Day 6, Oct-8, 2024

→ Intro to Linear Algebra

Linear Algebra is a branch of mathematics that deals with vector (collection set of number), vector Space (linear Space), linear transformation, matrices, determinants, eigen values, eigen vector.

→ Vector Space (linear Space) → if we add or multiply vectors by scalars (real or Complex numbers) with rules then it is a vector Space

A Linear Transformation → A function that maps vector from one space to another vector space while preserving (vector addition and scalar multiplication). It preserve structure and basic properties.

Example: Image scaling or resizing of image

Matrices

$$\begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix} \simeq \begin{aligned} 2x + 3y &= 0 \\ 5x + 6y &= 0 \end{aligned}$$

Bunch of numbers arranged in a rectangular formation.

Rows: Horizontal Entries

Columns: Vertical entries

$$\begin{array}{cl} & \quad C \\ & \quad \downarrow \\ R \rightarrow & \begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \\ 11 & 12 & 13 \end{bmatrix} \end{array}, \quad \begin{bmatrix} \sqrt{1}, & 3, & 8 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0.17 \\ 5 & 6.72 \end{bmatrix}$$

Vectors:

Column vector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Row Vector $\begin{bmatrix} 5 & 6 & 7 & 8 \end{bmatrix}$

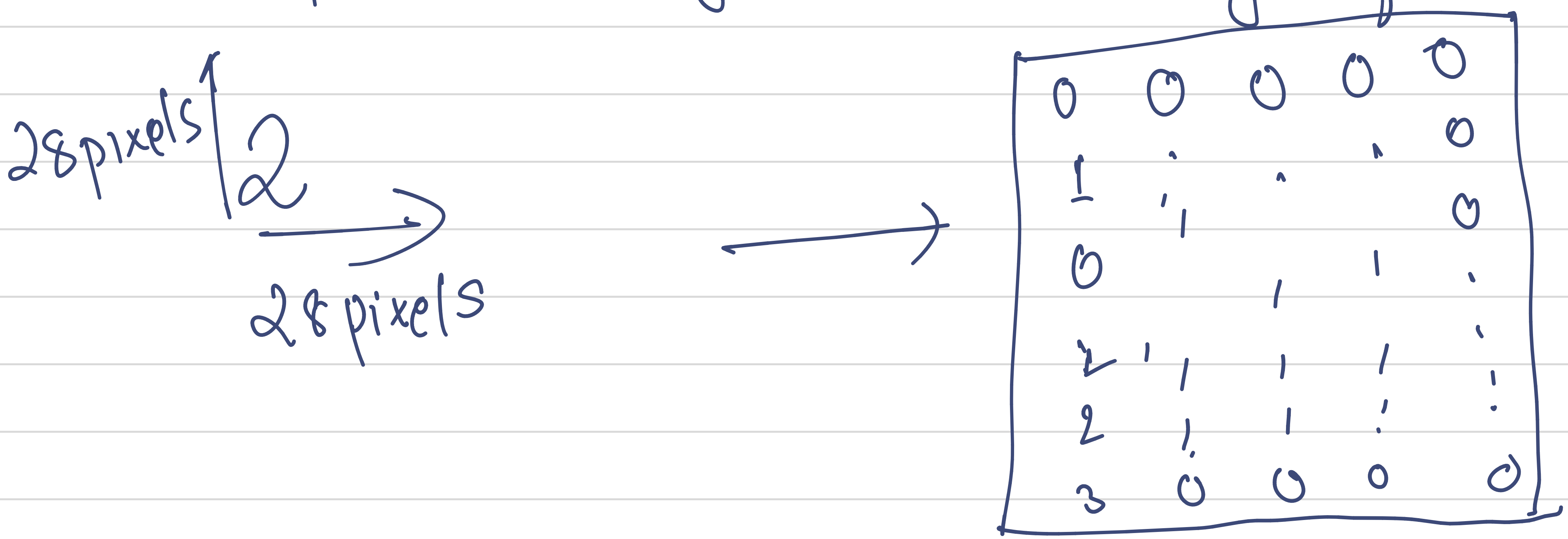
→ Matrices with Single Col or Row are Row|Col Vectors.

→ Matrices and Vectors are abstract mathematical Concept (language

Matrices and Vectors are also used to represent and draw insight in engineering, machine-learning, physics.

Matrices as Image Pixels

→ Represent Images as an array of pixel intensities



Matrices as Linear Transformations

Example of Linear Transformations

$$T(x, y) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

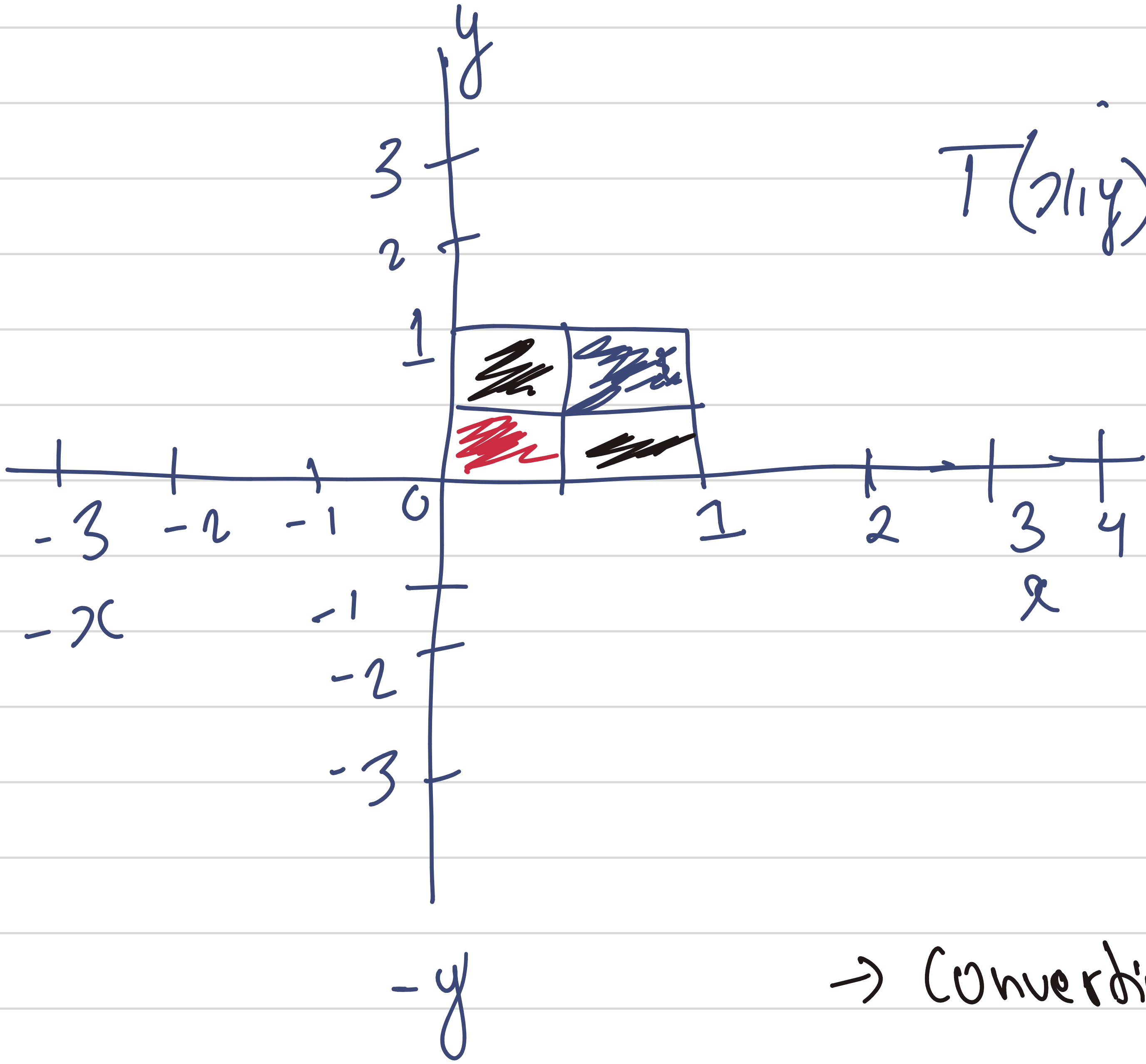
- Increasing each dimension by 2
- Rotate Image by π radians (counterclockwise).

