Algebra
Lecture Notes

Dima Trushin

2023

Contents

1 Sets

In modern math everything can be formulated in terms of Set Theory, that is in terms of sets and maps. Let me remind some basic facts about sets and maps.

1.1 Definition

Definition 1. A set is a collection of elements.

We denote sets by capital letters like X and Y. If an element x belongs to the collection X, we write $x \in X$. If y does not belong to X, we write $y \notin X$. There is a special set containing no elements. This set is called an empty set and is denoted by $\varnothing$.

If you think of a set you should imagine a sack full of elements. The sack is your set and the elements in the sack are the elements belonging to the set. An empty set becomes a sack with no elements inside.

1.2 Constructors

If you are given a definition of a new math object, the first question to ask is: "How do I construct such an object?" To define a set we need to specify the elements inside the set. For doing that, we use the following notation

$$X = \{x \mid \textbf{condition} \text{ on } x\}$$

Here, we mean that the set X consists of all elements x such that the **condition** on x holds. Let me demonstrate this on examples.

- The set of natural numbers

  $$\mathbb{N} = \{x \mid x \text{ is a natural number}\} = \{0, 1, 2, 3, \dots, n, \dots\}$$

- The set of integer numbers

  $$\mathbb{Z} = \{x \mid x \text{ is an integer number}\} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

- The set of real numbers

  $$\mathbb{R} = \{x \mid x \text{ is a real number}\}$$

  We think of the real numbers as a line containing all possible numbers we use in our calculations.

- The closed interval $[0, 1]$

  $$[0, 1] = \{x \in \mathbb{R} \mid 0 \leqslant x \leqslant 1\}$$

  Here, I use a slightly different notation. I specified that $x \in \mathbb{R}$ before $|$, this simply means that x must be a real number and the additional condition (the number is between zero and one) is written after $|$.

1.3 Operations on sets

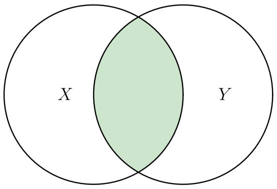
There are several useful procedures you can apply to sets in order to construct new sets. Let us discuss them.

1.3.1 Intersection

If we are given two sets X and Y, then we define the intersection of X and Y as follows

$$X \cap Y = \{z \mid z \in X \text{ and } z \in Y\}$$

If we denote the sets X and Y by discs on a plain then the intersection of X and Y is denoted as below

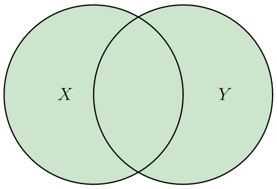


1.3.2 Union

If we are given two sets X and Y, then we define the union of X and Y as follows

$$X \cup Y = \{z \mid z \in X \text{ or } z \in Y\}$$

If we denote the sets X and Y by discs on a plain then the union of X and Y is denoted as below

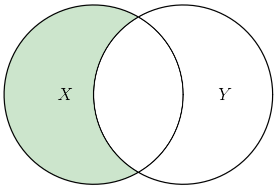


1.3.3 Difference

If we are given two sets X and Y, then we define the difference between X and Y as follows

$$X \setminus Y = \{z \mid z \in X \text{ and } z \notin Y\}$$

If we denote the sets X and Y by discs on a plain then the difference between X and Y is denoted as below



1.3.4 Cartesian product

If we are given two sets X and Y, then their Cartesian product is defined as follows

$$X \times Y = \{(x, y) \mid x \in X,\ y \in Y\}$$

The Cartesian product is simply the set of all possible pairs (x, y) where the first element is taken from X and the second from Y. We will use it every time when we need pairs of elements and not just elements.

1.4 Maps

Definition 2. Suppose we are given two sets X and Y, a map $f\colon X \to Y$ is a rule that takes elements of X to elements of Y. If $x \in X$, then its image in Y is denoted by $f(x)$. In this case, we will write $x \mapsto f(x)$.

The set X is called the source of f and the set Y is called the target of f.

Here is a way to think about maps. Suppose we are given a map $f\colon X \to Y$. Then f is a callable object with an operator $(-)$. You give it any element x of X, then it returns you some specific element $f(x)$ of Y. For each input $x \in X$, the result $f(x) \in Y$ will be the same every time you call it. So, a map is the same thing as a function.

Examples 3. Here are some examples of maps and non maps.

1. The rule $f\colon \mathbb{R} \to \mathbb{R}$ by $x \mapsto 2x + 3$ is a map.

2. The rule $f\colon \mathbb{R} \to \mathbb{R}$ by $x \mapsto \sin(x)$ is a map.

3. The rule $f\colon \mathbb{R} \to \mathbb{R}$ by $x \mapsto \frac{1}{x}$ is not a map because it is not defined at $x = 0$. It becomes a map if we change the source for f. If $\mathbb{R}^* = \{x \in \mathbb{R} \mid x \neq 0\}$, then $f\colon \mathbb{R}^* \to \mathbb{R}$ by $x \mapsto \frac{1}{x}$ is a map.

4. The rule $f\colon \mathbb{R} \to \mathbb{R}$ by $x \mapsto \ln x$ is not a map because it is only defined for positive x. It becomes a map if we change the source for f. If $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$, then $f\colon \mathbb{R}_+ \to \mathbb{R}$ by $x \mapsto \ln x$ is a map.

2 Binary operations

The simplest object in Algebra is a set with a good binary operation. I am going to explain what a binary operation is and the meaning of the word good. Our goal is to define an object called a Group.

