Day-24, Dec-8, 2024 (Mangshir-24, 2081 B.S.)

Techniques for Differentiation

Here we shall discuss some fundamental formulae of differentiation

1. The Sum Rule:

Let $f(x)$ and $g(x)$ be any two differentiable functions of 'x'.

Let $h(x) = f(x) \pm g(x)$

then $h'(x) = f'(x) \pm g'(x)$

or, $\frac{d}{dx} h(x) = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$.

Proof: Let 'a' be a fixed point. Then, by definition -

$$h'(a) = \lim_{x \to a} \frac{h(x) - h(a)}{x - a}$$

$$\Rightarrow \lim_{x \to a} \frac{[f(x) \pm g(x)] - [f(a) \pm g(a)]}{x - a}$$

$$\Rightarrow \lim_{x \to a} \frac{f(x) - g(a)}{x - a} \pm \lim_{x \to a} \frac{g(x) - g(a)}{x - a}$$

$$\Rightarrow f'(a) \pm g'(a).$$

But 'a' is an arbitrary fixed number so -

$$h'(a) = f'(a) \pm g'(a) \quad \text{or} \quad h'(x) = f'(x) \pm g'(x)$$

or $$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$$

Put $f(x) = u$ and $g(x) = v$, then we have -

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

From this Sum Rule, we can deduce a formula for the differentiation of 'a constant times a function'.

$$\frac{d(2u)}{dx} = \frac{d(u+u)}{dx}$$

$$= \frac{du}{dx} + \frac{du}{dx}$$

$$\Rightarrow 2 \cdot \frac{dy}{dx}$$

$$\frac{d(3u)}{dx} = \frac{d(2u+u)}{dx}$$

$$\Rightarrow \frac{d(2u)}{dx} + \frac{du}{dx}$$

$$\Rightarrow 2 \cdot \frac{dy}{dx} + \frac{dy}{dx}$$

$$\Rightarrow 3 \cdot \frac{du}{dx}$$

$$\frac{d(4u)}{dx} = \frac{d(3u+u)}{dx}$$

$$\Rightarrow \frac{d(3u)}{dx} + \frac{du}{dx}$$

$$\Rightarrow 3 \cdot \frac{du}{dx} + \frac{du}{dx}$$

$$\Rightarrow 4\frac{du}{dx}$$

So in General, we have –

$$\frac{d(nu)}{dx} = n \cdot \frac{du}{dx}$$ for any integer 'n'.

(Thus Rule not hold only for an integer, but also for an rational number).

Q. Derivative of $5x^3+4x^2-2x=7$?

Let $y = 5x^3+4x^2-2x=7$

$$\boxed{n\cdot\frac{du}{dx} = \frac{d(nu)}{dx}}$$

Differentiating both sides w.r.t 'x' we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(5x^3+4x^2-2x+7\right)$$

$$\Rightarrow \frac{d(5x^3)}{dx} + \frac{d(4x^2)}{dx} - \frac{d(2x)}{dx} + \frac{d(7)}{dx}$$

$$\Rightarrow 5\cdot\frac{dx^3}{dx} + 4\left(\frac{dx^2}{dx}\right) - 2\frac{dx}{dx} + 7\cancel{\frac{d}{dx}}$$

$$\boxed{\frac{dy}{dx} \Rightarrow 15x^2 + 8x - 2}$$

Product Rule:

Let $f(x)$ and $g(x)$ be any two differentiable functions of x. Let $h(x) = f(x) \cdot g(x)$. Then -

$$h'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

Or, $$\frac{d}{dx} h(x) = f(x) \cdot \frac{d}{dx} g(x) + g(x) \cdot \frac{d}{dx} f(x)$$