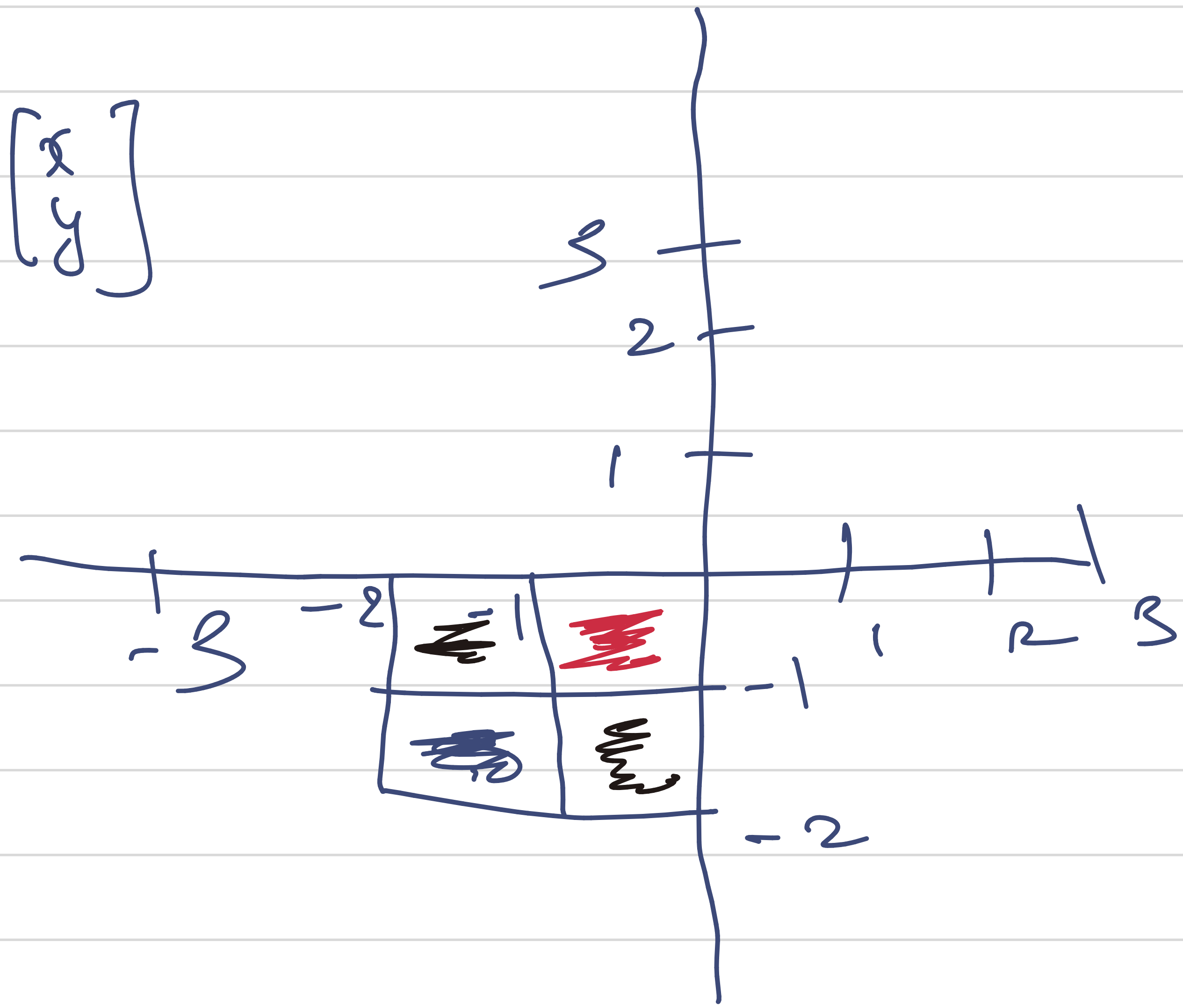
→ Matrices as Linear System of Equations

