Day - 26, Dec - 11, 2024 (Mangshir - 26, 2081)

Derivative as a function

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$ as the slope of tangent line to the curve $f(x)$ at $x = a$ or velocity of an object moving along $f(x)$ at $x = a$ or the marginal cost.

Definition: The derivative of a function f at a point 'a' is denoted by $f'(a)$ and is defined as :-

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{—— eqn I}$$

if the limit exists

Suppose $h = |x - a|$. Then eqn (i) can be rewrite as

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

The Symbol of derivative of $f(x)$ is often used $f'(x)$. Here, 'x' is independent variable and 'f' is dependent variable that depends upon 'x'. And, $f'(x)$ means f is derivable with respect to x.

The other common D or $\frac{d}{dx}$ are called differentiation operator.

Definition: A function 'f' is called differentiable at a point 'a' if $f'(a)$ exists.

Note that, the differentiability of any function exists (if the function is differentiable) only on open interval.

Q. Show that $|x| = f(x)$ or $f(x) = |x|$ is not differentiable at $x = 0$.

Here, for $x > 0$

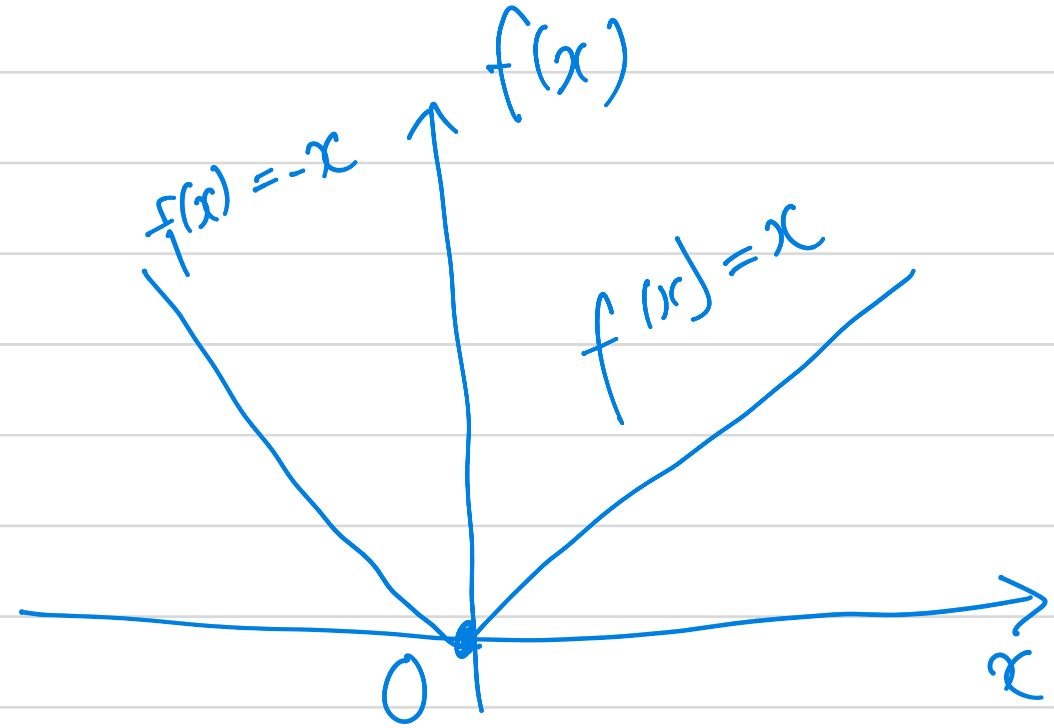
$$f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$

$$\Rightarrow \lim_{h\to 0}\left(\frac{|x+h|-|x|}{h}\right)$$

$$\Rightarrow \lim_{h\to 0}\left(\frac{x+h-x}{h}\right)$$

$$\Rightarrow \lim_{h\to 0}\left(\frac{h}{h}\right)$$

$$\Rightarrow 1.$$



This means f is differentiable for $x>0$

And for $x<0$, we have $|x| = -x$. Then choose 'h' is so small such that $|x+h|<0$ and $|x+h| = -(x+h)$

Here for $x < 0$,

$$f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$

$$\Rightarrow \lim_{h \to 0} \left(\frac{-(x+h) - (-x)}{h} \right)$$

$$\Rightarrow \lim_{h \to 0} \left(\frac{-h}{h} \right)$$

$$\Rightarrow -1$$

This means 'f' is differentiable for $x < 0$.

But,

$$f'(x)\Big|_{\text{for } x>0} \neq f'(x)\Big|_{\text{for } x<0}$$

So, f is not differentiable at $x=0$.

Geometrically, the curve $f(x)$ does not have a tangent line at the origin i.e. at $(0,0)$.

So, $x>0$ and $x<0$ $f(x)$ is differentiable but not differentiable at $x=0$

Theorem: If 'f' is differentiable at a point 'a' then f is continuous at that point 'a'.