Proof: Let 'a' be fixed point. Then, by definition –

$$h'(a) \Rightarrow \lim_{x \to a} \frac{h(x) - h(a)}{x - a}$$

$$\Rightarrow \lim_{x \to a} \frac{f(x) \cdot g(x) - f(a) \cdot g(a)}{x - a}$$

$$\Rightarrow \lim_{x \to a} \frac{f(x) \cdot g(x) - f(x) \cdot g(a) + f(x) \cdot g(a) - f(a) \cdot g(a)}{x - a}$$

$$\Rightarrow \lim_{x \to a} f(x) \cdot \lim_{x \to a} \frac{g(x) - g(a)}{x - a} + g(a) \cdot \lim_{x \to a} \frac{f(x) - g(a)}{x - a}$$

$$\left[\, h'(a) \Rightarrow f(a)\, g'(a) + g(a) \cdot f'(a) \,\right]$$

But 'a' is an arbitrary fixed number, So

$$h'(x) = f(x) \cdot g'(x) + g(x) \cdot f(x)$$

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot \frac{d}{dx} g(x) + g(x) \cdot \frac{d}{dx} f(x)$$

Put $f(x) = u$ and $g(x) = v$. Then we get,

$$\left[\frac{d}{dx}(u \cdot v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx} \right]$$

① Sum Rule $\frac{d(nu)}{dx} = n \cdot \frac{dy}{dx}$

② Product Rule $\frac{d}{dx}(u \cdot v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$.

Example: find the derivative of $(3x^2-5x)(2x+3)$.

Let $u = 3x^2 - 5x$ and $v = 2x+3$ then

$$\frac{d}{dx}(u \cdot v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

$$\Rightarrow 3x^2-5x \cdot \frac{d(2x+3)}{dx} + (2x+3) \cdot \frac{d(3x^2-5x)}{dx}$$

$$\Rightarrow (3x^2-5x) \cdot 2 + (2x+3) \cdot (6x-5)$$

$$\Rightarrow 6x^2 - 10x + 12x^2 - 10x + 18x - 15$$

$$\Rightarrow 6x^2 - 2x + 12x^2 - 15 \qquad \Rightarrow 18x^2 - 2x - 15$$

III) **Power Rule:**

If 'u' is the function of 'x', then

$$\frac{d}{dx}(u^n) = nu^{n-1} \cdot \frac{du}{dx} \quad \text{—— eqn (i)}$$

Proof: By the product Rule, we have —

$$\frac{d}{dx}(u^2) = \frac{d}{dx}(u \cdot u)$$

$$\Rightarrow u \cdot \frac{du}{dx} + u \cdot \frac{du}{dx}$$

$$\Rightarrow 2u \cdot \frac{du}{dx}$$

or, $\frac{d}{dx}(u^3) \Rightarrow \frac{d(u \cdot u^2)}{dx}$

$$\Rightarrow u \cdot \frac{du^2}{dx} + u^2 \cdot \frac{du}{dx}.$$

$$\Rightarrow u \cdot 2u \cdot \frac{du}{dx} + u^2 \cdot \frac{du}{dx}$$

$$\left[\frac{d}{dx}(u^3) \Rightarrow 3u^2 \frac{du}{dx} \right]$$

Similarly, for any rational number 'n', we have

$$\left[\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx} \right]$$

Example: Find the derivative of $(4x^3+5)^{3/2}$.

Let $u = 4x^3+5$

$$\frac{du}{dx} = \frac{d}{dx}(4x^3+5)^{3/2}$$

or, simply $\frac{du}{dx} = \frac{d}{dx}(4x^3+5)$

$\Rightarrow 12x^2$

$$\frac{d}{dx}(4x^3+5)^{3/2} = \frac{d\left(u^{3/2}\right)}{dx}$$

$$\Rightarrow \frac{3}{2} \cdot u^{3/2-1} \cdot \frac{du}{dx}$$

$$\Rightarrow \frac{3}{2} \cdot u^{1/2} \cdot \frac{du}{dx}.$$

$$\Rightarrow \frac{3}{2}(4x^3+5)^{1/2} \cdot 12x^2 \quad \left(\text{Since } \frac{du}{dx} = 12x^2\right)$$

$$\Rightarrow 18x^2\sqrt{4x^3+5}$$

The Quotient Rule:

Let $f(x)$ and $g(x)$ be any two differentiable function of x.

