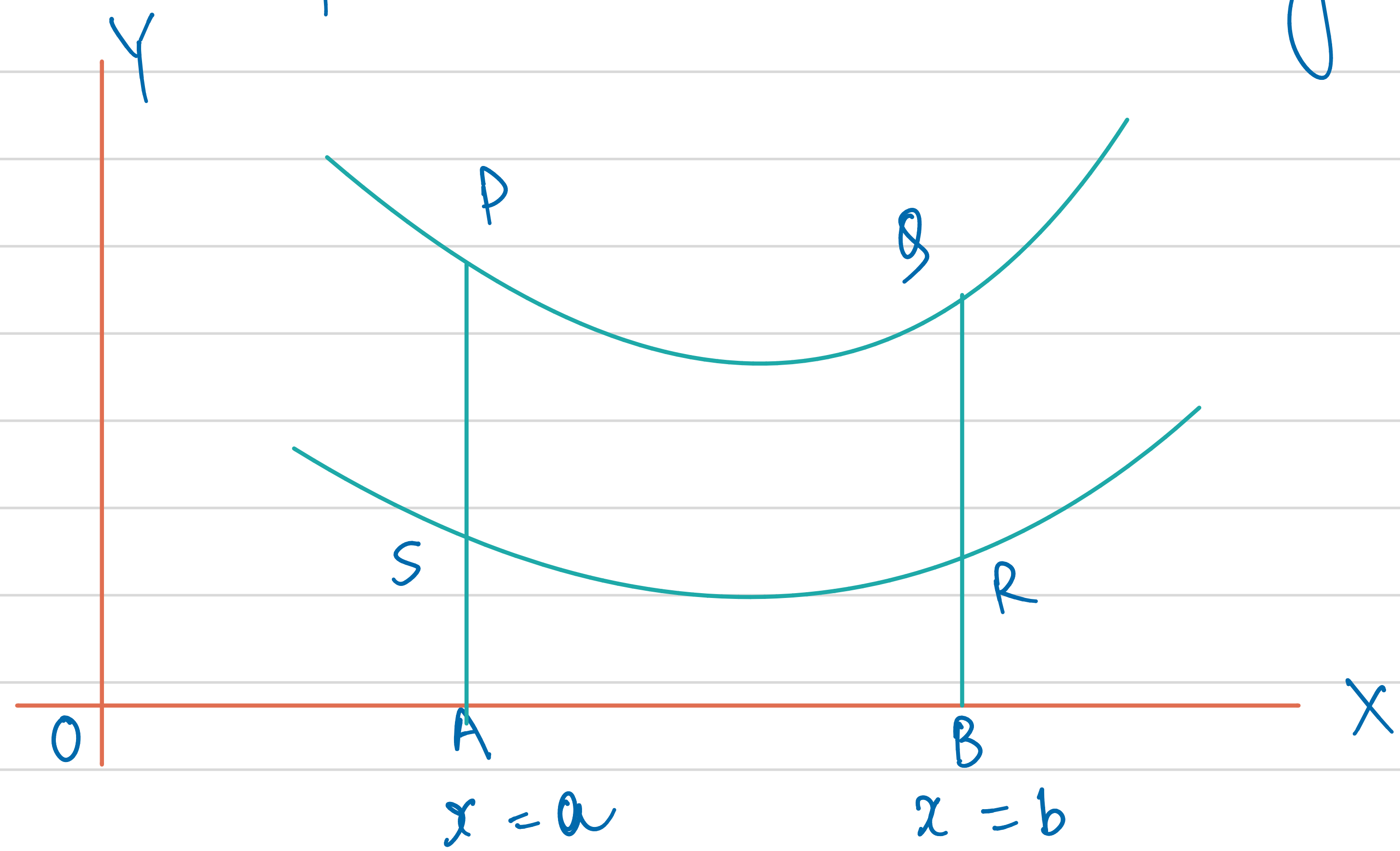
# Day-45, Jan-14, 2025 (Magh 1, 2081 B.S.)

## Area Between Two Curves

Now let us find out the area enclosed under the curves represented by two given functions $f_1$ and $f_2$ and the given ordinates $x = a$ and $x = b$. Let PQ be the curve given by the function $f_1$ and SR be the given by the function $f_2$.




