Day-35, Dec-23, 2024 (Poush-7, 2081 BS).

① Inverse of Matrix

Invertible Definition: An $n\times n$ matrix A is called invertible if there is an $n\times n$ matrix C such that

$$AC = I = CA$$

Where I be $n\times n$ identity matrix.

② Inverse: If 'A' is an invertible matrix then there is matrix C such that $AC = I = CA$. In such case, C is called inverse of A and write it as $C = A^{-1}$.

Definition (Singular and Non-Singular Matrix):

If A is an invertible matrix
then it is a non-singular Matrix
else:
A is a singular matrix.

Square Matrix A is called Singular
if $|A| = 0$
else it is called non-singular.

Trivial solution: $x=0,\ y=0$ (zero solutions in General solution)

Non-trivial solution: $x=\frac{1}{3},\ y=0$ (at least General solution variable have non-zero)

Q. Let $A=\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ and $C=\begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$

$$AC=\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}\cdot\begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -14+6 & -35+15 \\ 15-6 & +35-14 \end{bmatrix} \Rightarrow \begin{bmatrix} -8 & -20 \\ 9 & 21 \end{bmatrix}$$

$$CA = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -14+21 & -10+15 \\ 15- & -14 \end{bmatrix} \Rightarrow \begin{bmatrix} 7 & 5 \\ 1 & 1 \end{bmatrix}$$

If $AC = I = CA$ therefore A is invertible and C is the inverse matrix of A.

(Non-trivial solution has at least one free variable.)

Theorem: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ if $ad - bc \neq 0$ then A is invertible and

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

if $ad - bc = 0$ then A is not invertible matrix.

Soln;

Since $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Let $Ax = 0$ if A^{-1} exists then —

$\Rightarrow A^{-1} A x = A^{-1} 0$

$\Rightarrow \quad Ix = 0 \qquad (I = A^{-1}A).$

$\Rightarrow \quad x = 0$

Therefore, for existence of A^{-1}, $Ax = 0$ has trivial solution.

Here, the augmented matrix is

$$[A \;\; 0] = \begin{bmatrix} a & b & 0 \\ c & d & 0 \end{bmatrix}$$

Reducing in Echelon form $\sim \begin{bmatrix} 1 & b/a & 0 \\ c & d & 0 \end{bmatrix} - \begin{bmatrix} 1 & b/a & 0 \\ 0 & ad-bc/a & 0 \end{bmatrix}$

$Ax = 0$ has trivial solution only when there is no One free variable.

i.e. $\dfrac{ad - bc}{a} \neq 0$

$\Rightarrow \quad ad - bc \neq 0$

Hence, if $ad - bc \neq 0$ then A^{-1} exists.

Here, $A^{-1}A = \dfrac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow I$

Also $AA^{-1} = I$.

Thus, A is invertible and $A^{-1} = \frac{1}{ad-bC} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

if $ad - bc = 0$ then –

$$A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

which is not exist being $ad - bc = 0$.

This means A is not invertible when $ad - bc = 0$,

Theorem: If A is an invertible $n \times n$ matrix then for each b in $\mathbb{R}^n$, the equation $Ax = b$ has the unique solution $x = A^{-1} b$.

Proof: Let A is an invertible $n \times n$ matrix. Therefore, inverse of A (i.e. A^{-1}) exists. Take b is in such that $Ax = b$.

then we have to show

$$x = A^{-1} b$$

$$\Rightarrow Ax = A(A^{-1} b)$$

$$\Rightarrow (AA^{-1}) b = Ib$$

$$\Rightarrow b$$

This shows that $x = A^{-1} b$

$$\Rightarrow Ax$$
$$\Rightarrow A(A^{-1} b)$$
$$\Rightarrow (AA^{-1}) b$$
$$\Rightarrow Ib$$
$$\boxed{\Rightarrow Ax \Rightarrow b}$$

This shows that $x = A^{-1} b$ is a solution of $Ax = b$

For uniqueness of x, we suppose y be another solution $Ax = b$

if possible, then,

$$Ay = b$$

$$\Rightarrow A^{-1}(Ay) = A^{-1}b$$

$$\Rightarrow Iy = A^{-1}b$$

$$\Rightarrow y = A^{-1}b$$

from eqn (i) and eqn (ii) $x = y$.

This means x is an unique solution of $Ax = b$.

Theorem: (a) If A is an invertible matrix then A^{-1} is invertible and $(A^{-1})^{-1} = A$.

(b) If A and B are $n \times n$ matrix or invertible matrices, then so is AB and the inverse of AB is the product of the inverses of A and B in the reverse order. That is $(AB)^{-1} = B^{-1}A^{-1}$.

(c) If A is an invertible matrix, then so is A^T and the inverse of A^T is the transpose of A^{-1}. That is

$$(A^T)^{-1} = (A^{-1})^T$$

Proof:

(a) Let A is an invertible matrix then A^{-1} exists. And,

$$AA^{-1} = A^{-1}A$$

$$\Rightarrow I$$

This means A^{-1} is also invertible.

Therefore, $(A^{-1})^{-1} = A$.

(b) Let A and B are invertible matrices. So, A^{-1} and B^{-1} exists. Also,

$$AA^{-1} = A^{-1}A$$

$$\Rightarrow I$$

and $BB^{-1} = B^{-1}B \Rightarrow I$.

