Homework #3, Math 424, Spring 2024 Prof. Eugene Lerman
typeset February 1, 2024

Read Chapter 3, sections 5 and 6 of Rosenlicht. Problems:

1 Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three sequences of real numbers with $a_n \leq b_n \leq c_n$ for all n. Prove that if $\{a_n\}$, $\{c_n\}$ converge and if they converge to the same number L, then $\{b_n\}$ converges to L as well. Here and in the rest of this homework assignment $\mathbb{R}$ is given the standard metric unless noted otherwise.

2 Suppose $\{x_n\}$, $\{y_n\}$ are two Cauchy sequences of rational numbers (the set $\mathbb{Q}$ of the rationals is given the standard metric). Prove that their sum $\{x_n + y_n\}$ and product $\{x_n y_n\}$ are also Cauchy.

3 Suppose a sequence $\{s_n\}$ of real numbers is bounded: there is $M > 0$ so that $|s_n| < M$ for all n.

(a) Prove that there is a subsequence $\{s_{n_k}\}$ that converges to $\liminf s_n$.

(b) Prove that the limit L of any convergent subsequence of $\{s_n\}$ satisfies

$$\liminf s_n \leq L \leq \limsup s_n.$$

4 Recall that two metrics d, d' on a set E are equivalent if there are $c_1, c_2 > 0$ so that

$$c_1 d(x,y) \leq d'(x,y) \leq c_2 d(x,y)$$

for all $x, y \in E$ (see lecture 8).

(a) Prove that the relation of being equivalent is in fact an equivalence relation on the set of all metrics on the set E.

(b) Prove that if d, d' are two equivalent metrics on a set E then $C \subset E$ is bounded with respect to d if and only if it is bounded with respect to d'.

(c) Prove that if d, d' are two equivalent metrics on a set E then a sequence $\{s_n\}$ in E is Cauchy with respect to d if and only if it's Cauchy with respect to d'.

5 Let (E, d) be a metric space. Recall that the function $\bar{d} : E \times E \to [0, \infty)$ defined by

$$\bar{d}(x,y) = \min\{1, d(x,y)\}.$$

is a metric.

(a) Prove that a sequence $\{s_n\}$ is Cauchy with respect to d if and only if it's Cauchy with respect to $\bar{d}$.

(b) Show that in general the metrics d and $\bar{d}$ are not equivalent.

(c) Consider $\mathbb{R}$ with the standard metric d: $d(x,y) = |x - y|$. Is it true that every sequence $\{s_n\}$ in $\mathbb{R}$ which is bounded with respect to $\bar{d}$ has a convergent subsequence? Is $(\mathbb{R}, \bar{d})$ complete?

6 Let $\{s_n\}$ be a sequence in $\mathbb{R}^n$ which is bounded with respect to the Euclidean metric d_2 (and hence with respect to d_1 and d_∞ by problem **4**). Prove that $\{s_n\}$ has a convergent subsequence.

7 Let $f : X \to Y$ be a function between two sets and $d : Y \times Y \to [0, \infty)$ a metric. Prove that

$$d' : X \times X \to [0, \infty), \qquad d'(x,y) := d(f(x), f(y))$$

is a metric on X if and only if f is injective.