$$x - y = -1$$
$$3x + y = 9$$

$$\begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \end{bmatrix}$$

$$T(x, y) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$





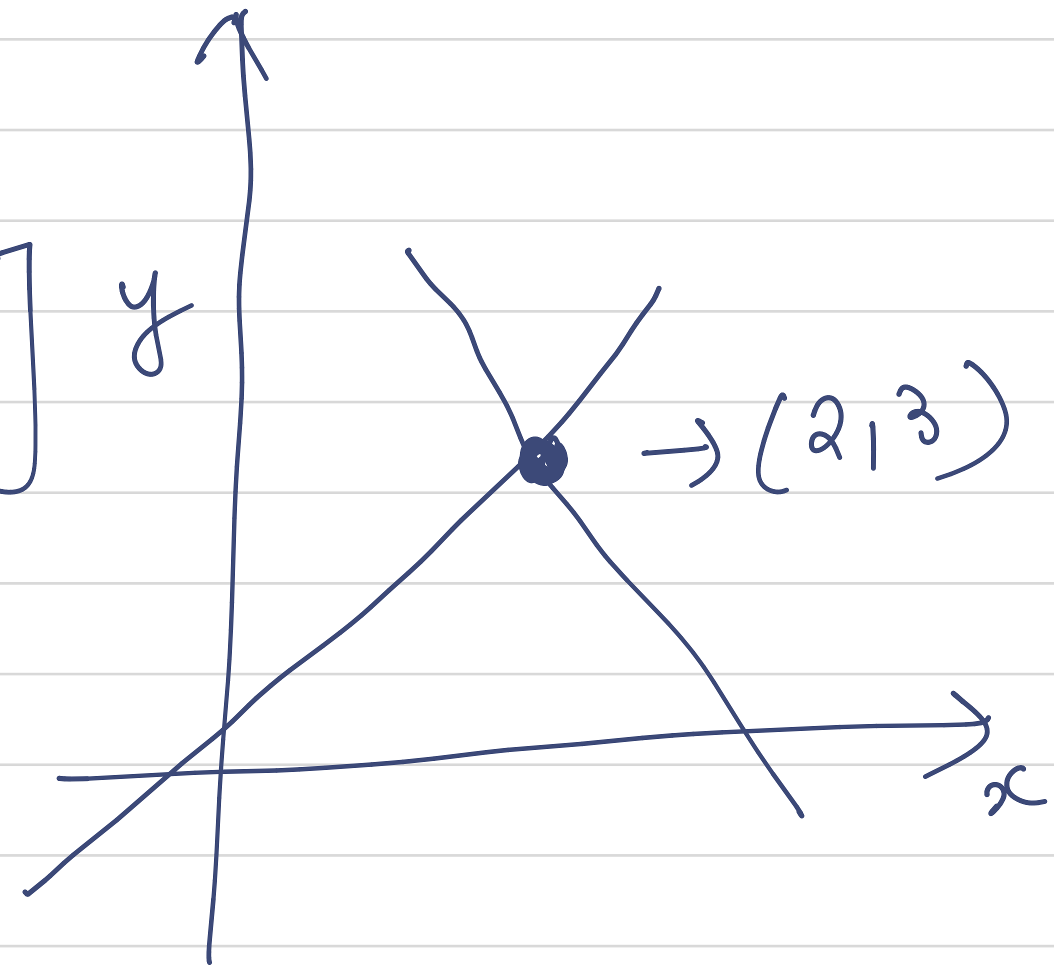
→ Converting Co-ordinates by rotation

→ As a System of equations (Linear System of Equations)

→ Graph

Linear System of Equations :

$$\begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} -1 \\ 9 \end{bmatrix}$$



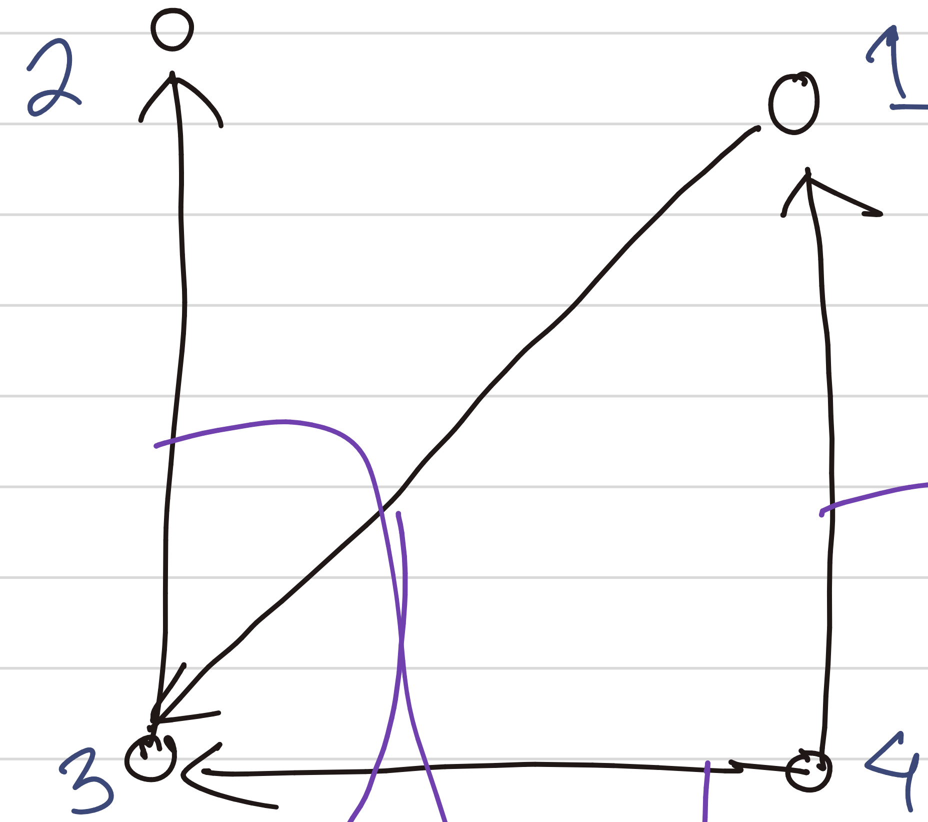
Solve you will get $x = 2$,
$y = (3)$.

$$\begin{bmatrix} 1 & -1 \\ 3 & +1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \times 2 + (-1) \times 3 \\ 3 \times 2 + 3 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 \\ 9 \end{bmatrix}$$

$\rightarrow$ Computationally Efficient

Matrices to Represent Graphs:

