Day-28. Dec-16, 2024 (Poush-1, 2081 B.S)

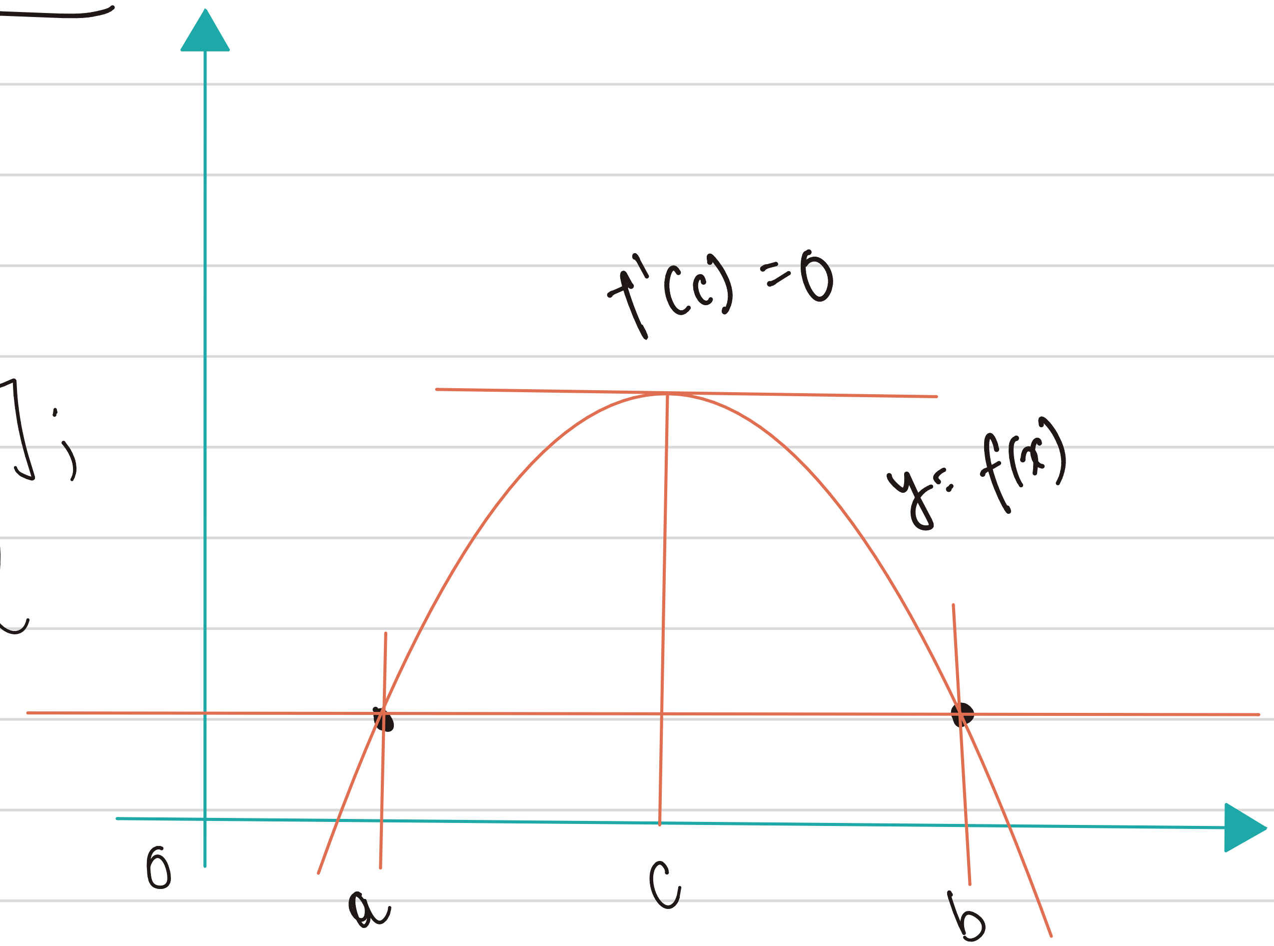
Geometrical Meaning of Rolle's Theorem:

If a function $f(x)$ is

i) Continuous in the closed interval $[a, b]$;

ii) Differentiable in the open interval (a, b); and

iii) $f(a) = f(b)$.



