MATH 424 HW1

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Q1

Consider the empty set $\varnothing$ as the subset of the reals $\mathbb{R}$. Give a convincing argument that every $x \in \mathbb{R}$ is an upper bound of $\varnothing$ and a lower bound of $\varnothing$. Does $\varnothing$ have the least upper bound? Explain.
Proof:
Let $x \in \mathbb{R}$ be arbitrary, then $s \leq x$ for all $s \in \varnothing$ since there are no elements in $\varnothing$, the statement is vacuously true. So every $x \in \mathbb{R}$ is an upper bound of $\varnothing$. Similarly, $s \geq x$ for all $s \in \varnothing$ and every $x \in \mathbb{R}$ is an upper bound of $\varnothing$. Suppose by contradiction that $\varnothing$ has a least upper bound, say $a' \in \mathbb{R}$ and let $x' = a' - 1$. We've proven that every element of $\mathbb{R}$ is an upper bound of $\varnothing$, so x' is an upper bound of $\varnothing$. But $x' < a'$, contradicting that a' is the least upper bound, so $\varnothing$ does not have the least upper bound.

Q2

Consider $\mathbb{Z}_3$, the integers modulo 3. It is a field and you don't need to prove this. Prove that $\mathbb{Z}_3$ cannot be an ordered field. Hint: p. 23 of the textbook.
Proof:
Let P be the positive subset of $\mathbb{Z}_3$. We first show that $1, 2$ are positive. $1 = 1 \times 1 = 1^2 \in P$. Also, $2 = 1 + 1 \in P$. Now suppose by contradiction that $\mathbb{Z}_3$ is an ordered field. Then $1, 2 \in P$ by above and $1 + 2 \in P$. But by the definition of $\mathbb{Z}_3$, $1 + 2 = 0 \notin P$, a contradiction, so $\mathbb{Z}_3$ is not an ordered field.

Q3

Let $a, b \in \mathbb{R}$ be two numbers with $a < b < 0$. Prove that

$$\frac{1}{b} < \frac{1}{a}$$

Hint: O8, p. 20.
Proof:
By O5 we know that the product of two negative numbers is positive, so we

have $ab > 0$, hence $(ab)^{-1} > 0$ by O7. Multiplying $a < b$ both sides by $(ab)^{-1}$ gives $(ab)^{-1}a < (ab)^{-1}b$, which simplifies to $\frac{1}{b} < \frac{1}{a}$.

Q4

Prove that for any $a, b \in \mathbb{R}$ $\max\{a, b\} = \frac{1}{2}(a + b + |b - a|)$.
Proof:
There are two cases. Either $a \geq b$ or $a < b$. If $a \geq b$, then $max\{a, b\} = a$ and $|b - a| = a - b$. Thus we have $\frac{1}{2}(a + b + |b - a|) = \frac{1}{2}(a + b + a - b) = \frac{1}{2}2a = a = max\{a, b\}$. If $a < b$, then $max\{a, b\} = b$ and $|b - a| = b - a$. So $\frac{1}{2}(a + b + |b - a|) = \frac{1}{2}(a + b + b - a) = \frac{1}{2}2b = b = max\{a, b\}$. In both cases, we've shown that $max\{a, b\} = \frac{1}{2}(a + b + |b - a|)$.

Q5

Give a careful proof that $1 = \sup([0,\ 1))$.
Proof:
It is obvious that 1 is an upper bound of 1 since for all $x \in [0, 1)$, $x \leq 1$. Now we show that any $a < 1$ cannot be an upper bound of [0,1). Let $x = \frac{1-a}{2} + a$, clearly $x > a$ because $\frac{1-a}{2} > 0$. But note that also $x < 1$. To see this, suppose by contradiction that $x \geq 1$, then $2x = 2 \cdot (\frac{1-a}{2} + a) = (1 - a) + 2a = 1 + a \geq 2$, which gives $a \geq 1$, a contradiction. Hence there is $x \in [0, 1)$ such that $x > a$, so a is not an upper bound of $[0, 1)$ and 1 is the supremum of $[0, 1)$.

Q6

Consider a function $d : \mathbb{R}^2 \times \mathbb{R}^2 \mapsto [0, \infty)$ defined by

$$d((x_1, x_2), (y_1, y_2)) := max\{|x_1 - y_1|, |x_2 - y_2|\}$$

Prove that the triangle inequality holds for d: for all $x = (x_1, x_2)$, $y = (y_1, y_2)$, $z = (z_1, z_2) \in \mathbb{R}^2, d(x, y) \leq d(x, z) + d(z, y)$.
Proof:
$\max\{|x_1 - y_1|, |x_2 - y_2|\} \leq d(x, y) + d(y, z) \iff |x_1 - z_1| \leq d(x, y) + d(y, z)$ and $|x_2 - z_2| \leq d(x, y) + d(y, z)$. But by the Triangle Inequality, $|x_1 - z_1| = |x_1 - y_1 + y_1 - z_1| \leq |x_1 - y_1| + |y_1 - z_1| \leq max\{|x_1 - y_1|, |x_2 - y_2|\} + max\{|y_1 - z_1|, |y_2 - z_2|\} = d(x, y) + d(y, z)$. Similarly, $|x_2 - z_2| = |x_2 - y_2 + y_2 - z_2| \leq |x_2 - y_2| + |y_2 - z_2| \leq d(x, y) + d(y, z)$, which ends our proof.