2.1 Definition

Definition 4. Suppose X is a set. A binary operation is a map $\circ\colon X \times X \to X$ by the rule $(x, y) \mapsto x \circ y$ for $x, y \in X$.

In this case the notation $\circ$ is the name of the operation. Simply speaking, the operation is a rule that takes two elements of the set X and produce a new element called $x \circ y$ of the same set X. This element $x \circ y$ is usually called the product of x and y.[1]

You should have noticed that we use the name of the operation in a quite unusual way. We write the name between the arguments and not before. This is just for convenience. However, there is a function-like notation (or map-like notation) for binary operations. Let me show you

Definition 5. Suppose X is a set. A binary operation is a map $\mu\colon X \times X \to X$ by the rule $(x, y) \mapsto \mu(x, y)$ for $x, y \in X$.

This is just a different notation for the same mathematical notion. You may denote an operation in operator-like stile (the first definition) or in a function-like style (the second definition). To clarify the situation, let me proceed to a series of examples.

Examples 6. Binary operations:

1. Addition of integral numbers. In an operator-like form

$$+\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \quad (m, n) \mapsto m + n$$

   In a function-like notation

$$\operatorname{add}\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \quad (m, n) \mapsto \operatorname{add}(m, n) = m + n$$

   Since we got used to addition of numbers in a form $m+n$, we want a general definition to be in a similar form. From the other hand, many programming languages allow us using operator-like and function-like notations for addition. Here I want to emphasize that $\operatorname{add}(m, n)$ and $m+n$ are the same things. These are just different notations of the same addition that we use with integers.

2. Integer multiplication. In an operator-like form

$$\cdot\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \quad (m, n) \mapsto m \cdot n$$

   In a function-like notation

$$\operatorname{mult}\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \quad (m, n) \mapsto \operatorname{mult}(m, n) = m \cdot n$$

   Again, these are just two different notations for exactly the same operation, that is, multiplication of integer numbers.

3. Integer maximum. In an operator-like form

$$\vee\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \quad (m, n) \mapsto m \vee n$$

   In a function-like notation

$$\max\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \quad (m, n) \mapsto \max(m, n) = m \vee n$$

   Just to clarify $\max(m, n) = m \vee n$ and this is the maximum between m and n.

[1] The operation could be usual addition of integer numbers or taking maximum between two numbers, but from the general point of view the name of the result is product. So, mathematics is the art to call different things in a similar way and similar things in a different way. I will clarify the situation every time when it may lead to confusion.

4. Integer minimum. In an operator-like form

$$\wedge\colon \mathbb{Z}\times\mathbb{Z}\to\mathbb{Z},\quad (m,n)\mapsto m\wedge n$$

In a function-like notation

$$\min\colon \mathbb{Z}\times\mathbb{Z}\to\mathbb{Z},\quad (m,n)\mapsto \min(m,n)=m\wedge n$$

Just to clarify $\min(m,n)=m\wedge n$ and this is the minimum between m and n.

5. Some random binary operation on integers

$$\phi\colon \mathbb{Z}\times\mathbb{Z}\to\mathbb{Z},\quad (m,n)\mapsto m^2-n^2$$

So, in general a binary operation on X is any map $f\colon X\times X\to X$. You are free to define it in a way you wish. But different operations have different properties. Some of the operations are better than the others in a certain way. Since we want to deal with good operations only, I am starting a discussion of operation properties.

2.2 Properties

There are several properties important for our goal. I am going to deal with them one-by-one explaining everything on examples.

2.2.1 Associativity

Definition 7. An operation $\circ\colon X\times X\to X$ is called associative if for every elements $x,y,z\in X$ we have $(x\circ y)\circ z = x\circ(y\circ z)$.

If you have a binary operation $\circ$ on a set X, you can compute the product of three elements x, y, and z in two different ways:

- first compute $w=x\circ y$ and then compute $w\circ z=(x\circ y)\circ z$.
- first compute $u=y\circ z$ and then compute $x\circ u=x\circ(y\circ z)$.

If an operation is arbitrary it may happen that these two products are different for some specific elements x, y, and z. Associativity means that the order of the operations does not matter. Moreover, if $(x\circ y)\circ z=x\circ(y\circ z)$ for any $x,y,z\in X$, then it does not matter how to place parentheses in any product of elements. In particular, we may not use parentheses to specify the order, because in general there is no difference between $(x\circ y)\circ(z\circ w)$, $x\circ(y\circ(z\circ w))$ and $((x\circ y)\circ z)\circ w$, we may call it simply $x\circ y\circ z\circ w$.

Examples 8. Here are examples of associative and non-associative operations.

1. Integer addition is associative. Our operation is

$$+\colon \mathbb{Z}\times\mathbb{Z}\to\mathbb{Z},\quad (m,n)\mapsto m+n$$

Let $m,n,k\in\mathbb{Z}$ be arbitrary. Then we know that $(m+n)+k=m+(n+k)$.

2. Integer subtraction is not associative. Our operation is

$$-\colon \mathbb{Z}\times\mathbb{Z}\to\mathbb{Z},\quad (m,n)\mapsto m-n$$

Then the equality $(m-n)-k=m-(n-k)$ does not hold for any integer numbers. Indeed, let us take $m=n=0$ and $k=1$. Then the left-hand side is equal to -1 and the right-hand side is equal to 1. So, $(0-0)-1\neq 0-(0-1)$.

