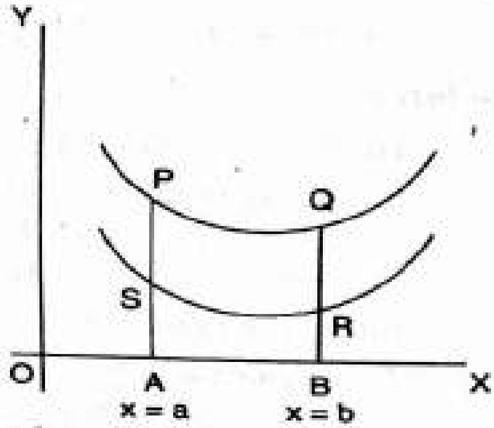
$$\int_a^b (-x)\,dy = -\int_a^b f(y)\,dy.$$

If the required area be negative, we consider the positive value because we need only the magnitude of the area.



## 19.7 Area between Two Curves

Now let us find out the area enclosed by the curves represented by two given functions $f_1$ and $f_2$ and the given ordinates $x = a$ and $x = b$. Let PQ be the curve given by the function $f_1$ and SR the curve given by the function $f_2$. Let PSA and QRB be the ordinates given by $x = a$ and $x = b$ respectively. We have to find out the area PQRS. Now




$$\text{Area PQRS} = \int_a^b f_1(x)\,dx - \int_a^b f_2(x)\,dx$$

$$= \int_a^b (y_1 - y_2)\,dx$$

### Worked out examples

***Example 1.***

Find the area enclosed by $y = 3x$, the x-axis and ordinates $x = 0$, $x = 4$.

***Solution:***

$$\text{The required area} = \int_0^4 y\,dx = \int_0^4 3x\,dx$$

$$= \frac{3x^2}{2}\Big|_0^4 = \frac{3.4^2}{2} = 24$$

***Example 2.***

Find the area bounded by the curve $y^2 = 4ax$, the $x$-axis and the ordinate which cuts the curve at the point $(a, 2a)$.

***Solution:***

The point (0, 0) satisfies the equation of the curve. So the curve passes through the origin. Hence, we have to find the area bounded by the curve $y^2 = 4ax$, the $x$-axis and the ordinates $x = 0$ and $x = a$. Therefore,

$$\text{The area} = \int_0^a y\,dx$$

$$= \int_0^a 2\sqrt{ax}\,dx = 2\sqrt{a}\int_0^a x^{1/2}\,dx$$

$$= 2\sqrt{a}\,\frac{x^{3/2}}{3/2}\Big|_0^a = \frac{4}{3}a^2$$

***Example 3***

Find the area enclosed by the axis of $x$ and the curve $y = x^2 - 4x + 3$.

***Solution :***

The given equation of the curve is $y = x^2 - 4x + 3$

This curve meets the $x$-axis at the point where $y = 0$

$$\therefore \quad x^2 - 4x + 3 = 0$$

$$\Rightarrow \quad (x - 1)(x - 3) = 0$$

Either $x = 1$ or $x = 3$.

The required area between the $x$-axis and the given curve

$$= \int_1^3 y\, dx = \int_1^3 (x^2 - 4x + 3)\, dx$$

$$= \frac{x^3}{3} - 2x + 3x \Big]_1^3$$

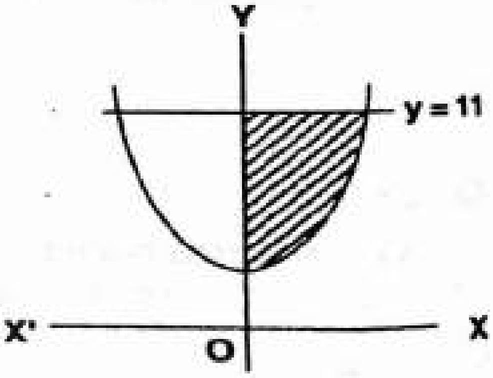
$$= (9 - 6 + 9) - \left(\frac{1}{3} - 2 + 3\right) = 10\frac{2}{3}$$

***Example 4***

Find the area bounded by the y-axis, the curve $x^2 = 4(y - 2)$ and the line $y = 11$.

***Solution :***

The curve $x^2 = 4(y - 2)$ meets the $y$-axis at the point where $x = 0$.




$$\therefore \quad 4(y - 2) = 0$$

$$\text{or,} \quad y = 2$$

Now, the required area $= \int_2^{11} x\, dy$

$$= \int_2^{11} 2\sqrt{y - 2}\, dy$$

$$= \frac{4}{3}(y - 2)^{3/2} \Big]_2^{11}$$

$$= \frac{4}{3}\{(9)^{3/2} - 0\} = 36$$

***Example 5***

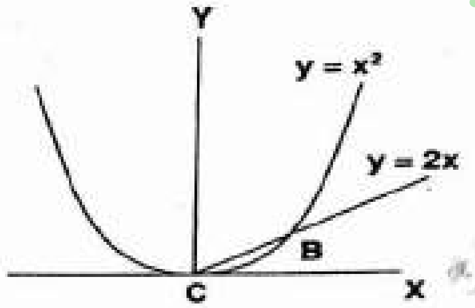
Obtain the area bounded by the curves $y = x^2$ and $y = 2x$.

***Solution:***

Eliminating $y$ between the given equations, we get

$$x^2 = 2x \quad \text{or} \quad x(x - 2) = 0$$

Therefore, $x = 0$ and $x = 2$ are the ordnates of the points at which the given curves intersect.




Therefore, the required area is given by

$$\int_0^2 (y_1 - y_2)\, dx$$

(consider $y_1 = 2x$ and $y_2 = x^2$)

$$\int_0^2 (2x - x^2)\, dx$$

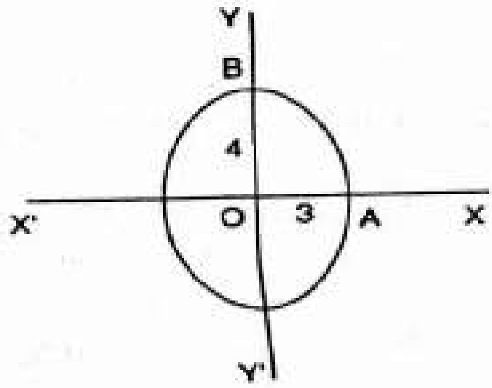
$$= x^2 - \frac{1}{3}x^3 \Big|_0^2 = 4 - \frac{8}{3} = \frac{4}{3} = 1\frac{1}{3}.$$

***Example 6***

Find the area of the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$

**Solution:**

The curve is symmetrical about $x$- and $y$-axis. So to find the area of the whole ellipse, first find the area of the portion lying in the first quadrant and then multiply it by 4.




Here OA = 3, OB = 4. The area of the portion lying in the first quadrant is bounded by the curve, $x$-axis and the ordinates $x = 0$, $x = 3$. So its area is

$$A = \int_0^3 y\, dx$$

$$= \int_0^3 \frac{4}{3}\sqrt{9 - x^2}\, dx$$

Put $x = 3\sin\theta$. Therefore $dx = 3\cos\theta\, d\theta$

$$\sqrt{9 - x^2} = \sqrt{9 - 9\sin^2\theta} = 3\cos\theta$$

When $x = 0$, $\theta = 0$; when $x = 3$, $\theta = \frac{1}{2}\pi$

$$A = \int_0^{\pi/2} \frac{4}{3} . 3\cos\theta . 3\cos\theta\, d\theta$$

$$= 12\int_0^{\pi/2} \cos^2\theta\, d\theta$$

$$= 12\int_0^{\pi/2} \frac{1 + \cos 2\theta}{2}\, d\theta$$

$$= 6\left(\theta + \frac{\sin 2\theta}{2}\right)_0^{\pi/2}$$

$$= 6.\frac{\pi}{2} = 3\pi$$

Therefore, the whole area $= 4A = 4.3\pi = 12\pi$.

## EXERCISE 19.7

1. Find the area bounded by the $x$-axis and the following curves and ordinates:
   a) $x^2 = 4by$; $x = a, x = b$  b) $y = 4x^3$; $x = 2, x = 4$
   c) $y = 3x^2 - 2$; $x = 1, x = 4$.  d) $y^2 - x - 4 = 0$; $x = 2, x = 5$
   e) $y = e^{ax}$; $x = b, x = c$  (HSEB 2050, 2052)
   f) $y = \log(1 + x)$; $x = 0, x = 1$.  $y^2 = x + 4$  $y = \sqrt{x+4}$
2. Find the areas bounded by the $x$-axis and following given curves and ordinates:
   (i) $y^2 = 8ax$ and the ordinate at the point $(4a, 0)$.
   (ii) $y^2 = 4a(x - a)$ and the ordinate at the point $(h, k)$.
   (iii) $x^2 = 4by$ and the ordinate at the point $(b, 0)$.
3. Find the area enclosed by the axis of $x$ and the following curves:
   (i) $y = 3x - 5x^2$  (HSEB 2055)
   (ii) $y = x^2 - 8x + 15$  (iii) $y = x^2 - 10x + 24$
   (iv) $y = x^2 - 25$  (v) $y = (x - 1)(x - 2)(x - 3)$
4. Find the area of the region between:
   (i) the curve $y^2 = 16x$ and line $y = 2x$.  (HSEB 2053, 056)
   (ii) the curve $y^2 = 4x$ and line $x = y$.
   (iii) the curve $y^2 = 4ax$ and line $x = a$.
   (iv) the curve $y = x^3$ and the line $x = y$ lying in the first quadrant.
   (v) the curve $y^2 = x^3$ and the line $x = 4$.
   (vi) the curves $y^2 = 4ax$ and $x^2 = 4ay$.  (HSEB 2058)
5. Find the areas of the circle:
   (i) $x^2 + y^2 = 16$  (ii) $x^2 + y^2 = a^2$

6. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
7. Find the area bounded by the axis of the coordinates, the curve $x^2 = 4a(y - 2a)$ and the ordinate at the point $(h, k)$.
8. Find the area bounded by $y$-axis, the curve $x^2 = 4by$ and $y = b$. (HSEB 2054)
9. Find the area bounded by $y$-axis, the curve $x^2 = 4a(y - 2a)$ and $y = 6a$.
10. Find the area bounded by the x-axis, the curve $x = at^2$, $y = 2at$ and the ordintes corresponding to $t = 0$ to $t = 1$.

