Algebra
Lecture Notes

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Contents

1 Sets

In modern math everything can be formulated in terms of Set Theory, that is in terms of sets and maps. Let me remind some basic facts about sets and maps.

1.1 Definition

Definition 1. A set is a collection of elements.

We denote sets by capital letters like X and Y. If an element x belongs to the collection X, we write $x \in X$. If y does not belong to X, we write $y \notin X$. There is a special set containing no elements. This set is called an empty set and is denoted by $\varnothing$.

If you think of a set you should imagine a sack full of elements. The sack is your set and the elements in the sack are the elements belonging to the set. An empty set becomes a sack with no elements inside.

1.2 Constructors

If you are given a definition of a new math object, the first question to ask is: "How do I construct such an object?" To define a set we need to specify the elements inside the set. For doing that, we use the following notation

$$X = \{x \mid \textbf{condition} \text{ on } x\}$$

Here, we mean that the set X consists of all elements x such that the **condition** on x holds. Let me demonstrate this on examples.

- The set of natural numbers
$$\mathbb{N} = \{x \mid x \text{ is a natural number}\} = \{0, 1, 2, 3, \ldots, n, \ldots\}$$
- The set of integer numbers
$$\mathbb{Z} = \{x \mid x \text{ is an integer number}\} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$
- The set of real numbers
$$\mathbb{R} = \{x \mid x \text{ is a real number}\}$$
We think of the real numbers as a line containing all possible numbers we use in our calculations.
- The closed interval $[0, 1]$
$$[0, 1] = \{x \in \mathbb{R} \mid 0 \leqslant x \leqslant 1\}$$
Here, I use a slightly different notation. I specified that $x \in \mathbb{R}$ before $|$, this simply means that x must be a real number and the additional condition (the number is between zero and one) is written after $|$.

1.3 Operations on sets

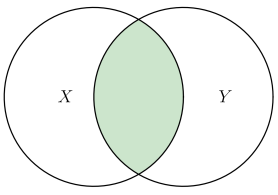
There are several useful procedures you can apply to sets in order to construct new sets. Let us discuss them.

1.3.1 Intersection

If we are given two sets X and Y, then we define the intersection of X and Y as follows

$$X \cap Y = \{z \mid z \in X \text{ and } z \in Y\}$$

If we denote the sets X and Y by discs on a plain then the intersection of X and Y is denoted as below

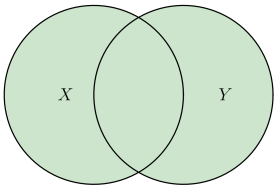


1.3.2 Union

If we are given two sets X and Y, then we define the union of X and Y as follows

$$X \cup Y = \{z \mid z \in X \text{ or } z \in Y\}$$

If we denote the sets X and Y by discs on a plain then the union of X and Y is denoted as below

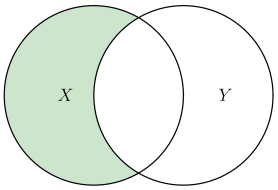


1.3.3 Difference

If we are given two sets X and Y, then we define the difference between X and Y as follows

$$X \setminus Y = \{z \mid z \in X \text{ and } z \notin Y\}$$

If we denote the sets X and Y by discs on a plain then the difference between X and Y is denoted as below



1.3.4 Cartesian product

If we are given two sets X and Y, then their Cartesian product is defined as follows

$$X \times Y = \{(x, y) \mid x \in X,\ y \in Y\}$$

The Cartesian product is simply the set of all possible pairs (x, y) where the first element is taken from X and the second from Y. We will use it every time when we need pairs of elements and not just elements.

1.4 Maps

Definition 2. Suppose we are given two sets X and Y, a map $f\colon X \to Y$ is a rule that takes elements of X to elements of Y. If $x \in X$, then its image in Y is denoted by $f(x)$. In this case, we will write $x \mapsto f(x)$.

The set X is called the source of f and the set Y is called the target of f.

Here is a way to think about maps. Suppose we are given a map $f\colon X \to Y$. Then f is a callable object with an operator $(-)$. You give it any element x of X, then it returns you some specific element $f(x)$ of Y. For each input $x \in X$, the result $f(x) \in Y$ will be the same every time you call it. So, a map is the same thing as a function.

Examples 3. Here are some examples of maps and non maps.

1. The rule $f\colon \mathbb{R} \to \mathbb{R}$ by $x \mapsto 2x + 3$ is a map.
2. The rule $f\colon \mathbb{R} \to \mathbb{R}$ by $x \mapsto \sin(x)$ is a map.
3. The rule $f\colon \mathbb{R} \to \mathbb{R}$ by $x \mapsto \frac{1}{x}$ is not a map because it is not defined at $x = 0$. It becomes a map if we change the source for f. If $\mathbb{R}^* = \{x \in \mathbb{R} \mid x \neq 0\}$, then $f\colon \mathbb{R}^* \to \mathbb{R}$ by $x \mapsto \frac{1}{x}$ is a map.
4. The rule $f\colon \mathbb{R} \to \mathbb{R}$ by $x \mapsto \ln x$ is not a map because it is only defined for positive x. It becomes a map if we change the source for f. If $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$, then $f\colon \mathbb{R}_+ \to \mathbb{R}$ by $x \mapsto \ln x$ is a map.

2 Binary operations

The simplest object in Algebra is a set with a good binary operation. I am going to explain what a binary operation is and the meaning of the word good. Our goal is to define an object called a Group.

2.1 Definition

Definition 4. Suppose X is a set. A binary operation is a map $\circ\colon X \times X \to X$ by the rule $(x, y) \mapsto x \circ y$ for $x, y \in X$.

In this case the notation $\circ$ is the name of the operation. Simply speaking, the operation is a rule that takes two elements of the set X and produce a new element called $x \circ y$ of the same set X. This element $x \circ y$ is usually called the product of x and y.[1]

You should have noticed that we use the name of the operation in a quite unusual way. We write the name between the arguments and not before. This is just for convenience. However, there is a function-like notation (or map-like notation) for binary operations. Let me show you

Definition 5. Suppose X is a set. A binary operation is a map $\mu\colon X \times X \to X$ by the rule $(x, y) \mapsto \mu(x, y)$ for $x, y \in X$.

This is just a different notation for the same mathematical notion. You may denote an operation in operator-like stile (the first definition) or in a function-like style (the second definition). To clarify the situation, let me proceed to a series of examples.

Examples 6. Binary operations:

1. Addition of integral numbers. In an operator-like form

$$+\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \quad (m, n) \mapsto m + n$$

   In a function-like notation

$$\mathrm{add}\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \quad (m, n) \mapsto \mathrm{add}(m, n) = m + n$$

   Since we got used to addition of numbers in a form $m+n$, we want a general definition to be in a similar form. From the other hand, many programming languages allow us using operator-like and function-like notations for addition. Here I want to emphasize that $\mathrm{add}(m, n)$ and $m+n$ are the same things. These are just different notations of the same addition that we use with integers.

2. Integer multiplication. In an operator-like form

$$\cdot\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \quad (m, n) \mapsto m \cdot n$$

   In a function-like notation

$$\mathrm{mult}\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \quad (m, n) \mapsto \mathrm{mult}(m, n) = m \cdot n$$

   Again, these are just two different notations for exactly the same operation, that is, multiplication of integer numbers.

3. Integer maximum. In an operator-like form

$$\vee\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \quad (m, n) \mapsto m \vee n$$

   In a function-like notation

$$\max\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \quad (m, n) \mapsto \max(m, n) = m \vee n$$

   Just to clarify $\max(m, n) = m \vee n$ and this is the maximum between m and n.

[1] The operation could be usual addition of integer numbers or taking maximum between two numbers, but from the general point of view the name of the result is product. So, mathematics is the art to call different things in a similar way and similar things in a different way. I will clarify the situation every time when it may lead to confusion.

4. Integer minimum. In an operator-like form

$$\wedge\colon \mathbb{Z}\times\mathbb{Z}\to\mathbb{Z},\quad (m,n)\mapsto m\wedge n$$

   In a function-like notation

$$\min\colon \mathbb{Z}\times\mathbb{Z}\to\mathbb{Z},\quad (m,n)\mapsto \min(m,n)=m\wedge n$$

   Just to clarify $\min(m,n)=m\wedge n$ and this is the minimum between m and n.

5. Some random binary operation on integers

$$\phi\colon \mathbb{Z}\times\mathbb{Z}\to\mathbb{Z},\quad (m,n)\mapsto m^2-n^2$$

So, in general a binary operation on X is any map $f\colon X\times X\to X$. You are free to define it in a way you wish. But different operations have different properties. Some of the operations are better than the others in a certain way. Since we want to deal with good operations only, I am starting a discussion of operation properties.

2.2 Properties

There are several properties important for our goal. I am going to deal with them one-by-one explaining everything on examples.

2.2.1 Associativity

Definition 7. An operation $\circ\colon X\times X\to X$ is called associative if for every elements $x,y,z\in X$ we have $(x\circ y)\circ z=x\circ(y\circ z)$.

If you have a binary operation $\circ$ on a set X, you can compute the product of three elements x, y, and z in two different ways:

- first compute $w=x\circ y$ and then compute $w\circ z=(x\circ y)\circ z$.
- first compute $u=y\circ z$ and then compute $x\circ u=x\circ(y\circ z)$.

If an operation is arbitrary it may happen that these two products are different for some specific elements x, y, and z. Associativity means that the order of the operations does not matter. Moreover, if $(x\circ y)\circ z=x\circ(y\circ z)$ for any $x,y,z\in X$, then it does not matter how to place parentheses in any product of elements. In particular, we may not use parentheses to specify the order, because in general there is no difference between $(x\circ y)\circ(z\circ w)$, $x\circ(y\circ(z\circ w))$ and $((x\circ y)\circ z)\circ w$, we may call it simply $x\circ y\circ z\circ w$.

Examples 8. Here are examples of associative and non-associative operations.

1. Integer addition is associative. Our operation is

$$+\colon \mathbb{Z}\times\mathbb{Z}\to\mathbb{Z},\quad (m,n)\mapsto m+n$$

   Let $m,n,k\in\mathbb{Z}$ be arbitrary. Then we know that $(m+n)+k=m+(n+k)$.

2. Integer subtraction is not associative. Our operation is

$$-\colon \mathbb{Z}\times\mathbb{Z}\to\mathbb{Z},\quad (m,n)\mapsto m-n$$

   Then the equality $(m-n)-k=m-(n-k)$ does not hold for any integer numbers. Indeed, let us take $m=n=0$ and $k=1$. Then the left-hand side is equal to -1 and the right-hand side is equal to 1. So, $(0-0)-1\neq 0-(0-1)$.

2.2.2 Neutral element

Definition 9. Let $\circ\colon X \times X \to X$ be an operation on X. An element $e \in X$ is called neutral (or identity element) if for every element $x \in X$ we have $x \circ e = x$ and $e \circ x = x$.

So, a neutral element $e \in X$ is such an element that does not change anything when we multiply by it.

Examples 10. A neutral element may exist or may not.

1. Integral addition has a neutral element. Our operation is

$$+\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \quad (m, n) \mapsto m + n$$

   Then it is clear that element $e = 0$ satisfies the required properties. Indeed, for every natural $m \in \mathbb{Z}$ we have $m + 0 = m$ and $0 + m = m$.

2. Integer subtraction has no neutral element. Our operation is

$$-\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \quad (m, n) \mapsto m - n$$

   Let us show that there is no element $e \in \mathbb{Z}$ such that $e - m = m$ for every $m \in \mathbb{Z}$. Indeed, if such e exists, then $e = 2m$ for any $m \in \mathbb{Z}$. But this is impossible because for $m = 0$, $e = 0$ and for $m = 1$, $e = 2$, a contradiction e must be a specific fixed element not depending on m. From the other hand, it is clear that $m - 0 = m$ for any $m \in \mathbb{Z}$.

The second example shows that it is not enough to check only one condition $x \circ e = x$ or $e \circ x = x$. This is a very common mistake to forget one of these conditions. You are warned!

A reasonable question is: "How many neutral elements may exist?" The answer is: "Not more than one." So, there may be no neutral element at all or just one.

Claim 11. *Let X be a set and $\circ\colon X \times X \to X$ be a binary operation. Then there exists at most one neutral element.*

Proof. If there is no neutral elements, we are done. Suppose that e and e' are neutral elements. We should show that they are the same. Consider the product $e \circ e'$. Since e is a neutral element $e \circ x = x$ for any $x \in X$. In particular, this holds for $x = e'$, that is, $e \circ e' = e'$. From the other hand, since e' is a neutral element $x \circ e' = x$ for any $x \in X$. In particular, this holds for $x = e$, that is, $e \circ e' = e$. Thus $e = e \circ e' = e'$. □

2.2.3 Inverse element

I want to start with a warning. This property depends on the previous one, that is, if an operation does not have a neutral element it is impossible to define inverse elements. This property does not make any sense in case the operation has no neutral element.

Definition 12. Let $\circ\colon X \times X \to X$ be an operation such that there is a neutral element $e \in X$. An element $y \in X$ is called inverse to an element $x \in X$ if $x \circ y = e$ and $y \circ x = e$.

I want to recall that a neutral element is unique if it exists. So, element e is well-defined and there is no confusion.

An excellent question is: "How many inverse elements are there for a particular element $x \in X$?" The answer is: "Not more than one if the operation is associative".

Claim 13. *Let $\circ\colon X \times X \to X$ be an associative binary operation and $e \in X$ is a neutral element. Then, every element $x \in X$ has at most one inverse element.*

Proof. Let us fix an element $x \in X$. If there is no inverse element for x, we are done. Now, suppose that y_1 and y_2 are inverse elements for x. The latter means that

$$\begin{cases} x \circ y_1 = e \\ y_1 \circ x = e \end{cases} \quad \text{and} \quad \begin{cases} x \circ y_2 = e \\ y_2 \circ x = e \end{cases}$$

Now consider the product $y_1 \circ x \circ y_2$. Since, $\circ$ is associative, it does not matter how to put parentheses, that is $(y_1 \circ x) \circ y_2 = y_1 \circ (x \circ y_2)$. Let us compute the left-hand side:

$$(y_1 \circ x) \circ y_2 = e \circ y_2 = y_2$$

And for the right-hand side, we get

$$y_1 \circ (x \circ y_2) = y_1 \circ e = y_1$$

So, $y_2 = (y_1 \circ x) \circ y_2 = y_1 \circ (x \circ y_2) = y_1$ and we are done. □

Hence in general, for every element x if an inverse y exists, then its the only inverse of x. In this case, we denote y as x^{-1}.

Examples 14. 1. Suppose our operation is an integer addition

$$+ \colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \quad (m, n) \mapsto m + n$$

Then the only neutral element is 0. If $n \in \mathbb{Z}$, then its inverse is $-n$. Indeed, $n + (-n) = 0$ and $(-n) + n = 0$. Hence, every element has inverse.

2. Suppose our operation is an integer multiplication

$$\cdot \colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \quad (m, n) \mapsto m \cdot n$$

The only neutral element is 1. If $n = 1$, then its inverse is 1. If $n = -1$, then its inverse is -1. If $n \neq \pm 1$, then there is no inverse in $\mathbb{Z}$. Indeed, if $n = 2$, then there is no integer m such that $nm = 2m = 1$. Hence, only two elements have inverse.

2.2.4 Commutativity

Definition 15. A binary operation $\circ \colon X \times X \to X$ is called commutative if, for every $x, y \in X$, we have $x \circ y = y \circ x$.

So, commutativity means that the order of operands does not matter.

Examples 16. 1. Integral addition is commutative. Our operation is

$$+ \colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \quad (m, n) \mapsto m + n$$

Let $m, n \in \mathbb{Z}$ be arbitrary. Then we know that $m + n = n + m$.

2. Integer subtraction is not commutative. Our operation is

$$- \colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \quad (m, n) \mapsto m - n$$

Then the equality $m - n = n - m$ does not hold for any integer m, n. Indeed, if $m = 0$ and $n = 1$, then the left-hand side is -1 and the right-hand side is 1.

3 Groups

3.1 Definition

Now we are ready to give the most important definition in Algebra, that is the definition of a Group. Before we proceed, I want to clarify the general structure of definitions in Algebra. Every definition of an abstract object consists of two parts: 1) in the first part we list all the data required for the definition, 2) in the second part we list all the axioms the data must satisfy.

Definition 17. Definition of a group.

- **Data:**
  1. A set G.
  2. An operation $\circ \colon G \times G \to G$.
- **Axioms:**
  1. The operation $\circ$ is associative.
  2. The operation $\circ$ has a neutral element.
  3. Every element $x \in G$ has an inverse.

In this case, we say that the pair $(G, \circ)$ is a group. In order to simplify the notation, we usually say simply that G is a group assuming that the operation in use is clear. If in addition we have

4. The operation $\circ$ is commutative.

Then the group G is called abelian or simply commutative.

In short, a group is a set with a good operation. Here, good means that we do not care about parentheses, we have neutral element and every element is invertible but the order of the elements still matters. Abelian group means that additionally the order of the elements does not matter.

Examples 18.

1. Integers with addition $(\mathbb{Z}, +)$ is an abelian group. Indeed, the operation $+$ is associative, has an identity element 0, every element n has inverse $-n$ and the order in addition does not matter, that is $n + m = m + n$. We usually call this group simply $\mathbb{Z}$ assuming the addition as our operation by default.
2. Integers with multiplication $(\mathbb{Z}, \cdot)$ is not a group. Indeed, the operation $\cdot$ is associative, has an identity element 1, but there are a lot of non-invertible elements (the only invertible elements are ± 1).
3. Non-zero real numbers with multiplication $(\mathbb{R}^*, \cdot)$ is an abelian group. Indeed, the operation $\cdot$ is associative, has an identity element 1, every element x as inverse $1/x$, and the order in multiplication does not matter, that is $xy = yx$.
4. Let n be any positive integer, then the set $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ with operation $a + b \pmod n$ is an abelian group. The operation on $\mathbb{Z}_n$ we will simply be denote by $+$.
5. Let n be any positive integer and $\mathbb{Z}_n^* = \{m \in \mathbb{Z}_n \mid (m, n) = 1\}$ (that is the set of all integers in $\mathbb{Z}_n$ coprime with n) with operation $a \cdot b \pmod n$ is an abelian group. The operation on $\mathbb{Z}_n^*$ will simply be denoted by $\cdot$.

3.2 Multiplicative and additive notations

If we are given a group G, we usually denote its operation by $\circ$. However, it is very cumbersome to use this notation. Instead, people use symbols for usual multiplication or addition and there are two different types of notation: multiplicative and additive. Let me introduce the notation

	Multiplicative	Additive
Operation	$\cdot\colon G \times G \to G$	$+\colon G \times G \to G$
On elements	$(x, y) \mapsto xy$	$(x, y) \mapsto x + y$
Neutral Element	1	0
Inverse Element	x^{-1}	$-x$
Power of Element	$x^n = \underbrace{x \cdot \ldots \cdot x}_{n}$	$nx = \underbrace{x + \ldots + x}_{n}$

Usually the multiplicative notation is used in case of an arbitrary non-abelian group and the additive notation is used in case of an abelian group. I will mostly stick to the multiplicative notation and use the additive only in case of abelian groups.

I want to emphasize that these are just two different notations for the operation $\circ$. That is $xy = x \circ y$ or $x + y = x \circ y$. You just denote $\circ$ by $\cdot$ or $+$ depending on your preferences. Do not confuse these notations with the usual multiplication and addition. In case of an arbitrary group G, there is no confusion because there is no addition and multiplication on an arbitrary set G. However, If we deal with integer numbers (real, rational, complex, etc.), the operations $+$ and $\cdot$ denote usual addition and multiplication.

3.3 Subgroups

Definition 19. Let G be a group.[2] We define a subgroup H of G.

- **Data:**
  1. A subset $H \subseteq G$.
- **Axioms:**

[2] Strictly speaking $(G, \cdot)$ but I am going to use the short notation all the time.

1. The neutral element 1 of G belongs to H.
2. $xy \in H$ whenever $x, y \in H$.
3. $x^{-1} \in H$ whenever $x \in H$.

In this case, we say that H is a subgroup of G.

It should be noted that if H is a subgroup of G, then $\cdot$ is a well-defined operation on H and $(H, \cdot)$ becomes a group.

Examples 20. Let $G = \mathbb{Z}$ with addition.

1. If $H \subseteq \mathbb{Z}$ is the set of even numbers, that is $H = 2\mathbb{Z}$, then H is a subgroup.
2. If $H \subseteq \mathbb{Z}$ is the set of odd numbers, that is $H = 1 + 2\mathbb{Z}$, then H is not a subgroup. For example, the neutral element 0 is not in H. Also, H is not closed under addition.

