Day - 47, Jan - 16, 2025 (Magh - 3, 2081 B.S.)

L - Hospital's Rule

If $f(x) = x^2 - 4$ and $g(x) = x - 2$ so that $\lim_{x \to 2} f(x) = 0$ and $\lim_{x \to 2} g(x) = 0$ giving $\frac{f(x)}{g(x)} = \frac{0}{0}$ when $x \to 2$, then the form is known as indeterminate form.

Other indeterminate form are $\frac{0}{0}$, $\frac{\infty}{\infty}$. We have L-Hospital's Rule for finding the limiting values of these functions which take the form $\frac{0}{0}$ and $\frac{\infty}{\infty}$ when $x \to a$ (say).

The form $\frac{0}{0}$ (L Hospital's Rule)

If $f(x)$ and $g(x)$ and also their derivatives $f'(x)$ and $g'(x)$ are continuous at $x=a$ and if $f(a) = g(a) = 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f'(x)}{\lim_{x \to a} g'(x)} = \frac{f'(a)}{g'(a)}$$

Provided that $g'(a) \neq 0$

If $f'(a)$ and $g'(a)$ are both zero then the above theorem can further be used. Thus,

$$\lim_{x \to a} \frac{f'(x)}{g'(x)}$$

$$\Rightarrow \lim_{x \to a} \frac{f'(x)}{g''(x)}$$

$$\Rightarrow \frac{f'(a)}{g''(a)}$$

Provided that $f'(x)$ and $g'(x)$ both are continuous at $x = a$ and $g''(a) \neq 0$.

The above theorem can further be used if $f'(a) = 0$ and $g''(a) = 0$.

Example:

$$\lim_{x \to \infty} \frac{x^3}{e^x}$$

$$\Rightarrow \lim_{x \to \infty} \frac{3x^2}{e^x}$$

$$\Rightarrow \lim_{x \to \infty} \frac{6x}{e^x}.$$

$$\Rightarrow \lim_{x \to \infty} \frac{6}{e^x} \qquad \boxed{\Rightarrow 0}$$

Thus, we have to change the form $\frac{\infty}{\infty}$ into the form $\frac{0}{0}$ after which L Hospital's Rule can be applied to evaluate the limit. $\lim_{x \to a} \frac{x^n - a^n}{x - a} = n \cdot a^{n-1}$

Ordinary Differential Equation

→ Calculus discovered by Newton (c. 1665) and Leibniz (1684)

↳ Differential Equations (mid 17th century)

Review of Ordinary Differential Equations (ODE)

A differential equation is an equation involving one dependent variable and its derivatives with respect to one or more independent variable then it is called Ordinary Differential Equation.

When a differential equation has two or more than independent variables, it is known as partial Differential Equation. The examples of ordinary differential equations are:

$$\frac{dy}{dx} = x \sin x \longrightarrow \text{Ordinary differential equation}$$

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0 \longrightarrow \text{Partial Differential Equation}$$

Order of Differential Equation:

The highest order derivative involve in the equation is called the order of differential equation for example,

$$\frac{d^4y}{dx^4} - \frac{d^2y}{dx^2} - 3y = 0$$

in ODE of order 4. And,

$$\left(\frac{dy}{dx}\right)^4 - \left(\frac{d^2y}{dx^2}\right) - 3y = 0$$

is an ODE of order 2.

Degree of Differential Equation

The power of the highest derivative term after it has been made free from fractional power, is called the degree of differential Equation. for example,

$$\frac{dy}{dx} - 3y = 0$$

is an ODE of first degree. And,

$$\left(\frac{dy}{dx}\right)^4 - \left(\frac{d^2y}{dx^2}\right) - 3y = 0$$ is an ODE of

ODE of first degree.

But the equation,

$$\left(\frac{d^2y}{dx^2}\right)^{3/2} + 3\frac{dy}{dx} = 0$$

is an ODE of degree 3 because, the equation can be written as

$$\left(\frac{d^2y}{dx^2}\right)^3 - 9\left(\frac{dy}{dx}\right)^2 = 0$$

Solution of Differential Equation: (Steps / Rules).

1. Integrate successively unless and until it is free from derivatives.

2. Total no. of arbitrary constants = Order of the corresponding differential equation

3. Solution of Differential Equation is a relation among the variables which is free from the derivative.

Definition: The solution of a differential equation is the value of dependent variable in terms of independent variables which is free from differential and that satisfy the differential equation.

Example: Show that $y = \frac{x^4}{3} + 3x^2 + 2$ is a solution of the differential equation $\frac{dy}{dx} = \frac{4x^3}{3} + 6x$.

Soln:

Let $y = \frac{x^4}{3} + 3x^2 + 2$

Differentiating both sides w.r.t. we get

$$\frac{dy}{dx} = \frac{4x^3}{3} + 6x.$$

This shows that $y = \frac{x^4}{3} + 3x^2 + 2$ is a Solution of

$$\frac{dy}{dx} = \frac{x^4}{3} + 3x^2 + 2.$$

Q. Show that the differential equation $\frac{d^2y}{dx^2} = 12x^2 + 6x + 5$ has a Solution $y = x^4 + x^3 + \frac{5}{2}x^2$.

Here,

$$y = x^4 + x^3 + \frac{5}{2}x^2$$

then, $$\frac{dy}{dx} = 4x^3 + 3x^2 + 5x$$

$$\frac{d^2y}{dx^2} = 12x^2 + 6x + 5$$

This means $y = x^4 + x^3 + \frac{5}{2}x^2$ is the solution of the differential equation

$$\left[\frac{d^2y}{dx^2} = 12x^2 + 6x + 5 \right]$$

Geometrical Meaning of Differential Equation and its Solutions:

A first order and first degree of ODE is,

$$y' = f(x, y)$$

$$\text{i.e. } \frac{dy}{dx} = f(x, y) \quad \text{——— eqn (i)}$$

We know that the derivative of $y'(x)$ of $y(x)$ = slope of $y(x)$.

So, the solution curve of the above eqn (i) that passes through a point (x_0, y_0) must have the slope $y'(x_0)$ equal to the value of f at that point (x_0, y_0) i.e. $y'(x_0) = f(x_0, y_0)$

The geometrical meaning of a particular example and its solution in the basis of above concept is given below.

Consider a differential Equation,

$$\frac{dy}{dx} = \frac{y}{x}$$

Here by the Separation of variables, we have

$$\frac{dy}{y} = \frac{dx}{x}$$

By Integration, we get -

$$\ln y = \ln x + \ln m$$

After using the property of log we have $y = mx$.

Here the solution of the differential equation represents all the family of straight lines passing through origin with slopes 'm'

Consider, the family of concentric circles whose center is at origin and different radii c is:

$$x^2 + y^2 = c^2$$

Now, differentiating both sides of equation with respect to x we have,

$$2x + 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -x/y$$

which is the corresponding differential equations of family of all concentric circles.

Some Special Types of Equations of first Order and First Degree (ODE).

A first order and first degree ODE is in the form

$$M.dx + N.dy = 0$$

where M and N are either constant or function of x or y or both that are free from the differentiation

As the nature of the above equation, it can be classify as (only some special forms are given below):

- Variable Separable Equation
- Exact Equation
- Linear Equation

Reference:

Binod Prasad Dhakal, Ramesh Gautam, Surendra et. al, 2075, Mathematics - I Second Edition, B.Sc. CSIT | first Semester, KEC Publication and Distribution (P). Ltd., Kathmandu.