Homework #4, Math 424, Spring 2024 Prof. Eugene Lerman
typeset February 8, 2024

Start reading Chapter 4 of Rosenlicht. Problems:

1 Let $\{s_n\}$ be a bounded sequence of real numbers. Prove that $\limsup |s_n| = 0$ if and only if $\{s_n\}$ converges to 0.

2 Suppose $\{z_n\}$, $\{w_n\}$ are two convergent sequences of complex numbers (the corresponding metric is $d(z,w) = |z - w|$). Prove that if $z_n \to L$ and $w_n \to M$ then $z_n w_n \to LM$.

Hint: the fact that convergent sequences are bounded may be useful.

3 Let $|| \cdot ||$ be a norm on $\mathbb{R}^n$ and $d(v,w) := ||v - w||$ the corresponding metric. Prove that if the sequences $\{v_n\}$, $\{w_n\}$ in $\mathbb{R}^n$ converge (with respect to d) then their sum $\{v_n + w_n\}$ converges as well and that

$$\lim(v_n + w_n) = \lim v_n + \lim w_n.$$

4 Denote the set of all bounded sequences in $\mathbb{R}$ by ℓ^∞. Then for any sequence $\{a_n\} \in \ell^\infty$

$$||\{a_n\}||_\infty := \sup\{a_n \mid n \in \mathbb{N}\}$$

is a well-defined nonnegative real number. Prove that the function $|| \cdot ||_\infty : \ell^\infty \to [0, \infty)$ is a norm. That is, prove that

(1) $||\{a_n\}||_\infty = 0$ if and only if $\{a_n\}$ is the zero sequence.

(2) $||\{ca_n\}||_\infty = |c| \, ||\{a_n\}||_\infty$ for all real numbers c and all sequences $\{a_n\} \in \ell^\infty$.

(3) $||\{a_n\} + \{b_n\}||_\infty \leq ||\{a_n\}||_\infty + ||\{b_n\}||_\infty$ for all sequences $\{a_n\}, \{b_n\} \in \ell^\infty$.

5 Let $\{s_n\}$, $\{t_n\}$ be two bounded sequences of real numbers.

(a) Prove that their sum $\{s_n + t_n\}$ is also bounded and that

$$\limsup(s_n + t_n) \leq \limsup s_n + \limsup t_n.$$

(b) Give an example of sequences $\{s_n\}$, $\{t_n\}$ with $\limsup(s_n + t_n) < \limsup s_n + \limsup t_n$.

6 Let $(X, \mathcal{T})$ be a topological space, $K_1, K_2 \subset X$ two compact subsets. Prove that their union $K_1 \cup K_2$ is also compact.

7 A topological space $(X, \mathcal{T})$ is Hausdorff if for any two points $x, y \in X$ with $x \neq y$ there are open sets U, V with $x \in U$, $y \in V$ and $V \cap U = \varnothing$. Prove that if the topology $\mathcal{T}$ comes from/ is defined by a metric d then $(X, \mathcal{T})$ is Hausdorff.

Hint: open balls are open sets.

The following problem will not be graded.

8 Prove that a compact set in a *Hausdorff* topological space is closed. Give an example to show that the condition of being Hausdorff is necessary (hint: it was briefly discussed in class, but not in so many words).