2.2.2 Neutral element

Definition 9. Let $\circ\colon X \times X \to X$ be an operation on X. An element $e \in X$ is called neutral (or identity element) if for every element $x \in X$ we have $x \circ e = x$ and $e \circ x = x$.

So, a neutral element $e \in X$ is such an element that does not change anything when we multiply by it.

Examples 10. A neutral element may exist or may not.

1. Integral addition has a neutral element. Our operation is
$$+\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \quad (m, n) \mapsto m + n$$
Then it is clear that element $e = 0$ satisfies the required properties. Indeed, for every natural $m \in \mathbb{Z}$ we have $m + 0 = m$ and $0 + m = m$.

2. Integer subtraction has no neutral element. Our operation is
$$-\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \quad (m, n) \mapsto m - n$$
Let us show that there is no element $e \in \mathbb{Z}$ such that $e - m = m$ for every $m \in \mathbb{Z}$. Indeed, if such e exists, then $e = 2m$ for any $m \in \mathbb{Z}$. But this is impossible because for $m = 0$, $e = 0$ and for $m = 1$, $e = 2$, a contradiction e must be a specific fixed element not depending on m. From the other hand, it is clear that $m - 0 = m$ for any $m \in \mathbb{Z}$.

The second example shows that it is not enough to check only one condition $x \circ e = x$ or $e \circ x = x$. This is a very common mistake to forget one of these conditions. You are warned!

A reasonable question is: "How many neutral elements may exist?" The answer is: "Not more than one." So, there may be no neutral element at all or just one.

Claim 11. *Let X be a set and $\circ\colon X \times X \to X$ be a binary operation. Then there exists at most one neutral element.*

Proof. If there is no neutral elements, we are done. Suppose that e and e' are neutral elements. We should show that they are the same. Consider the product $e \circ e'$. Since e is a neutral element $e \circ x = x$ for any $x \in X$. In particular, this holds for $x = e'$, that is, $e \circ e' = e'$. From the other hand, since e' is a neutral element $x \circ e' = x$ for any $x \in X$. In particular, this holds for $x = e$, that is, $e \circ e' = e$. Thus $e = e \circ e' = e'$. □

2.2.3 Inverse element

I want to start with a warning. This property depends on the previous one, that is, if an operation does not have a neutral element it is impossible to define inverse elements. This property does not make any sense in case the operation has no neutral element.

Definition 12. Let $\circ\colon X \times X \to X$ be an operation such that there is a neutral element $e \in X$. An element $y \in X$ is called inverse to an element $x \in X$ if $x \circ y = e$ and $y \circ x = e$.

I want to recall that a neutral element is unique if it exists. So, element e is well-defined and there is no confusion.

An excellent question is: "How many inverse elements are there for a particular element $x \in X$?" The answer is: "Not more than one if the operation is associative".

Claim 13. *Let $\circ\colon X \times X \to X$ be an associative binary operation and $e \in X$ is a neutral element. Then, every element $x \in X$ has at most one inverse element.*

Proof. Let us fix an element $x \in X$. If there is no inverse element for x, we are done. Now, suppose that y_1 and y_2 are inverse elements for x. The latter means that
$$\begin{cases} x \circ y_1 = e \\ y_1 \circ x = e \end{cases} \text{and} \quad \begin{cases} x \circ y_2 = e \\ y_2 \circ x = e \end{cases}$$
Now consider the product $y_1 \circ x \circ y_2$. Since, $\circ$ is associative, it does not matter how to put parentheses, that is $(y_1 \circ x) \circ y_2 = y_1 \circ (x \circ y_2)$. Let us compute the left-hand side:
$$(y_1 \circ x) \circ y_2 = e \circ y_2 = y_2$$

And for the right-hand side, we get

$$y_1 \circ (x \circ y_2) = y_1 \circ e = y_1$$

So, $y_2 = (y_1 \circ x) \circ y_2 = y_1 \circ (x \circ y_2) = y_1$ and we are done. □

Hence in general, for every element x if an inverse y exists, then its the only inverse of x. In this case, we denote y as x^{-1}.

Examples 14. 1. Suppose our operation is an integer addition

$$+ \colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \quad (m, n) \mapsto m + n$$

Then the only neutral element is 0. If $n \in \mathbb{Z}$, then its inverse is $-n$. Indeed, $n + (-n) = 0$ and $(-n) + n = 0$. Hence, every element has inverse.

2. Suppose our operation is an integer multiplication

$$\cdot \colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \quad (m, n) \mapsto m \cdot n$$

The only neutral element is 1. If $n = 1$, then its inverse is 1. If $n = -1$, then its inverse is -1. If $n \neq \pm 1$, then there is no inverse in $\mathbb{Z}$. Indeed, if $n = 2$, then there is no integer m such that $nm = 2m = 1$. Hence, only two elements have inverse.

2.2.4 Commutativity

Definition 15. A binary operation $\circ \colon X \times X \to X$ is called commutative if, for every $x, y \in X$, we have $x \circ y = y \circ x$.

So, commutativity means that the order of operands does not matter.

Examples 16. 1. Integral addition is commutative. Our operation is

$$+ \colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \quad (m, n) \mapsto m + n$$

Let $m, n \in \mathbb{Z}$ be arbitrary. Then we know that $m + n = n + m$.

2. Integer subtraction is not commutative. Our operation is

$$- \colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \quad (m, n) \mapsto m - n$$

Then the equality $m - n = n - m$ does not hold for any integer m, n. Indeed, if $m = 0$ and $n = 1$, then the left-hand side is -1 and the right-hand side is 1.

