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SCALE AND CONFORMAL INVARIANCE IN QUANTUM FIELD THEORY

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We study the relation between invariances under rigid and local changes of length scale. In two dimensions, we complete an argument of Zamolodchikov showing that the rigid invariance implies the local under broad conditions. In three or more dimensions we are unable to find either a general proof or a counterexample, but we find some new conformally invariant systems.

The invariance of a quantum field theory under a local change of length scale has a long history in high energy physics [1–4], general relativity [5], statistical mechanics [6–9] and string theory [10, 11]. In this paper we investigate the relation between this invariance and the invariance under a global change of scale. In particular we are interested in the conditions under which the global invariance implies the local one. Our main results are as follows. In three or more dimensions, a careful study of the literature shows that there is no known general argument to show that the global invariance implies the local. We are not able to find a general argument or a counterexample, but we do find some new nontrivial conformally invariant systems. In two dimensions we complete an argument of Zamolodchikov which shows that global scale invariance does imply local invariance under broad conditions. We also reconcile this with a counterexample due to Hull and Townsend.

We consider a Poincaré invariant quantum field theory in D flat dimensions of either euclidean or Minkowski signature. The extension to curved spacetime will be discussed later. The two transformation of interest are:

(a) scale transformations: $\delta x^{\mu} = \varepsilon x^{\mu}$;

(b) conformal transformations: $\delta x^{\mu} = \varepsilon v^{\mu}(x)$, such that

$$\partial_{\mu} v_{\nu}(x) + \partial_{\nu} v_{\mu}(x) = \frac{2}{D} g_{\mu\nu} \partial \cdot v(x) . \tag{1}$$

The condition (1) on the vector field $v^{\mu}(x)$ implies that the transformation is locally

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an isotropic change of scale, preserving angles. For $D \geqslant 3$, the conformal transformations include (beyond the assumed Poincaré invariance) the rigid scale transformation plus D special conformal transformations, and $\partial \cdot v(x)$ spans all linear plus constant functions of x^μ. For $D = 2$ there are an infinite number of special conformal transformations, and $\partial \cdot v(x)$ spans all harmonic functions of x^μ. In $D = 2$ one is sometimes interested in the smaller algebra of

(c) Möbius transformations: all conformal transformations in $D = 2$ such that $\partial \cdot v(x)$ is at most linear in x^μ.

A scale current must be of the form [1]

$$S^\mu(x) = x^\nu T_\nu{}^\mu(x) + K^\mu(x), \tag{2}$$

where $T_{\mu\nu}$ is the symmetric stress energy tensor and K^μ is a local operator without explicit dependence on the coordinate. The first term in (2) follows from the spacetime nature of the scale transformation. The second is an additional internal transformation: K^μ contributes to the scaling dimensions of fields. Conservation of S^μ implies

$$T_\mu{}^\mu(x) = -\partial_\mu K^\mu(x). \tag{Ca}$$

It might seem that we have more freedom in the choice of S^μ, since there are in general many symmetric conserved stress tensors giving rise to the same hamiltonian. However, if $T_{\mu\nu}$ and $T'_{\mu\nu}$ are two such tensors, then their difference is of the form

$$T'_{\mu\nu}(x) - T_{\mu\nu}(x) = \partial^\sigma \partial^\rho Y_{\mu\sigma\nu\rho}(x) \tag{3}$$

for some local operator $Y_{\mu\sigma\nu\rho}$ which is antisymmetric on $\mu\sigma$ and on $\nu\rho$, and symmetric under exchange of $\mu\sigma$ with $\nu\rho$. Eq. (3) may be understood by noting that it implies an additional coupling of the form $R^{\mu\sigma\nu\rho}Y_{\mu\sigma\nu\rho}$ when the system is coupled to gravity. If $T_{\mu\nu}$ satisfies (Ca), then so does $T'_{\mu\nu}$ with K^μ replaced by $K^\mu - \partial^\rho Y^{\mu\nu}{}_{\rho\nu}$. Thus, given any stress tensor, the necessary and sufficient condition for existence of a conserved scale current is (Ca), that its trace be the divergence of a local operator.

