Day - 41, Jan 10, 2024 (Poush 26, 2081 B.S.) .

Application of Derivative :

$$y = f(x)$$

① **Domain** : $f(x)$ value of x then $f(x)$ exists.

② **Intercepts**: It is where curve meets x-axis and y-axis

Put $y = 0$ find the x - intercept

Put $x = 0$, find the y - intercept

③ <u>Symmetry:</u> → Symmetric about y-axis, $f(x) = f(-x)$.

→ Symmetric about origin, $f(-x) = -f(x)$.

④ <u>Asymptote:</u> Rational function, logarithmic, exponential functions has asymptote.

⑤ <u>Horizontal Asymptote:</u> if either $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$ then $y = L$ is horizontal asymptote of curve $y = f(x)$.

⑥ <u>Vertical Asymptote:</u> A line $x = a$ is vertical asymptote of $y = f(x)$ if either $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$.

Interval of Increasing and Decreasing:

find the critical points by using $f'(x) = 0$ and $f'(x) = \infty$ and find the interval of increasing and decreasing.

Local Maxima and Minima Points:

→ Local Maxima → A point is occure at that point where sign is changed from +ve to -ve.

→ Local Minima → A point is obtained where sign from -ve to +ve

Concavity and Point of Inflection:

point of inflection → Point where concavity changes (changes from +ve to -ve or -ve to +ve)

We can find the interval of concave up and concave down by computing $f''(x) = 0$ and $f''(x) = \infty$.

