MATH 424 Honors Real Analysis

> This is the lecture notes based on Professor Eugene Lerman's MATH 424 Honors Real Analysis course at UIUC.

Reference:
Introduction to Analysis, Rosenlicht
Analysis I and Analysis II, Terence Tao

Chapter 1 Skipped

Chapter 2 The real number system

2.1 Field properties

Def (Field)

A set of numbers that satisfies the following properties is a field F equipped with two operations $\cdot: F \times F \to F, (x, y) \mapsto x + y$ and $+: F \times F \to F, (x, y) \mapsto x \cdot y$:

1. (COMMUTATIVITY) $\forall a, b \in F$, $a + b = b + a$ and $a \cdot b = b \cdot a$.
2. (ASSOCIATIVITY) $\forall a, b, c \in F$, $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
3. (DISTRIBUTIVITY) $\forall a, b, c \in F$, we have $a \cdot (b + c) = a \cdot b + b \cdot c$.
4. (EXISTENCE OF IDENTITY AND ZERO) There are distinct elements 0 and 1 of F s.t. $\forall a \in F$, $a + 0 = a$ and $a \cdot 1 = a$.
5. (EXISTENCE OF INVERSES) $\forall a \in F$, there is an element of F, denoted $-a$, s.t. $a + (-a) = 0$, and for all non zero $a \in F$, there is an element of F, denoted a^{-1} s.t. $a \cdot a^{-1} = 1$.
   And we say $(F,\ +,\ \cdot,\ 0,\ 1\ )$ is a field.

Field properties

#TODO

$\mathbb{R}$ is a special field because it is **ordered** and **complete**, so we introduce the concept of order...

2.2 Order

Def (Ordered field)

An ordered field is a field F with a subset $P \subseteq F$ (positive) s.t.

1. $0 \notin P$.
2. For any $a, b \in P$, $a + b$, $a \cdot b \in P$.
3. For any $a \in F$, $a \neq 0$, then either $a \in P$ or $-a \in P$. i.e, $F/\{0\} = P \cup -P$, where $-P := \{-x : \ x \in P\}$

Remark: $\mathbb{C}$ is not ordered. (see proof)

Def (Inequality)

If (F, P) is an ordered field, then there are relations $<, \leq$ $(>, \geq)$ defined by

$$a < b \iff a - b \in P$$
$$a \leq b \iff a = b \, or \, a < b$$

where $a, b \in F$

Order properties

O1 (Trichotomy) If $a, b \in \mathbb{R}$, then one and only one of the following statements is true:

$$a > b$$
$$a = b$$
$$a < b$$

Proof:
If we apply part iii) of order properties to a, b, then either $a - b \in P$, $a - b = 0$, or $a - b \in -P$, which is exactly what we want.

O2 (Transitivity) If $a > b$ and $b > c$ then $a > c$.
O3 If $a > b$ and $c > d$ then $a + c > b + d$.
O4 If $a > b > 0$
#TODO
Consequently, we have:

1. $a \leq b \iff -b \leq -a$
2. $\forall a, a^2 \geq 0$
3. If $a \in P$, $b \in -P$, then $ab \in -P$
   proof:

Note that $0 \leq a \iff a \in P \cup \{0\}$
i) $a \leq b \iff 0 \leq b - a = -(a-b) \iff -b \leq -a$
ii) If $a = 0$, $a^2 = 0 \geq 0$. If $a > 0,\ a \cdot a = a^2 \in P$.
If $a < 0,\ -a \in P$, which implies that $(-a)^2 = a^2 \in P \implies a^2 > 0$.
iii) Note that $-(ab) = a \cdot (-b) \in P$, so $ab \in -P$.

Corollary: $\mathbb{C}$ is not an ordered field
proof:
Suppose that it is, then $1 = 1 \cdot 1 > 0 \implies -1 < 0$, but $-1 = (\sqrt{-1})^2 > 0$, contradiction.

We can then define the **absolute value**

Def (Absolute value)

For any $a \in \mathbb{R}$, the absolute value $|a|$ is defined by

$$|a| = \begin{cases} a, & a \geq 0 \\ -a, & o.w. \end{cases}$$

Properties of absolute value

We define the function $|\cdot| : \mathbb{R} \to [0, \infty)$, $a \mapsto |a|$ to be the absolute value of a with the following properties:

1. $\forall a$, $|a| \geq 0$
2. $|ab| = |a||b|$
3. $|a^2| = |a|^2$
   Note that ii) directly implies iii) by taking $a = b$.

Proof: trivial

Lemma (Triangle Inequality) $\forall a, b \in \mathbb{R}$, $|a+b| \leq |a| + |b|$
Proof:
For $x \in \mathbb{R}$, we have $x \leq |x|$. Thus for $a, b \in \mathbb{R}$, $a \leq |a|$, $b \leq |b|$. By O3, we have $a + b \leq |a| + |b|$. Similarly, $-a \leq |a|$, $-b \leq |b|$ and $-a - b = -(a+b) \leq |a| + |b|$. Note that for any $x, y \in \mathbb{R}$, $x \leq y$ and $-x \leq y$ implies that $|x| \leq y$, thus $|a+b| \leq |a| + |b|$

Corollary $||a| - |b|| \leq |a - b|$
Proof:
$|a| = |a - b + b| \leq |a - b| + |b| \implies |a| - |b| \leq |a - b|$. Similarly, $|b| - |a| \leq |b - a| = |a - b|$, so $||a| - |b|| \leq |a - b|$

Def (Distance/ metric ...)

There is a function $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ defined by $d(a, b) = |a - b|$ that makes $\mathbb{R}$ a metric space.

2.3 Least upper bound Properties

Def (Bounded from above)

A subset $S \subseteq \mathbb{R}$ is **bounded from above** if $\exists\, a \in \mathbb{R}$ s.t. $s \leq a, \forall s \in S$. Any such a is an **upper bound** of S.

Def (Least Upper Bound/ Supremum)

Suppose $S \subseteq \mathbb{R}$ is bounded from above, then a number $a \in \mathbb{R}$ is a least upper bound if the following two conditions are satisfied:

1. a is an upper bound of S
2. If b is an upper bound of S, then $a \leq b$.
   We denote $a = sup(S) = lub(S)$. If S is not bounded from above, then we say $sup(S) = \infty$.

Completeness Axiom of LUB

Any nonempty subset $S \subseteq \mathbb{R}$ bounded from above has a l.u.b.

Similarly, we can define a lower bound and greatest lower bound/ infimum for S, where
$\inf(S) = -\sup(-S)$, and $-S = \{-s \mid s \in S\}$.
So does the completeness axiom work for lower bound.

Lemma 2.3.1 $\forall x \in \mathbb{R}, \exists\, n \in \mathbb{R}$ s.t. $x < n$.
Proof:
Suppose not, then $\exists\, x_0 \in \mathbb{R}$ s.t. $n \leq x_0, \forall n \in \mathbb{R}$. By Completeness, $a = \sup(\mathbb{R})$ exists and is not equal to ∞. Then $\forall n, n + 1 \leq a$. But then $\forall n, n \leq a - 1$, contradicting that a is a l.u.b.

Corollary 2.3.2 $\forall x \in \mathbb{R}, \exists\, n \in \mathbb{R}$ s.t. $\frac{1}{n} < \varepsilon$.
Proof:
By 2.1, $\exists\, n \in \mathbb{N}$ s.t. $\frac{1}{\varepsilon} < n$.

Corollary 2.3.3 $\forall x \in \mathbb{R}, \exists n \in \mathbb{Z}$ s.t. $n \leq x \leq n+1$.
Proof:
#TODO

Consequence $\forall a \in \mathbb{R}, a \geq 0$, if for all $\delta > 0, a < \delta$, then $a = 0$.
Proof:
If $a > 0, \exists\, n \in \mathbb{N}$ s.t. $\frac{1}{n} < a$, contradiction.

Theorem 2.3.4 ($\mathbb{Q}$ is dense in $\mathbb{R}$) $\forall x \in \mathbb{R}, \forall \varepsilon > 0, \exists\, r \in \mathbb{Q}$ s.t. $|x - r| < \varepsilon$.
Proof:
We fix $x, \varepsilon > 0$. By 2.2, $\exists\, N \in \mathbb{N}$ s.t. $\frac{1}{N} < \varepsilon$. By 2.3, $\exists\, n \in \mathbb{Z}$ s.t. $n < Nx \leq n+1$. So $\frac{n}{N} \leq x < \frac{n+1}{N}$, so $0 \leq x < \frac{1}{N} \leq \varepsilon$ by choice of N. Thus $|x - \frac{n}{N}| < \varepsilon$, where $\frac{n}{N}$ is the desired rational r.

Chapter 3 Metric spaces

3.1 Metric space

Def (Metric Space)

A metric space is a set E together with a function $d:\ E \times E \to [0, \infty)$ s.t. $\forall x, y \in E$

1. $d(x,y) = 0 \iff x = y$
2. $d(x,y) = d(y,x)$
3. $d(x,z) \leq d(x,y) + d(y,z)$ (Triangle Inequality)

The metric space is a pair (E, d), where d is called the metric.
ex.

1. $(E = \mathbb{R}, d = |x-y|)$
2. $(E : any\ set, d = \begin{cases} 1, x \neq y \\ 0, x = y \end{cases})$
3. $(E = \mathbb{C}, d((x+iy),(u+iy)) = |(x+iy)-(u+iv)| = \sqrt{(x-u)^2-(y-v)^2}$, and we define $d_2(x,y) = \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}$
4. $(E = \mathbb{R}, d_1(x,y) = \sum |x_i - y_i|)$, d_1 is also called the Taxi/ Manhattan metric. Also, we define $d_\infty(x,y) = max\{|x_1-y_1|, \cdots, |x_n-y_n|\}$
Remark: If (E,d) is a metric space, $\tilde{E} \subseteq E$, then $(\tilde{E}, \tilde{d} = d|_{\tilde{E}\times\tilde{E}})$ is also a metric space.