A conformal current must be of the form [1]

$$j_v^\mu(x) = v^\nu(x) T_\nu{}^\mu(x) + \partial \cdot v(x) K'^\mu(x) + \partial_\nu \partial \cdot v(x) L^{\nu\mu}(x), \tag{4}$$

where K'^μ is the same as K^μ up to the possible addition of a conserved current, corresponding to an ambiguity in the original choice of scale current, and $L^{\nu\mu}$ is some local operator. The form (4) can be found by trial and error or by the following general reasoning. The first term is again determined by the spacetime nature of the transformation, while the second appears because the conformal transformation is locally a scale transformation with scale factor $\varepsilon \partial \cdot v$. The third term is an additional correction due to the position dependence of the scale factor $\partial \cdot v$; it is not useful to include higher derivatives of v^μ. In $D \geqslant 3$, recalling that $\partial \cdot v$ is a general linear function of x^μ, conservation of (4) is equivalent to $T_\mu{}^\mu(x) =$

$-\partial_\mu K'^\mu(x)$ plus $K'^\mu(x) = -\partial_\nu L^{\nu\mu}(x)$. In $D=2$, $\partial\cdot v$ is a general harmonic function and conservation implies the additional relation $L^{\nu\mu}(x) = g^{\nu\mu}L(x)$ for some local operator $L(x)$.

These conditions combine to give

(Cb)
$$D \geqslant 3: \quad T_\mu^{\ \mu}(x) = \partial_\nu\partial_\mu L^{\nu\mu}(x),$$
$$D = 2: \quad T_\mu^{\ \mu}(x) = \partial^2 L(x).$$

Again, the form of (Cb) is independent of the particular choice of $T_{\mu\nu}$, and so the necessary and sufficient condition for the existence of conserved conformal currents is that the trace of the stress tensor have the form (Cb). Similarly, the necessary and sufficient condition for Mőbius invariance is

(Cc)
$$D = 2: \quad T_\mu^{\ \mu}(x) = \partial_\nu\partial_\mu L^{\nu\mu}(x).$$

If condition (Cb) holds for $T_{\mu\nu}$, then the equivalent stress tensor

$$\begin{aligned}\Theta_{\mu\nu}(x) &= T_{\mu\nu}(x)\\ &+ \frac{1}{D-2}\left(\partial_\mu\partial_\alpha L^\alpha{}_\nu(x) + \partial_\nu\partial_\alpha L^\alpha{}_\mu(x) - \partial^2 L_{\mu\nu}(x) - g_{\mu\nu}\,\partial_\alpha\partial_\beta L^{\alpha\beta}(x)\right)\\ &+ \frac{1}{(D-2)(D-1)}\left(g_{\mu\nu}\,\partial^2 L_\alpha^{\ \alpha}(x) - \partial_\mu\partial_\nu L_\alpha^{\ \alpha}(x)\right) \qquad \text{in } D\geqslant 3 \end{aligned} \tag{5}$$

or

$$\Theta_{\mu\nu}(x) = T_{\mu\nu}(x) + \frac{1}{D-1}\left(\partial_\mu\partial_\nu L(x) - g_{\mu\nu}\,\partial^2 L(x)\right) \qquad \text{in } D=2 \tag{6}$$

is traceless:

(Cb′)
$$\Theta_\mu^{\ \mu}(x) = 0.$$

Thus conformal invariance is equivalent to the existence of a traceless stress tensor, although an arbitrarily chosen stress tensor will have a trace of the form (Cb).

