MATH 424 HW4

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1 Q1

Let $\{s_n\}$ be a bounded sequence of real numbers. Prove that lim sup $|s_n| = 0$ if and only if $\{s_n\}$ converges to 0.
Proof:
$(\implies)$ Let $\{s_n\}$ be a bounded sequence of real numbers, then it is bounded above and below. Hence $\limsup s_n$ exists. Suppose $\limsup |s_n| = 0$, since $\liminf |s_n| \leq \limsup |s_n|$ and $|s_n| \geq 0, \forall n$, $\liminf |s_n| = 0$, and thus $\liminf |s_n| = \limsup |s_n| = 0 = \lim |s_n|$. Therefore, $||s_n| - 0| < \varepsilon$ for all $\varepsilon > 0$, so $|s_n| = |s_n - 0| < \varepsilon$, i.e., s_n converges to 0.
$(\impliedby)$ Now suppose that $\{s_n\}$ converges to 0, then $|s_n - 0| < \varepsilon$, $\forall \varepsilon > 0$. Thus $||s_n| - 0| < \varepsilon$, $\forall \varepsilon$ and $\lim |s_n| = 0$. Hence $\limsup |s_n| = \lim |s_n| = 0$.

2 Q2

Suppose $\{z_n\}, \{w_n\}$ are two convergent sequences of complex numbers (the corresponding metric is $d(z, w) = |z - w|$). Prove that if $z_n \to L$ and $w_n \to M$ then $z_n w_n \to LM$. Hint: the fact that convergent sequences are bounded may be useful.
Proof:
Suppose $z_n \to L$ and $w_n \to M$, then they are Cauchy sequences and hence bounded. Let a be greater than the least upper bound of the two sequences. Hence $\exists N', K$ s.t. $\forall n > N'$, $|z_n| \leq a$, and $\forall n > K$, $|w_n| \leq a$. Since $\{z_n\}, \{w_n\}$ are convergent, $\exists N''$ s.t. $n > N''$, $|z_n - L| < \frac{\varepsilon}{2a}$, $|w_n - a| < \frac{\varepsilon}{2a}$. Hence, $|z_n w_n - LM| = |z_n w_n - z_n M + z_n M - LM| \leq |z_n(w_n - M)| + |M(z_n - z_j)| = |z_i||w_i - w_j| + |w_j||z_i - z_j| < |z_i|\frac{\varepsilon}{2a} + |w_j|\frac{\varepsilon}{2a} \leq 2a \cdot \frac{\varepsilon}{2a} = \varepsilon$, so the product converges to LM.

3 Q3

Let $|| \cdot ||$ be a norm on $\mathbb{R}^n$ and $d(v, w) := ||v - w||$ the corresponding metric. Prove that if the sequences $\{v_n\}, \{w_n\}$ in $\mathbb{R}^n$ converge (with respect to d) then their sum $\{v_n + w_n\}$ converges as well and that

$$\lim(v_n + w_n) = \lim v_n + \lim w_n$$

Proof:
Suppose $\{v_n\}, \{w_n\}$ are convergent, then $\exists N, N'$ s.t. $\forall n > N$, $|v_i^{(i)} - L_i| < \varepsilon/2$ for all $\varepsilon > 0, i = 1, 2, \cdots, n$, where L_i is the limit of the sequences formed by the i^{th} coordinate of $\{v_n\}$, and $\forall n > N'$, $|w_i^{(i)} - M_i| < \varepsilon/2\ \forall \varepsilon > 0$, where M_i is the limit of the sequences formed by the i^{th} coordinate of $\{w_n\}$. Now let $K = \max\{N, N'\}$, then $\forall n > K$, $|v_n^{(i)} + w_n^{(i)} - L_i - M_i| \leq |v_n^{(i)} - L_i| + |w_n^{(i)} - M_i| = \varepsilon/2 + \varepsilon/2 = \varepsilon$, which ends our proof.

