Day-25, Dec-10, 2024 (Mangshir-25, 2081 B.S.)

Use of logarithm in differentiating a function:

Sometimes, in the process of differentiation, we come across a function of the types $a^{f(x)}$ or $\{f(x)\}^{g(x)}$ etc. where 'a' is a constant. In differentiating such a function we first take the logarithm and then only differentiate.

Example: Using first principles, the differential coefficient of a^x.

Let $f(x) = a^x$

$\Rightarrow e^{\log a^x}$ $\Rightarrow e^{x \log a}$

Now, $f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$\frac{d}{dx}(a^x) \Rightarrow \lim_{h\to 0} \frac{e^{(x+h)\log a} - e^{x \log a}}{h}$$

$$\Rightarrow \lim_{h\to 0} e^{x\log a} \frac{(e^{h\log a} - 1)}{h}$$

$$\Rightarrow \lim_{h\to 0} e^{x\log a} \left\{ \frac{e^{h\log a} - 1}{h\log a} \right\} . \log a$$

$$\Rightarrow e^{x\log a} . \log a$$

$$\boxed{\frac{d}{dx}(a^x) = a^x \log a}$$

Find the derivative of $y = x^y$.

Soln, Taking log on both sides

$$\log y = y \log x$$

Differentiating both sides w.r.t. x.

$$\frac{d}{dx}(\log y) = \frac{d}{dx}(y \log x)$$

$$\Rightarrow \frac{d(\log y)}{dy} \cdot \frac{dy}{dx} = y \cdot \frac{d}{dx}(\log x) + \frac{dy}{dx} \log x$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = y \cdot \frac{1}{x} + \log x \cdot \frac{dy}{dx}$$

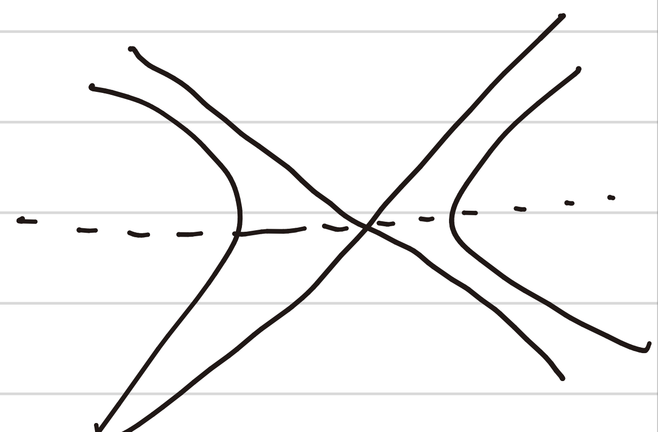
$$\Rightarrow \left(\frac{1}{y} - \log x\right)\frac{dy}{dx} = \frac{y}{x}$$

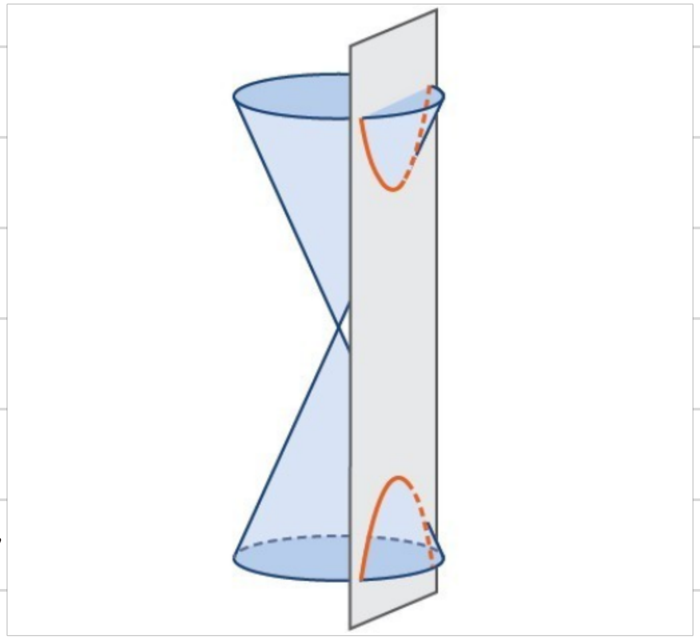
$$\left[\Rightarrow \frac{dy}{dx} = \frac{y^2}{x(1 - y\log x)}\right]$$

Derivatives of Hyperbolic functions

Q. What is Hyperbola?

→ A type of curve formed by the intersection of double cone and plane.