We see that a system will be scale invariant without being conformally invariant if the trace of the stress tensor is the divergence of a local operator $-K^\mu$ which is not itself a conserved current plus a divergence (in $D \geqslant 3$) or a gradient (in $D=2$). This immediately leads to an obvious condition under which scale invariance will imply conformal invariance, namely if there is no suitable candidate for K^μ. For example in four-dimensional ϕ^4 theory, in perturbation theory, the only vector of the correct dimensions is $\partial_\mu(\phi^2)$, so that (Ca) implies (Cb). Thus, the nontrivial fixed point in $4-\varepsilon$ dimensions is automatically conformally invariant. The same applies to $(\phi^3)_{6-\varepsilon}$ and $(\phi^6)_{3-\varepsilon}$. In abelian or non-abelian gauge theories coupled only to fermions, using the BRST invariance of the stress tensor, the only candidate is

$A_\mu \, \partial_\nu A^\nu + \alpha \bar{c} D_\mu c$ (where α is the gauge parameter), which is BRST trivial. Hence the perturbative fixed point which exists in SU(n_c) when $0 < 1 - 2n_f/11n_c \ll 1$ gives a conformally invariant theory. In addition, many statistical mechanical systems have only a small number of low dimension operators, and no candidate for K^μ.

In light of the above discussion, let us now review the literature on this subject. It is in fact very difficult to find systems which are scale invariant and not conformally invariant, so that many recent papers have asserted that the former implies the latter. One simple argument which is frequently given is that both invariances are equivalent to the tracelessness of the stress tensor. As we have seen, this is not correct. A second argument is that since a conformal transformation is locally a scale transformation, and the lagrangian is local, scale invariance is equivalent to conformal invariance. However, this ignores the effect of derivatives in the lagrangian; the same argument would imply that any global symmetry was also a local symmetry.

Since these trivial arguments are not adequate, we turn to more detailed studies of the question. Polyakov [6] and Migdal [7] showed that the Schwinger-Dyson equations satisfied by a statistical system at a critical point were consistent with a conformally invariant solution. This, however, gives no information about the existence of solutions which are not conformally invariant*. Wolsky, Green, and Gunton argued that a fluid is conformally invariant at its critical point [12,13]. The critical behavior of a fluid is described by a single scalar field, and there is no candidate for K^μ. The most complete treatment of conformal invariance in statistical mechanics is by Schäfer [14], who generalized Wilson's renormalization group from scale to conformal transformations. He finds that the conditions for conformal invariance are stronger than for scale invariance, parallel to our discussion of the stress tensor, but does not discuss any particular systems.

Turning to high energy physics, it is shown in refs. [2–4] that any renormalizable four dimensional field theory which is classically scale invariant is also classically conformally invariant. Of course, quantum effects alter both symmetries. Conformal invariance in perturbation theory was studied in refs. [15–22]. However, these papers consider only theories of a single scalar field, or gauge theories coupled to fermions. As we have noted, these theories have no nontrivial candidate for K^μ, and this plays an essential role in the arguments of [15–22]**.

Leaving aside for now arguments and examples which are particular to $D = 2$, the above discussion of the literature appears to be essentially complete. We see that

* We should note that in this context, different solutions in general correspond to different Lagrangians, not just different vacua.

** To be precise, refs. [18,22] report evidence of violation of conformal invariance. In [18], we believe that this is due to the incorrect substitution of an unintegrated for an integrated operator identity. In ref. [22] the result depends on the choice of regulator, suggesting an internal inconsistency. Our analysis shows that a consistent treatment would find the Ward identities of conformal invariance to be satisfied.

there is no evidence that a theory with a nontrivial candidate for K^μ will be conformally invariant. Let us consider the simplest such theory, namely two or more scalar fields with a quadratic interaction in $D = 4 - \varepsilon$. (We will be interested in $\varepsilon > 0$ so as to obtain a perturbative fixed point.) The action is

$$\int \mathrm{d}^d x \, \tfrac{1}{2} \, \partial^\mu \phi_i \, \partial_\mu \phi_i + \frac{1}{4!} \lambda_{ijkl} \phi_i \phi_j \phi_k \phi_l \tag{7}$$

(summation on $ijkl$ implied).

The candidate for K^μ is

$$K_\mu = \phi_i \, \partial_\mu \phi_j \, \zeta_{ij} \,, \tag{8}$$

with ζ_{ij} an arbitrary antisymmetric matrix.