Here, Suppose that f is differentiable at a point 'a'. Therefore,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{exists.}$$

Then to prove f is continuous at 'a', we have to show that –

$$\lim_{x \to a} f(x) = f(a)$$

Here,
$$f(x)-f(a)=\frac{f(x)-f(a)}{x-a}\cdot(x-a)$$

Therefore,
$$\lim_{x\to a}\left[f(x)-f(a)\right]=\lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}(x-a)\right]$$

$$\Rightarrow \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right)\lim_{x\to a}(x-a)$$

$$\Rightarrow f'(a)\cdot 0$$

$$\Rightarrow 0.$$

$$\boxed{\Rightarrow \lim_{x\to a} f(x)=f(a)}$$

This shows 'f' is continuous at 'a'.

Note: Remember that the converse of this theorem need not hold. For instance, consider function

$$f(x) = |x|$$

Since,

$$\lim_{x \to a} f(x) = \lim_{x \to a} |x| = 0.$$

So, f is continuous at $x=0$, but it is not differentiable at 0.

Review of Derivative. In the study of derivative of function,

Sometimes we observe that the functions will not in simple and singular form, we impose some rules to make simple to such functions. Here we'll study about such rules.

Constant Multiple Rule: If c be a constant and f is a differentiable function then

$$\frac{d}{dx}(c\,f(x)) = c\,\frac{d}{dx}(f(x))$$

Sum and Difference Rule:

If 'f' and 'g' are both differentiable functions then -

$$\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$$

and $$\frac{d}{dx}\left(f(x)-g(x)\right)=\frac{d}{dx}\left(f(x)\right)-\frac{d}{dx}\left(g(x)\right).$$

Product Rule: If f and g are both differentiable functions then

$$\frac{d}{dx}\left(f(x)\,g(x)\right)=f(x)\cdot\frac{d}{dx}\left(g(x)\right)+g(x)\,\frac{d}{dx}\left(g(x)\right)$$

Quotient Rule: If 'f' and 'g' are both are differentiable function and $g(x)\neq 0$, then

$$\frac{d}{dx}\left(\frac{e^x}{1+x^2}\right)=\frac{(1+x^2)\,\frac{d}{dx}(e^x)-e^x\,\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$\frac{d}{dx}\left(\frac{e^x}{1+x^2}\right) = \frac{(1+x^2)\frac{d}{dx}(e^x) - e^x\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

Chain Rule:

If 'g' is differentiable at 'x' and 'f' is differentiable at g(x). Then the Composite function f(g(x)) differentiable at 'x' and is differentiate function as

$$\frac{d}{dx}(f(g(x))) = \frac{d}{dx}(f(g(x))) \cdot \frac{d}{dx} g(x).$$

$$\Rightarrow f'(g(x)) \cdot g'(x)$$

Derivative of a function → How to differentiate the different types of functions.

1) Derivative of a Constant function:

Suppose $f(x) = c$ is a constant function. Then $f(x+h) = c$.

So that the derivative of $f(x)$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \lim_{h \to 0} \frac{c - c}{h} \Rightarrow 0$$

Thus the derivative of any constant function is zero. Therefore,

$$\boxed{\frac{d}{dx}(c) = 0}$$

B) Derivative of Power function

Suppose $f(x) = x^n$ where 'n' is any real number. Then the derivative of x^n is

$$f'(x) = \frac{d}{dx}(x^n)$$

$$\Rightarrow nx^{n-1}.$$

Justification:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$\Rightarrow \lim_{x \to a} \frac{x^n - a^n}{x - a}$$

$$\Rightarrow \lim_{x \to a} \frac{(x-a)\left(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \dots + xa^{n-2} + a^{n-1}\right)}{x-a}$$

$$\Rightarrow \lim_{x \to a} \left(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \dots + xa^{n-2} + a^{n-1}\right)$$

$$\Rightarrow \left(a^{n-1} + a^{n-2}a + a^{n-3}a^2 + \dots + a\,a^{n-2} + a^{n-1}\right)$$

$$\Rightarrow a^{n-1} + a^{n-1} + a^{n-1} + \dots + a^{n-1}$$

$$\Rightarrow na^{n-1}$$

$$\left[f(x) = x^n \text{ then } f'(x) = nx^{n-1}\right]$$

Intresting Example:

If $f(x) = e^{x+1} + 1$ then find f' and f''

Let,

$$f(x) = e^{x+1} + 1.$$

Then,

$$f'(x) = \frac{d}{dx}\left(e^{x+1} + 1\right)$$

$$\Rightarrow e^{x+1} + 0$$

$$\Rightarrow e^{x+1}$$

And,

$$f''(x) = e^{x+1}(1) \Rightarrow e^{x+1}.$$

Differentiation - Change # differentiable = $f(x)$ can apply derivative.

Q. At what point on the curve $y = e^x$ is the tangent line parallel to the line $y = 2x$?

Given curve is $y = e^x$. — eqn (i).

Then $y' = e^x$

Co-ordinate of the point be (a, e^a) where $y = e^x$ and let the x-coordinate of the point (a, e^a). And the slope of the tangent line to the curve at (a, e^a) is e^a.

