MATH 424 HW5 Dilys Wu

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February 28, 2024

1 Q1

Prove that there is no surjective continuous function from [0, 1] to $\mathbb{R}$.
Hint: there is a short solution.
Proof:
Suppose by contradiction that such a function $f : [0, 1] \to \mathbb{R}$ exists. Since $[0, 1]$ is closed and bounded, it is compact. Since f is continuous, its image $f[0, 1]$ is compact. Since f is subjective, $f[0, 1] = \mathbb{R}$ is compact, which is a contradiction.

2 Q2

Let $(S_1, d_1), (S_2, d_2)$ be two metric spaces. Prove that the function

$$d : (S_1 \times S_2) \times (S_1 \times S_2) \to [0, \infty),\ d((x_1, y_1), (x_2, y_2)) := d_1(x_1, x_2) + d_2(y_1, y_2)$$

for all $(x_1, y_1), (x_2, y_2) \in S_1 \times S_2$, is a metric on the product $S_1 \times S_2$.
Here d_1, d_2 may have nothing to do with the norms on $\mathbb{R}^n$; 1 and 2 are just indices.
Proof:
(1) $d((x_1, y_1), (x_1, y_1)) = d_1(x_1, x_1) + d_2(y_1, y_1) = 0 + 0 = 0$ since d_1, d_2 are metrics. Conversely, if $d((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2) = 0$, then $d_1(x_1, x_2) = d_2(y_1, y_2) = 0$ since $d_1(x, y), d_2(x, y) \geq 0$. Thus $x_1 = x_2, y_1 = y_2$, which gives $(x_1, y_1) = (x_2, y_2)$, i.e. if $d((x_1, y_1), (x_2, y_2)) = 0$ then $(x_1, y_1) = (x_2, y_2)$.
(2) Since d_1, d_2 are metrics, $d((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2) = d_1(x_2, x_1) + d_2(y_2, y_1) = d((x_2, y_2), (x_1, y_1))$.
(3) $d((x_1, y_1), (x_3, y_3)) = d_1(x_1, x_3) + d_2(y_1, y_3) \leq (d_1(x_1, x_2) + d_1(x_2, x_3)) + d_2(y_1, y_3) \leq (d_1(x_1, x_2) + d_1(x_2, x_3)) + (d_2(y_1, y_2) + d_2(y_2, y_3)) = (d_1(x_1, x_2) + d_2(y_1, y_2)) + (d_1(x_2, x_3) + d_2(y_2, y_3)) = d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3))$

3 Q3

Let $(S_1, d_1), (S_2, d_2)$ be two metric spaces and let d be the metric on the product $S_1 \times S_2$ constructed in problem 2 above.

(a) Prove that the function $p : S_1 \times S_2 \to S_1$ given by $p(x, y) = x$ is continuous.
(b) Is p uniformly continuous? Explain.
Proof:
(a) Let $(x, y), (x', y') \in S_1 \times S_2$. Take $\delta = \varepsilon$. Suppose that $d((x, y), (x', y')) = d((x, x'), (y, y')) < \delta$, then $d_1(x, x') < \delta = \varepsilon$. Since the points are arbitrary, p is continuous.
(b) It is uniformly continuous since $\forall\varepsilon$, we can always choose $\delta = \varepsilon$ s.t. if $d((x, y), (x', y') < \delta$, then $d'((x, y), (x', y')) = d_1(x, y) < \delta = \varepsilon$.