Curve sketching

$f(x) = x^3 - 3x + 3$

Given,

$$f(x) = x^3 - 3x + 3$$

$$f'(x) = 3x^2 - 3$$

$$\Rightarrow 3x(x-1)$$

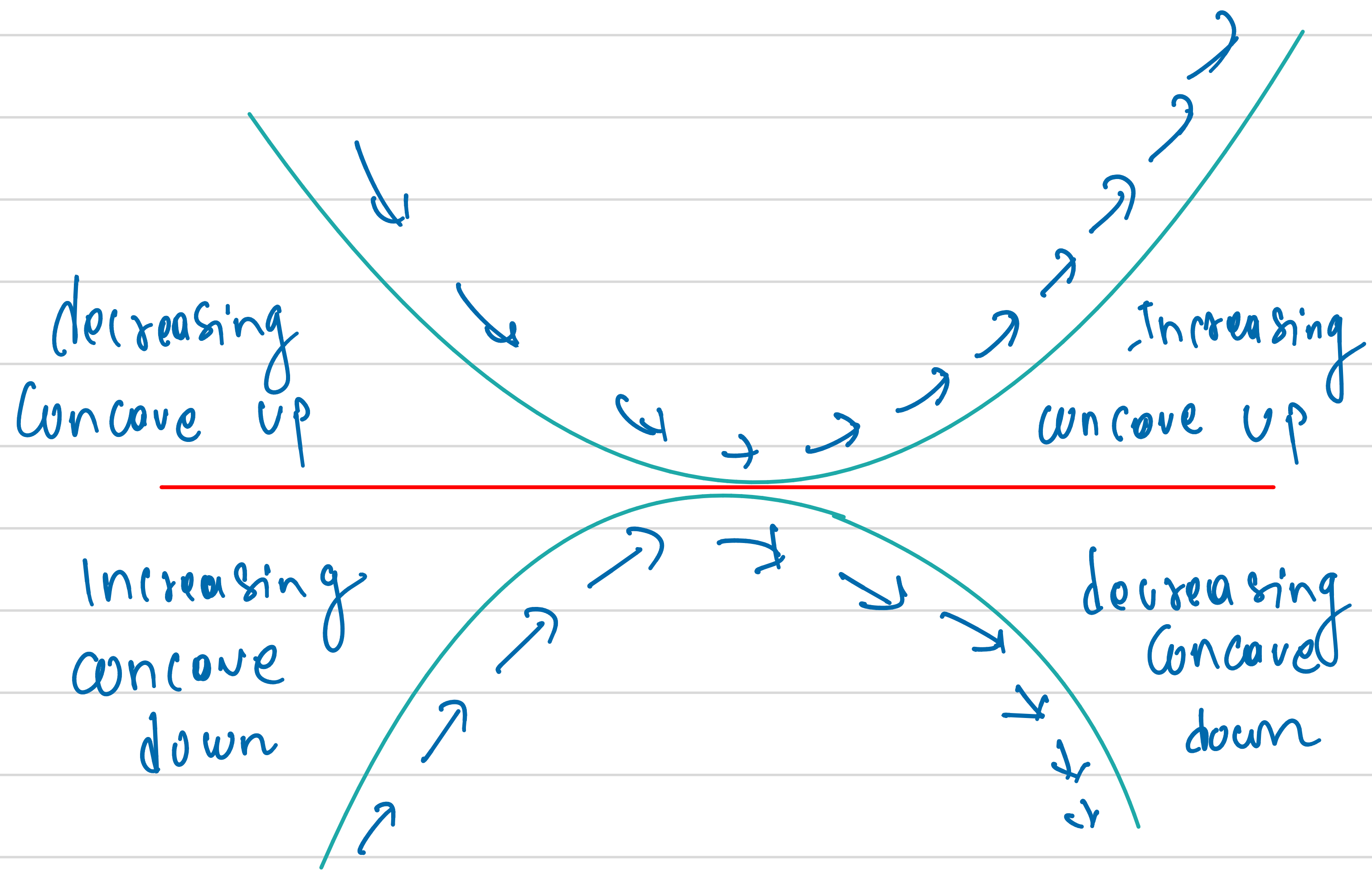
$$\Rightarrow 3(x-1)(x+1)$$

for critical points,

$$f'(x) = 0.$$

$$\Rightarrow 3(0)^2 - 3$$

$$\Rightarrow -3$$

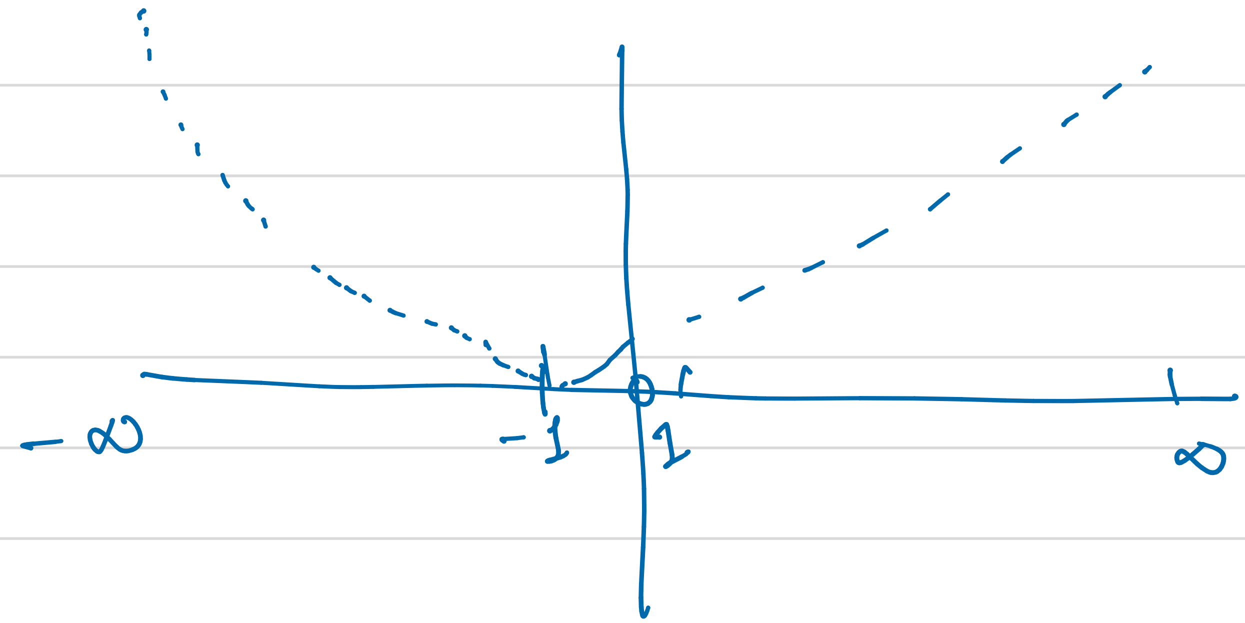


But we use

$f'(x) = 0$

$\Rightarrow 3(x-1)(x+1)$

$\Rightarrow \boxed{x = 1, -1}$



Interval	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
Sign of f'	+ve	-ve	+ve
Nature of f	Increasing	Decreasing	Increasing

Thus, the maxima occur at $x = -1$ and $(-1, f(-1)) = (-1, 5)$ is maximum point. And minima occur at $x = 1$ i.e. $(1, f(1)) = (1, 1)$ is minimum point.

for point of Inflection,

$$f''(x) = 0$$

$$\Rightarrow 6x = 0$$

$$\boxed{x = 0}$$

Interval	$(-\infty, 0)$	$(0, \infty)$
Sign of f''	$-ve$	$+ve$
Nature of f	Concave down	Concave up

Thus, point of inflection occur at $x = 0$ so $(0, f(0)) = (0, 3)$ is the point of inflection.

