#Day-48, Jan-17, 2025 (Magh 04, 2081 B.S.)

1. Variable Separable function:

Many first order and first degree ODE

$$M.dx + N.dy = 0 \quad \text{—— eqn (i)}$$

Can be solved by separating the variables equation (i),

We can reform as:

$$g(y) \cdot \frac{dy}{dx} = f(x)$$

i.e. $g(y) \cdot dy = f(x) \cdot dx$ —— eqn (ii)

On the integration (ii), we obtain the solution of given eqn (i)

This method is called the method of separating variables and the equation (i) is called the separable equation.

Example: Logistic Growth Model

$$\frac{dp}{dt} = rp\left(1 - \frac{p}{K}\right)$$ is the differential equation for Logistic Growth Model.

where $p(t)$ is the population at time 't'

r is the growth rate

K is the carrying capacity (Maximum Population)

This equation is separable as we can write,

$$\frac{1}{p(1 - p/K)} \, dp = r \, dt$$

We can integrate on both sides

$$\int \frac{1}{p(1 - p/K)} \cdot dp = \int r \, dt$$

The solution of this equation gives Sigmoid function which is essential for binary classification in Logistic Regression

2. Exact Ordinary Differential Equation:

A differential Equation,

$$M.dx + N.dy = 0$$

is called exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$ where ∂ indicates the partial differentiation.

Example: Given differential Equation is

$$\tan x \sec^2 y \cdot dy + \tan y \cdot \sec^2 x \cdot dx = 0$$

Comparing it with $M\,dx + N\,dy = 0$, we get

$$M = \tan y \sec^2 x + \tan y \sec^2 x \cdot dx = 0$$

Comparing it with $M\,dx + N\,dy = 0$, we get

$$M = \tan y \sec^2 x \quad \text{and} \quad N = \tan x \sec^2 y$$

So,

$$\frac{\partial M}{\partial y} = \sec^2 y \sec^2 x \qquad \frac{\partial N}{\partial x} = \sec^2 x \sec^2 y$$

This shows that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. That is the given equation is exact.

Solving Process to an Exact Differential Equation:

Let the differential equation $M.dx + N.dy = 0$ — eqn (i) be an exact –

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

then the solution of (i) is $\int M.dx + \int (\text{term of } N \text{ free from } x).dy = c$

Linear Equations

Linear Ordinary Differential Equation:

A differential equation is of the form

$$\frac{dy}{dx} + Py = Q$$

Where P and Q are either function of x or constant, Is called linear Ordinary differential equation in y.

The solution of such type of equation can be obtained by using the formula,

$$y \times \text{Integrating factor} = \int (Q \times \text{Integrating factor})\, dx + C$$

i.e. $y \times I.F = \int (Q \times I.F.) \cdot dx + C$

where Integrating factor (I.F) is –

$$I.F = e^{\int p\,dx}$$

Example: Find the Solution of the Initial value Problem:

$$x^2 y' + xy = 1, \quad x > 0, \; y(1) = 2$$

Here, $\dfrac{dy}{dx} + \dfrac{y}{x} = \dfrac{1}{x^2}$ —— eqn (i)

Here, given equation is linear in y, Comparing with $\dfrac{dy}{dx} + Py = Q$, we have

$$P = 1/x \text{ and } Q = 1/x^2$$

Now, $I.F = e^{\int p\,dx} = e^{\int \frac{1}{x} \cdot dx}$

$\Rightarrow e^{\log x}$

$\Rightarrow x.$

Since, $y \times x = \int x \cdot \frac{1}{x^2} \cdot dx$

$yx = \ln x + C$

or, $y = \frac{1}{x} \ln x + \frac{C}{x}.$

Now, $y(1) = 2$ we have,

$$2 = \frac{1}{1} \ln 1 + \frac{C}{1}$$

$$C = 2.$$

$$y = \frac{\ln x + C}{x}$$

or, $$\boxed{y = \frac{\ln x + 2}{x}}$$

ODE has various real-world applications like in Physics, Engineering, Chemistry, Robotics Control Theory, Machine learning.

