Day-49, Jan-18, 2025 (Magh 05, 2081 B.S.)

Convergence and Divergence of Infinite Sequence:

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \left(\frac{1}{n}\right) \to 0$$

Thus, $0 \le a_n \le 1$ for all n.

We observe that the sequence approaches to a single value 0 as 'n' tends to infinity. Such sequence is known as a convergent sequence.

$a_n = 2^n$ is bounded below but is unbounded above.

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} (2^n) \Rightarrow \infty.$$

This means it has no fixed greatest value

$\therefore$ The sequence $\{a_n\}$ is bounded below by 2 but not bounded above.

The sequence has no fixed value as $n \to \infty$

Such sequence is a divergent sequence.

Definition: Let $\{a_n\}$ be an infinite sequence. The sequence $\{a_n\}$ converges to a number L if for every positive number $\varepsilon > 0$ there corresponds an integer N such that for all n,

$$|a_n - L| < \varepsilon \quad \text{for } n > N.$$

Mathematically, the sequence $\{a_n\}$ converges to a number L. If $\forall\ \varepsilon > 0\ \exists\ N \in \mathbb{N}$ s.t.

$$|a_n - L| < \varepsilon \quad \forall\ n > N$$

In such condition, we may observe

$$\lim_{n \to \infty} a_n = L$$

Hence, the number L is called the limit of the sequence $\{a_n\}$

Definite: An Infinite Sequence $\{a_n\}$ of real numbers is called divergent if $\lim\limits_{n \to \infty} a_n$ has no fixed finite.

Example: Show that the sequence $\{1 + \frac{1}{n}\}$ Converges to 1.

Sol^n: Let $\epsilon > 0$ be given. Choose $N = \frac{1}{\epsilon}$.

Now, $\forall \ n > N = \frac{1}{\epsilon}$

$$\left|\left(1 + \frac{1}{n}\right) - 1\right| = \left|\frac{1}{n}\right| \Rightarrow \frac{1}{n} < N$$

$$\Rightarrow \frac{1}{\epsilon}$$

$[\because n > N \cdot$ so $1/n < N]$

This means the sequence $\{1+\frac{1}{n}\}$ converges to 1.

Alternatively,

Here $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right) = 1+0$

$= 1$

This means the sequence $\{1+\frac{1}{n}\}$ converges to 1.

Q. Prove that the sequence $\{u_n\} = \{(-1)^n\}$ diverges.

Sol^n: Let $\epsilon > 0$ be given. Choose $N = \epsilon = 1$. Take $L = 1$.

$\forall n > N = 1$ with n is even,

$$|a_n - L| = |(-1)^n - 1|$$

$$\Rightarrow |1 - 1|$$

$$\Rightarrow 0 < \varepsilon$$

But if 'n' is odd,

$$|a_n - L| = |(-1)^n - 1|$$

$$\Rightarrow |-1 - 1|$$

$$\Rightarrow |-2| \Rightarrow 2 > \varepsilon$$

This shows, the sequence $\{a_n\}$ has no fixed single value α such that $|a_n - \alpha| < \varepsilon$ for $\varepsilon > 0$. Therefore, the sequence is divergent.

Theorem: If $\lim_{n\to\infty} a_n = \alpha$ and the function f is continuous at α then $\lim_{n\to\infty} f(a_n) = f(\alpha)$.

Definition (Least Upper Bound):

Let $\{a_n\}$ be an infinite sequence. If there is a value M such that

$$a_n \leq M \quad \text{for all } n \text{ ---- eqn (i)}$$

and if there is no value less than M satisfies the relation (i) then M is called Least Upper bound of the sequence $\{a_n\}$.

Definition (Greatest Lower Bound):

Let $\{a_n\}$ be an infinite sequence. If there is a value Q such that

$$a_n \geq Q \quad \text{for all } n \text{ ——— eqn (i)}$$

and if there is no value greater than Q satisfies the relation (i) then M is called Least Lower Bound of the Sequence $\{a_n\}$.

Example:

a) The sequence $1, 2, 3, 4, \ldots, n, \ldots$ has no upper bound that but it has 1 as a greatest lower bound.

b) The sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$ has upper bound any value among $1, 2, 3, \ldots$ but the least upper bound of the sequence is 1.