Def (l^2 norm)

The l^2 norm on $\mathbb{R}^n$ is the function $\|\cdot\|_2 : \mathbb{R}^n \to [0, \infty)$ $\|x\|_2 = \sqrt{\sum x_i^2}$.
Note: the l^2 norm is actually the distance from the point to the origin. Also, recall the definition of d2 and we see $d_2(x, y) = \|x - y\|_2$

Cauchy-Schwarz Inequality $\forall x, y \in \mathbb{R}$, $|\sum x_i y_i| \leq \|x\|_2 \cdot \|y\|_2$
Proof:
If either $x = 0$ or $y = 0$, then we have $0 \leq 0$. Now suppose $x \neq 0, y \neq 0$. Then $\forall \alpha \in \mathbb{R}$, we have $x - \alpha y = (x_1 - \alpha y_1, \cdots, x_n - \alpha y_n)$. By definition of norm there is $0 \leq \|x - \alpha y\|^2 = \sum(x_i - \alpha y_i)^2 = \sum(x_i^2 - 2\alpha x_i y_i + \alpha^2 y_i^2) = \|x\|^2 - 2\alpha \sum x_i y_i + \alpha^2 \|y\|^2$. Now we take $\alpha = \pm \frac{\|x\|}{\|y\|}$, then the inequality becomes
$0 \leq \|x\|^2 \pm 2\frac{\|x\|}{\|y\|} \sum x_i y_i \pm (\frac{\|x\|}{\|y\|})^2 \|y\|^2 = 2\|x\|^2 \mp 2\frac{\|x\|}{\|y\|} \sum x_i y_i$
And we have $\pm \sum x_i y_i \leq \|x\|\|y\|$, and thus $|\sum x_i y_i| \leq \|x\|_2 \cdot \|y\|_2$.

d_2: the Euclidean metric
Theorem 3.1 (Triangle Inequality for d_2) $\forall x, y \in \mathbb{R}^n$, $\|x + y\|_2 \leq \|x\|_2 + \|y\|_2$. Consequently, $\forall x, y, z \in \mathbb{R}^n$, $d_2(x, z) \leq d_2(x, y) + d_2(y, z)$
Proof:

1. $\|x + y\|^2 = \sum(x_i + y_i)^2 = \sum(x_i^2 + 2x_i y_i + y_i^2) \leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + |y\|)^2$ (the inequality part comes from Cauchy-Schwarz).
2. $d(x, z) = \|x - z\|_2 = \|(x - y) + (y - z)\| \leq \|x - y\| + \|y - z\| = d(x, y) + d(y, z)$.

Corollary 3.1.2 $(\mathbb{R}^n, d_2)$ is a metric space.
Proof:
By Theorem 3.1 we've proven the triangle inequality for d_2. Now if $d_2(x, y) = 0$, then $\sqrt{\sum(x_i - y_i)^2)} = 0$, which implied that $x_i - y_i = 0, \forall i$, thus $x = y$.
$d_2(x, y) = \sqrt{\sum(x_i - y_i)^2)} = \sqrt{\sum(y_i - x_i)^2)} = d_2(y, x)$, which ends our proof.

3.2 Open and closed sets

Now we continue our discussion about metric space by introducing a new concept, the open/closed ball.

Def (Open ball and closed ball)

Let (E, d) be a metric space, an **open ball** centered at $x \in E$ of radius $r > 0$ is the set $B_r(x) \equiv B(x, r) := \{y \in E | d(x, y) < r\}$.
Similarly, a **closed ball** centered at $x \in E$ of radius $r > 0$ is the set $\overline{B_r(x)} \equiv \overline{B(x, r)} := \{y \in E | d(x, y) \leq r\}$.

ex

1. $E = \mathbb{R}^2, d = d_2$.
2. $E = \mathbb{R}^2, d = d_\infty = \max\{|x_1 - y_1|, |x_2 - y_2|\}$, $B_r(0) = \{y \in \mathbb{R}^2 | y_1| < r, |y_2| < r\}$, and this is a square of side length r.
3. $E = [0, 2]$, $d(x, y) = |x - y|$, $B_1(2) = (1, 2]$
4. E: a set, $d(x, y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$. $B_2(x) = E$, $B_1(x) = \{x\}$

Def (Open set)

A subset U of a metric space (E, d) is open if $\forall x \in U, \exists r > 0$ s.t. $B_r(x) \subseteq U$.

Ex. $E = \mathbb{R}$, $U = (a, b)$, $d = \min\{|a - x|, |b - x|\}$

Def (Closed set)

A subset C of a metric space (E, d) is closed if the complement of C, $C^c = E \setminus C := \{x \in E | x \notin C\}$ is open.

Ex. $E = \mathbb{R}$, $C = [0, 1]$.
Remark: $[0, 1) \in \mathbb{R}$ is neither closed nor open.

$E, \varnothing$ are both open and closed

Theorem 3.2.1

Let (E, d) be a metric space, then

1. $\forall$ collection $\{U_i\}_{i \in I}$ of open sets in E, $\cup_{i \in I} U_i$ is open.
2. $\forall$ k, $\forall$ open sets $U_1, \ldots, U_k$, $U_1 \cup \ldots \cup U_k$ is open, i.e., **finite** union of open sets are open.
3. Open balls are open.

Proof:
1) Suppose $x \in \cup_{i \in I} U_i$, then $x \in U_{i0}$ for some $i_0 \in I$. Since U_i is open, $\exists\, r > 0$ s.t. $B_r(x) \subseteq U_{i0} \subseteq \cup U_i$.
2) Suppose $U_1, \ldots, U_k$ are open and $x \in \cap_{i=1}^k U_i$. Then $\forall i$, $\exists r_i > 0$ s.t. $B_{ri}(x) \subseteq U_i$. Let $r = \min\{r_1, \ldots, r_k\}$, then $B_r(x) \subseteq \cap_{i=1}^k U_i$.
3) Given $y \in B_r(x)$, $d(x, y) = r$. We have $\delta = r - d(x, y) > 0$. Then $\forall z \in B_\delta(x)$, $d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + \delta = d(x, y) + (r - d(x, y)) = r$. So $z \in B_r(x)$ and

$B_\delta(x) \subseteq B_r(x)$, $B_r(x)$ is open. (see Figure 3.2.1 below).

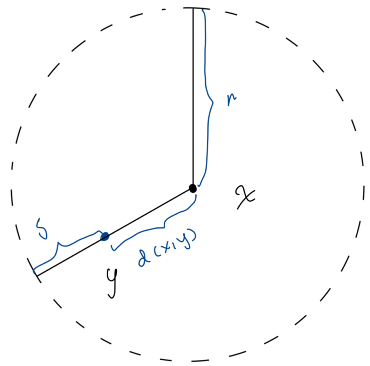

Figure 3.2.1

Remark: we need the number of intersections in (2) to be finite. If not, the infimum can be 0. Consider $E = \mathbb{R}$, $d(x,y) = |y-x|, U = (-\frac{1}{n}, \frac{1}{n})$. Then $\cup U_n = \{0\}$. But under the standard metric, any open ball centered at 0 *must* contain some number greater than and less than 0, which is a contradiction.

Remark. An **open rectangle** in $\mathbb{R}^n$ is a set $U = (a_1 \times b_1) \times (a_2 \times b_2) \times \cdots \times (a_n \times b_n), a_i < b_i$, $i = 1, 2, \cdots, n$.

It is open. $\forall x = (x_1, \cdots, x_n) \in U$, let $r = \frac{1}{2}\min\{|a_i - x_i|, |b_i - x_i|\}$, then $B_r(x) \subseteq U$.

Similarly, $F = [a_1 \times b_1] \times [a_2 \times b_2] \times \cdots \times [a_n \times b_n]$ is closed.

Def (bounded)

A subset $\varnothing \neq S \subseteq E$ of a metric space (E, d) is bounded if $\exists x \in E, r > 0$ s.t. $S \subseteq B_r(x)$.

Ex. $[a, b)$ is bounded, $x = 0, r = \max\{|a|, |b|\} + 1$, while $[0, \infty)$ is not.

Ex. $E = \{a\} \neq \varnothing$ s.t. $d(x,y) = \begin{cases} 1, x \neq y \\ 0, x = y \end{cases}$. Then for all $U \subseteq E$, for any $x \in E$, we have $U \subseteq B_2(x)$.

Remark. $\{d(x,s) | s \in S\}$ is bounded above by r.

Remark. Any subset of a bounded metric space E is bounded.

Theorem 3.2.2

Suppose $\varnothing \neq S \subseteq \mathbb{R}$ is closed and bounded. Then $\inf S$, $\sup S$ exist and are in S

Proof:

We show that $\sup S$ exists and $\sup S \in S$. Since S is bounded, S is bounded from above. Since also $S \neq \varnothing$, $L = \sup S$ exists. If $L \notin S$, then $L \in \mathbb{R} \setminus S$, which is open. But then $\exists r > 0$ s.t. $B_r(L) = (L - r, L + r) \subseteq \mathbb{R} \setminus S$, so we have $(L, \infty) \subseteq \mathbb{R} \setminus S$ and $(L - r, \infty) \subseteq \mathbb{R} \setminus S$. Hence, for all $s \in S$, $s \leq L - r < L - \frac{r}{2}$, which means that $L - \frac{r}{2}$ is also a lower bound, contradicting that L is the

supremum.
For inf$S$, argue for $-S$.

3.3 Convergent sequences

Def (Sequence)

A sequence in a set E is a function $S : \mathbb{N} \to E$.
Notation: $S = \{S_i\}_{i=1}^{\infty} = \{s_1, \ldots, s_n, \ldots\}$ or just (s_n).

Def (Convergent)

Let (E, d) be metric space and (S_n) a sequence. We say S_n converges to $L \in E$ if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. for all $n > N$, $d(s_n, L) < \varepsilon \equiv s_n \in B_\varepsilon(L)$.
We write $S_n \to L$ or $\lim_{n\to\infty} S_n = L$ and we say S_n is convergent, L is a limit of S_n.

Ex. $E = \mathbb{R}$, $S_n = \frac{1}{n}$, $S_n \to 0$. *proof:* Given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\frac{1}{\varepsilon} < N$. Then for all $n > N$, we have $\frac{1}{n} < \frac{1}{N} < \varepsilon$.

Lemma 3.3.1 A sequence $\{S_n\}$ in (E, d) converges to $L \iff \forall$ open set $U \subseteq E$ with $L \in U$, $\exists N \in \mathbb{N}$ s.t. for all $n > N$, $s_n \in U$.
Proof:
($\impliedby$) $\forall \varepsilon, B_\varepsilon(L)$ is open, so $\exists N \in \mathbb{N}$ s.t. $\forall n > N$, $s_n \in B_\varepsilon(L)$, i.e., $d(s_n, L) < \varepsilon$.
($\implies$) Suppose $S_n \to L$ and U is an open set with $L \in U$. Since U is open, $\exists\, r > 0$ s.t. $B_r(L) \subseteq U$. Since $S_n \to L$, $\exists N = N(r)$ s.t. for all $n > N$, $s_n \in B_r(L) \subseteq U$.