At one loop, the trace of the stress tensor is

$$T^\mu_\mu = \frac{1}{4!} \left(-\varepsilon \lambda_{ijkl} + \frac{3}{16\pi^2} \lambda_{ijmn} \lambda_{mnkl} \right) \phi_i \phi_j \phi_k \phi_l \,. \tag{9}$$

To lowest order,

$$\partial_\mu K^\mu = \frac{1}{3!} \zeta_{im} \lambda_{mjkl} \phi_i \phi_j \phi_k \phi_l \,. \tag{10}$$

The conditions for scale and conformal invariance are respectively

$$-\varepsilon \lambda_{ijkl} + \frac{1}{16\pi^2} \left(\lambda_{ijmn} \lambda_{mnkl} + 2\text{perms} \right) + \left(\zeta_{im} \lambda_{mjkl} + 3\text{perms} \right) = 0 \,, \tag{11}$$

$$-\varepsilon \lambda_{ijkl} + \frac{1}{16\pi^2} \left(\lambda_{ijmn} \lambda_{mnkl} + 2\text{perms} \right) = 0 \,. \tag{12}$$

The scale invariance condition (11) is in principle weaker, involving the additional unknowns ζ_{ij}. Remarkably, however, eq. (11) implies eq. (12), as is seen by contracting (11) with ($\zeta_{im} \lambda_{mjkl} + 3$ permutations). Thus, the order ε fixed points of the multiscalar theory are all conformally invariant, at least to one loop order. Here is an example of a system with a nontrivial candidate for K^μ but which is conformally invariant. A parallel result holds for $(\phi_i^3)_{6-\varepsilon}$ and $(\phi_i^6)_{3-\varepsilon}$. We do not know if this conformal invariance is accidental, or if it indicates the existence of a more general relation between scale and conformal invariance. We will return to this briefly after discussing the case of $D = 2$.

In $D \geqslant 3$, we have been unable to find either a counterexample to the relation between scale and conformal invariance, or a general proof. In $D = 2$, we have the opposite embarrassment, in that both a proof and a counterexample appear to exist.

We give first the proof, which is based on an argument by Zamolodchikov [23]. Consider the euclidean two-point function of the stress tensor $T_{\mu\nu}$, working in complex coordinates $z = x^1 + ix^2$:

$$C(x^2) = 2z^4\langle T_{zz}(x)T_{zz}(0)\rangle ,$$

$$E(x^2) = x^2z^2\langle T_{zz}(x)T_{z\bar{z}}(0)\rangle ,$$

$$F(x^2) = x^4\langle T_{z\bar{z}}(x)T_{z\bar{z}}(0)\rangle . \tag{13}$$

From conservation of $T_{\mu\nu}$, one obtains

$$F(x^2) = -\tfrac{1}{12}x^2\frac{\partial}{\partial x^2}\{C(x^2) - 4E(x^2) - 6F(x^2)\} . \tag{14}$$

If at a fixed point the stress tensor scales by its canonical dimension, then $C(x^2)$, $E(x^2)$ and $F(x^2)$ will be independent of x^2. In more detail, if the commutator of the scale generator S with the stress tensor takes its canonical form, which in D dimensions is

$$i[S, T_{\mu\nu}(x)] = x^\omega \partial_\omega T_{\mu\nu}(x) + DT_{\mu\nu}(x), \tag{15}$$

and if $\dot{S} = 0$ and $S|0\rangle = 0$, then $\partial_{x^2}C = \partial_{x^2}E = \partial_{x^2}F = 0$. From eq. (14), the two point correlation of $T^\mu_\mu = T_{z\bar{z}}$ is then zero, and positivity (unitarity) implies $T^\mu_\mu = 0$ as an operator equation.

