#Day-53, Jan 22, 2025 (Magh 09, 2081 B.S.)

Series Representations:

If we differentiate infinite times to a power series having center at a, the series of all such differentiation is known as a Taylor Series.

As a particular form of Taylor Series with centre at origin is popularly known as a Maclaurin Series.

Development of Taylor Series:

Consider a Power series,

$$f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n$$

$$\Rightarrow C_0 + C_1(x-a) + C_2(x-a)^2 + \cdots + C_n(x-a)^n + \cdots \cdots \longrightarrow \text{eqn } (i).$$

Now, differentiate $f(x)$ term by term then we obtain,

$$f'(x) = C_1 + 2C_2(x-a) + \cdots + nC_n(x-a)^{n-1} + \cdots$$

$$f''(x) = 2C_2 + 6C_3(x-a) + \cdots + n(n-1)C_n(x-a)^{n-2} + \cdots$$

$$f'''(x) = 6C_3 + 24C_4(x-a) + \cdots + n(n-1)(n-2)C_n(x-a)^{n-3} + \cdots$$

$$f^{(n)}(x) = n(n-1)\cdots 2 \cdot 1 \cdot C_n + \cdots$$

$$\Rightarrow n!\,C_n + \cdots$$

At $x = a$,

$$f(a) = c_0$$

$$f'(a) = c_1$$

$$f''(a) = 2c_2$$

$$\Rightarrow c_2 = \frac{f''(a)}{2!}$$

$$f'''(a) = 6c_3 \quad \Rightarrow c_3 = \frac{f'''(a)}{6} \quad \Rightarrow c_3 = \frac{f'''(a)}{3!}$$

$$\vdots$$

$$f^{n}(a) \Rightarrow n!\, c_n \quad \Rightarrow c_n = \frac{f^{(n)}(a)}{n!}$$

Replacing the coefficients $c_0, c_1, c_2 \ldots$ by its value in (i) then we get,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$\therefore$ This series is known as Taylor Series.

Definition (Taylor's Series and Maclaurin Series):

Let f be a continuously differentiable function of all orders throughout some interval containing 'a' as an interior point.

Then the taylor series generated by f at $x = a$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

and the Maclaurin Series generated by f is,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Definition (Taylor Polynomial) :

Let f be a function with derivatives of order k for in some interval containing 'a' as an interior point. Then for any integer 'n' from 0 through N, the Taylor polynomial of order n generated by f at $x = a$ is the polynomial,

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Example:

Use Maclaurin expansion, show that

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \quad \text{for } |x| < \infty.$$

Sol^n:

Let, $f(x) = e^x$

By Maclaurin expansion we have

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

Here,

$f(x) = e^x$

Differentiating we get,

$$f'(x) = e^x$$

$$\Rightarrow f''(x)$$

$$\Rightarrow f'''(x) = \ldots, \forall x$$

At $x=0$,

$$f(0) = e^0 \Rightarrow 1 = f'(0) = f''(0) = f'''(0) = \dots, \forall x$$

Then eqn (2) becomes,

$$f(x) = 1 + x \cdot 1 + \frac{x^2}{2!} \cdot 1 + \frac{x^3}{3!} + 1 + \dots, \forall x$$

$$\Rightarrow 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Use Maclaurin Expansion, show that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots$$

Let,

By Maclaurin's Series we have,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots \text{ eqn (i)}$$

Here, $f(x) = \sin x$

Differentiating we get,

$f(x) = \sin x$ $\qquad$ $f'(x) = \cos x$

$f''(x) = -\sin x$ $\qquad$ $f'''(x) = -\cos x$

$f^{iv}(x) = \sin x$ $\qquad$ $f^{v}(x) = \cos x$

$\vdots$

$f^{(2n)}(x) = (-1)^n \sin x$ $\qquad$ $f^{(2n+1)}(x) = (-1)^n \cos x$

$f^{(2n)}(0) = 0$ $\qquad$ $f^{(2n+1)}(0) = (-1)^n$

becomes $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$

Example: Find the taylor Series of $f(x) = x \sin x$ at $x = 0$.

As in examples,

$$\sin x = x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!} + \cdots\right)$$

Thus, $$x \sin x = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} + \cdots + (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

By Taylor's Series expansion of $f(x)$ is -

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots \quad \text{--- eqn (i)}$$

Here, $f(x) = \ln(x)$.

