# Day-71, Feb-9 2025 (Magh 27, 2081 B.S.)

# Random Variable is a real valued function defined on the sample space, usually denoted by a capital letter such as X, Y, Z.

- Range of a random variable, the set of values that it can take.
- A random variable is discrete if its range is either finite or countably infinite.
- Discrete Random Variable are different from Continuous random variable.

## Examples of Discrete Random Variables.

- Coin Flips

$$X = \begin{cases} 1 & \text{if Head} \\ 0 & \text{otherwise} \end{cases}$$

Range = $\{0, 1\}$ { finite }

- Calls in a Call Center

X = Number of daily calls.

Range = Set of all non-negative integers ( Countably infinite)

- Other Examples, Number of Table served in a Restaurant.

# Probability Mass Function (PMF) (for Discrete).

A random variable is characterized by its PMF

- Let $X$ be a random variable
- Let $x$ be a value in its range (not a point $\Omega$!)

# PMF.

PMF is a function $p_x$ that is defined as

$$p_x(x) = P(X = x)$$

where for short.

$$P(X = x) = P(\{X = x\})$$

For a subset $S$ in the range of $X$.

$$P(X \in S) = P(X \text{ takes value within a set } S)$$

# Key properties of PMF

- $\sum_x p_X(x) = 1.$
- $P(X \in S) = \sum_{x \in S} p_X(x)$

# Expectation

For a random variable $X$ with a PMF, the expected value or the expectation or the mean of $X$ is defined as

$$E[X] = \sum_x x\, p_X(x).$$

The expectation can be viewed as:

- Weight average of $x$, with $P_X(x)$ being the weights.
- The gravitational center.

# Properties of Mean and Variance

- $E[X] = \sum_x x P_X(x)$
- $E[g(X)] = \sum_x g(x) P_X(x)$
- $Var(X) = E[(X - E[X])^2]$

$\Rightarrow E[X^2] - (E[X])^2$

- Since $Var(X) \geq 0$, it follows that for any $X$.

$$(E[X])^2 \leq E[X^2]$$

- $E[aX+b] = aE[X] + b$
- $Var(aX+b) = a^2 Var(X)$.

# Example: Random Variable with Bernoulli Distribution.

- A coin is tossed. Let $X$ denote the random variable indicating whether a head was flipped

$$X = \begin{cases} 1 & \text{if head is flipped} \\ 0 & \text{otherwise} \end{cases}$$

- Let p denote the probability of coming heads.
- X has a Bernoulli distribution with probability p, denoted as

$$X \sim \text{Bernoulli}(p).$$

- PMF of X can be written as

$$P_r(X=1) = P_X(1) = p$$

$$P_r(X=0) = P_X(0) = 1-p$$

# Random Variable with Binomial Distribution

- Suppose a coin is tossed for n times.
- Let X denote the number of heads in the n times.
- Let p be the probability of head in each coin toss.

- $X \sim \text{Binomial}(n, p)$
- PMF of $X$ is

$$Pr(X=k) = p_X(k) \Rightarrow \binom{n}{k} p^k (1-p)^{n-k}$$

- $E(X) = np$

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- $X$ has a Geometric Distribution
- PMF of $X$ is

$$Pr(X=k) = p_X(k)$$

$$\Rightarrow (1-p)^{k-1} p \quad \text{with } k = 1, 2, 3 \ldots\ldots$$

The first $k-1$ trials are failures, followed by success nth trail

Example: Random Variable with Geometric Distribution

$$E(X) = \sum_{x} x\, p(x)$$

$$= \sum_{k=1}^{\infty} k(1-p)^{k-1} \cdot p$$

$$= 1/p$$

Source: NAAMII RESEARCH YT Statistics Part II video.