3 Groups

3.1 Definition

Now we are ready to give the most important definition in Algebra, that is the definition of a Group. Before we proceed, I want to clarify the general structure of definitions in Algebra. Every definition of an abstract object consists of two parts: 1) in the first part we list all the data required for the definition, 2) in the second part we list all the axioms the data must satisfy.

Definition 17. Definition of a group.

- **Data:**
  1. A set G.
  2. An operation $\circ \colon G \times G \to G$.
- **Axioms:**
  1. The operation $\circ$ is associative.
  2. The operation $\circ$ has a neutral element.
  3. Every element $x \in G$ has an inverse.

In this case, we say that the pair $(G, \circ)$ is a group. In order to simplify the notation, we usually say simply that G is a group assuming that the operation in use is clear. If in addition we have

4. The operation $\circ$ is commutative.

Then the group G is called abelian or simply commutative.

In short, a group is a set with a good operation. Here, good means that we do not care about parentheses, we have neutral element and every element is invertible but the order of the elements still matters. Abelian group means that additionally the order of the elements does not matter.

Examples 18.

1. Integers with addition $(\mathbb{Z}, +)$ is an abelian group. Indeed, the operation $+$ is associative, has an identity element 0, every element n has inverse $-n$ and the order in addition does not matter, that is $n + m = m + n$. We usually call this group simply $\mathbb{Z}$ assuming the addition as our operation by default.

2. Integers with multiplication $(\mathbb{Z}, \cdot)$ is not a group. Indeed, the operation $\cdot$ is associative, has an identity element 1, but there are a lot of non-invertible elements (the only invertible elements are ± 1).

3. Non-zero real numbers with multiplication $(\mathbb{R}^*, \cdot)$ is an abelian group. Indeed, the operation $\cdot$ is associative, has an identity element 1, every element x as inverse $1/x$, and the order in multiplication does not matter, that is $xy = yx$.

4. Let n be any positive integer, then the set $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ with operation $a + b \pmod{n}$ is an abelian group. The operation on $\mathbb{Z}_n$ we will simply be denote by $+$.

5. Let n be any positive integer and $\mathbb{Z}_n^* = \{m \in \mathbb{Z}_n \mid (m, n) = 1\}$ (that is the set of all integers in $\mathbb{Z}_n$ coprime with n) with operation $a \cdot b \pmod{n}$ is an abelian group. The operation on $\mathbb{Z}_n^*$ will simply be denoted by $\cdot$.

3.2 Multiplicative and additive notations

If we are given a group G, we usually denote its operation by $\circ$. However, it is very cumbersome to use this notation. Instead, people use symbols for usual multiplication or addition and there are two different types of notation: multiplicative and additive. Let me introduce the notation

	Multiplicative	Additive
Operation	$\cdot \colon G \times G \to G$	$+ \colon G \times G \to G$
On elements	$(x, y) \mapsto xy$	$(x, y) \mapsto x + y$
Neutral Element	1	0
Inverse Element	x^{-1}	$-x$
Power of Element	$x^n = \underbrace{x \cdot \ldots \cdot x}_{n}$	$nx = \underbrace{x + \ldots + x}_{n}$

Usually the multiplicative notation is used in case of an arbitrary non-abelian group and the additive notation is used in case of an abelian group. I will mostly stick to the multiplicative notation and use the additive only in case of abelian groups.

I want to emphasize that these are just two different notations for the operation $\circ$. That is $xy = x \circ y$ or $x + y = x \circ y$. You just denote $\circ$ by $\cdot$ or $+$ depending on your preferences. Do not confuse these notations with the usual multiplication and addition. In case of an arbitrary group G, there is no confusion because there is no addition and multiplication on an arbitrary set G. However, If we deal with integer numbers (real, rational, complex, etc.), the operations $+$ and $\cdot$ denote usual addition and multiplication.

3.3 Subgroups

Definition 19. Let G be a group.[2] We define a subgroup H of G.

- **Data:**

  1. A subset $H \subseteq G$.

- **Axioms:**

[2] Strictly speaking $(G, \cdot)$ but I am going to use the short notation all the time.

1. The neutral element 1 of G belongs to H.
2. $xy \in H$ whenever $x, y \in H$.
3. $x^{-1} \in H$ whenever $x \in H$.

In this case, we say that H is a subgroup of G.

It should be noted that if H is a subgroup of G, then $\cdot$ is a well-defined operation on H and $(H, \cdot)$ becomes a group.

Examples 20. Let $G = \mathbb{Z}$ with addition.

1. If $H \subseteq \mathbb{Z}$ is the set of even numbers, that is $H = 2\mathbb{Z}$, then H is a subgroup.
2. If $H \subseteq \mathbb{Z}$ is the set of odd numbers, that is $H = 1 + 2\mathbb{Z}$, then H is not a subgroup. For example, the neutral element 0 is not in H. Also, H is not closed under addition.