4 Q4

Denote the set of all bounded sequences in $\mathbb{R}$ by l_∞. Then for any sequence $\{a_n\} \in l_\infty$

$$||a_n||_\infty := \sup\{a_n | n \in \mathbb{N}\}$$

is a well-defined nonnegative real number. Prove that the function $||\cdot||_\infty : l_\infty \to [0, \infty)$ is a norm. That is, prove that
a) $||\{a_n\}||_\infty = 0$ if and only if $\{a_n\}$ is the zero sequence.
b) $||\{ca_n\}||_\infty = |c|\ ||\{a_n\}||_\infty$ for all real numbers c and all sequences $\{a_n\} \in l_\infty$
c) $||a_n + b_n||_\infty \leq ||a_n||_\infty + ||b_n||_\infty$ for all sequences $\{a_n\}, \{b_n\} \in l_\infty$.
Proof:
a) If $||\{a_n\}||_\infty = 0$, then $\sup a_n = 0$, and thus $a_n \leq 0$ for all n. Note that also $||a_n||_\infty \geq 0$ by the definition, so $\inf a_n = 0$. This could only happen when $a_n = 0$, $\forall n$, so n is the zero sequence. Conversely, if $\{a_n\}$ is the zero sequence, then $a_n = 0, \forall n$, thus $||a_n||_\infty = \sup\{a_n | n \in \mathbb{N}\} = 0$
b) $||ca_n||_\infty = \sup\{ca_n\} = |c| \sup\{a_n\} = |c|\ ||\{a_n\}||_\infty$
c) $||a_n + b_n||_\infty = \sup\{a_n + b_n | n \in \mathbb{N}\} \leq \sup\{a_n | n \in \mathbb{N}\} + \sup\{b_n | n \in \mathbb{N}\} = ||a_n||_\infty + ||b_n||_\infty$ since $a_n \leq \sup a_n, b_n \leq \sup b_n, \forall n$, $a_n + b_n \leq \sup a_n + \sup b_n$

5 Q5

Let $\{s_n\}, \{t_n\}$ be two bounded sequences of real numbers.
a) Prove that their sum $\{s_n + t_n\}$ is also bounded and that

$$\limsup(s_n + t_n) \leq \limsup s_n + \limsup t_n$$

b) Give an example of sequences $\{s_n\}, \{t_n\}$ with $\limsup(s_n + t_n) < \limsup s_n + \limsup t_n$.
Proof:
a) Since $\{s_n\}, \{t_n\}$ are bounded, their supremum exists. Let $\alpha = \sup s_n, \beta = \sup t_n$. Then we have $s_n \leq \alpha, t_n \leq \beta, \forall n$. Then $s_n + t_n \leq \alpha + \beta$, and hence is bounded.
b) $s_n = (-1)^n, t_n = (-1)^{n+1}$, then they are both bounded and $\limsup s_n =$

$\limsup t_n = 1$, but $\limsup(s_n + t_n) = \limsup(-1)^n(1+(-1)) = 0 < \limsup s_n + \limsup t_n = 2$

6 Q6

Let $(X, \mathcal{T})$ be a topological space, $K_1, K_2 \subset X$ two compact subsets. Prove that their union $K_1 \cup K_2$ is also compact.
Proof:
Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of $K_1 \cup K_2$. Then $K_1 \subseteq \cup U$ and $K_2 \subseteq \cup U$. Since K_1, K_2 are compact, $\exists \alpha_1, \cdots, \alpha_n$ s.t. $K_1 \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$ and $\exists \beta, \cdots, \beta$ s.t. $K_2 \subseteq U_{\beta_2} \cup \cdots \cup U_{\beta_1}$, then the union of $\cup U_{\alpha_i}$ and $\cup U_{\beta_i}$ is a finite subset of $\{U\}_\alpha$ that covers $K_1 \cup K_2$. Therefore, $K_1 \cup K_2$ is compact.

7 Q7

A topological space $(X, \mathcal{T})$ is Hausdorff if for any two points $x, y \in X$ with $x \neq y$ there are open sets U, V with $x \in U, y \in V$ and $V \cap U = \varnothing$. Prove that if the topology $\mathcal{T}$ comes from/is defined by a metric d then $(X, \mathcal{T})$ is Hausdorff. Hint: open balls are open sets.
Proof:
If $x \neq y$, then $r := d(x, y) > 0$ and the open balls $B_{r/2}(x)$ and $B_{r/2}(y)$ are disjoint. To see this, note that if $z \in B_{r/2}(x)$, then $d(z, y) + d(x, z) \geq d(x, y) = r$. So if $d(x, z) < r/2$ then $d(z, y) > r/2$ and $z \notin B_{r/2}(y)$

8 *Q8

Prove that a compact set in a Hausdorff topological space is closed. Give an example to show that the condition of being Hausdorff is necessary (hint: it was briefly discussed in class, but not in so many words).
Proof:
Let X be any space with the trivial topology which has more than one element. For each $x \in X$ the set $\{x\}$ is compact, but not closed.