MATH 424 HW9 Dilys wu

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1 Q1

Let $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$; it is a real valued function on $\mathbb{R}^2 \setminus \{(0,0)\}$. Consider the limits $\lim_{x\to 0}\lim_{y\to 0} f(x,y)$, $\lim_{y\to 0}\lim_{x\to 0} f(x,y)$ and $\lim_{(x,y)\to(0,0)} f(x,y)$. Compute these limits if they exist.
Proof:
$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0} \frac{x^2}{x^2} = 1$, while $\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{y\to 0} \frac{-y^2}{y^2} = -1$

2 Q2

Find a sequence $\{h_n : \mathbb{R} \to \mathbb{R}\}_{n\in\mathbb{N}}$ of continuous functions so that $\lim_{x\to 0}\lim_{n\to\infty} h_n(x)$ and $\lim_{n\to\infty}\lim_{x\to 0} h_n(x)$ exist and are unequal.
Hint: find a function $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ so that $\lim_{x\to 0}\lim_{y\to 0} f(x,y) \neq \lim_{y\to 0}\lim_{x\to 0} f(x,y)$.
Proof:
We take $f(x,y)$ in Q1 and define $h_n(x) = f(x, 1/n)$. Notice that $\lim_{x\to 0}\lim_{n\to\infty} h_n(x) = \lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0} \frac{x^2}{x^2} = 1$, which is not equal to $\lim_{n\to\infty}\lim_{x\to 0} h_n(x) = \lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{y\to 0} \frac{-y^2}{y^2} = -1$

3 Q3

Find a sequence of continuous functions $f_n : [0,1] \to \mathbb{R}$ that converge to the zero function so that the sequence $a_n = \int_{[0,1]} f_n$ diverges to $+\infty$ as $n \to \infty$, i.e., $\lim_{n\to\infty} a_n = +\infty$.
Hint: p. 139 of the textbook.
Proof:
Consider the sequence of functions defined by:

$$f_n(x) = \begin{cases} 4n^3 x & \text{for } 0 \le x \le \frac{1}{2n} \\ -2n^3 x + 2n^2 & \text{for } \frac{1}{2n} < x \le \frac{1}{n} \\ 0 & \text{for } \frac{1}{n} < x \le 1 \end{cases}$$

Note that $f = \lim_{n\to\infty} f_n = 0$ because $f(0) = 0$ and if $x \neq 0$, then $f_n(x) = 0$ if $n > 1/x$, so f_n converges to the zero function. Notice that the function draws a triangle with vertices at the origin, $(\frac{1}{2n}, n^2)$, and $(\frac{1}{n}, 1)$, and thus the area of the region under it is $\frac{1}{2} \cdot \frac{1}{n} \cdot 2n^2 = n$, so $\lim_{n\to} a_n = \lim_{n\to\infty} \int_{[0,1]} f_n = \lim_{n\to\infty} \int_{[0,1]} n = \infty$

4 Q4

Let m be a positive integer, $\{a_n\}_{n\geq 0}$ a sequence of real numbers. Prove that the series $\sum_{n=0}^{\infty} a_n$ converges if and only if $\sum_{n=0}^{\infty} a_{n+m}$ converges, and that in this case

$$\sum_{n=0}^{\infty} a_n = a_0 + \cdots + a_{m-1} + \sum_{n=0}^{\infty} a_{n+m}$$

Proof:
If $\sum_{n=0}^{\infty} a_n$ converges, then $\exists N$ s.t. for all $n > N$, $S_n = \sum_{k=0}^{n} a_k$ converges to some limit L. Now consider the partial sum of $\sum_{n=0}^{\infty} a_{n+m}$, then $S' = \sum_{k=0}^{n} a_{k+m} = S_{n+m} - \sum_{k=0}^{m-1} a_k$. Since S_n converges to L, then so is S_{n+m} for sufficiently large n, the S'_n converges to $L - \sum_{k=0}^{m-1} a_k$. Since $\sum_{k=0}^{m-1} a_k$ is finite, $\sum_{n=0}^{\infty} a_{n+m}$ converges.
Now suppose that $\sum_{n=0}^{\infty} a_{n+m}$ converges. Then it is just the series $\sum_{n=0}^{\infty} a_n$ without the first $m-1$ terms, and the convergence of the series is not affected by this finite number of terms. Hence $\sum_{n=0}^{\infty} a_n$ also converges.
To see the equality, notice that $S_n = S_{n+m} + \sum_{k=0}^{m-1} a_k$, which is the same as $\sum_{n=0}^{\infty} a_n = a_0 + \cdots + a_{m-1} + \sum_{n=0}^{\infty} a_{n+m}$

