Day-90, Feb-28, 2025 (Falgun 16, 2081)

1. Solving Non-Singular System of Linear Equations (Algorithm)
2. Solving Singular System of Linear Equations
3. Matrix Row Reduction, Row Operations that Represent Singularity
4. The Rank of a Matrix and Rank of Matrix in General
5. The Row Echelon form in with General form
6. Reduced Row Echelon form
7. The Gaussian Elimination Algorithm.

Goal: To Convert the System into Solved System.

Solving systems of equations

System

$a + b = 10$

$a + 2b = 12$

Some process

Manipulating Equations

↳ Swapping Equations

↳ Adding Equations

↳ Multiplying Equations by a Constant.

Solved system

$a = 8$

$b = 2$

That is, if the system is non singular and has exactly one solution.

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1. Steps to Manipulate Equations

Solving systems of equations

(1) *

$$\begin{array}{r} a+b=10 \\ \times\ 7 \\ \hline 7a+7b=70 \end{array}$$

System

$a + b = 10$

$a + 2b = 12$

$8+2=10$

$8+4=12.$

Solved system

$a = 8$

$b = 2$

(2)

$$\begin{array}{r} a+b=10 \\ 2a+3b=22 \\ \hline 3a+4b=32 \end{array}$$

That is, if the system is non singular and has exactly one solution.

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Systems of equations

System		Divide by coefficient of a		Solved system	
$5a + b = 17$	$\longrightarrow$	$a + 0.2b = 3.4$	— eqn (i)	$a = ?$	$\boxed{a = 3}$
$4a - 3b = 6$	$\longrightarrow$	$a - 0.75b = 1.5$	— eqn (ii)	$b = ?$	$\boxed{b = 2}$

$\uparrow$

Eliminate 'a'
from this equation

eqn (ii) − eqn (i)

$$a - 0.75b = 1.5$$
$$-\ a + 0.2b = 3.4$$
$$0a - 0.95b = -1.9$$
$$\boxed{b = 2}$$

$$a + 0.2(2) = 3.4$$
$$\boxed{a = 3}$$

So, if we do this,
the new equations are a +

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What if one of the coefficients of a is zero?

Divide by coefficient of a

System

5a + b = 17 → $a + 0.2b = 3.4$

3b = 6 → ??? b = 2.

Eliminate 'a' from this equation

b = 2

Solved system

a = ? → 3

b = ? → 2

Example

$$\begin{cases} 2a + 5b = 46 \\ 8a + b = 32 \end{cases} \rightarrow \text{put } a = 3,\ b = 8.$$

And you'd like to eliminate a from this equation.

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Infinitely many solutions eqn example - $\{5a+b=11,\ 10a+2b=22\}$

What if the system is singular (redundant)?

System

$a + b = 10$

$2a + 2b = 20$

Eliminate 'a' from this eqn

Divide by coeff of a

$a+b=10$ — eqn (i)

$a+b=10$ — eqn (ii)

Subtract eqn (i) from eqn (ii)

$$\begin{array}{r} a+b=10 \\ -\ a+b=10 \\ \hline 0=0 \end{array}$$

Deg of freedom of x

Solved system

$a = ?$ $a=x$

$b = ?$ $b=10-x$

Solved system

$a+b=10$

no other equation

Recall that the system was redundant since the second equation is equivalent to

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Solving System of Equations with More Variables

Elimination method

System

$a + b + 2c = 12$

$3a - 3b - c = 3$

$2a - b + 6c = 24$

→ Leave 'a' by itself

→ Divide each row by the coefficient of 'a'

$a + b + 2c = 12$

$a - b - \frac{1}{3}c = 1$

$a - \frac{b}{2} + 3c = 12$

Do Subtraction

$a + b + 2c = 12$

$-2b - \frac{7}{3}c = -11$

$-\frac{3}{2}b + c = 0$

Isolated a

It's actually very similar.

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Elimination method

System

$a + b + 2c = 12$

$-2b - 7/3\, c = -11$

$-3/2\, b + c = 0$

Divide last two rows by the coefficient of b

$a + b + 2c = 12$

$b + 7/6\, c = 11/2$

$b - 2/3\, c = 0$

Use the second equation to remove 'b' from the third

$a + b + 2c = 12$

$b + 7/6\, c = 11/2$

$-11/6\, c = -11/2$

Isolated 'b'

$c = 3$

$\boxed{b = 2}$

$a + 2 + 6 = 12$

$\boxed{a = 4}$

In this way, you have isolated b in the bottom two rows.