### Answers

1. (a) $\frac{b^3 - a^3}{12b}$ (b) 240 (c) 57 (d) $2(9 - 2\sqrt{6})$
   (e) $\frac{1}{a}(e^{ca} - e^{ba})$ (f) $2 \log 2 - 1$
2. (i) $\frac{32\sqrt{2}}{3} a^2$ (ii) $\frac{4}{3}\sqrt{a}(h - a)^{3/2}$ (iii) $\frac{b^2}{12}$
3. (i) $\frac{9}{50}$ (ii) $\frac{4}{3}$ (iii) $\frac{4}{3}$ (iv) $166\frac{2}{3}$ (v) $\frac{1}{2}$
4. (i) $5\frac{1}{3}$ (ii) $2\frac{2}{3}$ (iii) $\frac{8a^2}{3}$ (iv) $\frac{1}{4}$ (v) $25\frac{3}{5}$ (vi) $\frac{16}{3}a^2$
5. (i) $16\pi$ (ii) $\pi a^2$ 6. $\pi ab$
7. $\frac{h}{12a}(h^2 + 24a^2)$ 8. $\frac{4b^2}{3}$ 9. $\frac{32}{3}a^2$ 10. $\frac{4}{3}a^2$

## ADDITIONAL QUESTIONS

1. What is antiderivative ? Is it true that there are a number of antiderivatives of a function which are different from one another only by a constant. Show this relation by taking an example.
2. What does an indefinite integral mean ? How does it differ from definite integral ? What does $\int_a^b f(x)\,dx$ represent ?
3. Evaluate :
   (i) $\int \left(x + \frac{1}{x}\right)^3 dx$ (ii) $\int (2x + 3)(6x + 8)^5\,dx$

(iii) $\int \frac{x+1}{(5x+2)^3}dx$ (iv) $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}dx$

(v) $\int \frac{\cos(x-\alpha) - \cos(x+\alpha)}{\sin(x+\alpha) + \sin(x-\alpha)}dx$

(vi) $\int \frac{e^{2x}-1}{e^x+1}dx$

(vii) $\int \frac{\sec x}{\sec x + \tan x}dx$ (viii) $\int \frac{a\sin^3 x + b\cos^3 x}{\sin^2 x \cos^2 x}dx$

4. Evaluate :

(i) $\int \frac{2x+3}{\sqrt{3x+1}}$ (T.U. 2048)

(ii) $\int (7x+5)\sqrt{4x+3}\,dx$ (T.U. 2049)

(iii) $\int (5x+3)\sqrt{3x+2}\,dx$ (T.U. 2050)

(iv) $\int (3x+2)\sqrt{5x+1}\,dx$ (T.U. 2051)

(v) $\int \tan^3 x \sec^4 x\,dx$ (T.U. 2051)

(vi) $\int (2x+3)(4x+7)^3\,dx$ (T.U. 2052)

(vii) $\int (7x+9)\sqrt{4x+1}\,dx$ (T.U. 2053)

(viii) $\int \frac{1}{\sqrt{3x+a} - \sqrt{3x-b}}dx$ (HSEB 2050)

5. Find the following indefinite integrals :

(i) $\int \frac{1-\sin x}{x+\cos x}dx$ (ii) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx$

(iii) $\int \frac{e^x(1+x)}{\cos^2(x e^x)}dx$ (iv) $\int \frac{(x+1)(x+\log x)^3}{x}dx$

(v) $\int \frac{1}{1+e^{-x}}dx$ (vi) $\int \frac{\sin x \cos x\,dx}{a\cos^2 x + b\sin^2 x}dx$

(vii) $\int e^{\cos^2 x}\sin 2x\,dx$ (viii) $\int 2x.e^{x^2}\sec^2(e^{x^2})\,dx$

(ix) $\int \frac{e^{\sqrt{x}}\tan^2(e^{\sqrt{x}})}{2\sqrt{x}}dx$ (x) $\int \frac{\sqrt{\tan x}}{\sin x \cos x}dx$

(xi) $\int \frac{1+\sin x}{1+\cos x}dx$ (xii) $\int \frac{\sin x}{\sin(x-\alpha)}dx$

(xiii) $\int \frac{1}{1+e^x}dx$

6. Evaluate :

(i) $\int \frac{x}{1 + \cos x} dx$ (ii) $\int \frac{\sin x - x}{1 - \cos x} dx$

(iii) $\int e^{\sqrt{x}} dx$ (iv) $\int \left\{ \frac{1}{\log x} - \frac{1}{(\log x)^2} \right\} dx$

(v) $\int \sin \sqrt{x}\, dx$ (vi) $\int e^{\sin x} \sin 2x\, dx$

(vii) $\int x^2 e^x dx$ (HSEB 2054) (viii) $\int x \sin^3 x\, dx$

7. Evaluate the following integrals :

(i) $\int_0^{\pi/2} (a \cos^2 x + b \sin^2 x)\, dx$ (ii) $\int_0^1 \frac{1}{\sqrt{x} + x} dx$

(iii) $\int_1^{e^2} \frac{dx}{x(1 + \log x)^2}$ (iv) $\int_0^{\pi/2} \sin^2 x \cos^2 x\, dx$

(v) $\int_0^{\pi/2} \sin \theta \cos \theta \sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta}\, d\theta$

### Answers

3. (i) $\frac{x^4}{4} + \frac{3x^2}{2} + 3 \log x - \frac{1}{2x^2} + c$

(ii) $\frac{1}{756} (36x + 55) (6x + 8)^6 + c$

(iii) $-\frac{10x + 7}{50(5x + 2)^2} + c$ (iv) $2(\sin x + x \cos \alpha) + c$

(v) $x \tan \alpha + c$ (vi) $e^x - x + c$

(vii) $\tan x - \sec x + c$ (viii) $a \sec x - b \operatorname{cosec} x + c$

4. (i) $\frac{2}{27} (6x + 23) (3x + 1)^{1/2} + c$

(ii) $\frac{1}{60} (42x + 29) (4x + 3)^{3/2} + c$

(iii) $\frac{2}{27} (9x + 5) (3x + 2)^{3/2} + c$

(iv) $\frac{2}{375} (45x + 44) (5x + 1)^{3/2} + c$

(v) $\frac{\tan^4 x}{4} + \frac{\tan^6 x}{6} + c$

(vi) $\frac{1}{160} (16x + 23) (4x + 7)^4 + c$

(vii) $\frac{1}{60} (42x + 83) (4x + 1)^{3/2} + c$

(viii) $\frac{2}{9(a+b)}[(3x+a)^{3/2}+(3x-b)^{3/2}]+c$

5. (i) $\log(x+\cos x)+c$ (ii) $\log(e^x+e^{-x})+c$

(iii) $\tan(xe^x)+c$ (iv) $\frac{(x+\log x)^4}{4}+c$

(v) $\log(1+e^x)+c$

(vi) $\frac{1}{2(b-a)}\log(a\cos^2 x+b\sin^2 x)+c$

(vii) $-e^{\cos^2 x}+c$ (viii) $\tan(e^{x^2})+c$

(ix) $\tan(e^{\sqrt{x}})-e^{\sqrt{x}}+c$ (x) $2\sqrt{\tan x}+c$

(xi) $\tan\frac{x}{2}-\log(1+\cos x)+c$

(xii) $x\cos\alpha+\sin\alpha\log\sin(x-\alpha)+c$

(xiii) $-\log(1+e^{-x})+c$

6. (i) $x\tan\frac{x}{2}+2\log\cos\frac{x}{2}+c$

(ii) $\log(1-\cos x)+x\cot\frac{x}{2}-2\log\sin\frac{x}{2}+c$

(iii) $2e^{\sqrt{x}}(\sqrt{x}-1)+c$

(iv) $\frac{x}{\log x}+c$

(v) $2(\sin\sqrt{x}-\sqrt{x}\cos\sqrt{x})+c$

(vi) $2e^{\sin x}(\sin x-1)+c$

(vii) $x^2e^x-2xe^x+2e^x+c$

(viii) $-\frac{3}{4}x\cos x+\frac{3}{4}\sin x+\frac{1}{12}x\cos 3x-\frac{1}{36}\sin 3x+c$

7. (i) $\frac{1}{4}(a+b)\pi$ (ii) $2\log 2$ (iii) $\frac{2}{3}$

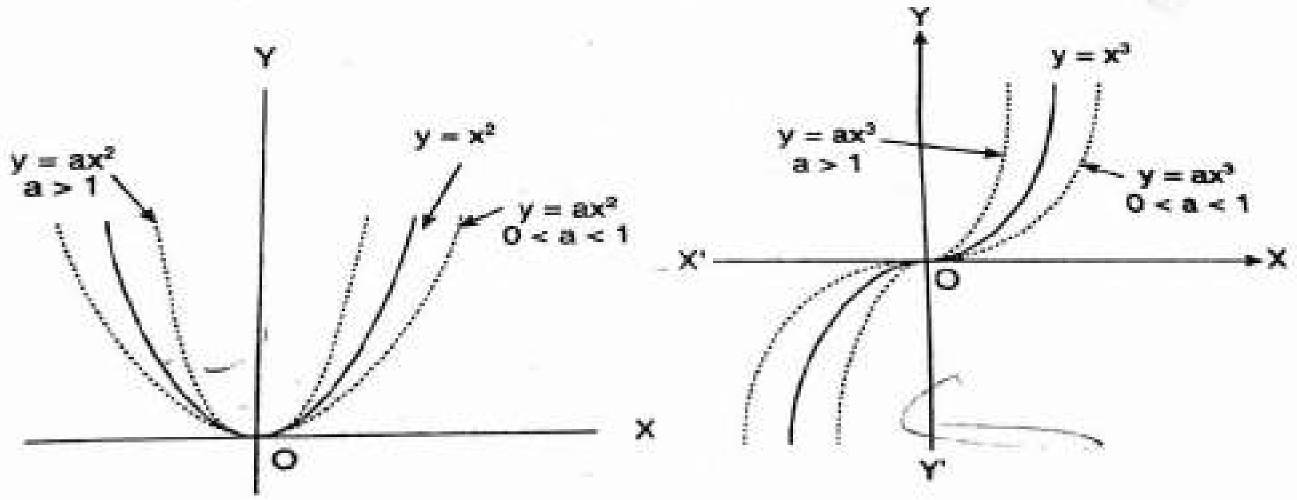
(iv) $\frac{\pi}{16}$ (v) $\frac{a^2+ab+b^2}{3(a+b)}$

# Appendix

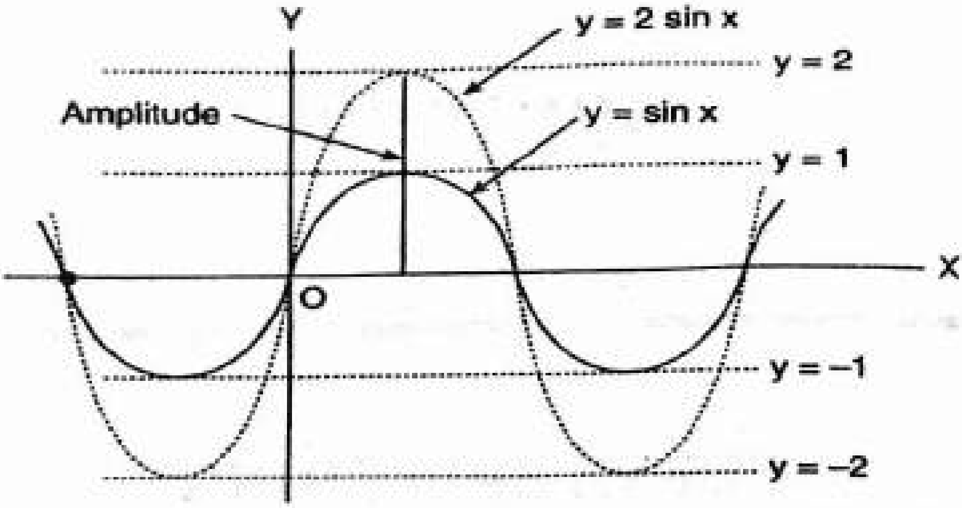
## CURVE SKETCHING

The graph of $y = ax^2$ is a parabola whose openness is narrower than that of $y = x^2$ when $|a| > 1$, which means the graph of $y = ax^2$ stretches towards the vertical line (i.e. y-axis) by the factor '$a$'. On the other hand, the openness will be wider when $|a| < 1$ which means the graph of $y = ax^2$ stretches away from the vertical line by the factor '$a$'.

Let us see the following graphs.