Summarizing above two tables:

$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
Increasing	Decreasing	Decreasing)	Increasing
Concave down	Concave down	Concave up	Concave up

Optimization Problems: [problems from business, mathematics and economics]

To optimize something means to maximize or minimize some aspect of it; So in this unit we have learned to find the maximum and minimum values of a function of interest (for example maximum profit, minimum cost)

In this section, we solve the optimization problems from business, mathematics and economics

Before that we must review how find the local and absolute maxima and minima.

To find the local maxima and minima of the function:

Step 1: find the critical point: Evaluate $f'(x)$ and

Solve $f'(x) = 0$. Say $x = a, b, c \ldots\ldots$ are solutions. (There are critical points)

Step 2: Evaluate $f''(x)$ and $f''(a)$.

→ If $f''(a) < 0$ then at $x = a$ maxima occurs and $f(a)$ is maximum value of $f(x)$.

→ If $f''(a) > 0$ then at $x = a$ minima and $f(a)$ is minimum value $f(x)$.

Repeat this for all critical points $x = a, b, c \ldots\ldots$

Critical points: A point where f' is undefined or zero.

To find the Absolute Maxima and Minima of function defined on $[a,b]$.

Step 1: find the critical point. Evaluate $f'(x)$ and solve $f'(x)=0$.
Take all those critical points which lies on $[a,b]$.

Step 2: find $f(a)$, $f(b)$ and $f(c)$, $\forall c \in [a,b]$, c is critical points. The greatest value among them is called absolute minimum value of $f(x)$.

Guideline for Solving Optimization Problems

Step 1
Understand the problem

$\rightarrow$

Step 2
Draw a diagram

$\downarrow$

Step 3
Introduce Variables

$\leftarrow$

Step 4
Write an equation for the unknown quantity

Example: $f(x) = 3x^3 - 2x^2$

$f'(x) = 9x^2 - 4x$ — eqn ?

Solving the eqn ? we get

$f'(x) = 9x^2 - 4x$

$\Rightarrow x(9 - x) - 4x$

$\Rightarrow x(9x - 4)$

$[x = 0]$ | $9x - 4 = 0$

$9x = 4$

$[x = 4/9]$

Critical points are at $x = 0$ and $x = 4/9$.

$$f'(x) = 9x^2 - 4x$$
$$f''(x) = 18x - 4$$

Evaluate for each $f''(x)$ at each critical points,

at $x = 0$,

$$f''(x) = 18 \times 0 - 4$$
$$= -4 < 0$$

Since $f''(x)$ or $f''(0) = -4 < 0$ local maxima $(x = 0)$.

At $x = 4/9$

$$f''(4/9) = 18 \times (4/9) - 4$$
$$\Rightarrow 4 > 0$$

Conclusion: The critical points are $x=0$ and $x=4$.
for $f'(x) = 9x^2 - 4x$ or $f(x) = 3x^3 - 2x^2$

Newton's Method:

Method to find the solution of polynomial $f(x)=0$ of degree 5 and more and type of $\sin x = x^2$ is called Newton's method or Newton-Raphson Method.