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Matrix Row Reduction (Gaussian Elimination Method):

Systems of equations to matrices

Original system	Intermediate System	Solved System
$5a + b = 17$	$a + 0.2b = 3.4$	$a = 3$
$4a - 3b = 6$	$b = 2$	$b = 2$

Original matrix

$$\begin{bmatrix} 5 & 1 \\ 4 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Upper Diagonal matrix

Row echelon form

Diagonal Matrix

Reduced Row echelon form.

in order to calculate
the value of b.

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Row echelon form provides lots of Information.

Systems of equations to matrices

Original system

$a + b = 10$

$2a + 2b = 20$

Intermediate System

$a + b = 10$

$0a + 0b = 0$

$0a + 0b = 0$

$0a + 0b = 0$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$5a + b = 11$

$10a + 2b = 22$

Original Matrix

$\begin{bmatrix} 5 & 1 \\ 10 & 2 \end{bmatrix}$

$a + 0.2b = 2.2$

$0a + 0b = 0$

$\begin{bmatrix} 1 & 0.2 \\ 0 & 0 \end{bmatrix}$ → upper Diagonal Matrix

Row echelon form

manipulated and turned into
this system over here,

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Row echelon form

Row Echelon form.

$$\begin{bmatrix} 1 & 5 & 6 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{bmatrix}$$

1	*	*	*	*
0	**1**	*	*	*
0	0	**0**	0	0
0	0	0	**0**	0
0	0	0	0	**0**

Reduced Row Echelon form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Following these rules in the
case of 2 by 2 matrices,

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Row Operations that Preserve Singularity.

Switching rows ?

So, the Row operations preserve the Singularity or Non-Singularity.

5	1
4	3

Determinant $= 5 \cdot 3 - 1 \cdot 4 = 11$

$\begin{bmatrix} 4 & 3 \\ 5 & 1 \end{bmatrix}$

$|D| \Rightarrow -11$

Non-Singular Matrix.

Multiplying by a row by number

$\begin{bmatrix} 5 & 1 \\ 4 & 3 \end{bmatrix}$ $\begin{bmatrix} 50 & 10 \\ 4 & 3 \end{bmatrix}$