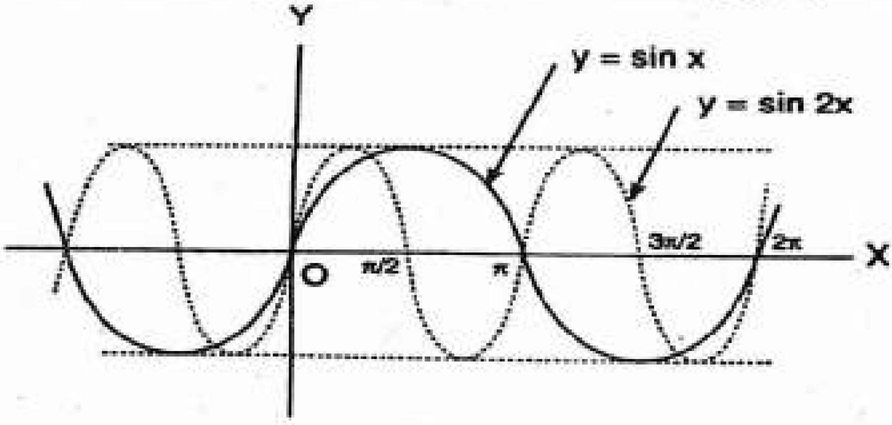


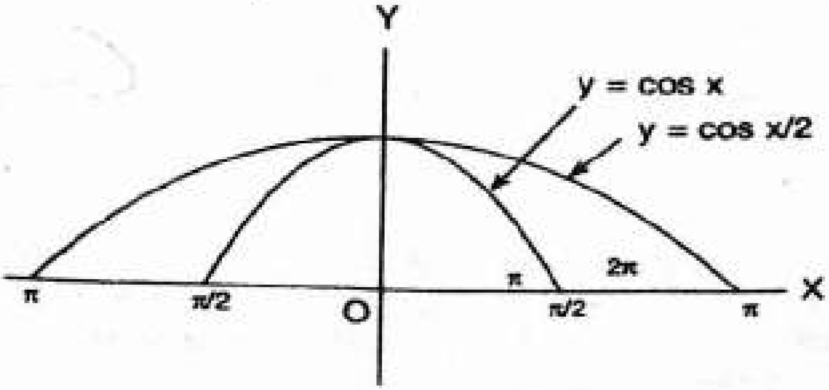

But in case of trigonometrical functions (i.e. $y = a \sin x$ or $y = a \cos x$) greater the value of $|a|$ ($|a| > 1$), greater will be the vertical stretchedness and lesser the value of $|a|$ ($|a| < 1$) less will be the vertical stretchedness or greater will be the vertical compressed.




The half of the distance between the maximum and the minimum values of the function $y = a \sin x$ or $y = a \cos x$ is known as the amplitude of the function. The amplitude of the function $y = a \sin x$ or $y = a \cos x$ is $|a|$. Thus the amplitude of $y = 2 \sin x$ is 2.

Again the graph of the function of the type $y = \sin ax$ or $y = \cos ax$, when $a > 1$, will be horizontally compressed in comparison to that of $y = \sin x$ or $y = \cos x$ respectively. On the other hand if $0 < a < 1$, then the graph will be stretched horizontally. Let us see the following graphs.



# Additional Questions

## Functions

1. Find the domain of the following functions:

a) $f(x) = \dfrac{1}{\sqrt{4 - x^2}}$

b) $f(x) = \sqrt{x^2 - 64}$

c) $f(x) = \sqrt{5 - 4x - x^2}$

d) $f(x) = \dfrac{1}{\sqrt{x^2 + 3x + 2}}$

e) $f(x) = \dfrac{x}{x^2 - 1}$

f) $f(x) = \sqrt{\dfrac{1}{x} - 1}$

g) $f(x) = \dfrac{\sqrt{x^2 - 9}}{x^2 - 5x + 6}$

h) $f(x) = \sqrt{x + 2} + \sqrt{x - 2}$

2. Find the domain and the range of the following functions:

a) $f(x) = \dfrac{1}{\sqrt{3 - x}}$

b) $f(x) = \dfrac{x - 1}{x^2 - 1}$

c) $f(x) = \dfrac{1}{x^2 - 1}$

d) $f(x) = \dfrac{x}{x^2 + 1}$

e) $y = \dfrac{x}{x^2 - x + 1}$

f) $y = (3 - x)^2 - 2$

g) $y = 3 - (x + 1)^2$

3. Find the domain and the range of the function

$$f(x) = \begin{cases} 1 + x & \text{for } -1 < x \le 0 \\ 2 - x & \text{for } 0 < x \le 1 \end{cases}$$

**Answers**

1. a) $D(f) = (-2, 2)$
b) $(-\infty, -8] \cup [8, \infty)$
c) $D(f) = [-5, 1]$
d) $D(f) = (-\infty, -2) \cup (1, \infty)$
e) $D(f) = R - \{-1, 1\}$
f) $D(f) = (0, 1]$
g) $D(f) = (-\infty, -3] \cup (3, \infty)$
h) $D(f) = [2, \infty)$ (i.e. $x \ge 2$)

2. a) $D(f) = (-\infty, 3)$, $R(f) = (0, \infty)$
b) $D(f) = R - \{-1, 1\}$, $R(f) = R - \{0, 1/2\}$
c) $D(f) = R - \{-1, 1\}$, $R(f) = (-\infty, -1] \cup (0, \infty)$

d) $D(f) = R = (-\infty, \infty)$, $R(f) = [-1/2, 1/2]$
e) $D(f) = R = (-\infty, \infty)$, $R(f) = [-1/3, 1]$
f) $D(f) = R = (-\infty, \infty)$, $R(f) = [-2, \infty)$
g) $D(f) = R = (-\infty, \infty)$, $R(f) = (-\infty, 3]$

3. $D(f) = (-1, 1]$, $R(f) = (0, 2)$

***Example***

Sketch the graph of the function
$y = f(x) = x(x + 1)(x - 2)^2$ using its different characteristics.

***Solution:***

**Characteristics**

i) When $x = 0$, $y = 0$ so the curve passes through the origin.

ii) When $y = 0$, then

$$0 = x(x + 1)(x - 2)^2$$

which gives $x = 0, -1, 2$
So, the curve cuts the x-axis at the points (0, 0), (–1, 0) and (2, 0).

iii) When $-1 < x < 0$, $y < 0$, so the part of the curve lies below the x-axis.

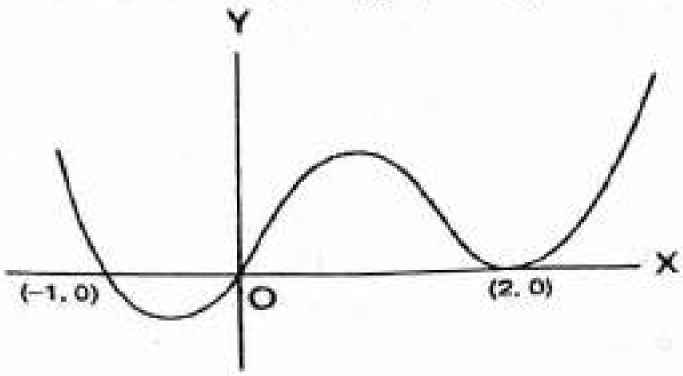
iv) When $0 < x < 2$, $y > 0$, so the part of the curve lies above the x-axis.

v) When $x < -1$, $y > 0$, so when x decreases, $y$ increases.

vi) When $x > 2$, $y > 0$, so when x increases, $y$ also increases.

vii) Since the curve cuts the x-axis at two coincident points $x = 2, 2$, so x-axis is tangent to the curve at the point $x = 2$.

With the above characteristics, the graph is presented below



# Curve Sketching

1. Sketch the graphs of the following functions

a) $y = \sqrt{16 - x^2}$  b) $y = \sqrt{x}$

c) $y = \frac{1}{x^2}$  d) $y = \frac{2x - 3}{x - 2}$

e) $y = \frac{1 - x^2}{x^2}$  f) $y = |x - 2|$

g) $y = \sqrt{x + 1}$

2. Sketch the graphs of

a) $y = x^2 - 3x$  b) $y = 2x - x^2$

c) $y = 2(x - 1)(x + 2)$  d) $y = 2 - 3x - 2x^2$

3. Sketch the graph of $y = x^2$ and hence sketch the graph of

a) $y = -x^2$  b) $y = 2x^2$

c) $y = \frac{1}{3}x^2$  d) $y = x^2 - 3$

e) $y = (x + 2)^2 + 3$  f) $y = x^2 - x - 1$

g) $y = 2 - (x - 1)^2$

4. Sketch the graphs of

a) $y = x(x - 2)^2$  b) $y = x^3 + 2x^2$

c) $y = x^3 - 2x^2 - 3x$  d) $y = x(x - 1)(x + 2)$

e) $y = (x + 1)(x + 2)(x - 3)$

f) $y = x^3 - x + 2$  g) $y = x^3 - x^2 - 3$

h) $f(x) = x^3 - 2x^2 + x - 3$  i) $f(x) = x(x - 1)^2(x + 2)$

5. Sketch the graphs of

a) $y = \frac{3x + 1}{x - 1}$  b) $y = \frac{3x^2 - x - 4}{x + 1}$

c) $y = \frac{x - 2}{x}$  d) $y = \frac{x^2 + 1}{x}$

6. Sketch the graphs of

a) $y = 2^x - 4$  b) $y = 2^{x+1}$

c) $y = 3^{x-2}$  d) $y = 3^x + 5$

7. Sketch the graphs of

a) $y = \log_2(x + 2)$  b) $f(x) = \log_3 x + 3$

c) $f(x) = \log_3(x - 1)$  d) $y = \log_2 x - 2$

8. Sketch the graphs of

a) $y = \sin 2x$  b) $y = 3 \cos x$

c) $y = \sin x - 3$  d) $y = \cos x + 1$

9. a) If $f(x) = \cos x + (3 - c)x + x^2$ is an even function, find the value of $c$.

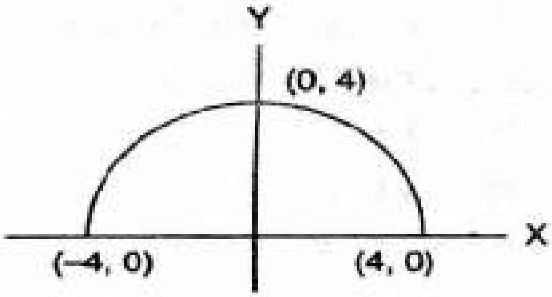
   b) If $f(x) = x^3 + (a + 2)x^2 + x$ is an odd function, show that $a = -2$

10. Show that $f(x) = \log\left(\sqrt{x^2 + 1} + x\right)$ is an odd function.

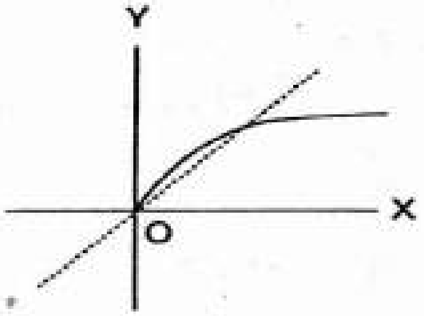
11. Show that $f(x) = \dfrac{x}{e^x + 1} - \dfrac{x}{2}$ is an even function.

