Homework #2, Math 424, Spring 2024 Prof. Eugene Lerman
typeset January 24, 2024

Read Chapter 3, sections 1-4 of Rosenlicht. Problems:

1 Let (E, d) be a metric space. Prove that:

(i) Arbitrary intersections of closed sets are closed

(ii) Finite unions of closed sets are closed.

(iii) Closed balls are closed.

2 Let (E, d) be a metric space. Prove that:

(i) For any $x \in E$ the singleton $\{x\}$ is closed.

(ii) Any finite subset of E is closed.

3 Consider a nonempty set E with the "discrete" metric d. That is, $d(x, y) = 1$ for all $x \neq y$. Prove that *all* subsets of E are open and closed.

Hint: Prove that for any $x \in E$ the singleton $\{x\}$ is open.

4 Consider the discrete metric space (E, d) of problem 3. Prove that a sequence $\{s_n\}$ in E converges to $L \in E$ if and only if there is $N \in \mathbb{N}$ so that for $n > N$, $s_n = L$. That is, the only convergent sequences are the sequences that are eventually constant.

5 Let (E, d) be a metric space. Define a function $\bar{d} : E \times E \to [0, \infty)$ by

$$\bar{d}(x, y) = \min\{1, d(x, y)\}.$$

Prove that $\bar{d}$ is a metric.

Hints: The hard part is the triangle inequality. Observe that for all $x, y, z \in E$

$$\bar{d}(x, z) = \min\{1, d(x, z)\} \leq d(x, z) \leq d(x, y) + d(y, z).$$

If $d(x, y), d(y, z) \leq 1$, $d(x, y) = \bar{d}(x, y)$ and $\bar{d}(y, z) = d(y, z)$ so you are done. Otherwise ...

6 Let (E, d) be a metric space and $\bar{d}$ the associated new metric constructed in problem 5 (so that $\bar{d}(x, y) = \min\{1, d(x, y)\}$ for all $x, y \in E$.

(a) Prove that any subset of $(E, \bar{d})$ is bounded.

(b) Prove that d and $\bar{d}$ give rise to exactly the same open set.

Hint: Let $B^d_r(x)$ and $B^{\bar{d}}_r(x)$ denote the open balls with respect to d and $\bar{d}$. Then if $r < 1$

$$B^d_r(x) = B^{\bar{d}}_r(x).$$

7 Let E be a metric space and $\varnothing \neq S \subset E$ a nonempty subset. Recall that we defined the boundary ∂S of S by

$$\partial S = \overline{S} \setminus S^\circ,$$

i.e., closure minus the interior.

Prove that $x \in \partial S$ if and only if for any $r > 0$, the ball $B_r(x)$ contains the points in S and the points in the complement $E \setminus S$.