Day-15, 30 Nov, 2024 (Mangshir 15, 2081 B.S)

Remarks:

1) If for a given function $f(x)$ and $x = a$ one of the following

i) $\lim_{x \to a} f(x) = \infty$ or $-\infty$

ii) $\lim_{x \to a^-} f(x) = \infty$ or $-\infty$

iii) $\lim_{x \to a^+} f(x) = \infty$ or $-\infty$

is true then the limit is called Infinite limit and the line $x = a$ is called vertical asymptote.

2) If for a given function $f(x)$ and if either $\lim_{x \to -\infty} f(x) = L$ or $\lim_{x \to \infty} f(x) = L_1$

then the lines $y = L$ or $y = L_1$ are called the horizontal asymptote curve.
If both cases hold good then both $y = L$ and $y = L_1$ are called limits at infinity and the lines $y = L$ and $y = L_1$ are called Horizontal Asymptotes of the curves

3) Infinite limits claim vertical asymptotes and limits at infinity claim horizontal asymptote.

Example: find $\lim\limits_{x \to \infty} \frac{1}{x}$ and $\lim\limits_{x \to -\infty} \frac{1}{x}$

Here we can observe that when x is larger then $\frac{1}{x}$ becomes Smaller. for instance $\frac{1}{100} \Rightarrow 0.01$, $\frac{1}{10{,}000} \Rightarrow 0.0001$, $\frac{1}{1{,}00{,}000} \Rightarrow 0.00001$

In fact, by taking 'x' large enough, we can make $\frac{1}{x}$ as close to '0' as we please. Therefore we have

$$\lim_{x \to \infty} \frac{1}{x} = 0.$$

Similar Reasoning shows that when x is large in magnitude but negative in sign then $\frac{1}{x}$ is small in magnitude but negative in sign i.e.

$$\left[\lim_{x \to -\infty} \frac{1}{x} = 0 \right]$$

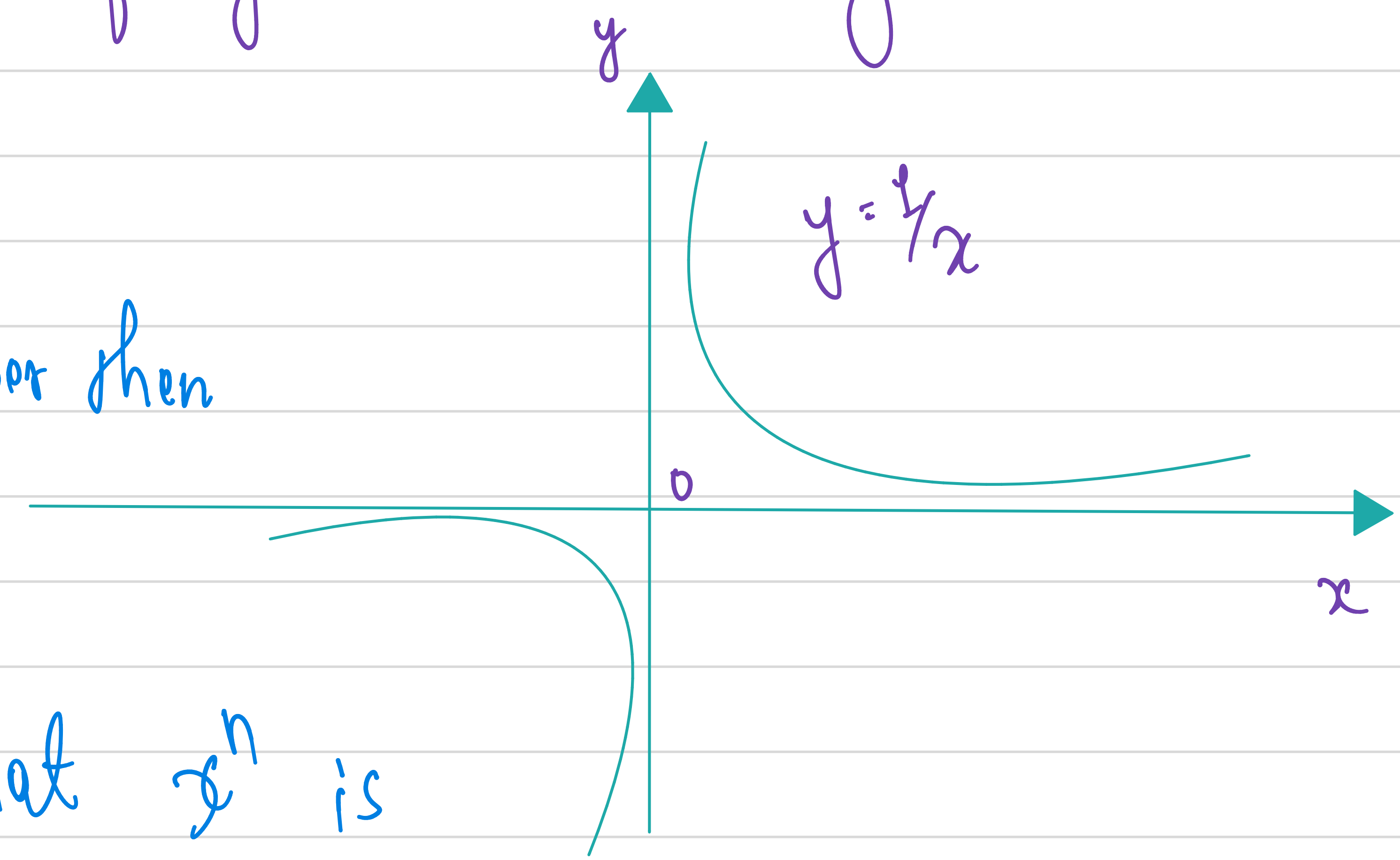
It follows that the line $y = 0$ (x-axis) is a horizontal asymptote of the curve $y = \frac{1}{x}$ (which is equilateral hyparabola).

