MATH 424 HW7 Dilys Wu

Dilys

March 21, 2024

1 Q1

Let $f : (0, \infty) \to \mathbb{R}$ be a function. Suppose $\lim_{x\to 0} f(x)$ exists and equals L. Let $g(u) = f(1/u)$. Prove that $\lim_{u\to\infty} g(u)$ exists and equals L as well. That is, prove that given $\varepsilon > 0$ there is $M \in \mathbb{R}$ so that $|g(u) - L| < \varepsilon$ for all $u > M$.
Proof:
When $u \to \infty, 1/u \to 0$. Then $\lim_{u\to\infty} g(u) = \lim_{v\to 0} f(v) = \lim_{v\to 0} \frac{f(v)-f(0)}{v-0} = L$, and since f is differentiable at 0, $\lim_{u\to\infty} g(u)$ exists and equal L.

2 Q2

(a) Suppose $f, g : (a, b) \to \mathbb{R}$ are two functions, $c \in (a, b)$, the limits $\lim_{x\to c} f(x)$, $\lim_{x\to c} g(x)$ both exist and equal L. Define $h : (a, b) \to \mathbb{R}$ by

$$h(x) := \begin{cases} f(x) & x \text{ is rational} \\ g(x) & x \text{ is irrational} \end{cases}$$

Prove that $\lim_{x\to c} h(x)$ exists and equals L.
(b) Assume further that $f(c) = g(c)$, that f, g are differentiable at c and that $f'(c) = g'(c)$. Prove that h is differentiable at c as well.
Proof:
(a) Since $\lim_{x\to c} f(x)$, $\lim_{x\to c} g(x)$ both exist and equal L, for all $\varepsilon > 0$, we have $0 < |x - c| < \delta_f \implies |f(x) - L| < \varepsilon_f$ and $0 < |x - c| < \delta_g \implies |f(x) - L| < \varepsilon_g$. Let $\delta = \min\{\delta_f, \delta_g\}$, now when x is rational, $h(x) = f(x)$, and since $0 < |x - c| < \delta$, $|f(x) - L| = |h(x) - L| < \varepsilon$. When x is irrational, $h(x) = g(x)$, and since $0 < |x - c| < \delta$, $|f(x) - L| = |g(x) - L| < \varepsilon$. In both case, we've shown that $\lim_{x\to c} f(x)$ exists and equals L.
(b) Since $f'(c) = g'(c) = k$, $\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = k$ when x is rational and $\lim_{x\to c} \frac{g(x)-g(c)}{x-c} = k$ when x is irrational. When x is rational, $h(x) = f(x)$, and thus $\lim_{x\to c} \frac{h(x)-h(c)}{x-c} = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = k$. When x is irrational, $h(x) = g(x)$, and thus $\lim_{x\to c} \frac{h(x)-h(c)}{x-c} = \lim_{x\to c} \frac{g(x)-g(c)}{x-c} = k$. Therefore $\lim_{x\to c} \frac{h(x)-h(c)}{x-c}$ exists and thus h is differentiable at c.

3 Q3

Consider

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Prove that f is differentiable at zero. Feel free to use problem 1 and l'Hopital's rule, if needed.

Proof:

Since $\lim_{x\to 0^+} e^{-\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{1}{e^{1/x^2}} = 0$, $f(x)$ is continuous at 0. $f'(0) = \lim_{x\to 0^+} \frac{e^{-1/x^2}-0}{x} = \lim_{x\to 0^+} \frac{e^{-1/x^2}-0}{x} = \lim_{x\to 0^+} \frac{1}{xe^{1/x^2}} = \lim_{y\to+\infty} \frac{y}{e^{y^2}} = \lim_{y\to+\infty} \frac{1}{2ye^{y^2}}$(L'Hopital) $= 0$. Similarly, $\lim_{x\to 0^-} \frac{e^{-1/x^2}-0}{x} = \lim_{x\to 0^-} \frac{e^{-1/x^2}-0}{x} = \lim_{x\to 0^-} \frac{1}{xe^{1/x^2}} = \lim_{y\to-\infty} \frac{y}{e^{y^2}} = \lim_{y\to-\infty} \frac{1}{2ye^{y^2}}$(L'Hopital) $= 0$. Hence f is differentiable at 0 and $f'(0) = 0$.

4 Q4

Prove directly from the definition of Darboux integral given in lecture 21 that the function

$$f(x) = \begin{cases} 1 & x \text{ is rational} \\ 0 & x \text{ is irrational} \end{cases}$$

is not integrable on the interval $[0, 1]$.

Proof:

For all $S \subseteq [0,1], M(f,S) = 1, m(f,S) = 0 \implies \forall P, L(f,P) = 0, U(f,P) = 1$. $\implies L(f) = \sup\{L(f,P)|P \text{ a partition}\} = 0$. $U(f) = \sup\{U(f,P)|P \text{ a partition}\} = 1$, i.e., $U(f) \neq L(f)$. Hence f is not (Darboux) integrable.

5 Q5

Prove directly from the definition of the Darboux integral (lecture 21) that the function $f : [0,b] \to \mathbb{R}, f(x) = x^2$ is integrable.