This argument is not yet complete, as the canonical commutator (15) is not in general correct. The general form of the commutator of the scale generator with the stress tensor is

$$i[S, T_{\mu\nu}(x)] = x^\omega \partial_\omega T_{\mu\nu}(x) + DT_{\mu\nu}(x) + \partial^\sigma\partial^\rho\tilde{Y}_{\mu\sigma\nu\rho}(x), \tag{16}$$

where $\tilde{Y}_{\mu\sigma\nu\rho}$ has the same symmetries as $Y_{\mu\sigma\nu\rho}$ in (3). The first term on the right in (16) is again fixed by the spacetime nature of the scale transformation. The rest is fixed by requiring that the right side, like the left, be symmetric and conserved, and by the integrated relation $i[S, H] = H$. The extra term in (16) corresponds to the fact that the stress tensor generally requires a logarithmic additive renormalization of the form (3), and so has off-diagonal mixing under the renormalization group [3, 24, 25]. We need to use the freedom to redefine $T_{\mu\nu}$ so as to bring the commutator to the form (15). Let $Y^a_{\mu\sigma\nu\rho}$ be a complete set of operators of the appropriate symmetry, and let $\tilde{Y} = y^aY^a$. The general commutator is

$$i[S, Y^a_{\mu\sigma\nu\rho}(x)] = x^\omega \partial_\omega Y^a_{\mu\sigma\nu\rho} + \gamma^{ab}Y^b_{\mu\sigma\nu\rho} . \tag{17}$$

Because $\tilde{Y}$ is differentiated in (16), it suffices to consider the reduced matrix $\hat{\gamma}^{ab}$, in which we omit the row and column associated with the constant operator $Y^0_{\mu\sigma\nu\rho} = g_{\mu\nu}g_{\sigma\rho} - g_{\mu\rho}g_{\nu\sigma}$. Then

$$\Theta'_{\mu\nu}(x) = T_{\mu\nu}(x) + y^a(D-2+\hat{\gamma})^{-1}_{ab}\,\partial^\sigma\partial^\rho Y^b_{\mu\sigma\nu\rho}(x) \tag{18}$$

has the desired commutator (15), provided the matrix inverse in (18) exists. The physical requirement that all correlations fall with distance (except for the constant operator which we have omitted) implies that $\hat{\gamma}$ is positive definite. Provided that the spectrum of $\hat{\gamma}$ is discrete, which is the case in compact σ-models and in all statistical systems, $\Theta'_{\mu\nu}(x)$ always exists in $D=2$, completing Zamolodchikov's argument. When the spectrum of $\hat{\gamma}$ goes continuously down to zero, there may be complications, as we discuss later*.

We have thus found that under very broad conditions, namely unitarity plus a discrete spectrum of operator dimensions, scale invariance implies conformal invariance in $D=2$. We consider now the counterexample of Hull and Townsend [26–29], which is the $D=2$ nonlinear σ-model. Taking for simplicity the case of vanishing torsion, the action is

$$\int \mathrm{d}^2x\, G_{ij}(\phi)\,\partial_z\phi^i\,\partial_{\bar{z}}\phi^j. \tag{19}$$

The one-loop trace of the stress tensor is

$$T_{z\bar{z}} = -\frac{1}{4\pi}R_{ij}\,\partial_z\phi^i\,\partial_{\bar{z}}\phi^j. \tag{20}$$

The candidate for K_μ is $v_i(\phi)\,\partial_\mu\phi^i/4\pi$, with v^i an arbitrary vector field. The scale invariance condition (Ca) then becomes at one loop

$$R_{ij} = \nabla_i v_j + \nabla_j v_i, \tag{21}$$

which defines the quasi-riemannian manifolds introduced by Friedan [30]. The stronger condition of conformal invariance, (Cb), implies

$$R_{ij} = \nabla_i\nabla_j\Phi \tag{22}$$

for some scalar field Φ.

The general proof we have given implies that any compact riemannian solution of (21) must in fact be of the more special form (22). A simple argument due to

* In the literature, an "improved" stress tensor is one whose trace vanishes in a conformally invariant theory [3], as in (5), (6). A distinct notion of improvement would be a stress tensor with canonical scaling properties [25], eq. (15). Zamolodchikov's argument shows that these are equivalent in $D=2$ unitary theories with a discrete spectrum of $\hat{\gamma}$. More generally, they may not coincide.