Let PSA and QRB be the ordinates given by $x = a$ and

$x = b$ respectively. We have to find out the Area of PQRS. Now,

$$PQRS = \int_a^b f_1(x)\,dx - \int_a^b f_2(x)\cdot dx$$

$$\Rightarrow \int_a^b (y_1 - y_2)\cdot dx$$

## Definition of a Definite Integral

If $f$ is function defined for $a \le x \le b$, we divide the interval $[a, b]$ into $n$ subintervals of equal width $\Delta x = \frac{(b-a)}{n}$. We let $x_0 (= a)$, $x_1, x_2, \ldots, x_n (= b)$

be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any sample points in the subintervals so $x_i^*$ lies in the $i$th subinterval $[x_{i-1}, x_i]$. Then the definite integral of $f$ from $a$ to $b$ is

$$\int_a^b f(x) \cdot dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x$$

The precise meaning of the limit that defines the integral is as follows. for every number $\epsilon > 0$ there is an integer $N$ such that

$$\left| \int_a^b f(x) \cdot dx - \sum_{i=1}^{n} f(x_i^*) \Delta x \right| < \epsilon$$

for every integer $n > N$ and for every choice of $x_i^*$ in $[x_{i-1}, x_i]$ the sum that occurs in definition is called a Riemann System.

Note 1: The symbol $\int$ was introduced by Leibniz and is called an integral sign. It is elongated S and was choosen because an integral is a limit of sums. In the notation is called the integrand and a and b are called the limits of integration; is the lower limit and b is the upper limit

$$\int_a^b f(x) \cdot dx$$

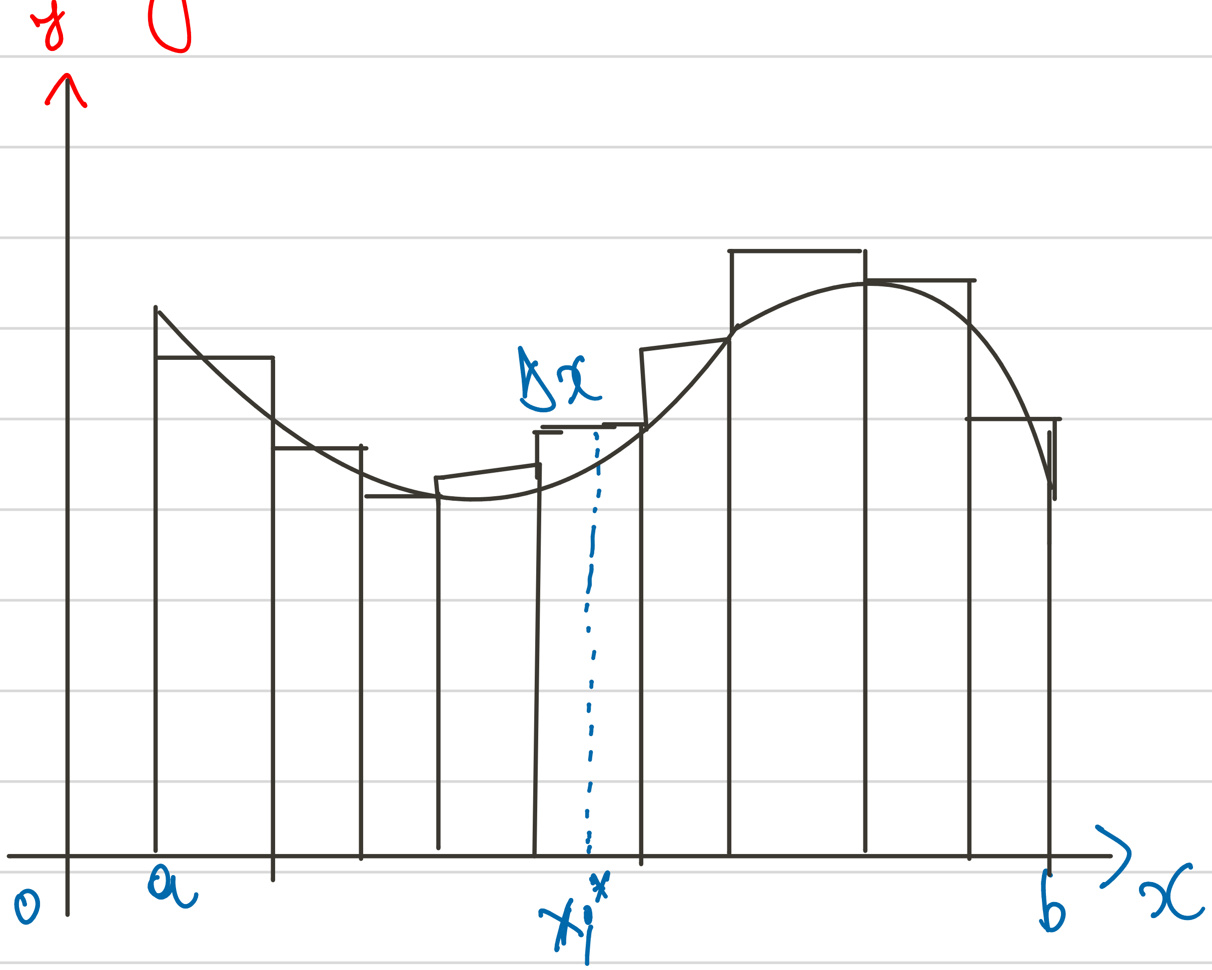
where dx is independent variable $x$
a and b are the limits of Integration.