3.4 Cyclic subgroups

Let G be a group and $g \in G$ be an arbitrary element. Then we may take any integer power of g as follows

Multiplicative notation	Additive notation
$g^n = \begin{cases} \underbrace{g \cdot \ldots \cdot g}_{n}, & n > 0 \\ 1, & n = 0 \\ \underbrace{g^{-1} \cdot \ldots \cdot g^{-1}}_{-n}, & n < 0 \end{cases}$	$ng = \begin{cases} \underbrace{g + \ldots + g}_{n}, & n > 0 \\ 0, & n = 0 \\ \underbrace{(-g) + \ldots + (-g)}_{-n}, & n < 0 \end{cases}$

Claim 21. *Let G be a group. Then*

1. *For any $x, y \in G$, $(xy)^{-1} = y^{-1}x^{-1}$.*
2. *For any $g \in G$, $(g^{-1})^n = (g^n)^{-1}$.*
3. *For any $g \in G$, $g^n g^m = g^{n+m}$ whenever $n, m \in \mathbb{Z}$.*

Proof. 1) We need to show that $(xy)^{-1} = y^{-1}x^{-1}$. Let us denote $y^{-1}x^{-1}$ by z. If we show that $(xy)z = z(xy) = 1$, this will mean that $z = (xy)^{-1}$ by definition. Now, we compute

$$(xy)z = xyz = xyy^{-1}x^{-1} = xx^{-1} = 1$$

In a similar way, we show the other equality.

2) We apply the previous property several times, that is

$$(g_1 \cdot \ldots \cdot g_n)^{-1} = g_n^{-1} \cdot \ldots \cdot g_1^{-1}, \quad \text{whenever } g_1, \ldots, g_n \in G$$

If we substitute $g_1 = \ldots = g_n = g$, this proves the required for $n > 0$.

If $n = 0$, then by definition $(g^{-1})^0 = 1$. From the other hand, $(g^0)^{-1} = 1^{-1} = 1$ because the inverse for 1 is 1.

If $n < 0$, then by definition

$$(g^{-1})^n = \underbrace{(g^{-1})^{-1} \cdot \ldots \cdot (g^{-1})^{-1}}_{-n}$$

On the other hand,

$$(g^n)^{-1} = (\underbrace{g^{-1} \cdot \ldots \cdot g^{-1}}_{-n})^{-1} = \underbrace{(g^{-1})^{-1} \cdot \ldots \cdot (g^{-1})^{-1}}_{-n}$$

The latter equality follows from the previous item.

3) We should consider 4 cases:

1. $n \geqslant 0$ and $m \geqslant 0$.
2. $n < 0$ and $m \geqslant 0$.
3. $n \geqslant 0$ and $m < 0$.

4. $n < 0$ and $m < 0$.

In the first case, we have

$$g^n g^m = \underbrace{g \cdot \ldots \cdot g}_{n} \cdot \underbrace{g \cdot \ldots \cdot g}_{m} = \underbrace{g \cdot \ldots \cdot g}_{n+m} = g^{n+m}$$

For convenience, we consider $g^{-n}g^m$ for $n > 0$ and $m \geqslant 0$ in the second case.

$$g^{-n}g^m = \underbrace{g^{-1} \cdot \ldots \cdot g^{-1}}_{n} \cdot \underbrace{g \cdot \ldots \cdot g}_{m}$$

We cancel the factors at the middle of the expression. If $n > m$, we get

$$\underbrace{g^{-1} \cdot \ldots \cdot g^{-1}}_{n-m} = g^{-n+m}$$

If $n < m$, we get

$$\underbrace{g \cdot \ldots \cdot g}_{m-n} = g^{m-n}$$

if $n = m$ we get $1 = g^{m-n}$. Other cases I leave as an exercise. □

Definition 22. Let G be a group and $g \in G$ be an arbitrary element. Then the set of all integer powers of g, that is,

$$\langle g \rangle = \{\ldots, g^{-2}, g^{-1}, 1, g, g^2, \ldots\}$$

is a subgroup of G. This group is called the cyclic subgroup generated by g. The element g is called a generator of $\langle g \rangle$.

The cyclic subgroup $\langle g \rangle$ is the smallest possible subgroup containing the element g.

Examples 23.

1. The group $(\mathbb{Z}, +)$ is cyclic. There are two different generators 1 and -1.
2. The group $(\mathbb{Z}_n, +)$ is cyclic.
3. The group of permutations on n elements S_n is not cyclic if $n > 2$.
4. The group $(\mathbb{R}, +)$ is not cyclic.

Claim 24. *Let G be a group and $g \in G$ be an arbitrary element. Then there are two options:*

- *If $\operatorname{ord} g = \infty$, then the elements g^n and g^m are different whenever $n, m \in \mathbb{Z}$ are different.*
- *If $\operatorname{ord} g = n < \infty$, then elements $1, g, g^2, \ldots, g^{n-1}$ are different. In this case, the powers are repeated in cycles, that is in the series*

$$\underbrace{\ldots, g^{-2}, g^{-1}}, \underbrace{1, g, g^2, \ldots, g^{n-1}}, \underbrace{g^n, g^{n+1}, \ldots, g^{2n-1}}, \underbrace{g^{2n}, \ldots}$$

$g^{kn}, g^{1+kn}, \ldots, g^{n-1+nk}$ are the same elements as $1, g, \ldots, g^{n-1}$ for any $k \in \mathbb{Z}$. In particular,

$$\langle g \rangle = \{1, g, \ldots g^{n-1}\}$$

Proof. If $g^n \neq g^m$ for all different $m, n \in \mathbb{Z}$, we are in the first case.

Now suppose that $g^n = g^m$ for some integer $m \neq n$. Then we may multiply this equality by g^{-m} and get $g^{n-m} = 1$. Hence, we may assume that for some $n \neq 0$, we have $g^n = 1$. If $n < 0$, multiply by g^{-n}. Thus, we may assume that for some positive integer n, we have $g^n = 1$.

