MATH 424 HW2

Dilys Wu

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1 Q1

Let (E, d) be a metric space. Prove that:
i) Arbitrary intersections of closed sets are closed
ii) Finite unions of closed sets are closed.
iii) Closed balls are closed.
Proof:
i) Note by De Morgan's Law that the complement of intersections of closed sets is a union of open sets, and thus is open by a theory in class. Since its complement is open, the arbitrary intersections of closed sets are closed by definition.
ii) Again, note that by De Morgan's Law the complement of the union of a finite number of closed sets is the intersection of the complements of this finite number of closed sets, i.e., the intersection of a finite number of open sets, which is open.
iii) Let S be the closed ball of center $x \in E$ and radius $r > 0$. Let $z \in S^C$, the complement of S. Then $d(x, z) > r$ and $d(x, z) - r > 0$. Consider the open ball centered at z and radius $d(x, z) - r$. Then for any point y in such a ball, we have $d(y, z) < d(x, z) - r$. Hence $d(x, y) = d(x, y) + d(y, z) - d(y, z) \geq d(x, z) - d(y, z) > r$. Therefore, the open ball of center z and radius $d(x, z) - r$ is entirely contained in S^C, so S^C is open, and S is closed.

2 Q2

Let (E, d) be a metric space. Prove that:
i) For any $x \in E$ the singleton $\{x\}$ is closed.
ii) Any finite subset of E is closed.
Proof:
i) Consider $S := E \setminus \{x\}$. Clearly, for any $y \in S$, $y \neq x$. Consequently, $r := d(x, y) > 0$. We claim that $B_r(y) \subset S$ and we only need to check that $x \notin B_r(y)$. Suppose by contradiction that $x \in B_r(y)$, then $d(y, x) < r = d(x, y)$, a contradiction. Thus $B_r(y) \subset S$, and since $y \in S$ is arbitrary, we have shown that S is open. Thus $\{x\}$ is closed.
ii) From i) we know that for any $x \in E$ the singleton $\{x\}$ is closed, and we can see a finite subset of E as a finite union of singletons, which is closed by $Q1.ii)$

3 Q3

Consider a nonempty set E with the "discrete" metric d. That is, $d(x, y) = 1$ for all $x \neq y$. Prove that *all* subsets of E are open and closed.
Hint: Prove that for any $x \in E$ the singleton $\{x\}$ is open.
Proof:
We show that for any $x \in E$ the singleton $\{x\}$ is open. Note that for any $y \in E$ either $d(x, y) = 0$ or $d(x, y) = 1$. Hence we can always pick $r = 2$ s.t. $B_r(x) = B_2(x) \subseteq E$, so $\{x\}$ is open. Therefore, for any subset U of E, we can consider U to be a union of any collection of singletons $\{x\}$, where $x \in E$. Hence, U is open.

4 Q4

Consider the discrete metric space (E, d) of problem 3. Prove that a sequence $\{s_n\}$ in E converges to $L \in E$ if and only if there is $N \in \mathbb{N}$ so that for $n > N$, $s_n = L$. That is, the only convergent sequences are the sequences that are eventually constant.
Proof:
Recall that a sequence $\{S_n\}$ converges to L if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. for all $n > N$, $d(s_n, L) < \varepsilon$. We first show that a sequence that is eventually constant is convergent after $S_N, N \in \mathbb{N}$. Suppose $s_n = L$ for some $n > N$, then $d(s_n, L) = 0$ by the definition of the metric, and it is clearly contained in a ball with radius $\varepsilon > 0$. Now we show that any sequence that is not eventually constant cannot be convergent. Note that for all s_n, if $L \in E$ is the limit, either $d(s_n, L) = 0$ or $d(s_n, L) = 1$. Since the sequence is not eventually constant, we have $s_n \neq L$ for some $n \in \mathbb{N}$. But then we have $d(s_n, L) = 1$. If we take $\varepsilon = 0.5$, clearly $d(s_n, L) > \varepsilon$ and thus $s_n \notin B_\varepsilon(L)$, contradicting that the sequence is convergent. Therefore, the only convergent sequences are the sequences that are eventually constant.