Again in this problem we can observe that for

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x \to 0^-} -\infty$$

Hence $x = 0$ is the vertical asymptote.

Moreover $y = 0$ is called the limit at infinity and $x = 0$ is gives the infinite limit for the given function.



Theorem: If $r > 0$ is a rational number then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0$$

if $r > 0$ is a rational number such that x^r is defined for all x, then $\lim_{x \to -\infty} \frac{1}{x^r} = 0$

Example: Find the horizontal and vertical asymptotes of the graph of the function $f(x) = \dfrac{\sqrt{2x^2+1}}{3x-5}$

Here,

$$\lim_{x \to \infty} \frac{\sqrt{2x^2+1}}{3x-5}$$

$$\Rightarrow \lim_{x \to \infty} \frac{x\sqrt{2+\frac{1}{x^2}}}{x\left(3-\frac{5}{x}\right)}$$

$$\Rightarrow \frac{\sqrt{2}}{3}$$

$\therefore y = \dfrac{\sqrt{2}}{3}$ is the limit at infinity and a horizontal asymptote of the curve.

In computing the limit of $x \to \infty$, we must remember for $x < 0$, we have $\sqrt{x^2} = |x| = -x$. So, when we divide the numerator by x, for $x < 0$, we get

$$\frac{1}{x}\sqrt{2x^2+1} \Rightarrow -\frac{1}{\sqrt{x^2}}\sqrt{2x^2+1}$$

$$\Rightarrow -\sqrt{2+\frac{1}{x^2}}$$

$$\therefore \lim_{x \to -\infty} \frac{\sqrt{2x^2+1}}{3x-5} \Rightarrow \lim_{x \to \infty} \frac{-\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}}$$

$$\Rightarrow \frac{-\sqrt{2}}{3}$$

$\therefore$ Thus, the line $y = -\frac{\sqrt{2}}{3}$ is also limit at infinity and a horizontal asymptote of the curve.

for the vertical asymptote we can test at the point for which the denominator is zero.

for $x = 5/3$. If x is close to $5/3$ and $x > \frac{5}{3}$ then the dominator is close to 0 and positive. The numerator is positive.