Differentiating we get,

$f'(x) = \frac{1}{x} \Rightarrow x^{-1}$ | $f''(x) = -x^{-2}$ | $f'''(x) = 2!\,x^{-3}$

$f^{iv}(x) = (-2) \times 3 x^{-4} \Rightarrow -3!\,x^{-4}$. | $f^{v}(x) = 4!\,x^{-5}$

At $x = a = 1$,

$f(1) = \ln(1)$ | $f'(1) = 1$ | $f''(1) = -1$ | $f'''(1) \Rightarrow 2!$

$\Rightarrow 0$

$f^{iv}(1) \Rightarrow -3!$ | $f^{v}(x) = 4!\, x^{-5}$

At $x = a = 1$,

$f(1) \Rightarrow \ln(1) \Rightarrow 0$, $f'(1) \Rightarrow 1$, $f''(1) \Rightarrow -1$, $f'''(1) \Rightarrow 2!$

$f^{iv}(1) \Rightarrow -3!$ and $f^{v}(1) \Rightarrow 4!$ and so on - - - - -

Therefore eqn (i) becomes at $a = 1$,

$$f(x) = 0 + (x-1) - \frac{(x-a)^2}{2!} + \frac{(x-1)^3}{3!} \times 2! - \frac{(x-1)^4}{4!} \times 3! + \frac{(x-1)^5}{5!} \times 4! - \ldots\ldots$$

$$f(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \cdots$$

Taylor's Theorem:

If f and its first n-derivatives are continuous on $[a,b]$ or on $[b,a]$ and $f^{(n)}$ is differentiable on (a,b) or on (b,a) then there exists a number c in between 'a' and 'b' such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(a)}{(n+1)!}(b-a)^{n+1}$$

if the fixed point b released as a variable x then the following formula is obtained.

Taylor's Formula:

If f has derivatives of all orders in an open interval I containing a then for each positive integer n and for each x in I.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

where the remainder $R_n(x)$ is,

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c lies in between a and x.

Note 1: The first $(n+1)$ terms of Taylor's formula is known as Taylor's polynomial of order n.

That is, $P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

Then the taylor's formula can be written as,

$$f(x) = P_n(x) + R_n(x).$$

Note 2: The remainder value $R_n(x)$ is also known as error term of Taylor's formula.

Note 3: If $R_n(x) \to 0$ as $n \to \infty$ for all x in I then the Taylor's formula

Can be written as,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

In such condition, we say the Taylor series generated by f at $x=a$, converges to f on I.

Theorem (The Remainder Estimation Theorem) (Taylor's Inequality)

If there are positive constant M and t such that $|f^{n+1}(t)| \leq M$ for all t between a and x then the remainder term $R_n(x)$ in Taylor's theorem satisfies the inequality

$$|R_n(x)| \le M \frac{|x-a|^{n+1}}{(n+1)!}$$

If these conditions hold for every x and all other conditions of Taylor's theorem are satisfied by f then the series converges to $f(x)$.

Example: Show that the series for $\cos x$ converges to $\cos x$ for all x.

Soln Let

$$f(x) = \cos x$$

Since we have the Maclaurin Series for $\cos x$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^{n+1}\frac{x^{2n}}{(2n)!} + R_{2n}(x) \quad \text{--- eqn (i)}$$

Since the cosine function and all the derivatives of cosine function have absolute value less than or equal to 1. So, with $M = 1$ and $r = 1$, the remainder estimation theorem is,

$$|R_{2n}(x)| \le 1 \cdot \frac{|x|^{2n+1}}{(2n+1)!}$$

Since $\frac{|x|^{2n+1}}{(2n+1)!} \to 0$ as $n \to \infty$ and for all value of x.

This implies that the series on the right of eqn (1) converges to $\cos x$ for every value of x.

Intuitive Understanding

① Taylor Series expresses a function as an infinite sum of terms derived from its derivatives at a specific point.

② Maclaurin Series → A special case of Taylor Series, where the expansion is around zero.

③ Taylor theorem → It guarantee that a function can be approximated by its Taylor Series within a range.

Tangent Planes and Normal Line:

Definition(Tangent plane and Normal Line):

The tangent plane of a point $P_0(x_0, y_0, z_0)$ on the surface $f(x,y,z) = c$, is the plane that is normal to ∇f and passes through P_0. Mathematically, the plane is,

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0.$$

And the normal line on the surface at P_0 is the line that is parallel to ∇f and passes through P_0.

Mathematically, the line is,

$$x = x_0 + f_x(P_0)t, \quad y = y_0 + f_y(P_0)t, \quad z = z_0 + f_z(P_0)t \quad \text{at } P_0 = P_0(x_0, y_0, z_0)$$

If $f(x,y)$ be given curve then $\vec{N} = \nabla f$ is gradient of f. Then the equation of tangent line to f at $P_0(x_0, y_0)$ is,

$$(\nabla f)_{P_0} \cdot \left((x - x_0)\vec{i} + (y - y_0)\vec{j} \right) = 0 \quad \text{where } P_0 = P_0(x_0, y_0)$$

i.e. $A(x - x_0) + B(y - y_0) = 0$ where $(\nabla f)_{P_0} = A\vec{i} + B\vec{j}$

when $\nabla f \big|_{\text{at } P_0(x_0, y_0)} = \vec{N}$

$$= A\vec{i} + B\vec{j}$$

$$\left(\nabla = \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right)$$

And, the equation of normal line to f at $P_0(x_0, y_0)$ is,

$$x = x_0 + \left(\frac{\partial f}{\partial x} \Big| \text{at } P_0\right) t \Rightarrow x_0 + (f_x)_{P_0} t$$

$$y = y_0 + \left(\frac{\partial f}{\partial y}\right) \text{at } P_0) t \Rightarrow y_0 + (f_y)_{P_0} \cdot t$$

Note: If $f(x, y)$ be given curve then extend the above formulae for tangent plane and normal lines in space.