$|D| \neq 11$

$|D| \Rightarrow 10 \cdot 11 \Rightarrow 110$

$\begin{bmatrix} 5 & 1 \\ 4 & 3 \end{bmatrix}$ $|D| = 11$

$\begin{bmatrix} 5 & 1 \\ +4 & 3 \end{bmatrix}$

$\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$

$\Rightarrow |D| \Rightarrow 11$

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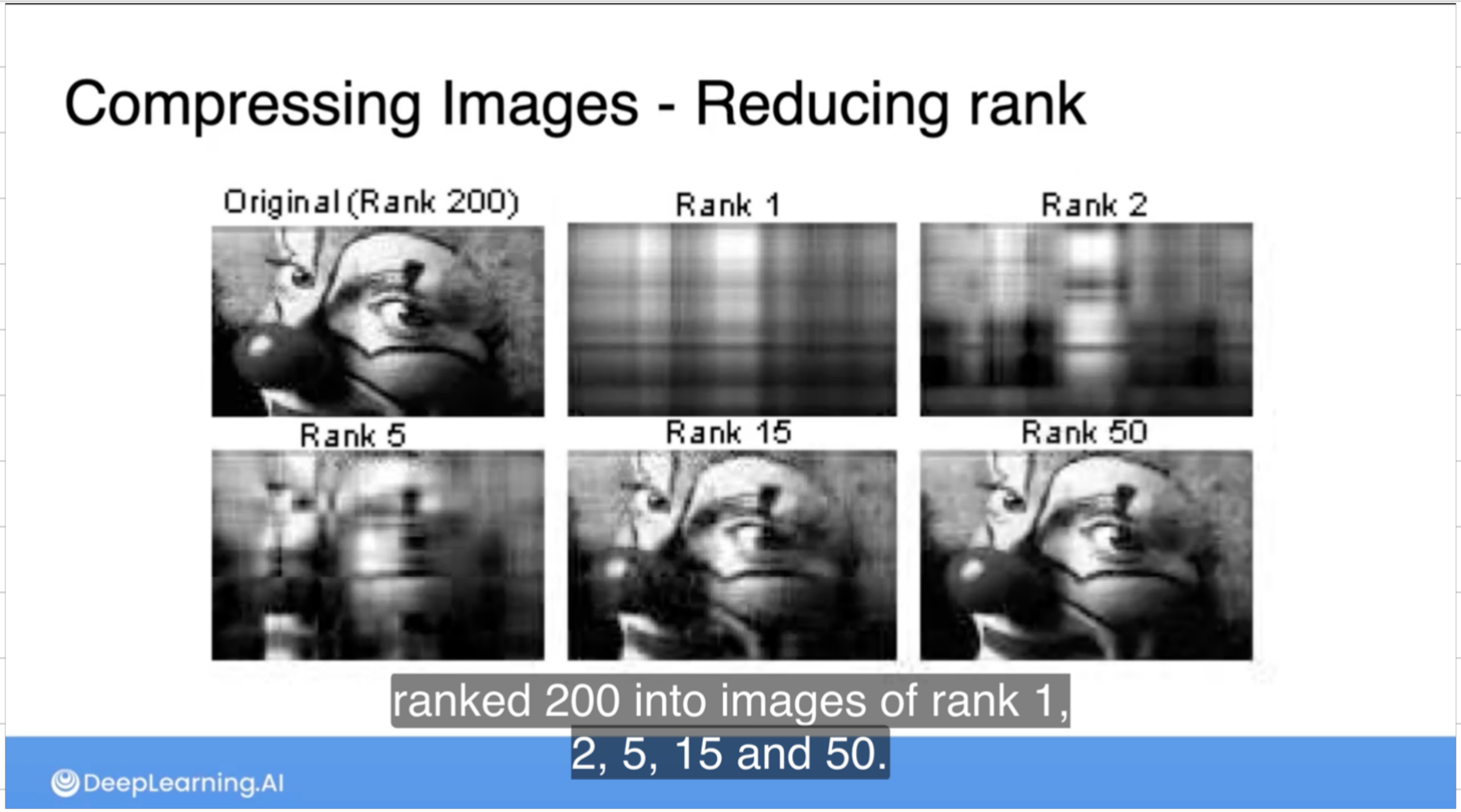
The determinant is 5 times 3

Rank of Matrix:

Measures How much Information that matrix or its Corresponding System linear equation is Carrying.

Blurring the Original Rank 200 Images to Rank 50 with lesser Storage and so on.

Has wide Applications.



Systems of information

(Rank → goal to find the colors of dogs)

System 1

The dog is **black**
The cat is **orange**

Two sentences

Two pieces of information

System 2

The dog is **black**
The dog is **black**

Two sentences

One piece of information

System 3

The dog
The dog

Two sentences

0 pieces of information.

System 2 also has two sentences,
but they're the same.

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Each Piece of Equation draw the information.

Systems of equations

System 1

$$a + b = 0$$
$$a + 2b = 0$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

Rank = 2

Two equations

Two pieces of Information

Rank = 2

System 2

$$a + b = 0$$
$$2a + 2b = 0$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

Rank = 1

Two Equations

1 piece of Information

Rank = 1

System 3

$$0a + 0b = 0$$
$$0a + 0b = 0$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Rank = 0.

Two Equations

0 piece of Information

Rank = 0.

That's why you're able to narrow down the solutions to one point.

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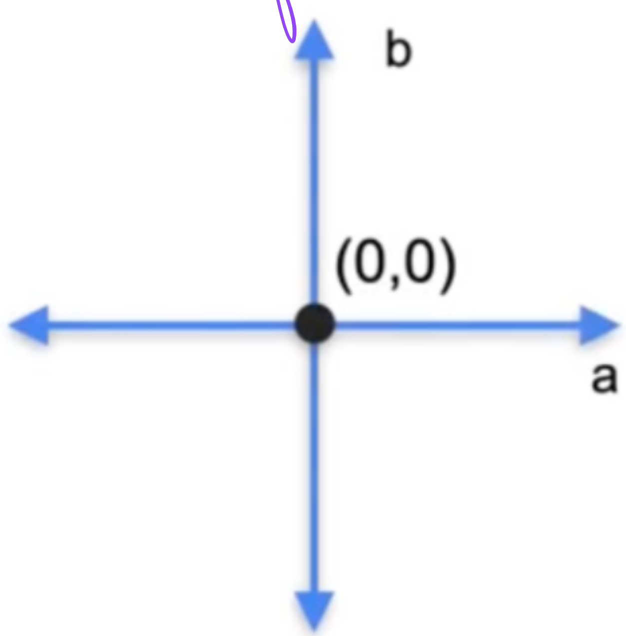
Solution Space for each matrices is the set of solutions to the System of equations when the constants are 0.

