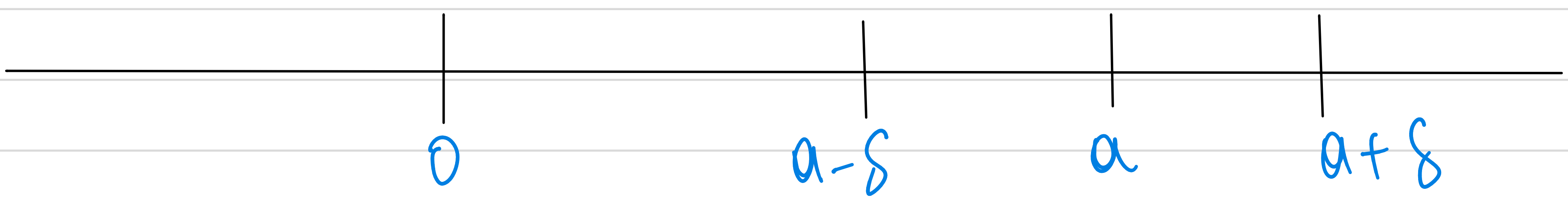
Day-4, Nov-19, 2024 (Mangshir 4, 2081 BS.)

A neighbourhood of a point 'a' is an open interval containing the point 'a'. It is generally denoted by $(a-\delta, a+\delta)$. Also.

$$x \in (a-\delta, a+\delta)$$

$$\Rightarrow \quad |x-a| < \delta$$

As we have $\lim_{x \to a} f(x) = l$ and $|f(x) - l|$ is very small difference between 'x' and 'a' ie. $|x-a|$ is very small.



Let $x \in (a-\delta, a+\delta)$ then

$$a-\delta < x < a+\delta$$

or, $-\delta < x-a < \delta$ So, $|x-a| < \delta$

So, $\lim_{x \to a} f(x) = l$

if to every positive number ϵ, however small, there corresponds a positive number δ, such that $|f(x) - l| < \epsilon$, whenever $|x-a| < \delta$.

take $\lim_{x \to 3} (3x-4) = 3 \times 3 - 4$

$\Rightarrow 5$

take $\epsilon > 0$ then there exists $|x-3| < \delta$

$|(3x-4)-5| < \epsilon$ whenever $|x-3| < \delta$

$|3x-9| < \epsilon$

Here δ is +ve and greater than $|x-3|$

$x-3 < \epsilon/3$

Hence, $\delta \leq \epsilon/3$

This notion and concept give us foundational knowledge and understanding for right and left hand limit.

Limit: Left Hand Limit:

A function is said to be the left hand limit L_2 at $x = a$ as 'x' approaches 'a' through values less than 'a' ($x < a$ i.e. x approaches from the left). So, the left hand limit of $f(x)$ at 'a' is written as

$$\lim_{x \to a-0} f(x) \quad \text{or} \quad f(a-0)$$

A_2 is said to be left-hand limit of f at $x = a$, if corresponding to any positive number ϵ, there exists a positive number δ such that $|f(x) - A_2| < \epsilon$, whenever $x \in (a-\delta, a)$.

Right-Hand Limit.

A function $f(x)$ is said to have the right hand limit l_1 at $x = a$ as 'x' approaches to 'a' through value greater than 'a' (i.e. x approaches 'a' from the right and Symbollically it is written as

$$\lim_{x \to a^+} f(x) = l_1.$$

The right hand limit of $f(x)$ at $x = a$ is also written as

$$\lim_{x \to a+0} f(x) \quad \text{or} \quad f(a+0)$$

Also, A_1 is said to be the right-hand limit of f at $x = a$

$$|f(x) - A_1| < \epsilon \text{, whenever } x \in (a, a+\delta)$$

It is not difficult to prove that necessary and Sufficient condition for a function 'f' to have a limit at $x = a$ is that the Left-Hand and the Right-Hand Limits of f at $x = a$ should exist and coincide.

So, $\lim_{x \to a} f(x)$ exists if and only if $\lim_{x \to a+0} f(x) = \lim_{x \to a-0} f(x)$.

Example: $f(x) = \begin{cases} 3x^2 - 1 & \text{when } x \le 2 \\ 4x + 3 & \text{when } x > 2 \end{cases}$ at $x = 2$

Here, Left Hand Limit $x = 2$ is

$\lim_{x \to 2-0} f(x) = \lim_{x \to 2-0} (3x^2 - 1) \Rightarrow 12 - 1$
$= 11$

Using Right-Hand Limit at $x=2$ is

$$\lim_{x \to 2+0} f(x) = \lim_{x \to 2+0} 4x+3 \Rightarrow 8+3$$
$$\Rightarrow 11.$$

Hence, $\lim_{x \to a+0} f(x) = \lim_{x \to a-0} f(x)$ and both are equal at $x=2$ so the limit exists.

