MATH 424 HW3

Dilys Wu

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1 Q1

Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three sequences of real numbers with $a_n \leq b_n \leq c_n$ for all n. Prove that if $\{a_n\}$, $\{c_n\}$ converge, and if they converge to the same number L, then $\{b_n\}$ converges to L as well. Here and in the rest of this homework assignment $\mathbb{R}$ is given the standard metric unless noted otherwise.

Proof:

Since $\{a_n\}, \{c_n\}$ converges to L, $\forall \varepsilon > 0$, $\exists N_1, N_2$ s.t. $d(a_n, L) < \frac{\varepsilon}{2}$ for $n > N_1$, and $d(c_n, L) < \frac{\varepsilon}{2}$ for $n > N_2$. Then for $n > \max(N_1, N_2)$, $d(a_n, c_n) \leq d(a_n, L) + d(c_n, L) = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, which is the same as $L - \varepsilon < a_n < L + \varepsilon$, $L - \varepsilon < c_n < L + \varepsilon$. Note that also $a_n \leq b_n \leq c_n$ for all n, hence $L - \varepsilon \leq a_n \leq b_n \leq c_n \leq L + \varepsilon$, which suggests that $\{b_n\}$ converges to L as well.

2 Q2

Suppose $\{x_n\}$, $\{y_n\}$ are two Cauchy sequences of rational numbers (the set $\mathbb{Q}$ of the rationals is given the standard metric). Prove that their sum $\{x_n + y_n\}$ and product $\{x_n \cdot y_n\}$ are also Cauchy.

Proof:

Since $\{x_n\}, \{y_n\}$ are Cauchy, $\forall \varepsilon > 0$, $\exists N_1, N_2$ s.t. for $n, m > N_1, k, l > N_2$, $d(x_n, x_m) < \frac{\varepsilon}{2}$, $d(y_k, y_l) < \frac{\varepsilon}{2}$. Let $N > \max(N_1, N_2)$, then for $r, s > N$, $d(x_r + y_r, x_s + y_s) = |(x_r + y_r) - (x_s + y_s)| \leq |x_r - x_s| + |y_r - y_s| = d(x_r, x_s) + d(y_r, y_s) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, so their sum is Cauchy.

Now consider the case for the product. Recall that Cauchy sequences are bounded, and let M be greater than the least upper bound of the two sequences. Hence $\exists N', N''$ s.t. $\forall n > N'$, $|x_n| \leq M$, and $\forall n' > N''$, $|y_n| \leq M$. Similarly as above, since $\{x_n\}, \{y_n\}$ are Cauchy, $\exists K$ s.t. $i, j > K$, $|x_i - x_j| < \frac{\varepsilon}{2M}$, $|y_i - y_j| < \frac{\varepsilon}{2M}$. Hence, $|x_i y_i - x_j y_j| = |x_i y_i - x_i y_j + x_i y_j - x_j y_j| \leq |x_i(y_i - y_j)| + |y_j(x_i - x_j)| = |x_i||y_i - y_j| + |y_j||x_i - x_j| < |x_i|\frac{\varepsilon}{2M} + |y_j|\frac{\varepsilon}{2M} \leq 2M \cdot \frac{\varepsilon}{2M} = \varepsilon$, so the product is Cauchy.

