Lecture 3

Lecturer: Giovanni Neglia *Scribe: Ioannis Mantas, Stefano Ponziani*

NOTE: The content of these notes has not been formally reviewed by the lecturer. It is recommended that they are read critically.

1 Introduction

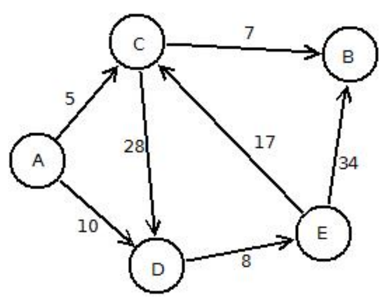
a simple

In the previous lesson, we looked into the example of an electrical network and showed that ~~the optimal~~ local policy of an agent, in our case an electron, can lead to the optimization of a large distributed problem which is the minimization of energy loss in the network. In this lesson, we consider ourselves with a road traffic model to demonstrate that choosing local policies is not trivial and how potential wrong choices can lead to poor system behaviour. Moreover we solve the more general problem which arises from our road network and present how local policies in such cases should be correctly chosen.

2 Problem Definition

Let us look into the following road-traffic model.

What do the number on the edges represent?



moreover, we denote by R the set of all the routes

From now on we can consider our problem as a routing problem. We have a set of directed links E representing the connections between the nodes and a set of *source-destination* pairs S where pair (A, B) rersents the agents wanting to travel from node A to node B.
For each $s \in S$ we define $R(s)$ to be the set of routes connecting the *source-destination* pair and $f(s)$ to be the amount of traffic on s. Since for a given s traffic from a source to a destination can be split into multiple routes, we define x_r to be the amount of traffic of s going through route r. Finally, we define y_l to be the amount of traffic on link l over all routes, i.e. $y_l = \sum_{r: l \in r} x_r$
On each link $l \in E$ there exists a delay which is a function of the traffic on that link, y_l, and is denoted $D_l(y_l)$. The delay function is assumed to be convex, increasing and its first derivative to exist.

Thus the problem consists of minimizing the overall delay experienced by the agents.
express the constraints
To formulate our problem ~~with a linear model~~ we introduce the following two matrices. Let A be the link-route incidence matrix where $A_{lr} = 1$ if link l belongs to route r and $A_{lr} = 0$ otherwise. Secondly, let H be the source-destination route incidence matrix where $H_{sr} = 1$ if route r connects s and $H_{sr} = 0$ otherwise. Note that the summation of each collumn of H is equal to 1, that means each route connects exactly one pair.

The model is not linear because the delays are general functions

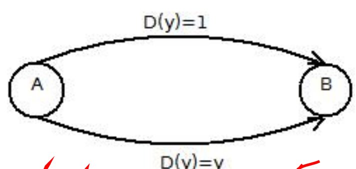
The optimization problem can be modelled as follows:

$$
\begin{aligned}
\underset{\mathbf{x},\mathbf{y}}{\text{minimize}} \quad & \sum_{l \in E} y_l D_l(y_l) \\
\text{subject to} \quad & y_l = \sum_{r | l \in r} x_r, \ \forall l \in E \\
& f_s = \sum_{r | s(r) = s} x_r, \ \forall s \in S \\
& x_r \geq 0, \ \forall r \in R\cancel{(s), \forall s \in S}
\end{aligned} \tag{1}
$$

where $s(r)$ returns the source-destination pair that the route r connects.

3 A first approach to the solution

In ~~the need~~ order to minimize the delay experience d ~~in total~~ by all drivers, we need to find the local policies that will be followed. In the previous lesson we observed how a simple local policy can lead to the optimal solution. In this section we illustrate, with a simple example, how a local policy which may seem logical and trivial may lead to a "bad" solution.



In our example, we have a flow of 1 car per time unit between ~~a number of car drivers who want to go from~~ ~~the~~ city A to ~~the~~ city B and they can choose between ~~from~~ two available routes. The upper route has a constant delay function of $D_{upper} = 1$ and the lower has a linear delay function which is equal to the percentage of drivers that take that route, $D_{lower} = y$.

The most natural approach would be for each driver to choose the route that has the lowest delay and in that way we ~~would~~ expect all the drivers to face equal delay. why would?
More formally, in the general case, we expect that $\forall r_1, r_2$ such that $s(r_1) = s(r_2), x_{r_1} > 0$ and $x_{r_2} > 0$ the delay will be $\sum_{l \in r_1} D(y_l) = \sum_{l \in r_2} D(y_l)$.