## *Answer*

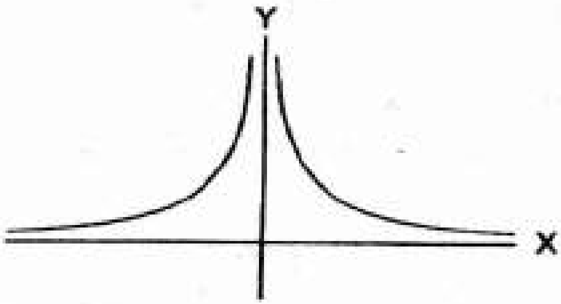
1. a)



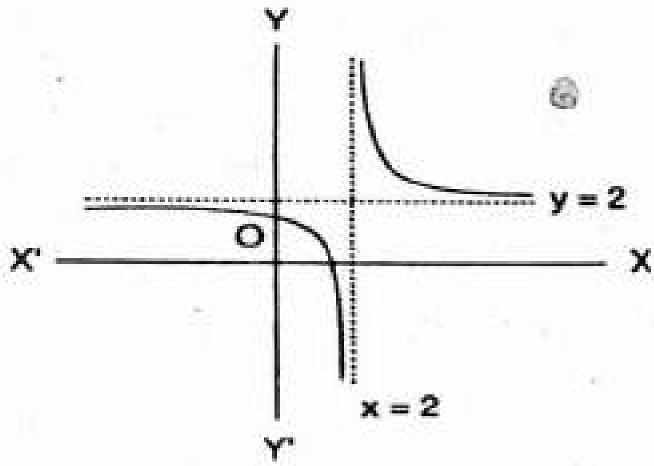

b)



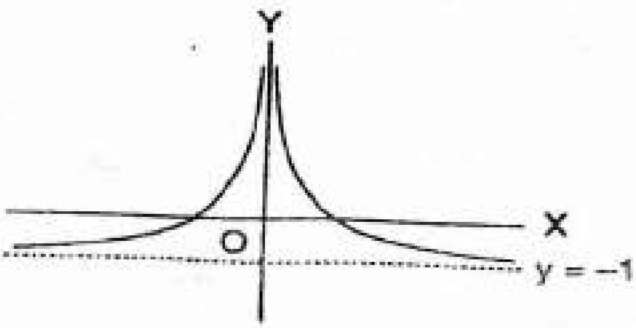

c)



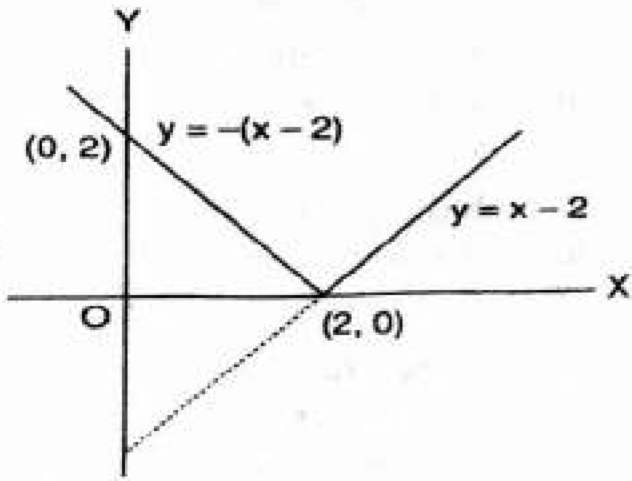

d)



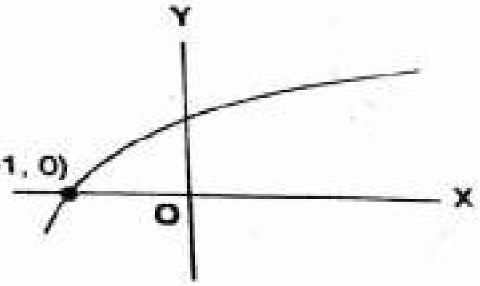

e)



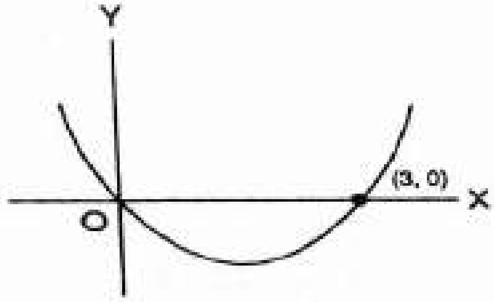

f)



g)


Y
(−1, 0)
O
X

2. a)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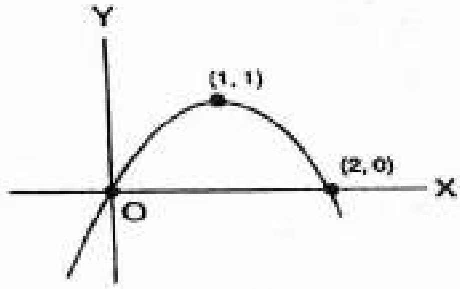
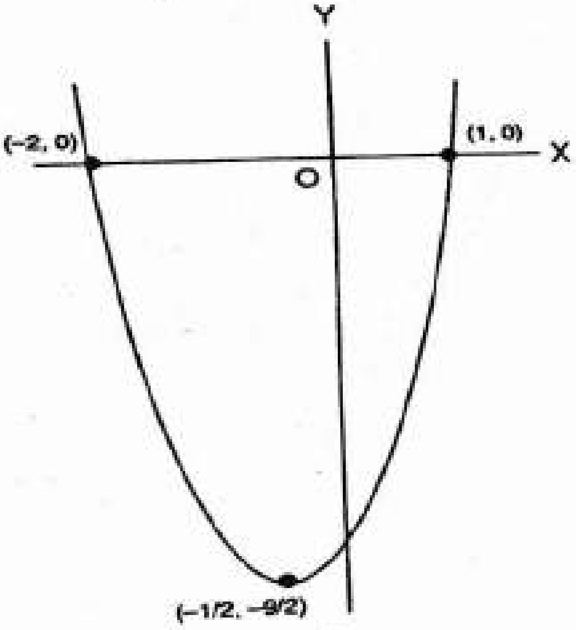
X

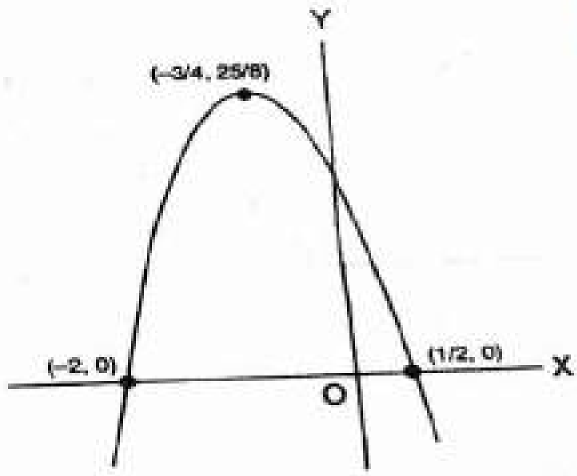
b)


Y
(1, 1)
(2, 0)
O
X

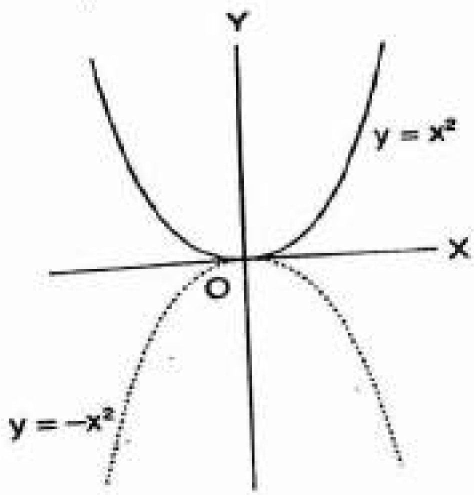
c)


Y
(−2, 0)
(1, 0)
O
X
(−1/2, −9/2)

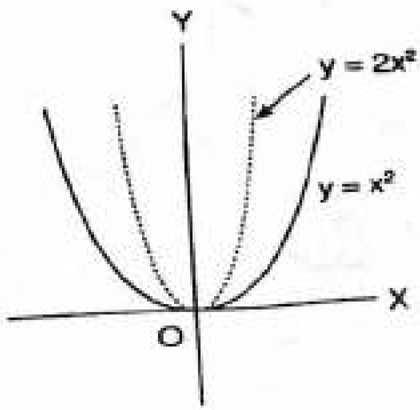
d)


Y
(−3/4, 25/8)
(−2, 0)
(1/2, 0)
O
X

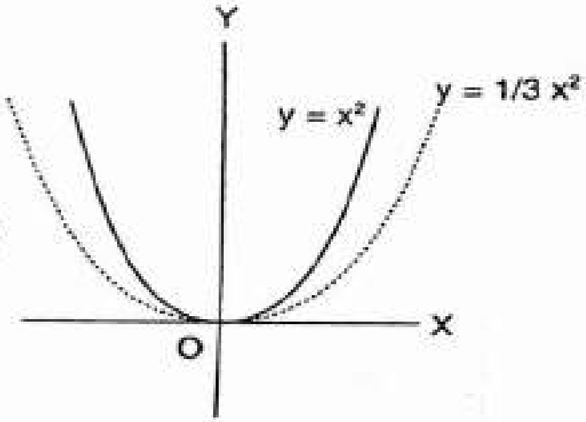
3. a)


Y
y = x²
X
O
y = −x²

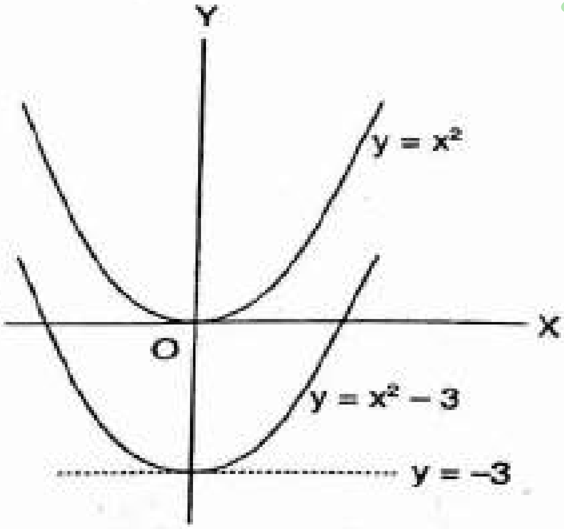
b)


Y
y = 2x²
y = x²
X
O

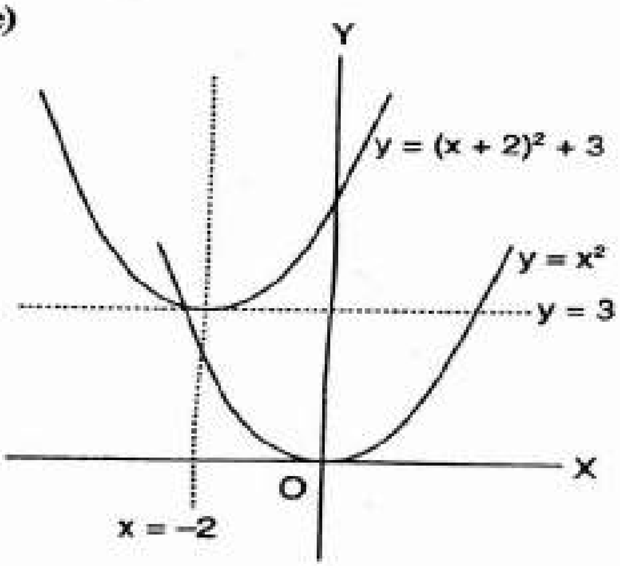
c)