Slope of tangent at $(x_1, f(x_1))$ is $f'(x_1)$ & it's equation is

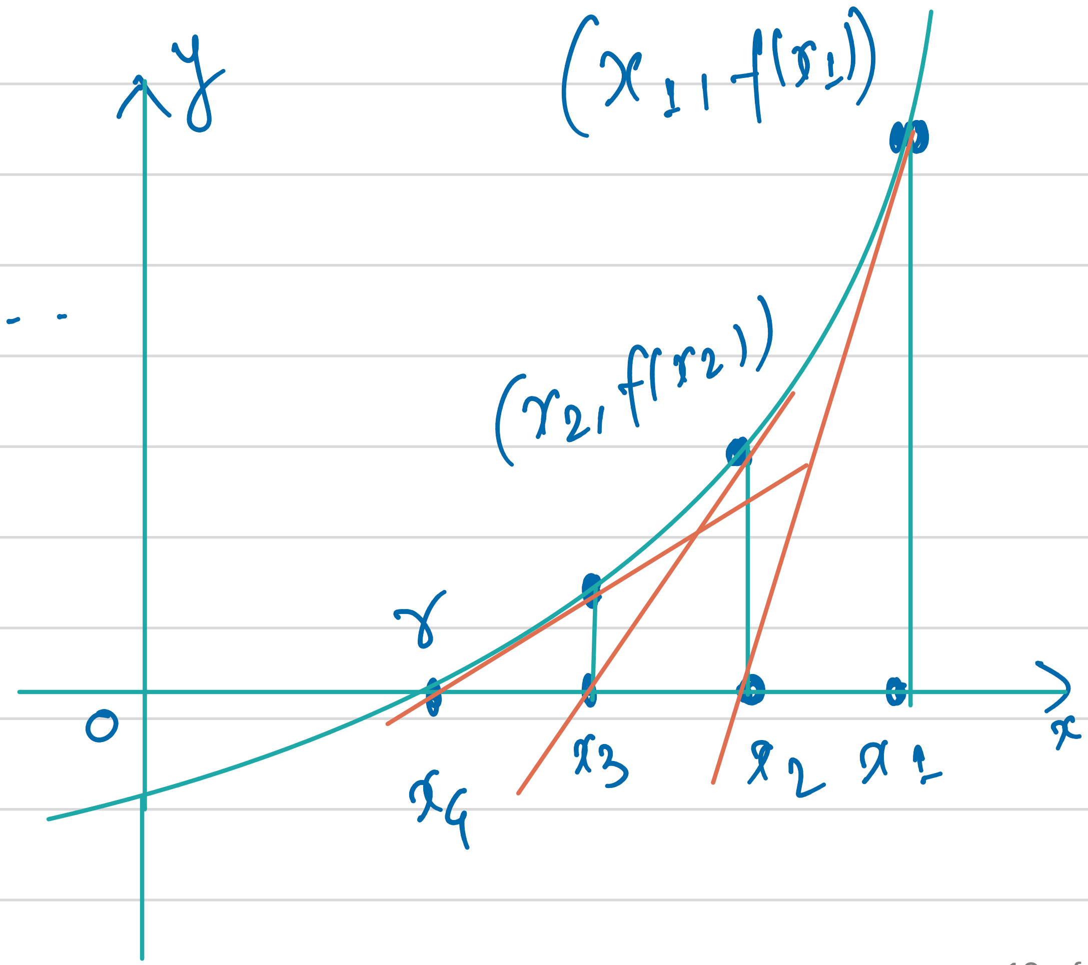
$$y - f(x_1) = f'(x_1)(x - x_1)$$

To find where it meet x-axis put $y=0$

$$0 - f(x_1) = f'(x_1)(x_2 - x_1)$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \quad \text{for } f'(x_1) \neq 0$$

If we repeat this process we obtain a sequence of approximation $x_1, x_2 \cdots \cdots$ and n^{th} approximation is x_n and for $f'(x_n) \neq 0$ the $n+1^{th}$ approximation is x_{n+1}



$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad ; \quad f'(x_n) \neq 0$$

Example: Use Newton's Method with initial approximation $x_1 = 1$ find the third approximation x_3 to the root of the equation.

$$f(x) = x^5 - x - 1 = 0$$

$$f'(x) = 5x^4 - 1$$

By N-R method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow x_n - \frac{x_n^5 - x_n - 1}{5x_n^4 - 1}$$

when $x_n = 1$

then $x_2 = x_1 - \frac{x_1^5 - x_1 - 1}{5x_1^4 - 1}$

$$\Rightarrow 1 - \frac{1 - 1 - 1}{5 - 1}$$

$$\left[x_2 \Rightarrow 1.25 \right]$$

then $x_3 = x_2 - \dfrac{x_2^5 - x_2 - 1}{5x_2^4 - 1}$

$$\Rightarrow 1.25 - \frac{(1.25)^5 - 1.25 - 1}{5(1.25)^4 - 1}$$

$$\boxed{x_3 \Rightarrow 1.1785}$$

$\therefore$ if we Repeat we get Convergence at some x_n iteration.