Then, there exists at least one real number $c \in (a, b)$ such that tangent at $(c, f(c))$ is parallel to tangent at $x = c$ is parallel to x-axis.

Example: Support with examples that the hypothesis of the Rolle's theorem, are essential to hold the theorem

Solution:

→ In figure a) given function $f(x)$ is discontinuous at end point $x = a$ of $[a,b]$, So it is discontinuous in $[a,b]$.

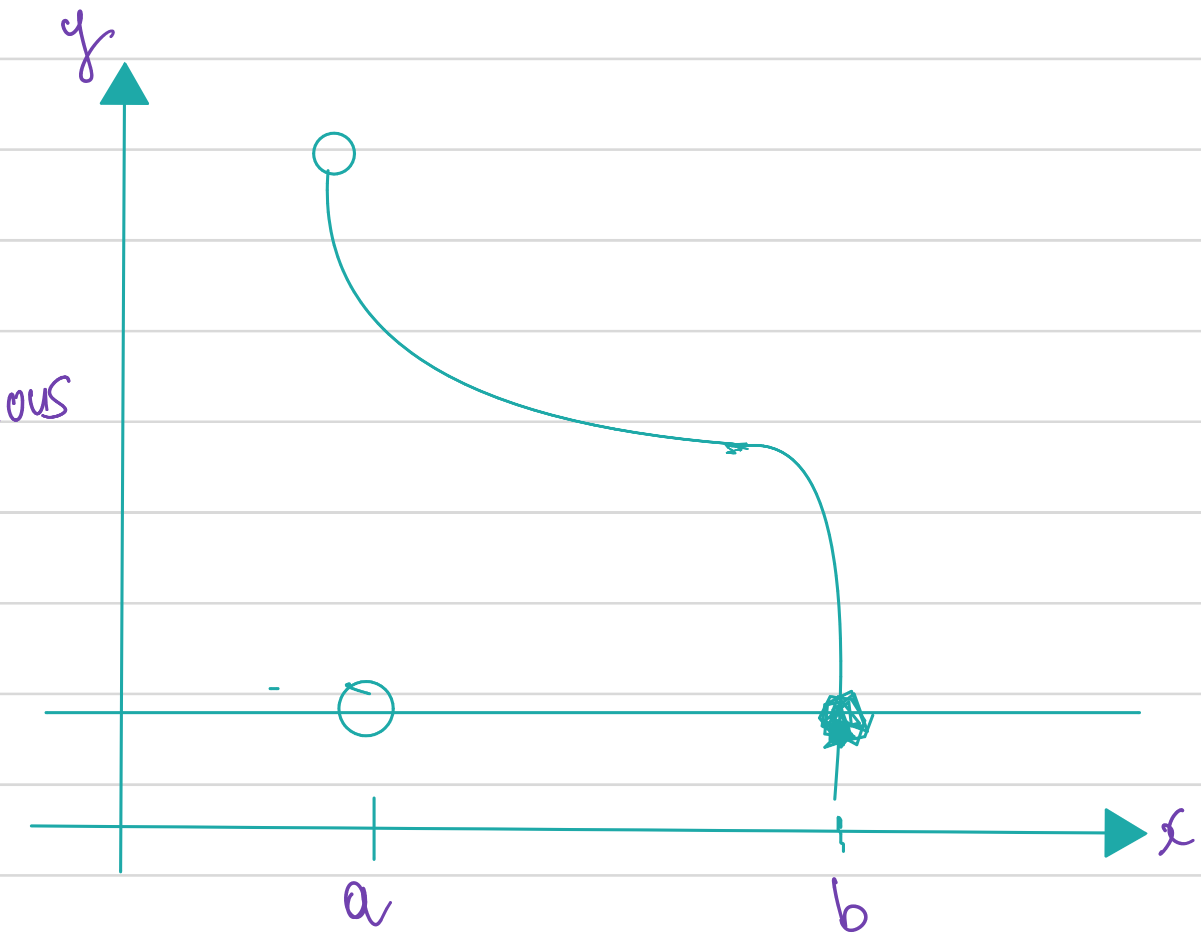
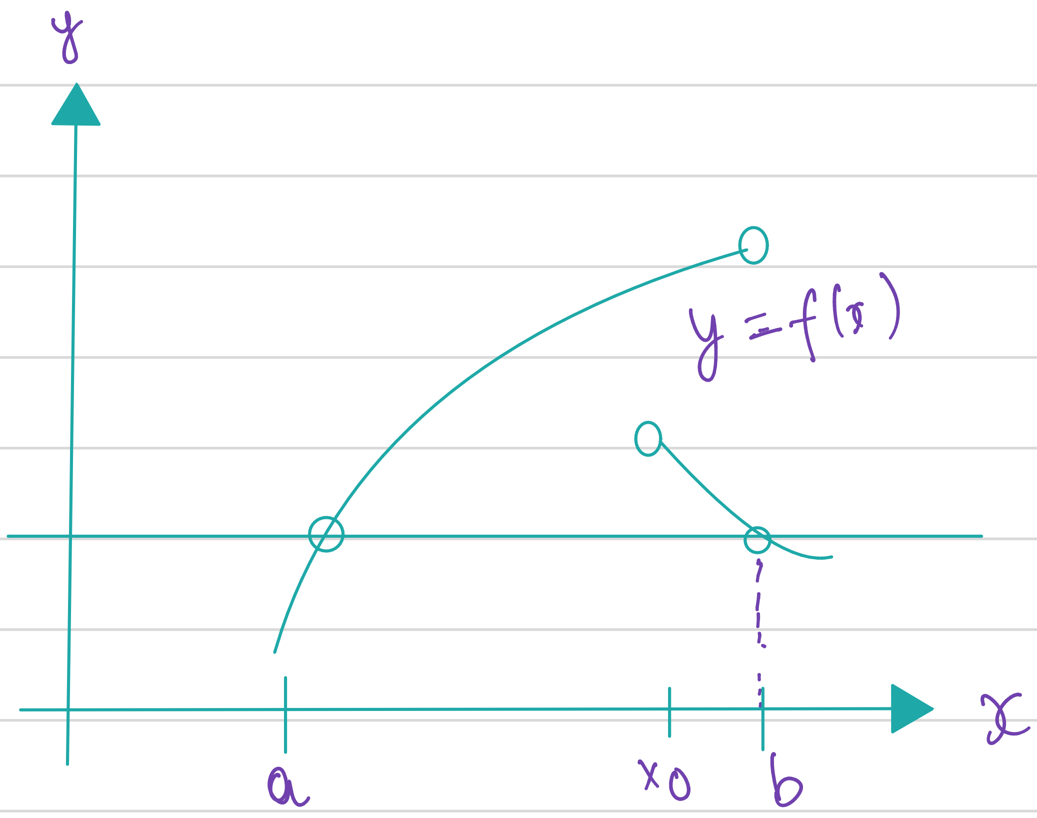