Y
y = x²
y = 1/3 x²
X
O

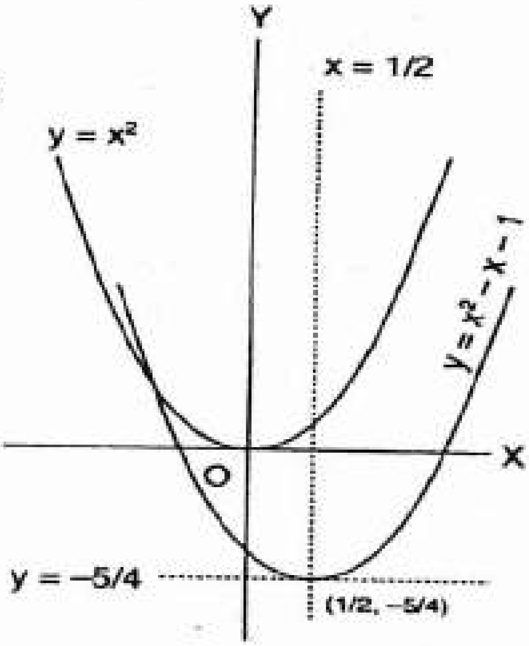
d)


Y
y = x²
X
O
y = x² – 3
y = –3

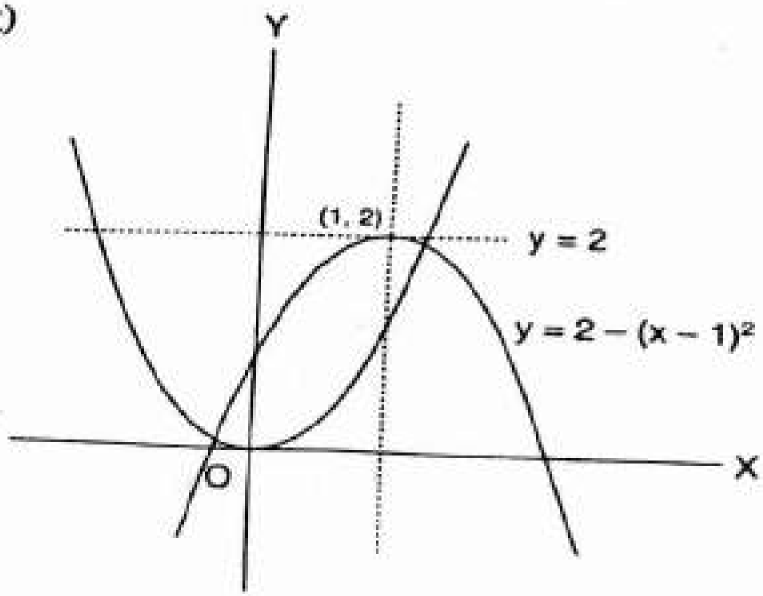
e)


Y
y = (x + 2)² + 3
y = x²
y = 3
X
O
x = –2

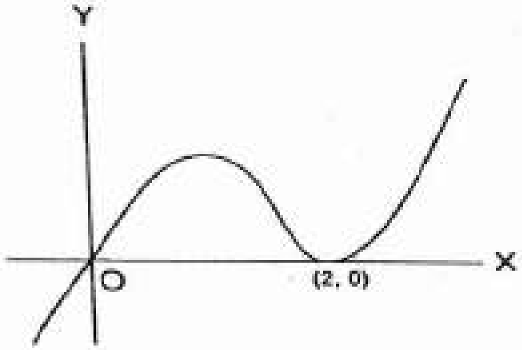
f)


Y
x = 1/2
y = x²
y = x² – x – 1
X
O
y = –5/4
(1/2, –5/4)

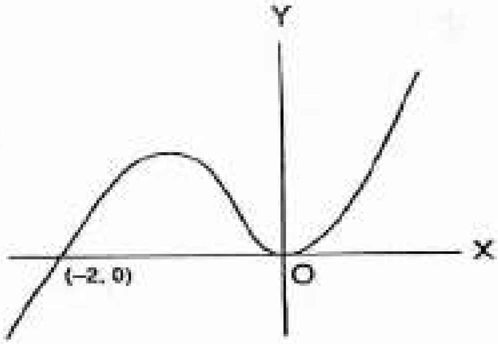
g)


Y
(1, 2)
y = 2
y = 2 – (x – 1)²
O
X

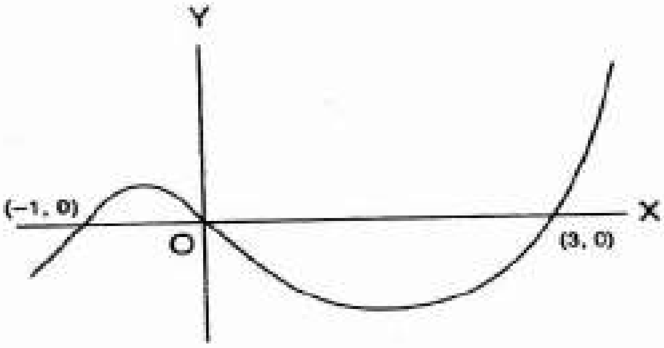
4. a)


Y
X
O
(2, 0)

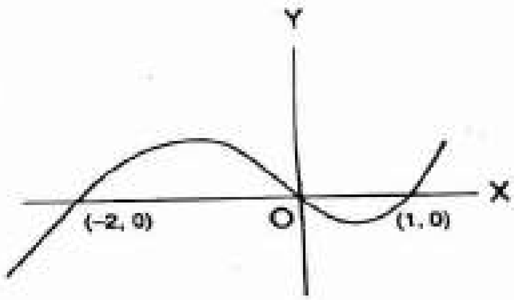
b)


Y
X
(–2, 0)
O

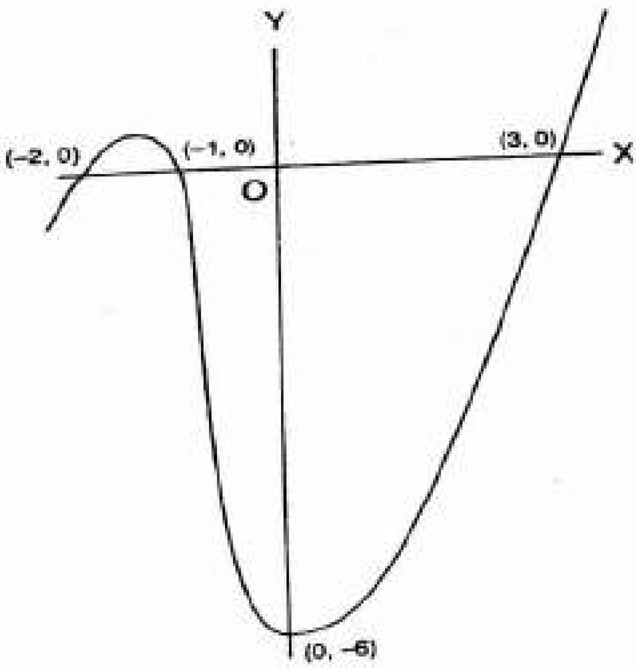
c)


Y
X
(–1, 0)
O
(3, 0)

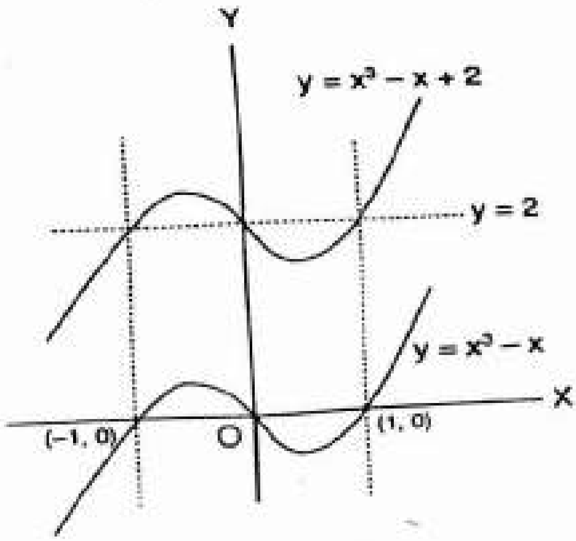
d)


Y
X
(–2, 0)
O
(1, 0)

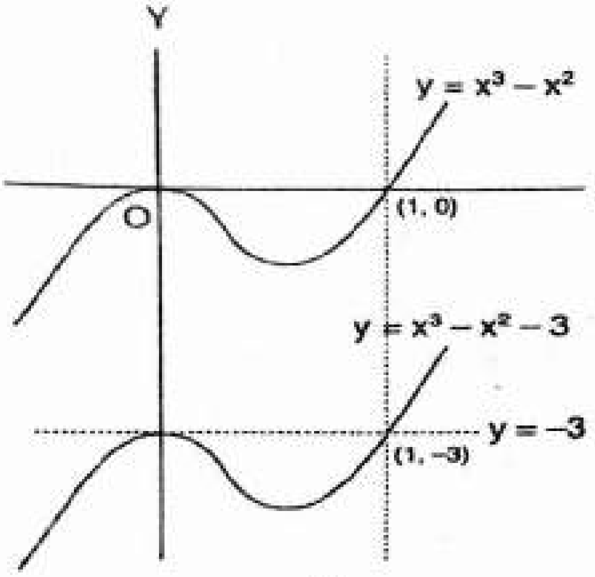
e)


Y
X
(–2, 0)
(–1, 0)
O
(3, 0)
(0, –6)

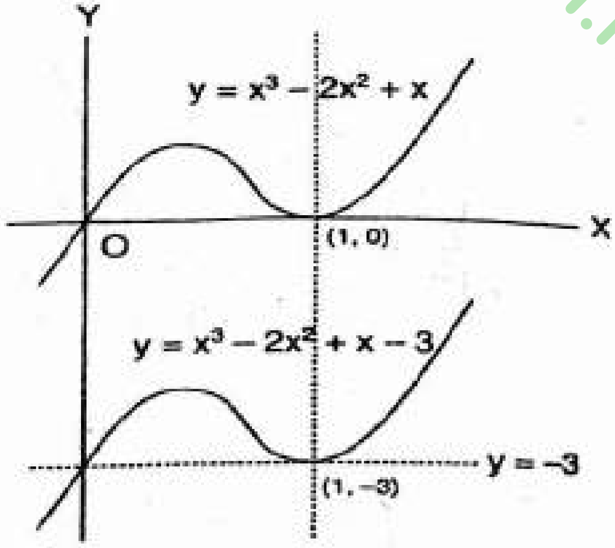
f)


Y
y = x³ – x + 2
y = 2
y = x³ – x
X
(–1, 0)
O
(1, 0)

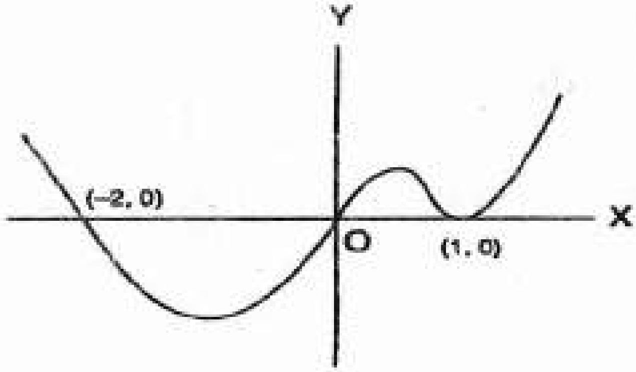
g)


Y
$y = x^3 - x^2$
X
O
(1, 0)
$y = x^3 - x^2 - 3$
$y = -3$
(1, −3)

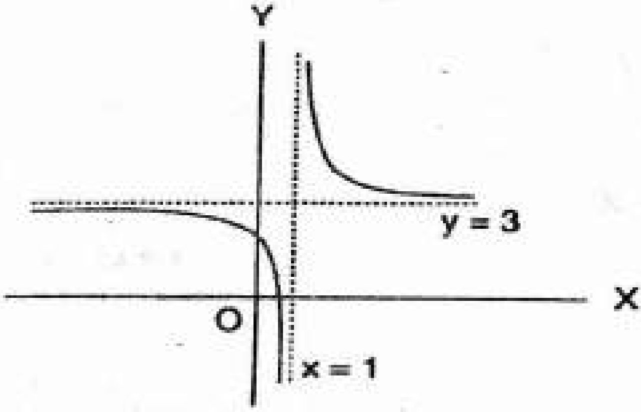
h)


