Day-51, Jan-20, 2025 (Magh 07, 2081 B.S).

Definition (Null Space):

Let A be $m\times n$ matrix then null-space of the matrix A is denoted by Nul A and defined by

$$\text{Nul } A = \{x : x \in \mathbb{R}^n : Ax = 0\}$$

Nul A is the set of all solutions to the Homogeneous equation $Ax=0$

Example: Let $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$ then determine if $u = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$ belongs to the null space of A.

Row x column

$(3\times 3)\times(3\times 1)$

$\Rightarrow (2\times 1)$

How,

$$Au = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1\times 5 - 3\times 3 + -2\times -2 \\ -5\times 5 + 9\times 3 + 1\times -2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5 - 9 + 4 \\ -25 + 27 - 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Theorem: The null space of $m\times n$ matrix A is subspace of R^n. Equivalently the set of all solutions to the system $Ax=0$ of m

homogeneous linear equations in 'n' unknown is a subspace of R^n.

Clearly, Nul A is non-empty since $A \cdot 0 = 0$ since A is $m \times n$ matrix so $0 \in$ Nul A. Moreover, let u and $v \in$ Nul A, then -

$$Au = 0, \quad Av = 0.$$

For every $\alpha, \beta \in K$ we have,

$$A(u+v) = Au + Av = 0$$

$$A(\alpha u + \beta v) = \alpha A(u) + \beta A(v)$$

$$\Rightarrow 0.$$

$$\alpha u + \beta v \in \text{Nul } A.$$

$\therefore$ Nul A is a subspace of R^n.

Definition (Column Space of a Matrix A)

Let A be $m \times n$ matrix $[a_1 \; a_2 \; a_3 \; \ldots \; a_n]$ then column space of A is denoted by Col A and defined by the space generated by the columns of A.

i.e. $$\text{Col } A = \text{Span}\{a_1, a_2 \; \ldots \; a_n\}$$

Example: Find a matrix A such that $W = \text{Col } A$ where

$$W = \left\{ \begin{bmatrix} 6a-b \\ a+b \\ -7a \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$

Here,
$$W = \left\{ a \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} - b \begin{bmatrix} +1 \\ -1 \\ 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

$$= \text{Span}\left\{ \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix}, \begin{bmatrix} -1 \\ +1 \\ 0 \end{bmatrix} \right\}$$

<u>Thus, Matrix A</u> $= \begin{bmatrix} 6 & -1 \\ 1 & +1 \\ -7 & 0 \end{bmatrix}$

<u>Theorem:</u> The column space of an $m \times n$ matrix A is subspace of $\mathbb{R}^m$.

Let,

$$\text{Col } A = \{ b \in \mathbb{R}^m : b = Ax \text{ for some } x \in \mathbb{R}^n \}$$

Here $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ then $\text{Col} A$ is non-empty because for $0 \in \mathbb{R}^m$ there is $0 \in \mathbb{R}^n$ such that $A \cdot 0 = 0$. Hence, $0 \in \text{Col} A$.

For any $b_1, b_2 \in \text{Col} A$ then there exists $x_1, x_2 \in \mathbb{R}^n$ such that

$b_1 = Ax_1 \quad , \; b_2 = Ax_2$

for any α and $\beta \in \mathbb{R}$ we have -

$$\alpha b_1 + \beta b_2 = \alpha A x_1 + \beta A x_2$$

$$\Rightarrow A(\alpha x_1 + \beta x_2).$$

Since, $\alpha b_1 + \beta b_2 \in \mathbb{R}^m$, $\alpha x_1 + \beta x_2 \in \mathbb{R}^n$. So,

$$A(\alpha x_1 + \beta x_2) = \alpha b_1 + \beta b_2 \in \text{Col } A.$$

This means, Col A is a subspace of $\mathbb{R}^m$.

Note: The column space of an $m \times n$ matrix A is all of $\mathbb{R}^m$ if and only if equation $Ax = b$ has a solution for each b in $\mathbb{R}^m$.

Definition (Kernel and Range of Linear Transformation)

Let $T : V \to W$ be linear Transformation then

$T(x) = Ax$ where A is matrix associate with

linear Transformation T.

$$\operatorname{Ker} T = \{ x \in V : T(x) = 0 \}$$

$$= \{ x \in V : Ax = 0 \}$$

$$\Rightarrow \operatorname{Nul} A$$

Kernel: The set of all input vectors mapped to the zero vector by a linear transformation.

Range: The set of all output vectors that a linear transformation can produce.

$\therefore$ Kernel of linear transformation T is Nul A and range of linear Transformation T is Col A, where A is matrix associate with the linear Transformation T.

Linearly Independent Sets, Bases:

An indexed set of vectors $\{v_1, v_2 \dots v_n\}$ is said to be linearly Independent if the vector equation,

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0 \quad \text{--- eqn (i)}$$

The set $\{v_1, v_2 \dots v_p\}$ is said to be linearly Dependent if

$$c_1 v_1 + c_2 v_2 + \dots + c_p v_p = 0 \quad \text{has non-trivial solution}$$

$$\text{--- eqn (ii)}$$

There are some weights $c_1, c_2 \dots c_p$ not all zero such that eqn ii) hold.