Rank and solutions to the system

1	1
1	2

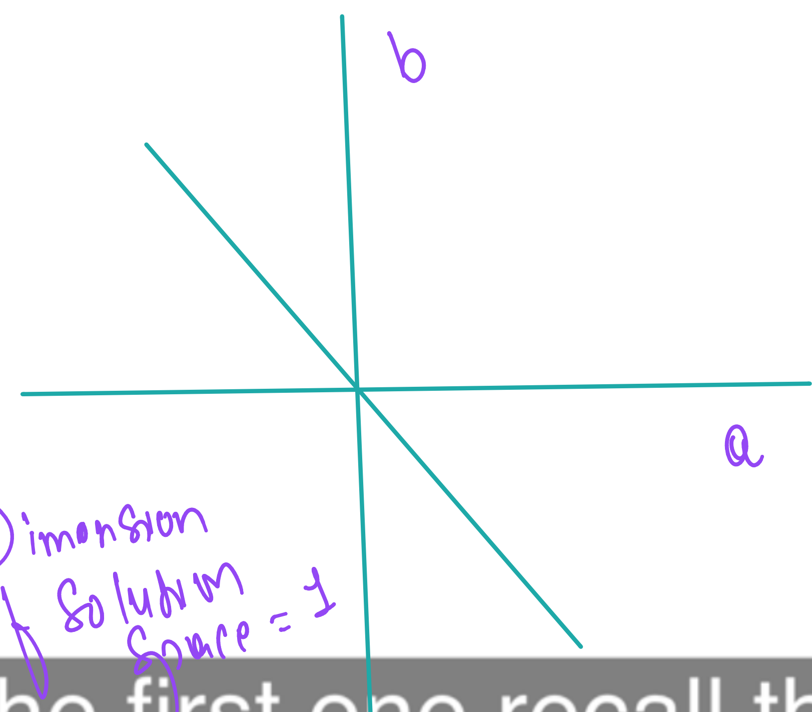
Rank = 2

Dimension of Solution Space = 0



1	1
2	2

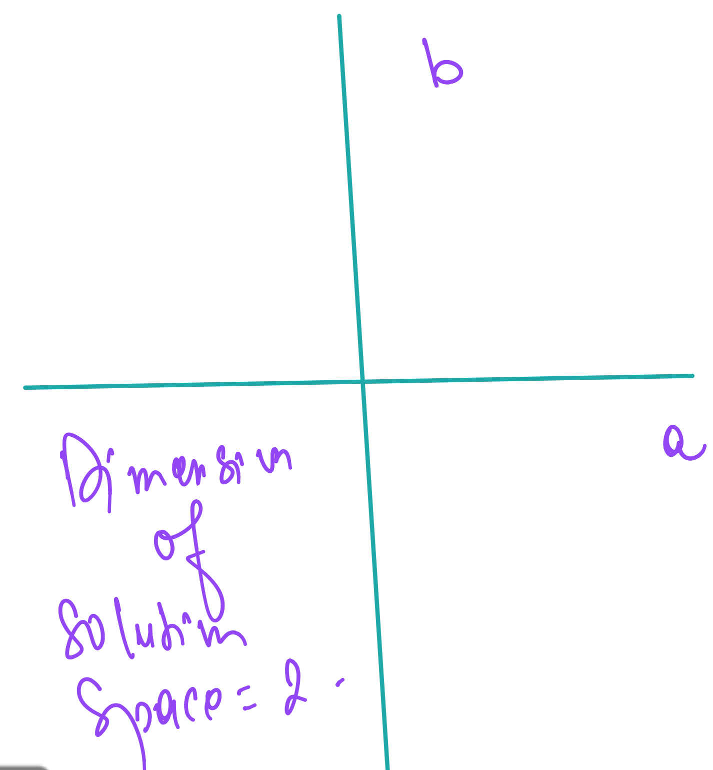
Rank = 1



Dimension of Solution Space = 1

0	0
0	0

Rank = 0



Dimension of Solution Space = 2.