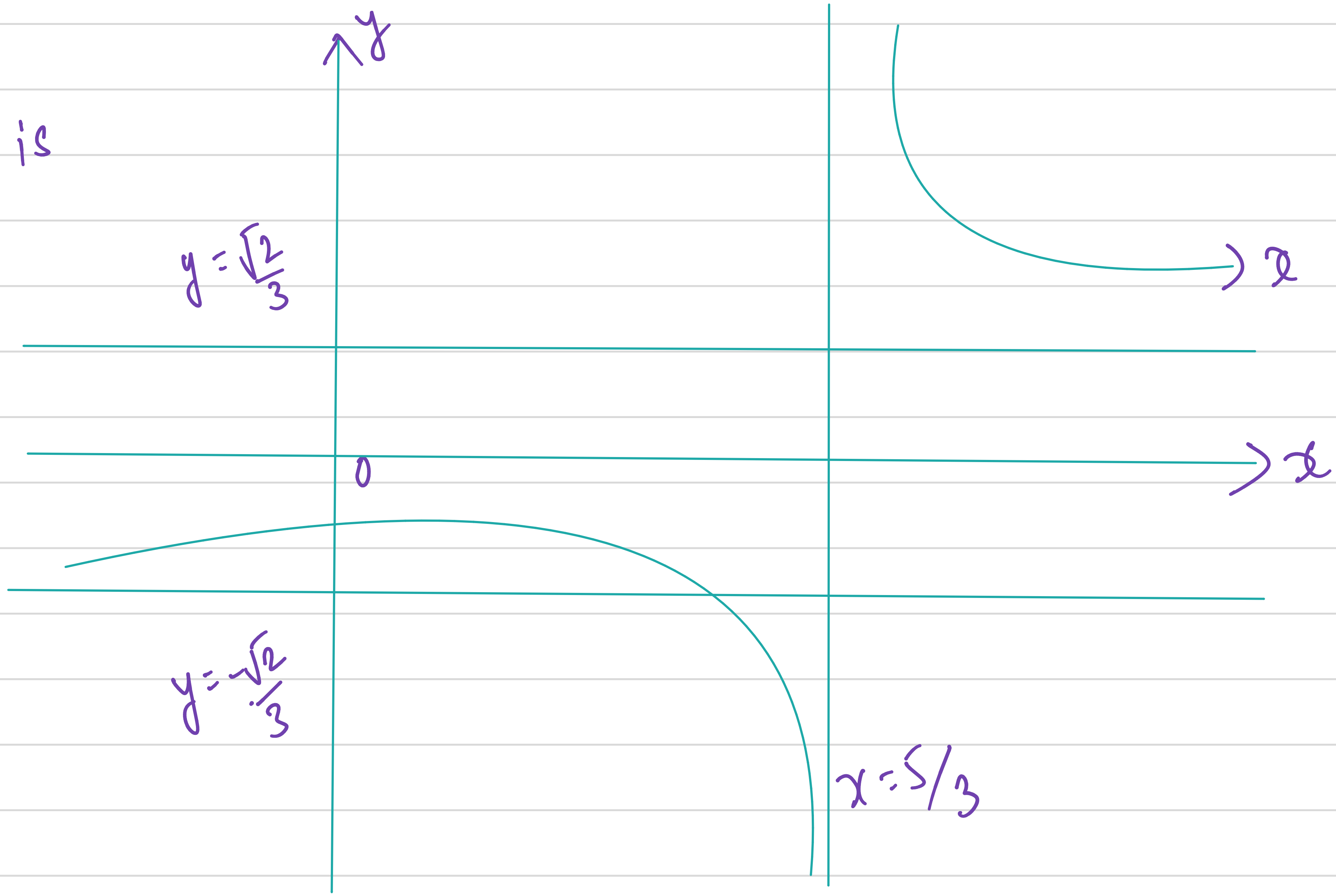
$$\therefore \lim_{x \to 5/3^+} \frac{\sqrt{2x^2+1}}{3x-5} \Rightarrow \infty$$

If x is close to $5/3$ but $x < 5/3$ then $3x-5 < 0$ and so $f(x)$ is large negative.