Bourguinion [31] shows this to be the case directly (with $\Phi = 0$). The double divergence of (21), using the Bianchi identity, gives

$$\nabla^2 R + v \cdot \nabla R = -2 R_{ij} R^{ij} . \tag{23}$$

Also, the integral over the σ-model target manifold of the trace of (21) gives

$$\int_{\mathrm{M}} \sqrt{g} R = 0 . \tag{24}$$

Consider the point p of M at which R takes its global minimum. At p, the left side of (23) is non-negative, and the right is non-positive. Hence both vanish, and $R_{ij}(p) = R(p) = 0$. But then $R \geqslant 0$ on all of M, and by (24) $R = 0$ everywhere. Finally (23) then implies $R_{ij} = 0$. Although this explicit argument is only for the one loop equations with vanishing torsion, the conformal invariance proof implies the nonexistence of compact riemannian solutions to all orders and with torsion.

With a target manifold of pseudo-riemannian signature, positivity is absent and the proof does not go through. Indeed, linearizing around Minkowski space, eq. (21) possesses plane wave solutions which are not of the form (22). Eq. (22) describes a propagating gravitational field and scalar, as expected in string theory, while (21) describes an additional propagating vector.

In the case of noncompact target manifolds of riemannian signature, the resolution of the counterexample and the proof is not completely clear. The nonexistence argument below eq. (24) used compactness in an essential way, and we expect that noncompact solutions do exist and that the proof breaks down. The first place to check is the existence of $\Theta'_{\mu\nu}$. The general $Y_{\mu\sigma\nu\rho}$ is of the form $(\varepsilon_{\mu\sigma}\varepsilon_{\nu\rho} - \varepsilon_{\mu\rho}\varepsilon_{\nu\sigma}) f(\phi)$, with $f(\phi)$ a scalar function. The one-loop anomalous dimensions is $\gamma = -\nabla_i \nabla^i / 4\pi$. On a compact manifold, $\hat{\gamma}$, which acts on the space orthogonal to the constant function, is indeed invertible. In the noncompact case, the invertibility depends on the boundary conditions. As long as γ is invertible for some boundary condition, as we would expect to be the case, then $\Theta'_{\mu\nu}$ can be constructed. We believe then that the breakdown of the proof occurs not here but elsewhere: Namely, the two point functions used in eq. (13) do not exist. Recall that due to infrared fluctuations, the fields in a $D = 2$ nonlinear sigma model spread over the whole target manifold, the familiar example being the absence of the Goldstone phenomenon in $D = 2$ [32, 33]. Any expectation value will be of the form $\int_{\mathrm{M}} f / \int_{\mathrm{M}} 1$, where f is some function of the geometry, possibly local on M. On a compact manifold this is well defined. On a noncompact but homogeneous manifold, the expectation value of any invariant will be defined, as $\int_{\mathrm{M}} 1$ cancels between numerator and denominator. On a noncompact inhomogenous manifold, however, in general no expectation value will be defined, even of innocuous quantities like the stress tensor. We believe this is where the proof fails, in the implicit assumption that these exist. (Note that this does not imply the nonexistence of the theory: such quantities as f are well defined.) Thus, the essential ingredients of the proof that scale invariance implies conformal invariance in $D=2$ are unitarity and the existence of $\Theta'_{\mu\nu}$ and of its two point function; we expect these always to exist when the spectrum of γ is discrete.

The $D=2$ proof does not generalize to $D \geqslant 3$: conservation of $T_{\mu\nu}$ does not seem to give sufficient information. Our failure to find a counterexample suggests that a proof may exist. It is likely that unitarity again plays a key role, as non-unitary counterexamples do exist: massless vector mesons without gauge invariance [3,4], and solutions to eq. (11) with complex couplings.

We now consider field theories in curved spacetime. We assume general coordinate invariance. It is convenient to take the length transformations to act on the metric. The global and local symmetries are:

(d) scale transformations: $\delta g_{\mu\nu} = -2\varepsilon g_{\mu\nu}$;

(e) Weyl transformations: $\delta g_{\mu\nu} = -2\varepsilon\phi(x) g_{\mu\nu}$ for arbitrary $\phi(x)$.