Y
$y = x^3 - 2x^2 + x$
X
O
(1, 0)
$y = x^3 - 2x^2 + x - 3$
$y = -3$
(1, −3)

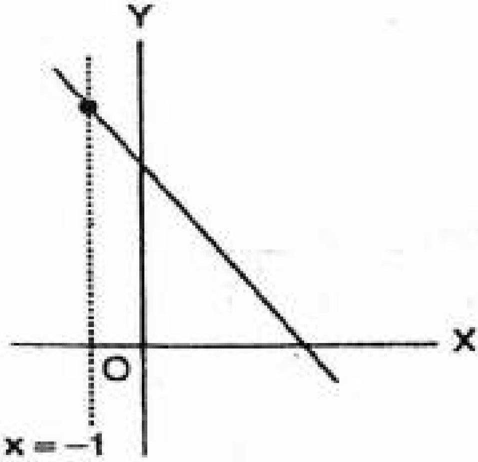
i)


Y
(−2, 0)
X
O
(1, 0)

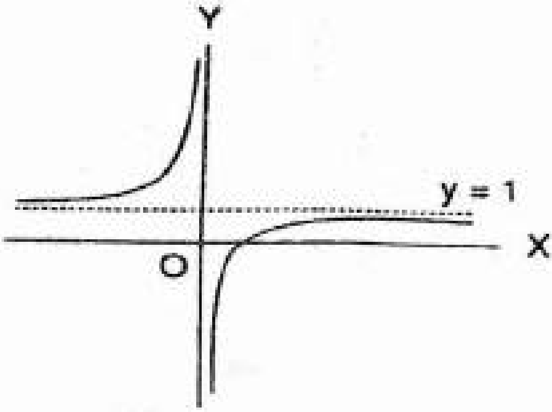
5. a)


Y
$y = 3$
X
O
$x = 1$

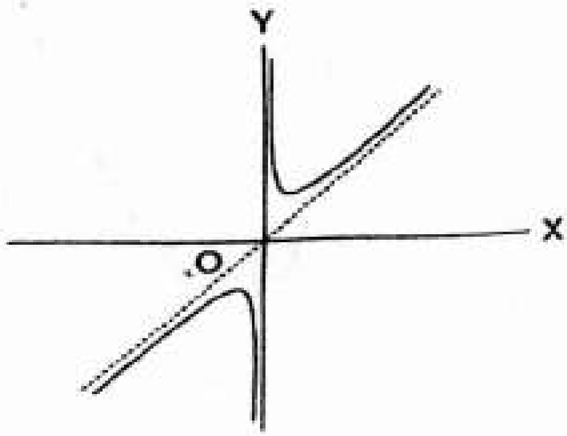
b)


Y
X
O
$x = -1$

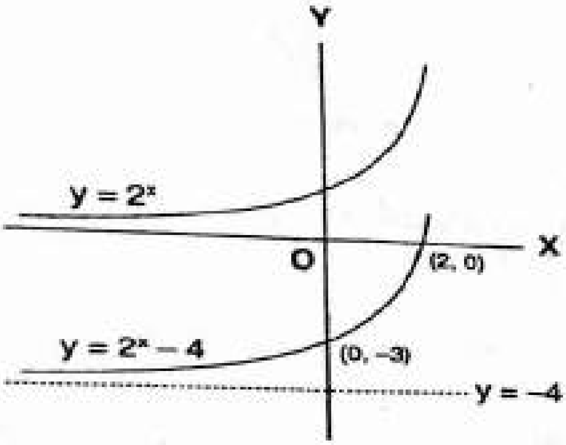
c)


Y
$y = 1$
X
O

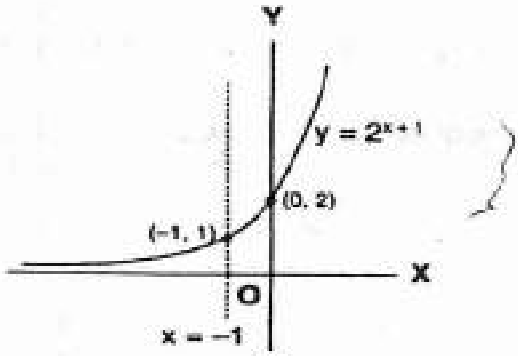
d)


Y
X
O

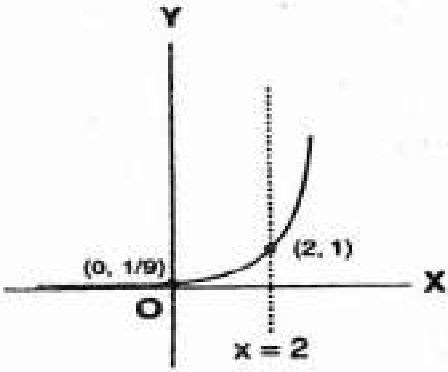
6. a)


Y
y = 2^x
O
(2, 0)
X
y = 2^x – 4
(0, –3)
y = –4

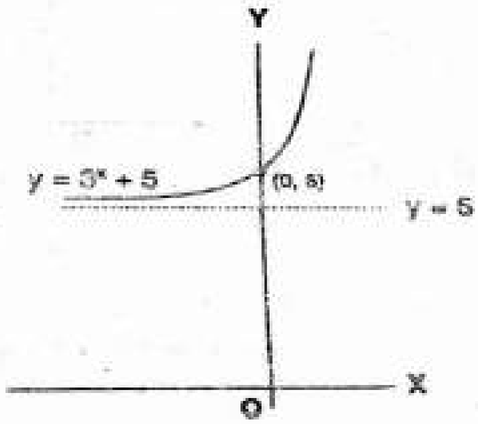
b)


Y
y = 2^{x+1}
(0, 2)
(–1, 1)
O
X
x = –1

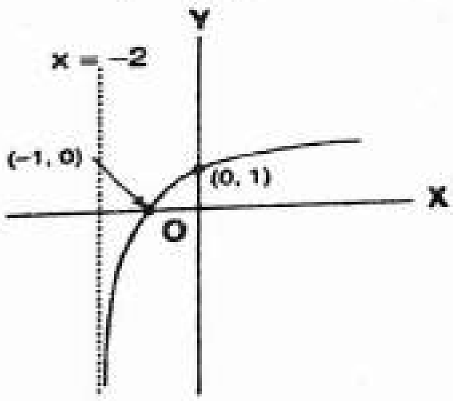
c)


Y
(2, 1)
(0, 1/9)
O
X
x = 2

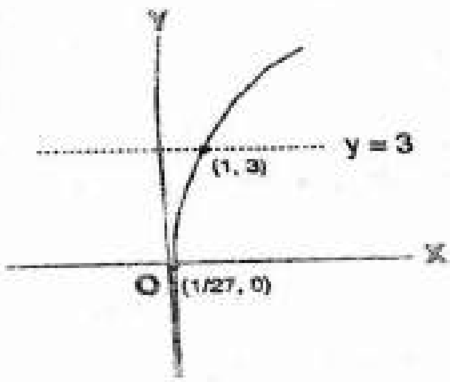
d)


Y
y = 3^x + 5
(0, 5)
y = 5
O
X

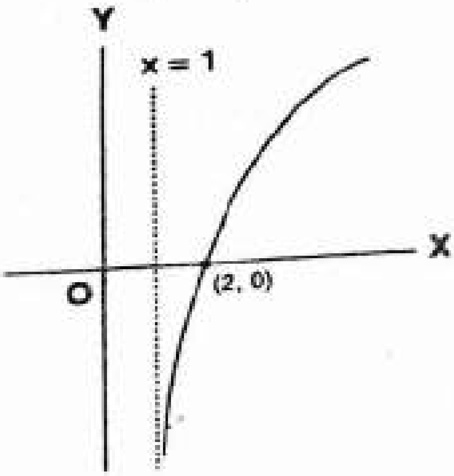
7. a)


x = –2
Y
(–1, 0)
(0, 1)
X
O

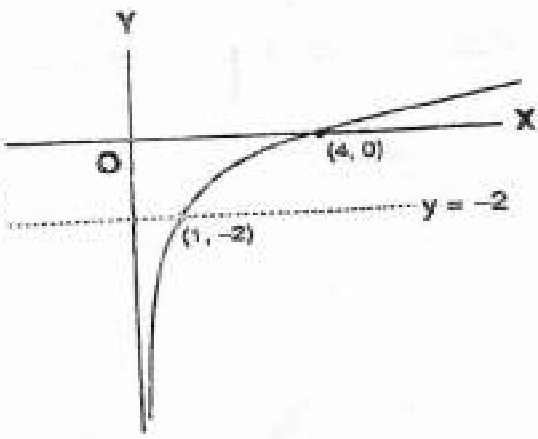
b)


Y
y = 3
(1, 3)
X
O
(1/27, 0)

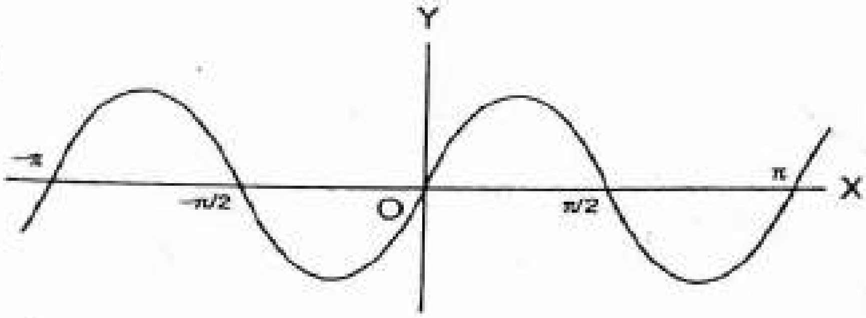
c)


Y
x = 1
X
O
(2, 0)

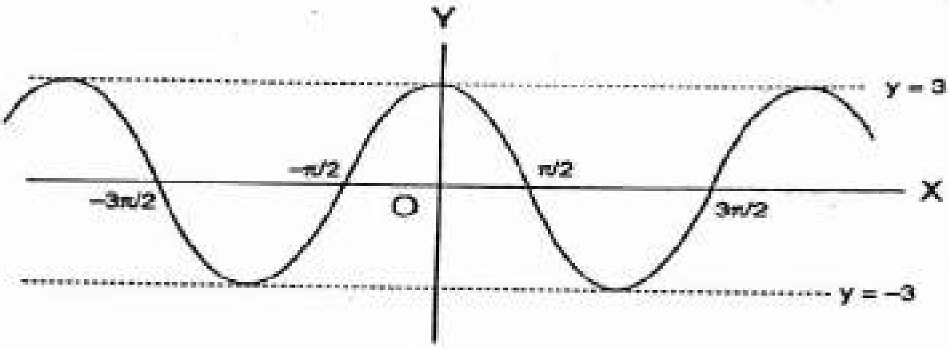
d)


Y
X
O
(4, 0)
y = –2
(1, –2)