# Independence of Random Variables

Two random variables X and Y are independent if

$$P_{X,Y}(x,y) = P_X(x)\,P_Y(y) \quad \text{for all } x, y$$

Since

$$P_{XY}(x,y) = P_Y(y)\,P_{X|Y}(x|y), \quad \text{for all } x, y$$

we conclude that when X and Y are independent

$$P_{X|Y}(x|y) = P_X(x) \cdot \quad \text{for any } Y \text{ at which } P_Y(y) > 0.$$

# Independence: Implication in the 2-Dimensional Table

When X and Y are independent then in the table of joint PMF, the value in each cell is the product of the sum of the corresponding row times the sum of the corresponding column!

| | $X=1$ | $X=2$ | $X=3$ |
|---|---|---|---|
| $Y=1$ | $1/3$ | $1/6$ | $1/6$ |
| $Y=2$ | $1/6$ | $1/12$ | $1/12$ |

# Conditional Independence

The notion can be further extended to the case where we condition $X$ and $Y$ on an event $A$.

**Definition:** We say $X$ and $Y$ are independent given an event $A$. $P(A) > 0$, if $P(X = x, Y = y | A)$

$$= P(X = x | A) \cdot P(Y = y | A)$$ or for

short, we write

$$P_{X,Y|A}(x) = P_{X|A}(x)\, P_{Y|A}(y).$$

Once again this is equivalent to that

$$P_{X|Y,A}(x) = P_{X|A}(x), \text{ for all } x \text{ and } y$$

with $P_{X|A_1}(y) > 0$.

# Properties of Independent Random Variables

① If $X$ and $Y$ are independent then

- $E[XY] = E[X] \cdot E[Y]$
- $\mathrm{var}(X+Y) = \mathrm{var}(X) + \mathrm{var}(Y)$

Remarks:

- The above property is usually not true when $X$ and $Y$ are not independent.

- As X and itself are usually not independent, it is not true that $E[X^2] = E[X] \cdot E[X]$.

# More than 2 Random Variable

Extensions to more than 2 random variables are straightforward

- Random Variables $X_1, X_2, \ldots X_n$ are independent if the joint PMF satisfies

$$p_{X_1, X_2, \ldots, X_n}(x_1, x_2 \ldots x_n) = p_{X_1}(x_1)\, p_{X_2}(x_2) \ldots p_{X_n}(x_n)$$

- $E[X_1 X_2 \ldots X_n] = E[X_1] \cdot E[X_2] \ldots E[X_n]$
- $\mathrm{var}(X_1 + X_2 + \ldots + X_n) = \mathrm{var}(X_1) + \mathrm{var}(X_2) + \ldots + \mathrm{var}(X_n)$.

## Example: Variance of the Binomial

If $X$ is Binomial Random variable with parameter $p$, $0 < p < 1$, then -

$$X = X_1 + X_2 + \cdots X_n$$

$$\sum_{i=1}^{n} X_i$$

Where $X_i$ are independent of each other, and each $X_i$ is Bernoulli Random Variable with Parameter $p$. We know that

$$E[X_i] = p, \qquad Var(X_i) = p(1-p)$$

and $$var(X) = var(X_1) + \cdots + var(X_n) \quad \boxed{\Rightarrow np(1-p)}$$

# Mean and Variance:

Mean = Average

Variance = How similar or different?

| | Mean | Variance |
|---|---|---|
| Bernoulli $(p)$ | $p$ | $p(1-p)$ |
| Binomial $(n, p)$ | $np$ | $np(1-p)$ |
| Geometric $(p)$ | $1/p$ | $(1-p)/p^2$ |
| Poisson $(\lambda)$ | $\lambda$ | $\lambda$ |

# Sample Mean

Suppose $X_i$; $i = 1, 2, 3, \ldots n$ are independent Bernoulli random variable with parameter $p$, $0 < p < 1$.

Q. If $p$ is not known to us. What is a good way to estimate $p$?

Let $\bar{X}$ be the Sample mean,

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i,$$

then

$$E[\bar{X}] = \frac{1}{n} E\left[\sum_{i=1}^{n} X_i\right]$$

$$= \frac{1}{n}(np)$$

$$= p$$

and $\operatorname{var}(\bar{X}) = \frac{1}{n^2} \operatorname{var}\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\left(np(1-p)\right) = \frac{p(1-p)}{n}$