So for the first one recall that the solutions are only a=0 and b=0.

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Rank of a matrix

1	1
1	2

Rank = 2

Dimension of solution space = 0

Non-Singular

Full Rank == Matrix Size.

1	1
2	2

Rank = 1

Dimension of solution space = 1

Singular

0	0
0	0

Rank = 0

Dimension of solution space = 2

Singular

Rank = 2 - (Dimension of solution space)

Eqn brings new piece of information.

The number of rows in the matrix minus the dimension of the solution space.

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Solutions: Rank of a matrix

Determine the rank of the following two matrices

Matrix 1: Since the solution space had dimension 0, the rank is **2**.

5	1
-1	3

Matrix 1: Since the solution space had dimension 1, the rank is **1**.

2	-1
-6	3

the same two matrices you've seen
recently and the solutions are in.

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Rank of Matrices in General form / View

↳ Use Row Echelon Form of the Matrix to calculate the Rank

Rank for matrices

System 1

$a + b + c = 0$ ✓
$a + 2b + c = 0$ ✓
$a + b + 2c = 0$ ✓

3 Eqn
3 pieces of information
Rank 3

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

System 2

$a + b + c = 0$ ✓
$a + b + 2c = 0$ ✗
$a + b + 3c = 0$ ✓

3 Eqn
2 pieces of information
Rank 2

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

System 3

$a + b + c = 0$ ✓
$2a + 2b + 2c = 0$ ✗
$3a + 3b + 3c = 0$ ✗

3 eqns
1 piece of information
Rank 1.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$$

System 4

$0a + 0b + 0c = 0$ ✗
$0a + 0b + 0c = 0$ ✗
$0a + 0b + 0c = 0$ ✗

3 Eqns.
0 piece of Information
Rank 0

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we'll look at the systems
of three equations

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Row echelon form

Row echelon form of a matrix ①

Original matrix	Row echelon form
$\begin{bmatrix} 5 & 1 \\ 4 & -3 \end{bmatrix}$	$\begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 5 & 1 \\ 10 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$
$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

used the systems of equations to find them.

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Row echelon form ②

Original matrix → **Row echelon form**

$$\begin{bmatrix} 5 & 1 \\ 4 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0.2 \\ 1 & -0.75 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0.2 \\ 0 & -0.95 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}$$

Divide each row by the leftmost coefficient

$$\begin{array}{rcc} & 1 & -0.75 \\ - & 1 & 0.2 \\ \hline & 0 & -0.95 \end{array}$$

Divide the second row by the leftmost non-zero coefficient

Now the matrix is in row echelon form.

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Row echelon form for singular matrices ③

Original matrix

$$\begin{bmatrix} 5 & 1 \\ 10 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0.2 \\ 1 & 0.2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0.2 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0.2 \\ ? & ? \end{bmatrix}$$

Divide each row by the leftmost coefficient

$$\begin{array}{rcc} & 1 & 0.2 \\ - & 1 & 0.2 \\ \hline & 0 & 0 \end{array}$$

Divide the second row by the leftmost non-zero coefficient

you'd be dividing 0 by 0, which is undefined.

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Row echelon form

Original matrix

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

Divide each row by the leftmost Coefficient

Row echelon form, singularity, and rank

Rank of the Matrix is the Diagonal One's in the matrix.

$$\begin{bmatrix} 5 & 1 \\ 4 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}$$

→ Rank 2
2 ones in diagonal

$$\begin{bmatrix} 5 & 1 \\ 10 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0.2 \\ 0 & 0 \end{bmatrix}$$

→ Rank 1

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

→ Rank 0

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It has two ones in the diagonal,

A matrix is Non-Singular if and only if there are only ones and no zeros in the diagonal.