3.4 Cyclic subgroups

Let G be a group and $g \in G$ be an arbitrary element. Then we may take any integer power of g as follows

Multiplicative notation	Additive notation
$g^n = \begin{cases} \underbrace{g \cdot \ldots \cdot g}_{n}, & n > 0 \\ 1, & n = 0 \\ \underbrace{g^{-1} \cdot \ldots \cdot g^{-1}}_{-n}, & n < 0 \end{cases}$	$ng = \begin{cases} \underbrace{g + \ldots + g}_{n}, & n > 0 \\ 0, & n = 0 \\ \underbrace{(-g) + \ldots + (-g)}_{-n}, & n < 0 \end{cases}$

Claim 21. *Let G be a group. Then*

1. *For any $x, y \in G$, $(xy)^{-1} = y^{-1}x^{-1}$.*
2. *For any $g \in G$, $(g^{-1})^n = (g^n)^{-1}$.*
3. *For any $g \in G$, $g^n g^m = g^{n+m}$ whenever $n, m \in \mathbb{Z}$.*

Proof. 1) We need to show that $(xy)^{-1} = y^{-1}x^{-1}$. Let us denote $y^{-1}x^{-1}$ by z. If we show that $(xy)z = z(xy) = 1$, this will mean that $z = (xy)^{-1}$ by definition. Now, we compute

$$(xy)z = xyz = xyy^{-1}x^{-1} = xx^{-1} = 1$$

In a similar way, we show the other equality.

2) We apply the previous property several times, that is

$$(g_1 \cdot \ldots \cdot g_n)^{-1} = g_n^{-1} \cdot \ldots \cdot g_1^{-1}, \quad \text{whenever } g_1, \ldots, g_n \in G$$

If we substitute $g_1 = \ldots = g_n = g$, this proves the required for $n > 0$.

If $n = 0$, then by definition $(g^{-1})^0 = 1$. From the other hand, $(g^0)^{-1} = 1^{-1} = 1$ because the inverse for 1 is 1.

If $n < 0$, then by definition

$$(g^{-1})^n = \underbrace{(g^{-1})^{-1} \cdot \ldots \cdot (g^{-1})^{-1}}_{-n}$$

On the other hand,

$$(g^n)^{-1} = (\underbrace{g^{-1} \cdot \ldots \cdot g^{-1}}_{-n})^{-1} = \underbrace{(g^{-1})^{-1} \cdot \ldots \cdot (g^{-1})^{-1}}_{-n}$$

The latter equality follows from the previous item.

3) We should consider 4 cases:

1. $n \geqslant 0$ and $m \geqslant 0$.
2. $n < 0$ and $m \geqslant 0$.
3. $n \geqslant 0$ and $m < 0$.

4. $n < 0$ and $m < 0$.

In the first case, we have

$$g^n g^m = \underbrace{g \cdot \ldots \cdot g}_{n} \cdot \underbrace{g \cdot \ldots \cdot g}_{m} = \underbrace{g \cdot \ldots \cdot g}_{n+m} = g^{n+m}$$

For convenience, we consider $g^{-n}g^m$ for $n > 0$ and $m \geqslant 0$ in the second case.

$$g^{-n}g^m = \underbrace{g^{-1} \cdot \ldots \cdot g^{-1}}_{n} \cdot \underbrace{g \cdot \ldots \cdot g}_{m}$$

We cancel the factors at the middle of the expression. If $n > m$, we get

$$\underbrace{g^{-1} \cdot \ldots \cdot g^{-1}}_{n-m} = g^{-n+m}$$

If $n < m$, we get

$$\underbrace{g \cdot \ldots \cdot g}_{m-n} = g^{m-n}$$

if $n = m$ we get $1 = g^{m-n}$. Other cases I leave as an exercise. □

Definition 22. Let G be a group and $g \in G$ be an arbitrary element. Then the set of all integer powers of g, that is,

$$\langle g \rangle = \{\ldots, g^{-2}, g^{-1}, 1, g, g^2, \ldots\}$$

is a subgroup of G. This group is called the cyclic subgroup generated by g. The element g is called a generator of $\langle g \rangle$.

The cyclic subgroup $\langle g \rangle$ is the smallest possible subgroup containing the element g.

Examples 23. 1. The group $(\mathbb{Z}, +)$ is cyclic. There are two different generators 1 and -1.

2. The group $(\mathbb{Z}_n, +)$ is cyclic.

3. The group of permutations on n elements S_n is not cyclic if $n > 2$.

4. The group $(\mathbb{R}, +)$ is not cyclic.

Claim 24. *Let G be a group and $g \in G$ be an arbitrary element. Then there are two options:*

- *If $\operatorname{ord} g = \infty$, then the elements g^n and g^m are different whenever $n, m \in \mathbb{Z}$ are different.*
- *If $\operatorname{ord} g = n < \infty$, then elements $1, g, g^2, \ldots, g^{n-1}$ are different. In this case, the powers are repeated in cycles, that is in the series*

$$\underbrace{\ldots, g^{-2}, g^{-1}}, \underbrace{1, g, g^2, \ldots, g^{n-1}}, \underbrace{g^n, g^{n+1}, \ldots, g^{2n-1}}, \underbrace{g^{2n}, \ldots}$$

$g^{kn}, g^{1+kn}, \ldots, g^{n-1+nk}$ are the same elements as $1, g, \ldots, g^{n-1}$ for any $k \in \mathbb{Z}$. In particular,

$$\langle g \rangle = \{1, g, \ldots g^{n-1}\}$$

Proof. If $g^n \neq g^m$ for all different $m, n \in \mathbb{Z}$, we are in the first case.

Now suppose that $g^n = g^m$ for some integer $m \neq n$. Then we may multiply this equality by g^{-m} and get $g^{n-m} = 1$. Hence, we may assume that for some $n \neq 0$, we have $g^n = 1$. If $n < 0$, multiply by g^{-n}. Thus, we may assume that for some positive integer n, we have $g^n = 1$.

Consider the minimal positive integer n such that $g^n = 1$. I claim that the elements $1, g, \ldots, g^{n-1}$ are different. Indeed, if $g^k = g^s$ for some $k, s \in [0, n-1]$ and $k \geqslant s$, then $g^{k-s} = 1$ and $k - s$ is not zero and is strictly less than n. The latter contradicts to the choice of n. □

It should be noted that n may equal 1 in case g is the neutral element 1.

Definition 25. Let G be a group and $g \in G$ be an arbitrary element. The order of g is the minimal positive natural number such that $g^n = 1$ and ∞ if there is no such a number. The order of g is denoted by $\operatorname{ord} g$.

From the previous Claim it follows that $\operatorname{ord} g$ equals the number of elements in $\langle g \rangle$. Note that $g = 1$ if and only if $\operatorname{ord} g = 1$.

If we use additive notation, that is the operation on the group G is denoted by $+$, then, the order of $g \in G$ is the small positive integer n such that $ng = 0$. The cyclic subgroup generated by g is

$$\langle g \rangle = \{\ldots, -2g, -g, 0, g, 2g, \ldots\}$$

Definition 26. Let G be a group. If there is an element $g \in G$ such that $\langle g \rangle = G$, then the group G is called cyclic.

Now, I want to describe all subgroups of the integers with addition.

Claim 27. *Every subgroup H of $\mathbb{Z}$, that is $(\mathbb{Z}, +)$, is of the form $k\mathbb{Z}$ for some natural k.*

Proof. Let us check that $k\mathbb{Z}$ is indeed a subgroup for any k. We need to check three properties of the subgroup. First, $k\mathbb{Z}$ is closed under addition. But this is clear by definition. Second, the neutral element, which is zero, belongs to $k\mathbb{Z}$. This is also clear since $0 = k \cdot 0$. Third, for each $m = kh \in k\mathbb{Z}$, its inverse $-m = k(-h)$ is also in $\mathbb{Z}$, and we are done with this part.

Now, let us check that every subgroup H is of the form $k\mathbb{Z}$. If H contains only the neutral element 0, then $H = 0\mathbb{Z}$ and we are done. Suppose H contains non-zero elements. Take an arbitrary non-zero $n \in H$. If $n < 0$, then $-n$ must belong to H by definition of a subgroup. And hence, we may assume that H contains some positive numbers. Let k be the smallest positive number in H. Let us show that $H = k\mathbb{Z}$.

First, $H \supseteq k\mathbb{Z}$. Indeed, if $k \in H$, then by definition of a subgroup every "power" of k is in H. For additive notation this means

$$mk = \underbrace{k + \ldots + k}_{m} \in H \quad \text{and} \quad (-n)k = \underbrace{(-k) + \ldots + (-k)}_{n} \in H \quad \text{for any} \quad m, n \in \mathbb{N}$$

Hence, $k\mathbb{Z} \subseteq H$.

Now, let us show that $H \subseteq k\mathbb{Z}$. If $n \in H$ is an arbitrary element, let us divide n by k: $n = qk + r$, where $q \in \mathbb{Z}$ and $0 \leqslant r < k$. We already know that $qk \in k\mathbb{Z} \subseteq H$, that is $qk \in H$. Hence, $r = n - qk \in H$. But r is a natural number in H smaller than k. Since k is the smallest positive in H, the only option is $r = 0$. Thus, $n = qk \in k\mathbb{Z}$ and we are done. □

Claim 28. *Every subgroup H of $\mathbb{Z}_n$, that is $(\mathbb{Z}_n, +)$, is of the form $k\mathbb{Z}_n = \{kh \in \mathbb{Z}_n \mid h \in \mathbb{Z}_n\}$ for some positive $k \mid n$.*

Proof. First, let us check that all numbers divisible by k such that $k \mid n$ form a subgroup in $\mathbb{Z}_n$. First, we need to check that $k\mathbb{Z}_n$ is closed under addition modulo n. Suppose $m_1 = kh_1$ and $m_2 = kh_2$ are elements of $k\mathbb{Z}_n$. Then their sum modulo n is a remainder r such that $m_1 + m_2 = r \pmod{n}$. In this case,

$$r = m_1 + m_2 + qn = kh_1 + kh_2 + qn$$

Since k divides n the whole expression above is divisible by k. Hence r is divisible by k. The latter means that $k\mathbb{Z}_n$ is closed under addition modulo n. Second, we need to check that $k\mathbb{Z}_n$ contains the neutral element. This is clear, since $0 = k \cdot 0 \in k\mathbb{Z}_n$. Third, if $m \in k\mathbb{Z}_n$ is a nonzero element, then its inverse is $n - m$. Since n is divisible by k, $n - m$ is divisible by k. Hence, it belongs to $k\mathbb{Z}_n$. In case $m = 0$ its inverse is 0 and is already in $k\mathbb{Z}_n$. Hence, for each $k \mod n$, $k\mathbb{Z}_n$ is a subgroup of $\mathbb{Z}_n$.

Now, let us show, that every subgroup H in $\mathbb{Z}_n$ coincides with a subgroup of the form $k\mathbb{Z}_n$ for $k \mid n$. The subgroup H must contain the neutral element 0. If this is the only element of H, then $H = \{0\} = n\mathbb{Z}_n$ and we are done. So, we may suppose there is a non-zero, and hence positive, element in H. Let k be the smallest positive element of H. By definition the cyclic subgroup of k, that is $k\mathbb{Z}_n$, belongs to H. Thus, we need to show, that $H \subseteq k\mathbb{Z}_n$ and k divides n.

First, let me show that k divides n. Let us divide n with remainder by k, we will get $n = qk + r$, where $0 \leqslant r < k$. Now, $r = n - qk$, hence $r = -qk \pmod{n}$. Since $k \in H$, the latter means that r is also in H. But this contradicts the choice of k (it was the smallest nonzero integer in H). Hence, r must be zero, thus k divides n. Second, let me show that every element of H is in $k\mathbb{Z}_n$. Suppose $h \in H$ is an arbitrary element. Let us divide h with remainder by k, we will get $h = qk + r$. Hence, $r = h - qk$. Since $h \in H$ and $k \in H$, the whole expression $h - qk$ is in H. Hence, $r \in H$. Since k was the smallest positive integer of H, we must have $r = 0$. The latter means that h is divisible by k, that is, h belongs to $k\mathbb{Z}_n$, and we are done. □

3.5 Cosets

Algebra usually tends to study groups using subgroups rather than elements. The main tool here is cosets.

Definition 29. Let G be a group, $H \subseteq G$ a subgroup and $g \in G$ an arbitrary element. Then the set

$$gH = \{gh \mid h \in H\}$$

is called the left coset of H with respect to g. In a similar way, we define right cosets. The set

$$Hg = \{hg \mid h \in H\}$$

is called the right coset of H with respect to g.

Remarks 30.
1. It should be noted that if G is commutative, then there is no difference between left and right cosets for any subgroup $H \subseteq G$.
2. The group H itself is a left coset as well as a right coset. Indeed, $H = 1 \cdot H = H \cdot 1$.
3. In general, the left cost gH need not be the same as the right coset Hg as an example below shows.

Examples 31. Here are some examples of cosets.

1. Let $G = (\mathbb{Z}, +)$ and $H = 2\mathbb{Z}$ the subgroup of even numbers. Then $2\mathbb{Z}$ and $1 + 2\mathbb{Z}$ are the only cosets of H.
2. Let $G = S_3$ and $H = \langle (1,2) \rangle$ the cyclic subgroup generated by the cycle $(1,2)$. We may list all the elements of G and H
$$G = \{1, (1,2), (1,3), (2,3), (1,2,3), (3,2,1)\},\ H = \{1, (1,2)\}$$
Then there are three different left cosets of H
$$H = \{1, (1,2)\},\ (1,3)H = \{(1,3), (1,2,3)\},\ (2,3)H = \{(2,3), (3,2,1)\}$$
And there are three different right cosets of H
$$H = \{1, (1,2)\},\ H(1,3) = \{(1,3), (3,2,1)\},\ H(2,3) = \{(2,3), (1,2,3)\}$$
This example shows that $(1,3)H \neq H(1,3)$. Also, it should be noted that
$$(1,2)H = H,\ (1,3)H = (1,2,3)H,\ (2,3)H = (3,2,1)H$$
So, cosets with respect to different elements may be the same.
3. Let $G = S_n$ be the group of permutations on n elements and $H = A_n$ be the subgroup of even permutations. Then, for any even permutation $\sigma \in A_n$, the set σA_n consists of all even permutations. Similarly, for any odd permutation $\sigma \in S_n \subseteq A_n$, the set σA_n consists of all odd permutations. Hence, there are only two left cosets of A_n
$$A_n \text{ and } (1,2)A_n$$
In a similar way, we can show that there are only two right cosets of A_n
$$A_n \text{ and } A_n(1,2)$$
Moreover, we have shown that $\sigma A_n = A_n \sigma$ for any $\sigma \in S_n$.

Definition 32. Let G be a group and H its subgroup. The subgroup H is normal if its left and right cosets are the same, that is, $gH = Hg$ whenever $g \in G$.

Claim 33. *Let G be a group and H its subgroup. Then, the following are equivalent*

1. *$gH = Hg$ for each $g \in G$.*
2. *$gHg^{-1} = H$ for each $g \in G$.*
3. *$gHg^{-1} \subseteq H$ for each $g \in G$.*

Proof. (1)$\Leftrightarrow$(2). Suppose $gH = Hg$. Multiply this on the right by g^{-1} and get $gHg^{-1} = H$. And if we are given $gHg^{-1} = H$, multiply this on the right by g and get $gH = Hg$.

(2)$\Leftrightarrow$(3). We should show that $gHg^{-1} \subseteq H$ for each $g \in G$ implies $gHg^{-1} = H$ for each $g \in G$. The other implication is clear. If $gHg^{-1} \subseteq H$ for each $g \in G$, then it holds for g^{-1} instead of g. Thus, $g^{-1}Hg \subseteq H$ for each $g \in G$. Multiply this on the left by g and get $Hg \subseteq gH$. Then, multiply the latter on the right by g^{-1} and get $H \subseteq gHg^{-1}$. Since $g \in G$ was arbitrary we are done.

□

3.6 The Lagrange Theorem

Properties of cosets Now, I want to prove several properties of the cosets. The important observation here is that all left cosets form a partition of the group G into non-overlapping subsets. The same is true for the right cosets. This observation provides us with some combinatorial tools.

Claim 34. *Let G be a group, $H \subseteq G$ a subgroup and $g_1, g_2 \in G$ be arbitrary elements. Then there are two options:*

1. *The cosets do not intersect each other: $g_1H \cap g_2H = \varnothing$.*
2. *The cosets coincide: $g_1H = g_2H$.*

This means that each element of the group G belongs to exactly one coset.

Proof. If g_1H does not intersect g_2H there is nothing to prove.

Now we assume that the intersection $g_1H \cap g_2H$ is not empty. We need to prove that $g_1H = g_2H$. Suppose $g \in g_1H \cap g_2H$. Since $g \in g_1H$, $g = g_1h_1$ for some $h_1 \in H$. Similarly, $g \in g_2H$ implies $g = g_2h_2$ for some $h_2 \in H$. Hence $g_1h_1 = g_2h_2$. Dividing by h_1 on the right, we get $g_1 = g_2h_2h_1^{-1}$. Since H is a subgroup $h = h_2h_1^{-1} \in H$. We have got $g_1 = g_2h$ for some $h \in H$.

Let us show that $g_1H \subseteq g_2H$. Suppose $g \in g_1H$, that is $g = g_1h'$ for some $h' \in H$. Then, $g = g_2hh' \in g_2H$ because $hh' \in H$. Now, suppose $g \in g_2H$, that is $g = g_2h'$ for some $h' \in H$. Then, $g = g_1h^{-1}h' \in g_1H$ because $h^{-1}h' \in H$. Hence, we have shown $g_2H \subseteq g_1H$. □

Remark 35. It should be noted that $g_1H = g_2H$ if and only if $g_1H \cap g_2H \neq \varnothing$. Moreover, this occurs if and only if there is an element $h \in H$ such that $g_1 = g_2h$. The latter is equivalent to the condition $g_2^{-1}g_1 \in H$. This provides us with a convenient way of checking if two cosets are the same.

Claim 36. *Let G be a group, $H \subseteq G$ be a finite subgroup and $g \in G$ an arbitrary element. Then $|gH| = |H| = |Hg|$.*

Proof. I will prove the claim for left cosets. Let us consider the map

$$\phi\colon H \to gH \quad x \mapsto gx$$

It takes elements of H to elements of gH. From the other hand, there is the inverse map

$$\psi\colon gH \to H \quad x \mapsto g^{-1}x$$

Thus ϕ and ψ are bijections and we are done. □

Claim 37. *Let G be a finite group and $H \subseteq G$ be a subgroup. Then*

1. *The amount of left cosets of H is equal to $|G|/|H|$.*
2. *The amount of right cosets of H is equal to $|G|/|H|$.*

In particular, the number of left and right cosets is the same.

Proof. We will prove the first item. Claim 34 shows that G is a disjoint union of some cosets, that is $G = g_1H \sqcup \ldots \sqcup g_kH$. From the other hand, Claim 36 shows that all the cosets $g_1H, \ldots, g_kH$ have the same amount of elements being equal to $|H|$. Hence

$$|G| = |g_1H| + \ldots + |g_kH| = |H| + \ldots + |H| = k|H|$$

Here k is the number of the distinct left cosets and we are done. □

Definition 38. Let G be a finite group and $H \subseteq G$ be a subgroup. Then the number of the left cosets of H is called index of H and is denoted by $(G : H)$. This number is also coincide with the number of the right cosets of H.

Using this notation, we can rewrite Claim 37 in the following way.

Claim 39 (The Lagrange Theorem)**.** *Let G be a finite group and $H \subseteq G$ be a subgroup. Then, $|G| = (G : H)|H|$*

Corollaries of The Lagrange Theorem

1. Let G be a finite group and $H \subseteq G$ be a subgroup. Then $|H|$ divides $|G|$.
2. Let G be a finite group and $g \in G$ be an arbitrary element. Then $\operatorname{ord}(g)$ divides $|G|$. Indeed, $\operatorname{ord}(g) = |\langle g \rangle|$. But $|\langle g \rangle|$ divides $|G|$ by the previous item.
3. Let G be a finite group and $g \in G$ be an arbitrary element. Then $g^{|G|} = 1$. Indeed, we already know that $|G| = \operatorname{ord}(g)k$. Hence,
$$g^{|G|} = g^{\operatorname{ord}(g)k} = \left(g^{\operatorname{ord}(g)}\right)^k = 1^k = 1$$
4. Let G be a group of prime order p. Then, G is cyclic. Indeed, since the order of G is prime, it is greater than 1. Hence, there is an element $g \in G$ such that $g \neq 1$. Hence $\langle g \rangle$ has order greater than 1. But $|\langle g \rangle|$ divides $|G| = p$. Since p is prime, the only option is $|\langle g \rangle| = p = |G|$. The latter means that $\langle g \rangle = G$ and we are done.
5. The Fermat Little Theorem. Let $p \in \mathbb{Z}$ be a prime number and $a \in \mathbb{Z}$. If p does not divide a, then p divides $a^{p-1} - 1$. Indeed, let us consider the group $(\mathbb{Z}_p^*, \cdot)$. For any element $b \in \mathbb{Z}_p^*$, we have $b^{|\mathbb{Z}_p^*|} = 1 \pmod{p}$ by item (3). But $\mathbb{Z}_p^*$ has $p-1$ elements. Now, let $a \in \mathbb{Z}$ be comprime with p. We denote its remainder modulo p by b. Then $a^{p-1} = b^{p-1} = 1 \pmod{p}$ and we are done.