Lemma 3.3.2 Convergent sequences are bounded.
Proof:
Suppose $S_n \to L$ in a metric space (E, d). Since $S_n \to L$, $\exists\, N \in \mathbb{N}$ s.t. $\forall n > N$, $s_n \in B_r(L)$. Let $r = \max\{d(s_1, L), \ldots, d(s_N, L), 1\} + 1$. Then $s_n \in B_r(L)$ for all n.

Proposition Let S_n be a sequence in the metric space (E, d). If $S_n \to L_1$ and $S_n \to L_2$, then $L_1 = L_2$.
Proof:
We have N, N' s.t. $\forall n > N, d(s_n, L_1) < \varepsilon/2, d(s_n, L_2) < \varepsilon/2$. Thus for $n > \max\{N, N'\}$, $d(L_1, L_2) \le d(L_1, s_n) + d(L_2, s_n) = \varepsilon/2 \cdot 2 = \varepsilon$.

Proposition Suppose $S_n \to L$, S_{nk} is a subsequence of S_n. Then $S_{nk} \to L$.
Proof:
Let s_{n_k} denote a subsequence of sn. Note that $n_k \ge k$ for all k. This easy to prove by induction: in fact, $n_1 \ge 1$ and $n_k \ge k$ implies $n_{k+1} > n_k \ge k$ and hence $n_{k+1} \ge k + 1$. Let $\lim s_n = L$ and let $\epsilon > 0$. Then

there exists N so that $n > N$ implies $|s_n - L| < \epsilon$. Now $k > N \implies n_k > N \implies |s_{n_k} - L| < \epsilon$. Therefore: $\lim_{k \to \infty} s_{n_k} = L$.

Lemma 3.3.3 Suppose C is closed in (E, d), $\{S_n\}$ a sequence in C ($\forall n, s_n \in C$) and $s_n \to L$. Then $L \in C$.
Conversely, if $\forall$ convergent $\{s_n\}$ in C, $\lim s_n \in C$, then C is closed.
Proof:
($\implies$) Suppose C is closed, $\{S_n\}$ is a sequence in C and $s_n \to L$. Suppose $L \notin C$, then $\exists r > 0$ s.t. $B_r(L) \subseteq E \setminus C$ since $E \setminus C$ is open. Since $s_n \to L$, $\exists N \in \mathbb{N}$ s.t. $s_n \in B_r(L)$ for all $n > N$, contradiction, since $s_n \in L$ and $B_r(L) \cap C = \varnothing$.
($\impliedby$) We prove the contrapositive. Suppose C is not closed, then $E \setminus C$ is not open. $\implies \exists p \in E \setminus C$ s.t. $\forall r > 0$, $B_r(p) \not\subset E \setminus C$, $B_r(p) \cap C \neq \varnothing$. In particular, $\forall n \in \mathbb{N}$, $\exists x_n \in B_{1/n}(p) \cap C$. Then $\{x_n\}$ is a sequence with $d(x_n, p) < \frac{1}{n}$. Given $\varepsilon > 0$, $\exists N$ s.t. $\frac{1}{N} < \varepsilon$. Hence for all $n > N$, $d(x_n, p) < \frac{1}{n} < \frac{1}{N} < \varepsilon$, which suggests that $x_n \to p \notin C$. So we prove that if C is not closed, then $\exists$ a sequence $\{x_n\} \subseteq C$ with $x_n \to p \notin C$.

Proposition Suppose $\{a_n\}, \{b_n\}$ are two sequences in $\mathbb{R}$, $a_n \to a$, $b_n \to b$. Then

1. $a_n + b_n \to a + b$
2. $a_n \cdot b_n \to a \cdot b$
3. $\forall c \in \mathbb{R}$, $ca_n \to ca$
4. If $b \neq 0$ and $b_n \neq 0$ for all n, then $\frac{a_n}{b_n} \to \frac{a}{b}$
5. If $a_n \leq b_n$ for all n, then $a \leq b$.
   #TODO
   Proof:
   For (2), take it with the fact that convergent sequences are bounded and take $\varepsilon = \varepsilon / 2M$, where M is and upper bound for both of the sequence.

Def (Inferior, Closure, Boundary)

Let (E, d) be a metric space, $S \subseteq E$ a subset, then:
Interior$(S) \equiv S^\circ = \cup_{O \subseteq S} O$, where O is an open set, is the largest open set contained in S.
Closure$(S) \equiv \overline{S} \equiv \cap_{S \subseteq C} C$, where C is a closed set, is the smallest closed set containing S.
Boundary$(S) \equiv \overline{S} \setminus C^\circ = \delta S$.

Ex. $E = \mathbb{R}, d(x, y) = |x - y|, S = \mathbb{Q}$. $\forall\, q \in \mathbb{Q}$, $\forall r > 0$, $B_r(q) = (q - r, q + r) \cap (\mathbb{R} \setminus \mathbb{Q}) \neq \varnothing$. So $\nexists r$ s.t. $B_r(q) \subseteq \mathbb{Q}$. Thus $\mathbb{Q}^\circ = \varnothing$. $\overline{\mathbb{Q}} = \mathbb{R}$. $\delta S = \mathbb{R}$.
Ex. $E = \{-1, 0, 1\} \subseteq \mathbb{R}, d(x, y) = |x - y|$. $B_1(0) = \{0\}$. $B_1(0)$ is open and closed. $\overline{B_1(0)} = B_1(0)$. But $\{y \in E \mid d(y, 0) \leq 1\} = E$.

Theorem

Let (E,d) be a metric space, $S \subseteq E$, then

1. $S^\circ = \{x \in S | \exists \varepsilon > 0,\ s.t.\ B_\varepsilon(x) \subseteq S\}, \varepsilon > 0$
2. $E \setminus \overline{S} = (E \setminus S)^\circ$
3. $\overline{S} = \{x \in E | \exists\ a\ sequence \{s_n\} \subseteq S\ with\ s_n \to x\}$
4. $\delta S = (E \setminus S^\circ) \cap (E \setminus (E \setminus S^\circ))$
5. $E = S^\circ \sqcup \delta S \sqcup (E \setminus S)^\circ$. E is the disjoint union of the interior of S, boundary of S, and exterior of S.

Ex. $S = \{\frac{1}{n} | n \in \mathbb{N}\} \subseteq \mathbb{R}$. $S^\circ = \varnothing, \overline{S} = S \cap \{0\}, \delta S = \overline{S} \setminus S^\circ = S \cup \{0\}, Ext(S) = (\mathbb{R} \setminus S)^\circ = \mathbb{R} \setminus \overline{S}$
Proof:
1) Suppose $x \in S^\circ$. Since S° is open, $\exists r > 0$ s.t. $B_r(x) \subseteq S^\circ \subset S \implies$ $x \in \{x' \in S | \exists r > 0\ s.t.\ B_r(x') \subseteq S\}$. Then $\exists r\ s.t. B_r(x) \subseteq S$. Note that $B_r(x)$ is an open set, so we have open sets $\subseteq S^\circ, x \in S$.
2) $E \setminus \overline{S} = E \setminus \cap_{C\ closed, S \subseteq C} C = \cup_{E \setminus C\ open, E \setminus C \subseteq E \setminus S}(E \setminus C) = \cup_{O\ open, O \subseteq E \setminus S} O = (E \setminus S)^\circ = Ext(S)$
3) Suppose $x \in \overline{S}$, then $E \setminus \overline{S} = (E \setminus S)^\circ$. If $x \notin (E \setminus S)^\circ$, then $\forall r,\ B_r(x) \nsubseteq E \setminus S$. Then $\forall r,\ B_r(x) \cup S \neq \varnothing$, so $\forall r, \exists S_r \in B_r(x) \cup S)$. Let $r = \frac{1}{n}$, then $\forall n, S_n \in B_{1/n}(x) \cap S$ and $S_n \to x$.
Conversely, if $\{S_n\} \subseteq S, S_n \to x$, then $x = \lim S_n \in \overline{S}$ since $\{S_n\} \subseteq \overline{S}$ and $\overline{S}$ is closed.
4) $\delta S = \overline{S} \setminus S^\circ = \overline{S} \cap (E \setminus S^\circ) = (E \setminus (E \setminus S^\circ)) \cap (E \setminus S^\circ) = (2) = (E \setminus Ext(S)) \cap (E \setminus S^\circ)$
5) $E = (E \setminus \overline{S}) \sqcup \overline{S} = (E \setminus S)^\circ \sqcup (\delta S) \sqcup (S^\circ)$

Monotone sequences in $\mathbb{R}$

Def (increasing/ decreasing sequence)

A sequence $\{a_n\} \subseteq \mathbb{R}$ is increasing if $a_1 \leq a_2 \leq \cdots \leq a_n \cdots$. Similarly, $\{a_n\} \subseteq \mathbb{R}$ is decreasing if $a_1 \geq a_2 \geq \cdots \geq a_n \cdots$. Note that the constant sequence is both increasing and decreasing.

Def (monotone sequence)

A sequence is monotone if it is increasing or decreasing.

Theorem (Bounded monotone sequences converges)

Any bounded monotone sequences in $\mathbb{R}$ converges.

Proof:
Suppose $\{a_n\}$ is increasing and bounded. Then $S = \{a_n | n \in \mathbb{N}\}$ is bounded above. Since $\mathbb{R}$ is complete, $L := \sup S$ exists. We claim that $a_n \to L$. To see this, notice that for all
$\varepsilon > 0, L - \varepsilon < L \implies \exists N \; s.t. \; L - \varepsilon \leq a_n \leq L$ for $n \geq N$, then
$L - \varepsilon \leq a_N \leq a_n \leq L \implies a_n \in B_\varepsilon(L)$.
For the cases when the sequence is bounded from below, we argue that for infimum.

Def (Divergent)

A sequence $\{a_n\} \subseteq \mathbb{R}$ diverges to $+\infty$ if $\forall M \in \mathbb{R}, \exists N \in \mathbb{N}$ s.t. for all $n > N$, $a_n > M$. A sequence $\{b_n\} \subseteq \mathbb{R}$ diverges to $-\infty$ if $\forall M \in \mathbb{R}, \exists N \in \mathbb{N}$ s.t. for all $n > N$, $a_n < M$.