Consider the minimal positive integer n such that $g^n = 1$. I claim that the elements $1, g, \ldots, g^{n-1}$ are different. Indeed, if $g^k = g^s$ for some $k, s \in [0, n-1]$ and $k \geqslant s$, then $g^{k-s} = 1$ and $k - s$ is not zero and is strictly less than n. The latter contradicts to the choice of n. □

It should be noted that n may equal 1 in case g is the neutral element 1.

Definition 25. Let G be a group and $g \in G$ be an arbitrary element. The order of g is the minimal positive natural number such that $g^n = 1$ and ∞ if there is no such a number. The order of g is denoted by $\operatorname{ord} g$.

From the previous Claim it follows that $\operatorname{ord} g$ equals the number of elements in $\langle g\rangle$. Note that $g = 1$ if and only if $\operatorname{ord} g = 1$.

If we use additive notation, that is the operation on the group G is denoted by $+$, then, the order of $g \in G$ is the small positive integer n such that $ng = 0$. The cyclic subgroup generated by g is

$$\langle g\rangle = \{\dots, -2g, -g, 0, g, 2g, \dots\}$$

Definition 26. Let G be a group. If there is an element $g \in G$ such that $\langle g\rangle = G$, then the group G is called cyclic.

Now, I want to describe all subgroups of the integers with addition.

Claim 27. *Every subgroup H of $\mathbb{Z}$, that is $(\mathbb{Z}, +)$, is of the form $k\mathbb{Z}$ for some natural k.*

Proof. Let us check that $k\mathbb{Z}$ is indeed a subgroup for any k. We need to check three properties of the subgroup. First, $k\mathbb{Z}$ is closed under addition. But this is clear by definition. Second, the neutral element, which is zero, belongs to $k\mathbb{Z}$. This is also clear since $0 = k \cdot 0$. Third, for each $m = kh \in k\mathbb{Z}$, its inverse $-m = k(-h)$ is also in $\mathbb{Z}$, and we are done with this part.

Now, let us check that every subgroup H is of the form $k\mathbb{Z}$. If H contains only the neutral element 0, then $H = 0\mathbb{Z}$ and we are done. Suppose H contains non-zero elements. Take an arbitrary non-zero $n \in H$. If $n < 0$, then $-n$ must belong to H by definition of a subgroup. And hence, we may assume that H contains some positive numbers. Let k be the smallest positive number in H. Let us show that $H = k\mathbb{Z}$.

First, $H \supseteq k\mathbb{Z}$. Indeed, if $k \in H$, then by definition of a subgroup every "power" of k is in H. For additive notation this means

$$mk = \underbrace{k + \dots + k}_{m} \in H \quad\text{and}\quad (-n)k = \underbrace{(-k) + \dots + (-k)}_{n} \in H \quad\text{for any}\quad m, n \in \mathbb{N}$$

Hence, $k\mathbb{Z} \subseteq H$.

Now, let us show that $H \subseteq k\mathbb{Z}$. If $n \in H$ is an arbitrary element, let us divide n by k: $n = qk + r$, where $q \in \mathbb{Z}$ and $0 \leqslant r < k$. We already know that $qk \in k\mathbb{Z} \subseteq H$, that is $qk \in H$. Hence, $r = n - qk \in H$. But r is a natural number in H smaller than k. Since k is the smallest positive in H, the only option is $r = 0$. Thus, $n = qk \in k\mathbb{Z}$ and we are done. □

Claim 28. *Every subgroup H of $\mathbb{Z}_n$, that is $(\mathbb{Z}_n, +)$, is of the form $k\mathbb{Z}_n = \{kh \in \mathbb{Z}_n \mid h \in \mathbb{Z}_n\}$ for some positive $k \mid n$.*

Proof. First, let us check that all numbers divisible by k such that $k \mid n$ form a subgroup in $\mathbb{Z}_n$. First, we need to check that $k\mathbb{Z}_n$ is closed under addition modulo n. Suppose $m_1 = kh_1$ and $m_2 = kh_2$ are elements of $k\mathbb{Z}_n$. Then their sum modulo n is a remainder r such that $m_1 + m_2 = r \pmod n$. In this case,

$$r = m_1 + m_2 + qn = kh_1 + kh_2 + qn$$

Since k divides n the whole expression above is divisible by k. Hence r is divisible by k. The latter means that $k\mathbb{Z}_n$ is closed under addition modulo n. Second, we need to check that $k\mathbb{Z}_n$ contains the neutral element. This is clear, since $0 = k \cdot 0 \in k\mathbb{Z}_n$. Third, if $m \in k\mathbb{Z}_n$ is a nonzero element, then its inverse is $n - m$. Since n is divisible by k, $n - m$ is divisible by k. Hence, it belongs to $k\mathbb{Z}_n$. In case $m = 0$ its inverse is 0 and is already in $k\mathbb{Z}_n$. Hence, for each $k \mod n$, $k\mathbb{Z}_n$ is a subgroup of $\mathbb{Z}_n$.

Now, let us show, that every subgroup H in $\mathbb{Z}_n$ coincides with a subgroup of the form $k\mathbb{Z}_n$ for $k \mid n$. The subgroup H must contain the neutral element 0. If this is the only element of H, then $H = \{0\} = n\mathbb{Z}_n$ and we are done. So, we may suppose there is a non-zero, and hence positive, element in H. Let k be the smallest positive element of H. By definition the cyclic subgroup of k, that is $k\mathbb{Z}_n$, belongs to H. Thus, we need to show, that $H \subseteq k\mathbb{Z}_n$ and k divides n.