Getting back to the example, the overall delay experienced by drivers will be:

$$(1 - \alpha) \times D_{upper}(1 - \alpha) + \alpha \times D_{lower}(\alpha) = (1 - \alpha) \times 1 + \alpha \times \alpha \tag{2}$$

where α is the percentage of drivers that choose the lower route.
According to the aforementioned approach all drivers should choose the lower route as D_{lower} would be lower than D_{upper}. In this way α will be 1 and the total delay will be 1. But is this the optimal solution? Minimizing the equation (2) results to $\alpha^* = \frac{1}{2}$ and an overall experienced delay of $\frac{3}{4} = 0.75$.

Problem (1)

4 Solving by the Lagrange multipliers method

optimization problem IT'S NOT LINEAR

We can observe that the objective function of our linear program (1) is convex. Indeed:

$$f(x) = y_l D(y_l) \Rightarrow f'(x) = y_l D'(y_l) + D(y_l) \Rightarrow f''(x) = y_l D''(y_l) + D'(y_l) + D'(y_l) \tag{3}$$

$D''(y_l)$ and $D'(y_l)$ are ~~greater or~~ equal to zero, for the hypotesis that $D(y_l)$ is convex. So every terms in $f''(x)$ is ~~positive~~ non negative and thus the objective function is convex.
Another observation that we make is that each constraint is linear and that they define a set which is compact. Indeed, each variable x_r is bounded:

$$0 \leq x_r \leq \max_{s \in S} f_s \forall r \in R\cancel{(s)\ and\ \forall s \in S} \tag{4}$$

so the set ~~set~~ defined is closed and bounded.

Considering the previous observations we can apply ~~the *lagrange multipliers method* to apply~~ theorem (1) from the lesson 2. We define two set of multipliers:

1. one for each source-destination pair, λ_s, $\forall s \in S$ and
2. one for each link, μ_l, $\forall l \in E$.

We can claim that $z^* = \binom{x^*}{y^*}$ is a global minimum for this optimization problem if and only if there exist λ_s^*, $\forall s \in S$ and μ_l^*, $\forall l \in E$ such that:

1. $\mathbf{x}^*$ and $\mathbf{y}^*$ are feasible,
2. $\nabla_{\mathbf{x}} L(\cancel{z}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)^T (\cancel{z} - \cancel{z}^*) \geq 0, \ \ \forall z \in \{\binom{x}{y},\ x \geq 0\}$.

The lagrangian function results:

$$L(z, \lambda, \mu) = \sum_{l \in E} y_l D(y_l) + \sum s \in S \lambda_s \left(f_s - \sum_{r:s(r)=s} x_r\right) + \sum_{l \in E} \mu_l \left(\sum_{r:l \in r} x_r - y_l\right) \tag{5}$$

When we differentiate it, we obtain:

$$\frac{\partial L}{\partial y_l} = D(y_l) + y_l D'(y_l) - \mu_l \qquad \qquad \frac{\partial L}{\partial x_r} = -\lambda_{s(r)} + \sum_{l \in r} \mu_l \tag{6}$$

At the optimum, we have:

$$\lambda_s^*(r) \begin{cases} = \sum_{l \in r} \mu_l^* & \text{if } x_r > 0, \\ \leq \sum_{l \in r} \mu_l^* & \text{if } x_r = 0, \end{cases} \tag{7}$$

WHY?

and the quantity μ_l results:

$$\mu_l^* = D(y_l) + y_l D'(y_l) \tag{8}$$

We can interpret the result for the multiplier μ_l like the cost that the drivers ~~of the~~ traversing link l must pay. That cost is composed of the delay (i.e. $D(y_l)$) and a congestion toll to use the link l (i.e. $y_l D'(y_l)$). Furthermore, we can see $\lambda_{s(r)}$ the minimal cost available to source-destination pair $s(r)$.

this is a possible way to change the utility function of the drivers so that they take into account this additional term

With the previous observation, if we add tolls on the link, the drivers are encouraged to have a more desirable behaviour. Indeed, we ~~keep the~~ go back to example in the previous section and we add the toll $y_l D'(y_l)$ at the cost of each link, so $\tilde{D}_{upper} = 1 + y_{upper} D'(y_{upper}) = 1$ and $\tilde{D}_{lower} = y_{lower} + y_{lower} D'(y_l) = 2y_{lower}$. Now the drivers choose the link where the total cost, delay + toll, is minimum and the final solution is that half of the drivers choose the lower link and the others choose the upper link, that is the optimal solution for the formulation (1).

References

[1] Frank Kelly and Elena Yudovina, Stochastic Networks. *Cambridge Press*, 2014.