3.7 Homomorphisms and Isomorphisms

We have a lot of different groups in algebra. And we will study a couple of ways to produce new groups from given ones. In this situation, we need a way to compare the groups. How to recognize that we have constructed the same group that we already know? In order to answer this question, we need to explain what it means that two groups are the same. So, we need a way to compare two groups. This leads us to the notions of homomorphism (a way to compare groups) and isomorphism (a way to say that groups are the same). Let me proceed with formal definitions.

Definition 40. Let G and H be groups. We define a homomorphism $\varphi\colon G \to H$.

- **Data** A map $\varphi\colon G \to H$.
- **Axiom** $\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)$ for any $g_1, g_2 \in G$.

In this case φ is called a homomorphism from G to H.

Remark 41. Let me explicitly repeat the definition. We are given a group $(G, \circ)$ and $(H, \cdot)$. A homomorphism $\varphi\colon G \to H$ is a map such that $\varphi(g_1 \circ g_2) = \varphi(g_1) \cdot \varphi(g_2)$. On the left-hand side we take elements g_1 and g_2 from G and multiply them using the operation on G and then send the resulting element to H. On the right-hand side, we send the elements g_1 and g_2 to the group H first and then multiply the images using operation on H.

Examples 42.
1. Let $G = (\mathbb{Z}, +)$ and $H = (\mathbb{Z}_n, +)$, then the map $\pi\colon \mathbb{Z} \to \mathbb{Z}_n$ by the rule $k \mapsto k \pmod{n}$ is a homomorphism.
2. Let $G = S_n$ be the group of permutations and $H = \mu_2 = \{\pm 1\}$ with multiplication. Then the map $\operatorname{sgn}\colon S_n \to \mu_2$ taking each permutation to its sign (even one goes to 1 and odd one goes to -1) is a homomorphism.
3. Let $G = (\mathrm{GL}_n(\mathbb{R}), \cdot)$ and $H = (\mathbb{R}^*, \cdot)$ be the set of non-zero real numbers with multiplication. Then the map $\det\colon \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}^*$ by the rule $A \mapsto \det(A)$ is a homomorphism.
4. Let $G = (\mathbb{R}, +)$ and $H = (\mathbb{R}^*, \cdot)$. Then the map $\exp\colon \mathbb{R} \to \mathbb{R}^*$ by the rule $x \mapsto e^x$ is a homomorphism.
5. Let $G = (\mathbb{Z}, +)$, H be an arbitrary group and $h \in H$ be an arbitrary element. Then the map $\phi\colon \mathbb{Z} \to H$ by the rule $k \mapsto h^k$ is a homomorphism.
6. Let $G = (\mathbb{Z}_n, +)$, H be an arbitrary group and $h \in H$ be an element such that $h^n = 1$. Then the map $\phi\colon \mathbb{Z}_n \to H$ by the rule $k \mapsto h^k$ is a group homomorphism.

Let us prove several properties of homomorphisms.

Claim 43. *Let $\varphi\colon G \to H$ be a homomorphism of groups. Then*

1. *$\varphi(1) = 1$, that is the neutral element of G goes to the neutral element of H.*

2. $\varphi(g^{-1}) = \varphi(g)^{-1}$ *whenever* $g \in G$.

Proof. 1) We know that $1 = 1 \cdot 1$. Let us apply φ to this equality. Then we get

$$\varphi(1) = \varphi(1 \cdot 1) = \varphi(1)\varphi(1) \in H$$

Now, multiply this equality by $\varphi(1)^{-1}$, we will get $1 = \varphi(1)$.

2) Let $g \in G$ be an arbitrary element. Then, $gg^{-1} = 1$. Let us apply φ to this equality. Then we get

$$\varphi(g)\varphi(g^{-1}) = \varphi(gg^{-1}) = \varphi(1) = 1$$

Now multiply this on the left by $\varphi(g)^{-1}$. We will get $\varphi(g^{-1}) = \varphi(g)^{-1}$. □

Definition 44. Let G and H be groups. We define an isomorphism $\varphi\colon G \to H$.

- **Data** A homomorphism $\varphi\colon G \to H$.
- **Axiom** φ is bijective.

In this case, φ is called an isomorphism between G and H. If there is an isomorphism between G and H, the groups G and H are called isomorphic.

Let me clarify the definition. First let me explain what it means that $\varphi\colon X \to Y$ is a bijection (between sets). Suppose $X = \{1, 2, 3\}$, $Y = \{a, b, c\}$, and φ works as follows $1 \mapsto a$, $2 \mapsto b$, and $3 \mapsto c$. Then you may think of it like this. The set X is a set of names for your elements and the set Y is a set of some other names for your elements. Than the map φ is a renaming map it just switches the names of the elements. Thus, you may think that Y is the same set as X but with elements named differently.

Now, if $\varphi\colon G \to H$ is an isomorphism of groups, then it is at least a bijection. Hence, it identifies elements of G and H saying that the underlying sets of the groups are the same. Also, the condition $\varphi(g_1g_2) = \varphi(g_1)\varphi(g_2)$ means that after this identification the operation on G becomes an operation on H. The latter means that you rename the elements and the operation. Hence, you may think that H is exactly the same group as G but with different set of names for the elements and a different notation for the operation. However, this is essentially the same group. As a corollary, isomorphic groups have exactly the same properties.

Examples 45.

1. Let $G = (\mathbb{Z}_n, +)$ and $H = \mu_n \subseteq \mathbb{C}$ be the set of complex roots of unity with multiplication as an operation. Let us fix a primitive root $\xi \in \mu_n$. Then the map $\mathbb{Z}_n \to \mu_n$ by the rule $k \mapsto \xi^k$ is an isomorphism.
2. Let $G = (\mathbb{Z}, +)$ and
$$H = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \;\middle|\; n \in \mathbb{Z} \right\}$$
with multiplication as an operation. Then the map $\varphi\colon \mathbb{Z} \to H$ by the rule $k \mapsto \left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right)$ is an isomorphism.
3. Let $G = (\mathbb{C}, +)$ and $H = (\mathbb{R}^2, +)$. Then the map $\varphi\colon \mathbb{C} \to \mathbb{R}^2$ by the rule $z \mapsto (\operatorname{Re} z, \operatorname{Im} z)$ is an isomorphism.
4. Let $G = (\mathbb{C}^*, \cdot)$ and
$$H = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \;\middle|\; a, b \in \mathbb{R} \text{ such that } a^2 + b^2 \neq 0 \right\}$$
with multiplication as an operation. Then the map $\varphi\colon \mathbb{C}^* \to H$ by the rule $a + bi \mapsto \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$ is an isomorphism.
5. Claim 24 says that a cyclic group $G = \langle g \rangle$ is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}_n$ depending on the order of a generator. If $\operatorname{ord} g = \infty$, then $G \simeq \mathbb{Z}$. If $\operatorname{ord} g = n$, then $G \simeq \mathbb{Z}_n$.

With each homomorphism we may associate several subgroups: kernel and image.

Definition 46. Let $\varphi\colon G \to H$ be a homomorphism of groups. Then

1. The kernel of φ is $\ker \varphi = \{g \in G \mid \varphi(g) = 1\} \subseteq G$.
2. The image of φ is $\operatorname{Im} \varphi = \{\varphi(g) \mid g \in G\} = \varphi(G) \subseteq H$.

It should be noted that the kernel is a subset of G (belongs to the source of the map φ) and the image is a subset of H (belongs to the target of the map φ).

Claim 47. *Let $\varphi\colon G \to H$ be a homomorphism of groups. Then*

1. $\operatorname{Im}\varphi \subseteq H$ *is a subgroup.*
2. $\ker\varphi \subseteq G$ *is a normal subgroup.*
3. *The map φ is surjective if and only if* $\operatorname{Im}\varphi = H$.
4. *The map φ is injective if and only if* $\ker\varphi = \{1\}$.

Proof. 1) Let us check all the requirements for being subgroup. First, $1 = \varphi(1) \in \operatorname{Im}\varphi$, hence we have the identity. Second, $\varphi(g_1)\varphi(g_2) = \varphi(g_1g_2) \in \operatorname{Im}\varphi$, thus it is closed under the operation. Third, $\varphi(g)^{-1} = \varphi(g^{-1}) \in \operatorname{Im}\varphi$, therefore it contains the inverse element for every element.

2) Let us check the requirements for the subgroup. First, $\varphi(1) = 1$, hence $1 \in \ker\varphi$ by definition. Second, if $x, y \in \ker\varphi$, then $\varphi(xy) = \varphi(x)\varphi(y) = 1\cdot 1 = 1$. Therefore, $xy \in \ker\varphi$. Third, if $x \in \ker\varphi$, then $\varphi(x^{-1}) = \varphi(x)^{-1} = 1^{-1} = 1$. Hence, $x^{-1} \in \ker\varphi$. We have just verified that $\ker\varphi$ is a subgroup. We should show that $g\ker\varphi = \ker\varphi g$ for every $g \in G$. By Claim 33, we need to show that $g\ker\varphi g^{-1} \subseteq \ker\varphi$ for each $g \in G$. That is we should show that $\varphi(g\ker\varphi g^{-1}) = 1$ for each $g \in G$. Indeed, for each $h \in \ker\varphi$, we have

$$\varphi(ghg^{-1}) = \varphi(g)\varphi(h)\varphi(g^{-1}) = \varphi(g)\cdot 1\cdot\varphi(g^{-1}) = \varphi(g)\varphi(g^{-1}) = \varphi(gg^{-1}) = \varphi(1) = 1$$

3) This holds trivially by the definition.

4) Suppose φ is injective and $x \in \ker\varphi$. The latter means that $\varphi(x) = 1$. From the other hand, we know that $\varphi(1) = 1$. Hence x and 1 go to the same element 1. This means that $x = 1$ by the injectivity.

Now suppose that $\ker\varphi = \{1\}$. Consider two elements $x, y \in G$ such that $\varphi(x) = \varphi(y)$. Multiplying by $\varphi(x)^{-1}$, we get

$$1 = \varphi(y)\varphi(x)^{-1} = \varphi(y)\varphi(x^{-1}) = \varphi(yx^{-1})$$

Hence $yx^{-1} \in \ker\varphi = \{1\}$. Thus $yx^{-1} = 1$. Therefore $y = x$ and we are done. □

3.8 Product of groups

In general, we do not want to produce new groups from scratch. We want to construct a group using already given ones. There are many different operations in algebra to produce new groups. We are not going to learn all of them. Instead, we discuss the simplest one, which is the most useful one at the same time.

Definition 48. Let G and H be groups, we define a new group $G \times H$ as follows

1. As a set it is a product of underlying sets of the groups: $G \times H = \{(g, h) \mid g \in G,\ h \in H\}$.
2. The operation
$$\cdot\colon (G \times H) \times (G \times H) \to G \times H$$
is given by the rule
$$(g_1, h_1)(g_2, h_2) = (g_1g_2, h_1h_2), \quad g_1, g_2, \in G,\ h_1, h_2 \in H$$

The group $G \times H$ is called the product of the groups G and H.

It should be noted that we must show that $G \times H$ is indeed a group. We have just defined the required data for a group. However, we need to check the axioms. Let me recall them.

- The operation is associative, that is
$$(g_1, h_1)\Big((g_2, h_2)(g_3, h_3)\Big) = \Big((g_1, h_1)(g_2, h_2)\Big)(g_3, h_3)$$
- There is an identity, $1 = (1, 1)$.
- Each element has inverse, $(g, h)^{-1} = (g^{-1}, h^{-1})$.

All the properties are verified by a direct computation. I am leaving this as an exercise. If we have several groups $G_1, \ldots, G_k$, we may produce the group $G_1 \times \ldots \times G_k$ in a similar way.

3.9 Finite Abelian Groups

Now I want to focus on the most important class of groups, that is the class of finitely generated abelian groups. Let me start with the definition.

Definition 49. A finite abelian group is a commutative (abelian) group G with finitely many elements.

The definition is not a surprise, the name of the term is clear enough. But pay attention to the next result.

Claim 50. *Let G be a finite abelian group, the G is isomorphic to a group $\mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_k}$.*

I am not going to prove this result. The proof is not hard but requires some technical tools that we are not going to learn because of the lack of time. Also the proof itself does not reveal any important technique. So it is better to spend our time mastering the way of using the result instead of proving it. Now, I want to show you several examples.

Examples 51.

1. Let $G = \mathbb{Z}_8^*$ with multiplication as an operation. It is obviously a finite abelian group, hence it must be a product of cyclic groups. Indeed, we can check that
$$\mathbb{Z}_8^* \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$$
and the operation preserving bijection (that is isomorphism) is given by
$$1 \leftrightarrow (0,0),\ 3 \leftrightarrow (1,0),\ 5 \leftrightarrow (0,1),\ 7 \leftrightarrow (1,1)$$
This is not the only way to identify these two groups. We may take a different isomorphism
$$1 \leftrightarrow (0,0),\ 3 \leftrightarrow (1,0),\ 7 \leftrightarrow (0,1),\ 5 \leftrightarrow (1,1)$$
I am not going to describe all possible ways, but it must be clear that there are many different isomorphisms serving our purpose. Note also that the group is not cyclic because there is no element of order 4 in it.
2. Let $G = \mathbb{Z}_9^*$ with multiplication as an operation. This is also a finitely generated abelian group. In this case, we have
$$\mathbb{Z}_9^* \simeq \mathbb{Z}_6$$
and there are two different isomorphisms
$$\begin{array}{c} \mathbb{Z}_6 \to \mathbb{Z}_9^* \\ k \mapsto 2^k \end{array} \quad \text{and} \quad \begin{array}{c} \mathbb{Z}_6 \to \mathbb{Z}_9^* \\ k \mapsto 5^k \end{array}$$
Also note that the group here is cyclic. The elements 2 and 5 are different generators of the group. The isomorphisms above correspond to the choice of a generator.

Claim 52 (The Chinese Remainder Theorem)**.** *Let $m, n \in \mathbb{N}$ be two coprime positive integers, that is $(m,n) = 1$. Then the map*
$$\Phi\colon \mathbb{Z}_{mn} \to \mathbb{Z}_m \times \mathbb{Z}_n, \quad k \mapsto (k \mod m,\ k \mod n)$$
is an isomorphism of groups.

Proof. First, we should check that the map is a homomorphism. We need to show that $\Phi(k+d) = \Phi(k) + \Phi(d)$, that is
$$\Phi(k+d) = ((k+d) \mod m,\ (k+d) \mod n) = ((k \mod m) + (d \mod m),\ (k \mod n) + (d \mod n)) =$$
$$= (k \mod m,\ k \mod n) + (d \mod m,\ d \mod n) = \Phi(k) + \Phi(d)$$

Now, I claim that the homomorphism is injective. Claim 47 item (4) ensures that it is enough to show that the kernel of the homomorphism consists of the identity element only. By definition,
$$\ker \Phi = \{k \in \mathbb{Z}_{mn} \mid k = 0 \pmod m,\ k = 0 \mod n\}$$
Hence $k \in \ker \Phi$ if and only if m divides k and n divides k. Since m and n are coprime, this implies that mn divides k. The latter means that $k = 0$ in $\mathbb{Z}_{mn}$.

In order to prove that Φ is an isomorphism, we need to show that it is surjective. Let us compute the number of elements in both groups. By definition $|\mathbb{Z}_{mn}| = mn$. From the other hand, $|\mathbb{Z}_m \times \mathbb{Z}_n| = |\mathbb{Z}_m| \cdot |\mathbb{Z}_n| = mn$. Hence, Φ is an injective map between two sets of the same size. Hence, it must be bijective and we are done. □

From the previous claim it is clear how to take elements from $\mathbb{Z}_{mn}$ to the product $\mathbb{Z}_m \times \mathbb{Z}_n$. However, It is worth mentioning who to produce the map in the other direction. Since m and n are coprime, we have $1 = um + vn$ for some $u, v \in \mathbb{Z}$ by the Euclidean algorithm. Now consider the element $a_1 = um = 1 - vn$. It is clear that $a_1 \mapsto (0, 1)$ under the action of Φ. Similarly, the element $a_2 = vn = 1 - um$ goes to $(1, 0)$. Hence, the element (a, b) corresponds to the element $aa_1 + ba_2 \pmod{mn}$ in $\mathbb{Z}_{mn}$.

Examples 53.

1. In case $m = 3$ and $n = 2$, we have $\mathbb{Z}_6 \simeq \mathbb{Z}_3 \times \mathbb{Z}_2$. Here element 1 goes to $(1, 1)$. Hence $(1, 1)$ is the generator of the cyclic group $\mathbb{Z}_3 \times \mathbb{Z}_2$. Since $1 = 3 - 2$, we see that 3 goes to $(0, 1)$ and -2 goes to $(1, 0)$ (note that $-2 = 4$ in $\mathbb{Z}_6$). Hence the inverse map is given by $(a, b) \mapsto -2a + 3b = 4a + 3b \pmod 6$.
2. From the other hands, the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not cyclic. Hence there is no isomorphism with $\mathbb{Z}_4$.
3. Another example of different presentations of a finite abelian group

   $$\mathbb{Z}_{30} \simeq \mathbb{Z}_6 \times \mathbb{Z}_5 \simeq \mathbb{Z}_3 \times \mathbb{Z}_{10} \simeq \mathbb{Z}_2 \times \mathbb{Z}_{15} \simeq \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$$

   So, all five different constructions give us the same cyclic group.
4. In general, if $m = p_1^{k_1} \dots p_r^{k_r}$, where p_i are prime, then

   $$\mathbb{Z}_m = \mathbb{Z}_{p_1^{k_1}} \times \dots \times \mathbb{Z}_{p_r^{k_r}}$$

As we saw above, the same finite abelian group may be written in many different ways. How to check quickly that two different representations give us the same group? The answer is given in the next result.

Claim 54. *Let G be a finite abelian group. Then*

1. *G is uniquely presentable in the following form*

   $$G = \mathbb{Z}_{d_1} \times \dots \times \mathbb{Z}_{d_k}, \quad \text{where } 1 < d_1 | d_2 | \dots | d_k \text{ are positive integers}$$

2. *G is uniquely (up to permutation of factors) presentable in the following form*

   $$G = \mathbb{Z}_{p_1^{k_1}} \times \dots \times \mathbb{Z}_{p_r^{k_r}}, \quad \text{where } p_i \text{ are (not necessarily distinct) primes, } k_i \text{ are positive integers}$$

It is important to mention that primes p_i may repeat in the second presentation, that is $\mathbb{Z}_2 \times \mathbb{Z}_4$ is one of the possible cases.

Examples 55.

1. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_6$ and $H = \mathbb{Z}_{12}$. These groups are presented in the first form. Since such a presentation is unique G and H are not isomorphic.
2. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_6$ and $H = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$. We see that G is presented in the first form and H is presented in the second one. Let us recompute G into the second form using the Chinese Remainder Theorem

   $$G = \mathbb{Z}_2 \times \mathbb{Z}_6 = \mathbb{Z}_2 \times (\mathbb{Z}_2 \times \mathbb{Z}_3) = H$$

   Hence, the groups are isomorphic.

Now I want to formulate a second version of the Chinese Remainder Theorem.

Claim 56. *Let $m, n \in \mathbb{N}$ be two coprime positive integers, that is $(m, n) = 1$. Then the map*

$$\Phi \colon \mathbb{Z}_{mn}^* \to \mathbb{Z}_m^* \times \mathbb{Z}_n^*, \quad k \mapsto (k \mod m, k \mod n)$$

is a well-defined isomorphism of groups.

Proof. We already know that $\Phi \colon \mathbb{Z}_{mn} \to \mathbb{Z}_m \times \mathbb{Z}_n$ is a bijection. It is clear that k is coprime with mn if and only if k is coprime with m and k is coprime with n. The latter means that $\Phi \colon \mathbb{Z}_{mn}^* \to \mathbb{Z}_m^* \times \mathbb{Z}_n^*$ is a bijection.

Second, we should show that Φ preserves multiplication. On the one hand

$$\Phi(k_1 k_2) = (k_1 k_2 \mod m, k_1 k_2 \mod n)$$

From the other hand

$$\Phi(k_1)\Phi(k_2) = (k_1 \mod m, k_1 \mod n)(k_2 \mod m, k_2 \mod n) = (k_1 k_2 \mod m, k_1 k_2 \mod n)$$

□

This result means that we can reduce computation of $\mathbb{Z}_n^*$ to the computation of groups $\mathbb{Z}_{p^k}^*$ where p is prime. Indeed, if $n = p_1^{k_1} \dots p_r^{k_r}$, then

$$\mathbb{Z}_n^* \simeq \mathbb{Z}_{p_1^{k_1}}^* \times \dots \times \mathbb{Z}_{p_r^{k_r}}^*$$

In order to complete the computation, we need to know the answer for the powers of primes. Here is the required result without proof.