→ Represent Graphs and Networks



Connections

1 vertex → 3 vertex

4 vertex → 1 vertex

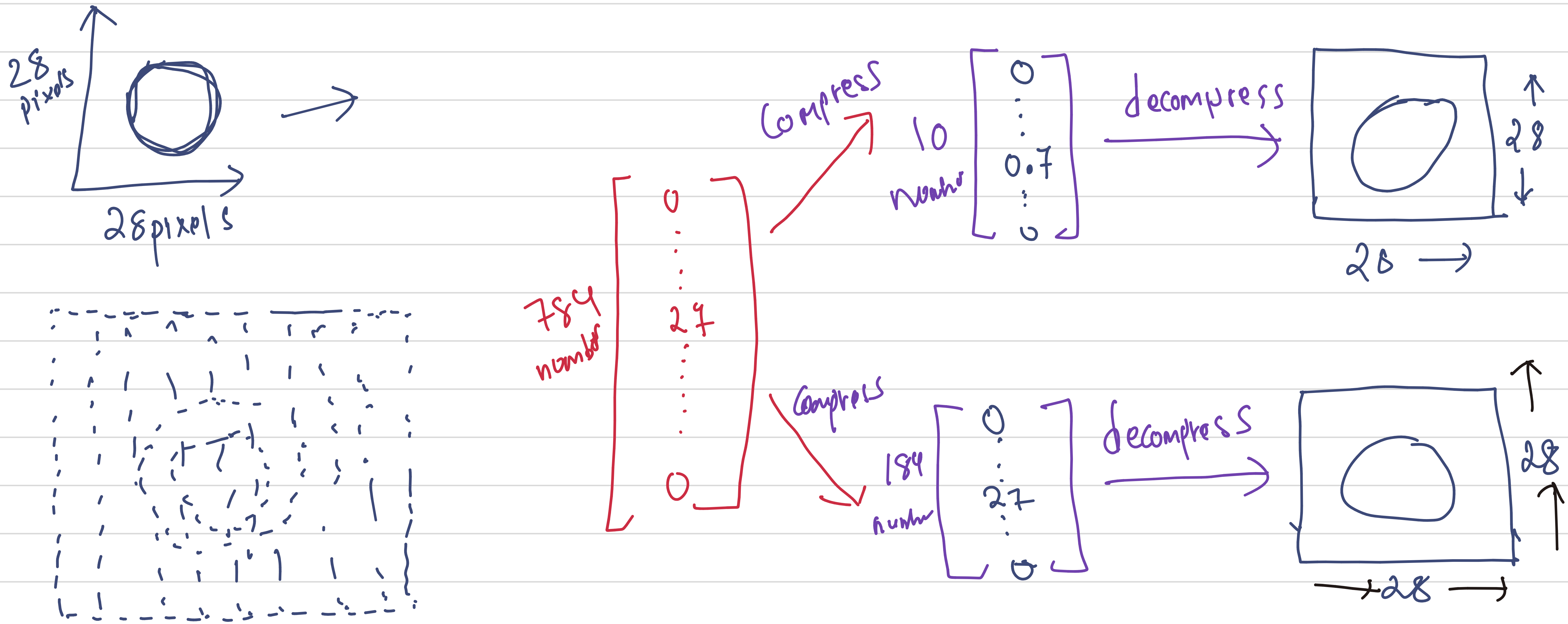
3 " → 2 vertex

4 vertex → 3 vertex

$$
\begin{array}{l}
\text{Top 1} \\
\text{Top 2} \\
\text{Top 3} \\
\text{Top 4}
\end{array}
\begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

All we have done here is represent matrices from the graph vertex.

Matrices for Dimensionality Reduction



Number 0 m x n size Matrix

→ We can use PCA to the Image Matrix,

→ flatten the original Image, Compressing 784 → Decompress 184 number

→ 784 pixels Decomposes → 184 pixes (Single Value Decomposition) PCA

and still we can get meaningful image appearance

Vectors

- an ordered list of numbers
- has a magnitude and direction
- follows Rules for
  - Addition
  - Multiplication

Vector Notations:

denoted by → bar arrow above letter $\vec{a}$

$\vec{u} \in \mathbb{R}^n$ where n-dimensional vector (i.e. n-entries)

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \Rightarrow u$$

Transpose $u^T \Rightarrow \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}^T$

$\Rightarrow [u_1, u_2 \cdots u_n]$

$\vec{u} \neq \{u_1, u_2 \cdots u_n\}$

$[u \neq u^T]$

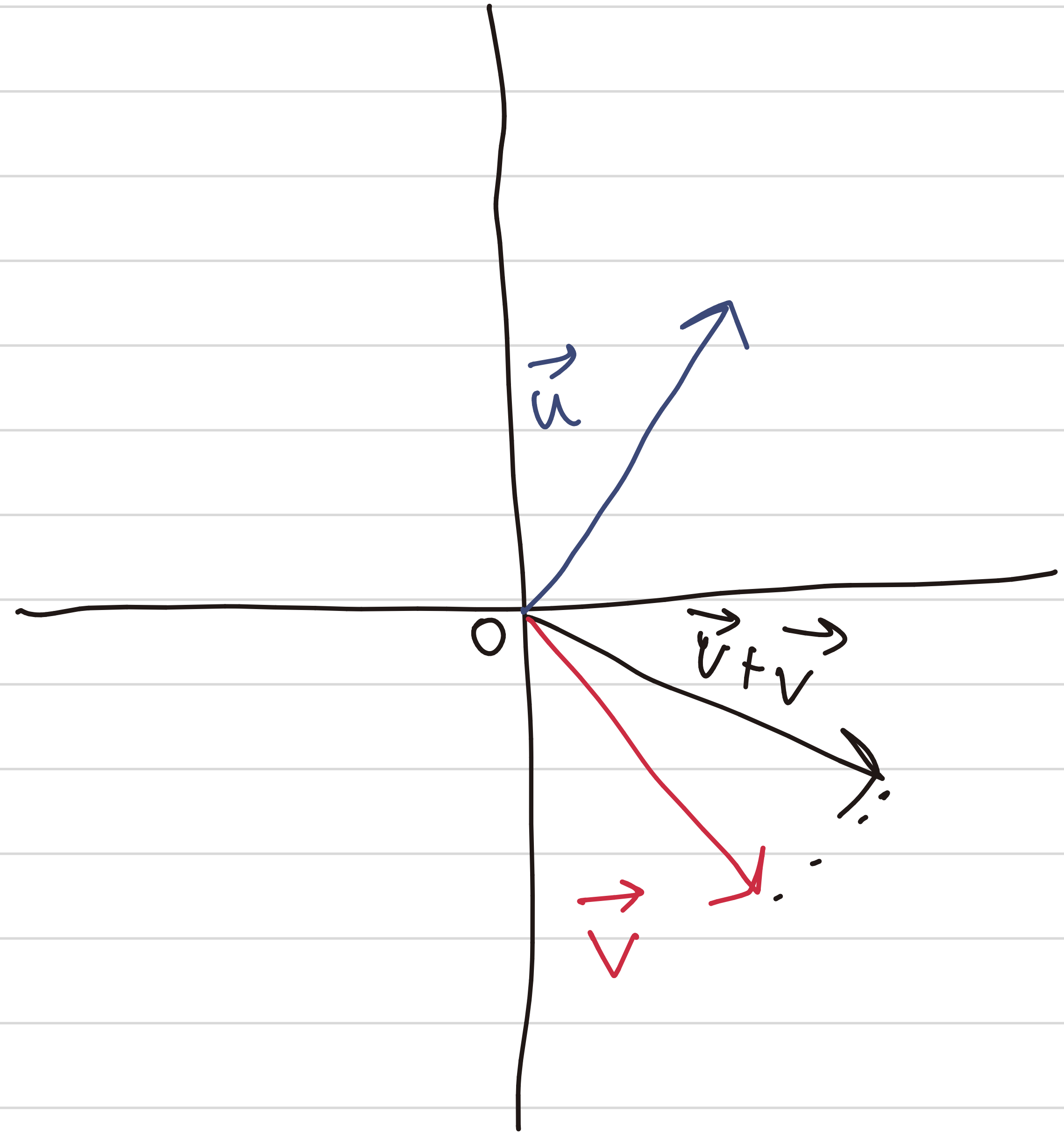
→ n-dimensional means n-different numbers makeup vector