First, let me show that k divides n. Let us divide n with remainder by k, we will get $n = qk + r$, where $0 \leqslant r < k$. Now, $r = n - qk$, hence $r = -qk \pmod n$. Since $k \in H$, the latter means that r is also in H. But this contradicts the choice of k (it was the smallest nonzero integer in H). Hence, r must be zero, thus k divides n. Second, let me show that every element of H is in $k\mathbb{Z}_n$. Suppose $h \in H$ is an arbitrary element. Let us divide h with remainder by k, we will get $h = qk + r$. Hence, $r = h - qk$. Since $h \in H$ and $k \in H$, the whole expression $h - qk$ is in H. Hence, $r \in H$. Since k was the smallest positive integer of H, we must have $r = 0$. The latter means that h is divisible by k, that is, h belongs to $k\mathbb{Z}_n$, and we are done. □

3.5 Cosets

Algebra usually tends to study groups using subgroups rather than elements. The main tool here is cosets.

Definition 29. Let G be a group, $H \subseteq G$ a subgroup and $g \in G$ an arbitrary element. Then the set

$$gH = \{gh \mid h \in H\}$$

is called the left coset of H with respect to g. In a similar way, we define right cosets. The set

$$Hg = \{hg \mid h \in H\}$$

is called the right coset of H with respect to g.

Remarks 30.

1. It should be noted that if G is commutative, then there is no difference between left and right cosets for any subgroup $H \subseteq G$.

2. The group H itself is a left coset as well as a right coset. Indeed, $H = 1 \cdot H = H \cdot 1$.

3. In general, the left cost gH need not be the same as the right coset Hg as an example below shows.

Examples 31. Here are some examples of cosets.

1. Let $G = (\mathbb{Z}, +)$ and $H = 2\mathbb{Z}$ the subgroup of even numbers. Then $2\mathbb{Z}$ and $1 + 2\mathbb{Z}$ are the only cosets of H.

2. Let $G = S_3$ and $H = \langle(1, 2)\rangle$ the cyclic subgroup generated by the cycle $(1, 2)$. We may list all the elements of G and H
$$G = \{1,(1, 2),(1, 3),(2, 3),(1, 2, 3),(3, 2, 1)\},\ H = \{1,(1, 2)\}$$
Then there are three different left cosets of H
$$H = \{1,(1, 2)\},\ (1, 3)H = \{(1, 3),(1, 2, 3)\},\ (2, 3)H = \{(2, 3),(3, 2, 1)\}$$
And there are three different right cosets of H
$$H = \{1,(1, 2)\},\ H(1, 3) = \{(1, 3),(3, 2, 1)\},\ H(2, 3) = \{(2, 3),(1, 2, 3)\}$$
This example shows that $(1, 3)H \neq H(1, 3)$. Also, it should be noted that
$$(1, 2)H = H,\ (1, 3)H = (1, 2, 3)H,\ (2, 3)H = (3, 2, 1)H$$
So, cosets with respect to different elements may be the same.

3. Let $G = S_n$ be the group of permutations on n elements and $H = A_n$ be the subgroup of even permutations. Then, for any even permutation $\sigma \in A_n$, the set σA_n consists of all even permutations. Similarly, for any odd permutation $\sigma \in S_n \subseteq A_n$, the set σA_n consists of all odd permutations. Hence, there are only two left cosets of A_n
$$A_n \text{ and } (1, 2)A_n$$
In a similar way, we can show that there are only two right cosets of A_n
$$A_n \text{ and } A_n(1, 2)$$
Moreover, we have shown that $\sigma A_n = A_n\sigma$ for any $\sigma \in S_n$.

Definition 32. Let G be a group and H its subgroup. The subgroup H is normal if its left and right cosets are the same, that is, $gH = Hg$ whenever $g \in G$.

Claim 33. *Let G be a group and H its subgroup. Then, the following are equivalent*

1. *$gH = Hg$ for each $g \in G$.*

2. *$gHg^{-1} = H$ for each $g \in G$.*

3. *$gHg^{-1} \subseteq H$ for each $g \in G$.*

Proof. (1)⇔(2). Suppose $gH = Hg$. Multiply this on the right by g^{-1} and get $gHg^{-1} = H$. And if we are given $gHg^{-1} = H$, multiply this on the right by g and get $gH = Hg$.

(2)⇔(3). We should show that $gHg^{-1} \subseteq H$ for each $g \in G$ implies $gHg^{-1} = H$ for each $g \in G$. The other implication is clear. If $gHg^{-1} \subseteq H$ for each $g \in G$, then it holds for g^{-1} instead of g. Thus, $g^{-1}Hg \subseteq H$ for each $g \in G$. Multiply this on the left by g and get $Hg \subseteq gH$. Then, multiply the latter on the right by g^{-1} and get $H \subseteq gHg^{-1}$. Since $g \in G$ was arbitrary we are done.

□

3.6 The Lagrange Theorem

Properties of cosets Now, I want to prove several properties of the cosets. The important observation here is that all left cosets form a partition of the group G into non-overlapping subsets. The same is true for the right cosets. This observation provides us with some combinatorial tools.