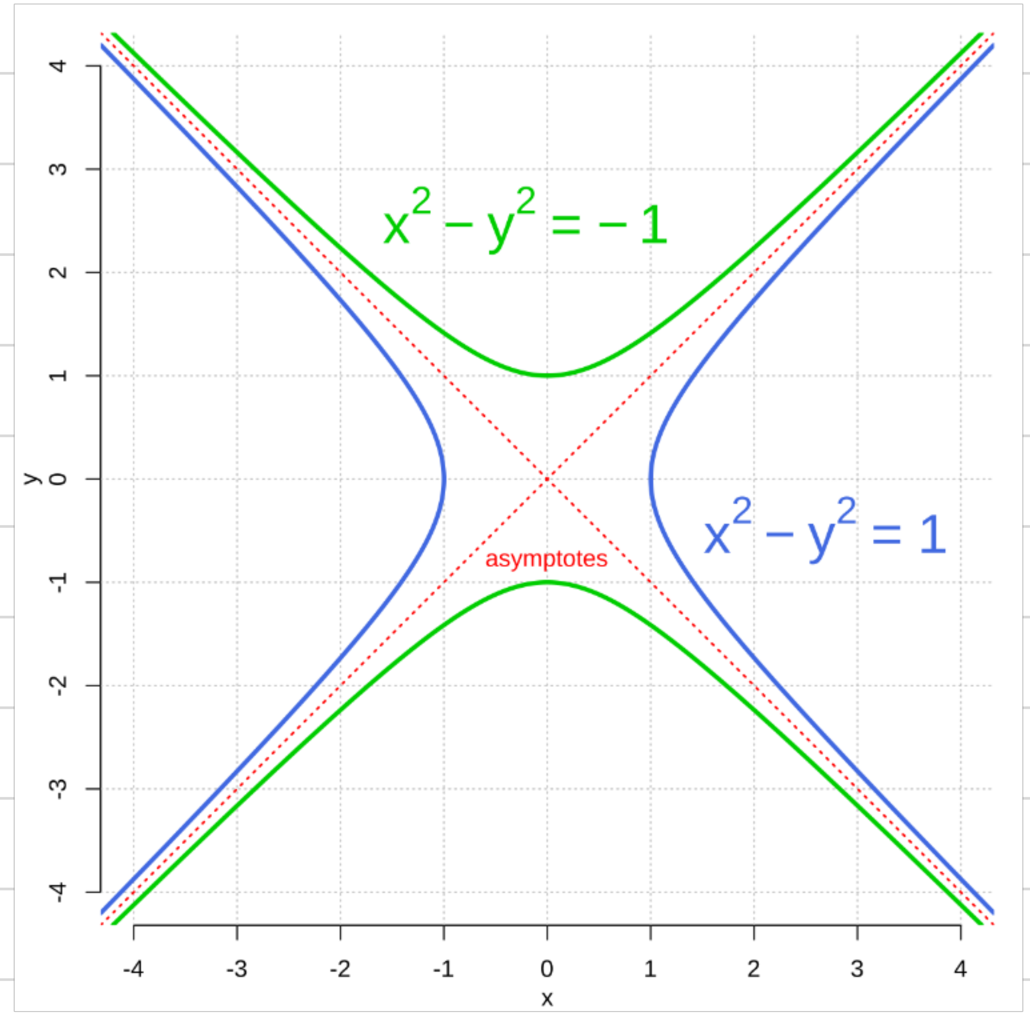




Unit Hyperbola:

A Special Kind of Hyperbola with a Simplified equation and geometric Properties. It is a hyperbola centered at the origin with

$$\left(x^2 - y^2 = -1 \right)$$



Hyperbola functions are defined for a real number 'x' as follows:

$$\sinh x = \frac{1}{2} \left(e^{x} - e^{-x} \right)$$

$$\cosh x = \frac{1}{2}\left(e^{-x} + e^{-x}\right)$$

$$\tan x = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

Cosechx, Secx. and coth x are respectively the reciprocals of sinhx, coshx and tanhx and are defined accordingly. In the cases of cosech x and cothx, x cannot be zero.

From these relations, we can easily derive the following

$\sinh(-x) = -\sinh x$.	$\cosh^2 x - \sinh^2 x = 1$
$\cosh(-x) = \cosh x$	$\cosh^2 x + \sinh^2 x = \cosh 2x$.

