Homework #5, Math 424, Spring 2024 Prof. Eugene Lerman
typeset February 15, 2024

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Finish reading Chapter 4 of Rosenlicht. Problems:

1 Prove that there is no surjective continuous function from $[0,1]$ to $\mathbb{R}$.
Hint: there is a short solution.

2 Let (S_1, d_1), (S_2, d_2) be two metric spaces. Prove that the function

$$\mathbf{d} : (S_1 \times S_2) \times (S_1 \times S_2) \to [0, \infty), \qquad \mathbf{d}((x_1, y_1), (x_2, y_2)) := d_1(x_1, x_2) + d_2(y_1, y_2)$$

for all $(x_1, y_1), (x_2, y_2) \in S_1 \times S_2$, is a metric on the product $S_1 \times S_2$.
Here d_1, d_2 may have nothing to do with the norms on $\mathbb{R}^n$; 1 and 2 are just indices.

3 Let (S_1, d_1), (S_2, d_2) be two metric spaces and let $\mathbf{d}$ be the metric on the product $S_1 \times S_2$ constructed in problem **2** above.

(a) Prove that the function $p : S_1 \times S_2 \to S_1$ given by

$$p(x, y) = x$$

is continuous.

(b) Is p uniformly continuous? Explain.

4 Let (E, d) be a metric space. Then by problem **2** the product $E \times E$ is also a metric space with the metric $\mathbf{d}$ given by

$$\mathbf{d}((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + d(y_1, y_2).$$

(a) Prove that the function $d : (E \times E, \mathbf{d}) \to [0, \infty)$ is continuous.
Hint: you need to show that for any $(x_0, y_0) \in E \times E$, any $\varepsilon > 0$ there is $\delta > 0$ so that if $\mathbf{d}((x_0, y_0), (x, y)) < \delta$ then $|d(x_0, y_0) - d(x, y)| < \epsilon$.

(b) Is the function $\mathbf{d}$ uniformly continuous? Explain.

5(a) Prove that the function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x + y$ is continuous.

(b) Let E be a metric space, $g, h : E \to \mathbb{R}$ two continuous functions. Use part **(a)** to give another proof that $g + h : E \to \mathbb{R}$ is continuous.

6 Let (E, d), (E', d') be two metric spaces and $\{f_n : E \to E'\}_{n \in \mathbb{N}}$ a sequence of functions converging uniformly to a function f. Prove that if f_n's are all bounded then so is f.

7 Let (E, d), (E', d') be two metric spaces, $f, g : E \to E'$ two bounded functions.

(a) Prove that the set $\{d'(f(x), g(x)) \mid x \in E\}$ is bounded in E'.

(b) As in lecture 14, let $C(E, E')$ denote the set of all bounded continuous functions from E to E'. Prove that the function

$$D : C(E, E') \times C(E, E') \to [0, \infty), \qquad D(f, g) := \sup\{d'(f(x), g(x)) \mid x \in E\}$$

is a metric.

The following problem will not be graded.

8 Let $(X, \mathcal{T})$ be a topological space and $A \subset X$ a subset.

(a) Check that the collection of sets $\mathcal{T}_A := \{U \cap A \mid U \in \mathcal{T}\}$ is a topology on A. $\mathcal{T}_A$ is called the subspace topology.

(b) Let (E, d) be a metric space, $\mathcal{T}_d$ the corresponding topology and $A \subset E$ a subset. We then have two ways to define a topology on A: (1) we can give A the subspace topology. (2) Alternatively the metric d on E gives us a metric d_A on A given by the same formula: $d_A(x, y) = d(x, y)$ for all $x, y \in A$. Consequently we get a topology $\mathcal{T}_{d_A}$ on A. Show that these two topologies ($\mathcal{T}_{d_A}$ and $(\mathcal{T}_d)_A$) are the same.