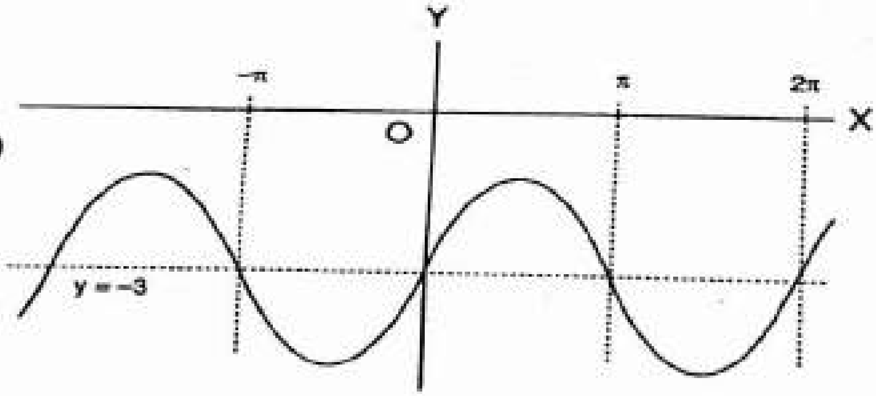
8. a)


Y
−π
−π/2
O
π/2
π
X

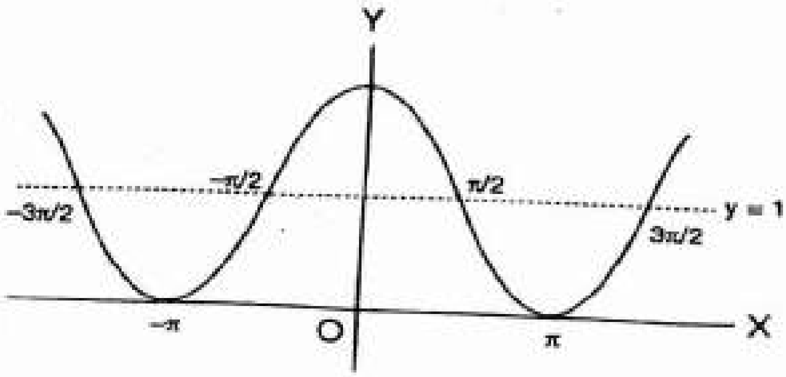
b)


Y
y = 3
−π/2
π/2
−3π/2
O
X
3π/2
y = −3

c)


Y
−π
π
2π
X
O
y = −3

d)


Y
−π/2
π/2
−3π/2
y = 1
3π/2
−π
O
π
X

9. a) 3

# Sequence and Series

1. If the $m$th term of an A.P. is $n$ and the $n$th term is $m$, show that the $p$th term is $m + n - p$.

2. If $p$th, $q$th and $r$th terms of an A.P. are $x$, $y$ and $z$ respectively, prove that $$x(q-r) + y(r-p) + z(p-q) = 0$$

3. If A, G and H are the arithmetic mean, geometric mean and harmonic mean between two unequal positive real numbers, prove that
   a) A, G, H are in G.P.
   b) A, G, H are in descending order

4. If $a$, $b$ and $c$ are the $p$th, $q$th and $r$th terms of a G.P., prove that
$$a^{q-r}.b^{r-p}.c^{p-q} = 1.$$

5. If the sum of $p$ terms of an A.P. is equal to the sum of $q$ terms, show that the sum of its $(p + q)$ terms is zero.

6. If the sums of $p$, $q$ and $r$ terms of an A.P. are $a$, $b$ and $c$ respectively, prove that
$$\frac{a}{p}(q-r) + \frac{b}{q}(r-p) + \frac{c}{r}(p-q) = 0$$

7. If S is the sum, P the product and R the sum of the reciprocals of $n$ terms of a G.P., prove that $P^2 = \left(\frac{S}{R}\right)^n$

8. If $a$, $b$, $c$ are in H.P., prove that $a^2 + c^2 > 2b^2$

9. Using geometric series, express each of the following as a rational number a) $0.\dot{3}$ b) $0.\overline{24}$ c) $0.1\overline{23}$

10. The sum of first two terms of an infinite G.S. is 12 and each term is equal to twice the sum of all terms following it. Find the G.S.

11. Find the $n$th terms and the sum of $n$ terms of the following series
   a) $2 + 5 + 10 + 17 + 26 + \ldots\ldots$
   b) $5 + 11 + 19 + 29 + 41 + \ldots\ldots$
   c) $5 + 7 + 13 + 31 + 85 + \ldots\ldots$

### *Answers*

9. i) $\frac{1}{3}$ ii) $\frac{8}{33}$ iii) $\frac{61}{495}$ 10. $9, 3, 1, \frac{1}{3}, \ldots\ldots$

11. a) $\frac{1}{6}n(2n^2 + 3n + 7)$ b) $\frac{1}{3}n(n+2)(n+4)$ c) $\frac{1}{2}(3^n + 8n - 1)$

# Mathematical Induction

1. Let a and d be the first term and the common difference of an A.P. Using the principle of mathematical induction prove that

$$a + (a + d) + (a + 2d) + \ldots\ldots + a(n - 1)d = \frac{n}{2}\{2a + (n - 1)d\}$$

2. If $a$ and $r$ be the first term and the common ratio of a G.P., prove by the method of induction that

$$a + ar + ar^2 + ar^3 + \ldots\ldots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$$

3. Using principle of mathematical induction prove that
   a) $n^2 > 2n$ for all $n \geq 3$.
   b) $(1 + x)^n \geq 1 + nx$ for $x > -1$
   c) $2^n \geq n^2$ for all $n \geq 4$

# Circle

1. Find the equation of a circle with centre on the x-axis, radius 5 and passing through the point (1, 4).
2. Find the equation of a circle lying in the first quadrant touching both axes and also the line $x = c$.
3. Find the equation of a circle concentric with the circle $x^2 + y^2 - 2x + 4y + 1 = 0$ and
   i) passing through the centre of the circle $x^2 + y^2 + 2x - 6y + 5 = 0$
   ii) touching the y-axis.
4. i) If the centre of the circle be at (2, 3) and the one end of the diameter has the coordinates (–1, –1), find the equation of the circle.
   ii) If the coordinates of one end of the diameter of the circle $x^2 + y^2 - 2x + 6y = 10$ be (2, –1), find the coordinates of the other end of the diameter.
5. Find the equation of the circle through the intersection of two circles $x^2 + y^2 - 3x - 5y + 18 = 0$ and $x^2 + y^2 + 5x - 3y - 6 = 0$ and passing through the origin.
6. Find the point of intersection of the line $x + y = 3$ and the circle $x^2 + y^2 - 2x - 3 = 0$. Also find the length of the intercept made by the line with the circle.

7. Find the equation of the tangent to the circle $x^2 + y^2 = 40$ at the point where (i) it meets the line $y - 2 = 0$ ii) the abscissa is 6.

8. Show that the locus of the point from which the length of the tangents to the circles $x^2 + y^2 - 3x + 4y - 7 = 0$ and $x^2 + y^2 - 2x - 5y + 1 = 0$ are equal, is a straight line perpendicular to the line joining the centres of the circles.

9. Find the equation of the chord of the circle $x^2 + y^2 = 9$ which is bisected at the point $(-1, 2)$.

**Answers**

1. $x^2 + y^2 - 8x - 9 = 0$, $x^2 + y^2 + 4x - 21 = 0$
2. $x^2 + y^2 - c(x + y) + \frac{c^2}{4} = 0$
3. i) $x^2 + y^2 - 2x + 4y - 24 = 0$ ii) $x^2 + y^2 - 2x + 4y + 4 = 0$
4. i) $x^2 + y^2 - 4x - 6y - 12 = 0$ ii) $(0, -5)$
5. $2(x^2 + y^2) + 6x - 7y = 0$
6. $(1, 2), (3, 0)$; $2\sqrt{2}$
7. i) $3x + y = 20$, $3x - y + 20 = 0$ ii) $3x + y = 20$, $3x - y = 20$
9. $2y = x + 5$

# HSEB Questions

## 2069 (2012)
### New Course

Full Marks : 100
Time : 3 hrs
Pass Marks : 35

**Attempt ALL questions.**

**Group A** $5 \times 3 \times 2 = 30$

1. a) Define negation of a statement. Construct a truth table for the compound statement $\sim(p \vee (\sim q))$.
   b) Find the domain of the function $y = \sqrt{x - 2}$
   c) Test the periodicity of the function $f(x) = \sin 2x$ and find its period.

2. a) Solve: $\sin x - \cos x = \sqrt{2}$
   b) Use the principle of mathematical induction:
   $$2 + 4 + 6 + \ldots\ldots + 2n = n(n + 1)$$
   c) If $A = \begin{pmatrix} 4 & -5 \\ 3 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$ find $(AB)^T$.

3. a) Using Cramer's rule, solve the following equations:
   $$x - 2y = -7$$
   $$3x + 7y = 5$$
   b) If $w$ be a complex cube root of unity, find the value of
   $$(1 - w + w^2)^4 (1 + w - w^2)^4$$
   c) For what values of $p$ will the equation $5x^2 - px + 45 = 0$ have equal roots?

4. a) Find the equation of a line through (5, 4) and perpendicular to the line $4x - 3y = 10$.
   b) Find the equation of the circle concentric with the circle $x^2 + y^2 - 8x + 12y + 15 = 0$ and passing through (5, 4).

c) Evaluate : $\lim_{x \to \infty} \left(\sqrt{x} - \sqrt{x-3}\right)$

5. a) Find $\frac{dy}{dx}$ if $x^3 + y^3 - 3axy = 0$

b) Evaluate : $\int \cot x (\log \sin x)^3 \, dx$

c) The side of a square sheet is increasing at the rate of 5 cm/min. At what rate is the area increasing when the side is 12 cm long?

## Group B

$5 \times 2 \times 4 = 40$

6. a) Define union and intersection of two sets. If A, B and C are any three non-empty sets, prove that:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Or

If $x \in R$ and $a$ is any positive real number, prove that :

$$|x| < a \Rightarrow -a < x < a \quad \text{and conversely.}$$

b) Draw the graph of the function $y = x^2 - 4x + 3$ using its different characteristics.

7. a) If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$, show that :

$$x^2 + y^2 + z^2 + 2xyz = 1$$

Or

State sine law. Prove that: $\tan \frac{1}{2}(B - C) = \frac{b-c}{b+c} \cot \frac{A}{2}$

b) Show that : $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ b+c & c+a & a+b \end{vmatrix} = (b-c)(c-a)(a-b)(a+b+c)$

8. a) Using row equivalent matrix method or inverse matrix method, solve the following equations

$$x - 2y - z = 7$$
$$2x + y + z = 0$$
$$3x - 5y + 8z = 13$$

b) Prove that a quadratic equation cannot have more than two roots.

9. a) Find the equation of the tangent to the circle $x^2 + y^2 - 2x - 4y - 4 = 0$ which are perpendicular to $3x - 4y = 1$

b) Evaluate : $\lim_{x \to \theta} \frac{x \cos \theta - \theta \cos x}{x - \theta}$

Or

Defind continuity of a function at a point. A function is defined as follows:

$$f(x) = \begin{cases} \dfrac{2x^2 - 18}{x - 3} & \text{for } x \neq 3 \\ k & \text{for } x = 3 \end{cases}$$

Find the value of $k$ so that $f(x)$ is continuous at $x = 3$.

10. a) Find from first principle the derivative of $\sqrt{2x + 3}$
    b) Find the area of the region between the curve $y^2 = 16x$ and the line $y = 2x$

## Group C

$5 \times 6 = 30$

11. Define one to one function and onto function. Let a function $f : A \to B$ be defined by $f(x) = \frac{x^2}{6}$ with $A = \{-2, -1, 0, 1, 2\}$ and $B = \{0, \frac{1}{6}, \frac{2}{3}\}$. Find the range of $f$, Is the function $f$ one to one and onto both?