3 Q3

Suppose a sequence $\{s_n\}$ of real numbers is bounded: there is $M > 0$ so that $|s_n| < M$ for all n.
(a) Prove that there is a subsequence $\{s_{n_k}\}$ that converges to lim inf s_n.
(b) Prove that the limit L of any convergent subsequence of $\{s_n\}$ satisfies $\liminf s_n \leq L \leq \limsup s_n$.
Proof:
(a) Since $\{s_n\}$ is bounded, it is bounded from above and from below. Since it is nonempty, it has a supremum and an infimum. By Bolzano-Let $L = \liminf s_n$. Let $V_N := \inf\{s_n | n \geq N\}$, then by definition $L = \inf_{N\to\infty} V_N$. Then $\forall \varepsilon > 0, \exists k$ s.t. $N \geq k \implies L-\varepsilon < V_N < L+\varepsilon$. Also, $\exists i$ s.t. $L-\varepsilon < s_i \leq V_N < L+\varepsilon$. When $\varepsilon = 1$, $\exists K_1$ s.t. for $n_1 \geq K_1$, $L - 1 < s_{n_1} < L + 1$. When $\varepsilon = 1/2$, $\exists K_2$ s.t. for $n_2 \geq K_2$, $L - 1/2 < s_{n_2} \leq V_{n_2} < L + 1/2$. If we replace K_2 by $\min\{K_2, n_1 - 1\}$, we may assume that $n_2 \leq K_2 < n_1$. Then continuing the construction by taking $\varepsilon = 1/2^k, k \in \mathbb{N}$ we will get a sequence $s_{n_1} > s_{n_2} > \cdots > s_{n_k} > s_{n_{k+1}} > \cdots$ so that $L - 1/2^k < s_{n_k} < L + 1/2^k$, $\forall k$, hence $s_{n_k} \to L = \liminf s_n$.
(b) Let $\{s_{n_k}\}$ be a convergent subsequence of $\{s_n\}$ that converges to L, then lim sup of the subsequence exists and is equal to L. We look at the lim sup of these two. We have

$$\begin{aligned}\limsup s_{n_k} &= \lim s_{n_k} = L = \limsup\{S_{n_k} | n \geq N\}\\ \limsup s_n &= \limsup\{S_n | n \geq N\}\end{aligned}$$

and since $\{S_{nk}\} \subseteq \{S_n\}$, we have $\sup\{S_{nk}\} \leq \sup\{S_n\}$ by a theorem in class, and thus $L = \limsup s_{n_k} \leq \limsup s_n$. The proof for $\liminf s_n \leq L$ is similar.

4 Q4

Recall that two metrics d, d' on a set E are equivalent if there are $c_1, c_2 > 0$ so that $c_1 d(x, y) \leq d'(x, y) \leq c_2 d(x, y)$ for all $x, y \in E$ (see lecture 8).
(a) Prove that the relation of being equivalent is in fact an equivalence relation on the set of all metrics on the set E.
(b) Prove that if d, d' are two equivalent metrics on a set E then $C \subset E$ is bounded with respect to d if and only if it is bounded with respect to d'.
(c) Prove that if d, d' are two equivalent metrics on a set E then a sequence $\{s_n\}$ in E is Cauchy with respect to d if and only if it's Cauchy with respect to d'.
Proof:
(a) We prove reflexivity, symmetry, and transitivity.
(i) Reflexivity. Take $c_1 = c_2 = 1$, then clearly $c_1 d(x, y) \leq d(x, y) \leq c_2 d(x, y)$ for all $x, y \in E$, and thus $d \sim d$.
(ii) Symmetry. If $d \sim d'$, then $\exists c_1, c_2 > 0$ s.t. $c_1 d(x, y) \leq d'(x, y) \leq c_2 d(x, y)$ for all $x, y \in E$. Take $c'_1 = \frac{1}{c_2}, c'_2 = \frac{1}{c_2}$, then we have $d(x, y) \leq c'_2 d'(x, y)$ and $c'_1 d(x, y) \leq d(x, y)$, which combines to $c'_1 d'(x, y) \leq d(x, y) \leq c'_2 d'(x, y)$ for all $x, y \in E$.

(iii) Transitivity. If $d \sim d'$ and $d' \sim d''$, then $\exists c_1, c_2, c_3, c_4 > 0$ s.t. $c_1 d(x,y) \leq d'(x,y) \leq c_2 d(x,y)$ and $c_3 d'(x,y) \leq d''(x,y) \leq c_4 d'(x,y)$ for all $x, y \in E$. Combining these inequalities gives $c_1 d(x,y) + c_3 d'(x,y) \leq d'(x,y) + d''(x,y)(1)$ and $d'(x,y) + d''(x,y) \leq c_2 d(x,y) + c_4 d'(x,y)$ (2). For (1), simplifying gives $c_1 d(x,y) + (c_3 - 1)d'(x,y) \leq d''(x,y)$, so we choose c_1' s.t $c_1' d(x,y) \leq c_1 d(x,y) + (c_3 - 1)d'(x,y) \leq d''(x,y)$. For (2), similarly, we choose c_2' s.t. $d''(x,y) \leq c_2 d(x,y) + (c_4 - 1)d'(x,y) \leq c_2' d(x,y)$. Combining these two gives $c_1' d(x,y) \leq d''(x,y) \leq c_2' d(x,y)$