Let $y = \sin x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$

$$\therefore \frac{dy}{dx} = \frac{1}{2}\frac{d\left(e^{x} - e^{-x}\right)}{dx}$$

$$= \frac{1}{2}\left(e^{x} + e^{-x}\right)$$

$$\left[\frac{dy}{dx} = \cosh x\right]$$

i.e. $\frac{d(\sinh x)}{dx} = \cosh x$

Similarly $\frac{d(\cosh x)}{dx} = \sin x$.

$$\therefore \frac{d(\tanh x)}{dx} = \operatorname{sech}^2 x$$

$$y = \tanh x = \frac{\sinh x}{\cosh x}$$

$\frac{d(\coth x)}{dx} = -\operatorname{cosech}^2 x$

$\frac{d(\operatorname{sech} x)}{dx} = -\operatorname{sech} x \cdot \tanh x.$

Since we know Derivative can be interpret as slopes and rate of change so let's find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1,1)$.

Here,

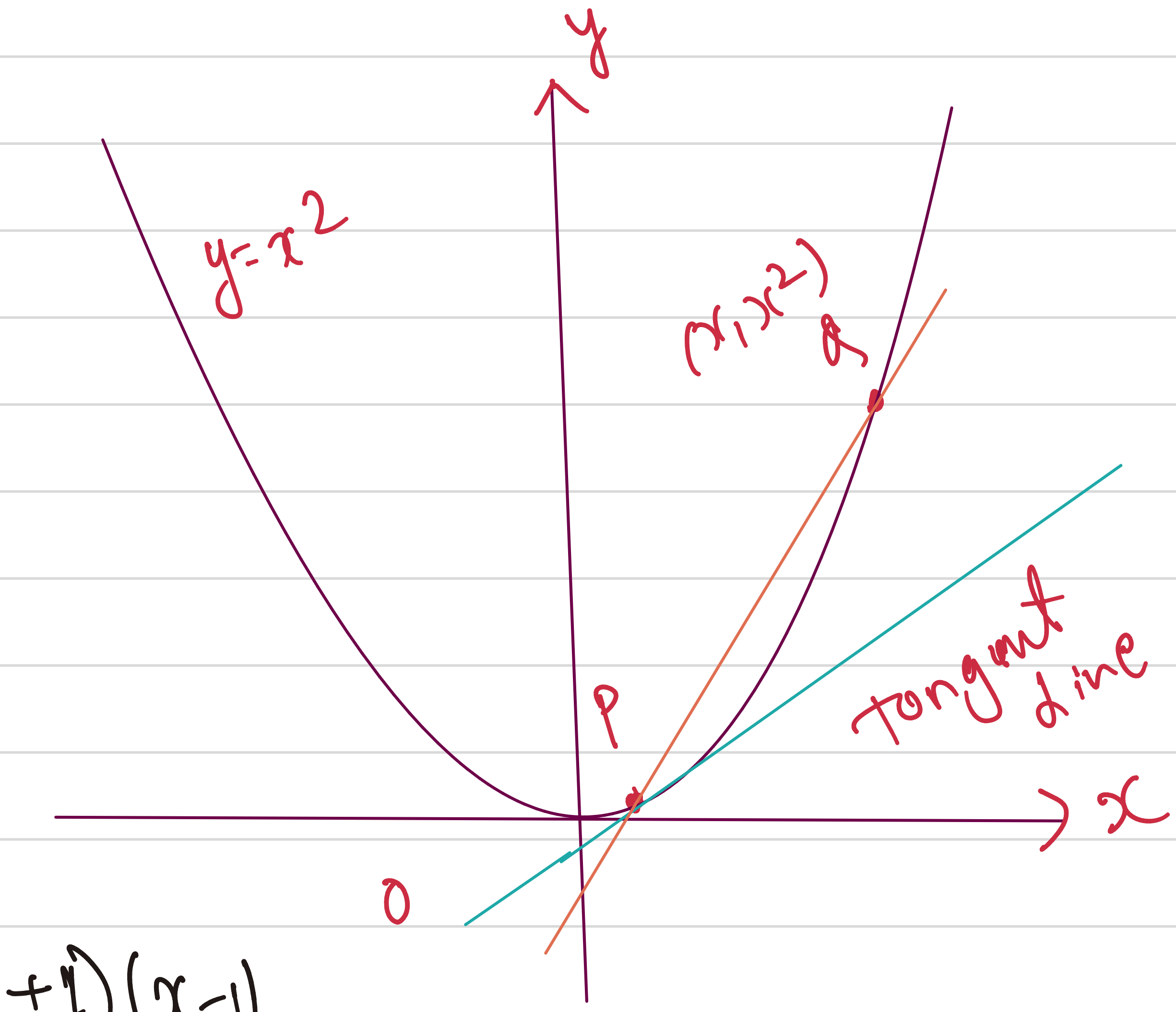
$y = x^2$

Choose a point $Q(x, y) = Q(\overset{x_1}{x}, \overset{y_1}{x^2})$ near to $P(\overset{x_2}{1}, \overset{y_2}{1})$

Then the slope of the line PQ is

$$m_{PQ} = \frac{x^2 - 1}{x - 1}$$

$$\left[m = \frac{y_2 - y_1}{x_2 - x_1} \right]$$



So, the limit of the slope of PQ is $\lim_{Q \to P} m_{PQ} = m = \lim_{Q \to P} \frac{(x+1)(x-1)}{(x-1)}$

$$\left[\because x^2 - 1 = (x+1)(x-1) \right]$$

$$\Rightarrow \lim_{Q \to P} (x+1)$$

$$\boxed{m = 2}$$

$\therefore$ The eqn of tangent line passing through the point $P(1,1)$ and having slope 2.

Slope $= 2$

Point $= (1,1)$.

ie - $y - 1 = 2(x-1)$

ie $y = 2x - 1$

Note: A straight line to the given curve is known as Secant line if it cuts the curve into two points (may more than two) like as in above figure (the line PQ is secant line) the word Secant is originated from Latin word Secans means cutting.

Definition: The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P and having the slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that the limit exists.