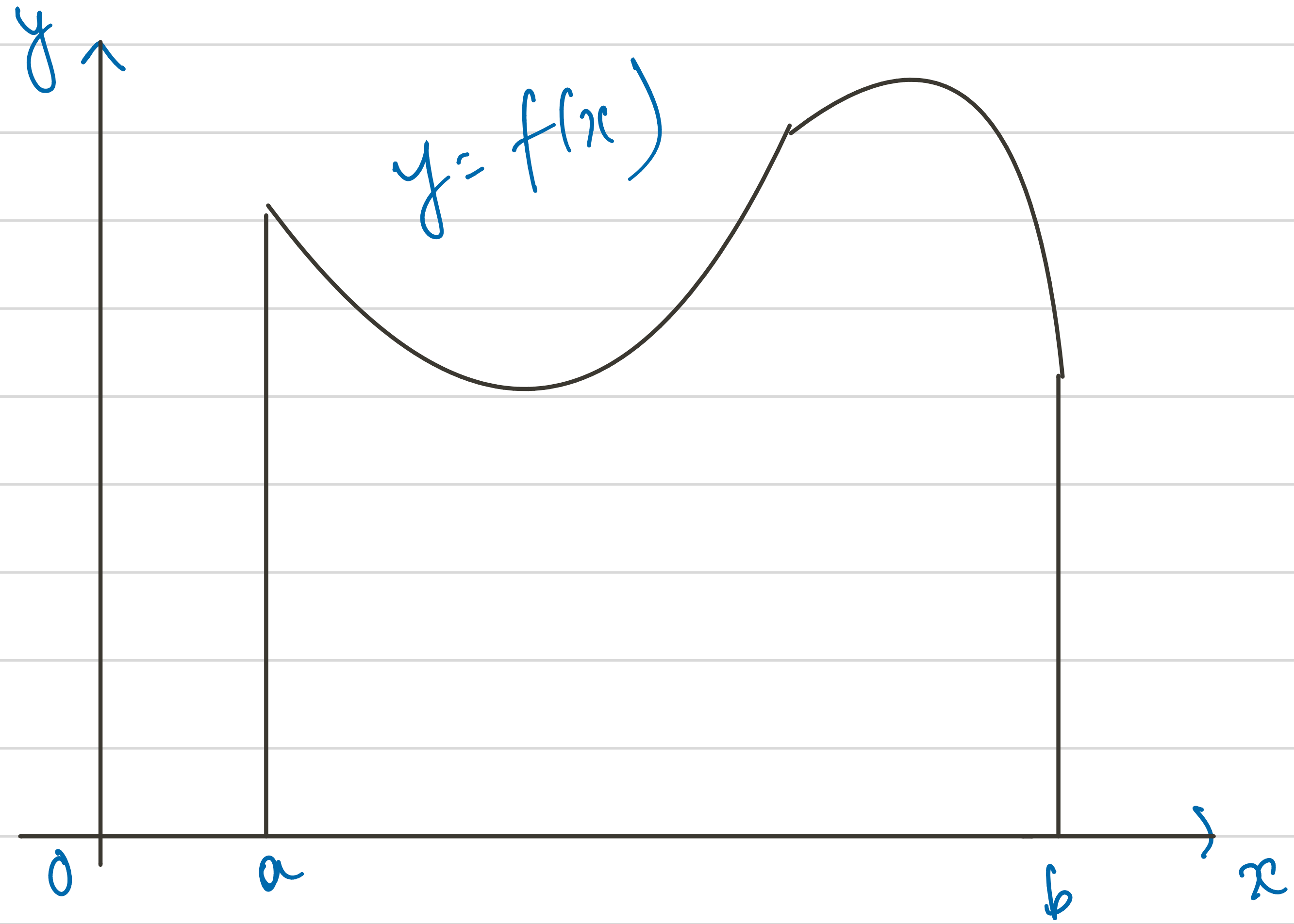
**Note 2:** The definite integral $\int_a^b f(x) \cdot dx$ is a number; it does not depend upon $x$. In fact, we could use any letter in place of $x$ without changing the value of the integral.

$$\int_a^b f(x) \cdot dx = \int_a^b f(t) \cdot dt$$

$$\Rightarrow \int_a^b f(r) \cdot dr$$

**Note 3:** If $f$ takes on both positive and negative values then the Riemann Sum is the sum of the area of the rectangles that lie above the x-axis and the negatives of the area of rectangles that lies below the x-axis.




if $f(x) \geq 0$ the Riemann sum $\sum f(x_i^*) \Delta x$ is the sum of areas of rectangles.

if $f(x) \geq 0$, the integral $\int_a^b dx$ is the area under the curve $y = f(x)$ from a to b.