Row Echelon form in General form

Row echelon form

System		System
$a + b + 2c = 12$	→	$a + b + 2c = 12$
$3a - 3b - c = 3$		$-6b - 7c = -33$
$2a - b + 6c = 24$		$6c = 18$

Matrix

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & -3 & -1 \\ 2 & -1 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 2 \\ 0 & -6 & -7 \\ 0 & 0 & 6 \end{bmatrix}$$

and the third one
only has variable

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Row echelon form

$$\begin{bmatrix} 2 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 3 & * & * \\ 0 & 0 & 0 & -5 & * \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Rank = 5

$$\begin{bmatrix} 3 & * & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & -4 & * \\ \boxed{0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank 3

- Zero rows at the bottom
- Each row has a pivot (leftmost non-zero entry)

- Every pivot is to the right of the pivots on the rows above
- Rank of the matrix is the number of pivots

and there's a rule about pivots which said that

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Note:

- In general, pivots different than 1 are allowed
- For this class, pivots are 1. This makes no mathematical difference.

Important note

3	*	*	*	*
0	0	1	*	*
0	0	0	-4	*
0	0	0	0	0
0	0	0	0	0

$\div 3$, $\div 1$, $\div(-4)$ →

1	*	*	*	*
0	0	1	*	*
0	0	0	1	*
0	0	0	0	0
0	0	0	0	0

p p p

1

Obviously the stars are

Another example

2

Matrix

1	1	1
1	2	1
1	1	2

Subtract the first row from the second and the third ones

Row echelon form

1	1	1
0	1	0
0	0	1

Notice that the one on the right is in row echelon form.

What if the matrix is singular?

3

Matrix

1	1	1
1	1	2
1	1	3

Subtract the first row from the second and the third ones

1	1	1
0	0	1
0	0	2

Subtract twice the second row from the third one

Row echelon form

1	1	1
0	0	1
0	0	0

That is the row echelon form of that matrix.

Rank for matrices

Matrix 1

1	1	1
1	2	1
1	1	2

Matrix 2

1	1	1
1	1	2
1	1	3

Matrix 3

1	1	1
2	2	2
3	3	3

Matrix 4

0	0	0
0	0	0
0	0	0

Row echelon forms

1	1	1
0	1	0
0	0	1

Rank = 3

1	1	1
0	0	1
0	0	0

Rank = 2

1	1	1
0	0	0
0	0	0

Rank = 1

0	0	0
0	0	0
0	0	0

Rank = 0.

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If you look at the pivot ones,

Row Reduced Echelon Form

Systems of equations to matrices

Original system	Intermediate System	Solved system
$5a + b = 17$	$a + 0.2b = 3.4$	$a = 3$
$4a - 3b = 6$	$b = 2$	$b = 2$

Original matrix → **Upper diagonal matrix** → **Diagonal matrix**

$$\begin{bmatrix} 5 & 1 \\ 4 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Rank = 2.