Example: $\lim_{x \to 2} \frac{x-2}{|x-2|}$

Let $f(x) = \frac{x-2}{|x-2|}$

By definition,

$$|x-2| = \begin{cases} x-2 & \text{if } x > 2 \\ -(x-2) & \text{if } x < 2 \end{cases}$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{x-2}{x-2}$$

$$\Rightarrow 1$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{x-2}{-(x-2)}$$

$$\Rightarrow -1$$

Since, $\lim_{x \to 2^+} f(x) \neq \lim_{x \to 2^-} f(x)$, so $\lim_{x \to 2} f(x)$ doesn't exist

Example: $\lim_{x \to \infty} \frac{3x^2+2x+1}{4x^2+x+5}$

Putting $x \to \infty$ gives $\frac{\infty}{\infty}$, when $x = \infty$ Indeterminate form

$$\lim_{x \to \infty} \frac{3 + \frac{2}{x} + \frac{1}{x^2}}{4 + \frac{1}{x} + \frac{5}{x^2}}$$

$$\Rightarrow \frac{3+0+0}{4+0+0} \quad \Rightarrow \frac{3}{4}$$

Example: $\lim_{x \to \infty} \left(\sqrt{x+a} - \sqrt{x}\right) = 0$

The given function takes $\infty - \infty$ when $x = \infty$ so rationalizing,

$$\lim_{x \to \infty} \frac{\left(\sqrt{x+a} - \sqrt{x}\right)\left(\sqrt{x+a} + \sqrt{x}\right)}{\left(\sqrt{x+a} + \sqrt{x}\right)}$$

$$\Rightarrow \lim_{x \to \infty} \frac{\cancel{x} + a - \cancel{x}}{\sqrt{x+a} + \sqrt{x}} \qquad \Rightarrow \lim_{x \to \infty} \frac{a}{\sqrt{x+a} + \sqrt{x}}$$

$$\Rightarrow \frac{a}{\sqrt{a+\infty} + \sqrt{\infty}}$$

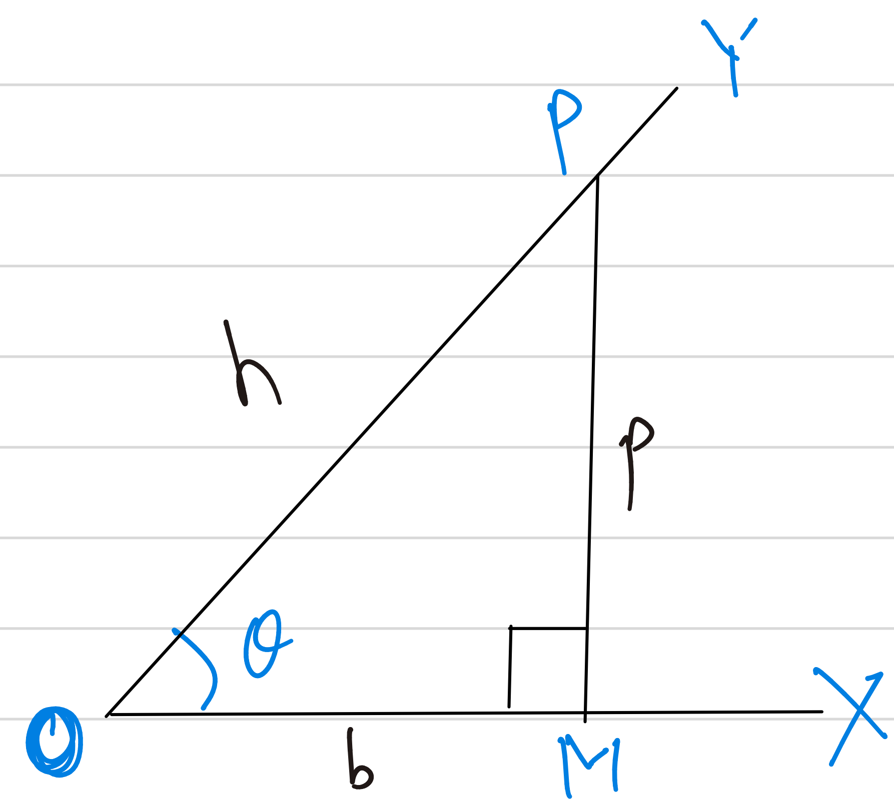
$$\Rightarrow \frac{a}{\infty + \infty} \qquad \Rightarrow \frac{a}{\infty} \qquad \Rightarrow 0.$$

Limits of Trigonometric functions

Standard Results:

i) $\lim_{\theta \to 0} \sin\theta = 0$

ii) $\lim_{\theta \to 0} \cos\theta = 1$



Let OX be the initial line and $\angle XOY = \theta$. Take any point 'P' on the limit OY. From P draw PM ⊥ to OX. Then -

$$\sin\theta = \frac{MP}{OP} \Rightarrow \frac{p}{h}$$

and $$\cos\theta = \frac{OM}{OP} \Rightarrow \frac{b}{h}$$

When θ is small, MP will be small and P will be near to M.

When θ is small enough, MP will be small enough and P will be very close to M.

This implies that as $\theta \to 0$, $MP \to 0$ and $OP \to OM$

$$\therefore \lim_{\theta \to 0} \sin\theta = \lim_{\theta \to 0} \frac{MP}{OP} \Rightarrow 0$$

$$\therefore \lim_{\theta \to 0} \cos\theta = \lim_{\theta \to 0} \frac{OM}{OP} \Rightarrow 1$$

References:

1) D.R. Bajaracharya, R.M. Shrestha et. la, 2014, "Basic Mathematics Grade XI (3rd. Edition)", Sukunda Pustak Bhawan, Kathmandu.