**Homework #9**, Math 424, Spring 2024 Prof. Eugene Lerman
typeset March 28, 2024

Please report any typos/issues on Campuswire.

**1** Let $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$; it is a real valued function on $\mathbb{R}^2 \smallsetminus \{(0,0)\}$. Consider the limits $\lim_{x\to 0}\lim_{y\to 0} f(x,y)$, $\lim_{y\to 0}\lim_{x\to 0} f(x,y)$ and $\lim_{(x,y)\to(0,0)} f(x,y)$. Compute these limits if they exist.

**2** Find a sequence $\{h_n : \mathbb{R} \to \mathbb{R}\}_{n\in\mathbb{N}}$ of continuous functions so that $\lim_{x\to 0}\lim_{n\to\infty} h_n(x)$ and $\lim_{n\to 0}\lim_{x\to\infty} h_n(x)$ exist and are unequal.

Hint: find a function $f : \mathbb{R}^2 \smallsetminus \{(0,0)\} \to \mathbb{R}$ so that $\lim_{x\to 0}\lim_{y\to 0} f(x,y) \neq \lim_{y\to 0}\lim_{x\to 0} f(x,y)$.

**3** Find a sequence of continuous functions $f_n : [0,1] \to \mathbb{R}$ that converge to the zero function so that the sequence $a_n = \int_{[0,1]} f_n$ diverges to $+\infty$ as $n \to \infty$, i.e., $\lim_{n\to\infty} a_n = +\infty$.

Hint: p. 139 of the textbook.

**4** Let $m$ be a positive integer, $\{a_n\}_{n\geq 0}$ a sequence of real numbers. Prove that the series $\sum_{n=0}^{\infty} a_n$ converges if and only if $\sum_{n=0}^{\infty} a_{n+m}$ converges, and that in this case

$$\sum_{n=0}^{\infty} a_n = a_0 + \cdots + a_{m-1} + \sum_{n=0}^{\infty} a_{n+m}.$$

**5** Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous, $h : \mathbb{R} \to \mathbb{R}$ differentiable. Consider

$$G(x) := \int_0^{h(x)} f(t)\, dt.$$

Explain why $G$ is differentiable and find its derivative in terms of $f$, $h$ and $h'$.

**6** Suppose $f : [0,\infty) \to (0,\infty)$ is a decreasing function. Prove that $\sum_{n=0}^{\infty} f(n)$ converges if and only if the limit $\lim_{n\to\infty} \int_{[0,n]} f$ exists.

**7** Consider the sequence of functions $f_n(x) := \frac{1}{\sqrt{n}} \sin nx$ on the interval $[0, 2\pi]$. Prove that the sequence $\{f_n\}$ converges *uniformly* to the zero functions. Does the sequence of derivatives $\{f_n'\}$ converge? Prove your answer.