Proof:

$[0,b](b > 0), P = \{0 = t_0 < t_1 < \cdots < t_n = b\}$. $U(f,P) = \sum_{k=1}^n M(f,[t_{k-1},t_k])\cdot(t_k, t_{k-1}) = \sum_{k=1}^n t_k^2(t_k - t_{k-1})$. Similarly, $L(f,P) = \sum_{k=1}^n t_{k-1}^2(t_k - t_{k-1})$. In particular let P_n be the partition with $t_k = \frac{kb}{n}$, Then $U(f,P_n) = \sum_{k=1}^n (\frac{kb}{n})^2 \cdot (\frac{kb}{n} - \frac{(k-1)b}{n}) = \sum_{k=1}^n (\frac{b}{n})^3 \cdot k^2 = (\frac{b}{n})^3 \cdot \frac{n(n+1)(2n+1)}{6}$ Similarly, $L(f,P_n) = \sum_{k=1}^n (\frac{b}{n})^3 \cdot (k-1)^2 = (\frac{b}{n})^3 \cdot \frac{n(n-1)(2n-1)}{6}$. Since $U(f) = \inf\{U,(f,P)\} \leq U(f,P) = (\frac{b}{n})^3 \cdot \frac{n(n+1)(2n+1)}{6}$, $U(f) \leq \lim_{n\to\infty} (\frac{b}{n})^3 \cdot \frac{n(n+1)(2n+1)}{6} = \frac{b^3}{6} \cdot 2 = \frac{b^3}{3}$. Similarly, $L(f) = \sup\{L,(f,P)\} \geq \lim_{n\to\infty} (\frac{b}{n})^3 \cdot \frac{n(n-1)(2n-1)}{6} = \frac{b^3}{6} \cdot 2 = \frac{b^3}{3}$. Now we have $\frac{b^3}{3} \leq L(f) \leq U(f) \leq \frac{b^3}{3}$, so $L(f) = U(f) = \frac{b^3}{3}$ and f is integrable.

6 Q6

Let $f : [0, b] \to \mathbb{R}$ be a function that is identically 0 everywhere except at the points $x_1, ..., x_n \in [a, b]$ $(n \geq 1)$. Prove directly from the definition that f is integrable on $[a, b]$.

Proof:

Let $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ be a partition of $[a, b]$, $t_k := a + \frac{b-a}{n} \cdot k$. Consider one point $x_i \in [t_{k-1}, t_k]$ with $f(x_i) = c$. When $c < 0$ it is obvious that $U(f, P) = 0 = L(f, P)$, and thus $L(f) = \sup(L, (f, P)) = 0 = U(f) = \inf(U, (f, P))$. Now if $c > 0$, $U(f, P) = c \cdot (t_k - t_{k-1}) = \frac{c}{n}(b-a)$. When we make $n \to \infty$, we have $U(f, P) = 0 = U(f)$. $L(f, P) = 0$. $L(f) = \sup(L, (f, P)) = 0, U(f) = \inf(U, (f, P))$. Hence, $L(f) = U(f) = 0$ and thus f is integrable.

7 Q7

Let X be a metric space. Define a relation $\sim$ on X by $x \sim y$ if and only if there is a continuous map (a path) $\gamma : [0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. Prove that $\sim$ is an equivalence relation. The equivalence classes of $\sim$ are called path components of X.

Hint: problem 2 from homework 6 may be useful for a proof of transitivity of the relation $\sim$. The fact that $f : [0, 1] \to [0, 1], f(x) = 1 - x$ is continuous may be useful for a proof of symmetry of $\sim$.

Proof:

(Reflexivity) Consider the map $\gamma : [0, 1] \to X, \gamma(t) = x, \forall t \in [0, 1]$. Note that it is continuous since the preimage of any open sets of X is either $[0, 1]$ or $\varnothing$, both of which are open, and thus the map is continuous.

(Symmetry) Suppose $x \sim y$, we want to show that $y \sim x$. Since $x \sim y$, $\exists \gamma(t)$ with $\gamma(0) = x$ and $\gamma(1) = y$. Now consider $\gamma'(t) = \gamma(1 - t)$ for $t \in [0, 1]$, and it is continuous by the hint. Now we have $\gamma'(0) = \gamma(1 - 0) = \gamma(1) = y, \gamma'(1) = \gamma(1 - 1) = \gamma(0) = x$. Therefore we find a continuous map with $\gamma'(0) = y, \gamma'(1) = x$, thus $y \sim x$.

(Transitivity) Suppose $x \sim y, y \sim z$, we want to show that $x \sim z$. Since $x \sim y, y \sim z$, $\exists \gamma_1 : [0, 1] \to X, \gamma_2 : [0, 1] \to X$ continuous satisfying the requirements of $\sim$. Now define $\gamma : [0, 2] \to X$ by $\gamma(t) = \begin{cases} \gamma_1(2t), & t \in [0, \frac{1}{2}] \\ \gamma_2(2t - 1), & t \in [\frac{1}{2}, 1] \end{cases}$ and by HW 6 Q2 $\gamma(t)$ is continuous. Note that $\gamma(0) = \gamma_1(0) = x$, $\gamma(1) = \gamma_2(2 \cdot 1 - 1) = \gamma_2(1) = z$, thus $x \sim z$.