Day-42, Jan-11, 2025 (Poush-27, 2081)

Anti-Derivatives (Indefinite Integral)

Let f be continuous function defined in an open interval (a, b). The function F is said to be an antiderivative of f on the interval, if the derivative of F is equal to f on the interval, i.e. if,

$$\frac{dF(x)}{dx} = f(x), \quad x \in (a, b)$$

As the derivative of a constant 'c' is zero, $F(x) + c$

is also an anti-derivative of f, whenever the function F is so

Actually, when -

$$\frac{dF(x)}{dx} = f(x)$$

We have,

$$\frac{d\,[F(x) + c]}{dx} = \frac{dF(x)}{dx} + \frac{dc}{dx}$$

$$\Rightarrow f(x) + 0$$

$$\Rightarrow f(x)$$

The converse is also true any two antiderivatives of a function differ a constant. Let F and G be antiderivatives of a function f. Then—

$$\frac{d\left[F(x) - G(x)\right]}{dx} = \frac{dF(x)}{dx} - \frac{dG(x)}{dx}$$

$$\Rightarrow f(x) - f(x)$$

$$\Rightarrow 0$$

From this it follows that there exists a constant C such that,

$$\left[F(x) - G(x) = C\right]$$

All these go to establish the fact that if F is an antiderivative of f, $F(x)+C$ gives all the possible antiderivatives of f, when c runs through the real numbers

General form of All Anti-derivatives which we call Indefinite Integral of f denoted by:

$$\int f\,dx \quad \text{or} \quad \int f(x)\cdot dx$$

If F is an antiderivative of f, we have

$$\int f(x) \cdot dx = F(x) + C$$

The basic property of the indefinite integral is -

$$\int \{ c_1 f(x) + c_2 g(x) \} dx = c_1 \int f(x) \cdot dx + c_2 \int g(x) \cdot dx + C$$

Where f and g are continuous functions in an interval (a, b) and c_1 and c_2 are some constants

Integration is the reverse process of differentiation. Thus if $\frac{d}{dx}[f(x)] = F(x)$, then $f(x)$ is called the anti-derivative or integral of $F(x)$.

It is denoted by $\int F(x) \cdot dx = f(x)$

for example: $\frac{d}{dx}(\sin x) = \cos x$ then $\int \cos x \cdot dx = \sin x$

Generally, if $\frac{d}{dx}[f(x)] + C = F(x)$ then, $\int F(x) \cdot dx = f(x) + C$

where C is arbitrary constant.

Anti-Derivative

Evaluation of the Area Under a plane Curve or we can say a limit of a sum when the number of terms in the sum tends to infinity and each term tends to zero. $\frac{d}{dx} f(x)$

→ The inverse of differentiation is Anti-Derivative or Indefinite Integral

$d\,F(x)$ or $\int f(x)\,dx$

Example:

$$\frac{d(\sin x)}{dx} = \cos x$$

$$\therefore \int \cos x \cdot dx = \sin x$$

Standard Integrals (I) → Directly Related to Standard Differentiation formulae

Eg.:

$$\int \frac{1}{a^2 + x^2} \cdot dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

Put $x = a \tan\theta$ $\qquad dx = a\sec^2\theta \cdot d\theta$

$$a^2 + x^2 \Rightarrow a^2 + a^2 \tan^2\theta$$

$$\Rightarrow a^2(1 + \tan^2\theta)$$

$$\Rightarrow a^2 \sec^2\theta$$

$$\therefore \int \frac{1}{a^2 + x^2} \cdot dx = \int \frac{1}{a^2 \sec^2\theta} \cdot a \sec^2\theta \cdot d\theta$$

$$\Rightarrow \frac{1}{a} \int d\theta$$

$$\Rightarrow \frac{1}{a} \theta + C$$

$$\left[\therefore \int \frac{1}{a^2 + x^2} \cdot dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C \right]$$

Standard Integrals (II)

In this section we considered the second set of standard integrals which can be evaluated by a technique known as integration by parts. This is a formula which can be deduced as given below:

We know,

$$\frac{d}{dx}(uv_1) = \frac{du}{dx}.v_1 + u.\frac{dv_1}{dx}$$

for any two differentiable functions u and v_1 of x. Integrating both sides, we get –

We Know,

$$\frac{d}{dx}(u \cdot v_1) = \frac{du}{dx} \cdot v_1 + u \cdot \frac{dv_1}{dx}$$

for any two differentiable functions u and v_1 of x. Integrating both sides, we get –

$$uv_1 = \int \left(\frac{du}{dx} \cdot v_1 \right) \cdot dx + \int \left(u \cdot \frac{dv_1}{dx} \right) \cdot dx$$

Let $\frac{dv_1}{dx} = v$ So that $v_1 = \int v \, dx$. Then find

$$\int (uv) \cdot dx = u \int v \cdot dx - \int \left(\frac{du}{dx} \int v \cdot dx \right) \cdot dx$$

which is the required formula.

Recovering the function F from its known derivative.
If such function F exists it is called an anti-derivative of f.

Example: (a) $f(x) = 2x$ (b) $g(x) = \cos x$ (c) $h(x) = 2x + \cos x$

(a) $f(x) = 2x$ $F(x) = x^2$

(b) $g(x) = \cos x$ $G(x) = \sin x$

(c) $h(x) = 2x + \cos x$ $H(x) = x^2 + \sin x$

Example:

$f(x)$ if $f'(x) = e^x + 20(1+x^2)^{-1}$ and $f(0) = 5$

Here,

$$f'(x) = e^x + 20(1+x^2)^{-1}$$

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$

$$f(x) = e^x + 20\tan^{-1}x + C$$

Since $f(0) = 5$

$$f(0) = e^0 + 20\tan^{-1}0 + C$$

$$5 = 1 + C$$

$$\boxed{C = 4}$$

$$\boxed{f(x) = e^x + 20\tan^{-1}x + 4}$$