Example: Find the equation of tangent line to the curve $y = x^2$ at $x = 1$.

Soln: Given that $x = 1$ and $f(x) = y = x^2$. Then the slope of the curve at the point is –

$$m = \lim_{x \to 1} \left(\frac{f(x) - f(1)}{x - 1} \right)$$

$$\Rightarrow \lim_{x \to 1} \left(\frac{x^2 - 1^2}{x - 1} \right)$$

$$\Rightarrow \lim_{x \to 1} (x + 1) \Rightarrow 2$$

Now, the equation of the tangent line to $y = x^2$ at $p(1,1)$ is,

$$y - 1 = 2(x-1)$$

Alternatively, Given that $x = 1$ and $f(x) = y = x^2$. Then the slope of the curve at the point is

$$m = \lim_{h \to 0} \left(\frac{f(1+h) - f(1)}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{(1+h)^2 - (1)^2}{h} \right)$$

$$\Rightarrow \lim_{h\to 0} \left(\frac{2h+h^2}{h}\right)$$

$$\Rightarrow \lim_{h\to 0} (2+h) = 2.$$

Now, the equation of the tangent line to $y = x^2$ at $P(1,1)$ is,

$$y - 1 = 2(x-1)$$

$$(\text{i.e. } y = 2x - 1)$$

Example: Show that the line $y = mx + b$ is its own tangent at any point (x_0, y_0) is itself.

Given the Curve is -

$$y = mx + b \longrightarrow \text{eqn (i)}$$

Here, the slope of eqn (i) at $x = x_0$ is -

$$m = \lim_{h \to 0} \left(\frac{f(x_0 + h) - f(x_0)}{h} \right)$$

Now, the equation of the tangent line to (i) at (x_0, y_0) is,

$$y - y_0 = m(x - x_0)$$

$$\Rightarrow y - (mx_0 + b) = m(x - x_0)$$

$$\Rightarrow y = mx_0 + b + mx - mx_0$$

$$\Rightarrow y = mx + b \longrightarrow \text{Same to eqn (i)} \#.$$

Rate of Change:

Suppose 'y' be a function of x and we write $y = f(x)$. If 'x' Changes from a point 'x_1' to another point 'x_2' then the Change in 'x' is defined as:

$$\Delta x = x_2 - x_1$$

which we called the increment of x.