4 Q4

Let (E, d) be a metric space. Then by problem 2 the product $E \times E$ is also a metric space with the metric d given by $d((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + d(y_1, y_2)$.
(a) Prove that the function $d : (E \times E, d) \to [0, \infty)$ is continuous.
Hint: you need to show that for any $(x_0, y_0) \in E \times E$, any $\varepsilon > 0$, there is $\delta > 0$ so that if $d((x_0, y_0), (x, y)) < \delta$ then $|d(x_0, y_0) - d(x, y)| < \varepsilon$.
(b) Is the function d uniformly continuous? Explain.
Proof:
(a) Let $(x_0, y_0), (x, y) \in E, \varepsilon > 0$ be arbitrary. We choose $\delta = \varepsilon$. Suppose that $d((x_0, y_0), (x, y)) = d(x_0, x) + d(y_0, y) < \delta = \varepsilon$, then $|d(x_0, y_0) - d(x, y)| = |d(x_0, x) + d(y_0, y)| < \varepsilon$.
(b) Yes. Since $\forall\varepsilon$, we can always choose $\delta = \varepsilon$ s.t. if $d((x_0, y_0), (x, y)) < \delta$ then $|d(x_0, y_0) - d(x, y)| = |d(x_0, x) + d(y_0, y)| < \varepsilon$ by (a).

5 Q5

(a) Prove that the function $f : \mathbb{R}^2 \to \mathbb{R}, f(x, y) = x + y$ is continuous.
(b) Let E be a metric space, $g, h : E \to \mathbb{R}$ two continuous functions. Use part (a) to give another proof that $g + h : E \to \mathbb{R}$ is continuous.
Proof:
(a) Let $(x, y) \in \mathbb{R}^2$, we construct two convergent sequences $\{x_n\}, \{y_n\}$ with $x_n \to x, y_n \to y$, then we have $x_n + y_n \to x + y$. Now $f(x_n, y_n) \to x + y = f(x, y)$, and thus it is pointwise continuous at (x, y). Since (x, y) is arbitrary, f is continuous.
(b) Consider the function $p : E \to \mathbb{R}p(x) = f(g(x), (h(x)) = g(x) + h(x)$. Since f is continuous by (a), g, h are continuous, $p(x) = f(g(x) + h(x)) = g(x) + h(x)$ is continuous.

6 Q6

Let $(E, d), (E', d')$ be two metric spaces and $\{f_n : E \to E'\}_{n\in\mathbb{N}}$ a sequence of functions converging uniformly to a function f. Prove that if f_n's are all bounded then so is f.

Proof:
Suppose that f_n's are bounded, then $\exists M$ s.t. $\forall n, f_n \leq M$. Since f_n's converge uniformly to f, $\exists N$ s.t. $\forall \varepsilon > 0$, $d(f_n(x), f(x)) < \varepsilon$ for $n > N$. We now want to show that $\exists M'$ s.t. $|f(x)| \leq < M'$. To see this, let $\varepsilon = 1$, then for $n > N$,

$$
\begin{aligned}
d(f_n(x), f(x)) &< 1 \\
|f(x)| < 1 + f_n(x) &\leq 1 + M
\end{aligned}
$$

So we choose any $M' \geq 1 + M$ and that it will be a bound of $f(x)$, since for $n > N$, $|f(x)| < 1 + M \leq M'$.

7 Q7

Let $(E, d), (E, d')$ be two metric spaces, $f, g : E \to E'$ two bounded functions.
(a) Prove that the set $\{d'(f(x), g(x)) | x \in E\}$ is bounded in E'.
(b) As in lecture 14, let $C(E, E')$ denote the set of all bounded continuous functions from E to E'. Prove that the function

$$D : C(E, E') \times C(E, E') \to [0, \infty), \;\; D(f, g) := \sup\{d'(f(x), g(x)) | x \in E\}$$