5 Q5

Let (E, d) be a metric space. Define a function $\bar{d} : E \times E \to [0, \infty)$ by $d(x, y) = min\{1, d(x, y)\}$.
Hints: The hard part is the triangle inequality. Observe that for all $x, y, z \in E$ Prove that $\bar{d}$ is a metric. $\bar{d}d(x, z) = min1, d(x, z) \leq d(x, z) \leq d(x, y) + d(y, z)$.. If $d(x, y), d(y, z) \leq 1, d(x, y) = \bar{d}(x, y)$ and $d(y, z) = \bar{d}(y, z)$ so you are done. Otherwise ...
Proof:
(1) If $\bar{d}(x, y) = min\{1, d(x, y)\} = 0$, then $\bar{d}(x, y) = d(x, y) = 0$. But this happens iff $x = y$.
(2) $\bar{d}(x, y) = min\{1, d(x, y)\} = min\{1, d(y, x)\} = \bar{d}(y, x)$ since d is a metric.
(3) Note that we always have $\bar{d}(x, y) = min\{1, d(x, y)\} \leq d(x, y)$ and $\bar{d}(x, y) = min\{1, d(x, y)\} \leq 1$. Hence for $x, y, z \in E$, $\bar{d}(x, z) = min\{1, d(x, z)\} \leq d(x, z) \leq$

$d(x, y) + d(y, z)$.
a) If $d(x, y), d(y, z) \leq 1, \bar{d}(x, y) = d(x, y)$ and $\bar{d}(y, z) = d(y, z)$, so we are done.
b) If $d(x, y) > 1, d(y, z) > 1$, then $\bar{d}(x, y) = \bar{d}(y, z) = 1$. We have $\bar{d}(x, z) \leq 1 < 1 + 1 = \bar{d}(x, y) + \bar{d}(y, z)$.
c) If $d(x, y) \leq 1, d(y, z) > 1$, then $\bar{d}(x, y) = d(x, y), \bar{d}(y, z) = 1$. $\bar{d}(x, z) \leq 1 \leq 1 + d(x, y) = \bar{d}(y, z) + \bar{d}(x, y)$.
The case when $d(x, y) > 1, d(y, z) \leq 1$ is similar to (c) so we are done for Triangle inequality.

6 Q6

Let (E, d) be a metric space and $\bar{d}$ the associated new metric constructed in problem 5 (so that $\bar{d}(x, y) = min\{1, d(x, y)\}$ for all $x, y \in E$.
(a) Prove that any subset of (E, d) is bounded.
(b) Prove that d and $\bar{d}$ give rise to exactly the same open set.
Hint: Let $B_r^d(x)$ and $B_r^{\bar{d}}(x)$ denote the open balls with respect to d and $\bar{d}$. Then if $r < 1$

$$B_r^d(x) = B_r^{\bar{d}}(x)$$

Proof:
(a) We have $\bar{d}(x, y) \leq 1$ for all $x, y \in E$. With this in mind, let $\varnothing \neq S \subseteq E$ be a subset of E. We choose $r = 2$. then every $x \in S$ is in the open ball $B_2(x)$ since $\bar{d}(x, y) \leq 1$ for all $x, y \in E$, hence S is bounded. Since S is arbitrary, any subset of (E, d) is bounded.
(b) We first follow the hint. Let $B_r^d(x)$ and $B_r^{\bar{d}}(x)$ denote the open balls with respect to d and $\bar{d}$. Then if $r < 1, B_r^d(x) = B_r^{\bar{d}}(x)$. Now consider the case when $r \geq 1$, note that we need not show d and $\bar{d}$ give rise to the same open ball but the same open set. Hence when $r \geq 1$, we only need to choose $0 < r' < 1$ and construct $B_{r'}^{\bar{d}}(x)$ and $B_{r_0}^d(x)$, so they give rise to the same open set.

7 Q7

Let E be a metric space and $\varnothing \neq S \subset E$ a nonempty subset. Recall that we defined the boundary δS of S by $\delta S = \overline{S} \setminus S^\circ$, i.e., closure minus the interior. Prove that $x \in \delta S$ if and only if every ball centered at x contains some point(s) of S and some point(s) in the complement of S.
Proof:
If $x \in \delta S$, then $x \in \overline{S}$ and $x \notin S^\circ$. Then we must have $B_r(x) \nsubseteq S$ and $B_r(x) \nsubseteq S^c$ for all $r > 0$. But this happens iff $B_r(x) \cap S \neq \varnothing$ and $B_r(x) \cap \neq S^c \neq \varnothing$. To see this, suppose by contradiction that $\exists r_0 > 0$ s.t. $B_{r_0}(x) \cap (E \setminus S) = \varnothing$. But this suggests that $B_{r_0}(x) \subseteq S^\circ$, contradicting that $x \in \delta S$. So $B_{r_0}(x) \cap (E \setminus S) \neq \varnothing$. Similarly, we can show that $B_r(x) \cap S \neq \varnothing$. Then any ball centered at δS must contain some points in the interior and the complement of S.