And, the corresponding change in y is

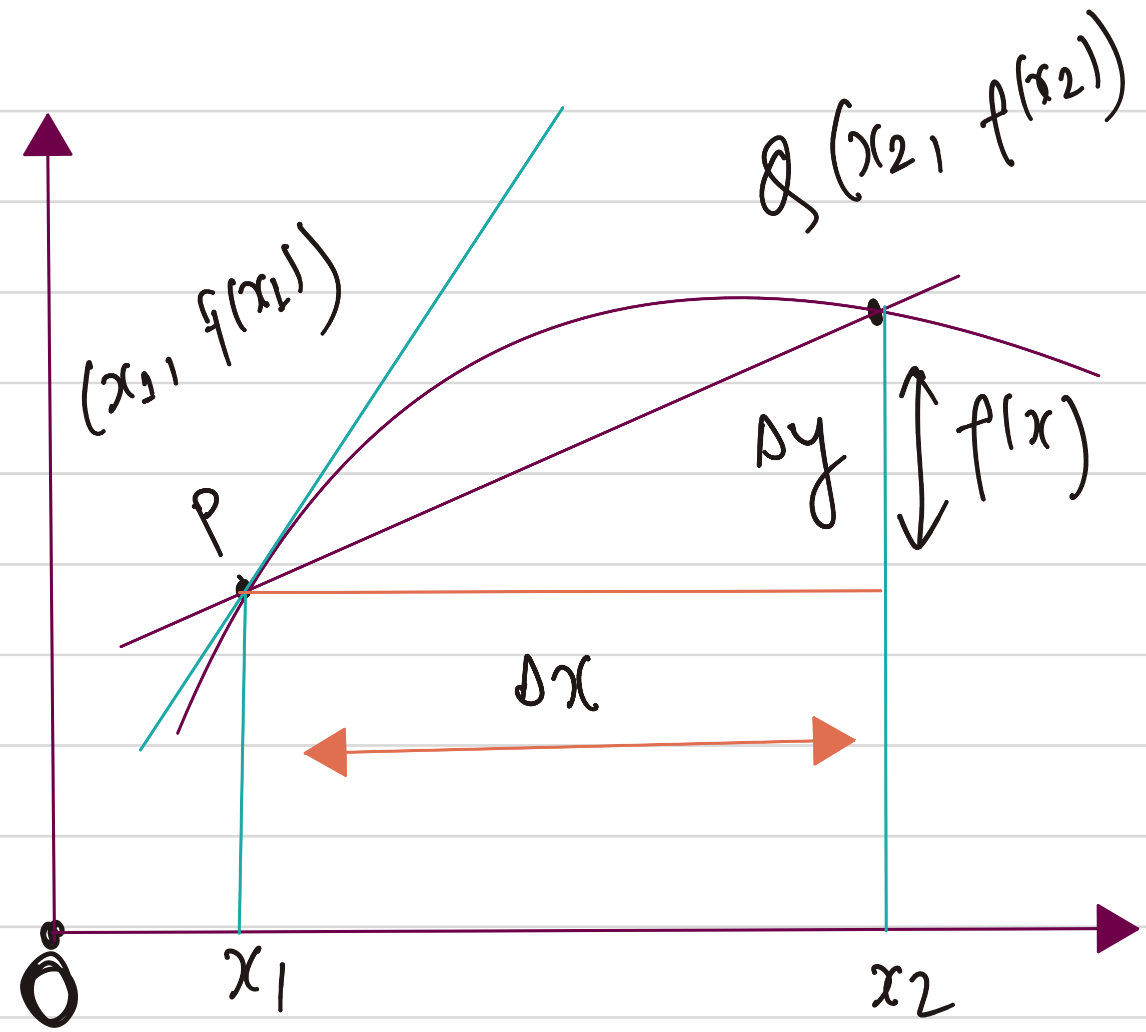
$$\Delta y = f(x_2) - f(x_1)$$

Then, the quotient of these increments ie.

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad \text{—— eqn (i)}$$

is the average rate of change of y with respect to 'x' over the interval $[x_1, x_2]$.



And, the limit rate of change of y with respect to 'x' over the interval $[x_1, x_2]$.

Geometrically, the average rate (i) be the slope of the secant line PQ.

And the limit of average rates (i) is called the rate of

Change of y with respect to x at $x = x_1$.

Mathematically,
Rate of Change of y with respect to x at $x = x_1$ is,

$$\text{Rate of Change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$\Rightarrow \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad \text{— eqn (ii)}$$

We recognize this limit (ii) as the derivative of $f'(x_1)$.

So, Derivative $f'(x_1)$ is the slope of the tangent line to the curve $y = f(x)$ at $x = x_1$. And also $f'(x_1)$ is the rate of change of y