$$\therefore \lim_{x \to 5/3^-} \frac{\sqrt{2x^2+1}}{3x-5} \Rightarrow -\infty$$

$\therefore$ The vertical asymptote is

$x = 5/3$

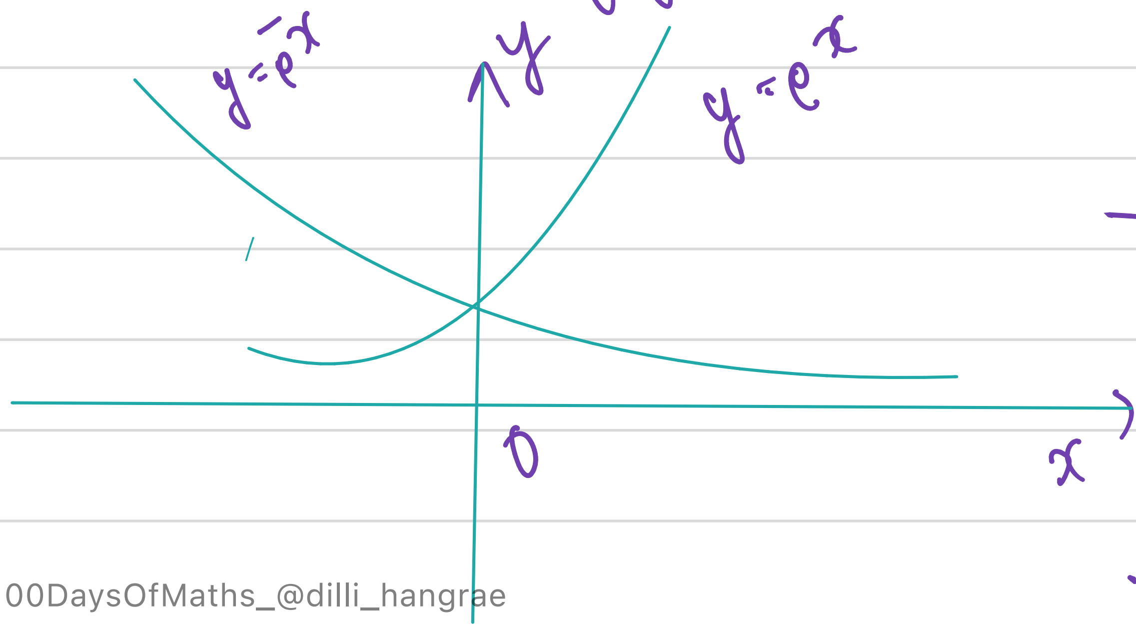


Note: $\lim\limits_{x\to\infty} e^{x} = \infty$,

$\lim\limits_{x\to-\infty} e^{x} = 0$, $\lim\limits_{x\to\infty} e^{-x} = 0$,

$\lim\limits_{x\to-\infty} e^{-x} = \infty$ which is

shown in the figure (A) below:



$\longrightarrow \lim\limits_{x\to\infty} e^{-x} = 0$ i.e. $y=0$ is the limit at infinity for the function $y = e^{+x}$ and hence horizontal asymptote of $y = e^{x}$.

$\therefore y = 0$ is the common horizontal asymptote of the $y = e^{x}$ and $y = e^{-x}$

Definition: Infinite Limits at Infinity

Let $f(x)$ be a given function of x then the notation $\lim_{x \to \infty} f(x) = \infty$ is used to indicate that the values of $f(x)$ become large as x becomes large.

Similarly $\lim_{x \to \infty} f(x) = \infty$, $\lim_{x \to \infty} f(x) = -\infty$, $\lim_{x \to -\infty} f(x) = -\infty$.

Above all limits are called Infinite Limits at Infinity. For example $\lim_{x \to \infty} e^{x} = \infty$, $\lim_{x \to -\infty} e^{-x} = \infty$ are called infinite limits at Infinity.

Similarly $\lim_{x \to \infty} x^3$ and $\lim_{x \to -\infty} x^3$ are also called Infinite Limits at Infinity.

Precise Definition of Limit At Infinity:

Let f be a function defined on some interval (a, ∞). Then $\lim_{x \to \infty} f(x) = L$ means that for every $\varepsilon > 0$ there corresponds a number N sufficiently large positive number such that

$$\text{if } x > N \text{ then } |f(x) - L| < \varepsilon.$$