Claim 57. *If p is an odd prime and n is an arbitrary positive integer, then*

$$\mathbb{Z}_{p^n}^* \simeq \mathbb{Z}_{p^{n-1}(p-1)}$$

is a cyclic group. An integer $a \in \mathbb{Z}_{p^n}$ is a generator of $\mathbb{Z}_{p^n}^*$ if and only if a is a generator in $\mathbb{Z}_p^*$ and $a^{p-1} \neq 1 \pmod{p^2}$. Hence, every element of $\mathbb{Z}_{p^n}^*$ is uniquely presented in the form a^k, where $0 \leqslant k < p^{n-1}(p-1)$.

In case of a power of 2, the answer is the following

$$\mathbb{Z}_{2^n}^* \simeq \begin{cases} 0, & n \leqslant 1 \\ \mathbb{Z}_2, & n = 2 \\ \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}, & n \geqslant 3 \end{cases}$$

In case $n = 2$, the group is generated by the element $3 = -1$. In case $n \geqslant 3$, the first factor is generated by $2^n - 1 = -1$ and the second factor is generated by 5. Hence, every element of $\mathbb{Z}_{2^n}^*$ is uniquely presented in the form $\pm 5^k$, where $0 \leqslant k < 2^{n-2}$.

In particular, the group $\mathbb{Z}_p^*$ is cyclic of order $p-1$ for any prime p. We will show this result latter using some abstract algebra.

Claim 58. *An element $m \in \mathbb{Z}_n$ is a generator if and only if m and n are coprime.*

Proof. ($\Rightarrow$). Suppose $(m, n) = d > 1$. Then all elements of $\langle m \rangle$ are divisible by d. In particular, we will never get an element 1. Hence m is not a generator, a contradiction. Therefore m and n are coprime.

($\Leftarrow$). We want to show that $\langle m \rangle = \mathbb{Z}_n$. Since 1 is a generator of $\mathbb{Z}_n$, it is enough to show that $1 \in \langle m \rangle$. Since m and n are coprime, there exist elements $a, b \in \mathbb{Z}$ such that $1 = am + bn$. Hence $1 = am \pmod{n}$. The latter means that 1 is a-th power of m, thus, $1 \in \langle m \rangle$. □

4 Cryptography

4.1 The setting

Suppose there are you, your spouse, and your lover. And you want to send a message to your lover suggesting a private meeting. Also you have seen recently a receipt from a gun store that someone bought a shotgun and you are one hundred percent sure that it was not you. What to do in such a subtle situation? Cryptography to the rescue! The basic idea behind any cryptographic method is this: there are some procedures that are easy to compute in one direction but are very hard to compute in the opposite one, that is the inverse map is difficult to compute. So, it is easy to encrypt the data but hard to decrypt them. But we do not use these procedures as they are because no one will be able to extract the original data. However, using such procedure, we produce a different one. This new procedure is also easy to compute but the inverse map is hard to compute unless you know some secret information.

Before going straight into details, I need to give you an example of such a procedure. There are two most popular ones.

- The direct procedure is the multiplication of integer numbers and the inverse one is the factorization of an integer.
- The direct procedure is raising to a power in an abelian group and the inverse one is taking logarithm. The inverse one is hard to compute if the abelian group is chosen well enough.

Since we are here to utilize some abstract algebra, I am going to focus on the second case. Suppose G is a group (usually finite abelian, but it does not matter for now), $g \in G$ is an arbitrary element, and $n \in \mathbb{N}$ is a natural number. Then we may compute the element $h = g^n$. It turns out that raising to a power can be done in $O(\log n)$ operations (the algorithm will be explained below). However, suppose now that h is known and either g or n is not. Now, we want to solve one of the following problems

- $h = (?)^n$ for some $? \in G$.
- $h = g^?$ for some $? \in \mathbb{N}$.

Whether these problems are hard to compute or not depends on the choice of the group G and the element h in the first case or g in the second one. But if we chose everything carefully, these problems become hard. The first problem is utilized in RSA problem and the second one in Diffie-Hellman exchange procedure. RSA problem is related to factorization problem of composite numbers and is a very popular method. However, I am going to discuss Diffie-Hellman approach only.

4.2 Exponentiation by squaring

First I want to explain why raising to a power is a very fast operation. Suppose G is a group, $g \in G$ is an element and $n \in \mathbb{N}$ is a positive integer. Then, using multiplicative or additive notation, we have

$$g^n = \begin{cases} g(g^2)^k, & n = 2k+1 \\ (g^2)^k, & n = 2k \end{cases} \quad \text{or} \quad ng = \begin{cases} g + k(2g), & n = 2k+1 \\ k(2g), & n = 2k \end{cases}$$

So, in multiplicative case, the problem is raising to a power and, in additive case, the problem is multiplying by an integer. I will describe the algorithm for the multiplicative notation only.

Input: $g \in G$, $n \in \mathbb{N}$.

Output: $g^n \in G$.

We use three internal variables $r, d \in G$ and $k \in \mathbb{N}$. We maintain the invariant $rd^k = g^n$ all the time. The result will be stored in r. The algorithm terminates when $k = 0$.

Algorithm

1. Set $r = 1 \in G$, $d = g \in G$, $k = n \in \mathbb{N}$.
2. In a loop, check if k is odd or even. Terminate the loop if $k = 0$.
   (a) If k is even. Assign $r = r$, $d = d^2$, $k = k/2$.
   (b) If k is odd. Assign $r = r \cdot d$, $d = d^2$, $k = (k-1)/2$.

Remarks During the procedure we have $rd^k = g^n$. At the beginning $r = 1$, $d = g$, $k = n$, so this holds. At each step of the loop we have two cases:

- $k = 2m$. Then, $rd^{2m} = r(d^2)^m$. And we update $r = r$, $d = d^2$, and $k = m = k/2$.
- $k = 2m + 1$. Then, $rd^{2m+1} = (rd)(d^2)^m$. And we update $r = rd$, $d = d^2$, and $k = m = (k-1)/2$.

There is a similar procedure as follows. Suppose for simplicity that $n = 11$. Then $11 = 1+2+2^3 = 1+2(1+2(0+2))$. Then

$$g^{11} = g^{1+2(1+2(0+2))} = g(g^{1+2(0+2)})^2 = g(g(g^{0+2})^2)^2 = g(g(g^2)^2)^2$$

If $n = 2^k$, then you need exactly k operations. For example $n = 8 = 2^3$, then $g^8 = ((g^2)^2)^2$. So, there $\log_2 n$ operations. In general, the number of operations is proportional to $\log_2 n$. But I do not want to explain this carefully.

4.3 The Discrete logarithm problem

Let G be a group, $g \in G$, and $h \in \langle g \rangle$. Then the problem to find $n \in \mathbb{N}$ such that $g^n = h$ is called the Discrete logarithm problem. Sometimes this procedure is fast sometimes is not. Here are some examples.

Examples 59.
1. Let $G = \mathbb{Z}$ with addition, $g = 1$, and $h = k$. Then it is clear to everyone that the required $n = k$. Indeed, $ng = h$. The problem here is trivial.
2. Let $G = \mathbb{Z}_m$ with addition, $g = a \in \mathbb{Z}_m$, and $h = b \in \mathbb{Z}_n$. Then, the problem is to find $n \in \mathbb{N}$ such that $na = b \pmod m$. This can be solved effectively using Euclidean division algorithm.

3. Let p be a prime number, $G = \mathbb{Z}_p^*$ with multiplication and $g = a \in \mathbb{Z}_p^*$ be a generator of the group, and $h = b \in \mathbb{Z}_p^*$. Then the problem is to find $n \in \mathbb{N}$ such that $a^n = b \pmod p$. Well, the experience of the humankind tells us that this problem should be extremely complicated and there is only on option to solve it: the brute force approach.

4.4 Diffie-Hellman

Here I am going to explain the communication process using Diffie-Hellman approach. First, we need to fix a cyclic group G, its generator $g \in G$, and we denote the order of G by n. A natural choice for G is $\mathbb{Z}_p^*$, where p is prime. Finding a generator is unpleasant but we should do this only once.

Let me recall the situation we are in. We have three participants: you, your spouse, and your lover. The communication process consists of several steps:

1. Transform a message (or a part of a message) into an element t of the group G.
2. Encrypt the element t, that is apply some transformation and get $t' \in G$. The element t' is then broadcasted.
3. The element t' is decrypted using some special information to recover the element t.
4. The element t is transformed to the initial message (or the part of the message).

The steps (1) and (4) are usually performed using some table and the table is known to every participant. Do not worry, your spouse knows how to do these steps.

Key exchange Before sending any messages you and your lover must do some preparations to produce a private key and only then the communication begins.

On the diagram below, we show what each participant knows at any step of the process. Let me recall that $G = \langle g \rangle$ and $n = |G|$.

Participants	You	Your spouse	Your lover
Knowledge	G, g, n	G, g, n	G, g, n

You randomly generate a number $a \in \mathbb{Z}_n^*$ and compute $r = g^a \in G$. Your lover randomly generates a number $b \in \mathbb{Z}_n^*$ and computes $s = g^b \in G$.

Participants	You	Your spouse	Your lover
Knowledge	G, g, n $r{=}g^a$	G, g, n	G, g, n $s{=}g^b$

You and your lover broadcast elements r and s. Hence, everyone knows r and s as the result. But no one knows the elements a and b because this is the discrete logarithm problem in G and we have chosen G and $g \in G$ such that the problem is hard to solve.

Participants	You	Your spouse	Your lover
Knowledge	G, g, n $r{=}g^a$, s	G, g, n r, s	G, g, n $s{=}g^b$, r

You raise element s to the power a and get $s^a = (g^b)^a = g^{ab}$. Your lover raises r to the power b and gets $r^b = (g^a)^b = g^{ab}$. Now you and your lover know the secret key $k = g^{ab}$.

Participants	You	Your spouse	Your lover
Knowledge	G, g, n $r{=}g^a$, s $k{=}s^a$	G, g, n r, s	G, g, n $s{=}g^b$, r $k{=}r^b$

As the result you and your lover know the secret key $k \in G$ and no one even your spouse has a way of finding the key. But in order this to be robust we need to choose G and $g \in G$ carefully. No one wants to find out who bought the shotgun.

Broadcast Now its time to send sweet messages to each other. As I have described already, we should translate all the messages to the elements of the group G. Suppose we use English alphabet with 26 symbols. We will use period, comma, exclamation mark, and space symbol as well. Hence, we have 30 symbols at all. There are 30^m sequences with m symbols. If $30^m \leqslant n$, we may map all the sequences to the elements of the group G. This allows us to transform messages to sequences of elements of the group G.

From now, I am going to ignore the translation stage. Our goal is to send an element of the group G. Suppose you have an element $h \in G$ and want to send it to your lover.

Participants	You	Your spouse	Your lover
Knowledge	G, g, n $r{=}g^a$, s $k{=}s^a$ h	G, g, n r, s	G, g, n $s{=}g^b$, r $k{=}r^b$

Now you encrypt your element h multiplying it by the secret k and send the result $m = hk$ to your lover.

Participants	You	Your spouse	Your lover
Knowledge	G, g, n $r{=}g^a$, s $k{=}s^a$ h, $m = hk$	G, g, n r, s m	G, g, n $s{=}g^b$, r $k{=}r^b$ m

Your lover decrypts the message by computing $h = mk^{-1} = mk^{n-1}$. It should be noted that the inverse k^{-1} is computed using Corollary 3 of the Lagrange Theorem, that is $k^{-1} = k^{n-1}$, because $n = |G|$.[3]

Participants	You	Your spouse	Your lover
Knowledge	G, g, n $r{=}g^a$, s $k{=}s^a$ h, $m = hk$	G, g, n r, s m	G, g, n $s{=}g^b$, r $k{=}r^b$ m, $h = mk^{n-1}$

And voilà. No one got shot and the private meeting was worth it.

ElGamal encryption In the system described above, the private key k remains the same during the communication process.This fact can be used to compromise the system. In order to increase robustness of the system we may change the secret key after each d messages or even after each message.

Let me describe a one-way communication system based on this idea. First, you must publish your public key. This stage is considered as an invitation to communicate. Now, the lover can send messages to you. On the next stage the lover sends a sequence of messages such that each of the messages is encrypted by its own private key. Below, I will explain the whole process in details.

In order to initiate the incoming transmission, you should invent a secret integer $a \in \mathbb{Z}_n^*$ and publish the open key $r = g^a$.

Participants	You	Your spouse	Your lover
Knowledge	G, g, n $r{=}g^a$	G, g, n r	G, g, n r

Now, assume that the lover has a sequence of messages $h_1, \ldots, h_k$. For each message h_i she or he should randomly choose a secret integer $b_i \in \mathbb{Z}_n^*$. Then the lover generates the corresponding public and private keys as follows $s_i = g^{b_i}$ and $k_i = r^{b_i}$. Then she or he encrypts each message using the private key $m_i = h_i k_i$. And finally, the lover transmits the sequence $(m_1, s_1), \ldots, (m_k, s_k)$.

Participants	You	Your spouse	Your lover
Knowledge	G, g, n $r{=}g^a$ (m_i, s_i)	G, g, n r (m_i, s_i)	G, g, n r $h_1, \ldots, h_k \in G$ $b_1, \ldots, b_k \in \mathbb{Z}_n^*$ $s_i = g^{b_i}$, $k_i = r^{b_i}$ $m_i = h_i k_i$

[3] If the group G has an effective way to compute the inverse, you should apply that specific algorithm. For example, in case $G = \mathbb{Z}_p^*$ such an algorithm exists and is based on the extended Euclidean algorithm.

In order to decrypt the messages, we should produce the corresponding private key using the public key of the lover $k_i = s_i^a$. Now one else can do this, because now one knows a but you. Then, we recover the original message using the rule $h_i = m_i k_i^{-1}$.

Participants	You	Your spouse	Your lover
Knowledge	G, g, n $r{=}g^a$ (m_i, s_i) $k_i = {s_i}^a$, $h_i = m_i {k_i}^{-1}$	G, g, n r (m_i, s_i)	G, g, n r $h_1, \ldots, h_k \in G$ $b_1, \ldots, b_k \in \mathbb{Z}_n^*$ $s_i = g^{b_i}$, $k_i = r^{b_i}$ $m_i = h_i k_i$

If we want to communicate in the opposite direction, we should repeat the process from the beginning changing the roles. That is, your lover should publish her or his public key as an invitation to receive messages. And then you repeat the whole process from your side.

4.5 RSA

Let me explain the principals of a different encryption system. This one is based on the fact that it is very easy to multiply numbers but hard to factor them. The system is called RSA. This is a one-way communication system.

Suppose we are given $n = pq$, where p and q are distinct prime numbers. Let us take $G = \mathbb{Z}_n^*$. Using the multiplicative version of the Chinese Remainder Theorem we know that

$$\mathbb{Z}_n^* \simeq \mathbb{Z}_p^* \times \mathbb{Z}_q^* \simeq \mathbb{Z}_{p-1} \times \mathbb{Z}_{q-1}$$

Hence $|G| = (p-1)(q-1)$. Usually $|\mathbb{Z}_m^*|$ is denoted by $\varphi(m)$ and is called the Euler function.

Suppose we are given two integers $e, d \in \mathbb{Z}$ and an element $h \in \mathbb{Z}_n^*$. If we raise h to the power of e, we will get h^e. Now we want to recover h by raising h^e to the power d. That is, we want $h^{ed} = h$ whenever h is in $\mathbb{Z}_n^*$. In order to ensure that this happens, it is enough to have $ed = 1 \pmod{\varphi(n)}$. Indeed,

$$h^{ed} = h^{1+|\mathbb{Z}_n^*|k} = h\left(h^{|\mathbb{Z}_n^*|}\right)^k = h \text{ in } \mathbb{Z}_n^*$$

This means that the map $\mathbb{Z}_n^* \to \mathbb{Z}_n^*$ by the rule $h \mapsto h^e$ is a permutation of elements of the group. And if e was chosen randomly, we expect that it is hard to solve the problem $x^e = m$ in $\mathbb{Z}_n^*$. For example it is a bad idea to take e to be 1 or 2 or something like that. If we want to generate e we choose e from $\mathbb{Z}_{\varphi(n)}^*$. And if e was chosen appropriately solving $x^e = m$ is a hard task.

Establishing a connection If you want to make an invitation to receive incoming transmission you need some preparation work to be done. First you need to generate two huge prime numbers p and q and then compute their product $n = pq$. We will use the group $G = \mathbb{Z}_n^*$ as the set of messages. Thus we publish n and G.

Participants	You	Your spouse	Your lover
Knowledge	p, q, $n = pq$	n, $\mathbb{Z}_n^*$	n, $\mathbb{Z}_n^*$

Now, we produce an open key as follows. We generate $e \in \mathbb{Z}_{\varphi(n)}^*$. Then the open key is the pair (e, n).

Participants	You	Your spouse	Your lover
Knowledge	p, q, $n = pq$ e	n, $\mathbb{Z}_n^*$ (e, n)	$\mathbb{Z}_n^*$ (e, n)

The next step is to generate the private key. We find $d \in \mathbb{Z}_{\varphi(n)}^*$ such that $de = 1$ in $\mathbb{Z}_{\varphi(n)}^*$. This can be done using the extended Euclidean algorithm applied to e and $\varphi(n)$. The private key is the pair (d, n).

Participants	You	Your spouse	Your lover
Knowledge	p, q, $n = pq$ e $de = 1 \pmod{\varphi(n)}$	n, $\mathbb{Z}_n^*$ (e, n)	$\mathbb{Z}_n^*$ (e, n)

Communication Suppose the lover wants to send a message $h \in \mathbb{Z}_n^*$. She or he encrypts the message using the rule $m = h^e \pmod{n}$ and sends m by the network.

Participants	You	Your spouse	Your lover
Knowledge	p, q, $n = pq$ e $de = 1 \pmod{\varphi(n)}$ m	n, $\mathbb{Z}_n^*$ (e, n) m	$\mathbb{Z}_n^*$ (e, n) $h \in \mathbb{Z}_n^*$ $m = h^e \pmod{n}$

In order to decrypt the message we apply the map $h = m^d \pmod{n}$. This method works because of the choice of e and d.

Participants	You	Your spouse	Your lover
Knowledge	p, q, $n = pq$ e $de = 1 \pmod{\varphi(n)}$ m $h = m^d \pmod{n}$	n, $\mathbb{Z}_n^*$ (e, n) m	$\mathbb{Z}_n^*$ (e, n) $h \in \mathbb{Z}_n^*$ $m = h^e \pmod{n}$

Let us discuss why this approach is reliable. As we can see at the beginning of the communication process p, q, $\varphi(n) = (p-1)(q-1)$, and d is a secret information. However, everyone knows $n = pq$ and e such that $ed = 1 \pmod{\varphi(n)}$. If we know n, then it is hard to recover p and q because the factorization is a hard problem. One can show, that if you know $n = pq$, then computation of $\varphi(n)$ is equivalent to knowing p and q. Hence, no one knows $\varphi(n)$. Hence, it is impossible to recover d because we do not know two numbers out of three in the equality $ed = 1 \pmod{\varphi(n)}$. Thus we believe that it is impossible to recover the secret key. When we send a message h^e, we believe that solving the equation $x^e = m$ is also hard (there are some requirements on e for this to happen). This more or less explains why the system is robust. If you want to implement this system you should read the official standard explaining details of how to make a good choice of all the parameters.

5 Rings and Fields

5.1 Definitions

As time passes, we are no longer satisfied with an algebraic structure having one operation only. We want more. We are now mature enough to deal with two operations simultaneously. So, let us go straight into the abyss.

Definition 60. We are going to define a ring $(R, +, \cdot)$ or simply R.

- **Data:**
  1. A set R of elements.
  2. An operation $+ \colon R \times R \to R$ called addition.
  3. An operation $\cdot \colon R \times R \to R$ called multiplication.
- **Axioms:**
  1. $(R, +)$ is an abelian group.
  2. Multiplication is left and right distributive over addition:
  $$a(b+c) = ab + ac \quad \text{and} \quad (a+b)c = ac + bc \quad \text{for all } a, b, c \in R$$
  3. Multiplication is associative: $(ab)c = a(bc)$ for all $a, b, c \in R$.
  4. Multiplication has a neutral element denoted by 1.