Theorem

A monotone sequence in $\mathbb{R}$ either converges or diverges to $\pm\infty$

Proof:
If $\{a_n\}$ is increasing and not bounded, it diverges to $+\infty$.

lim sup and lim inf

Remark: Suppose $\{S_n\}, \{T_n\}$ are two bounded sequences, $T \subseteq S$. then $\sup T \leq \sup S, \; \inf T \geq \inf S$.
Proof:
Suppose $\{S_n\}$ is a sequence bounded above. Then $V_N := \sup\{S_n | n \geq N\}$ exists and since $\{S_n | n \geq N+1\} \subseteq \{S_n | n \geq N\}$, $V_{N+1} \leq V_N, \forall N$, so $\{V_n\}$ is monotone, hence either converges or diverges to infinity.

lim sup and lim inf

Let $\{S_n\}$ be a sequence bounded above. Then $\forall N$, $V_N := \sup\{S_n | n \geq N\}$. We define

$$\limsup S_n := \lim V_n = \lim_{n\to\infty} \sup\{S_n | n \geq N\}$$

which may be infinity or a number.
Similarly, if the sequence is bounded below, we define

$$\liminf S_n := \lim V_n = \lim_{n\to\infty} \inf\{S_n | n \geq N\}$$

Ex. $S_n = (-1)^n, V_N = \sup\{(-1)^n | n \geq N\} = 1$. $S_n = -n, V_N = \sup\{-n | n \geq N\} = -N, V_N \to -\infty$
Remark. $\inf\{S_n | n \geq N\} \leq S_N \leq \sup\{S_n | n \geq N\}$

Remark. Given $\{S_n\}$ not bounded above, $\sup\{S_n|n \geq N\} = +\infty$. $\limsup S_n := +\infty$. Similar definition for lim inf.
Ex. $S_n = (-1)^n n$, $\limsup S_n = +\infty, \liminf S_n = -\infty$

Theorem

Let $\{S_n\}$ be a sequence in $\mathbb{R}$

1. If $\{S_n\} \to \pm\infty$ or converges, then $\liminf S_n = \lim S_n = \limsup S_n$.
2. If $\liminf S_n = \limsup S_n$ (could be $\pm\infty$), then $\liminf S_n = \lim S_n = \limsup S_n$.

Proof:
1) Let $L = \lim S_n$. Then $\forall \varepsilon > 0, \exists N$ s.t. $\forall n \geq N, |S_n - L| < \varepsilon/2$, or, equivalently, $L - \varepsilon/2 < L < L + \varepsilon/2$. Then $\forall M \geq N$,
$L - \varepsilon < L - \varepsilon/2 \leq \inf\{S_n|n \geq M\} \leq \sup\{S_n|n \geq M\} \leq L + \varepsilon/2 < L + \varepsilon$. Hence we have $\liminf S_n = L = \limsup S_n$.
2) If $\liminf S_n = L = \limsup S_n$, then $\forall \varepsilon > 0, \exists N$ s.t. for all $M \geq N$,
$L - \varepsilon < \inf\{S_n|n \geq M\} \leq S_M \leq \sup\{S_n|n \geq M\} < L + \varepsilon \implies S_n \to L$

3.4 Completeness

Def (Cauchy Sequences)

A sequence $\{S_n\}$ in a metric space (E, d) is Cauchy if $\forall \varepsilon > 0, \exists N$ s.t. $\forall n, m > N, d(s_n, s_m) < \varepsilon$

Lemma 3.4.1 Any convergent sequence $\{S_n\}$ in (E, d) is Cauchy.
Proof: Suppose $s_n \to L$ in (E, d), then given $\varepsilon > 0, \exists N$ s.t. for $n > N, d(s_n, L) < \varepsilon/2$. For $n, m > N$, $d(s_n, s_m) \leq d(s_n, L) + d(s_m, L) = \varepsilon/2 \cdot 2 = \varepsilon$.
Ex. $S_n = \sum_{k=1}^{n} \frac{1}{k}$, $\lim S_n = \sum_{k=1}^{\infty} \frac{1}{k}$. Claim: it is not Cauchy.
$s_{2n} - s_n = \frac{1}{n+1} + \cdots + \frac{1}{n+n} \geq \frac{1}{2n} + \cdots + \frac{1}{2n} = \frac{1}{2}$. Hence $\nexists N$ s.t. for $2n, n > N, |s_{2n} - s_n| < 1/2$. It is not epsilon-close.

Remark. There are (E, d) where Cauchy sequence need not have a limit in E.
Consider $E = \mathbb{R} \setminus \{0\}, d = |x - y|$. Then $S_n = \frac{1}{n} \to L = 0 \in \mathbb{R}$, but $L \notin \mathbb{R} \setminus \{0\}$

Def (Completeness)

A metric space (E, d) is complete if every Cauchy sequences converges.

We want to show that. 1) $\mathbb{R}, \mathbb{R}^n$ are complete. 2) Any subset of a complete metric space is complete. (see below)

Lemma 3.4.2 Let (E, d) be a metric space and $\{S_n\}$ a Cauchy sequence in E. Then $\{S_n\}_{n\in\mathbb{N}}$ is bounded.
Proof:
Since it is a Cauchy sequence, $\exists N$ s.t. for $m, n \geq N, d(s_n, s_m) < 1$. Let
$r = \max\{d(s_1, s_n), \cdots d(s_{N-1}, s_N), 1\} + 1$, then $\forall k, d(s_n, s_k) < r$.

Lemma 3.4.3 Suppose $\{S_n\}$ is a Cauchy sequence and $\{S_{n_k}\}$ is a convergent subsequence with $S_{n_k} \to L$. Then $S_n \to L$ as well.
Proof:
Since $S_{n_k} \to L, \forall \varepsilon > 0, \exists k$ s.t. $d(S_{n_l}, L) < \varepsilon/2, \forall l > k$. Since $\{S_n\}$ is a Cauchy sequence. Then $\exists N$ s.t. for $n, m > N, d(s_n, s_m) \leq \varepsilon/2$. Then for $n > \max\{N, k\} = M$, $d(s_n, s_{n_M}) \leq \varepsilon/2$ since $n_M \geq M \geq N$ and $d(s_{n_M}, L) < \varepsilon/2$. Then $d(s_n, L) \leq d(s_n, s_{n_M}) + d(s_{n_M}, L) = \varepsilon$

Lemma 3.4.4 Any closed subsets C of a complete metric space (E, d) is complete.
Proof:
Let $\{S_n\}$ be a Cauchy sequence in E with respect to metric d. Then $\{S_n\}$ is a Cauchy sequence in E. Since E is complete, $\{S_n\}$ converges to some limit $L \in E$. Since $\{S_n\} \subseteq C$ and C is closed, we have $L \in C$ and thus C is complete.

We then show that $\mathbb{R}, \mathbb{R}^n$ are complete. Let's start with $\mathbb{R}$.

> **Theorem (Bolzano–Weierstrass in $\mathbb{R}$)**
>
> Let $\{S_n\}$ be a bounded sequence in $\mathbb{R}$, $L = \limsup S_n$. Then there is a subsequence $\{S_{n_k}\}$ s.t. $S_{n_k} \to L$.

Proof:
Let $V_N := \sup\{s_n | n \geq N\}$, then by definition $L = \lim_{N\to\infty} V_N$. Then $\forall \varepsilon > 0, \exists k$ s.t.
$N \geq k \implies L - \varepsilon < V_N < L + \varepsilon$. Also, $\exists i$ s.t. $L - \varepsilon < s_i \leq V_N < L + \varepsilon$.
When $\varepsilon = 1$, $\exists K_1$ s.t. for $n_1 \geq K_1, L - 1 < s_{n_1} < L + 1$
When $\varepsilon = 1/2$, $\exists K_2$ s.t. for $n_2 \geq K_2, L - 1/2 < s_{n_2} \leq V_{n_2} < L + 1/2$
If we replace K_2 by $\max\{K_2, n_1 + 1\}$, we may assume that $n_2 \geq K_2 > n_1$.
Then continuing the construction we will get a sequence $n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$ so that $L - 1/k < s_{n_k} < L + 1/k, \forall k$, hence $s_{n_k} \to L$.
Note: 1/k comes from the construction of $\varepsilon = 1/k$

Corollary 3.4.6 $(\mathbb{R}, d)$ is complete.
Proof:
Let $\{S_n\} \subseteq \mathbb{R}$ be Cauchy. Then by Lemma 3.4.2, it is bounded. By Bolzano–Weierstrass $\{S_n\}$ has a convergent subsequence $\{S_{n_k}\}$. By Lemma 3.4.3, $S_n \to \lim S_{n_k} \in \mathbb{R}$.

Now recall some useful metric in $\mathbb{R}^n$:
$d_2(x,y) := \sqrt{\sum_i (x_i - y_i)^2}$
$d_1(x,y) := \sum_i |x_i - y_i|$
$d_\infty(x,y) = \max_{1\le i\le n} |x_i - y_i|$

Lemma 3.4.7 $(\mathbb{R}^n, d_1)$ is complete.
Proof:
Let $x^{(k)} = \{(x_1^{(k)}, x_2^{(k)}, \cdots x_n^{(k)})\}_{k=1}^\infty$ be a Cauchy sequence in $\mathbb{R}^n$ with respect to d_1. Note that $\forall j$ we have $|x_j - y_j| \le \sum_{i=1}^n |x_i - y_i| = d_1(x,y)$ for all $x, y \in \mathbb{R}^n$. Hence $\forall j$, $\exists N$ s.t.
$k, l > N \implies \varepsilon > d_1(x^{(k)}, x^{(l)}) \ge |x_j^{(k)} - x_j^{(l)}|$, so $\{x_1^{(k)}\}, \cdots, \{x_n^{(k)}\}$ are a convergent sequences in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $\forall j, \exists L_j$ s.t. $x_j^{(k)} \to L_j$. Hence $\exists N_j$ s.t. $k > N_j \implies |x_j^{(k)} \to L_j| < \varepsilon/n$. Then $\forall N > \max\{N_1, \cdots, N_n\}$

$$d((L_1, L_2, \cdots, L_n), x^{(k)}) = \sum_{j=1}^n |L_j - x_j^{(k)}| < \frac{\varepsilon}{n} + \frac{\varepsilon}{n} + \cdots \frac{\varepsilon}{n} = \varepsilon$$

Hence $x^{(k)} \to L = (L_1, \cdots, L_n) \in \mathbb{R}^n$ with respect to d_1 and thus complete.

Theorem (Bolzano–Weierstrass in $\mathbb{R}^n$)

Let $\{s_n\}$ be a bounded sequence in $\mathbb{R}^n$, then $\{s_n\}$ has a convergent subsequence.