→ Vector ≠ Set

Vector Addition:

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$



* put (−) sign for vector subtraction.

Why is this valid technique?

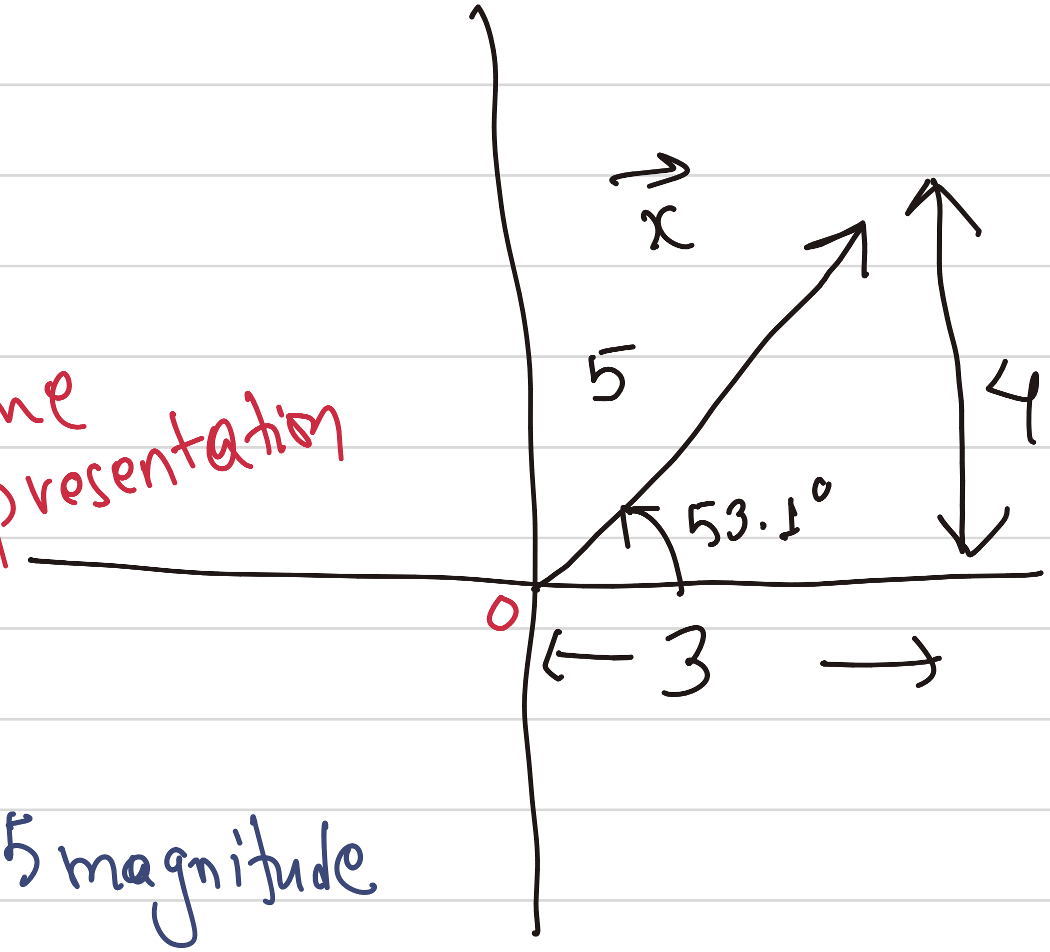
Can we do this with polar Co-ordinate?

Polar Coordinates ??