Now, $(AB)\cdot(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$

$\Rightarrow AIA^{-1}$

$\Rightarrow AA^{-1}$

$[(AB)(B^{-1}A^{-1}) \Rightarrow I]$

and

$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B$

$\Rightarrow B^{-1}IB$

$\Rightarrow B^{-1}B$

$\Rightarrow I.$

$\therefore B^{-1}A^{-1}$ is an Inverse of AB. That is $[B^{-1}A^{-1} = (AB)^{-1}]$

c) Let A is an invertible matrix. So, A^{-1} exists.

Here,

$$A^T (A^{-1})^T = (A^{-1} A)^T = I^T = I$$

$$[\because B^T A^T = (AB)^T]$$

and

$$(A^{-1})^T A^T = (AA^{-1})^T$$

$$= I^T$$

$$= I.$$

This means A^T is invertible and an inverse matrix of A^T is $(A^T)^{-1}$. That is $(A^T)^{-1} = (A^{-1})^T$

Let $A = \begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 7 & 4 \end{bmatrix}$. Then verify $(AB)^{-1} = B^{-1}A^{-1}$.

Here,

$$\det(A) = \begin{vmatrix} 8 & 6 \\ 5 & 4 \end{vmatrix} \Rightarrow 2$$

$$\det(B) = \begin{vmatrix} 3 & 2 \\ 7 & 4 \end{vmatrix} \Rightarrow -2$$

Hence, A^{-1} and B^{-1} exist.

$$B^{-1}A^{-1} = \frac{\text{Adj}.B}{\det(B)} \cdot \frac{\text{Adj}.A}{\det(A)}$$

$$\Rightarrow \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -7 & 3 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 4 & -6 \\ -5 & 8 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -13/2 & 10 \\ 43/4 & -33/2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 7 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 66 & 40 \\ 43 & 28 \end{bmatrix}$$

$$\det(AB) = \begin{vmatrix} 66 & 40 \\ 43 & 28 \end{vmatrix} \Rightarrow -4$$

So $(AB)^{-1}$ exists. And

$$(AB)^{-1} = \frac{[\text{Adj} \cdot AB]}{\det(AB)}$$

$$\Rightarrow \frac{1}{-4}\begin{bmatrix} 26 & -40 \\ -43 & 66 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -13/2 & 10 \\ 43/4 & -33/2 \end{bmatrix}$$

Thus, $\underline{(AB)^{-1} = B^{-1}A^{-1}}$

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case any sequence of elementary row equations that reduces A to I_n also transform I_n into A^{-1}.

[An Algorithm for finding A^{-1}]

Algorithm: If A is row equivalent to I $[A \quad I]$ is row equivalent to $[I \quad A^{-1}]$. Otherwise, A^{-1} does not exist.

Find the inverse of the matrix.

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix},$$

by using elementary row reduce Augmented Matrix.

if we solve this we get $-$ A^{-1} $\begin{bmatrix} 9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$

Characterizations of Invertible Matrices:

Let A be a square $n \times n$ matrix. Then the following statements are equivalent.

i) A is an invertible matrix.

ii) A is row equivalent to the $n \times n$ identity matrix.

iii) A has n pivot positions

iv) The equation $Ax = 0$ only has the trivial solution

v) The columns of A form a linearly Independent set.

vi) The linear transformation $x \to Ax$ is one-to-one.

vii) The equation $Ax = b$ has at least one-solution for each b in $\mathbb{R}^n$.

viii) The columns of A span $\mathbb{R}^n$.

ix) The linear transformation $x \rightarrow Ax$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

x) There is an $n \times n$ matrix C such that $CA = I$.

xi) There is an $n \times n$ matrix C such that $AD = I$.

xii) A^T is an invertible Matrix.

Partitioned Matrices:

The horizontal and vertical dividing rules to a matrix, such rule is partitions of the matrix.

Let $A = 3\times5$ matrix.

$$A = \begin{bmatrix} 2 & 1 & 0 & 4 & 9 \\ 3 & 2 & -1 & 2 & 1 \\ 5 & 3 & 3 & 6 & 3 \end{bmatrix}$$

Divide as,

$$A_{11} = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & -1 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 5 & 3 & 3 \end{bmatrix} \qquad A_{22} = \begin{bmatrix} 6 \end{bmatrix}, \quad A_{23} = \begin{bmatrix} 3 \end{bmatrix}.$$

Then A can be written as -

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Here, A is called partitioned or block matrix whose entries are blocks $A_{11}, A_{12}, A_{13}, A_{21}, A_{22}, A_{23}$.

Addition and Scalar Multiplication:

If two matrices A and B are the same size and are partitioned in exactly the same way so that the addition is possible. Then, the sum of A and B is same to the given matrix. Similarly, the scalar multiple of A is the multiplication of a partitioned matrix by a scalar.

Multiplication of Partitioned Matrices

The multiple of partitioned matrices by usual row-column rule, produces a multiple (i.e. product) of the matrices.

Theorem: Column-Row Expansion of B

If A is m×n and B is n×p matrices then

$$AB = \begin{bmatrix} col_1(A) & col_2(A) & \cdots & col_n(A) \end{bmatrix} \begin{bmatrix} row_1(B) \\ \vdots \\ row_n(B) \end{bmatrix}$$

$$\Rightarrow \sum_{k=1}^{n} col_k(A) \cdot row_k(B).$$

Definition (Block Upper and Lower Triangular Matrices):

A matrix of a form $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$

is called block upper triangular matrix.

A matrix of a form $B = \begin{bmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{bmatrix}$

is called Block Lower Triangular Matrix.

[Numerical Importance, Computational Simplicity]