MATH 424 HW7

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1 Q1

Let $S \subset \mathbb{R}$ be a bounded set. Define

$$-S := \{(-1)x \in \mathbb{R} | x \in S\}.$$

Prove that $\sup(-S) = -\inf(S)$ and that $\inf(-S) = -\sup(S)$
Proof:
Since S is bounded, it is bounded above and from below, and thus $\inf S = \alpha, \sup S = \beta$ exists. Also, we have $x \geq \alpha, x \leq \beta, \forall x \in S$. Hence $-x \leq -\alpha, -x \geq -\beta, \forall x$. Then $-\alpha = -\inf S$ becomes an upper bound of $-S$, $-\beta = -\sup S$ becomes a lower bound of S. We will show that $-\inf S$ is the supremum of $-S$ and the prove that $-\sup S$ is the infimum of $-S$ is similar. Assume $a < -\alpha$ is also an upper bound of $-S$. Now take $x = \frac{-\alpha-a}{2} + a$, clearly $x > a$, but also we have $x < -\alpha$. To see this, suppose by contradiction that $x \geq -\alpha$, then $2x = -\alpha - a + 2a = -\alpha + a \geq 2(-\alpha)$, which gives $-\alpha \leq a$, a contradiction. Therefore, we have a point x with $a < x \leq -\alpha$, thus $-\alpha = -\inf S$ is the supremum of $-S$. Similarly, $\inf(-S) = -\sup(S)$.

2 Q2

Let $S \subset \mathbb{R}$ be a bounded set, as in problem 1. For $c > 0$ define

$$cS := \{cx \in \mathbb{R} | x \in S\}.$$

Prove that $\sup(cS) = c \sup S$.
Proof:
Since S is bounded, it is bounded above and from below, and thus $\sup S = \alpha$ exists and $x \leq \alpha, \forall x \in S$, thus $cx \leq c\alpha = c \sup S$. $c\alpha$ is an upper bound of cS. Now we show that it is the supremum of cS. Suppose $a < c\alpha$ is also an upper bound of cS. Similar as Question 1 we can take $x = \frac{c\alpha-a}{2} + a$ and it follow that $x > a$ and $x < c\alpha$. Hence $c\alpha$ is the supremum of cS.

3 Q3

Suppose $f : [a, b] \to \mathbb{R}$ is integrable (remember this presupposes that f is bounded). Prove that for any $c, d \in [a, b]$ with $c < d$ the restriction $f|_{[c,d]} : [c, d] \to \mathbb{R}$ is integrable.

Proof:

Let $P = \{c = t_0 < t_1 < \cdots < t_n = d\}$ be a partition of $[c, d]$. Now since f is integrable, $\exists$ a partition P s.t. $U(f, P) - L(f, P) < \varepsilon, \forall \varepsilon > 0$. Note that also $U(f, P) - L(f, P) < U(f, P') - L(f, P') < \varepsilon$, so by Cauchy's criteria f is integrable.

4 Q4

A function $f : [a, b] \to \mathbb{R}$ is a step function if there is a partition $a = t_0 < t_1 < t_{n-1} < t_n = b$ of $[a, b]$ so that $f|_{[t_{i-1},t_i]}$ is constant for each i. Prove that a step function is integrable.

Proof:

Since f is a step function, for any interval $[t_{i-1}, t_i] \subseteq$ desired partition P, we have $M_i(f) = \sup\{f|_{[t_{i-1},t_i]}\} = \inf\{f|_{[t_{i-1},t_i]}\} = m_i(f)$. Now $U(f, P) - L(f, P) = \sum_i (M_i(f) - m_i(f))(t_i - t_{i-1}) = 0 < \varepsilon$ for all $\varepsilon > 0$. Hence the step function f is integrable by Cauchy's criteria of integrability.

5 Q5

Suppose $f : [a, b] \to \mathbb{R}$ is integrable and $g : [a, b] \to \mathbb{R}$ is a function such that $g(x) = f(x)$ except at some points $x_1, \cdots, x_n \in [a, b]$. Prove that g is integrable and that

$$\int_{[a,b]} g = \int_{[a,b]} f.$$

Hint: homework 7, problem 6.

Proof:

Define $h = f(x) - g(x)$, then h is identically 0 everywhere except for some point $x_1, x_2 \cdots x_n \in [a, b]$, where $f(x) \neq g(x)$. By HW7, Problem 6, $h(x)$ is integrable and $\int_{[a,b]} h(x) = 0$ (while proving integrability, we prove that $U(f) = L(f) = 0$ and thus $\int_{[a,b]} h(x) = 0$). Note that $g(x) = h(x) + f(x)$, the sum of two integrable functions, then g is integrable. It follows that $\int_{[a,b]} g(x) = \int_{[a,b]} f(x) + \int_{[a,b]} h(x) = \int_{[a,b]} f(x)$

6 Q6

Suppose $f : [a, b] \to \mathbb{R}$ is continuous, $f(x) \geq 0$ for all $x \in [a, b]$ and that $\int_{[a,b]} f = 0$. Prove that $f(x) = 0$ for all x.

Proof:

Pick any point $c \in [a,b]$, then $\int_{[a,b]} f = \int_{[a,c]} f + \int_{[c,b]} f = 0$. Since f is continuous on $[a,b]$, by Mean Value Theorem, $\exists d \in [a,c], e \in [c,b]$ s.t. $\int_{[a,c]} f = f(d)(c-a), \int_{[c,b]} f = f(e)(b-c)$. Since $f(x) \geq 0, \forall x \in [a,b]$, $c-a > 0, b-c > 0$, $f(d)(c-a) \geq 0, f(e)(b-c) \geq 0$. Since $f(d)(c-a) + f(e)(b-c) = 0$, we must have $f(d) = f(e) = 0$. Since c is arbitrary, $f(x) = 0$ for all x.

7 Q7

Suppose $f : [a,b] \to \mathbb{R}$ is a continuous function with the property that $\int_{[a,b]} fg = 0$ for all continuous functions $g : [a,b] \to \mathbb{R}$. Prove that f is identically 0. (A version of this fact is used in the calculus of variation.)

Proof:

Assume by contradiction that there exists some point $c \in [a,b]$ s.t. $f(c) > 0$. Since f is continuous, if $|x-c| < \delta$ then $f(x) - f(c) < \varepsilon$ for all $\varepsilon > 0$, i.e., $f(x) > 0$. Define $g(x)$ by $g(x) > 0$ on $[a,b]$ and $g(x) = 0$ other wise. then $\int_{[a,b]} fg = 0 + \int_{[c-\delta,c+\delta]} fg = 0$, which suggests that $\int_{[a,b]} fg = \int_{[c-\delta,c+\delta]} fg > 0$, a contradiction, hence $f(x)$ is identically 0.