Tribhuvan University

Institute of Science and Technology

2074

Bachelor Level / First Year /Second Semester/Science

Computer Science and Information Technology (MTH. 155)

(Linear Algebra)

Full Marks: 80
Pass Marks: 32
Time: 3 hours.

Candidates are required to give their answers in their own words as for as practicable.
The figures in the margin indicate full marks.
Attempt all questions:

Group A

(10×2=20)

1. What are the criteria for a rectangular matrix to be in echelon form?

2. Prove that (a) $(A^T)^T = A$ (b) $(A + B)^T = A^T + B^T$, Where A and B denote matrices whose size are appropriate for the above mentioned operations.

3. Define square matrix. Can a square matrix with two identical columns be invertible? Why or why not?

4. Let A and B be two square matrices. By taking suitable examples, show that even though AB and BA may not be equal, it is always true that detAB = detBA.

5. Using Cramer's rule solve the following simultaneous equations:
$$5x + 7y = 3$$
$$2x + 4y = 1$$
5/6, -1/6

6. Define vector space with suitable examples.

7. Let **W** be the set of all vectors of the form $\begin{bmatrix} 5b+2c \\ b \\ c \end{bmatrix}$, where b and c are arbitrary. Find vector **u** and **v** such that **W** = Span {**u**, **v**}.

8. What are necessary and sufficient conditions for a matrix to be invertible?

9. Determine whether the pair of vectors $\mathbf{u} = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$ are orthogonal or not?

10. What do you understand by least square line? Illustrate.

Group B

(5×4=20)

11. What are the criteria for a transformation T to be linear? If T: $\mathbf{R}^2 \to \mathbf{R}^2$ is defined by T(x) = 3x, show that T is a linear transformation. Also give a geometric description of the transformation $x \mapsto Ax$, where $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

12. Prove that if A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}.

13. Define subspace of a vector space **V**. Given $\mathbf{v}_1$ and $\mathbf{v}_2$ in a vector space **V**, let $\mathbf{H} = \text{span}\ \{\mathbf{v}_1, \mathbf{v}_2\}$. Show that **H** is a subspace of **V**.

OR

If $\mathcal{B} = \{b_1 \ldots, b_n\}$ is a basis for a vector space V and x is in V, define the coordinate of x relative to the basis $\mathcal{B}$ Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$, $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$. Then $\mathcal{B}$ is a basis for H = Span $\{\mathbf{v}_1, \mathbf{v}_2\}$. Determine is X is in **H**, and if it is, find the coordinate vector of **x** relative to $\mathcal{B}$

14. The mapping T: $\mathbf{P}_2 \rightarrow \mathbf{P}_2$ defined by $T(a_0 + a_1t + a_2t^2) = a_1 + 2a_2t$ is a linear transformation.

    a) Find the $\mathcal{B}$ - matrix for T, when $\mathcal{B}$ is the basis $\{1, t, t^2\}$.

    b) Verify that $[T(\mathbf{p})]_{\mathcal{B}} = [T]_{\mathcal{B}}[\mathbf{p}]_{\mathcal{B}}$ for each **p** in $\mathbf{P}_2$.

15. Let $A = \begin{bmatrix} 1 & -3 & -3 \\ 1 & 5 & 1 \\ 1 & 7 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 5 \\ -3 \\ -5 \end{bmatrix}$. Find a least square solution of Ax = b, and compute the associated least square error.

Group C

(5×8=40)

16. Let T: $\mathbf{R}^n \rightarrow \mathbf{R}^m$ be a linear transformation and let A be the standard matrix for T. Then prove that: T map $\mathbf{R}^n$ on to $\mathbf{R}^m$ if and only if the columns of A span $\mathbf{R}^m$; and T is one - to - one if and only if the columns of A are linearly independent. Let $T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$. Show that T is a one - to - one linear transformation. Does T map $\mathbf{R}^2$ onto $\mathbf{R}^3$?

17. Compute the multiplication of partitioned matrices for

$$A = \left[\begin{array}{ccc|cc} 2- & 3 & -1 & 0 & -2 \\ 1 & 5 & -2 & 3 & -1 \\ \hline 0 & 4 & 1 & 7 & -3 \end{array}\right]$$

and $$B = \left[\begin{array}{cc} 6 & 4 \\ -2 & 3 \\ -3 & 7 \\ \hline -1 & 2 \\ 5 & 3 \end{array}\right]$$

18. Let $\mathcal{B} = \{b_1, \ldots, b_n)$ be a basis for a vector space V. Then the coordinate mapping $x \mapsto [X]_{\mathcal{B}}$ is one-to-one linear transformation from V into R^n. Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$, and

$$\mathbf{x} = \begin{bmatrix} -8 \\ 2 \\ 3 \end{bmatrix}.$$

a) Show that the set $\mathcal{B} = \{\mathbf{b1}, \ldots, \mathbf{bn}\}$ is a basis of $\mathbf{R}^3$.

b) Find the change-of-coordinates matrix for $\mathcal{B}$ to the standard basis.

c) Write the equation that relates x in $\mathbf{R}^3$ to $[x]_{\mathcal{B}}$

d) Find $[x]_{\mathcal{B}}$ for the x give above.

19. Diagonalize the matrix $\begin{bmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{bmatrix}$ if possible.

OR

Suppose $A = PDP^{-1}$, where D is a diagonal n x n matrix. If $\mathcal{B}$ is the basis for $\mathbf{R}^n$ formed for the columns of P, then prove that D is the $\mathcal{B}$ matrix for the transformation $\mathbf{x} \mapsto A\mathbf{x}$. Define $T: \mathbf{R}^2 \to \mathbf{R}^2$ by $T(x) = Ax$, where $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Find a basis $\mathcal{B}$ for $\mathbf{R}^2$ with the property that the $\mathcal{B}$- matrix for T is a diagonal matrix.

20. What is a least squares solution? Find a least squares solution of Ax = b, where $A = \begin{pmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix}$,

$b = \begin{pmatrix} 2 \\ 0 \\ 11 \end{pmatrix}$. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

OR

What do you understand by orthonormal set? Show that $\{v_1, v_2, v_3\}$ is an orthonormal basis of $\mathbf{R}^3$, where

$$v_1 = \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \text{ and } v_3 = \begin{bmatrix} -1/\sqrt{66} \\ -4/\sqrt{66} \\ 7/\sqrt{66} \end{bmatrix}.$$

Prove that an m x n matrix U has orthonormal columns if and only if $U^TU = 1$.