Proof:
Let $\{x^m\}$ be a bounded sequence in $\mathbb{R}^n$. The sequence $\{x_1^m\}$ of first components of the terms of $\{x^m\}$ is a bounded real sequence, which has a convergent subsequence $\{x_1^{mk}\}$. Let $\{x^{mk}\}$ be the corresponding subsequence of $\{x^m\}$. Then the sequence $\{x_2^{mk}\}$ of second components of $\{x^{mk}\}$ is a bounded sequence of real numbers, so it too has a convergent subsequence, and we again have a corresponding subsequence of $\{x^{mk}\}$(and therefore of $\{x^m\}$), in which the sequences of first and second components both converge. Continuing for n iterations, we end up with a subsequence $\{z^m\}$ of $\{x^m\}$ in which the sequences of first, second, ..., nth components all converge, and therefore the subsequence $\{z^m\}$ itself converges in $\mathbb{R}^n$.

Def (Norm)

A norm on $\mathbb{R}^n$ is a function $\mathbb{R}^n \to \mathbb{R}, x \mapsto \|x\|$ so that

1. $\|x\| \ge 0$ and $\|x\| = 0 \iff x = 0$
2. $\lambda\|x\| = \lambda\|x\|, \forall \lambda \in \mathbb{R}, x \in \mathbb{R}^n$
3. $\|x + y\| \le \|x\| + \|y\|$

Lemma 3.4.8 Let $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ be a norm. Then $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$, $d(x,y) = \|x - y\|$ is a metric.

Def (Equivalent norm and metric)

Two norms $\|\cdot\|, \|\cdot\|'$ are equivalent on $\mathbb{R}^n$ if $\exists c_1, c_2 > 0$ s.t. $\forall x \in \mathbb{R}^n$

$$c_1\|x\| \leq \|x\|' \leq c_2\|x\|$$

two metrics d, d' are equivalent on a set E if $\exists c_1, c_2 > 0$ s.t. $\forall x, y \in E$

$$c_1 d(x,y) \leq d'(x,y) \leq c_2 d(x,y)$$

Note that c_1, c_2 here are constants.

Theorem 3.4.9 $\frac{1}{n}\|x\|_1 \leq \|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$ for all $x \in \mathbb{R}^n$
Proof:
Fix $x \in \mathbb{R}^n$, then $\|x\|_\infty = \max\{|x_1|, \cdots, |x_n|\}$, so $\exists j$ s.t. $\|x\|_\infty = |x_j|$. But then $|x_j| = \sqrt{x_j^2} \leq \sqrt{x_i^2} = \|x\|_2$. $(\|x\|)^2 = \sum_j |x_j|^2 \leq (\sum_j |x_j|)^2 = (\|x\|_1)^2$, so we have $\|x\|_2 \leq \|x\|_1$.
$n\|x\|_\infty = |x_j| + |x_j| + \cdots + |x_j| \geq |x_1| + |x_2| + \cdots + |x_n| = \|x\|_1$, which ends our proof.
Remark. Suppose $\|\cdot\|, \|\cdot\|'$ are two equivalent norms, then $\exists c_1, c_2 > 0$ s.t. $\forall z \in \mathbb{R}^n$, $c_1\|z\| \leq \|z\|' \leq c_2\|z\|$. Then $\forall x, y \in \mathbb{R}^n$, $\forall x \in \mathbb{R}^n$, $c_1\|x-y\| \leq \|x-y\|' \leq c_2\|x-y\|$, so the two metric $d = \|x-y\|$, $d' = \|x-y\|'$ are equivalent.

Lemma 3.4.10 Suppose $d, d' : E \to [0, \infty)$ are two equivalence metric.

1. $\{S_n\}$ a Cauchy sequence with respect to $d \iff \{S_n\}$ is a Cauchy sequence with respect to d'.
2. $\{S_n\} \subseteq E$ converges with respect to $d \iff \{S_n\}$ converges with respect to d'.
   Proof:
   1) Since d, d' are equivalent, $\exists c_1, c_2 > 0$ s.t. $c_1 d(x,y) \leq d'(x,y) \leq c_2 d(x,y), \forall x, y$. Suppose $\{S_n\}$ is a Cauchy sequence with respect to d. Then given $\varepsilon > 0$, $\exists N$ s.t.
   $n, m \geq M \implies d'(s_n, s_m) < c_1\varepsilon$, and then $n, m \geq N$,
   $d(s_n, s_m) \leq \frac{1}{c} d'(s_n, s_m) < c_1 d'(s_n, s_m) < c_1 < \frac{1}{c_1} c_1 \varepsilon = \varepsilon$. The converse is similar.
   3) #TODO

Corollary 3.4.11 $(\mathbb{R}^n, d_2), (\mathbb{R}^n, d_\infty)$ are complete: Cauchy sequences converge.
Proof:
Suppose $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ is a metric equivalent to d_1, and $\{S_n\}$ is a Cauchy sequence with respect to d. Then $\{S_n\}$ is a Cauchy sequence with respect to d_1, hence converges, hence converges with respect to d. Since d_2, d_∞ are equivalent to d_1, we are done.

*Topology

Recall if (E, d) is a metric space, then we have a notion of an open set
$O : E$ is open $\iff \forall x \in O, \exists r(x) = r > 0$ s.t. $B_{r(x)}(x) \subseteq O$.

We prove 3 properties of open sets: 1) $\varnothing, E$ are open. 2) If O, O' are open then so is $O \cap O'$. 3) If $\{O_\alpha\}_{\alpha \in A}$ is a collection of open sets, then $\cup_{\alpha \in A} O_\alpha$ is open.

Def (Topology)

A topology $\mathcal{T}$ on a set X is a collection of subset of X (so $\mathcal{T} \subseteq \mathbb{P}(X)$,where $\mathbb{P}(X)$ is the power set) whose elements are called "open sets" s.t.

1. $\varnothing, X \in \mathcal{T}$
2. If $O, O' \in \mathcal{T}$, then $O \cap O' \in \mathcal{T}$.
3. $\forall \{O_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}$, then $\cup_{\alpha \in A} O_\alpha \in \mathcal{T}$.

Note: $\mathbb{R}^n$ comes with the standard topology and metric (d_2)
Note: the "open set" here does not necessarily have anything to do with metric.

We proved: if (E, d) is a metric space, then there is a topology $\mathcal{T}_d$ with associated to d.

Def (Topological space)

A topological space is a pair $(X, \mathcal{T})$, where $\mathcal{T}$ is a topology on a set X.

Lemma 3.4.12 Let d, d' be two equivalent metric on a set E, then $\mathcal{T}_d = \mathcal{T}_{d'}$, i.e., they give rise to the same topology.
Proof:
It's enough to show that $\mathcal{T}_d \subseteq \mathcal{T}_{d'}$, then by the same argument $T_{d'} \subseteq T_d$.
Since d, d' are equivalent, $\exists c > 0$ s.t. $cd(x, y) \leq d'(x, y), \forall x, y$. Suppose $O \subseteq \mathcal{T}_d$ an open set, then $\forall x \in O, \exists r(x)$ s.t. $B^d_{r(x)}(x) \subseteq O$. If $y \in B^{d'}_{cr}(x)$ (another ball risen by d' centered at x). then $cr > d'(x, y)(def\ of\ y \in B^{d'}_{rc}(x)) \geq cd(x, y) \implies d(x, y) < r \implies y \in B^d_{r(x)}(x)$ $\implies B^{d'}_{rc}(x) \subseteq B^d_{r(x)}(x) \subseteq O$. Since x is arbitrary, $O \in \mathcal{T}_{d'}$, which gives $\mathcal{T}_d \subseteq \mathcal{T}_{d'}$

Def (Convergent sequence in a topological space)

Let $(X, \mathcal{T})$ be a topological space, $\{S_n\} \subseteq X$ a sequence. Then $S_n \to L \in X$ if $\forall$ open set $U \subseteq X$ with $L \in U, \exists N$ s.t. $n > N \implies S_n \in U$.

Exercise: If $\mathcal{T} = \mathcal{T}_d$ for some metric d, then the two notions of convergence agree.

Corollary 3.4.13 Let E be a set, d, d' two equivalent metric, then $S_n \to L \in E$ with respect to d $\iff S_n \to L \in E$ with respect to d'.

3.5 Compactness

note: def different from the book

Def (Open cover)

Let $(X, \mathcal{T})$ be a topological space, $K \subseteq X$ a subset, then an open cover of K is a collection of open sets $\{O_\alpha\}_{\alpha \in A} \subseteq \mathbb{P}(X)$ s.t. $K \subseteq \cup_{\alpha \in A} O_\alpha$.

Ex. $\{(n, n+2)\}_{n \in \mathbb{Z}}$ is an open cover of $\mathbb{R}$
Ex. (E, d) metric space, $\{B_{1-1/n}(x)\}_{n \in \mathbb{N}}$ is an open cover in $\mathbb{R}$.

Def (Compactness)

A subset K of a topological space $(X, \mathcal{T})$ is compact if for every open cover $\{U_\alpha\}_{\alpha \in A}$ of K, $\exists n, \alpha_1, \ldots \alpha_n \in A$ s.t. $K \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$

Any open over has a finite open subcover that covers the subset

Ex. any finite set $\{x_1, \ldots, x_n\}$ is compact. if $\{U_\alpha\}_{\alpha \in A}$ is an open cover, then $\forall i$, $\exists \alpha_i$ s.t. $x_i \in U_{\alpha_i}$ and then $\{x_1, \ldots, x_n\} \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$
Counter(?) example: $\mathbb{R}$ is not compact: $\{(n, n+2)\}_{n \in \mathbb{Z}}$ is an open cover with no finite subcover.
Counter(?) example 2: $\mathbb{N} \subseteq \mathbb{R}$ is not compact. $\{U_i = (i - 1/2, i + 1/2)\}_{i \in \mathbb{N}}$ is an open cover of $\mathbb{N}$ no finite subcover.
Counter(?) example 3: E = a set. $d(x, y) = 1$ for $x \neq y$. This is the "discrete metric". Then every set is bounded. If $K \subseteq E$ is a set, then $\{\{x\}\}_{x \in K}$ is an open cover of K. So K is compact $\implies$ K is finite.

Remark. to prove compactness is to prove an open subcover.