Claim 34. *Let G be a group, $H \subseteq G$ a subgroup and $g_1, g_2 \in G$ be arbitrary elements. Then there are two options:*

1. *The cosets do not intersect each other: $g_1H \cap g_2H = \varnothing$.*

2. *The cosets coincide: $g_1H = g_2H$.*

This means that each element of the group G belongs to exactly one coset.

Proof. If g_1H does not intersect g_2H there is nothing to prove.

Now we assume that the intersection $g_1H \cap g_2H$ is not empty. We need to prove that $g_1H = g_2H$. Suppose $g \in g_1H \cap g_2H$. Since $g \in g_1H$, $g = g_1h_1$ for some $h_1 \in H$. Similarly, $g \in g_2H$ implies $g = g_2h_2$ for some $h_2 \in H$. Hence $g_1h_1 = g_2h_2$. Dividing by h_1 on the right, we get $g_1 = g_2h_2h_1^{-1}$. Since H is a subgroup $h = h_2h_1^{-1} \in H$. We have got $g_1 = g_2h$ for some $h \in H$.

Let us show that $g_1H \subseteq g_2H$. Suppose $g \in g_1H$, that is $g = g_1h'$ for some $h' \in H$. Then, $g = g_2hh' \in g_2H$ because $hh' \in H$. Now, suppose $g \in g_2H$, that is $g = g_2h'$ for some $h' \in H$. Then, $g = g_1h^{-1}h' \in g_1H$ because $h^{-1}h' \in H$. Hence, we have shown $g_2H \subseteq g_1H$. □

Remark 35. It should be noted that $g_1H = g_2H$ if and only if $g_1H \cap g_2H \neq \varnothing$. Moreover, this occurs if and only if there is an element $h \in H$ such that $g_1 = g_2h$. The latter is equivalent to the condition $g_2^{-1}g_1 \in H$. This provides us with a convenient way of checking if two cosets are the same.

Claim 36. *Let G be a group, $H \subseteq G$ be a finite subgroup and $g \in G$ an arbitrary element. Then $|gH| = |H| = |Hg|$.*

Proof. I will prove the claim for left cosets. Let us consider the map

$$\phi\colon H \to gH \quad x \mapsto gx$$

It takes elements of H to elements of gH. From the other hand, there is the inverse map

$$\psi\colon gH \to H \quad x \mapsto g^{-1}x$$

Thus ϕ and ψ are bijections and we are done. □

Claim 37. *Let G be a finite group and $H \subseteq G$ be a subgroup. Then*

1. *The amount of left cosets of H is equal to $|G|/|H|$.*

2. *The amount of right cosets of H is equal to $|G|/|H|$.*

In particular, the number of left and right cosets is the same.

Proof. We will prove the first item. Claim 34 shows that G is a disjoint union of some cosets, that is $G = g_1H \sqcup \ldots \sqcup g_kH$. From the other hand, Claim 36 shows that all the cosets $g_1H, \ldots, g_kH$ have the same amount of elements being equal to $|H|$. Hence

$$|G| = |g_1H| + \ldots + |g_kH| = |H| + \ldots + |H| = k|H|$$

Here k is the number of the distinct left cosets and we are done. □

Definition 38. Let G be a finite group and $H \subseteq G$ be a subgroup. Then the number of the left cosets of H is called index of H and is denoted by $(G : H)$. This number is also coincide with the number of the right cosets of H.

Using this notation, we can rewrite Claim 37 in the following way.

Claim 39 (The Lagrange Theorem)**.** *Let G be a finite group and $H \subseteq G$ be a subgroup. Then, $|G| = (G : H)|H|$*

Corollaries of The Lagrange Theorem

1. Let G be a finite group and $H \subseteq G$ be a subgroup. Then $|H|$ divides $|G|$.
2. Let G be a finite group and $g \in G$ be an arbitrary element. Then $\mathrm{ord}(g)$ divides $|G|$. Indeed, $\mathrm{ord}(g) = |\langle g \rangle|$. But $|\langle g \rangle|$ divides $|G|$ by the previous item.
3. Let G be a finite group and $g \in G$ be an arbitrary element. Then $g^{|G|} = 1$. Indeed, we already know that $|G| = \mathrm{ord}(g)k$. Hence,
$$g^{|G|} = g^{\mathrm{ord}(g)k} = \left(g^{\mathrm{ord}(g)}\right)^k = 1^k = 1$$
4. Let G be a group of prime order p. Then, G is cyclic. Indeed, since the order of G is prime, it is greater than 1. Hence, there is an element $g \in G$ such that $g \neq 1$. Hence $\langle g \rangle$ has order greater than 1. But $|\langle g \rangle|$ divides $|G| = p$. Since p is prime, the only option is $|\langle g \rangle| = p = |G|$. The latter means that $\langle g \rangle = G$ and we are done.
5. The Fermat Little Theorem. Let $p \in \mathbb{Z}$ be a prime number and $a \in \mathbb{Z}$. If p does not divide a, then p divides $a^{p-1} - 1$. Indeed, let us consider the group $(\mathbb{Z}_p^*, \cdot)$. For any element $b \in \mathbb{Z}_p^*$, we have $b^{|\mathbb{Z}_p^*|} = 1 \pmod p$ by item (3). But $\mathbb{Z}_p^*$ has $p-1$ elements. Now, let $a \in \mathbb{Z}$ be comprime with p. We denote its remainder modulo p by b. Then $a^{p-1} = b^{p-1} = 1 \pmod p$ and we are done.