It might also be of interest to consider intermediate symmetries corresponding to locally or globally defined conformal Killing transformations. From the definitions, the symmetries considered are nested so that some automatically imply others:

$$\begin{tikzcd} & e \arrow[dl] \arrow[r] & b \arrow[d] \\ d \arrow[dr] & & (c) \arrow[dl] \\ & a & \end{tikzcd} \tag{25}$$

In addition, we have discussed conditions under which $a \to b$. We now consider the relation between the flat and curved spacetime symmetries in $D=2$. (We leave $D \geqslant 3$ to the reader.) The condition for Weyl invariance is $T_\mu{}^\mu = 0$. In two dimensional curved spacetime, dimensional analysis gives the general form

$$T_\mu{}^\mu(x) = T_{f\mu}{}^\mu(x) + \frac{1}{24\pi}\Gamma(x)R(x), \tag{26}$$

where $T_{f\mu\nu}$ is the flat space stress tensor, $R(x)$ is the curvature scalar, and $\Gamma(x)$ is an operator. Given (Cb), flat space conformal invariance, we have seen that it is possible to find a stress tensor $\Theta_{\mu\nu}$ which is traceless in flat spacetime, (Cb′). Thus, for the particular choice of curvature coupling corresponding to $T_{f\mu\nu} = \Theta_{\mu\nu}$, we have $T_{f\mu}{}^\mu = 0$. Further, from the conservation of the two point function of $T_{zz}(x)$ one finds $\Gamma(x) = $ c-number $\equiv$ central charge. Thus $T_\mu{}^\mu(x)$ is a c-number times the curvature. In expectation values of operators, this is an overall normalization which drops out, so b $\to$ e in expectation values *always*. (For b $\to$ e in the partition function the extra condition of vanishing central charge is needed [34].) We see that in $D=2$ our entire diagram (25) of logically distinct symmetries collapses to a single point, provided the conditions for a $\to$ b are met.

To summarize, the relation between global and local scale invariance is not as trivial as is sometimes assumed. We believe the situation in $D=2$ is now largely resolved. In $D \geqslant 3$ it would be desirable to find either a counterexample or a general proof.

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Note added in proof

(1) Several additional examples of four-dimensional classical field theories which are scale invariant but not conformally invariant have been brought to my attention. These are the fourth derivative scalar theory with lagrangian $(\Box\varnothing)^2/2$, generic fourth derivative gravitational theories, and the lagrangian [35] $\phi^4 f(\partial_\mu\phi\,\partial^\mu\phi/\phi^4)$ for f more general than constant plus linear. These examples are all consistent with the extension of the two-dimensional theorem to four dimensions, since the fourth derivative theories are not unitary, while in the last example, even leaving aside the nonrenormalizability, the scale invariance is spontaneously broken. So we still lack either a proof or a counterexample. We should point out that in the case of a curved background, the issue is whether, given a conformally invariant flat-space theory, there exists any choice of curvature coupling which preserves the conformal invariance. Of course, generic choices of curvature coupling do spoil the conformal invariance. I thank S. Deser, R. Jackiw, and M. Duff for communications on these points.

(2) Regarding references, I should emphasize that I have cited only papers which deal with the relation between scale and conformal invariance, not the much larger literature on consequences of conformal invariance. Additional references which should be included are: Jackiw [35], who carries out (for $d = 4$) the same analysis of the conditions on $T_{\mu\nu}$ as we have given; Ferrara et al. [36], who study the first solvable but intrinsically quantum-mechanical example of conformal invariance (the Thirring model); and Capper and Duff [37], who first discussed conformal anomalies in general relativity (more recent references appear in ref. [5]). In addition, Tseytlin [38] has also discussed the dependence of Zamolodchikov's argument on the choice of flat-space stress tensor (I thank S.P. de Alwis for explaining ref. [38] to me).

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