Lemma 3.5.1 Let $(X, \mathcal{T})$ be a topological space, $K \subseteq X$ compact, $C \subseteq X$ closed. Then $K \cap C$ is compact.
Proof:
Suppose $\{U_\alpha\}_{\alpha \in A}$ is an open cover of $K \cap C$. Then $X \setminus C \cup \{U_\alpha\}_{\alpha \in A}$ is an open cover of K. Since K is compact, $\exists \alpha_1, \cdots, \alpha_n$ s.t. $K \subseteq (K \setminus C) \cup U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$,
$(K = (K \cap C) \cup (K \setminus C) \subseteq (X \setminus C) \cup (X \setminus C) \cup_{\alpha \in A} U_\alpha)$ which implies that $K \cap C \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$, and we are done.

Theorem 3.5.2 Let (E, d) be a metric space. If $K \subseteq E$ is compact, then K is closed and bounded.
Proof:
(K is bounded) Choose any $x \in E$, then $E = \cup_{n=1}^{\infty} B_n(x)$, so $\{B_n(x)\}_{n=1}^{\infty}$ is an open cover of K. So $\exists n_1 < n_2 < \cdots < n_k$ s.t. $K \subseteq B_{n1}(x) \cap \cdots \cap B_{nk}(x) = B_{nk}(x)$

(K is closed) If $K = E$, we are done, K is closed since $\varnothing = E \setminus K$ is open. If $K \neq E$, let $x \in E \setminus K$, then $K \subseteq E \setminus \{x\}$. $U_r = E \setminus \overline{B_r}(x)$ is open for all r. Now consider the union.
$\cup_{r>0}(E \setminus \overline{B_r}(x)) = E \setminus \cap_{r>0}\overline{B_r}(x) = E \setminus \{x\} \supseteq K \implies \{U_r\}_{r>0}$ is an open cover of K. Since K is compact, $\exists r_1 < \cdots < r_k$ s.t. $K \subseteq U_{r1} \cup \cdots \cup U_{rk} = E \setminus (\overline{B_{r1}}(x) \cap \cdots \cap \overline{B_{rk}}(x)) = E \setminus \overline{B_{r1}}(x)$ (since it is closed) $\implies B_{r1}(k) \cap K = \varnothing \implies E \setminus K$ is open.
The converse is false by simply taking any metric that is bounded: d = min(|x-y|,1), R: closed and bounded, but not compact. Since every singletons $\{x\}$ is closed and open, there is no finite subcover of any infinite set.
Remark. In $(\mathbb{R}^n, d_2)$ a metric space is compact $\iff$ it is closed and bounded

Remark. In general, compact sets need not be closed.
Ex. $X = \{a, b\}, \mathcal{T} = (X, \varnothing, \{a\})$. $K = \{a\}$ is compact (finite) but not closed since $X \setminus K = \{b\} \notin \mathcal{T}$
Note that singletons are compact since they are finite.

Theorem 3.5.3 Let X be a topological space, $K_1 \supseteq K_2 \supseteq \cdots \supseteq K_n \supseteq \cdots$ a sequence of compact sets, then $K = \cap_{n=1}^{\infty} K_n \neq \varnothing$ (and compact: since the intersection sets in K_1)
Note: $X = \mathbb{R}, K_n = [0, \infty), \cap_{n \geq 1} K_n = \varnothing$. $U_n = (0, 1/n),\ \cap U_n = \varnothing$. Such examples don't hold because they are not closed.

Theorem 3.5.4 Let X be a topological space, $K_1 \supseteq K_2 \supseteq \cdots \supseteq K_n \supseteq \cdots$ sequences of nonempty nested closed compact sets. Then $\cap_{i=1}^{\infty} K_i \neq \varnothing$.
Proof:
Suppose $\cap_{i=1}^{\infty} K_i = \varnothing$. Then $E = E \setminus \cap_{i=1}^{\infty} K_i = \cup_{i=1}^{\infty}(E \setminus K_i)$. We get an open cover $\{E \setminus K_n\}_{n=1}^{\infty}$ of E and of K_1. Since K_1 is compact, $\exists n_1 < n_2 < \cdots < n_l$ s.t.
$K \subseteq (E \setminus K_1) \cup \cdots \cup (E \setminus K_{n_l}) = E \setminus \cap_{i=1}^{l} K_{n_i} = E \subseteq K_{n_l}$. But $K_{n_l} \subseteq K_1$, contradiction.

Def (Sequentially compact)

A subset K of a topological space is sequentially compact if every sequence in K has a convergent subsequence whose limit is in K.

Ex 3.5.1. Suppose $K \subseteq \mathbb{R}^n$ is closed and bounded. Then K is sequentially compact:
Proof:
Every bounded sequence $\{s_n\}$ in K has a convergent subsequence $\{s_{n_k}\}$ by Bolzano-Weierstrass since $K is bounded. Since$K is closed, $L = \lim_{n \to \infty}\{s_{n_k}\} \in K$.

Lemma 3.5.5 Suppose (E, d) is a metric space and $K \subseteq E$ is compact, then K is sequentially compact.
Proof:
Let $\{S_n\}$ be a sequence in K. We argue that (1) $\exists x \in K$ s.t. $\forall \varepsilon > 0, \{n | s_n \in B_\varepsilon(x)\}$ is infinite, and this (1) implies that there is a subsequence #question $\{S_{n_k}\}$ that converges to x. Take $\varepsilon = 1, 1/2, 1/3 \cdots$ and find $n_1 < n_2 < \cdots < n_k \cdots$ s.t. $S_{n_k} \in B_{1/n}(x)$. To see (1), suppose it is false. Then $\forall x \in K$,

$\exists\varepsilon = \varepsilon(x)$ s.t. $\{n|s_n \in B_{\varepsilon(x)}(x)\}$ is finite. Then $\{B_{\varepsilon(x)}(x)\}_{x\in K}$ is an open cover of K. Since K is compact, $\exists x_1, \cdots x_l \in K$ s.t. $K \subseteq B_{\varepsilon(x_1)}(x_1) \cap \cdots \cap B_{\varepsilon(x_l)}(x_l)$. But then $\{n|s_n \in B_{\varepsilon(x_1)}(x_1) \cup \cdots \cup B_{\varepsilon(x_l)}(x_l)\}$ is finite, a contradiction.

Def (Totally bounded)

A subset K of a metric space (E, d) is totally bounded if $\forall s > 0$, $\exists x_1, \cdots, x_n \in K$ s.t. $K \subseteq B_\varepsilon(x_1) \cap \cdots \cap B_\varepsilon(x_n)$. i.e., $\forall\varepsilon > 0$, K can be covered by finitely many balls of radius ε.

Lemma 3.5.6 Suppose (E, d) is a metric space, $K \subseteq E$ sequentially compact. Then (K, d) is complete and totally bounded.
Proof:
Suppose $\{S_n\} \in K$ is Cauchy. Since K is sequentially compact, $\{S_n\}$ has a convergent subsequence and since $\{S_n\}$ is Cauchy, $\{S_n\}$ converges to $L = \lim S_{n_k}$ #question . Suppose K is not totally bounded, then $\exists\varepsilon > 0$ s.t. K cannot be covered by finitely many ε balls $\implies \exists x_1 \in K$ s.t. $K \setminus B_\varepsilon(x_1) \neq \varnothing$. Then $\exists x_2 \in K \setminus B_\varepsilon(x_1)$ s.t. $K \setminus (B_\varepsilon(x_1) \cup B_\varepsilon(x_2)) \neq \varnothing$.... $\exists x_n \in K \setminus (B_\varepsilon(x_1) \cup \cdots \cup B_\varepsilon(x_{n-1}))$ s.t. $K \setminus (B_\varepsilon(x_1) \cup \cdots \cup B_\varepsilon(x_{n-1})) \neq \varnothing$ (*). We get a sequence $\{x_n\}$ in K with $d(x_n, x_m) \geq \varepsilon$ for all $n \neq m$. Then $\{x_n\}$ has no Cauchy subsequence $\implies$ $\{x_n\}$ has no convergent subsequences. This contradicts sequential compactness.
*: This is a recursive definition, $\{x_n\}$ is infinite.

Lemma 3.5.7 Let (E, d) be a metric space, $K \subseteq E$ complete and totally bounded, then K is compact.
Proof:
Suppose $\exists$ an open cover $\{U_\alpha\}_{\alpha\in A}$ of K with no finite subcover. Since K is totally bounded, K can be covered with finitely many balls of radius 1. $\implies$ $\exists x_0 \in K$ s.t. $B_1(x_0)$ cannot be covered by finitely many U_α. There is a finite cover of K by balls of radius $1/2$ $\implies$ $\exists x_1$ s.t. $B_{1/2}(x_1) \cap B_1 x_0 \neq \varnothing$ and $B_{1/2}(x_1)$ cannot be covered by finitely many U_α. Proceeding this way we get a sequence $x_0, x_1, \cdots, x_n, \cdots$ s.t. $B_{1/2^n}(x_n) \cap B_{1/2^{n-1}}(x_{n-1})$, and each $B_{1/2^n}(x_n)$ cannot be covered by finitely manny U_αs. Then
$d(x_n, x_{n+k}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+k-1}, x_{n+k}) < \frac{1}{2^{n-1}}(1 + \frac{1}{2} + \cdots + \frac{1}{2^k}) < \frac{1}{2^{n-2}}$
Thus $\{x_n\}$ is Cauchy. Since K is complete, $x_n \to y$ for some $y \in K$. Since $\{U_\alpha\}_{\alpha\in A}$ is a cover, $\exists\alpha_0$ s.t. $y \in U_{\alpha_0}$. Since U_{α_0} is open, $\exists r > 0$ s.t. $B_r(y_0) \subseteq U_{\alpha_0}$. Since $x_n \to y$, $\exists n$ s.t. $x_n \in B_{r/2}(y)$ and $\frac{1}{2^n} < \frac{r}{2}$. $B_{1/2^n}(x_n) \subseteq B_r(y) \subseteq U_{\alpha_0}$. But according to the construction of $B_{1/2^n}(x_n)$, we get a contradiction.

Summary

For a subset K of a metric space, (1) $\implies$ (2), (2) $\implies$ (3), (3) $\implies$ (1), thus TFAE

1. K is compact
2. K is sequentially compact

3. K is compete and totally bounded.

Theorem (Heine-Borel)

A subset K of $\mathbb{R}^n$ is compact $\iff$ K is closed and bounded.

Proof:
($\implies$) is true for any metric space.
($\impliedby$) Suppose K is closed and bounded, then by Ex 3.5.1, K is sequentially compact and hence compact.

