**Homework #1**, Math 424, Spring 2024 Prof. Eugene Lerman
typeset January 17, 2024

Read Chapter 2 of Rosenlicht. Problems:

**1** Consider the empty set $\varnothing$ as the subset of the reals $\mathbb{R}$. Give a convincing argument that every $x \in \mathbb{R}$ is an upper bound of $\varnothing$ and a lower bound of $\varnothing$. Does $\varnothing$ have the least upper bound? Explain.

**2** Consider $\mathbb{Z}_3$, the integers modulo 3. It is a field and you don't need to prove this. Prove that $\mathbb{Z}_3$ cannot be an ordered field.
Hint: p. 23 of the textbook.

**3** Let $a, b \in \mathbb{R}$ be two numbers with $a < b < 0$. Prove that

$$\frac{1}{b} < \frac{1}{a}.$$

Hint: O8, p. 20.

**4** Prove that for any $a, b \in \mathbb{R}$

$$\max\{a, b\} = \frac{1}{2}(a + b + |b - a|).$$

**5** Give a careful proof that $1 = \sup([0, 1))$.

**6** Consider a function $d : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ defined by

$$d((x_1, x_2), (y_1, y_2)) := \max\{|x_1 - y_1|, |x_2 - y_2|\}$$

Prove that the triangle inequality holds for $d$: for all $x = (x_1, x_2)$, $y = (y_1, y_2)$, $z = (z_1, z_2)$ in $\mathbb{R}^2$,

$$d(x, y) \leq d(x, z) + d(z, y).$$