12. Find the sum of n terms of the series $3.1^2 + 4.2^2 + 5.3^2 + \ldots\ldots$

13. If p and p' be the lengths of the perpendiculars from origin upon the straight lines whose equations are $x \cos \theta + y \operatorname{cosec} \theta = a$ and $x \cos \theta - y \sin \theta = a \cos 2\theta$ prove that : $4p^2 + p'^2 = a^2$

Or

Show that the homogeneous equation of degree two always represents a pair of straight lines passing through the origin. Also, find the angle between them.

14. If $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$, prove that $z_1z_2 = r_1r_2\{\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)\}$ and $\frac{z_1}{z_2} = \frac{r_1}{r_2} (\cos (\theta_1 - \theta_2) + i \sin (\theta_1 - \theta_2)$

Or

Define complex number. Express a complex number into polar form. State De-Moivre's theorem. Using De'Moivre's theorem, find the cube roots of unity.

15. What are the criteria for a function $y = f(x)$ to have the local maxima and local minima at a point ? Find the local maxima and local minima of the function $f(x) = 4x^3 - 6x^2 - 9x + 1$ on the interval $(-1, 2)$. Also find the point of inflection.

# 2070 (2013)

## New Course

Full Marks : 100
Time : 3 hrs

**Attempt ALL questions.**

Pass Marks : 35

### Group A

$5 \times 3 \times 2 = 30$

1. a) Prepare a truth table for the compound statement $p \vee \sim (p \wedge q)$. What would you conclude from the truth table.

   b) Let $A = \{1, 2, 3, 4\}$. Find the relation on A satisfying the condition $x + y \leq 4$.

   c) Examine whether the function $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ is even or odd. Also examine for its symmetricity.

2. a) Prove that: $\tan^{-1}a - \tan^{-1}c = \tan^{-1}\frac{a-b}{1+ab} + \tan^{-1}\frac{b-c}{1+bc}$

   b) Using principle of mathematical induction, prove that :
   $$1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2$$

   c) If $A = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$, find $AA^T$.

3. a) Using Cramer's rule, solve the following equations:
   $$3x - 2y = 8, \qquad 5x + 3y = 7$$

   b) If $\alpha = \frac{1}{2}(-1 + \sqrt{-3})$, $\beta = \frac{1}{2}(-1 - \sqrt{-3})$, show that
   $$\alpha^4 + \alpha^2\beta^2 + \beta^4 = 0$$

   c) If the equation $x^2 + 2(k + 2)x + 9k = 0$ has equal roots, find $k$.

4. a) Find the distance between the two parallel lines $3x + 5y = 11$ and $3x + 5y = -23$.

   b) Find the equation of the circle whose two of the diameters are $x + y = 6$ and $x + 2y = 8$ and radius 10.

   c) Evaluate : $\lim\limits_{x \to 4} \frac{x^3 - 64}{x^2 - 16}$

5. a) Find $\frac{dy}{dx}$ when $x = t + \frac{1}{t}$ and $y = t - \frac{1}{t}$

   b) Evaluate : $\int \frac{1}{\sqrt{2x+1} - \sqrt{2x-3}}\, dx$

   c) Find the interval in which the function $f(x) = 3x^2 - 6x + 5$ is increasing or decreasing.

## Group B

$5 \times 2 \times 4 = 40$

6. a) Define union and intersection of two sets. If A, B and C are any three non-empty sets, prove that:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Or

Let $A = [-3, 1]$ and $B = [-2, 4]$. Find $A \cup B$, $A \cap B$, $A - B$ and $B - A$.

b) Using different characteristics, sketch the graph of

$$y = -x^2 + 4x - 3$$

7. a) Solve : $\sin x + \cos x = \sqrt{2}\ (-2\pi \le x \le 2\pi)$

Or

State sine law. Using sine law, prove that:

$$\tan \frac{1}{2}(C - A) = \frac{c - a}{c + a} \cot \frac{B}{2}$$

b) Prove that : $\begin{vmatrix} a^2 & bc & c^2 + ac \\ a^2 + ab & b^2 & ac \\ ab & b^2 + bc & c^2 \end{vmatrix} = 4a^2b^2c^2$

8. a) Using row equivalent matrix method or inverse matrix method, solve the following equations.

$x + 4y + z = 18 \qquad 3x + 3y - 2z = 2 \qquad -4y + z = -7$

b) Find the condition under which the two quadratic equations $ax^2 + bx + c = 0$ and $a'x^2 + b'x + c' = 0$ may have one root common.

9. a) Find the equations of the tangent and normal to the circle $x^2 + y^2 - 3x + 10y - 5 = 0$ at the point $(4, -11)$.

b) Evaluate : $\lim\limits_{x \to \theta} \dfrac{x \cot \theta - \theta \cot x}{x - \theta}$

Or

A function $f(x)$ is defined as follows:

$$f(x) = \begin{cases} 3 + 2x & \text{for} \quad -\frac{3}{2} \le x < 0 \\ 3 - 2x & \text{for} \quad 0 \le x < \frac{3}{2} \\ -3 - 2x & \text{for} \quad x \ge \frac{3}{2} \end{cases}$$

Show that $f(x)$ is continuous at $x = 0$ but discontinuous at $x = \frac{3}{2}$.

10. a) Find from first principles the derivative of $\sqrt{2-3x}$

b) Find the area bounded by $y$-axis, the curve $x^2 = 4a(y-2a)$ and $y = 6a$.

## Group C

$5 \times 6 = 30$

11. Let a function $f : A \to B$ be defined by $f(x) = \frac{x+1}{2x-1}$. Find the range of $f$. Is the function $f$ one to one and onto both? If not, how can the function be made one to one and onto both?

12. Show that A.M., G.M. and H.M. between any two unequal positive numbers satisfy the following relations.

a) $(G.M.)^2 = A.M. \times H.M.$ b) A.M. > G.M. > H.M.

13. Find the length of the perpendicular drawn from the point $(x', y')$ on the line whose equation is $Ax + By + c = 0$

Or

Find the equation to the pair of lines joining the origin to the intersection of the straight line $y = mx + c$ and the curve $x^2 + y^2 = a^2$. Prove that they are at right angles if $2c^2 = a^2(1 + m^2)$

14. Find the square root of the complex number $-5 + 12i$.

15. List the criteria for the function $y = f(x)$ to have the local maxima and local minima at a point? Find the local maxima and local minima of the function $f(x) = 4x^3 - 15x^2 + 12x + 7$. Also, find the point of inflection.

Or

A spherical ball of salt is dissolving in water in such a way that the rate of decrease in volume at any instant is proportional to the surface. Prove that the radius is decreasing at the constant rate.

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 HIGHER SECONDARY LEVEL BASIC MATHEMATICS : GRADE XI



# 2071 (2014)

## New Course

*Candidates are required to give their answer in their own words as far as practicable. The figures in the margin indicate full marks.*

Full marks: 100 Pass marks: 35

Time 3 hrs

**Attempt ALL questions.**

### Group A

$5\times3\times2 = 30$

1. a) Define a statement. If $p$ is true, $q$ is true and r is true, find the truth value of $(p \vee q) \wedge (\sim q)$.

   b) For any two real numbers $x$ and $y$, prove that: $|x + y| \leq |x| + |y|$

   c) Determine whether the function $f(x) = \sqrt{x^2 - 1}$ is even or odd. Also test its symmetricity.

2. a) Solve for $x$: $\tan 2x = \tan x$.

   b) Prove by mathematical induction:
   $1 + 3 + 5 + \ldots\ldots + n$ terms $= n^2$.

   c) Define singular matrix. Test whether the matrix
   $A = \begin{pmatrix} 3 & 1 & 0 \\ -2 & 1 & -1 \\ -1 & 3 & -2 \end{pmatrix}$ is singular or not.

3. a) Solve by inverse matrix method:
   $2x + y = 7, \qquad x + 3y = 11$

   b) Prove that the modulus of a complex number and its conjugate are equal.

   c) Find a quadratic equation whose roots are twice the roots of
   $4x^2 + 8x - 5 = 0$.

4. a. Find the equation of the line through (4, 2) which is parallel to $x - 2y - 4 = 0$.

   b. Find the equation of the circle which touches the axes at (2, 0) and (0, 2).

   c. Evaluate $\lim_{x \to \infty} (\sqrt{x+a} - \sqrt{x})$

5. a. Find $\frac{dy}{dx}$ when $x + y = \sin(x + y)$.

b. Evaluate $\int \frac{\cos x - \sin x}{\cos x + \sin x} dx$.

c. Find the intervals in which $f(x) = x^2 - 2x + 10$ is increasing or decreasing.

## Group B

$5 \times 2 \times 4 =$

6. a. For any three non-empty sets A, B, C prove that: $(A - B) - C = A - (B \cup C)$.

Or

b. Define absolute value of a real number. Also, a is a positive real number and $x \in R$ then prove that:

$|x| < a$ if and only if $-a < x < a$.

c. Sketch the graph of $y = -x^2 + 4x - 3$ indicating its characteristics.

7. a. If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi/2$, show that: $xy + yz + zx = 1$.

Or

State sine law. Use this law to prove the projection law.

b. Without expanding the determinant show that:

$$\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \begin{vmatrix} a^2 & bc & a \\ b^2 & ca & b \\ c^2 & ab & c \end{vmatrix}$$

8. a. Using Cramer's rule or row-equivalent method solve the following system:

$x + y + z = 6,\ 2x + 3y + 5z = 23,\ 7x + 5y - 2z = 11.$

b. If the roots of the equation $x^2 + px + q = 0$ are in the same ratio as those of the equation $x^2 + lx + m = 0$, show that $p^2m = l^2q$.

9. a. Prove that the tangent to the circle $x^2 + y^2 = 5$ at the point $(1, -2)$ also touches the circle $x^2 + y^2 - 8x + 6y + 20 = 0$.

b. Prove geometrically: $\lim_{\theta \to 0} \sin\theta = 0$.

Or

Show that the function $f(x) = \begin{cases} x, \text{ when } 0 \le x < 1/2 \\ 1, \text{ when } x = 1/2 \\ 1 - x, \text{ when } 1/2 < x < 1 \end{cases}$

is discontinuous at $x = \frac{1}{2}$,

Also, redefine $f(x)$ so as to $f(x)$ be continuous at $x = \frac{1}{2}$.

10. a. Find, from definition, the derivative of $\sqrt{\frac{1}{1-x}}$.

b. Using integration, find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

## Group C

5×6 = 30

11. Define inverse function. If f: R → R be defined by $f(x) = x^2 - 3$, Find $f^{-1}(x)$. Also determine whether $fof^{-1}(x) = f^{-1}of(x)$.

12. Sum to n terms of the series: $\frac{2}{5} - \frac{6}{5^2} + \frac{10}{5^3} - \frac{14}{5^4} + \ldots$

13. Find the angle between the pair of lines represented by a homogeneous equation of second degree. Also derive the condition of parallelism and perpendicularity of the lines. Find the angle between the lines represented by $x^2 + 9xy + 14y^2 = 0$.

Or

Find the bisectors of the angles between the lines $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$. Also determine the condition that the bisector of the angle in which the origin lies.

14. State De Moivre's theorem. Using De Moivre's theorem find the square roots of $4 + 4\sqrt{3}\,i$.

15. A window is in the form of a rectangle surmounted by a semi circle. If the total perimeter is 9 m, find the radius of the semicircle for the greatest windows' area.

Or

Water flows into an inverted conical tank at the rate of 42 $cm^3$/sec. When the depth of the water is 8 cm, how fast is the level rising? Assume that the height of the tank is 12 cm and the radius of the top is 6 cm.

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