$$\vec{x} \Rightarrow (3,4) \Rightarrow \begin{bmatrix}3\\4\end{bmatrix}$$

$$\vec{x} = 5\angle 53.1^\circ \neq \begin{bmatrix}5\\53.1\end{bmatrix}$$

$\vec{x}$ No Some Representation $\vec{x}$



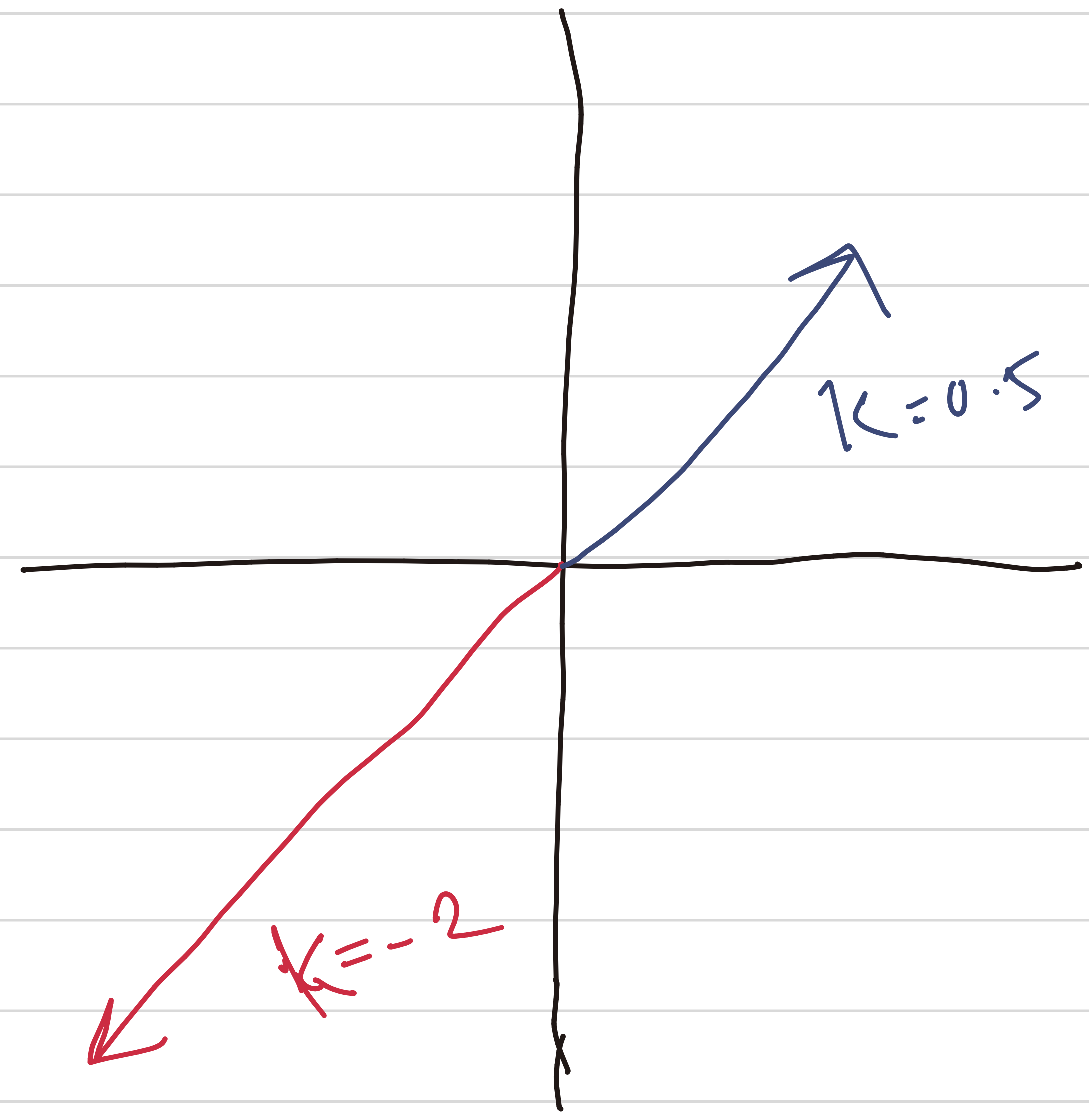
We cannot use polar Co-ordinates of 5 magnitude and $\angle 53.1$, represent lines of points in 2D Space. Polar-Cordinates cannot be used to represent vector.

and polar Co-ordinates itself doesn't follow vector Space and its rules.

polar Co-ordinates → uses distance and angle instead of Cartesian Co-ordinates (x,y) that uses horizontal & vertical distance

Vector Scaling

- Stretching and Squishing a vector
- only change length
- Direction remains the Same

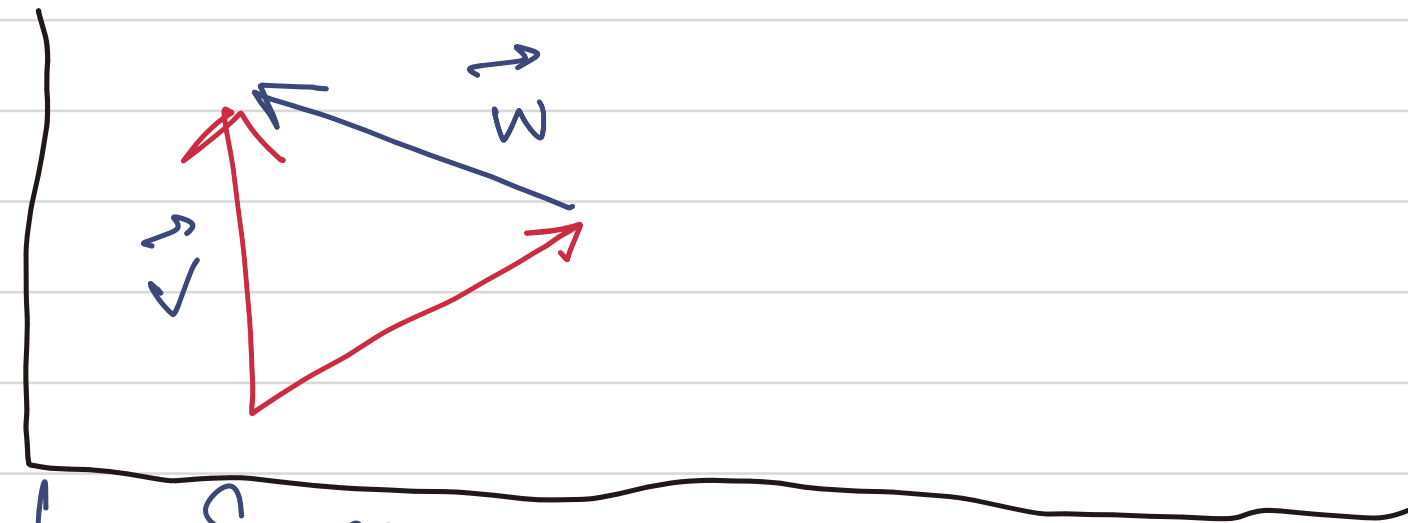


$$\vec{x} \Rightarrow (3,4) \Rightarrow \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$K\vec{x} = K \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 3K \\ 4K \end{bmatrix}$$

- Multiply Vector by Scalar gives vector Scaling Since Scaling word Comes from Scalar

Span of Vectors:



S ⇒ Set of vectors

These set of vectors make up the vector Space.

Span ⇒ all combinations of vectors in S.

The span of $\vec{v}$ and $\vec{w}$ is the set of all their linear Combination.

i.e. $a\vec{v} + b\vec{w}$

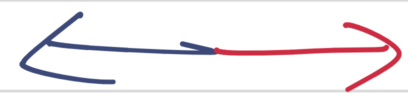
Set of vectors, Span (Combinations of Vectors) ⇒ Span of vectors do represent all different $\vec{v}$ and $\vec{w}$ in a 2D Space.

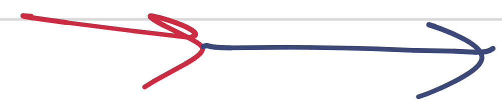
Span:

These sets of vectors Span the 2D Space.

Note: We cannot Span the vectors all over in 2D Space even if we have v and u

these Spans is always straight line despite having two different Span.





