Day-38, Dec-27, 2024 (Poush 12, 2081)

Example: find $\left(\frac{\partial w}{\partial x}\right)_{y,z}$ if $w = x^2 + y - z + \sin t$

and $x + y = t$.

Let,

$w = x^2 + y - z + \sin t$ and $x + y = t$

$w = x^2 + y - z + \sin t$

$\Rightarrow x^2 + y - z + \sin(x+y)$

So, $\left(\frac{\partial w}{\partial x}\right)_{yz} = 2x + \cos(x+y)$

Example: find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ where $w = x + 2y + z^2$,

$x = r/s$, $y = r^2 + \sin s$, $z = 2r$

Let,

$$w = x + 2y + z^2 \text{ ——— eqn (i)}$$

$$\boxed{x = r/s \qquad y = r^2 + \sin s \qquad z = 2r. \text{ — eqn (ii)}}$$

then,

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r} \text{ - eqn (iii)}$$

$$\Rightarrow (1)(1/s) + (2)(2r) + (2z)(2)$$

Because, $\frac{\partial w}{\partial r} = ?$ $\frac{\partial w}{\partial s} = ?$

Partial Derivatives of w — eqn (i)

$$\frac{\partial w}{\partial x} = 1 \qquad \frac{\partial w}{\partial y} = 2 \qquad \frac{\partial w}{\partial z} = 2z$$

Partial Derivatives of x, y, z w.r.t to r — eqn (ii)

$$\frac{\partial x}{\partial r} = 1/s, \quad \frac{\partial y}{\partial r} = 2r, \quad \frac{\partial z}{\partial r} = 2$$

Now substitute in eqn (iii) we get -

$$\boxed{\frac{\partial w}{\partial r} \Rightarrow \left(\frac{1}{s}\right) + 4r + 4z}$$

Now,

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial s}$$ → eqn (iv)

Similarly

$$\frac{\partial x}{\partial s} = -\frac{1}{s^2}, \quad \frac{\partial y}{\partial s} = \cos s, \quad \frac{\partial z}{\partial s} = 0$$ (Since $z = 2r$ doesn't depend upon s)

So, eqn (iv) becomes–

$$\Rightarrow (1)\left(-\frac{1}{s^2}\right) + (2)(\cos s) + (2z)(0) \Rightarrow \boxed{2\cos s - 1/s^2}$$

Implicit Differentiation:

Normally, the process of differentiation of an equation $f(x, y) = 0$ to obtain the value of $\frac{dy}{dx}$, is the implicit differentiation -

if $\omega = f(x, y) = 0$ be an equation. Then by using the partial differentiation, we find,

$$0 = \frac{d\omega}{dx}$$

$$= \frac{\partial f}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

$$① \Rightarrow f_x + f_y \cdot \frac{dy}{dx}$$

$$\Rightarrow 0 = f_x + f_y \cdot \frac{dy}{dx}.$$

$$\Rightarrow \frac{dy}{dx} = -\frac{f_x}{f_y} \quad [\text{for } f_y \neq 0]$$

Definition: If $f(x,y) = 0$ be an equation then the differentiation of 'y' w.r.t to 'x' at the point where $f_y \neq 0$ is,

$$\left[\frac{dy}{dx} = -\frac{f_x}{f_y}\right]$$

Examples: using partial differentiation, find $\frac{dy}{dx}$ if $x^2 + \sin y - 2y = 0$.

Let, $f(x,y) = x^2 + \sin y - 2y = 0$.

Then, $f_x = \frac{\partial f}{\partial x} \Rightarrow 2x$

$f_y = \frac{\partial f}{\partial y} \Rightarrow \cos y - 2 \neq 0$

So, $\frac{dy}{dx} = -\frac{f_x}{f_y}$

$\Rightarrow \frac{-2x}{\cos y - 2}$

Example: Using partial derivatives, find $\frac{dy}{dx}$ if $x^2 + \cos y - y^2 = 0$.