manipulation we'll
get it to the metrics

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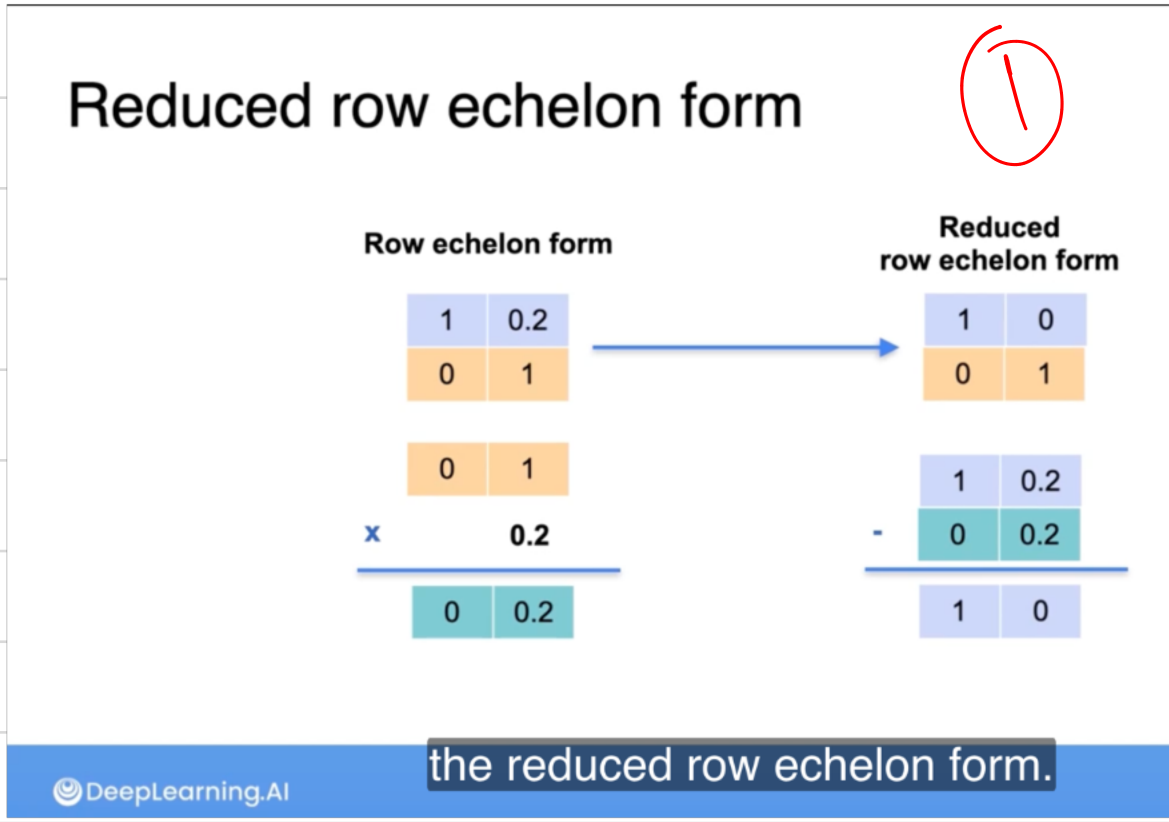
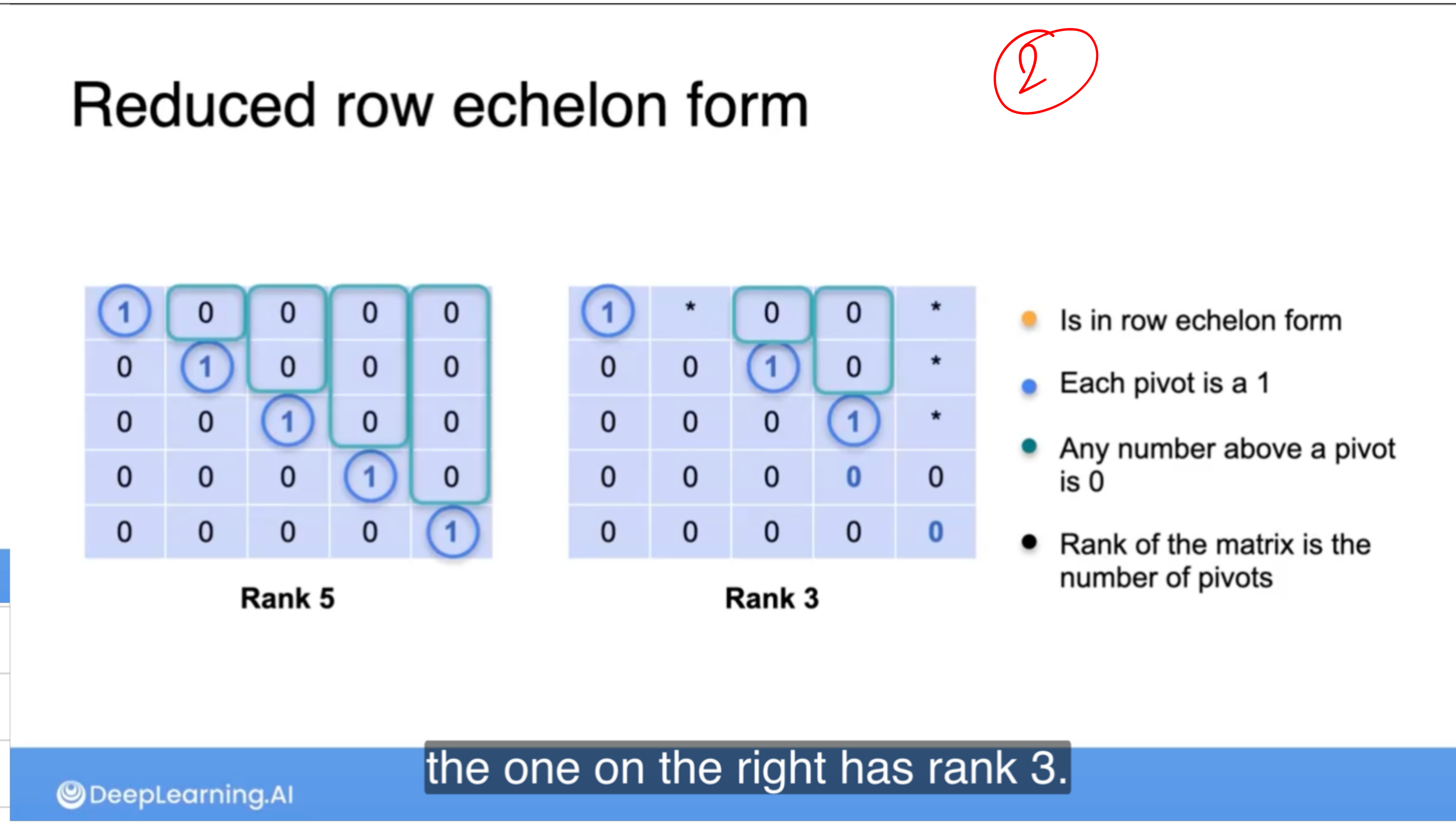
1