Let $f(x) = \frac{f(x)}{g(x)}$. then-

$$h'(x) = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{[g(x)]^2}$$

Proof: We have, by definition —

$$h'(a) = \lim_{x \to a} \frac{h(x) - h(a)}{x-a}$$

$$= \lim_{x \to a} \frac{\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}}{x-a} \qquad [g(a) \neq 0]$$

$$= \lim_{x \to a} \frac{g(a)\cdot f(x) - f(a)\cdot g(x)}{(x-a)\, g(x)\cdot g(a)}$$

$$\Rightarrow \lim_{x \to a} \frac{g(a) \cdot f(x) - g(a) f(a) + g(a) \cdot f(a) - f(a) \cdot g(x)}{(x-a)\, g(x)\, g(a)}$$

$$\Rightarrow \lim_{x \to a} \left[\frac{1}{g(a)\, g(x)} \left\{ g(a) \cdot \frac{f(x) - f(a)}{x-a} - f(a) \cdot \frac{g(x) - g(a)}{x-a} \right] \right.$$

$$\Rightarrow \frac{1}{g(a) \cdot g(a)} \cdot \left[g(a) f'(a) - f(a) \cdot g'(a) \right]$$

$$\Rightarrow \frac{g(a) \cdot f(a) - f(a) \cdot g'(a)}{[g(a)]^2}$$

$\Rightarrow$ In general, we have

$$h'(x) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$$

or,
$$\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x) \cdot \frac{d}{dx} f(x) - f(x) \cdot \frac{d}{dx} g(x)}{[g(x)]^2}$$

Put $u = f(x)$ and $v = g(x)$. Then we have

$$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \cdot \frac{du}{dx} - u \frac{dv}{dx}}{v^2}}$$

Example: Find the derivative of $\dfrac{4x^2+3}{3x^2-2}$

Let $u = 4x^2+3$ and $v = 3x^2-2$. Then

$$\frac{du}{dx} = g(x) \quad \text{and} \quad \frac{dv}{dx} = 6x$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$$

$$\Rightarrow \frac{(3x^2-2) \cdot 8x - (4x^2+3) \cdot 6x}{(3x^2-2)^2}$$

$$\Rightarrow \frac{-34x}{(3x^2-2)^2}$$

Chain Rule:

If $y = f(u)$ and $u = g(x)$, where f and g are differentiable functions then $\frac{dy}{dx}$ exists and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\left[\frac{dy}{dx} = f'(u) \cdot \frac{du}{dx} \right]$$

Proof: We have $u = g(x)$. Let Δx be a small increment in x and Δu be corresponding small increment in 'u'. Then -

$$u + \Delta u = g(x + \Delta x)$$

Since, $g(x)$ is differentiable, it is continuous. So,

$$\lim_{\Delta x \to 0} \Delta u = \lim_{\Delta x \to 0} [g(x+\Delta x) - g(x)]$$

$$\Rightarrow g(x) - g(x)$$

$$\Rightarrow 0$$

Thus, $\Delta u \to 0$ as $\Delta x \to 0$. Then

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[\frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}\right]$$

$$= \left(\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u}\right)\left(\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}\right)$$

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

Example: find $\frac{dy}{dx}$, if $y = 4u^2 - 3u + 5$ and $u = 2x^2 - 3$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{d(4u^2 - 3u + 5)}{d(2x^2 - 3)} \cdot \frac{d(2x^2 - 3)}{d(4u^2 - 3u + 5)}$$

Instead of above, $\frac{dy}{du} = 8u - 3$ and $\frac{du}{dx} = 4x$.

$$\frac{d(4u^2 - 3u + 5)}{dx} \text{ and } \frac{d(2x^2 - 3)}{dx} = 4x.$$

$$\Rightarrow 8u - 3$$

So, $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\Rightarrow (8u - 3) \cdot 4x$$

$$\Rightarrow [8(2x^3 - 3) - 3] \cdot 4x$$

$$\left(\frac{dy}{dx} \Rightarrow 64x^3 - 108x \right)$$