Also,

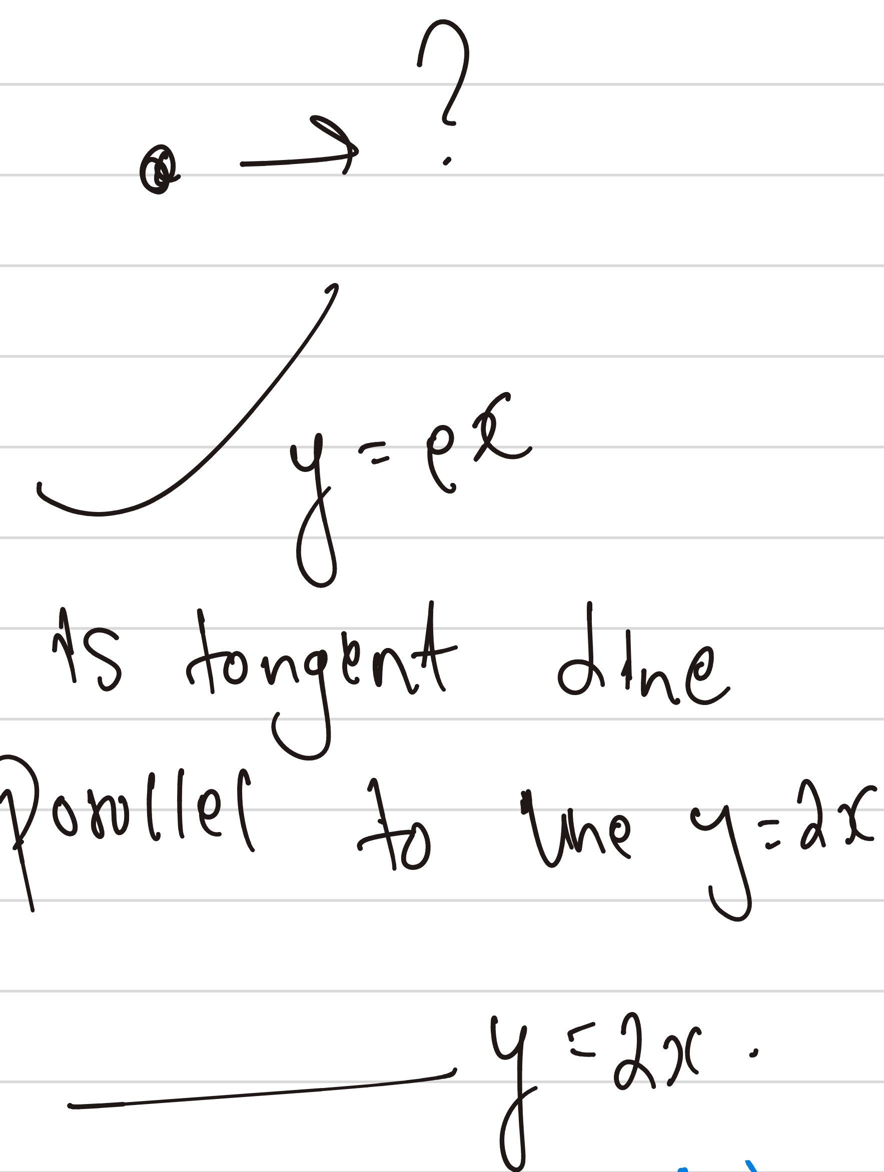
$$y = 2x \quad \text{—— eqn (ii)}$$

Clearly slope of the line eqn (ii) is 2.

Since the tangent line to (i) is parallel to (ii). So, the slope of these lines should be equal that is,

$$e^{a} = 2$$

$$\Rightarrow e = \ln(2).$$



Therefore, the coordinate of the point on $y = e^x$ is the tangent line parallel to $y = 2x$ is $(\ln(2), 2)$.

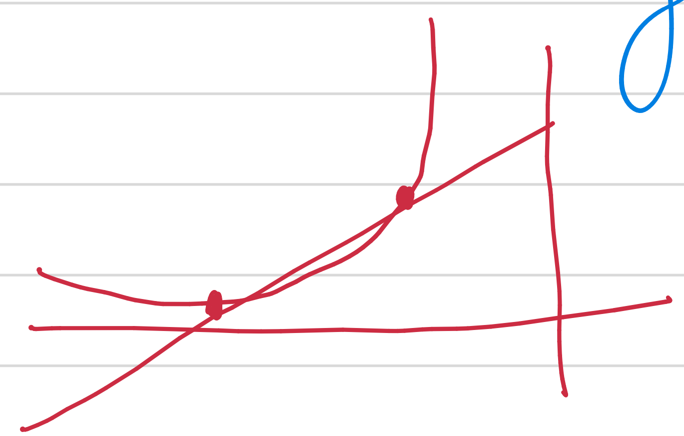
① Relation of tangent and slope

② Relation of tangent and slope with derivative.

③ Derivative with logarithmic. (Simplifying the complicated problems)

④ Relation Between Derivative and Velocity. ($\frac{\Delta d}{\Delta t}$).

① tangent line → touches the points.



slope → Rate of change

② derivative with slope and tangent gives how steepest.