2



Source: Linear Algebra for Machine Learning and Data Science Coursera course offered by DeepLearning.AI.

Reduced row echelon form

(1) **Row echelon form**

3	*	*	*	*
0	0	2	*	*
0	0	0	-4	*
0	0	0	0	0
0	0	0	0	0

(2)

1	*	*	*	*
0	0	1	*	*
0	0	0	1	*
0	0	0	0	0
0	0	0	0	0

Divide each row by the value of the pivot

(3) **Reduced row echelon form**

1	*	0	0	*
0	0	1	0	*
0	0	0	1	*
0	0	0	0	0
0	0	0	0	0

Turn anything above a pivot to 0

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For example, if ab

Gaussian Elimination.

(1) Create Augmented Matrix.

(2) Get Matrix into Reduced Row echelon form

(3) Complete Back Substitution

(4) Stop if you encounter a row of 0s.

Reduced row echelon form

(4) **Row echelon form**

1	2	3
0	1	4
0	0	1

Subtract 2 times the second row from the first one

(5)

1	0	-5
0	1	4
0	0	1

Add 5 times the third row to the first one

(6)

1	0	0
0	1	4
0	0	1

Subtract 4 times the third row from the second one

(7) **Reduced row echelon form**

1	0	0
0	1	0
0	0	1

the reduced row echelon form of the matrix.

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Gaussian Elimination Algorithm

Augmented matrix

$$
\begin{aligned}
2a - b + c &= 1 \\
2a + 2b + 4c &= -2 \\
4a + b &= -1
\end{aligned}
\;\longrightarrow\;
\begin{array}{c}
R_1 \\ R_2 \\ R_3
\end{array}
\left[\begin{array}{ccc|c}
2 & -1 & 1 & 1 \\
2 & 2 & 4 & -2 \\
4 & 1 & 0 & -1
\end{array}\right]
$$

Augmented matrix

① Pivoting

$R_1 \leftarrow \frac{1}{2} R_1$

$R_2 \leftarrow R_2 - 2R_1$

$R_3 \leftarrow R_3 - 4R_1$

$$
\sim
\begin{array}{c}
R_1 \\ R_2 \\ R_3
\end{array}
\begin{array}{cccc}
1 & -1/2 & 1/2 & 1/2 \\
0 & 3 & 3 & -3 \\
0 & 3 & -2 & -3
\end{array}
$$

The vertical line is used
to separate the constants,

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Final Answer:

What if the System is Singular? Does Gaussian Elimination works?

The result

a = 0	
b = -1	
c = 0	

Solution to the system

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

1	0	0
0	1	0
0	0	1

Identity matrix

and by simplifying the matrix to
this form using Gaussian elimination,

Checking if it has infinitely many or no solutions

Look at the column of constants

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 2 & 4 & 5 & 1 \\ 3 & 6 & 4 & 6 \end{array}\right] \xrightarrow{\text{After row reduction...}} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & 0 & -7 & 9 \\ \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} \end{array}\right]$$

$0a + 0b + 0c = 0$

Infinitely many solutions

If the constant value in the
row of zeros is also zero,

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$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & 0 & -7 & 9 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

No Solutions.