In this case, we say that $(R, +, \cdot)$ is an associative ring with an identity. We will use the term ring to denote an associative ring with an identity. As before, we usually say that R is a ring assuming that the operations in use are clear. The neutral element with respect to addition is denoted by 0 and is called zero. If $a \in R$ is an arbitrary element, its inverse with respect to addition is denoted by $-a$. For any $a, b \in R$, the expression $a + (-b)$ will be denoted by $a - b$ for short. If in addition we have

5. Multiplication is commutative: $ab = ba$ for all $a, b \in R$.

The ring is said to be commutative. And if we add the following conditions

6. Every non-zero element is invertible with respect to multiplication: for every $a \in R \setminus \{0\}$, there exists an element $b \in R$ such that $ab = ba = 1$.
7. $1 \neq 0$.

The ring is said to be a field. In this case, the inverse element for a is denoted by a^{-1}.[4]

Examples 61.
1. Let $R = \{*\}$ be the set with one element only. Then there is only one possible binary operation on this set. We use this operation as addition and multiplication, that is $+\colon R \times R \to R$ is given by $* + * = *$ and $\cdot\colon R \times R \to R$ is given by $* \cdot * = *$. Then this is a commutative ring. Its only element $*$ is the zero element (neutral with respect to addition) and the identity element (neutral with respect to multiplication). Every nonzero element is invertible because there is no nonzero elements, hence condition (6) holds. But $1 = 0$ in this ring.
2. The integer numbers with the usual addition and multiplication $(\mathbb{Z}, +, \cdot)$ is a commutative ring.
3. The ring of matrices with the usual matrix addition and multiplication $(\mathrm{M}_n(\mathbb{R}), +, \cdot)$ is a ring.
4. The real numbers with the usual addition and multiplication $(\mathbb{R}, +, \cdot)$ is a field.
5. The set of remainders modulo natural number n with the usual addition and multiplication modulo n, that is $(\mathbb{Z}_n, +, \cdot)$, is a commutative ring.
6. If in the previous example the modulus p is prime, then $(\mathbb{Z}_p, +, \cdot)$ is a field.
7. Let A be any commutative ring and x be a variable. Then, $A[x] = \{a_0 + a_1 x + \ldots + a_n x^n \mid n \in \mathbb{N},\ a_i \in A\}$ is the set of all polynomial with coefficients in A. The set $A[x]$ with usual addition and multiplication becomes a commutative ring.

Remark 62. If we are given a ring R, then there are some natural properties of operations that are not included in the list of axioms but easily follow from them. I do not want to torture you proving these formal tricks but it makes sense to list the useful properties:

1. For any element $x \in R$, we have $0x = x0 = 0$.
2. For any elements $x, y \in R$, we have $x - (-y) = x + y$.
3. For any element $x \in R$, we have $(-1)x = -x$.
4. For any invertible elements $x, y \in R$, $(xy)^{-1} = y^{-1}x^{-1}$.

Definition 63. Let R be a ring. We are going to define a subring $T \subseteq R$.

- **Data:**
  1. A subset $T \subseteq R$.
- **Axioms:**
  1. $(T, +) \subseteq (R, +)$ is a subgroup.
  2. T is closed under multiplication.
  3. T contains 1.

Examples 64.
1. Consider $\mathbb{Z} \subseteq \mathbb{R}$. The set $\mathbb{Z}$ is a subring.
2. Upper triangular matrices is a subring of all square matrices.
3. Scalar matrices is a subring of all square matrices.

[4] It should be noted that it is not enough to check one of the conditions $ab = 1$ or $ba = 1$ if the multiplication is not commutative.

5.2 Elements of a ring

There are many approaches to study a ring. The simplest one is the element-wise approach. In this case, we study elements with different properties.

Definition 65. Let R be a ring and $x \in R$ be an element of R.

- The element x is called invertible if there exists $y \in R$ such that $xy = yx = 1$. In this case y is denoted by x^{-1}. The set of all invertible elements of R is denoted by R^*.
- The element x is called left zero divisor if there exists a nonzero $y \in R$ such that $xy = 0$. Similarly, x is called right zero divisor if there exists a nonzero $y \in R$ such that $yx = 0$. The sets of left and right zero divisors will be denoted by $D_l(R)$ and $D_r(R)$, respectively. The set $D(R) = D_l(R) \cup D_r(R)$ is the set all all zero divisors of R.
- The element x is called nilpotent if $x^n = 0$ for some $n \in \mathbb{N}$. The set of all nilpotent elements is denoted by $\mathrm{nil}(R)$.
- The element x is called idempotent if $x^2 = x$. The set of all idempotents of R is denoted by $E(R)$.

Examples 66.

1. Let $R = \mathbb{Z}$ be the ring of integers. Then, $\mathbb{Z}^* = \{\pm 1\}$, $D(\mathbb{Z}) = 0$, $\mathrm{nil}(\mathbb{Z}) = 0$, $E(\mathbb{Z}) = \{1, 0\}$.
2. Let $R = \mathbb{R}$ be the field of real numbers. Then, $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$, $D(\mathbb{R}) = 0$, $\mathrm{nil}(\mathbb{R}) = 0$, $E(\mathbb{R}) = \{1, 0\}$.
3. Let $R = \mathrm{M}_n(\mathbb{R})$ be the ring of square matrices. Then, $\mathrm{M}_n(\mathbb{R})^* = \mathrm{GL}_n(\mathbb{R})$, $D(\mathrm{M}_n(\mathbb{R}))$ is the set of degenerate matrices, $\mathrm{nil}(\mathrm{M}_n(\mathbb{R})$ is the set of matrices with zero complex eigenvalues, $E(\mathrm{M}_n(\mathbb{R}))$ is the set of matrices of the form $C^{-1}DC$, where D is diagonal with elements 1 and 0 on the diagonal.
4. Let $R = \mathbb{Z}_n$ and $n = p_1^{k_1} \dots p_r^{k_r}$, where p_i are distinct prime numbers and $k_i > 0$. Then, $\mathbb{Z}_n^* = \{k \in \mathbb{Z}_n \mid (k, n) = 1\}$, $D(\mathbb{Z}_n) = \{k \in \mathbb{Z}_n \mid (k, n) \neq 1\}$, $\mathrm{nil}(\mathbb{Z}_n) = \{k \in \mathbb{Z}_n \mid p_1 | k, \dots, p_r | k\}$. By the Chinese Remainder Theorem, there exist elements $e_i \in \mathbb{Z}_n$ such that $e_i = 1 \pmod{p_i^{k_i}}$ and $e_i = 0 \pmod{p_j^{k_j}}$ if $j \neq i$. Then, $E(\mathbb{Z}_n)$ consists of the sums $\sum_t e_{i_t}$. The empty sum denotes the zero and the sum through all the elements e_i gives the identity.

5.3 Ideals

Another approach to study a ring is to study its special subsets. It turns out that this approach is more convenient than the elements-wise one. There are two interesting types of subsets: subrings and ideals. The word ideal sounds much cooler, so I am going to deal with the ideals right now.

Definition 67. Suppose that $(R, +, \cdot)$ is a ring. I am going to define an ideal I in the ring R.

- **Data:**
  1. A subset $I \subseteq R$.
- **Axioms:**
  1. $(I, +) \subseteq (R, +)$ is a subgroup.
  2. For any $r \in R$ we have

$$rI = \{rx \mid x \in I\} \subseteq I \quad \text{and} \quad Ir = \{xr \mid x \in I\} \subseteq I$$

In this case, we say that I is an ideal of R. The subsets 0 and R are always ideals and are called the trivial ideals of R.

It should be noted that it is not enough to check that $rI \subseteq I$ for any r or that $Ir \subseteq I$ for any r. If the ring R is not commutative, it may happen that on of the conditions holds while the other one does not.

Claim 68. *Let $R = \mathbb{Z}$, then every ideal is of the form $n\mathbb{Z}$ for some $n \in \mathbb{Z}$.*

Proof. Let $I \subseteq \mathbb{Z}$ be an ideal. Then $(I, +)$ is at least a subgroup in $(\mathbb{Z}, +)$. By Claim 27, we already know that $I = n\mathbb{Z}$ for some $n \in \mathbb{Z}$. From the other hand, let us take an arbitrary $I = n\mathbb{Z}$ and $k \in \mathbb{Z}$. Then,

$$kI = \{kx \mid x \in n\mathbb{Z}\} = \{knm \mid m \in \mathbb{Z}\} = kn\mathbb{Z} \subseteq n\mathbb{Z}$$

Hence, every additive subgroup is an ideal. □

Claim 69. *Let $R = \mathbb{Z}_n$, then every ideal is uniquely presented in the form $k\mathbb{Z}_n$ for some $k|n$.*

Proof. First, we should show that the set $k\mathbb{Z}_n$ is an ideal. By Claim 28, $k\mathbb{Z}_n$ is an additive subgroup of $\mathbb{Z}_n$. Hence, we only need to show that for each $a \in \mathbb{Z}_n$ and each $x \in k\mathbb{Z}_n$ their product $ax \pmod n$ is in $k\mathbb{Z}_n$. Suppose $ax = r \pmod n$. Then $ax = qn + r$. Since $k|x$ and $k|n$, r is divisible by k. The latter means that r belongs to $k\mathbb{Z}_n$ and we are done.

Let $I \subseteq \mathbb{Z}_n$ be any ideal. By Claim 28, every subgroup of $\mathbb{Z}_n$ is of the form $k\mathbb{Z}_n$ for some $k|n$. Since I must be subgroup with respect to addition, I must be of the form $k\mathbb{Z}_n$. □

5.4 Homomorphisms of Rings

We used homomorphisms of groups in order to compare different groups and isomorphisms provides us with a way of saying that to groups are the same. We can extend this approach to the case of rings.

Definition 70. Let $(R, +, \cdot)$ and $(S, +, \cdot)$ be rings. We are going to define a homomorphism $\phi\colon R \to S$.

- **Data:**
  1. A map $\phi\colon R \to S$.
- **Axioms:**
  1. $\phi(a + b) = \phi(a) + \phi(b)$ for all $a, b \in R$.
  2. $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in R$.
  3. $\phi(1) = 1$.

In this case, we say that ϕ is a homomorphism from R to S. If in addition we have

4. ϕ is bijective

then ϕ is called an isomorphism. In this case R and S are called isomorphic.

As before, this is not the most general notion of a homomorphism. But I am going to stick to this convenient and commonly used case.

Remarks 71.
1. It should be noted that if $\phi\colon R \to S$ is a homomorphism of rings, then at least $\phi\colon (R, +) \to (S, +)$ is a homomorphism of abelian groups. In particular, $\phi(0) = 0$ and $\phi(-a) = -\phi(a)$ by Claim 43.
2. If R and S are isomorphic rings, then they are basically the same. We already discussed how to understand isomorphisms in case of arbitrary groups, see discussion after Definition 44. Briefly, isomorphism rename elements of R into elements of S and switch addition and multiplication of R into addition and multiplication of S, respectively. Isomorphic rings have exactly the same properties.

Examples 72.
1. The map $\mathbb{Z} \to \mathbb{Z}_n$ via $k \mapsto k \pmod n$ is a ring homomorphism.
2. The map $\mathbb{R} \to \mathrm{M}_n(\mathbb{R})$ via $\lambda \mapsto \lambda E$, where E is the identity matrix, is a ring homomorphism.
3. The map $\mathbb{R}[x] \to \mathbb{C}$ via $f(x) \mapsto f(i)$, where $i^2 = -1$, is a ring homomorphism.
4. The map $\mathbb{C} \to \mathrm{M}_n(\mathbb{R})$ via $a + bi \mapsto \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$ is a ring homomorphism.

Claim 73 (The Chinese Remainder Theorem)**.** *Let n and m be coprime natural numbers, that is $(n, m) = 1$. Then the map*

$$\Phi\colon \mathbb{Z}_{mn} \to \mathbb{Z}_m \times \mathbb{Z}_n, \quad k \mapsto (k \mod m, k \mod n)$$

is a ring isomorphism.

Proof. We already know that $\Phi\colon (\mathbb{Z}_{mn}, +) \to (\mathbb{Z}_m \times \mathbb{Z}_n, +)$ is an isomorphism of abelian groups, see Claim 52. Also, we checked that Φ preserves the multiplication and the identity element, see the proof of Claim 56. □

Definition 74. Let $\phi\colon R \to S$ be a homomorphism of rings. Then

- The kernel of ϕ is $\ker\phi = \{r \in R \mid \phi(r) = 0\} \subseteq R$.
- The image of ϕ is $\operatorname{Im}\phi = \{\phi(r) \mid r \in R\} = \phi(R) \subseteq S$.

Claim 75. *Let $\phi\colon R \to S$ be a homomorphism of rings. Then*

1. *$\operatorname{Im}\phi \subseteq S$ is a subring.*
2. *$\ker\phi \subseteq R$ is an ideal.*
3. *The map ϕ is surjective if and only if $\operatorname{Im}\phi = S$.*
4. *The map ϕ is injective if and only if $\ker\phi = \{0\}$.*

Proof. 1) We already know that $\operatorname{Im}\phi$ is an additive subgroup, see Claim 47. By definition of the homomorphism $1 = \phi(1) \in \operatorname{Im}\phi$. Hence, we need to show that it is closed under multiplication. Indeed, if $x, y \in \operatorname{Im}\phi$, then $x = \phi(a)$ and $y = \phi(b)$ for some $a, b \in R$. Then,

$$xy = \phi(a)\phi(b) = \phi(ab) \in \operatorname{Im}\phi$$

2) We already know that $\ker\phi$ is an additive subgroup, see Claim 47. Hence, we should show that it is stable under multiplication by any element of R. Let $x \in \ker\phi$ and $r \in R$, we need to show that $rx, xr \in \ker\phi$, that is $\phi(rx) = 0$ and $\phi(xr) = 0$. Indeed, $\phi(rx) = \phi(r)\phi(x) = \phi(r)0 = 0$ and similarly $\phi(xr) = 0$.

3) The captain Obvious told us that this is true.

4) Since $\phi\colon (R, +) \to (S, +)$ is a homomorphism of additive groups, the result follows from Claim 47.

□

6 Polynomials in one variable

6.1 Definition

Let F be a field. A polynomial f in a variable x is a picture

$$f = a_0 + a_1 x + \ldots + a_n x^n, \quad \text{where } a_i \in F$$

Here operations $+$ and $\cdot$ are just symbols. So, we take some elements a_i of the field F and write down a string of symbols as above on the sheet of paper. So, formally a polynomial is just an ordered sequence of coefficients $(a_0, \ldots, a_n)$. We also assume that coefficients $a_{n+1}, a_{n+2}, \ldots$ are zero for the polynomial f. Hence, we may assume that there are countably many coefficients but only finitely many of them are nonzero. This is very convenient if we want to compare polynomials or perform arithmetic operations.

Suppose we are given two polynomials

$$f = a_0 + a_1 x + \ldots + a_n x^n \text{ and } g = b_0 + b_1 x + \ldots + b_m x^m$$

We say that f and g are equal if $a_k = b_k$ for all $k \in \mathbb{N}$.[5] We can add and multiply polynomials using the following rules

$$f = \sum_{k=0}^{n} a_k x^k \quad \text{and} \quad g = \sum_{k=0}^{m} x^k$$

$$f + g = \sum_{k \geqslant 0} (a_k + b_k) x^k \qquad fg = \sum_{k \geqslant 0} \Big(\sum_{u+v=k} a_u b_v \Big) x^k$$

The set of all polynomials with coefficients in F is denoted by $F[x]$. A straightforward computation shows that $F[x]$ with the addition and multiplication defined above is a commutative ring.

[5] Here, I assume that $a_k = 0$ for $k > n$ and similarly $b_k = 0$ for $k > m$.

Remark 76. It should be noted that each polynomial $f \in F[x]$ defines a function $\hat{f}\colon F \to F$ by the rule $a \mapsto f(a)$. Namely, if $f = a_0 + a_1x + \ldots + a_nx^n$, then $f(a) = a_0 + a_1a + \ldots + a_na^n$. However, in general case f is not uniquely determined by the function $\hat{f}$. Indeed, suppose $F = \mathbb{Z}_2$. Then, polynomials $f_n = x^n$ define exactly the same function $\hat{f}_n\colon \mathbb{Z}_2 \to \mathbb{Z}_2$ sending $0 \mapsto 0$ and $1 \mapsto 1$. So, the polynomials f_n are all distinct but they define the same function. This explains why we give such a tricky definition of polynomials. We just want them to be pictures and not functions. And if we want, we can always produce a function for each polynomial.

If $f \in F[x]$ is presented in the form $f = a_0 + a_1x + \ldots + a_nx^n$, where $a_n \neq 0$, that is n is the largest index of the nonzero coefficient present in f. Then n is called the degree of f and is denoted by $\deg f$. The coefficient a_n is called the leading coefficient. The degree of the zero polynomial is not well-defined. We will assume that $\deg(0) = -\infty$. A polynomial $f \in F[x]$ is an element of the field F if and only if $\deg(f) \leqslant 0$ and f is a nonzero constant if and only if $\deg f = 0$.[6]

Claim 77. *Let F be a field. Then for any $f, g \in F[x]$, we have $\deg(fg) = \deg(f) + \deg(g)$.*[7]

Proof. If at least one of the polynomials is zero the required equality contains $-\infty$ on both sides and hence is true. Therefore, we may assume that f and g are not zero. Let

$$f = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \quad \text{and} \quad g = b_mx^m + b_{m-1}x^{m-1} + \ldots + b_1x + b_0$$

such that $a_n \neq 0$ and $b_m \neq 0$. Then

$$fg = a_nb_mx^{m+n} + h(x)$$

where $\deg h < m + n$. Since F is a field, $a_nb_m \neq 0$. Hence, $\deg fg = n + m = \deg f + \deg g$. □

Claim 78. *Suppose F is a field. Then the only zero divisor of $F[x]$ is the zero polynomial.*

Proof. If $f, g \in F[x]$, $f \neq 0$, and $g \neq 0$, then $\deg(f) \geqslant 0$ and $\deg(g) \geqslant 0$. Hence $\deg(fg) = \deg(f) + \deg(g) \geqslant 0$. In particular, $fg \neq 0$. □

Claim 79. *Suppose F is a field. Then a polynomial $f \in F[x]$ is invertible if and only if $f \in F^*$.*

Proof. If f is invertible, then $fg = 1$ for some polynomial $g \in F[x]$. Then, $0 = \deg(1) = \deg(fg) = \deg(f) + \deg(g)$. Since the degree of non-zero polynomials is not negative, $\deg(f) = \deg(g) = 0$. The latter means that f and g are constants. □

6.2 Euclidean algorithm

If F is a field, then $F[x]$ has a division with remainder. If $f, g \in F[x]$ and $g \neq 0$, then we may divide f by g with remainder. The latter means that there exist unique $q, r \in F[x]$ such that $f = qg + r$ and $\deg(r) < \deg(g)$. The polynomial q is called the quotient and r is the remainder.

Suppose $f, g \in F[x]$, we say that f divides g if $g = fh$ for some $h \in F[x]$. It should be noted that any polynomial divides 0. Also, if $a \in F^*$, then f divides af and vise versa because F is a field.

Definition 80. Suppose F is a field. A polynomial $f \in F[x]$ is called monic if its leading coefficient is equal to 1.

Definition 81. Let F be a field and $f, g \in F[x]$ be some polynomials. A polynomial $d \in F[x]$ is called a greatest common divisor of f and g if

1. d divides both f and g.
2. if h divides both f and g, then h divides d.
3. d is monic.

Claim 82. *Let F be a field and $I \subseteq F[x]$ be an ideal. Then $I = fF[x] = \{fh \mid h \in F[x]\}$ for some $f \in F[x]$.*

Proof. If I consists of zero only, then $I = 0F[x]$. Let $f \in I$ be a non-zero polynomial of the least degree. Then for every $h \in I$, we divide h by f with remainder and get $h = qf + r$, where $\deg r < \deg f$. Then, $r = h - qf \in I$ and has the degree smaller than f. Since f is not zero and has the smallest possible degree, r must be zero. This completes the proof. □

[6] A constant here means an element of F.

[7] It should be noted that we assume that $-\infty + a = a + (-\infty) = -\infty$.

The ideal $fF[x]$ is usually denoted by (f) for short. It is very convenient and I will stick to this notation.

Claim 83. *Let F be a field and $f, g \in F[x]$. Then*

1. *There exist a greatest common divisor d of f and g and polynomials $u, v \in F[x]$ such that $d = uf + vg$.*
2. *The greatest common divisor for f and g is unique.*

Proof. (1) Consider the following set of polynomials

$$I = \{af + bg \mid a, b \in F[x]\} \subseteq F[x]$$

This subset is an ideal of $F[x]$. By Claim 82, there is a monic polynomial $r \in I$ such that $I = (r)$. Since $r \in I$, $r = uf + vg$ for some $u, v \in F[x]$. Since $f, g \in I = (r)$, $f = sr$ and $g = tr$ for some $s, t \in F[x]$.

Now, lets prove that r is a greatest common divisor. Equalities $f = sr$ and $g = tr$ mean that r divides f and g, that is d is a common divisor. If h divides f and g, then h divides both summands in the right-hand side of $r = uf + vg$. Hence h divides r. All the properties of the gcd are satisfied.