Theorem (Non-Decreasing Sequence Theorem):

A non-decreasing sequence of real numbers converges if and only if it is bounded from above if a non-decreasing sequence converges, it converges to its least upper bound.

Infinite Series:

Given a sequence of numbers $\{a_n\}$ an expression,

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n$$

is called Infinite Series.

The summing form of infinite sequence as an infinite series.
The number a_n is known as n^{th} term (or general form) of the series.

Partial Sum:

Given a series with finite K-terms in the form $\sum_{n=1}^{K} a_n$ is called K^{th} partial sum of the series $\sum_{n=1}^{\infty} a_n$ and denoted by S_k i.e. $S_k = \sum_{i=1}^{k} a_i$.

Definition (Convergence and Divergence of an Infinite Series)

Let $\sum_{n=1}^{\infty} a_n$ be an infinite series. If there is a finite value L such that

$$\lim_{n\to\infty} \sum_{i=1}^{n} a_i = L \quad \text{i.e.} \quad \lim_{n\to\infty} S_n = L.$$

Then we say the given series is convergent to L.

Otherwise, the series is divergent.

Definition (Telescoping Series)

A Series in which on expansion of nth partial Sum, every term except first and last term is cancelled out is called telescoping series.

Example The series $\sum_{n=1}^{\infty} \left(\frac{1}{n(n+1)} \right)$ is convergent and its Sum is 1.

[Geometric Series · Definition (Geometric Series)

A Series of the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

Where a is the non-zero first term r is fixed ratio a such series is known as geometric series.

Power Series, Taylor and Maclaurin's Series

Definition (power series):

A power series about $x=0$ is a series of the form

$$\sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n + \cdots$$

A power series about $x=a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

in which is a centre and the coefficients $c_0, c_1, c_2 \ldots c_3 \ldots c_n \ldots$ all are constants.

Interval, Centre and Radius of Convergence of a Power Series.

Consider a power series,

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

if there exists a positive number r such that the series converges for $|x| < r$ and diverges for $|x| > r$, then $(-r, r)$ is called the interval, $\frac{r + (-r)}{2} = 0$ is center and r is called the radius of convergence.

<u>Note</u>: If there exists a positive number r such that the series converges for $|x| < r$ and the diverges for $|x| > r$ then $(-r, r)$ is called the interval, $\frac{r + (-r)}{2} = 0$ is center and r is called the radius of convergence.

<u>Remarks</u>: If r is the radius of convergence of the above power series, the interval of convergence (or region of convergence)

is any one of the following intervals:

a) $(-r, r)$ b) $(-r, r]$ c) $[-r, r)$ d) $[-r, r]$.

Theorem (The Convergence Theorem for Power Series)

If the series,

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

converges for $x = a \neq 0$ then it converges absolutely for all $|x| < |a|$. If the series diverges for $x = b$ then it diverges for all $|x| > |b|$.

Theorem (The term-by-term Differentiation Theorem)

If $\sum c_n (x-a)^n$ converges for $(a-R) < x < (a+R)$ for some $R > 0$, and define

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad \text{for } (a-R) < x < (a+R)$$

Such a function f has derivatives of all orders inside the interval of convergence — By differentiating the series term by term then we obtain the derivatives,

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2}$$ and so on.

Each of these converges at every interior point of the interval of convergence of the original series.

Theorem(The term-by-term Integration Theorem):

Suppose that,

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

Converges for $(a-R) < x < (a+R)$ (for $R > 0$). Then,

$$\sum_{n=0}^{\infty} C_n \left(\frac{(x-a)^{n+1}}{n+1} \right)$$

Converges for $(a-R) < x < (a+R)$ and

$$\int f(x) \cdot dx = \sum_{n=0}^{\infty} C_n \left(\frac{(x-a)^{n+1}}{n+1} \right) + C$$

for $(a-R) < x < (a+R)$

17 $f''(x) = 0 + 0 + 2 + 3 \cdot 2 x^2 + \ldots + n(n-1) x^{n-2} + \ldots = \sum_{n=2}^{\infty} n(n-1) x^{n-2}$

Series Representations

If we differentiate infinite times to a power series having centre at a, the series of all such differentiation is known as Taylor Series.

As a particular form of Taylor Series with centre at origin is popularly known as Maclaurin Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

and the Maclaurin Series generated by f is keep $a=0$.