Chapter 1

1.1 System of Linear Equations:

Linear Equations:

A linear equation in the variables $x_1, x_2, x_3, x_4, \ldots\ldots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + \ldots. + a_n x_n = b$$

Where the coefficients $a_1, a_2, a_3, a_4, \ldots\ldots, a_n$ and b are real or complex numbers

Example: $2x_1 - 5x_2 + 3x_3 = 4$

System of Linear Equations:

A collection of one or more linear equations is called a System of Linear Equations or linear system.

Example: $2x_1 - 5x_2 + 3x_3 = 4,$

$$x_1 - 6x_2 + 7x_3 = 0$$

Solution of System of Linear Equations:

A solution of System of Linear Equations is a list of values

$(x_1, x_2, x_3, x_4, \ldots\ldots, x_n)$ **of numbers that satisfies the given system of equations.**

For example: **The system**

$$2x_1 - 5x_2 = 1,$$

$$x_1 - 3x_2 = 0$$

is satisfied by a list $(x_1, x_2) = (3, 1)$.

So, (3, 1) is the solution of the system.

Consistent and inconsistent :

A system of linear equations is called consistent if it has one or more than one or infinitely many solutions and called inconsistent if it has no solution.

Matrix notation of the system:

A matrix form of coefficients and the constant values of a linear system of equations is known as matrix notation of the system.

If the matrix notation involves only the coefficients of variable then the system is called coefficient matrix. And if, the matrix notation involves the coefficients of linear system as well as constant value then the matrix is called augmented matrix of the system.

Example: **Consider a linear system**

$$2x_1 - 5x_2 = 1,$$
$$x_3 - 3x_2 = 0$$
$$3x_1 - 4x_2 + 6x_3 = -2,$$

The coefficients matric is

$$\begin{bmatrix} 2 & -5 & 0 \\ 0 & -3 & 1 \\ 3 & -4 & 6 \end{bmatrix}$$

and the augmented matrix of the system is

$$\begin{bmatrix} 2 & -5 & 0: & 1 \\ 0 & -3 & 1: & 0 \\ 3 & -4 & 6: & -2 \end{bmatrix}$$

Elementary row operations:

1. Replacement (replace one row by the sum of itself and a multiple of another row)
2. Interchange (interchange of two rows)
3. Scaling (Multiply all entries in a row by a nonzero constant.

Echelon form or Row Echelon form:

A rectangular matrix is in Echelon form (or row echelon form) if it has the following three properties:

1) All non-zero rows are above any rows of all zeros.
2) Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3) All entries in a column below a leading entry are zeros.

If a matrix is in echelon form and satisfies the conditions

4) The leading entry in each non-zero row is 1.
5) Each leading 1 is the only non-zero entry in its column.

$$\begin{bmatrix} 2 & 4 & 5 & 7 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 3 & 9 \end{bmatrix}$$ it is echelon form.

$$\begin{bmatrix} 0 & 4 & 5 & 7 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$ it is echelon form.

$$\begin{bmatrix} 2 & 4 & 5 & 7 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ it is echelon form.

$$\begin{bmatrix} 2 & 4 & 5 & 7 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 3 & 9 \end{bmatrix}$$ it is not echelon form.

$$\begin{bmatrix} 0 & 4 & 5 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$ it is not echelon form.

$$\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 9 \end{bmatrix}$$ it is reduced echelon form.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ it is reduced echelon form.

$$\begin{bmatrix} 1 & 4 & 5 & 7 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ it is not reduced echelon form.

$$\begin{bmatrix} 1 & 4 & 0 & 7 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$ it is not reduced echelon form.

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ it is echelon form.

Note:

1) Different operations of row operation gives different echelon matrix of any non-zero matrix. This means a matrix may have more than one echelon matrix as the operated row operations.
2) Remembered that a non-zero matrix have one and only one i.e. exactly one reduced echelon matrix

Theorem 1 (Uniqueness of the Reduced echelon form)

Each matrix is row equivalent to one and only one reduced echelon matrix.

Pivot position in reduced echelon form:

A pivot position in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A pivot column is a column of A that contains a pivot position.

Row reduction Algorithm:

The algorithm helps to obtain a matrix in echelon form and in reduced echelon form.

The process of the algorithm is as follows:

1) Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.
2) Select the nonzero entry in the pivot column as a pivot. If necessary , interchange rows to move this entry into the pivot position.
3) Use row replacement operations to create zeros in all positions below the pivot.
4) Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1-3 to the sub matrix that remains. Repeat the process until there are no more nonzero rows to modify.
5) Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by scaling operation.