## Theorems:

**Theorem 1:** (The existence of definite integrals) If $f$ is continuous on $[a,b]$ or if $f$ has only a finite number of jump discontinuities then $f$ is integrable on $[a,b]$ i.e. the definite integral $\int_a^b f(x)\cdot dx$ exists.

**Theorem 2:** If $f$ is integrable on $[a,b]$ then

$$\int_a^b f(x)\cdot dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x \quad \text{where } \Delta x = \frac{b-a}{n}$$

$$x_i = a + i\Delta x$$

# The Midpoint Rule:

Instead of using the right end point of the ith subinterval it is convenient to choose $x_i^*$ to be the midpoint of the interval which we denote by $\bar{x}_i$.

Any Riemann Sum is an approximation to an integral, but if we use midpoints we get the following approximation.

## Midpoint Rule:

$$\int_a^b f(x)\cdot dx = \sum_{i=1}^{n} f(\bar{x}_i)\cdot \Delta x$$

$$\Rightarrow \Delta x \left[ f(\bar{x}_1) + \cdots + f(\bar{x}_n) \right]$$

where $\Delta x = \dfrac{b-a}{n}$

and $\bar{x}_i = \dfrac{1}{2}(x_{i-1} + x_i)$

$\Rightarrow$ midpoint of $[x_{i-1}, x_i]$

Example: use the Midpoint Rule with $n=5$ to approximate

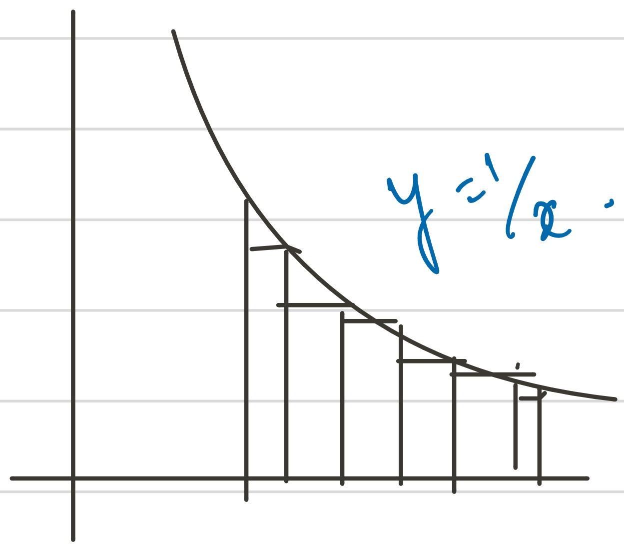
$$\int_1^2 \frac{1}{x}\, dx.$$

Here, the endpoints of the five sub-intervals are $1, 1.2, 1.4, 1.6, 1.8$ and $2.0$ so the midpoint are $1.1, 1.3, 1.5, 1.7,$ and $1.9$.

The width of the Subintervals is $\Delta x = \frac{(2-1)}{5} \Rightarrow 1/5$

So, the midpoint Rule -

$$\int_1^2 \frac{1}{x}\, dx = \Delta x \left[ f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9) \right]$$




$$\Rightarrow \frac{1}{5}\left( \frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9} \right)$$

$$\Rightarrow 0.691908$$

$f(x) = 1/x \geq 0$ for $1 \leq x \leq 2$ the Integral represents an area.

## Properties of the definite Integral:

If $f$ and $g$ are integrable, and $K$ is any number then -

1. Order of Integration: $\int_a^b f(x) \cdot dx = -\int_b^a f(x) \cdot dx$

2. $\int_a^b K \cdot dx = K(b-a)$

3. Zero width Interval: $\int_a^a f(x) \cdot dx = 0$

4. Constant Multiple: $\int_a^b K \cdot f(x) \, dx = K \int_a^b f(x) \cdot dx$

5. Sum and Difference: $\int_a^b (f(x) \pm g(x)) \, dx = \int_a^b f(x) \cdot dx \pm \int_a^b g(x) \cdot dx$

6. Additivity: $\int_a^b f(x) \cdot dx + \int_b^c f(x) \cdot dx = \int_a^c f(x) \cdot dx$

# Application of Derivatives

1. Geometrical Interpretation of Derivative of a function
2. Increasing and Decreasing functions
3. Maxima and Minima
4. Absolute Maxima and Minima
5. Stationary Point $\frac{dy}{dx} = f'(x) = 0$.
6. Concavity and Convexity of Curves
7. Point of Inflection, Critical points, Rate of Measure