5 Q5

Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous, $h : \mathbb{R} \to \mathbb{R}$ differentiable. Consider

$$G(x) := \int_0^{h(x)} f(t)dt.$$

Explain why G is differentiable and find its derivative in terms of f, h and h'.
Proof:
Since f is continuous, f is integrable. Since also that h is differentiable. The continuity of f ensures that the integral of f on $[0, h(x)]$ is well-defined for all x, and the differentiability of h implies that small changes in x lead to predictable changes in $h(x)$, which can be related back to changes in $G(x)$ using the chain rule. Notice that $G' = \frac{d}{dx} \int_0^{h(x)} f(t)dt$. By the Fundamental Theorem of Calculus, version 2 and the chain rule, we have $G'(x) = \frac{d}{dx} f(h(x)) = h(x)' \cdot f(h(x))$.

6 Q6

Suppose $f : [0, \infty) \to (0, \infty)$ is a decreasing function. Prove that $\sum_{n=0}^{\infty} f(n)$ converges if and only if the limit $\lim_{n\to\infty} \int_{[0,n]} f$ exists.

Proof:

Suppose that $\sum_{n=0}^{\infty} f(n)$ converges, then its partial sum $S = \sum_{k=0}^{n} a_k$ converges for some $n > k-1 > N$. Since f is decreasing, it is integrable, i.e., $\int f_{[0,n]}$ exists. we want to show that $\lim_{n\to\infty} \int_{[0,n]} f$ exists. to see this, notice that

$$\sum_{n=1}^{N} f(n) \leq \int_0^N f \leq \sum_{n=0}^{N} f(n)$$

since the range of f is $(0, \infty)$, hence $\int_0^N f$ is bounded, and thus it convergent by the Monotone Convergent Theorem, so $\lim_{n\to\infty} \int_{[0,n]} f$ exists.

Conversely, suppose that $\lim_{n\to\infty} \int_{[0,n]} f$ exists. Then similarly we will have

$$\int_0^N f \leq \sum_{n=0}^{N} f(n) \leq \int_0^{N+1} f$$

Then $f(n)$, the sequence of monotonically decreasing positive numbers is bounded by $\int_0^N f$ and $\int_0^{N+1} f$, so by the Monotone Convergence Theorem, $\sum_{n=0}^{\infty} f(n)$ converges.

7 Q7

Consider the sequence of functions $f_n(x) := \frac{1}{\sqrt{n}} \sin nx$ on the interval $[0, 2\pi]$. Prove that the sequence $\{f_n\}$ converges *uniformly* to the zero functions. Does the sequence of derivatives $\{f_n'\}$ converge? Prove your answer.

Proof:

Consider $|\frac{1}{\sqrt{n}} \sin nx - 0| = |\frac{1}{\sqrt{n}} \sin nx|$. Since $\sin(x) \leq 1, \forall x$, $|\frac{1}{\sqrt{n}} \sin nx| \leq \frac{1}{\sqrt{n}}$. Now given $\varepsilon > 0$, we want to show that there $\exists N$ s.t. $\forall n > N, |\frac{1}{\sqrt{n}} \sin nx| \leq \frac{1}{\sqrt{n}} < \varepsilon$. Then $n > \frac{1}{\varepsilon^2} \implies N > \frac{1}{\varepsilon^2}$, and thus $f_n(x)$ converges uniformly to the zero functions. Now consider the sequence of derivatives $\{f_n'\}$. By the chain rule, we have $f_n'(x) = n \cdot \frac{1}{\sqrt{n}} \cos nx = \sqrt{n} \cos nx$. Since $\cos x \leq 1, \forall x$, $\sqrt{n} \cos nx \leq \sqrt{n}$. since $\lim_{n\to\infty} \sqrt{n} = \infty$, the sequence of derivatives $\{f_n'\}$ does not converge.