fig a) Discontinuous at an endpoint of $[a,b]$.



→ In fig b) given function $f(x)$ is discontinuous at interior point $x = x_0$ of $[a, b]$

fig b) Discontinuous at an interior point of $[a, b]$.

This means function does not behave smoothly (i.e. it has a "jump", "break", or "undefined" behaviour) at that specific point even it is inside interval.

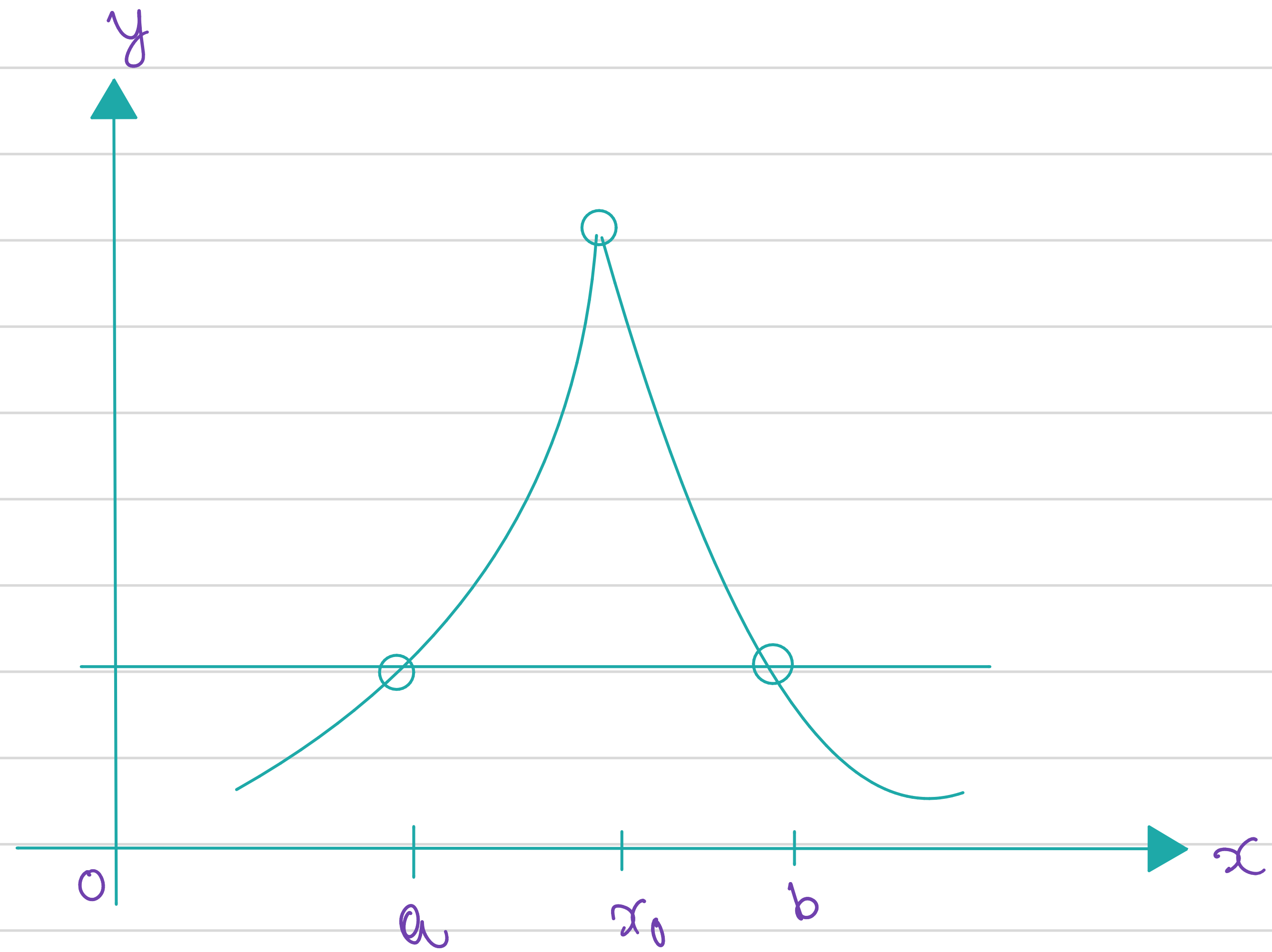


fig c) Continuous on $[a, b]$ but not differentiable at an interior point.

Example: Verify the Rolle's Theorem for the function $y = \sqrt{1-x^2}$ $[-1, 1]$.

Here Given,

$$f(x) = \sqrt{1-x^2}$$

Since $\lim\limits_{x \to 1} f(x) = f(-1)$ and $\lim\limits_{x \to 1^-} f(x) = f(1)$

Thus, $f(x)$ is continuous at end points and all interior points of $[-1, 1]$.

i) $f(x)$ is continuous in $[-1, 1]$

ii) $f'(x) = \dfrac{-2x}{2\sqrt{1-x^2}} \Rightarrow \dfrac{-x}{\sqrt{1-x^2}}$ exist in $(-1, 1)$

So, $f(x)$ is differentiable in $(-1, 1)$.

iii) $f(1) = f(-1) \Rightarrow 0$.

