MATH 424 HW6 Dilys Wu

Dilys Wu

March 7, 2024

1 Q1

Suppose X is a topological space and $A, B \subset X$ are two connected subsets with $A \cap B \neq \varnothing$. Prove that their union $A \cup B$ is connected.
Proof:
Suppose $A \cup B$ is not connected, then it can be partitioned into two disjoint, non-empty, relatively open subsets U, V. Let $x \in A \cap B$. Without loss of generality assume $x \in U$ and $x \notin V$. Then $A \subseteq U$ and $B \subseteq U$ cannot hold at the same time, or else $A \cup B \subseteq U, V = \varnothing$. Suppose $A \subset U$, then $A \cap V$ and $A \cap U$ is a partition of A. But since $x \in A \cap U$, $A \cap U \neq \varnothing$, and thus A is the union of two disjoint open sets, contradicting that A is connected. Therefore, $A \cup B$ must be connected.

2 Q2

Suppose (E, d) is a metric space, $f : [0, 1] \to E, g : [0, 1] \to E$ two continuous maps with $f(1) = g(0)$, i.e., f, g are two paths with the second starting where the first ended. Define $h : [0, 2] \to E$ by

$$h(t) = \begin{cases} f(t), & \text{if } t \in [0,1] \\ g(t-1), & \text{if } t \in [1,2] \end{cases}$$

Prove that h is continuous. (The fact that E is a metric space should not matter; the same result holds if E is a topological space.)
Proof:
Since f, g are continuous, any closed sets $C \subseteq E$, the preimages $f^{-1}(C), g^{-1}(C)$ are closed in $[0, 1]$ and $[1, 2]$, respectively. Since $f^{-1}(C), g^{-1}(C)$ are closed, their union $f^{-1}(C) \cup g^{-1}(C) = h^{-1}(C)$ is also closed, and thus suggesting that h is continuous since $[0, 2]$ is closed as well.

3 Q3

We will assume in this homework that the function $g(u) = \sin(u)$ is differentiable. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Prove that f is differentiable at every point of $\mathbb{R}$.

Proof:

We've shown in class that x^2 is differentiable and $(x^2)' = 2x$. When $x \neq 0, f'(x) = \frac{dx^2}{dx}\sin(\frac{1}{x}) + x^2 \frac{d\sin(\frac{1}{x})}{dx} = 2x\sin(\frac{1}{x}) + x^2\frac{d\sin(\frac{1}{x})}{dx}$. Note that $\sin(u)$ is differentiable, and thus $x^2\sin(1/x)$ where $x \neq 0$ is differentiable since it's the composite of two differentiable functions.

When $x = 0, f'(0) = 0$ since $f(0)$ is constant. Thus f is differentiable.

4 Q4

Prove that the function

$$f(x) = \begin{cases} x^2, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$$

is differentiable at 0.

Proof:

When x is rational, $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{x^2-a^2}{x-a} = \lim_{x\to a} \frac{(x+a)(x-a)}{x-a} = \lim_{x\to a} x + a = 2a$. When x is irrational, $f'(x) = 0$ since $f(x)$ is constant. Thus f is differentiable.

5 Q5

Prove that the function

$$f(x) = \begin{cases} x^2 & x \geq 0 \\ 0, & x \leq 0 \end{cases}$$

is differentiable at 0.

Proof:

When $x \geq 0$, $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{x^2-a^2}{x-a} = \lim_{x\to a} \frac{(x+a)(x-a)}{x-a} = \lim_{x\to a} x + a = 2a$. When $x \leq 0$, $f'(x) = 0$ since $f(x)$ is constant. Thus f is differentiable.

6 Q6

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Assume that its derivative f' is also differentiable (i.e., suppose f is twice differentiable). Show that if $f''(x) = 0$ for all $x \in \mathbb{R}$ then

$$f(x) = f(0) + f'(0)x$$

for all $x \in \mathbb{R}$
Proof:
Since $f''(x) = 0, \forall x \in \mathbb{R}, f'(x) = c$, where c is a constant. In particular, $f'(0) = c$. Consider the function $h(x) = f(x) - f'(0)x$ and look at its derivative. Notice that $h'(x) = f'(x) - f'(0) = c - c = 0$, so $h(x)$ must be a constant function. To find the value of $h(x)$, let $x = 0$, then $h(0) = f(0) - 0 = f(0)$, so $h(x) = f(0)$ for all x. Rearranging the expression of $h(x)$ and we get $f(x) = h(x) + f'(0)x = f(0) + f'(0)x$.

7 Q7

Consider the function $f : \mathbb{R} \to \mathbb{R}$ from problem 3:

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Let $g(x) = f(x) + x/2$. Prove that $g'(0)$ is positive but g is not increasing on any open interval containing 0.
Proof:
$g'(0) = f'(0) + \lim_{x\to 0} \frac{x/2+0/2}{x-0} = f'(0) + \lim_{x\to 0} \frac{x/2}{x} = \lim_{x\to 0} f'(0) + 1/2 = \lim_{x\to 0}(2x\sin(\frac{1}{x}) + x^2 \frac{d\sin(\frac{1}{x})}{dx}) + 1/2 = \lim_{x\to 0}(2x\sin(\frac{1}{x}) - \cos(\frac{1}{x})) + 1/2 = 1/2$, so $g'(0)$ is positive. To see that g is not increasing, consider the interval $(-\delta, \delta)$ where $\delta > 0$. We want to show that $\exists x_1, x_2 \in (-\delta, \delta)$ s.t. $x_1 < x_2, g(x_1) > g(x_2)$, which is equivalent to say that $\exists u \in (-\delta, \delta)$ s.t. $g'(u) < 0$. To find these values, notice that $\sin(1/x), \cos(1/x)$ oscillates between ± 1. Now consider x values where $1/x$ corresponds to points where the cosine function is close to its maximum and minimum. Since $x \to 0, 2x\sin(1/x)$ does not affect the sign of $g'(u)$ that much, thus $-\cos(\frac{1}{x})$ will become the dominant term and cause $g'(u)$ to be negative.