(2) Suppose we have two gcds d_1 and d_2. Then d_1 divides d_2 because d_2 is a gcd. And vice versa, d_2 divides d_1 because d_1 is a gcd. Thus

$$d_1 = ad_2 \quad d_2 = bd_1$$

In particular, $d_1 = abd_1$. Hence, $d_1(1 - ab) = 0$ in $F[x]$. But there is no zero-divisors in $F[x]$ by Claim 78. Hence, $1 - ab = 0$ and, thus, $1 = ab$. Then a and b are invertible elements of F. □

Remarks 84.
- Let me explicitly state that if $f = g = 0$, then the greatest common divisor is 0 by definition. Indeed, in this case every polynomial divides f and g and 0 is the divisor that is divisible by any other divisor.
- If $f \neq 0$ and $g = 0$, then the greatest common divisor is f divided by the leading coefficient because f divides g in this case.

Claim 85. *Let F be a field and $f, g, h \in F[x]$ are some polynomials. Then $(f, g) = (f, g - hf)$.*

Proof. Indeed, the set of divisors for the pair $\{f, g\}$ is the same as for the pair $\{f, g - hf\}$. In particular, the maximal elements are also the same, that is the greatest common divisors are the same. □

The latter claim enables us to compute a greatest common divisor in an effective way using Euclidean algorithm.

Input: Two polynomials $f, g \in F[x]$. Here F is a field.

Output: A greatest common divisor $d \in F[x]$.

Algorithm We use two temporary variables $u, v \in F[x]$.

1. Initialize $u = f$, $v = g$ in case $\deg f \geqslant \deg g$ and $u = g$, $v = f$ otherwise.
2. While $v \neq 0$ do the following:
   (a) Divide u with reminder by v and get $u = qv + r$.
   (b) Replace $u = v$, $v = r$.
3. When $v = 0$, u becomes a greatest common divisor of the initial f and g.

6.3 Unique Factorization Domain

Definition 86.
- A polynomial $f \in F[x] \setminus F$ is reducible if there exist $g, h \in F[x]$ such that $f = gh$ and $0 < \deg(g) < \deg(f)$ and $0 < \deg(h) < \deg(f)$.
- A polynomial $f \in F[x] \setminus F$ is irreducible if for any $g, h \in F[x]$ such that $f = gh$, either g or h is a nonzero constant.

It should be noted that all nonzero polynomials are divided into three classes: 1) invertible polynomials, that is, F^*, 2) reducible polynomials, 3) irreducible polynomials.

Claim 87 (UFD)**.** *Let F be a field. Then every element $f \in F[x] \setminus F$ is uniquely presented in the form $f = ap_1^{k_1} \dots p_n^{k_n}$, where $a \in F$ is a nonzero constant, k_i are positive integers, and p_i are distinct irreducible monic polynomials.*

I do not want to waste our time on proving this result. An important remark is that the behavior of the polynomials resembles that of integer numbers. The key argument in the proof of the claim is item (1) of Claim 83, that is the fact, that a greatest common divisor of polynomials is a linear combination of the polynomials. However, it is worth mentioning the following partial case of the claim.

Claim 88. *Let F be a field, $f, g \in F[x]$ are coprime polynomials, and $h \in F[x]$ is divisible by f and g. Then, h is divisible by fg.*

Proof. Since f and g are coprime, we have $1 = uf + vg$ for some $u, v \in F[x]$ by Claim 83 item (1). Multiplying the equality by h, we get $h = uhf + vhg$. Since g divides h, gf divides uhf. Since f divides h, fg divides vhg. Thus, fg divides h. □

6.4 Ring of remainders

Now we are ready to meet one of the most important objects in algebra, that is the ring of remainders in case of polynomials.

Let F be a field and $f \in F[x]$ be any polynomial. I am going to define the ring $F[x]/(f)$. First, I need to specify a set, then two operations: addition and multiplication, and finally, I should check all the axioms. If $f = 0$, we define $F[x]/(f)$ to be the polynomial ring itself $F[x]$. The interesting case is when $f \neq 0$:

- $F[x]/(f) = \{g \in F[x] \mid \det g < \deg f\}$ the set of reminders with respect to f.
- $+\colon F[x]/(f) \times F[x]/(f) \to F[x]/(f)$ is the usual addition of polynomials.
- $\cdot\colon F[x]/(f) \times F[x]/(f) \to F[x]/(f)$ is the multiplication modulo f, namely: for every $g, h \in F[x]/(f)$, we define $gh \pmod f$. The latter means, we divide gh by f with reminder and get $gh = qf + r$. Then the product of g and h is r.

Claim 89. *If F is a field and $f \in F[x]$ is a polynomial, the set $F[x]/(f)$ with the given operations is a commutative ring.*

Proof. If $f = 0$, this is clear because $F[x]/(f) = F[x]$ by definition. Suppose that $f \neq 0$. The set $F[x]/(f) = \{g \in F[x] \mid \deg g < \deg f\}$ with addition is an abelian group because its a subgroup in $(F[x], +)$.

Now we need to show: 1) the distributivity law, 2) associativity of multiplication, 3) existence of a neutral element for multiplication, 4) commutativity of multiplication.

1) If $g, h, p \in F[x]/(f)$, we should show that

$$(g+h)p \mod f = gp \mod f + hp \mod f \quad \text{and} \quad g(h+p) \mod f = gh \mod f + gp \mod f$$

We will show the first one. The second one follows from item (4). We divide gp and hp by f with reminder and get

$$gp = q_1 f + r_1,\ \deg r_1 < \deg f,\ \text{and } hp = q_2 f + r_2,\ \deg r_2 < \deg f$$

How, the right-hand side is $r_1 + r_2$ by definition. From the other hand the expression

$$(g+h)p = (q_1 + q_2)f + r_1 + r_2$$

is a division of $(g+h)p$ by f and the remained here is $r_1 + r_2$. Hence, the left-hand side is the same.

2) If $g, h, p \in F[x]/(f)$, then

$$(g \cdot (h \cdot p \mod f)) \mod f = (g \cdot h \cdot p) \mod f = ((g \cdot h \mod f) \cdot p) \mod f$$

If am going to leave this as an exercise in abstract nonsense.

3) The polynomial 1 is the neutral element by definition.

4) The multiplication is commutative by definition.

□

Remarks 90.
- It should be noted that we may consider $F[x]/(f)$ as a subset (even a subgroup with respect to addition) in $F[x]$. However, this inclusion does not preserve the multiplication. The latter means that $F[x]/(f)$ is NOT a subring of $F[x]$.
- From the other hand, the map $F[x] \to F[x]/(f)$ by the rule $g \mapsto g \bmod f$ is a surjective ring homomorphism.

Claim 91. *Let F be a field, $f \in F[x]$ be a polynomial, and $I \subseteq F[x]/(f)$ an ideal. Then there is a polynomial $g \in F[x]$ dividing f such that $I = (g) = \{gh \bmod f \mid h \in F[x]\}$.*

Proof. The case of $f = 0$ is covered in Claim 82. Now we assume that $f \neq 0$. If I consists of the zero only, then $I = (f)$ and we are done.

Let $h \in I$ be a nonzero polynomial of the least possible degree. Then for any $g \in I$, we divide g by h with remainders and get $g = qh + r$ and $\deg r < \deg h$. Also, $r = g - qh \in I$. Since h was nonzero with least possible degree in I, the only option for r is 0. Hence, h divides any $g \in I$. The latter means $I = (h)$.

Now, we need to show that h divides f. Let us divide f by h with remainder, we get $f = qh + r$, where $\deg r < \deg h$. This means that $r = -qh$ in $F[x]/(f)$. In particular, $r \in I$ and has degree smaller than h. The only possible option here is $r = 0$ and we are done. □

Claim 92 (The Chinese Remainder Theorem)**.** *Let $f, g \in F[x]$ be coprime polynomials, that is $(f, g) = 1$. Then the map*

$$\Phi\colon F[x]/(fg) \to F[x]/(f) \times F[x]/(g) \quad h \mapsto (h \mod f, h \mod g)$$

is an isomorphism of rings.

Proof. First, we check that the map is a homomorphism of rings. We need to show that it preserves the addition, the multiplication and the identity. This is a straightforward computation and I skip this.

Now, we should show that it is injective. By Claim 75 it is enough to show that the kernel of Φ is zero. Suppose $h \in \ker \Phi$. This means $h = 0 \pmod f$ and $h = 0 \pmod g$, that is h is divisible by f and g. Since f and g are coprime the latter means that h is divisible by fg. But $\deg h < \deg fg$. The only possible case is $h = 0$.

Now, we should show the surjectivity. Since F is a field $F[x]/(fg)$ and $F[x]/(f) \times F[x]/(g)$ are vector spaces over F. Moreover, the map Φ is a linear map because it preserves the addition and multiplication by any polynomial, hence it preserves the addition and multiplication by any constant. Since Φ is injective, it is enough to show that both spaces have the same dimension. It is clear that $\dim_F F[x]/(fg) = \deg(fg)$. Also the dimension of the right-hand side is $\deg f + \deg g$. Now the result follows from Claim 77. □

7 Fields

7.1 Characteristic

Definition 93. Let F be a field. The characteristic of F is the minimial positive integer p such that

$$\underbrace{1 + \ldots + 1}_{p} = 0$$

If there is no such p the characteristic is said to be zero. The characteristic of F is denoted by $\operatorname{char} F$.

I want to introduce a convenient notation, if we add an element $x \in F$ n time, where $n \in \mathbb{N}$, we may denote the sum as follows

$$nx = \underbrace{x + \ldots + x}_{n}$$

In particular, the characteristic of F is the smallest positive integer p such that $p \cdot 1 = 0$.

Examples 94.
1. If $F = \mathbb{Q}$, then the sum $1 + \ldots + 1$ is never zero. Hence, $\operatorname{char} \mathbb{Q} = 0$.
2. If $F = \mathbb{Z}_p$, then p is the smallest positive integer such that $1 + \ldots + 1 = p \cdot 1 = 0$ in $\mathbb{Z}_p$. Hence, $\operatorname{char} \mathbb{Z}_p = p$.

Claim 95. *If F is a field, then $\operatorname{char} F$ is either 0 or a prime number.*

Proof. Suppose the characteristic is not zero. Assume that $\operatorname{char} F = p = nm$ is not prime. Then

$$0 = \underbrace{1 + \ldots + 1}_{nm} = (\underbrace{1 + \ldots + 1}_{n})(\underbrace{1 + \ldots + 1}_{m}) = (n \cdot 1)(m \cdot 1)$$

Moreover, the elements $n \cdot 1$ and $m \cdot 1$ are nonzero because $p = nm$ was the smallest one. But the product of the elements $n \cdot 1$ and $m \cdot 1$ is zero. This contradicts to F being a field. □

Remark 96. Suppose F is a field. We have a unique ring homomorphism $\phi\colon \mathbb{Z} \to F$ by the rule $n \mapsto n \cdot 1$. Then the kernel of this homomorphism is an ideal of $\mathbb{Z}$. Every ideal of the ring is of the form (p) for some $p \in \mathbb{N}$ (see Claim 68). In this case, p equals the characteristic of F. This explains the relation between the characteristic of a field and the ideals of the ring of integers.

Claim 97. *Let R be a commutative ring. Then, R is a field if and only if there are only two ideals 0 and R.*

Proof. Suppose that R is a field and $I \subseteq R$ is an ideal. If $I = 0$, then we are done. We may assume that I contains a nonzero element and we should show that $I = R$. Let $x \in I$ be a nonzero element. Since R is a field, there exists $x^{-1} \in R$. Since I is an ideal $1 = x^{-1}x \in I$. Hence, for any $y \in R$, the element $y = y1 \in I$.

Now, suppose that R contains only trivial ideals 0 and R. Let $x \in R$ be a nonzero element. Then the set $I = \{rx \mid r \in R\}$ is an ideal of R.[8] Since there are only two ideals 0 and R, I must be one of them. Since I contains a nonzero element, I coincides with R. In particular, $1 \in I$, that is $1 = rx$ for some $r \in R$. Since R is commutative, x is invertible and we are done. □

The latter claim means that we should study fields elements-wise because ideals distinguish nothing in this case. However, ideals somehow measure the difference between an arbitrary ring and a field. Less ideals a ring has, the closer the ring is to a field.

Claim 98. *A ring $\mathbb{Z}_n$ is a field if and only if n is prime.*

Proof. This is an immediate consequence of Claim 69 and Claim 97. □

Claim 99. *Suppose F is a field of prime characteristic p. Then F contains $\mathbb{Z}_p$ as a subfield.*

Proof. The field F contains the identity element $1 \in F$. Let us consider the set $\{0, 1, 2 \cdot 1, \ldots, (p-1) \cdot 1\} \subseteq F$. Then there is an obvious bijection between $\mathbb{Z}_p$ and the subset. We should show that the bijection is an isomorphism of rings. In order to do that, we need to show that

$$n \cdot 1 + m \cdot 1 = (n + m \mod p) \cdot 1, \quad (n \cdot 1)(m \cdot 1) = (mn \mod p) \cdot 1$$

Indeed, if $m + n = qp + r$, then

$$n \cdot 1 + m \cdot 1 = (m + n) \cdot 1 = (qp) \cdot 1 + r \cdot 1 = q(p \cdot 1) + r \cdot 1 = r \cdot 1$$

The second statement is shown in a similar way. □

7.2 Field extension

Suppose F is a field and K is another field containing F as a subfield. Then we will say that K is an extension of F. In this case, K may be considered as a vector space over F. Hence, there is a dimension of K over F. The dimension of K over F is called the degree of K over F and is denoted by $[K : F] = \dim_F K$.

We need a method to produce fields. The following claim explains one of the most useful constructions.

Claim 100. *Let F be a field, $f \in F[x] \setminus F$ be a polynomial. The ring $F[x]/(f)$ is a field if and only if f is irreducible nonzero polynomial.*

Proof. Suppose that f is reducible. Then $f = gh$, where $\deg g < \deg f$ and $\deg h < \deg f$. Then $h \neq 0$ in $F[x]/(f)$ as well as $g \neq 0$ in $F[x]/(f)$. But $gh = f = 0$ in $F[x]/(f)$. Hence, g and h are nonzero zero divisors. But zero divisors are not invertible. The latter contradicts to the definition of a field.

If f is irreducible we should show that any nonzero $g \in F[x]/(f)$ is invertible. Since $\deg g < \deg f$ and $g \neq 0$, f and g are coprime. Hence, $1 = (g, f)$. By Claim 83 item (3), it follows that $1 = ug + vf$ for some $u, v \in F[x]$. But the latter means that $1 = ug \pmod f$ and hence, $u = g^{-1}$ in $F[x]/(f)$. □

[8] Here, it is important that R is commutative.

Remarks 101.

- It is also worth mentioning that the element $x \in F[x]/(p)$ is a root of the polynomial p in the field $F[x]/(p)$. Indeed, $p(x) = 0$ in $F[x]/(p)$ be definition.
- Let us consider the field of real numbers $\mathbb{R}$. Then the polynomial $x^2 + 1 \in \mathbb{R}[x]$ is irreducible. Hence $\mathbb{R}[x]/(x^2 + 1)$ is a field and the element x becomes a root of $x^2 + 1$. Explicitly elements of $\mathbb{R}[x]/(x^2 + 1)$ are of the form $a + bx$, where $a, b \in \mathbb{R}$ and we also know that $x^2 = -1$. Hence, this is the usual model for the complex numbers.
- If $\deg f = n$, then the elements $1, x, \ldots, x^{n-1}$ form a basis of $F[x]/(f)$ over the field F. In particular, $\dim_F F[x]/(f) = \deg f$.

Extension by a root Here I am discussing a standard way to produce a larger field containing a root of a given polynomial. Suppose F is a field and $f \in F[x]$ is a polynomial. Now, I want to produce a field L containing F and an element $\alpha \in L$ such that $f(\alpha) = 0$. If we are lucky enough and F already has such an element, then there is nothing to do. However, the problem is not trivial in case there is no such element inside F. We already know (see Claim 87) that f is a product of irreducible polynomials, that is $f = p_1 \ldots p_n$ (p_i need not be different here). It is enough to solve the problem for some p_i (if we find an extension L containing a root of p_1, then it will be an extension containing a root of f). Hence, we may assume that f is irreducible.

Let us define L to be the ring of remainders modulo f, that is $L = F[x]/(f)$. By Claim 100, L is a field. The element $x \in F[x]/(f)$ will be denoted by α. Let us show that α is a root of f. Indeed, if we consider $f(\alpha)$ in L, we have

$$f(\alpha) = f(x) = 0 \pmod{f}$$

That is $f(\alpha) = 0$ in L. It should be noted that the degree of L over F coincides with $\deg f$.

7.3 Finite fields

Now, I want to deal with finite field only, that is with fields consisting of finitely many elements.

Claim 102. *If F is a finite field, then its characteristic is not zero. In particular, it contains $\mathbb{Z}_p$ for $p = \operatorname{char} F$.*

Proof. If F is finite, then the ring homomorphism $\mathbb{Z} \to F$ by the rule $n \mapsto n \cdot 1$ cannot be injective. Hence its kernel is of the form (p) for some nonzero p. The last statement follows from Claim 99. □

Claim 103. *Suppose F is a finite field and $\operatorname{char} F = p$. Then, $|F| = p^n$, where $n = [F : \mathbb{Z}_p]$.*

Proof. By Claim 99, F contains $\mathbb{Z}_p$ as a subfield. Now, we consider F as a vector space over $\mathbb{Z}_p$. Since F is finite, it has finite dimension. Therefore, F is isomorphic to $\mathbb{Z}_p^n$ as a vector space. But, $\mathbb{Z}_p^n$ has exactly p^n elements, where n is the dimension of F over $\mathbb{Z}_p$. □

Claim 104. *Let F be a finite field. Then the group F^* is cyclic of order $|F| - 1$.*

Proof. The group F^* is a finite abelian group. Hence, it is isomorphic to $\mathbb{Z}_{d_1} \times \ldots \times \mathbb{Z}_{d_k}$, where $d_1 | \ldots | d_k$ (see Claim 54). In particular, for every element of the group F^* we have $x^{d_k} = 1$. Thus, all elements of the group F^* are the roots of the polynomial $x^{d_k} - 1$. Because of the unique factorization property, each polynomial f has at most $\deg f$ roots. Hence, $|F^*| \leqslant d_k$. From the other hand $|F^*| = d_1 \ldots d_k$. The only way this is possible is if $d_1 = \ldots = d_{k-1} = 1$. That is F^* is isomorphic to $\mathbb{Z}_{d_k}$. □

There is a very important classification result (I am not going to prove it).

Claim 105. *For any prime number p and any positive integer n, there exists a unique (up to isomorphism) field F consisting of p^n elements.*

Since there is a unique field of p^n elements, it has a special name $\mathbb{F}_{p^n}$. In particular, $\mathbb{F}_p = \mathbb{Z}_p$. However, it should be noted that $\mathbb{F}_{p^n} \not\simeq \mathbb{Z}_{p^n}$, for example, because $\mathbb{Z}_{p^n}$ contains zero divisors.

A good question is how to produce all finite fields. Suppose we want to produce a field F such that $|F| = p^n$ for some prime p and positive integer n. A starting point is the field $\mathbb{F}_p = \mathbb{Z}_p$. We should find an irreducible polynomial $f \in \mathbb{Z}_p[x]$ of degree n. Then, the required field is $\mathbb{F}_{p^n} = \mathbb{Z}_p[x]/(f)$. It should be noted that there could be many different irreducible polynomials of degree n over $\mathbb{Z}_p$. However, the resulting fields will be isomorphic. How to choose f? Simply take anyone or the most convenient one for your purposes.

Example 106. Let us produce $\mathbb{F}_4$. The base field is $\mathbb{Z}_2$. There is only one irreducible polynomial of degree 2, that is $x^2+x+1 \in \mathbb{Z}_2[x]$. Then the required field is

$$\mathbb{F}_4 = \mathbb{Z}_2[x]/(x^2+x+1) = \{a+bx \mid a,b \in \mathbb{Z}_2\}$$

The addition is given by the coordinate-wise addition. In order to compute a product of any two elements of $\mathbb{F}_4$, it is enough to compute products of all powers of x. The products $1 \cdot 1 = 1$ and $1 \cdot x = 1$ are simple to compute. Now, we need to compute $x \cdot x$. It equals $x^2 = 1 + x \pmod{x^2+x+1}$. In general, a product is computed like this

$$(a+bx)(c+dx) = ac + adx + bcx + bdx^2 = ac + adx + bcx + bd(1+x) = ac + bd + (ad+bc+bd)x$$

Remarks 107.