Example: Reduced the augmented matrix of the system in reduced echelon form

$$\begin{bmatrix} 0 & 1 & 0 & -3 & 3 \\ 1 & 0 & 3 & 0 & 2 \\ 0 & -2 & 3 & 2 & 1 \\ 3 & 0 & 0 & 7 & -5 \end{bmatrix}$$

Solution: Here, Interchanging R_1 and R_2 to make non-zero pivot then,

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 3 & 0 & 0 & 7 & -5 \end{bmatrix}$$

Applying $R_3 \rightarrow R_3 + 2R_2$ then the above matrix reduces to

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 3 & 0 & 0 & 7 & -5 \end{bmatrix}$$

Applying $R_4 \rightarrow R_4 - 3R_1$ then the above matrix reduces to

$$\sim\begin{bmatrix}1 & 0 & 3 & 0 & 2\\ 0 & 1 & 0 & -3 & 3\\ 0 & 0 & 3 & -4 & 7\\ 0 & 0 & -9 & 7 & -11\end{bmatrix}$$

Applying $R_4 \to R_4 + 3R_3$ then the above matrix reduces to

$$\sim\begin{bmatrix}1 & 0 & 3 & 0 & 2\\ 0 & 1 & 0 & -3 & 3\\ 0 & 0 & 3 & -4 & 7\\ 0 & 0 & 0 & -5 & 10\end{bmatrix}$$

Which is a echelon form.

Now, for reduced echelon form

Applying $R_3 \to \frac{R_3}{3}$ and $R_4 \to \frac{R_4}{-5}$ then the above matrix reduces to

$$\sim\begin{bmatrix}1 & 0 & 3 & 0 & 2\\ 0 & 1 & 0 & -3 & 3\\ 0 & 0 & 1 & -\frac{4}{3} & \frac{7}{3}\\ 0 & 0 & 0 & 1 & -2\end{bmatrix}$$

Applying $R_2 \to R_2$ + 3R_4 and $R_3 \to R_3 + \frac{3}{4}R_4$ then the above matrix reduces to

$$\sim\begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & \frac{7}{3} \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

Applying $R_1 \to R_1 - 3R_2$ then the above matrix reduces to,

$$\sim\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & \frac{7}{3} \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

Which is the reduced echelon form.

Solution of Linear system:

The row reduction algorithm leads to the solution set of a linear system.

For example suppose that the augmented matrix of a linear system has been changed into the equivalent reduced echelon form

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Which is reduced echelon form.

There are three variables because the augmented matrix has four columns. Here, 1st and 2nd columns are pivot column. So, the variables x_1 and x_2 are basic variables and 3rd column is not a pivot column so x_3 is free variable.

Thus the associated system of equations is

$$x_1 + 0.x_2 - 5x_3 = 1$$

$$0.x_1 + x_2 + x_3 = 4$$

$$0 = 0$$

Hence, the general solution is
$$\begin{cases} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \\ x_3 \; is \; free \end{cases}$$

Theorem (Existence and Uniqueness theorem)

A linear system is consistence if and only if the rightmost column of the augmented matrix is not a pivot column.

That is, a linear system has solution if and only if an echelon form of the augmented matrix has no row of the form [o o o o … o b] with b$\neq$ 0

If the system is consistence then the solution set contains either unique solution when there is no free variable or infinitely many solutions when there is at least one free variable.

Exercise 1.1

1. **Determine the values of h such that the matrix is the augmented matrix of a consistence linear system.**

i) $\begin{bmatrix} 2 & 3 & h \\ 4 & 6 & 7 \end{bmatrix}$

Solution: Here,

$$\begin{bmatrix} 2 & 3 & h \\ 4 & 6 & 7 \end{bmatrix}$$

Operating $R_2 \rightarrow R_2 - 2R_1$

$$\sim \begin{bmatrix} 2 & 3 & h \\ 0 & 0 & 7-2h \end{bmatrix}$$

For consistence,

$$7 - 2h = 0$$

$$\therefore \quad h = \frac{7}{2}$$

2. Find the values of h and k so that the system has (a) no solution (b) a unique solution c) many solutions.

i) $x_1 + hx_2 = 2$

$4x_1 + 8x_2 = k$

Solution: Here, given system of linear equations is

$$x_1 + hx_2 = 2$$

$$4x_1 + 8x_2 = k$$

Augmented matrix is

$$\begin{bmatrix} 1 & h & 2 \\ 4 & 8 & k \end{bmatrix}$$

Operating $R_2 \to R_2 - 4R_1$

$$\sim \begin{bmatrix} 1 & h & 2 \\ 0 & 8-4h & k-8 \end{bmatrix}$$

a) For no solution,

$$8 - 4h = 0 \text{ and } k - 8 \neq 0$$

Or, $h = 2$ and $k \neq 8$

b) For a unique solution,

$$8 - 4h = 0 \text{ and } k - 8 = 0$$

Or, $h = 2$ and $k = 8$