Homework #8, Math 424, Spring 2024 Prof. Eugene Lerman
typeset March 21, 2024

Please report any typos/issues on Campuswire.

1 Let $S \subset \mathbb{R}$ be a bounded set. Define

$$-S := \{(-1)x \in \mathbb{R} \mid x \in S\}.$$

Prove that $\sup(-S) = -\inf(S)$ and that $\inf(-S) = -\sup S$.

2 Suppose $S \subset \mathbb{R}$ be a bounded set, as in problem **1**. For $c > 0$ define

$$cS := \{cx \in \mathbb{R} \mid x \in S\}.$$

Prove that $\sup(cS) = c \sup S$.

3 Suppose $f : [a, b] \to \mathbb{R}$ is integrable (remember this presupposes that f is bounded). Prove that for any $c, d \in [a, b]$ with $c < d$ the restriction $f|_{[c,d]} : [c, d] \to \mathbb{R}$ is integrable.

4 A function $f : [a, b] \to \mathbb{R}$ is a step function if there is a partition $a = t_0 < t_1 < \cdots t_{n-1} < t_n = b$ of $[a, b]$ so that $f|_{[t_{i-1},t_i]}$ is constant for each i.
Prove that a step function is integrable.

5 Suppose $f : [a, b] \to \mathbb{R}$ is integrable and $g : [a, b] \to \mathbb{R}$ is a function such that $g(x) = f(x)$ except at some points $x_1, \dots, x_n \in [a, b]$. Prove that g is integrable and that

$$\int_{[a,b]} g = \int_{[a,b]} f.$$

Hint: homework 7, problem **6**.

6 Suppose $f : [a, b] \to \mathbb{R}$ is continuous, $f(x) \geq 0$ for all $x \in [a, b]$ and that $\int_{[a,b]} f = 0$. Prove that $f(x) = 0$ for all x.

7 Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function with the property that $\int_{[a,b]} fg = 0$ for *all* continuous functions $g : [a, b] \to \mathbb{R}$. Prove that f is identically 0. (A version of this fact is used in the calculus of variation.)