Ex. $\mathbb{R}, d = \min(1, |x-y|)$. $(\mathbb{R}, d)$ is bounded but not totally bounded since it's complete but not compact. $B^d_{1/2}(x) = (x - 1/2, x + 1/2), \forall x$ and $\mathbb{R}$ cannot be covered by finitely many balls of radius $1/2$. Thus only in $(\mathbb{R}, d_2)$ or (d_1 or d_∞) bounded $\implies$ totally bounded.

Chapter 4 Continuous functions

4.1 Continuity

Def (Continuous at a point)

Let $(E, d), (E', d')$ be two metric space. A function $f : E \to E'$ is continuous at $p \in E$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ so that $\forall x \in E$, $d(x, p) < \delta \implies d'(f(x), f(p)) < \varepsilon$. i.e. $f(B^d_\delta(p)) \subseteq B^{d'}_\varepsilon(f(p))$

Remark. think of "$\delta - \varepsilon$ close"

Def (Continuous function)

A function $f : E \to E'$ is continuous if it is continuous at every point p of E.

Ex 1. (E, d) a metric space, $q \in E$ a point, $f : E \to \mathbb{R},\ f(p) = d(p, q)$. Then f is continuous at every $p \in E$: $|f(x) - f(p)| = |d(x, q) - d(p, q)| \leq d(x, q)$. So
$\forall \varepsilon > 0, d(x, p) < \varepsilon \implies |f(x) - f(p)| < \varepsilon = \delta$.
Ex 2. $f : \mathbb{R} \to \mathbb{R}, f(x) = \begin{cases} 1, x \ \ rational \\ 0, x \ \ irrational \end{cases}$ is not continuous at any $p \in \mathbb{R}$. To see this,
$\forall p \in \mathbb{R}, \delta > 0, B_\delta(p) = (p - \delta, p + \delta)$ contains both rationals and irrationals. If p is rational, then $f(p) = 1$ and $\forall \delta > 0$, $\exists x \in B_\delta(1)$ s.t. $f(x) = 0$. Then for all $\varepsilon < 1$ (1/2, for example), no matter which δ we choose, $|x - p| < \delta \not\Longrightarrow |f(x) - 1| < 1/2$. Similar problem happens if $f(p) = 0$.

Theorem 4.1.1

$f:(E,d)\to(E',d')$ is continuous $\iff \forall U\subseteq E', U$ open, $f^{-1}(U)$ is open.

Proof:
($\implies$) Suppose f is continuous, $U\subseteq E'$ open, then $\forall p\in f^{-1}(U)$, $f(p)\in U \implies \exists\varepsilon>0$ s.t. $B_\varepsilon(f(p))\subseteq U$. Since f is continuous at p, $\exists\delta>0$ s.t. $f(B_\delta(p))\subseteq B_\varepsilon(f(p))$, which implies that $f(B_\delta(p))\subseteq U$, and thus $B_\delta(p)\subseteq f^{-1}(U)$. Since $p\in f^{-1}(U)$ is arbitrary, $f^{-1}(U)$ is open.
($\impliedby$) Suppose $\forall U\subseteq E', U$ open, $f^{-1}(U)$ is open. Given $p\in E$ and $\varepsilon>0$, $B_\varepsilon(f(p))$ is open in E. Since $p\in f^{-1}(f(p))\subseteq f^{-1}(B_\varepsilon(f(p)))$ and $f^{-1}(B_\varepsilon(f(p)))$ is open, $\exists\delta>0$ s.t. $f(B_\delta(p))\subseteq B_\varepsilon(f(p))$ and f is continuous at p.

some helpful extensions:
($\implies$) Suppose f is continuous and U is an open subset of E'. We want to show that $f^{-1}(U)$ is open. Let $p\in f^{-1}(U)$, then $f(p)\in U$ since f is continuous. Since U is open, $\exists\varepsilon>0$ s.t. $B_\varepsilon(p)\subseteq E'$ is open. Since f is continuous at p, $\exists\delta>0$ s.t. if $d(p,x)<\delta$, then $d'(f(p),f(x))<\varepsilon$, i.e., if $x\in B_\delta(p)$, then $f(x)\in B_\varepsilon(p)\subseteq E'$. Hence $f(p)\in B_\varepsilon(p)\subseteq U$ and that $B_\delta(p)\subseteq f^{-1}(U)$. Since p is any point of $f^{-1}(U)$, the set $f^{-1}(U)$ is open.

Corollary 4.1.2 $f:(E,d)\to(E',d')$ is continuous $\iff \forall C\subseteq E', C$ closed, $f^{-1}(C)$ is closed.

Def (Continuous function)

A map/function $f:(X,\mathcal{T})\to(X',\mathcal{T}')$ between two topological spaces is continuous if $\forall U\in\mathcal{T}'$, U open, $f^{-1}(U)\in\mathcal{T}$. i.e., preimages of open sets are open.

Remark. Theorem 4.1.1 says that the notion of continuity of a map depends only on the topologies: If d_1,d_2 are two metrics s.t. $\mathcal{T}_{d_1}=\mathcal{T}_{d_2}$, d_1',d_2' are two metrics s.t. $\mathcal{T}_{d_1'}=\mathcal{T}_{d_2'}$. Then $f:(E,d_1)\to(E',d_1')$ is continuous iff $f:(E,d_2)\to(E',d_2')$ is continuous.

Theorem 4.1.3 The composite of two continuous maps is continuous. If $f:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$, $g:(Y,\mathcal{T}_Y)\to(Z,\mathcal{T}_Z)$ are continuous, then $g\circ f:(X,\mathcal{T}_X)\to(Z,\mathcal{T}_Z)$ is continuous.
Proof:
Suppose $W\subseteq Z$ is open, then $g^{-1}(W)\subseteq Y$ is open $\implies f^{-1}(g^{-1}(W))$ is open in X. But $f^{-1}(g^{-1}(W))=(f\circ g)^{-1}(W)$, so $(f\circ g)^{-1}(W)$ is open in X and hence $f\circ g$ is continuous.

Theorem 4.1.4 Images of compact sets under continuous functions are compact. If $f:X\to Y$ is continuous and $K\subseteq X$ is compact, then $f(X)\subseteq Y$ is compact.
Proof:

Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of $f(K)$, then $\{f^{-1}(U_\alpha)\}_{\alpha\in A}$ is an open cover of K. Since K is compact, $\exists\alpha_1\cdots\alpha_n$ s.t. $K\subseteq f^{-1}(U_{\alpha_1})\cup\cdots\cup f^{-1}(U_{\alpha_n}) \implies f(K)\subseteq U_{\alpha_1}\cup\cdots\cup U_{\alpha_n}$ and $f(K)$ is compact.

Corollary 4.1.5 Let (E,d) be a metric space, X a topological space, $K\subseteq X$ compact and $f:X\to E$ is continuous, then $f(k)$ is complete, totally bounded, sequentially compact and closed.

Corollary 4.1.6 Suppose X is a topological space, $f:X\to\mathbb{R}$ continuous and $K\subseteq X$ compact, then $\exists x_1,x_2\in X$ s.t. $\forall x\in X$, $f(x_1)\le f(x)\le f(x_2)$. i.e., f achieves maximum and minimum on X.
Proof:
$f(K)$ is closed and bounded in $\mathbb{R}$, hence $\exists x_1,x_2\in\mathbb{R}$ s.t. $f(x_1)=\inf(f(K)), f(x_2)=\sup(f(K))$

4.2 Continuity and Limits

Def (Cluster point of a topological space)

Let X be a topological space, $S\subseteq X$ a subspace, then $x\in X$ is a cluster point of S if $\forall$ open set U with $x\in U$, $(U\setminus\{x\})\cap S$ is nonempty. If E is a metric space, x is a cluster point of S iff $\exists$ a sequence $\{S_n\}\subseteq S\setminus\{x\}$ s.t. $s_n\to x$.

Ex. $S=\{0\}\cup[1,2]\subseteq\mathbb{R}$, $d(x,y)=|x-y|$. 0 is not a cluster point. Every $x\in[1,2]$ is a cluster point of S.

Def (Cluster point of a metric space)

Suppose (E,d), (E',d') are metric spaces, $A\subseteq E$, $f:A\to E'$, p is a cluster point of A. Then

$$\lim_{x\to p} f(x)=q$$

If $\forall\varepsilon>0$, $\exists\delta>0$ s.t. $x\in A\cap B_\delta(p), x\ne p, d'(f(x),q)<\varepsilon$

Remarks.
(1) $p\in A$ is not necessary. i.e., $f(p)$ need not be defined.
(2) Even if $p\in A$ we are requiring $f(p)=\lim_{x\to p}f(x)$
Ex. $f:\mathbb{R}\to\mathbb{R}$, $f(x)=1, x=0, f(x)=0, o.w.$ Then $\lim_{x\in\to 0}f(x)=0\ne 1=f(0)$.

Lemma 4.2.1 E,E' metric spaces, p cluster point of E. Then $f:E\to E'$ is continuous at $p\iff\lim_{x\to p}f(x)=f(p)$.
Proof:
#TODO
Note that if p is not a cluster point, then $\exists r$ s.t. $B_r(p)\setminus\{p\}\cap E=\varnothing$. i.e. $B_r(p)=\{p\}$. And then any $f:E\to E'$ is continuous at p.

Theorem 4.2.2 E, E' metric spaces, $f : E \to E'$ is continuous at $p \in E \iff \forall$ sequences $\{S_n\} \in E$ with $S_n \to p$, we have $f(s_n) \to f(p)$.
Proof:
($\implies$) Suppose $S_n \to p$ and f is continuous. Then given $\varepsilon > 0$, $\exists \delta > 0$ s.t.
$d(x,p) < \delta \implies d'(f(x), f(p)) < \varepsilon$. Since $S_n \to p$, $\exists N$ s.t. if $n \geq N$,
$d(x,p) < \delta \implies d'(f(x), f(p)) < \varepsilon \implies f(p) \to f(x)$
($\impliedby$) We prove the contrapositive. Suppose f is not continuous at p. We construct $\{S_n\}$ s.t. $S_n \to p$ but $f(S_n) \nrightarrow f(p)$. Since f is not continuous at p, $\exists \varepsilon_0 > 0$ s.t. $\forall \delta > 0$, $\exists x_r \in B_\delta(p)$ with $f(x_r) \notin B_{\varepsilon_0}(f(p))$.
Let $S_n = x_{1/n}$. Then $s_n \in B_{1/n}(p)$ (hence $s_n \to p$) and $f(s_n) \notin B_{\varepsilon_0}(f(p))$ (hence $f(s_n) \nrightarrow f(p)$)

Theorem 4.2.3 Suppose $f, g : (E, d) \to \mathbb{R}$ are continuous at $p \in E$. Then $f + g$, $f \cdot g$ are continuous at p. If $g(p) \neq 0$, f/g is also continuous.
Proof:
Suppose $S_n \to p$, then $f(s_n) \to f(p), g(s_n) \to g(p)$. Hence $(f+g)(s_n) = f(s_n) + g(s_n)$
$\to f(p) + g(p) \implies f + g$ is continuous. Other proofs are similar.