Let,

$$f(x, y) = x^2 + \cos y - y^2 = 0$$

$$f_x = \frac{\partial f}{\partial x} = 2x \qquad f_y = \frac{\partial f}{\partial y}$$

$$\Rightarrow \sin y - 2y.$$

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

$$\Rightarrow \frac{-2x}{-\sin y - 2x}$$

$$\boxed{\frac{dy}{dx} \Rightarrow \frac{2x}{\sin y + 2x}}$$

Example: Using implicit differentiation, find the value of $\frac{dy}{dx}$ at $(0, \ln(2))$ when $xe^y + y\sin xy - \ln(2) = 0$.

Let,

$$f(x,y) = xe^y + y\sin xy - \ln(2) = 0$$

Then,

$$f_x = \frac{\partial f}{\partial x} \quad \text{and} \quad f_y = xe^y + x\cos xy$$

$$\Rightarrow e^y + y\cos xy$$

$$\frac{dy}{dx} = \frac{-f_x}{f_y}$$

at point $(0, \ln(2))$,

$$\frac{dy}{dx}\Big| \text{ at } (0, \ln(2))$$

$$\Rightarrow \frac{-\left(e^{\ln(2)} + \ln(2)\right) \cdot \cos(0)}{0+1+0}$$

$$\Rightarrow \frac{-(2+\ln(2))}{1}$$

$$\left[\frac{dy}{dx} \Rightarrow -(2+\ln(2))\right] \text{ at } \begin{pmatrix} x, & y \\ 0, & \ln(2) \end{pmatrix}$$

Hession Matrix: (H or ∇^2)

Hession Matrix or Hession is a square matrix of second-order partial derivatives of scalar-valued function or scalar-field.

$\nabla \rightarrow$ nabla operator used to denote gradient

Definitions and Properties:

Suppose $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is a function taking as input vector $x \in \mathbb{R}^n$ and outputing scalar $f(x) \in \mathbb{R}$.

If all second-order partial derivatives of f exist, then the Hession matrix of H of f is a square $n \times n$ matrix

Usually defined and arranged as —

$$H_f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_2 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_2 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & & \vdots & \vdots & & \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

That is, the entry of the ith row and the jth column

$$(H_f)_{ij} = \frac{\partial^2 f}{\partial x_i \cdot \partial x_j}$$

(Hessian Matrix is Symmetric by the Symmetry of second derivative if the second-order partial derivatives are all continuous.).

Determinant of Hessian Matrix = Hessian Determinant

The Hessian Matrix of a function f is the transpose of the Jacobian Matrix of the gradient of the function f; that is

$$H(f(x)) = J(\nabla f(x))^T$$

Example: Hessian for a Simple function

Lets calculate the Hessian Matrix for a simple function of two variables.

$$f(x,y) = x^2 + 3xy + y^2$$

① find the first-order derivative (∇)

$$\frac{\partial f}{\partial x} = 2x + 3y$$

$$\frac{\partial f}{\partial y} = 3x + 2y$$

② find the second-order derivative (∇^2) → elements of Hessian Matrix.

$$\frac{\partial^2 f}{\partial x^2} \Rightarrow 2$$

$$\frac{\partial^2 f}{\partial y^2} \Rightarrow 2$$

(∵ verified Euler's theorem)

$$\frac{\partial^2 f}{\partial x \partial y} \Rightarrow 3 \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y} \Rightarrow 3$$

(4) · Construct the Hessian Matrix:

$$H(f) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$$

Optimization (Choosing the step size and direction)

Convexity → if all eigenvalues are +ve, then function is convex.

Convergence → ∇^2 helps to determine the curvature of f to adjust the update steps more accurately

Hessian Matrix in Optimization

Update Rule for Newton's Method is :

$$\theta_{k+1} = \theta_k - H(f)^{-1} \nabla f(\theta_k)$$

where

θ_k is the parameter vector of k iteration

- $\nabla f(\theta_k)$ is the gradient at θ_k,
- $H(f)^{-1}$ is the inverse of the Hessian Matrix.

In Newton's Method we use both the gradient and the Hessian Matrix. Curvature Adjustments $\rightarrow$ where curvature is steep it takes smaller steps and in flatter regions it takes larger steps.