Example. find an equation for tangent to the ellipse $x^2 + 4y^2 = 8$ at the point $(-2, 1)$.

Let, $f(x,y) = x^2 + 4y^2 - 8$

Then,

$f_x = 2x \qquad f_y = 8y$

At $P_0(-2,1)$ $\quad (f_x)_{P_0} = -4 \quad , (f_y)_{P_0} = 8$

Now, the tangent line to f at $(-2,1)$ is,

$$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) = 0$$

where $P_0 = P_0(-2,1)$

Maximum and Minimum Values:

Continuous function defined over a closed and bounded regions in xy-plane take absolute maximum and minimum values on these domains. In this section, we will learn the method to find the absolute maxima and minima by using partial derivatives.

Definition (Local Maxima and Minima):

Let $f(x,y)$ be defined on a region R containing the point (a,b). Then

i) $f(a,b)$ is a local maximum value of f if $f(a,b) \geq f(x,y)$ for all domain points (x,y) in an open disk centered at (a,b).

ii) $f(a,b)$ is a local minimum value of f if $f(a,b) \leq f(x,y)$ for all domain points (x,y) in an open disk centered at (a,b).

* Maxima or Minima are also called extrema.

Theorem (First Derivative Test for Local Extreme Values):

If $f(x,y)$ has a local maximum or local minimum value at an interior point (a,b) of its domain and if the first partial derivatives exist there then $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

As in the single variable case, the above theorem tells that the only places a function $f(x,y)$ can have an extreme value are:-

i) Interior points at $f_x = f_y = 0$

ii) Interior points where one or both of f_x and f_y do not exist,

iii) Boundary points of the functions domain.

In at some interior points $f(x,y)$ do not attains maxima and minima. Such point is known as saddle point.

And, at the point at where the first partial derivatives are zero is called a critical point.

<u>Definition (Critical point)</u>:

An interior point (a,b) of the domain of a function $f(x,y)$ where $f_x(a,b)=0$ and $f_y(a,b)=0$ or one or both f_x and f_y do not exist then the point (a,b) is a critical point of f.

It is the point where f is zero

Definition (Saddle Point):

A differentiable function $f(x,y)$ has a saddle point at a critical point (a,b) if in every open disk centre at (a,b) there exists some domain points (x,y) satisfying $f(x,y) > f(a,b)$ and $f(x,y) < f(a,b)$. Then the point (a,b) is called saddle point.

The fact that $f_x(a,b) = 0 = f_y(a,b)$ of R doesnot ensure f has a local extreme values there.

The following theorem helps to finding the extreme values on R (if they exist) if f and its first and second partial derivatives are continuous on R.

Theorem (Second Derivative Test for Local Extreme Values)

Suppose $f(x,y)$ and its first and second partial derivatives are continuous in a disk center at (a,b) and that $f_x(a,b) = f_y(a,b) = 0$. Then for

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} \cdot f_{yy} - (f_{xy})^2$$

i) f has a local maximum at (a,b) if $f_{xx} < 0$ and $D > 0$ at (a,b)

ii) f has a local minimum at (a,b) if $f_{xx} > 0$ and $D > 0$ at (a,b).

iii) f has a saddle point at (a,b) if $D < 0$ at (a,b).

iv) The test is inconclusive at (a, b) if $D = 0$. In this case, note that we may search other way to learn the behavior of f at (a, b).

Q. Find the local extreme values of the function $f(x, y) = xy - x^2 - y^2 - 2x + 4$.

Let, $f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4$

Then,

$$f_x = \frac{\partial f}{\partial x} = y - 2x - 2$$

and $f_y = \frac{\partial f}{\partial y} \Rightarrow x - 2y - 2$

$f_{xx} = -2, \quad f_{yy} = -2, \quad f_{xy} = 1.$

for critical points set -

$$f_x = 0, \quad f_y = 0$$

i.e. $y - 2x - 2 = 0, \quad x - 2y - 2 = 0$

Solving these equations we get $x = y = -2$.

$f_{xx} = -2 \qquad f_{yy} = -2$ and $f_{xy} = 1$.

$f_{xx} = -2 < 0$

Also, and

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = 4 - 1 = 3 > 0$$

$\therefore f$ has local maxima at $(-2, 2)$.

And, the maximum value of f at the point is,

$$f(-2,2) = (-2)(-2) - (-2)^2 - (-2)^2 - 2(-2) - 2(-2) + 4$$

$$\Rightarrow \cancel{4} - \cancel{4} - \cancel{4} + \cancel{4} + \cancel{4} + 4 + 4$$

$$= 8$$

Thus, f(x,y) attains local maxima at $(-2,-2)$ and its maximum value at the point is 8.