These sets of vectors Span the 1D Space

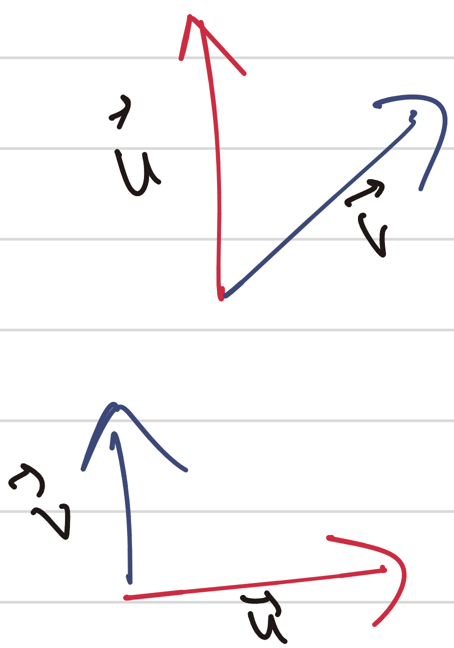
• Even if we have two "different" vectors

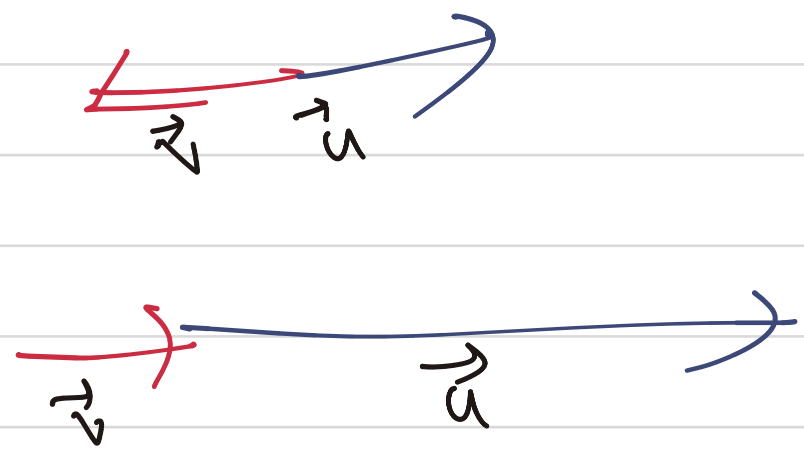
Linearly dependent Vector: $\vec{u} = a\vec{v} + b\vec{w}$

Linearly independent vector: $\vec{u} \neq a\vec{v} + b\vec{w}$

Concept of Dependenty

Linearly Dependent → Linear Combinations of two vectors and contrast concept to Independent.



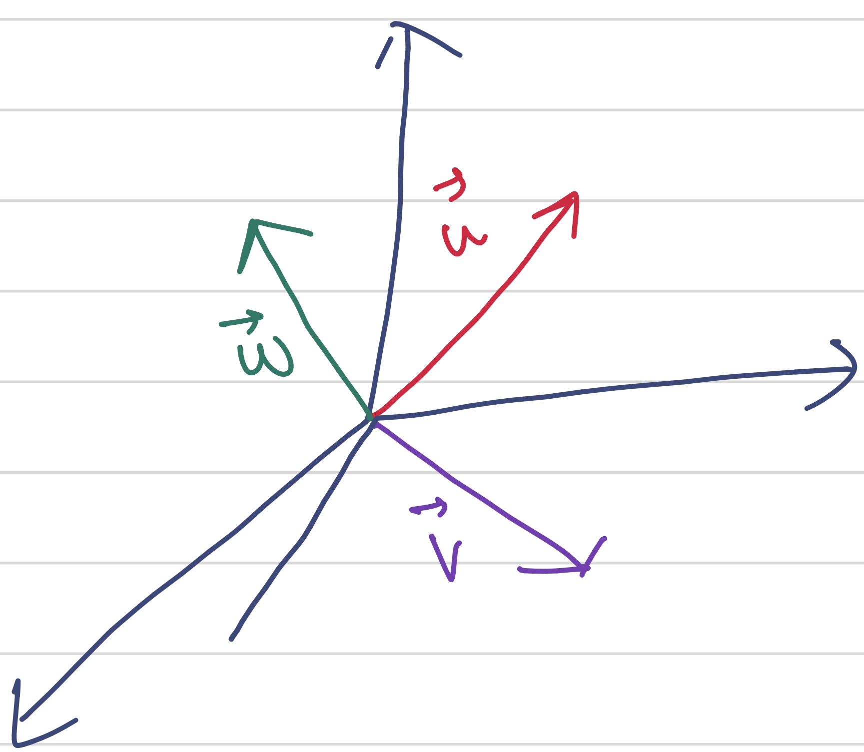


$\vec{u} = a\vec{v} + b\vec{w}$ → Linearly dependent $\vec{v}$ is depending upon $\vec{u}$

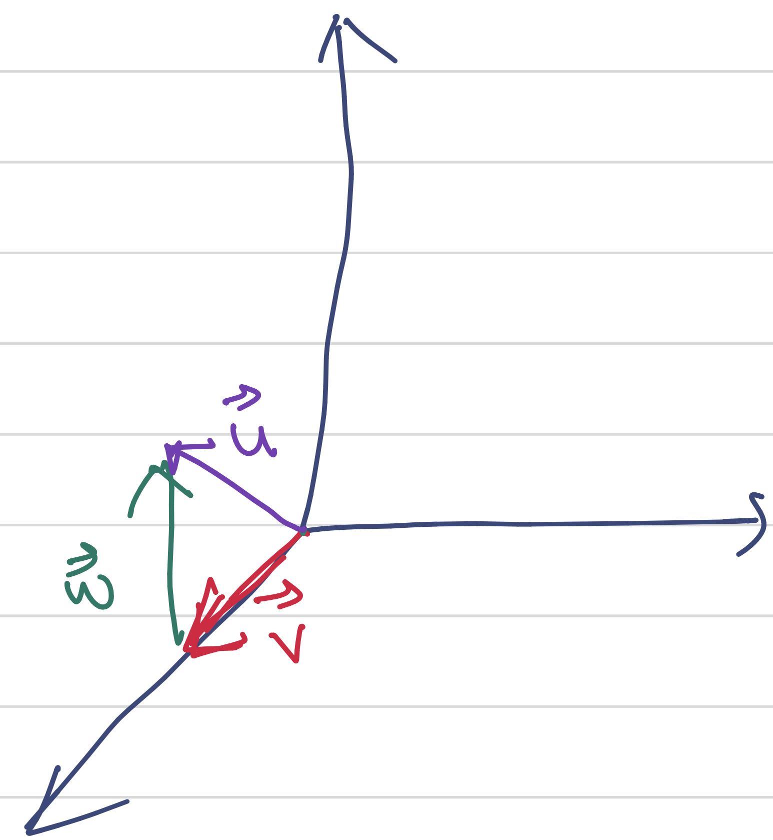
$\vec{u} \neq a\vec{v} + b\vec{w}$ → Linearly Independent

3D Vectors:

→ We may not be be able to always span 3D if we have 3 vectors.