Thus, all three Conditions are holded by $f(x)$. Hence $\exists$ a point $c \in (-1,1)$ such that

$$f'(c) = 0$$

$$\Rightarrow \frac{-c}{\sqrt{1-c^2}} = 0$$

$$\Rightarrow c = 0 \in (-1,1).$$

So, Rolle's Theorem is Verified.

Applications: (1) Mechanical System (Explains point where velocity is 0) (2) Optimization Problem (3) Regression Analysis (4) Signal Processing (5) Climate Modeling (change Rate is Zero?) #

Mean Value Theorem:

Let f be a function that satisfies the following hypothesis:

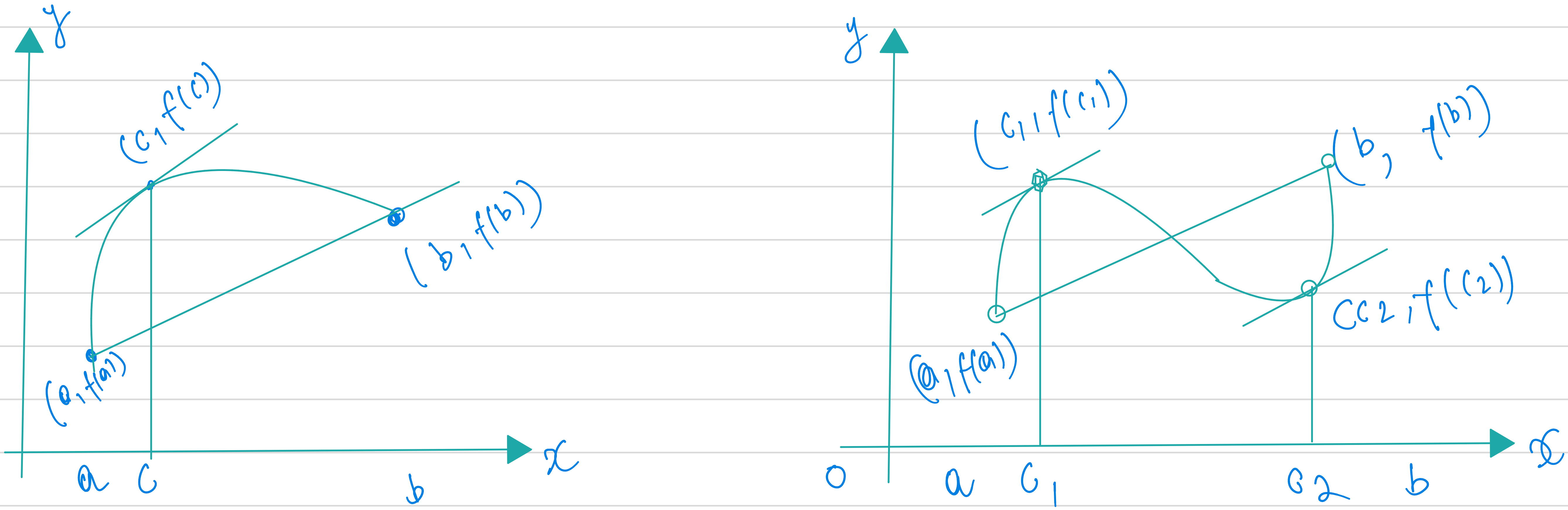
i) f is continuous on the closed interval $[a,b]$.

ii) f is differentiable on the open interval (a,b)

Then there is a number c in (a,b) such that

$$f(c) = \frac{f(b) - f(a)}{b-a}$$

i.e $f(b) - f(a) = (b-a)f'(c)$



① Rolle's Theorem

→ $f(a) = f(b)$

→ $f(x)$ continuous closed interval $[a, b]$

→ $f(x)$ is differentiable on the open interval (a, b)

→ 3 cases.

→ Mean Value Theorem

→ f is continuous closed interval $[a, b]$

→ f is differentiable in open interval (a, b)

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$