is a metric.
Proof:
(a) Since f, g is bounded, $d'(f(x), f_0) \leq M_f, d'(g(x), g_0) \leq M_g$ for some M_f, M_g. We aim to show that there exists a real number M such that for all $x \in E$, $(f(x), g(x)) \leq M$. Given that f and g are bounded, we use the triangle inequality property of the metric space (E', d'). For any $x \in E$, we have $d'(f(x), g(x)) \leq d'(f(x), f_0) + d'(f_0, g_0) + d'(g_0, g(x))$. The term (f_0, g_0) is a fixed distance between two points in E' and hence is a constant, say K. Hence for all $x \in E$, $d'(f(x), g(x)) \leq M_f + K + M_g$, and thus the set is bounded.
(b)
(i) $D(f, f) = \sup\{d'((f(x), f(x)) | x \in E\} = 0$ since d' is a metric. Note that if $\sup\{d'(f(x), g(x)) | x \in E\} = 0$ then $f = g$ again since d' is a metric.
(ii) $D(f, g) = \sup\{d'((f(x), g(x)) | x \in E\} = \sup\{d'((g(x), f(x)) | x \in E\} = D(g, f)$
(iii) $D(f, h) = \sup\{d'(f(x), h(x)) | x \in E\}$. Note that we have $d'(f(x), h(x)) \leq d'(f(x), g(x)) + d'(g(x), h(x))$, and thus $\sup d'(f(x), h(x)) \leq \sup d'(f(x), g(x)) + \sup d'(g(x), h(x))$, where $x \in E$. Therefore, we have $D(f, h) = \sup\{d'(f(x), h(x)) | x \in E\} \leq \sup\{d'(f(x), g(x)) | x \in E\} + \sup\{d'(g(x), h(x)) | x \in E\} = D(f, g) + D(g, h)$
Hence, D is a metric.

8 *Q8

Let $(X, \mathcal{T})$ be a topological space and $A \subset X$ a subset.
(a) Check that the collection of sets $\mathcal{T}_A := \{U \cap A | U \in T\}$ is a topology on A. $\mathcal{T}_A$ is called the subspace topology.

(b) Let (E, d) be a metric space, $\mathcal{T}_d$ the corresponding topology and $A \subset E$ a subset. We then have two ways to define a topology on A: (1) we can give A the subspace topology. (2) Alternatively the metric d on E gives us a metric d_A on A given by the same formula: $d_A(x, y) = d(x, y)$ for all $x, y \in A$. Consequently, we get a topology $\mathcal{T}_{d_A}$ on A. Show that these two topologies ($\mathcal{T}_{d_A}$ and $(\mathcal{T}_d)_A$) are the same.

Proof:

(a) It contains $\varnothing$ and A since $\varnothing = \varnothing \cap \mathcal{T}$ and $A = A \cap X$, and $\varnothing, X$ are elements of $\mathcal{T}_A$. Suppose $U_1, U_2, \cdots \subseteq \mathcal{T}$ are open sets. Notice that $(U_1 \cap Y) \cap (U)2 \cap Y) \cap \cdots \cap (U_n \cap Y) = (U_1 \cap U_2 \cap \cdots \cap U_n) \cap Y \subseteq \mathcal{T}_Y$. For arbitrary intersections, we have $\cup_{\alpha \in A}(U_\alpha \cap Y) = (\cup_{\alpha \in A} U_\alpha) \cap Y \subseteq \mathcal{T}_Y$.

(b) Suppose $U \subseteq \mathcal{T}_{d_A}, U$ open, we will show that $U \subseteq (\mathcal{T}_d)_A$. And then we show the converse. Since U is open, $\forall x \in U, \exists \varepsilon > 0$ s.t. $B_\varepsilon^{d_A}(x) \subseteq U$. Since $d_A = d$, it is also an open ball in $(\mathcal{T}_d)_A$ intersecting A. Hence, $B_\varepsilon^{d_A}(x) = B_\varepsilon^{d}(x) \cap A$, and we can write U as a union of intersections of open sets in E with A, hence U is open in the subspace topology $(\mathcal{T}_d)_A$. Conversely, suppose that $U \subseteq (\mathcal{T}_d)_A$. Then there exists an open set $U = V \cap A$. For all $x \in V, \exists \varepsilon > 0$ s.t. , $B_\varepsilon^{d}(x) \subseteq V$. Since $U = V \cap A$, we have $B_\varepsilon^{d_A}(x) = B_\varepsilon^{d}(x) \cap A \subseteq U$. Therefore, U is open in $(\mathcal{T}_d)_A$.