Theorem 4.2.4 Suppose $f = (f_1, \cdots, f_n) : E \to \mathbb{R}^n$ is a function, $p \in E$. Then f is continuous at $p \iff f_1 \cdots f_n$ are all continuous at p.
Proof:
A sequence $t_k = (t_k^{(1)}, \cdots, t_n^{(n)}) \to \mathbb{R}^n$ converges to $q = (q_1, \cdots, q_n) \in \mathbb{R}^n \iff$
$t_k^{(k)} \to q^{(i)}, i = 1, 2, \cdots n$.

Uniform continuity

Def (Uniformly continuous)

$f : (E, d) \to (E', d')$ is uniformly continuous if $\forall \varepsilon > 0$, $\exists \delta = \delta_\varepsilon > 0$ s.t.

$$d(x,p) < \delta \implies d'(f(x), f(p)) < \varepsilon, \forall x, p$$

Ex. $f(x) = x^2$. $f : [0, \infty) \to \mathbb{R}$ is not uniformly continuous.
Proof: $|f(x) - f(y)| = x^2 - y^2 = |x-y||x+y| \geq 2 \cdot \min(x,y) \cdot |x-y|$. There for $\forall \delta$ if $x, y > \frac{1}{\delta}$ and $|x - y| = \delta/2$. We have $|f(x) - f(y)| \geq 2 \cdot \frac{1}{\delta} \cdot \frac{\delta}{2} = 1$.

Lemma 4.2.5 Suppose $f : E \to E'$ is uniformly continuous, then for any Cauchy sequences $\{S_n\}$ in E, $f(s_n)$ is Cauchy.
Proof:
Since f is uniformly continuous, $\forall \varepsilon > 0$, $\exists \delta$ s.t. $d(x,y) < \delta \implies d'(f(x), f(p)) < \varepsilon$. Fix $\varepsilon > 0$ and choose δ. Since $\{S_n\}$ is Cauchy, $\exists N$ s.t. $n, m \geq N \implies d(s_n, s_m) < \delta$. And then $d'(f(s_n), f(s_m)) < \varepsilon$.

Ex. $f:(0,1)\to\mathbb{R}, f(x)=\sin(1/x)$. Claim: f is not uniformly continuous. Reason: $S_n=\frac{1}{\pi/2+n\pi}\to 0$,not in $(0,1)$ but still Cauchy. $f(s_n)=\sin(\frac{\pi}{2}+n\pi)=(-1)^n$. So f is not uniformly continuous.

Theorem 4.2.6 Suppose $f:E\to E'$ is continuous and E is compact, then f is uniformly continuous.
Proof:
Given ε we want $\delta=\delta_\varepsilon$ s.t. $d(x,y)<\delta \implies d(f(x),f(y))<\varepsilon$. Since f is continuous, $\forall x, \exists\delta_x$ s.t. $d(x,y)<\delta_x \implies d'(f(x),f(y))<\varepsilon/2$. $\{B_{\delta_{x/2}}(x)\}_{x\in E}$ is an open cover of $E \implies \exists n, x_1,\cdots,x_n$ s.t. $E=B_{\delta_{x1}/2}\cup\cdots\cup B_{\delta_{x_n}/2}(x_n)$. Let $\delta=\min(\delta x_1/2,\cdots\delta x_n/2)$. Suppose $d(p,q)<\delta$. Then $q\in B_{\delta x_i/2}(x)$ for some i. Then $d(p,x_i)\le d(p,q)+d(q,x_i)<\delta+\delta_{x_1}/2\le\delta_{x_i}$. $p\in B_{\delta_{x_i}}(x_i)$. Since $p,q\in B_{\delta_{x_i}}(x_i)$, $d'(f(p),f(q))\le d'(f(p),f(x_i))+d'(f(x_i),f(p))<\varepsilon/2+\varepsilon/2=\varepsilon$.

Def (Pointwise continuity)

$\{f_n:(E,d)\to(E',d')\}_{n=1}^\infty$ sequence of functions between two metric spaces. The sequence $\{f_n\}$ converges pointwise to $f:E\to E'$ if $\forall p\in E$, $f_n(p)\to f(p)$

Pointwise limit of continuous functions need not be continuous.

Def (Uniform convergence)

$\{f_n:(E,d)\to(E',d')\}_{n=1}^\infty$a sequence of functions between two metric spaces, $A\subseteq E$ a subspace. $f_n\to f$ uniformly on A if $\forall\varepsilon>0, \exists N$ s.t. $n\ge N \implies d'(f_n(p),f(p))<\varepsilon, \forall p\in A$.
Equivalently, $\forall\varepsilon>0, \exists N$ s.t. $n\ge N \implies \sup\{d'(f_n(p),f(p))|p\in A\}<\varepsilon$.
$\lim_{n\to\infty}\sup\{d'(f_n(p),f(p))|p\in A\}=0$

Ex. $f_n(x)=x^n, f_n:[0,1]\to[0,1]$ on $A=[0,a]$. $a<1$, f_n converges uniformly. Check $\sup\{|x^n-0|:0\le x\le a\}=a^n\to 0$. $\{f_n\}$ converges to 0 on $[0,1)$ but not uniformly, $\sup\{|x^n-0|:0\le x\le 1\}=1\nrightarrow 0$
Ex. $f_n(x)=\frac{nx}{1+n^2x^2}, f_n:\mathbb{R}\to\mathbb{R}$. Since $f_n(0)=0,\forall n$, and for $x\ne 0, |\frac{nx}{1+n^2x^2}|\le|\frac{nx}{n^2x^2}|=\frac{1}{n|x|}\underset{n\to\infty}{}\to 0$. Note that $f_n(\frac{1}{n})=\frac{n/n}{1+n^2/n^2}=\frac{1}{2}\nrightarrow 0$, so $\{f_n\}$ is not uniformly convergent.

Def (Uniformly Cauchy)

A sequence of functions $\{f_n:E\to E'\}_{n\in\mathbb{N}}$ is uniformly Cauchy on $A\subseteq E$ if $\forall\varepsilon>0,\exists N$ with $n,m\ge N \implies \sup\{d'(f_n(x),f_m(x))|x\in A\}<\varepsilon$

Theorem Let $\{f_n:E\to E'\}_{n\in\mathbb{N}}, E'$ complete, then $\{f_n\}$ converges uniformly on $A \iff \{f_n\}$ is uniformly Cauchy.

Proof:
($\Longrightarrow$) Suppose $f_n \to f$ uniformly on A. Then $\forall \varepsilon > 0, \exists N$ s.t. for $n \geq N$,
$\sup\{d'(f_n(x), f(x)) | x \in A\} < \varepsilon/3$. Then $\forall n, m \geq N$, $\forall x \in A$,
$d'(f_n(x), f_m(x)) \leq d'(f_n(x), f(x)) + d'(f_m(x), f(x)) < \varepsilon/3 + \varepsilon/3$, which suggests that
$\sup\{d'(f_n(x), f_m(x)) | x \in A\} \leq \frac{2}{3}\varepsilon < \varepsilon$.
($\Longleftarrow$) Suppose $\{f_n\}$ is uniformly Cauchy on A. Then $\forall x \in A, \{f_n(x)\}$ is Cauchy. Since E' is complete, we can define $f : A \to E'$ by $f(x) = \lim_{n\to\infty} f_n(x)$. We now argue: $f_n \to f$ is uniformly on A. Recall that $\forall x \in E', h : E' \to [0, \infty), h(p) = d'(x, p)$ is continuous. Since $\{f_n\}$ is uniformly Cauchy on A, given $\varepsilon > 0, \exists N$ s.t. if $m, n \geq N$, then $\sup\{d'(f_n(x), f_m(x)) | x \in A\} < \varepsilon/2$. Fix $n \geq N$, then $d'(f_n(x), f(x)) = d'(f_n(x), \lim_{m\to\infty} f_m(x)) = \lim_{m\to\infty} d'(f_n(x), f_m(x)) \leq \sup_{m\geq n} d'(f_m(x), f_n(x))$ $< \varepsilon$. Hence $\forall n \geq N, \sup\{d'(f_n(x), f(x)) | x \in A\} \leq \varepsilon/2 < \varepsilon$.

Theorem Uniform limit of continuous functions is continuous.
Proof:
Suppose $\{f_n : E \to E'\}$ converges uniformly on all of E. Fix $p \in E$, we prove that f is continuous on p. For $\forall x \in E, \forall n \in \mathbb{N}$, $d'(f(p), f(x)) \leq d'(f(p), f_n(p)) + d'(f_n(p), f_n(x)) + d'(f_n(x), f(x))$. Given $\varepsilon > 0$, we want to show: $\exists \delta > 0$ s.t. if $d(x, p) < \delta$, then $d'(f(x), f(p)) < \varepsilon$. Since $f_n \to f$ converges uniformly, $\exists N$ s.t. for $n \geq N$, $d'(f_n(y), f(y)) < \varepsilon/3, \forall y \in E$. Since f_N is continuous at p, $\exists \delta > 0$ s.t. if $d(x, p) < 0, d'(f_N(x), f_N(p)) < \varepsilon/3$. Then $\forall x$ with $d(x, p) < \delta$,
$d'(f(p), f(x)) \leq d'(f(p), f_n(p)) + d'(f_n(p), f_n(x)) + d'(f_n(x), f(x)) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$

Def (Bounded function)

Let $(E, d), (E', d')$ be two metric spaces, a function $f : E \to E'$ is bounded if $f(E) \subseteq E'$ is bounded.
Notation: $C(E, E') = \{f : E \to E' | f \textit{ bounded and continuous}\}$

Exercise. if $f, g : E \to E'$ are bounded, then $\{d'(f(x), g(x)) | x \in E\}$ is bounded. Define $D : C(E, E') \times C(E, E') \to [0, \infty)$ by $D(f, g) = \sup\{d'(f(x), g(x)) | x \in E\}$
Exercise. D is a metric. Then $f_n \to f$ in $(C(E, E'), D) \iff f_n \to f$ uniformly convergent.

Chapter 5 Differentiation

Chapter 6 Riemann Integrals

Chapter 7 Interchange of limit operations