(b) We prove the forward direction since the converse is similar. If $C \subset E$ is bounded with respect to d, then $\exists x \in E, r > 0$, s.t. $C \subseteq B_r^d(x)$. Since d, d' are two equivalent metrics, $\exists c_1, c_2 > 0$ s.t. $c_1 d'(x,y) \leq d(x,y) \leq c_2 d'(x,y)$ for all $x, y \in E$. Now consider $B_{r'}^{d'}(x)$, where $r' = c_2 r$. Choose $z \in E$ s.t. $d'(x,z) = r'$. Let $a \in B_r^d(x)$, then $d(a,x) < r \leq c_2 r = r' = d'(x,z)$, which suggests that $B_r^d(x) \subseteq B_{r'}^{d'}(x)$, so $C \subseteq B_r^d(x) \subseteq B_{r'}^{d'}(x)$ and thus C is bounded with respect to d'.

(c) We prove the forward direction and the converse is similar. Suppose $\{s_n\} \subseteq E$ is Cauchy with respect to d, then $\forall \varepsilon > 0$, $\exists N$ s.t. $\forall n, m > N$, $d(s_n, s_m) < \varepsilon$. Since d, d' are two equivalence relations, $\exists c_1, c_2 > 0$ s.t. $c_1 d(x,y) \leq d'(x,y) \leq c_2 d(x,y)$ for all $x, y \in E$. Now we choose M s.t. $\forall r, t > M$, $d(s_r, s_t) < \frac{\varepsilon}{c_2}$. Then we have $d'(s_r, s_t) \leq c_2 d(s_r, s_t) < c_2 \cdot \frac{\varepsilon}{c_2} = \varepsilon$, so $\{s_n\}$ is Cauchy with respect to the metric d'.

5 Q5

Let (E, d) be a metric space. Recall that the function $\bar{d} : E \times E \to [0, \infty)$ defined by $\bar{d}(x,y) = \min\{1, d(x,y)\}$. is a metric.

(a) Prove that a sequence $\{s_n\}$ is Cauchy with respect to d if and only if it's Cauchy with respect to $\bar{d}$.

(b) Show that in general, the metrics d and $\bar{d}$ are not equivalent.

(c) Consider $\mathbb{R}$ with the standard metric $d : d(x,y) = |x - y|$. Is it true that every sequence $\{s_n\}$ in $\mathbb{R}$ which is bounded with respect to $\bar{d}$ has a convergent subsequence? Is $(\mathbb{R}, \bar{d})$ complete?

Proof:

(a) Suppose $\{s_n\}$ is Cauchy with respect to d, then $\forall \varepsilon > 0$, $\exists N$ s.t. $\forall n, m > N$, $d(s_n, s_m) < \varepsilon$. Note that we have $\bar{d}(x,y) \leq d(x,y)$ for all $x, y \in E$, so $\bar{d}(s_n, s_m) \leq d(s_n, s_m) < \varepsilon$, and thus $\{s_n\}$ is Cauchy with respect to d'. Conversely, suppose that $\{s_n\}$ is Cauchy with respect to $\bar{d}$, then $\forall \varepsilon > 0$, $\exists N$ s.t. $\forall k, l > N$, $\bar{d}(s_k, s_l) < \varepsilon$. When $\varepsilon > 1$, we have $\bar{d}(s_k, s_l) \leq 1 < \varepsilon$

(b) Note that $\bar{d}$ is bounded by 1. We choose x, y s.t. $d(x,y) = 2$. Now consider the inequality $cd(x,y) \leq \bar{d}(x,y) = 1$. We can choose $c = 1/2$, then $cd(x,y) = 1/2 \cdot 2 = 1 \leq \bar{d}(x,y)$. But for the two metrics to be equivalent, this inequality must hold for all $x, y \in E$ for a constant c, which is impossible since it clearly fails if we pick y, s.t. $d(x,y) = 3$.