A set of three linearly independent vectors span the 3D space.



A set of two linearly independent vector span a plane in the 3D space.

Basis of a Vector Space:

→ The basis of a vector Space is a set of linearly independent vectors that span the full vector space.

→ A vector space can have several bases

→ All bases have the same number of dimensions [i.e. the dimension of the vector Space]

if we can span full vector space using two vectors then it is the basis of a Vector Space → 2D vector + 2D Basis Vector Spac

Matrices in ML

→ A collection of vectors $\begin{bmatrix} \vdots & \vdots & \vdots \\ \vec{u} & \vec{v} & \vec{w} \\ \vdots & \vdots & \vdots \end{bmatrix}$

→ function that transform Matrices

Vector Products

- Element-by-Element (Hadamard Product)
- Outer Product
- Inner Product

Vector Addition : $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 7 \\ 5 & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 9 \\ 8 & 13 \end{bmatrix}$

Hadamard Product → Element-wise multiplication

→ useful in Convolutional Neural Network

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \odot \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 b_1 \\ a_2 b_2 \end{bmatrix}$$

→ element-wise pixel wise in filtering / Masking at IP

Outer Product:

Given two vectors of size $m \times 1$ and $n \times 1$ respectively

$$u \Rightarrow \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}, \quad v \Rightarrow \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

→ Outer product $u \otimes v$

$$u \otimes v = A = \begin{bmatrix} u_1 v_1 & u_1 v_2 & \cdots & u_1 v_n \\ u_2 v_1 & u_2 v_2 & \cdots & u_2 v_n \\ \vdots & \vdots & & \\ u_m v_1 & u_m v_2 & \cdots & u_m v_m \end{bmatrix}$$

Or in index notation:

$$(u \otimes v)_{ij} \Rightarrow u_i v_j$$

Dot Product → element-wise but with Summation

$$\vec{x} \cdot \vec{y} \Rightarrow [x_1 \; x_2 \cdots x_N] \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} \Rightarrow [x_1 \cdot y_1 + x_2 y_2 + \cdots + x_N y_n]$$

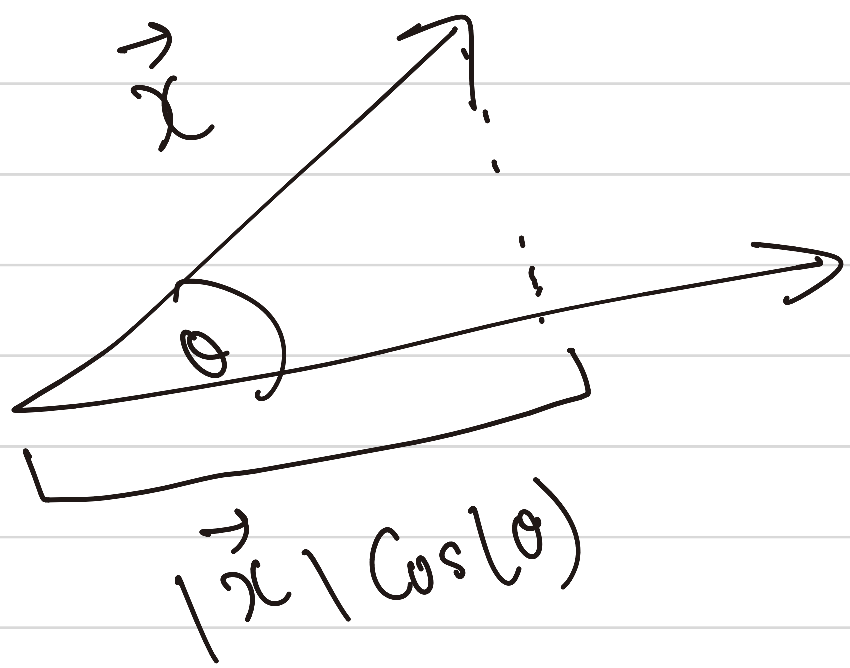
$1 \times N \quad\quad N \times 1 \quad\quad 1 \times 1$

Inner Matrix dimensions must agree

Outer dimensions give size of resulting matrix

Dot Product gives Scalar.

Geometric Interpretation Dot Product



$\vec{x} \cdot \vec{y}$ is maximized when $\vec{x}$ and $\vec{y}$ are Parallel to each other i.e $\theta = 0$

$$\vec{x} \cdot \vec{y} \Rightarrow |\vec{x}|\,|\vec{y}| \cdot \cos(\theta)$$

You can find θ or the angle.

Scalar → have only Magnitude, used to scale vectors or multiply them, Scalars are the elements of (Field) usualy Real ($\mathbb{R}$) or Complex numbers ($\mathbb{C}$).

Eg: 5, -2 could be scalar so,

$$5\begin{bmatrix}2\\3\end{bmatrix} \Rightarrow \begin{bmatrix}10\\3\end{bmatrix}$$ → vector is stretched by factor 5.

Vector :

list of numbers, both magnitude and direction in a space, elements of vector space, can exist in any dimensions, can be added or scaled together by a scalar.

Summary

① Linear Algebra is a branch of mathematics that deals with Matrices, Linear transformations, Determinant, Vectors, Vectors Space, Scalars, very useful in Computer Engineering, Science, Physics.

② Matrices are collection of vectors $\vec{v_1}, \vec{v_2}$ or $\vec{u_1} \cdots \vec{u_n}$, $m \times n$ size row x column.

③ Vectors are the list of numbers (set of ordered numbers) that have both magnitude and direction, can have any dimensional in vector Space.

(4) Matrices can be used to represent System of linear Equations, Graphs, dimensionality Reduction (PCA), as a linear transformations.

(5) Span of vectors is the set of collections of vectors that make up vector space. Linear Combination of set of vectors.

(6) Vectors can Span in vector Space, in 2D or 3D space with 2D or 3D vectors but not always. Linearly Independent vectors don't depend on eachother $\vec{u}$ and $\vec{v}$ whereas linearly dependent vectors depend in vector Space.

7. Linearly Independent → a drone navigating example, the movements are independent allowing for unique paths; whereas in economic-analysis the revenue streams are dependent upon the contributor.

8. Basis of vector is defined as the set of linearly Independent vectors spanning fall in the vector space.

9. Vectors form Matrices, Vectors can be added, subtracted, there are 3 methods for vector product

- Dot product → element wise scaling and summation
- Outer product → $1 \times M$ $1 \times N$ $\begin{bmatrix} u_1v_1 & u_1v_2 & u_1v_3 & u_1v_n \\ u_2v_1 & u_2v_2 & u_2v_3 & u_2v_N \end{bmatrix}$