Homework #6, Math 424, Spring 2024 Prof. Eugene Lerman
typeset February 29, 2024

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Start reading Chapter 5 of Rosenlicht. Problems:

1 Suppose X is a topological space and $A, B \subset X$ are two connected subsets with $A \cap B \neq \varnothing$. Prove that their union $A \cup B$ is connected.

2 Suppose (E, d) is a metric space, $f : [0,1] \to E$, $g : [0,1] \to E$ two continuous maps with $f(1) = g(0)$, i.e., f, g are two paths with the second starting where the first ended. Define $h : [0,2] \to E$ by

$$h(t) := \begin{cases} f(t) & \text{if } t \in [0,1] \\ g(t-1) & \text{if } t \in [1,2]. \end{cases}$$

Prove that h is continuous. (The fact that E is a *metric* space should not matter; the same result holds if E is a topological space.)

3 We will assume in this homework that the function $g(u) = \sin(u)$ is differentiable. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) := \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Prove that f is differentiable at every point of $\mathbb{R}$.

4 Prove that the function

$$f(x) := \begin{cases} x^2 & x \geq 0 \\ 0 & x \leq 0 \end{cases}$$

is differentiable at 0.

5 Prove that the function

$$f(x) := \begin{cases} x^2 & x \text{ is rational} \\ 0 & x \text{ is irrational} \end{cases}$$

is differentiable at 0.

6 Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Assume that its derivative f' is also differentiable (i.e., suppose f is twice differentiable). Show that if $f''(x) = 0$ for all $x \in \mathbb{R}$ then

$$f(x) = f(0) + f'(0)x$$

for all $x \in \mathbb{R}$.

7 Consider the function $f : \mathbb{R} \to \mathbb{R}$ from problem **3**:

$$f(x) := \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Let $g(x) = f(x) + x/2$. Prove that $g'(0)$ is positive but g is not increasing on any open interval containing 0.