A set containing a single vector v is linearly independent if $v \neq 0$. Also, a set of two vectors is linearly dependent if one can be expressed as a multiple of other.

Basis of vector → A set of linearly independent vector that span a vector space. Every vector in a space can be expressed as linear combination of the vectors in a set, we call it spaning of vector.

Basis : Let H be a subspace of a vector space V. An indexed set of vectors $B = \{b_1, b_2, \dots b_n\}$ in V is a basis of H.

If

i) the set $\{b_1, b_2 \dots b_p\}$ is linearly independent

ii) $H = \text{Span}\{b_1, b_2 \dots b_p\}$

Spanning Set Theorem:

Let $S = \{v_1, v_2 \dots v_p\}$ be a set in V and let $H = \text{Span}\{v_1, v_2, \dots v_p\}$.

a) If one of the vectors is S- say, v_k is a linear combination of the remaining vectors in S, then the set formed from S

by removing v_k still spans H

b) If $H \neq \{0\}$, some subset of S is a basis for H.

Bases for Nul A and Col A:

The pivot columns of a matrix A form a basis for Col A.

Some Intuitive Understanding of Basis of Vector:

1. Building blocks for creating any vector in a given vector space.
2. Also allow to express any vector (being unique combination of themselves) so they provide co-ordinate system

(3) They are the combination of linearly Independent vector and the spanning over the area.

Eigen values and Eigen Vectors:

How Ax is related to x ?

where A is $n \times n$ matrix and x is a column vector in R^n.

If A is 2×2 matrix

and x is non-zero vector in R^2 such that $Ax = \lambda x$

for some scalar λ, then each vector in the line through origin determined by x gets mapped back on to the same line under the multiplication by matrix A.

Eigen Value and Eigen Vectors have numerous applications.

Definition (Eigenvalue):

If A is $n \times n$ matrix, then a scalar λ is called an eigenvalue of matrix A if equation $Ax = \lambda x$ has a non-trivial solution. Such an 'x' is called eigenvector corresponding to eigenvalue 'λ'.

Definition (Eigenvector): If A is $n \times n$ matrix, then a non-zero vector $x \in \mathbb{R}^n$ is called an eigenvector of matrix A if

$$Ax = \lambda x, \text{ where } \lambda \text{ is scalar.}$$

Q. Is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ an eigen vector of $\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$?

Since, $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ and $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$Ax = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3+0 \\ 8-2 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$\Rightarrow 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow 3x.$$

So, $Ax = 3x$.

where $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a eigen-vector of $\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$

Q. Show that -2 is eigen value of $\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$.

Let $\lambda = -2$ and $A = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$.

$$Ax = \lambda x$$

$$Ax = -2x.$$ I = Identity Matrix.

$$(A + 2I)x = 0$$

So, $A + 2I = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$$

$\begin{bmatrix} A+2I & 0 \end{bmatrix}$

So row reduced the Augmented Matrix is $\begin{bmatrix} 3 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

So, Homogeneous system has free variable (Here x_2 is free variable). So equation i) has non-trivial solution. Thus $\lambda = -2$ is given value of given matrix A.

Theorem: If $v_1, v_2, \dots v_r$ are eigenvectors that correspond to distinct eigenvalue $\lambda_1, \lambda_2, \dots \lambda_r$ of an $n \times n$ matrix A, then the set of $\{v_1, v_2, \dots v_r\}$ is linearly Independent.

Theorem: The eigenvalues of a triangular Matrix are the entries on its main diagonal.

Proof: Consider 3×3 upper triangular Matrix A and λ be eigenvalue of

λ.

So, $Ax = \lambda x$

$(A - \lambda I)x = 0$ ⟶ eqn (i).

has no trivial solution.

Here,

$$A - \lambda I = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ 0 & a_{22}-\lambda & a_{23} \\ 0 & 0 & a_{33}-\lambda \end{bmatrix}$$

So its Augmented Matrix is

$$\left[\begin{array}{ccc:c} a_{11}-\lambda & a_{12} & a_{13} & 0 \\ 0 & a_{22}-\lambda & a_{23} & 0 \\ 0 & 0 & a_{33}-\lambda & 0 \end{array}\right]$$

Justification: Now, the eqn (i) has non-trivial solution if and only if in augmented matrix of $A - \lambda I$ one of entries $a_{11}-\lambda$, $a_{22}-\lambda$, $a_{33}-\lambda$ is zero.

This will happen only when λ equal to one of the entries $a_{11}-\lambda$, $a_{22}-\lambda$, $a_{33}-\lambda$ in A. Hence, eigenvalue of triangular matrix A are on the entries of the main diagonal.

REFERENCE:

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