Day-37, Dec-26, 2024 (Poush 11, 2081 B.S.)

Second Order Partial Derivatives

Let $z = f(x,y)$ be a function of two variables x and y. Then $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are first order partial derivatives (being the forms involve derivative of first time). If we again produce partial derivatives in $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ then the differential forms are known as second order partial derivative of $f(x,y)$.

The following process help to understand the form(s);

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial \cdot \partial f}{\partial \cdot x \partial x}.$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow (f_x)_x \quad \Rightarrow f_{xx}$$

$\frac{\partial^2 f}{\partial y^2} = \frac{\partial . \partial f}{\partial y . \partial y}$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$\Rightarrow (f_y)_y$$

$$\boxed{\Rightarrow f_{yy}}$$

$\frac{\partial^2 f}{\partial x \, \partial y} = \frac{\partial . \partial f}{\partial x . \partial y} \Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

$$\boxed{\# \quad \frac{\partial^2 f}{\partial x \cdot \partial y} \Rightarrow \left(f_y\right)_x \Rightarrow f_{yx}.}$$

$$\# \quad \frac{\partial^2 f}{\partial y \cdot \partial x} = \frac{\partial \cdot \partial f}{\partial y \cdot \partial x}$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \left(f_x\right)_y \qquad \boxed{\frac{\partial^2 f}{\partial y . \partial x} \Rightarrow f_{xy}}$$

Second Order Partial Derivative of a function:

The differentiation partially of $f(x,y)$ twice, the derivatives are known as second order partial derivatives of $f(x,y)$ and it is usually denoted by $f_{xx}, f_{xy}, f_{yy}, f_{yx}$.

Example: If $f(x,y) = x\cos y + ye^x$ then produce second order partial derivatives of $f(x,y)$.

Soln:

$$f(x,y) = x\cos y + ye^{x}$$

Then, $$\frac{\partial f}{\partial x} = \cos y + y e^{x}$$

and $$\frac{\partial f}{\partial y} = -x\sin y + e^{x}$$

Also, $$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \Rightarrow y e^{x}.$$

$$\frac{\partial^2 f}{\partial y^2} \Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \Rightarrow -x\cos y.$$

$$\frac{\partial^2 f}{\partial x \cdot \partial y} \Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \Rightarrow -\sin y + e^x.$$

$$\frac{\partial^2 f}{\partial y \cdot \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \Rightarrow -\sin y + e^x.$$

Convergence.

$$\left[\begin{array}{l} f_{xx} \Rightarrow ye^x \\ f_{yy} \Rightarrow -x\cos y \end{array}\right] \quad \left[\begin{array}{l} f_{xy} = -\sin y + e^x. \\ f_{yx} = -\sin y + e^x. \end{array}\right]$$

Euler's Theorem:

In the above examples, we observe the second order partial derivatives are f_{xx}, f_{xy}, f_{yy}, f_{yx}. Among them

f_{xy}, f_{yx} are mixed second order partial derivatives

The above examples, we observe the second order partial derivatives f_{xy} and f_{yx} are equal. However, this will not occur forever. If f, f_x, f_y, f_{xy}, f_{yx} are exist and continuous then the fixed derivatives are always equal.

Euler's Theorem (The Mixed Derivative Theorem):

If $f(x,y)$ and its partial derivatives f_x, f_y, f_{xy} and f_{yx} are defined on an open region containing a point (a,b) and are all continuous at (a,b) then

$$\therefore f_{xy}(a,b) = f_{yx}(a,b)$$

Verify Euler's theorem for the function

$$f(x,y) = x^3 + x^2y^3 - 2y^2$$

Soln:

Let $f(x,y) = x^3 + x^2y^3 - 2y^2$

Then, $f_x = 3x^2 + 2xy^3$ and $f_y = x^2 3y^2 - 4y$

$f_{xy} = (f_x)_y$

$\Rightarrow 2x\, 3y^2$

$\Rightarrow 6xy^2$

$f_{yx} = (f_y)_x$

$\Rightarrow 6xy^2$

This means the function $f(x,y)$ verifies the Euler's theorem.

Example: Verify the Euler's theorem for mixed partial derivatives

$w = x\sin y + y\sin x + xy.$

Let $w = x\sin y + y\sin x + xy$

Then,
$$w_x = \sin y + y\cos x + y$$
and $$w_y = x\cos y + \sin x + x$$

And,
$$w_{xy} = (w_x)_y \Rightarrow \cos y + \cos x + 1$$
$$w_{yx} = (w_y)_x \Rightarrow \cos y + \cos x + 1$$

$\therefore$ Hence, this also verifies $w_{xy} = w_{yx}$ for the mixed partial derivatives.

Total Differential:

Let $w=f(x,y)$ be a function and its first order partial differential at a point (x_0,y_0) are $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ exist. Then the total differential of its,

$$\boxed{df = f_x(x_0,y_0)\,dx + f_y(x_0,y_0)\,dy.}$$

The Chain Rule:

Definition (The chain Rule of Two Independent Variables):

If $w = f(x,y)$ is differentiable and $x = x(t)$ and $y = y(t)$ are differentiable functions of 't' then the derivative of 'w' with respect to 't' is,

$$\frac{dw}{dt} = \frac{d f(x,y)}{dt}$$

$$\Rightarrow \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$$

If $y = f(x)$ and $x = g(t)$ are differentiable functions of t then,

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

Definition (The Chain Rule of Three Independent Variables):

If $w = f(x, y, z)$ is differentiable and $x = x(t)$, $y = y(t)$ and $z = z(t)$ are differentiable function of 't' then the derivative of 'w' with respect to 't' is –

$$\frac{\partial w}{dt} = \frac{d\, f(x, y, z)}{dt}$$

$$\Rightarrow \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}$$

Definition (The Chain Rule of two and three Independent Variables of Two variables):

If $w = f(x, y)$ where $x = g(r, s)$, $y = h(r, s)$ are differentiable functions then—

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r}$$

and

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s}$$

If $w = f(x, y, z)$ where $x = g(r, s)$, $y = h(r, s)$, $z = i(r, s)$ are differentiable functions then,

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r}$$

and,

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial s}$$