Homework #7, Math 424, Spring 2024 Prof. Eugene Lerman
typeset March 7, 2024

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Start reading Chapter 6 of Rosenlicht. Problems:

1 Let $f : (0, \infty) \to \mathbb{R}$ be a function. Suppose $\lim_{x\to 0} f(x)$ exists and equals L. Let $g(u) = f(1/u)$. Prove that $\lim_{u\to+\infty} g(u)$ exists and equals L as well. That is, prove that given $\varepsilon > 0$ there is $M \in \mathbb{R}$ so that $|g(u) - L| < \varepsilon$ for all $u > M$.

2(a) Suppose $f, g : (a, b) \to \mathbb{R}$ are two functions, $c \in (a, b)$, the limits $\lim_{x\to c} f(x)$, $\lim_{x\to c} g(x)$ both exist and equal L. Define $h : (a, b) \to \mathbb{R}$ by

$$h(x) := \begin{cases} f(x) & x \text{ is rational} \\ g(x) & x \text{ is irrational} \end{cases}.$$

Prove that $\lim_{x\to c} h(x)$ exists and equals L.

(b) Assume further that $f(c) = g(c)$, that f, g are differentiable at c and that $f'(c) = g'(c)$. Prove that h is differentiable at c as well.

3 Consider

$$f(x) := \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0. \end{cases}.$$

Prove that f is differentiable at zero. Feel free to use problem **1** and l'Hopital's rule, if needed.

4 Prove directly from the definition of Darboux integral given in lecture 21 that the function

$$f(x) := \begin{cases} 1 & x \text{ is rational} \\ 0 & x \text{ is irrational} \end{cases}.$$

is not integrable on the interval $[0, 1]$.

5 Prove directly from the definition of the Darboux integral (lecture 21) that the function $f : [0, b] \to \mathbb{R}$, $f(x) = x^2$ is integrable.

6 Let $f : [a, b] \to \mathbb{R}$ is a function that is identically 0 everywhere except at the points $x_1, \dots, x_n \in [a, b]$ ($n \geq 1$). Prove directly from the definition that f is integrable on $[a, b]$.

7 Let X be a metric space. Define a relation $\sim$ on X by $x \sim y$ if and only if there is a continuous map (a path) $\gamma : [0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. Prove that $\sim$ is an equivalence relation. The equivalence classes of $\sim$ are called *path components* of X.

Hint: problem **2** from homework #6 may be useful for a proof of transitivity of the relation $\sim$. The fact that $f : [0, 1] \to [0, 1]$, $f(x) = 1 - x$ is continuous may be useful for a proof of symmetry of $\sim$.