- The field $\mathbb{Z}_p$ is contained in $\mathbb{F}_{p^n}$. In particular, the group $\mathbb{Z}_p^*$ is contained in the group $\mathbb{F}_{p^n}^*$. As we mentioned already in Section 4.3, the discrete logarithm problem is hard to solve in $\mathbb{Z}_p^*$. This implies that the discrete logarithm problem must be hard for $\mathbb{F}_{p^n}^*$. Indeed, if it were simple to solve $g^x = h$ for x, when we are given $g, h \in \mathbb{F}_{p^n}^*$, we could take $g, h \in \mathbb{Z}_p^* \subseteq \mathbb{F}_{p^n}^*$. This would imply that the discrete logarithm problem for $\mathbb{Z}_p^*$ were simple. The immediate consequence of this observation is that we can use $\mathbb{F}_{p^n}^*$ in the Diffie-Hellman approach discussed before.

- If we want to use $\mathbb{F}_{p^n}^*$ in cryptography, we have to find a generator of the group. If we produced the field in the form $\mathbb{Z}_p[x]/(f)$ for some irreducible $f \in \mathbb{Z}_p[x]$ of degree n, a good candidate for the generator of $\mathbb{F}_{p^n}^*$ would be x. However, the element x of $\mathbb{F}_{p^n}^*$ need not be a generator. It depends on the choice of f.

  For example, if we take $\mathbb{Z}_3$, there are three irreducible polynomials of degree 2: $f_1 = x^2+1$, $f_2 = x^2+x-1$, $f_3 = x^2-x-1$. If we use the first polynomial, we get $\mathbb{F}_9 = \mathbb{Z}_3[x]/(x^2+1)$. The group $\mathbb{F}_9^* \simeq \mathbb{Z}_8$, hence the generator must have order 8. However, $x^4 = (x^2)^2 = (-1)^2 = 1$ in this case. Thus, the order of x is 4 and it is not a generator. From the other hand, we can use f_2 and get $\mathbb{F}_9 = \mathbb{Z}_3[x]/(x^2+x-1)$ or f_3 and get $\mathbb{F}_9 = \mathbb{Z}_3[x]/(x^2-x-1)$. In both cases, a direct computation shows that x is a generator of $\mathbb{F}_9^*$.

7.4 Galois random generator

There is a notion of Linear-feedback shift register. This is a scheme to generate random numbers. It is usually presented in two different forms: Fibonacci and Galois. They are dual to each other in some sense that I am not going to explain here. The Galois scheme is a bit more complicated but we can easily describe it using the language of finite fields.

Suppose we have a finite alphabet $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$ and we want to produce a random sequence of length n consisting of these elements, that is we want randomly choose an element of $\mathbb{Z}_p^n$. A key observation here is that we can identify $\mathbb{Z}_p^n$ with $\mathbb{F}_{p^n}$ and choose randomly an element of a finite field instead. Let me describe the whole process.

Suppose p is a prime number and n is a positive integer. We choose an irreducible polynomial $f \in \mathbb{Z}_p[x]$ of degree n and produce $\mathbb{F}_{p^n} = \mathbb{Z}_p[x]/(f)$. By definition

$$\mathbb{F}_{p^n} = \{a_0 + a_1 x + \ldots + a_{n-1}x^{n-1} \mid a_i \in \mathbb{Z}_p\}$$

Hence, every element of the field is described as a sequence $(a_0, a_1, \ldots, a_{n-1})$. Now, we choose a generator $g \in \mathbb{F}_{p^n}^*$. This is an unpleasant step but we have to make it only once. Usually, we choose f in such a way that x is a generator and take g to be x. Now, we generate a sequence $g, g^2, g^3, g^4, \ldots$. Each power of g corresponds to a sequence of coefficients as above. We can also start from some fixed power of g, that is, we fix a power $k \in \mathbb{N}$, and generate a sequence $g^k, g^{k+1}, g^{k+2}, \ldots$. In practice, we usually choose a some $h \in \mathbb{F}_{p^n}^*$ and generate $hg, hg^2, hg^3, hg^4, \ldots$. Since $h = g^k$ for some k, this is exactly the same approach.

Matrix form As before, we assume that

$$\mathbb{F}_{p^n} = \mathbb{Z}_p[x]/(f) = \{a_0 + a_1 x + \ldots + a_{n-1}x^{n-1} \mid a_i \in \mathbb{Z}_p\}$$

In this case, $1, x, x^2, \ldots, x^{n-1}$ is a basis of $\mathbb{F}_{p^n}$ over $\mathbb{Z}_p$. Suppose, that the polynomial $f = x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0$ is chosen in such a way that x is a generator of $\mathbb{F}_{p^n}^*$. The map $\phi\colon \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$ given by $h \mapsto xh$ is a linear

map. Let us write the matrix for ϕ in the basis above

$$\phi(1, x, \ldots, x^{n-2}, x^{n-1}) = (1, x, \ldots, x^{n-2}, x^{n-1}) \begin{pmatrix} 0 & 0 & \ldots & 0 & -c_0 \\ 1 & 0 & \ldots & 0 & -c_1 \\ 0 & 1 & \ddots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -c_{n-1} \end{pmatrix}$$

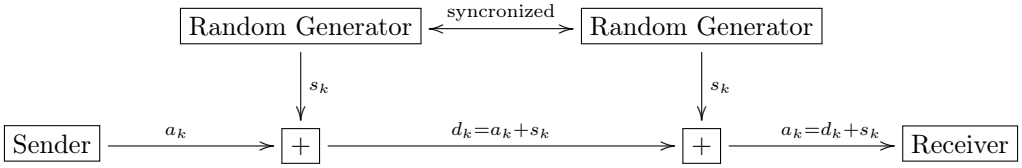
We denote this matrix by A. An element $h = a_0 + a_1 x + \ldots + a_{n-1}x^{n-1}$ is described by the vector $v = (a_0, a_1, \ldots, a_{n-1})^t$. Then, the element xh has coordinates described by the vector Av. Hence, the random generator is described as follows. We fix some vector $v = (a_0, a_1, \ldots, a_{n-1})^t$ and produce a sequence $v, Av, A^2v, A^3v, \ldots$. The Fibonacci approach is similar but we use A^t instead of A. There is a way to describe Fibonacci approach in terms of the field $\mathbb{F}_{p^n}$ but I do not want to do this.

7.5 Stream cipher

There is an application of random generators in cryptography. I want to discuss stream ciphers. Usually, if you want to communicate, you have a secure channel and an open channel. And it is more expensive to transfer data via the secure channel because of the computational overhead. Hence, you want to minimize using the secure channel. However, you still want to have some security transferring the data via the open channel. You want it to be chip, fast, and secure. Here is the method.

We fix a two symbol alphabet $\mathbb{Z}_2 = \{0, 1\}$. Our message is a sequence $a_0, a_1, a_2, a_3, \ldots$. Suppose, we have a random generator $S\colon \mathbb{N} \to \mathbb{Z}_2$, then we can generate a sequence $s_0 = S(0), s_1 = S(1), s_2 = S(2), \ldots$. Then, we can encode the data replacing a_k by $d_k = a_k + s_k \pmod 2$ and broadcast d_k instead. In order to decode the message, we compute $a_k = d_k + s_k \pmod 2$. However, this means that the other person must know the whole sequence s_k. Instead of transferring the sequence s_k using a secure channel, we send a copy of the generator via the secure channel and then broadcast the information using above method via the open channel.

Let me demonstrate the basic transferring process using the following diagram



This method can be absolutely insecure if all the components are not well chosen. From the other hand, modern standards of GSM use some enhanced variants of this idea. So, the appropriate use of this approach leads to very reliable results.

8 Gröbner bases

Now I want to discuss polynomials in several variables. One of the main reason why we love polynomials in one variable is the Euclidean division algorithm. It turns out that we can solve almost any problem effectively using this simple division algorithm. However, if we switch to polynomials in several variables there is no Euclidean division algorithm (one can even prove this rigorously). Instead of rubbing our tearing eyes regretting about switching to several variables we will follow a better plan. We can find a procedure very similar to Euclidean division algorithm, it is called reduction. Moreover, we will be able to reduce a polynomial with respect to a family of polynomials, that is, we will divide one polynomials by a family of polynomials with remainder. There will be one problem. The division process is not unique and we may get different remainders sometimes. At this point we call Groëbner basis to the rescue. A Gröbner basis will provide us with a good reduction algorithm and ensure the uniqueness of all remainders.

8.1 Polynomials in several variables

Let F be a field. A polynomial f in variables $x_1, \ldots, x_n$ is a picture

$$f = \sum_{k_1, \ldots, k_n \geqslant 0} a_{k_1 \ldots k_n} x_1^{k_1} \ldots x_n^{k_n}, \quad a_{k_1 \ldots k_n} \in F$$

where only finitely many coefficients are nonzero. The addition and multiplication are given by the usual rules. The set of all polynomials in variables $x_1, \ldots, x_n$ with coefficients in F will be denoted by $F[x_1, \ldots, x_n]$. The set $F[x_1, \ldots, x_n]$ with addition and multiplication is a commutative ring.

The expression $m = x_1^{k_1} \ldots x_n^{k_n}$ is called monomial. The degree of the monomial is $\deg m = k_1 + \ldots + k_n$. The degree of a polynomial f is the maximum of the degrees of all monomials appearing in f with nonzero coefficient. The degree of the zero polynomial is $-\infty$ as in the case of one variable. A straightforward computation shows that, if $f, g \in F[x_1, \ldots, x_n]$, then $\deg(fg) = \deg f + \deg g$. A monomial multiplied by an element of the field F is called a term. Hence, a polynomial is always a sum of terms.

8.2 Monomial orderings

The first ingredient for the general division algorithm is the ordering on monomials. We need to know when a monomial is less than another one. There are many different orderings. However, I am going to simplify your life and will stick to lexicographical order only.

Definition 108. We want to define a lexicographical order on monomials.

1. We need to fix an ordering on the variables $x_1, \ldots, x_n$. For example $x_1 > x_2 > \ldots > x_n$. However, we can take any permutation of the variables.

2. Suppose we fixed the ordering $x_1 > \ldots > x_n$ on the variables. Now, we are ready to define the corresponding lexicographical order $\mathrm{Lex}(x_1, \ldots, x_n)$ on the monomials.

Let $m = x_1^{k_1} \ldots x_n^{k_n}$ and $m' = x_1^{k_1'} \ldots x_n^{k_n'}$ be two monomials. Then we compare k_1 and k_1'. If $k_1 > k_1$, then $m > m'$. If $k_1 < k_1'$, then $m < m'$. If $k_1 = k_1'$, then we compare k_2 and k_2' and repeat the algorithm above. In particular, $m > m'$ if and only if there exists $1 \leqslant j \leqslant n$ such that $k_1 = k_1', \ldots, k_{j-1} = k_{j-1}'$ and $k_j > k_j'$.

Examples 109. Consider the ring $F[x, y, z]$ and monomials $m_1 = x^2yz^4$, $m_2 = xy^3z$, and $m_3 = xyz^5$. There are 6 ways to order the variables x, y, z. In each case, we define the corresponding lexicographical order and compare the monomials.

1. If $x > y > z$, then $m_1 > m_2 > m_3$.

2. If $x > z > y$, then $m_1 > m_3 > m_2$.

3. If $y > x > z$, then $m_2 > m_1 > m_3$.

4. If $y > z > x$, then $m_2 > m_3 > m_1$.

5. If $z > x > y$, then $m_3 > m_1 > m_2$.

6. If $z > y > x$, then $m_3 > m_1 > m_2$.

Remarks 110. Here I want to list several properties of any lexicographical order. All monomials below depend on n variables and we are given a lexicographical order.

1. For any monomial $m \neq 1$, we have $1 < m$.

2. If m and m' are monomials such that m' divides m, that is, $m = m't$ for some other monomial t, then $m' \leqslant m$ and if $t \neq 1$, the inequality is strict.

3. if m, t, s are monomials such that $m \leqslant t$, then $ms \leqslant ts$ and if $m < t$, then $ms < ts$.

Claim 111. *Suppose we fixed a lexicographical order on monomials in n variables and $m_1 > m_2 > \ldots > m_k > \ldots$ is a strictly descending chain of monomials in variables $x_1, \ldots, x_n$. Then the chain is finite.*

Proof. First of all, we may rename the variables $x_1, \ldots, x_n$ such that they go in the decreasing order $x_1 > x_2 > \ldots > x_n$. Now, suppose the contrary holds and there is an infinite sequence $m_1 > m_2 > \ldots > m_k > \ldots$. I will

write the monomials explicitly on the following diagram.

$$
\begin{array}{lccccccccc}
\text{First} & x_1^{k_1(1)} & x_2^{k_2(1)} & \dots & x_n^{k_n(1)} & \text{Second} & x_1^{k_1} & x_2^{k_2(r_1)} & \dots & x_n^{k_n(r_1)} \\
& \vee\!\!/ & & & & & \| & \vee\!\!/ & & \\
& x_1^{k_1(2)} & x_2^{k_2(2)} & \dots & x_n^{k_n(2)} & & x_1^{k_1} & x_2^{k_2(r_1+1)} & \dots & x_n^{k_n(r_1+1)} \\
& \vee\!\!/ & & & & & \| & \vee\!\!/ & & \\
& \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
& \vee\!\!/ & & & & & \| & \vee\!\!/ & & \\
& x_1^{k_1(r_1)} & x_2^{k_2(r_1)} & \dots & x_n^{k_n(r_1)} & & x_1^{k_1} & x_2^{k_2(r_2)} & \dots & x_n^{k_n(r_2)} \\
& \| & & & & & \| & \| & & \\
& x_1^{k_1(r_1+1)} & x_2^{k_2(r_1+1)} & \dots & x_n^{k_n(r_1+1)} & & x_1^{k_1} & x_2^{k_2(r_2+1)} & \dots & x_n^{k_n(r_2+1)}
\end{array}
$$

First, we look at the degree of x_1. By the definition of the lexicographical order the degrees must go in the descending order, that is, $k_1(1) \geqslant k_1(2) \geqslant \dots$. However, this is a sequence of natural numbers. Hence, must stabilize in the sense there is a number r_1 such that $k_1(r_1) = k_1(r_1+1) = \dots$. We denote this number by k_1. Hence, for all monomials m_i with $i \geqslant r_1$, the degree of x_1 is k_1.

Now, we look at the degree of x_2 and the sequence of monomials $m_{r_1} > m_{r_1+1} > \dots$. Since the degree of x_1 is the same for all monomials, we must have $k_2(r_1) \geqslant k_2(r_1+1) \geqslant \dots$. Again, this is a descending chain of natural numbers. Hence, there is a number $r_2 > r_1$ such that $k_2(r_2) = k_2(r_2+1) = \dots$. We denote this number by k_2. Hence, for all monomials m_i with $i \geqslant r_2$, the degrees of x_1 and x_2 are k_1 and k_2, respectively.

We may proceed in the similar way and find $r_3 > r_2$ such that all degrees of x_3 are the same for m_i whenever $i \geqslant r_3$. Then we find $r_4 > r_3$ such that all degrees of x_4 are the same for m_i whenever $i \geqslant r_4$ and so on. Finally, we find $r_n > r_i$ such that all the degrees of all variables are the same for the monomials m_i such that $i \geqslant r_n$. But then all such monomials are equal. In particular, the comparison $m_{r_n} > m_{r_n+1}$ is impossible. This contradiction completes the proof. □

Remark 112. There is a common misunderstanding here. There is no infinite strictly descending chain of monomials. However, for a given monomial, there are infinitely many strictly smaller ones as the following example shows. Let us consider $\mathbb{Q}[x,y]$ with the order $x > y$ and the corresponding lexicographical order. Then there are infinitely many monomials strictly smaller than x^2. Indeed, all monomials of the form xy^n are strictly smaller than x^2. Hence, we may produce a sequence

$$x^2 > xy^n > xy^{n-1} > \ldots > xy > x > y^m > y^{m-1} > \ldots > y > 1$$

Each descending chain of monomials starting at x^2 is finite but the length of the chain can be arbitrary long.

8.3 Reduction

Since we know how to compare the monomials, we are ready to define the reduction algorithm. First we need to define an elementary reduction and then an arbitrary reduction. Let me recall the context. We are given a field F, a polynomial ring $F[x_1,\dots,x_n]$, and a lexicographic ordering on the monomials in n variables.

Definition 113. Suppose F is a field, we fix a lexicographical order on the monomials of $F[x_1,\dots,x_n]$, and $f \in F[x_1,\dots,x_n]$ is an arbitrary nonzero polynomial. Then, the polynomial f can be written as

$$f = c_1m_1 + c_2m_2 + \ldots + c_km_k, \quad c_i \in F,\ m_i \text{ are monomials such that } m_1 > m_2 > \ldots > m_k$$

The term c_1m_1 is called the leading term and will be denoted by $T(f)$. The monomial m_1 is called the leading monomial of f and will be denoted by $M(f)$. The coefficient c_1 is called the leading coefficient of f and is denoted by $C(f)$. It is clear by definition that $T(f) = C(f)M(f)$. The part $c_2m_2 + \ldots + c_km_k$ is called the tail of f and will be denoted by f_0. Hence, f can be written as $f = T(f) + f_0$.

Definition 114. Suppose $g \in F[x_1,\dots,x_n]$ is a nonzero polynomial and $f \in F[x_1,\dots,x_n]$ is any polynomial. Assume that

$$f = c_1m_1 + \ldots + c_im_i + \ldots + c_km_k, \quad c_i \in F,\ m_i \text{ are monomials such that } m_1 > m_2 > \ldots > m_k$$

and

$$g = C(g)M(g) + g_0 = T(g) + g_0$$

We take m to be m_i, that is a monomial in f and assume that m is divisible by the leading monomial of g, that is $m = tM(g)$. We define an elementary reduction of f with respect to g as

$$f \xrightarrow{g} f' = f - \frac{c_i}{C(g)} tg$$

The polynomial f' is the result of the elementary reduction.

In short, the elementary reduction works as follows: we find a monomial m_i of f divisible by $M(g)$ and replace it by the tale of g multiplied by $-c_i m_i / T(g)$.

Example 115. Let us consider the ring $\mathbb{Q}[x, y, z]$, $f = xyz$, $g_1 = xy - z$, and $g_2 = yz - 1$. Then

$$f \xrightarrow{g_1} xyz - z(xy - z) = z^2, \quad \text{or} \quad f \xrightarrow{g_2} xyz - x(yz - 1) = x$$

Definition 116. Suppose $G \subseteq F[x_1, \dots, x_n] \setminus \{0\}$ is a set of polynomials and $f, f' \in F[x_1, \dots, x_n]$ are any polynomials. We say that f is reducible to f' with respect to G if there is a finite sequence of elementary reductions as below[9]

$$f \xrightarrow{g_1} f_1 \xrightarrow{g_2} f_2 \xrightarrow{g_3} \dots \xrightarrow{g_k} f_k = f' \quad \text{where } g_i \in G$$

In this case we will write $f \overset{G}{\rightsquigarrow} f'$.

If the polynomial f' is not reducible by any $g \in G$, we say that f' is a remainder of f with respect to G.

Remarks 117.

1. A polynomial f' is not reducible by any $g \in G$ if and only if every monomial of f' is not divisible by $M(g)$ for any $g \in G$.
2. It should be noted that there could be different remainders of a polynomial f with respect to a set G. Here is an example. We deal with the ring $\mathbb{Q}[x, y, z]$ and the lexicographical order is given by $x > y > z$. Suppose $f = xyz$, $g_1 = xy - 1$, $g_2 = yz - 1$, and $G = \{g_1, g_2\}$. Then,

$$f \xrightarrow{g_1} xyz - z(xy - 1) = z \quad \text{and} \quad f \xrightarrow{g_2} xyz - x(yz - 1) = x$$

Then by definition $f \overset{G}{\rightsquigarrow} z$ and $f \overset{G}{\rightsquigarrow} x$. Moreover, the polynomials z and x are not reducible with respect to G. Hence, they are different remainders of f with respect to G.

Hence, in general a remainder of a polynomial f with respect to some set of polynomials G is not unique. The latter happens because the set G was not good enough. This leads us to the notion of Gröbner basis.

Definition 118 (Gröbner basis)**.** Suppose F is a field, $G \subseteq F[x_1, \dots, x_n] \setminus \{0\}$, and we fix a lexicographical order on the monomials.[10] We say that G is a Gröbner basis if for every $f \in F[x_1, \dots, x_n]$ all its remainders are the same.

There is a hidden problem here. Suppose we are given a polynomial f and a set of polynomials G. If f is reducible with respect to G we can make an elementary reduction and get f_1. Then if f_1 is reducible, we can make another elementary reduction and get f_2 and so on. A natural question is: does the process stop? The answer to this question is yes as the Claim 122 shows.

Examples 119. We will see a lot of examples of Gröbner bases latter. Now, I want to give some simple examples without explanations because we are not ready to prove that the examples below are Gröbner bases.