Second Order Linear Equations:

A differential equation is of the form

$$P(x)\frac{d^2y}{dx^2} + Q(x)\cdot\frac{dy}{dx} + R(x)\,y = G(x) \quad \text{— eqn (i)}$$

Where P, Q, R and G are continuous functions, is called second order linear differential equation.

If $G(x) = 0$ for all x, then the equation

$$P(x)\frac{d^2y}{dx^2} + Q(x)\cdot\frac{dy}{dx} + R(x) = 0 \quad \text{— eqn (ii)}$$ is called

Homogeneous Second order linear differential equation.

① If $Q(x) \neq 0$ for some x eqn ① is called non-homogeneous second order linear differential equation.

Solving Homogeneous Second Order linear Differential Equation

→ if we know solutions y_1 and y_2 of such equation

↳ linear combination $y = c_1 y_1 + c_2 y_2$ is also a solution of the equation.

Theorem (1):

if $y_1(x)$ and $y_2(x)$ are both solutions of the linear Homogeneous Equation (2) and C_1 and C_2 are any constants then the function $y(x) = C_1 y_1(x) + C_2 y_2(x)$ is also a solution of equation (2).

In AI:

$$\theta_{new} = \theta_{old} - H^{-1} \nabla L(\theta)$$

where H is the Hessian Matrix (Second order dx)

$\nabla L \theta$ (gradient descent first order dx)

$L\theta$ is the loss function.

Non-Homogeneous Linear Equation:

$$a x'' + by' + cy = G(x)$$

where $G(x) \neq 0$ for some x

where a, b, c are constants and G is a continous function

the corresponding Homogeneous Equation is —

$$ay'' + by' + cy = 0$$ is called the Complementary Equation and plays an important role solving the original non-Homogeneous Equation

$$a(x) \frac{d^2y}{dx^2} + b(x) \frac{dy}{dx} + C(x) \cdot y = f(x)$$

Infinite Sequence and Series:

A sequence is an ordered list of things. Such thinks may finite or infinite. In a sequence, the terms are separated by commas (,).

Infinite Sequence:

A sequence is a list of numbers which are written as a definite order

$$a_1, a_2, a_3 \ldots\ldots a_n, \ldots$$

a_1 = 1st term a_2 = 2nd term a_n = a is the nth term

$\{1, 2, 3, 4 \ldots\} \rightarrow N$ $\{\sqrt{n}\} \rightarrow \sqrt{n}$ $2, 4, 6, 8 \rightarrow 2n$, $1, 8, 27, 64 \rightarrow n^3$

Definition (Constant Sequence):

A sequence with every term is some fixed value is called a constant sequence. for example $\{7, 7, 7 \ldots \}$

Bounded and Unbounded Sequence:

- A sequence $\{a_n\}$ is bounded if there are two fixed value k, $K \in \mathbb{R}$ such that $k \le a_n \le K$ for all n.
- A sequence $\{a_n\}$ is bounded above if there is a real value $K \in \mathbb{R}$ such that $a_n \le K$ for all n.

- A Sequence $\{a_n\}$ is bounded below if there is a real value $K \in \mathbb{R}$ such that $K \leq a_n$ for all n.

- A Sequence $\{a_n\}$ is called unbounded if it is no bounded above and bounded below.

Q. Consider a sequence $a_n = 1/n$ show that sequence is bounded.

Soln:

$$a_n = 1/n.$$

$$\{a_n\} = \{1, 1/2, 1/3, \ldots\}$$

$u_n \leq 1$ for all n.

And,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1/n\right)$$

$$\Rightarrow 0.$$

So, $0 \le a_n \le 1$ for all n.

This shows that the sequence $\{a_n\}$ is bounded below by 0 and above by 1.

\# $a_n = 2^n$ is bounded below and is not unbounded below.

$a_n = 2^n$ for all n.

$$\{a_n\} = \{2, 2^2, 2^3, 2^4 \ldots\}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (2^n) = \infty$$

bounded below by 2 but not bounded above.