(c) If we have a sequence $\{s_n\}$ that is bounded in $\mathbb{R}$ with respect to $\bar{d}$, then it is Cauchy. It is also Cauchy with respect to d and thus bounded with respect to d. Now it will have a convergent subsequence $\{s_{n_k}\}$ by Bolzano-Weierstrass. Since it is induced from $\{s_n\}$, a sequence bounded with respect to $\bar{d}$ indeed has a convergent subsequence $\{s_{n_k}\}$. Now let $\{a_n\}$ be a convergent sequence with respect to $\bar{d}$. Then it is Cauchy and thus Cauchy with respect to d by part (a). Since $(\mathbb{R}, d)$ is complete, all Cauchy sequences with respect to d converges, and so is $\{a_n\}$. Since $\{a_n\}$ is arbitrary, we conclude that $(\mathbb{R}, \bar{d})$ is complete.

6 Q6

Let $\{s_n\}$ be a sequence in $\mathbb{R}^n$ which is bounded with respect to the Euclidean metric d_2 (and hence with respect to d_1 and d_∞ by Q4). Prove that $\{s_n\}$ has a convergent subsequence.

Proof:

This question is to prove the Bolzano-Weierstrass Theorem in $\mathbb{R}^n$ for d_2. Note that being bounded with respect to d_2 is the same as being bounded with respect to d_1 since by Q4. We have shown in class that when $n = 1$, a bounded sequence will have a convergent subsequence. Let $\{x^m\}$ be a bounded sequence in Rn. The sequence $\{x_1^m\}$ of first components of the terms of xm is a bounded real sequence, which has a convergent subsequence $\{x_1^{mk}\}$. Let $\{x^{mk}\}$ be the corresponding subsequence of $\{x^m\}$. Then the sequence $\{x_2^{mk}\}$ of second components of $\{x^{mk}\}$ is a bounded sequence of real numbers, so it too has a convergent subsequence, and we again have a corresponding subsequence of $\{x^{mk}\}$(and therefore of $\{x^m\}$), in which the sequences of first and second components both converge. Continuing for n iterations, we end up with a subsequence $\{z^m\}$ of $\{x^m\}$ in which the sequences of first, second, ..., nth components all converge, and therefore the subsequence $\{z^m\}$ itself converges in $\mathbb{R}^n$.

7 Q7

Let $f : X \to Y$ be a function between two sets, and $d : Y \times Y \to [0, \infty)$ a metric. Prove that

$$d' : X \times X \to [0, \infty), d'(x, y) := (f(x), f(y))$$

is a metric on X if and only if f is injective.

Proof:

If d' is a metric, then $d'(x, y) = d'(y, x) \iff x = y$. Then we have $f(x) = f(y)$ by definition of d', which suggests that f is injective. Conversely, suppose f is injective. We define a function $h : X \times X \to [0, \infty), (x, y) \mapsto (f(x), f(y)) \mapsto [0, \infty)$ by $h(x, y) = d(f(x), f(y))$. Since d is a metric, $d(f(x), f(y)) = 0 \iff f(x) = f(y)$, and by injectivity this suggest that $x = y$. Hence $d'(x, y) = 0 \iff x = y$. Also, $h(x, y) = h(y, x) \equiv d(f(x), f(y)) = d(f(y), f(x))$ when $x \neq y$ since d is a metric, and thus $d'(x, y) = d(f(x), f(y)) = d(f(y), f(x)) = d'(y, x)$. For the Triangle Inequality part, again we have $h(x, z) = d'(x, z) = d(f(x), f(z)) \leq d(f(x), f(y)) + d(f(y), f(z)) = h(x, y) + h(y, z) = d'(x, y) + d'(y, z)$ since d is a metric.