1. For any polynomial ring $F[x_1, \dots, x_n]$ with any lexicographical order the set $G = \{g\}$ consisting of one nonzero polynomial is always a Gröbner basis.
2. Suppose $F[x, y, z, w]$ and we are given any lexicographical order, then the set $G = \{xy - 1, zw + 1\}$ is a Gröbner basis. This happens because the leading monomials of the polynomials of G are coprime.

[9] The polynomials $g_1, \dots, g_k$ need not be distinct.

[10] It should be noted the the notion of a Gröbner basis highly depends on the choice of the order. If we change the ordering on the variables, the set G may become a Gröbner basis even if it was not one with respect to the old ordering and vice verse.

Definition 120. Suppose F is a field and we fix some lexicographical order on the monomials in n variables. As before, each polynomial $f \in F[x_1, \ldots, x_n]$ can be written as

$$f = c_1 m_1 + c_2 m_2 + \ldots + c_k m_k, \quad c_i \in F,\ m_i \text{ are monomials such that } m_1 > m_2 > \ldots > m_k$$

We denote its i-th largest monomial m_i by $M_i(f)$. In particular, $M_1(f)$ is the leading monomial of f, $M_2(f)$ is the next largest monomial of f etc. The i-th largest monomial need not exist if f contains less than i monomials.

Claim 121. *Suppose F is a field, we are given $F[x_1, \ldots, x_n]$, and some lexicographical order is fixed. Suppose $f, f', g \in F[x_1, \ldots, x_n]$ are such that $f \xrightarrow{g} f'$ and $M_1(f) = M_1(f'), \ldots, M_{k-1}(f) = M_{k-1}(f')$, and monomials $M_k(f)$ and $M_k(f')$ exist. Then, $M_k(f) \geqslant M_k(f')$.*

Proof. Suppose $f = c_1 m_1 + \ldots + c_{k-1} m_{k-1} + c_k m_k + \ldots + c_s m_s$, where $c_i \in F$ and m_i are monomials written in decreasing order with respect to the lexicographical order. The polynomial f' is obtained from f via one elementary reduction. The latter means there is a monomial m_i in f divisible by $M(g)$ and we replace m_i by a linear combination of smaller monomials. By the hypothesis the first $k-1$ monomials of f and f' are the same. Hence, $i \geqslant k$. If $i > k$ then, the k-th monomial of f' is the same as in f. This means that $M_k(f) = M_k(f')$. If $i = k$, then we replaced m_k by a linear combination of smaller monomials. For example the result may look like this:

$$f \longleftrightarrow m_1 \quad \ldots \quad m_{k-1} \quad m_k \qquad m_{k+1} \qquad \ldots \quad m_s$$

$$f' \longleftrightarrow m_1 \quad \ldots \quad m_{k-1} \qquad m'_k \quad m'_{k+1} \quad m'_{k_2} \quad \ldots \qquad m'_{s'}$$

But then, the monomial $m'_k = M_k(f')$ and is strictly smaller than $m_k = M_k(f)$ and we are done. □

Claim 122. *Suppose F is a field, we are given $F[x_1, \ldots, x_n]$, and some lexicographical order is fixed. Then, for every polynomial $f \in F[x_1, \ldots, x_n]$ and any subset $G \subseteq F[x_1, \ldots, x_n] \setminus \{0\}$, any sequence of elementary reductions of f with respect to G is finite.*

Proof. Suppose the contrary holds and there is an infinite sequence of elementary reductions

$$f \xrightarrow{g_1} f_1 \xrightarrow{g_2} f_2 \xrightarrow{g_3} \ldots \xrightarrow{g_k} f_k \xrightarrow{g_{k+1}} \ldots$$

Let us consider the sequence of leading monomials $M_1(f), M_1(f_1), M_1(f_2), \ldots$. Applying Claim 121 in case $k = 1$, we see that $M_1(f) \geqslant M_1(f_1) \geqslant M_1(f_2) \geqslant \ldots$. By Claim 111, the sequence cannot be strictly decreasing and there is r_1 such that $M_1(f_{r_1}) = M_1(f_{r_1+1}) = \ldots$. We denote this common monomial by m_1. Since the sequence of reductions is infinite, there must exist the second monomial in all polynomials f_i. By Claim 121 in case $k = 2$, we have $m_1 > M_2(f_{r_1}) \geqslant M_2(f_{r_1+1}) \geqslant \ldots$. This descending chain of monomials cannot be strictly decreasing by Claim 111. Hence, there is $r_2 > r_1$ such that $m_1 > M_2(f_{r_2}) = M_2(f_{r_2+1}) = \ldots$. We denote this common monomial by m_2 and get a sequence $m_1 > m_2$.

Now we repeat the arguments and consider the third largest monomials, then find m_3 such that $m_1 > m_2 > m_3$. Proceeding in such a way we construct an infinite decreasing sequence of monomials $m_1 > m_2 > m_3 > \ldots > m_s > \ldots$. The latter contradicts Claim 111. This contradiction completes the proof. □

8.4 The Buchberger Criterion

There is another hidden problem. Is there at least one Gröbner basis? Or how to check if G is a Gröbner basis? We are going to address the problem with the Buchberger criterion. I am going to postpone the proof until the next lecture and to focus on the criterion with its applications.

Definition 123. Suppose F is a field, $f_1, f_2 \in F[x_1, \ldots, x_n]$ are some nonzero polynomials, and we are given a lexicographical order on monomials. Assume that $f_1 = c_1 m_1 + f'_1$, where $c_1 m_1$ is the leading term, and $f_2 = c_2 m_2 + f'_2$, where $c_2 m_2$ is the leading term. Let m be the least common multiple of m_1 and m_2, then $m = m_1 t_1 = m_2 t_2$. Then, the polynomial

$$S_{f_1 f_2} = c_2 t_1 f_1 - c_1 t_2 f_2 = c_2 t_1 f'_1 - c_1 t_2 f'_2$$

is called S-polynomial of f_1 and f_2.

Example 124. Consider $\mathbb{Q}[x, y, z]$, $f_1 = xy - 1$, $f_2 = yz - 1$, and $\mathrm{Lex}(x, y, z)$ is the order. Then, the leading terms are $m_1 = xy$ and $m_2 = yz$ and the least common multiple is $xyz = m_1 z = m_2 x$. Then,

$$S_{f_1 f_2} = z f_1 - x f_2 = z(xy - 1) - x(yz - 1) = x - z$$

Claim 125 (The Buchberger criterion)**.** *Suppose F is a field, we are given a lexicographical order on monomials in n variables, and $G \subseteq F[x_1, \ldots, x_n] \setminus \{0\}$ is any set. Then, the following conditions are equivalent*

1. *G is a Gröbner basis.*
2. *For every $g_1, g_2 \in G$, $S_{g_1\, g_2}$ is reducible to zero with respect to G.*[11]

Remark 126. In Example 124, we see that the S-polynomial of f_1 and f_2 does not reduce to zero with respect to $\{f_1, f_2\}$. Hence, the set is not a Gröbner basis.

There is a special case ensuring that an S-polynomial will always reduce to zero.[12]

Claim 127. *Suppose F is a field, $g_1, g_2 \in F[x_1, \ldots, x_n] \setminus \{0\}$, and we are given a lexicographical order. Assume that the leading monomials of g_1 and g_2 are coprime. Then, $S_{g_1\, g_2}$ reduces to zero with respect to $\{g_1, g_2\}$.*

Proof. Suppose $g_1 = c_1 m_1 + g_1'$ and $g_2 = c_2 m_2 + g_2'$, where m_i is the leading monomial of g_i, c_i is the leading coefficient of g_i and g_i' is the tail of g_i. Since m_1 and m_2 are coprime the S polynomial of g_1 and g_2 is

$$S_{g_1\, g_2} = c_2 m_2 g_1 - c_1 m_1 g_2 = c_2 m_2 (c_1 m_1 + g_1') - c_1 m_1 (c_2 m_2 + g_2') = c_2 m_2 g_1' - c_1 m_1 g_2'$$

Now we use equality $c_i m_i = g_i - g_i'$ and get

$$S_{g_1\, g_2} = c_2 m_2 g_1' - c_1 m_1 g_2' = (g_2 - g_2') g_1' - (g_1 - g_1') g_2' = g_2' g_1 - g_1' g_2$$

So, we expressed the S-polynomial in the form $ag_1 + bg_2$ such that each monomial of a is strictly smaller than m_2 and each monomial of b is strictly smaller than m_1.

Now the goal is to show that the polynomial $ag_1 + bg_2$ is either zero or we can reduce it at least once with $\{g_1, g_2\}$. In order to do that, we show that the leading monomial $M(ag_1 + bg_2)$ is either $M(ag_1)$ or $M(bg_2)$. If $M(ag_1) > M(bg_2)$, then $M(ag_1)$ is the leading monomial of $ag_1 + bg_2$. If $M(ag_1) < M(bg_2)$, then $M(bg_2)$ is the leading monomial of $ag_1 + bg_2$. So, we may assume that $M(ag_1) = M(bg_2)$. However, $m_1 = M(g_1)$ divides $M(ag_1)$ and $m_2 = M(g_2)$ divides $M(bg_2)$. Since the m_1 and m_2 are coprime, $m_1 m_2$ divides $M(ag_1) = M(a)M(g_1) = M(a)m_1$. Therefore m_2 divides $M(a)$. This contradicts to the fact that all monomials of a are smaller then m_2. Hence, a must be zero. The same argument shows that b is zero also. Now assume that $ag_1 + bg_2$ is not zero. Then its leading monomial is either $M(a)m_1$ or $M(b)m_2$. In either case, it is reducible with respect to g_1 or g_2.

Let us go back to the S-polynomial of g_1 and g_2. We rewrote the S-polynomial as follows $ag_1 + bg_2$ such that monomials of a are smaller than m_2 and monomials of b are smaller than m_1. In the previous paragraph we showed that the leading monomial of $ag_1 + bg_2$ is either $M(a)m_1$ or $M(b)g_2$. Without loss of generality, we may assume that the leading term is $M(a)m_1$. Let us reduce the leading term using g_1.

$$S_{g_1, g_2} \xrightarrow{f} ag_1 + bg_2 - C(a)M(a)g_1 = (a - C(a)M(a))g_1 + bg_2$$

As we can see, the result of reduction is also a polynomial of the form $ag_1 + bg_2$ such that all monomials of a are smaller than m_2 and all monomials of b are smaller than m_1. Hence, we can repeat the reduction process. However, by Claim 122 the process must terminate. But the latter means that the result of the reduction is zero (otherwise we would be able to proceed with the reduction). This completes the proof. □

Example 128. Suppose we are given $\mathbb{Q}[x, y, z]$ with $\mathrm{Lex}(x, y, z)$, $g_1 = x^2 y - z$, and $g_2 = z^2 + 1$. Then the leading monomials $M(g_1) = x^2 y$ and $M(g_2) = z^2$ are coprime. Hence, their S-polynomial reduces to zero with respect to $\{g_1, g_2\}$. We can check this manually, this is not hard. But it is much easier to apply the claim and avoid computation at all.

8.5 Ideals in polynomial rings

It turns out that a lot of problems about polynomials can be solved using ideals. Hence, it is worth talking about ideals in a polynomial ring in several variables for a bit.

Definition 129. Suppose F is a field and we are given a finite set of polynomials $g_1, \ldots, g_k \in F[x_1, \ldots, x_n]$. Then the set

$$(g_1, \ldots, g_k) = \{g_1 h_1 + \ldots + g_k h_k \mid h_1, \ldots, h_k \in F[x_1, \ldots, x_n]\}$$

is an ideal of $F[x_1, \ldots, x_n]$ and is called the ideal generated by $g_1, \ldots, g_k$. If take $G = \{g_1, \ldots, g_k\}$, then the ideal $(g_1, \ldots, g_k)$ is also denoted by (G) for short.

[11] There is at least one way to reduce each S-polynomial to zero.
[12] There is one additional trick but it is quite complicated and not worth mentioning.

There is a nontrivial result saying that every ideal of $F[x_1, \ldots, x_n]$ is generated by a finite set. The result was obtained by Hilbert and is known as the Hilbert basis theorem.

Claim 130 (The Hilbert Basis Theorem)**.** *Suppose F is a field and $I \subseteq F[x_1, \ldots, x_n]$ is an ideal. Then there are polynomials $g_1, \ldots, g_k \in F[x_1, \ldots, x_n]$ such that $I = (g_1, \ldots, g_k)$.*

If we want to solve problems about ideals effectively, we need to generate the ideals by Gröbner bases. Hence, we need to solve the following problem. Suppose an ideal I is generated by $g_1, \ldots, g_k$ and we fix a lexicographical order. It may happen that the initial set $\{g_1, \ldots, g_k\}$ is not a Gröbner basis. Hence, we want to replace it by a different set G such that G is a Gröbner basis and generates I, that is, $I = (G)$. Additionally, we want G to be finite, otherwise it is almost useless from the effectiveness point of view. The latter problem can be solved by the Buchberger algorithm.

The Buchberger Algorithm As usual, we have a field F, a polynomial ring $F[x_1, \ldots, x_n]$, and we are given a lexicographical order on monomials.

Input A finite set of polynomials $G = \{g_1, \ldots, g_k\} \subseteq F[x_1, \ldots, x_n]$.

Output A finite set of polynomials $G_0 \subseteq F[x_1, \ldots, x_n]$ such that

1. G_0 is a Gröbner basis.
2. $(G) = (G_0)$.

Algorithm

1. At the beginning we initialize $G_0 = G$.
2. For each $g_i, g_j \in G_0$ we compute $S_{g_i\, g_j}$ and reduce it by G_0 to a remainder $S_{g_i\, g_j} \overset{G_0}{\rightsquigarrow} r_{ij}$.
3. If all r_{ij} are zero, then G_0 is the required Gröbner basis. If not, then we update G_0 to be $G_0 \cup \{r_{ij} \mid r_{ij} \neq 0\}$ and repeat the step (2).

If we want to show that the algorithm works, then we need to show several things: 1) it halts at a finite step, 2) the set G_0 is finite, 3) the resulting set G_0 is a Gröbner basis, 4) the resulting set generates the same ideal as G. The most difficult item is the first one. I am going to postpone the proof until the next lecture and explain all the other items.

2) Suppose the loop in the algorithm worked n times. Then, we constructed a set G_1 from the set G, then the set G_2, next G_3 and so on. At the end we terminated at the set G_n. As you can see at each step the set G_k is obtained from the set G_{k-1} by adding finitely many polynomials. Hence, all sets G_k are finite. In particular, the final set $G_0 = G_n$ is finite.

3) Since algorithm terminates on G_n, all S-polynomials of elements of G_n reduces to zero with respect to G_n. Hence, the Buchberger Criterion shows that G_n is a Gröbner basis.

4) We need to show that $(G) = (G_n)$. It is enough to show that on each step $(G_{k-1}) = (G_k)$. But $G_k = G_{k-1} \cup \{r_{ij} \mid r_{ij} \neq 0\}$. In particular, $G_{k-1} \subseteq G_k$ and thus $(G_{k-1}) \subseteq (G_k)$. In order to show the other inclusion it is enough to show that each r_{ij} belongs to (G_{k-1}). Indeed, for each $g_i, g_j \in G_{k-1}$ the S-polynomial $S_{g_i\, g_j}$ is of the form $c_2 t_1 g_i - c_1 t_2 g_j \in (G_{k-1})$ by definition. Since, $S_{g_i\, g_j} \overset{G_{k-1}}{\rightsquigarrow} r_{ij}$. This means there exist a sequence of reductions: $S_{g_i\, g_j} \xrightarrow{g'_1} f_1 \xrightarrow{g'_2} f_2 \xrightarrow{g'_3} \ldots \xrightarrow{g'_k} f_k = r_{ij}$. Hence, it is enough to show that if $f \in (G)$, $g \in G$, and $f \xrightarrow{g} f'$, then $f' \in (G)$. But this is clear by the definition, because $f' = f - \lambda t g$ for some $\lambda \in F$ and a monomial t.

8.6 Rings of remainders

Suppose F is a field, we consider $F[x_1, \ldots, x_n]$, and a lexicographical order is fixed. Suppose $G = \{g_1, \ldots, g_k\} \subseteq F[x_1, \ldots, x_n]$ and $I = (G)$. We compute a Gröbner basis of I and denote it by G_0. Hence, we also have $I = (G_0)$.

Now I want to define the ring of remainders using the Gröbner basis G_0 as follows. As a set

$$F[x_1, \ldots, x_n]/(G_0) = \{f \in F[x_1, \ldots, x_n] \mid f \text{ is a remainder with respect to } G_0\}$$

The addition is the usual addition of polynomials. The multiplication is defined as follows: suppose $f, f' \in F[x_1, \ldots, x_n]/(G_0)$, we compute the usual product of f and f' as polynomials and reduce it with respect to G_0, that is $ff' \overset{G_0}{\rightsquigarrow} r$. Then, we define $f \cdot f' = r$ in $F[x_1, \ldots, x_n]/(G_0)$. One can show that we have defined a commutative ring.

Remarks 131. By the definition of the remainder ring $F[x_1, \ldots, x_n]/(G_0)$ its construction depends on the choice of the Gröbner basis G_0. In particular, it depends on the choice of the lexicographical order. It turns out that the remainder ring depends only on the choice of the ideal $I = (G_0)$ in the following sense:

1. Let us fix a lexicographical order and find two different Gröbner bases of the ideal I, say, G_1 and G_2. Suppose $f \in F[x_1, \ldots, x_n]$ and $f \overset{G_1}{\rightsquigarrow} r_1$ and $f \overset{G_2}{\rightsquigarrow} r_2$, where r_1 and r_2 are remainders with respect to G_1 and G_2. Then, one can show that $r_1 = r_2$. In particular, the sets of remainders with respect to G_1 and G_2 are the same
$$F[x_1, \ldots, x_n]/(G_1) = F[x_1, \ldots, x_n]/(G_2)$$
Moreover, this also implies that the operations are the same. Hence, the rings for different Gröbner bases are exactly the same.

2. Now suppose that we take two different lexicographical orders Lex_1 and Lex_2 and compute Gröbner bases G_1 and G_2 of I corresponding to Lex_1 and Lex_2, respectively. Then, the sets of remainders can be different. However, the rings of remainders will be isomorphic.
$$F[x_1, \ldots, x_n]/(G_1) \simeq F[x_1, \ldots, x_n]/(G_2)$$
The isomorphisms are given by the following rules. If $f \in F[x_1, \ldots, x_n]/(G_1)$, we compute its remainder with respect to G_2, that is $f \overset{G_2}{\rightsquigarrow} f'$. Then the map $f \mapsto f'$ is an isomorphism of rings. The inverse map is given as follows. We take $g \in F[x_1, \ldots, x_n]/(G_2)$ and compute its remainder with respect to G_1, that is $g \overset{G_1}{\rightsquigarrow} g'$. Then, the map $g \mapsto g'$ is the inverse isomorphism.

8.7 Membership problem and variable elimination

There are two very popular problems that can be solved using Gröbner bases. I want to discuss them both.

Membership problem Suppose we have a field F, a polynomial ring $F[x_1, \ldots, x_n]$, an ideal $I = (g_1, \ldots, g_k)$ in the polynomial ring, and a polynomial f. The question is how to check if f belongs to I. Here is an algorithm.

1. Fix a lexicographical order Lex.
2. Find a Gröbner basis G of I.
3. Compute the remainder of f with respect to G, that is $f \overset{G}{\rightsquigarrow} r$.
4. The polynomial f belongs to I if and only if $r = 0$.

It should be noted that if G is not a Gröbner basis, then only a half of the algorithm works correctly. If the remainder r is zero, then f is guaranteed to be in I. The proof is similar to the one we used to show that the Buchberger algorithm is correct. However, in case the remainder r is not zero the polynomial f may still be in I. Indeed, in Example 124 S-polynomial $S_{f_1 f_2}$ belongs to the ideal (f_1, f_2). However, $S_{f_1 f_2}$ is not reducible with respect to $\{f_1, f_2\}$.

Variable elimination Suppose we have a field F, a polynomial ring $F[x_1, \ldots, x_n, x_{k+1}, \ldots, x_n]$, and an ideal $I = (g_1, \ldots, g_k)$ in the polynomial ring. The question is how to compute $I \cap F[x_1, \ldots, x_k]$. This means, we want to eliminate all polynomials from I depending on $x_{k+1}, \ldots, x_n$. Here is an algorithm.

1. Fix a lexicographical order such that $x_1, \ldots, x_k < x_{k+1}, \ldots, x_n$. The ordering of $x_1, \ldots, x_k$ or $x_{k+1}, \ldots, x_n$ does not matter.
2. Compute a Gröbner basis G_0 of I.
3. Define $G = \{g \in G_0 \mid g \text{ does not depend on } x_{k+1}, \ldots, x_n\}$.
4. Then G is a Gröbner basis of $I \cap F[x_1, \ldots, x_k]$ with respect to the lexicographical order given by the chosen ordering of $x_1, \ldots, x_k$.