Paper review

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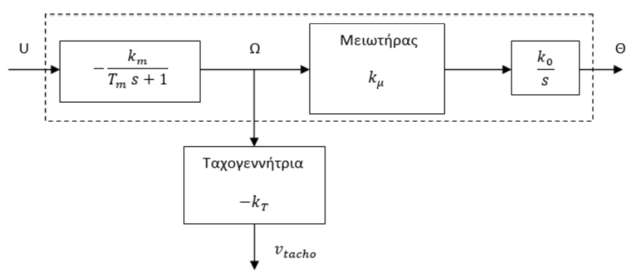
1 System model

The following paper refers to simulations and theorytical analysis of the system below.



1.1 Parameters calculations

The transfer function of Vtacho is given from

$$\frac{V_{\text{tacho}}(s)}{U(s)} = \frac{k_m k_t}{T_m s + 1} \tag{1}$$

In the steady state we get the limit to 0 of the transfer function and we get

$$\frac{V_{\text{tacho,ss}}}{U} = k_m k_t \tag{2}$$

By substituting the measurements

$$k_m k_t = 1.44 \tag{3}$$

For the calculation of Tm we calculate 0.63*Vtacho and throuth the oscilloscope the time that the transfer value get this value is

$$Tm = 568ms$$

The parameter k_μ of the reducer corresponds to the ratio of the angle of rotation of the output axis to the angle of rotation of the motor axis.

$$k_\mu = \frac{1}{36} \tag{4}$$

As shown above we can calculate the transfer function of the Output $\Theta within input \Omega$

$$\frac{\Theta(s)}{\Omega(s)} = \frac{k\mu \cdot k_0}{s} \tag{5}$$

We set the power on and we connect the channel of the oscilloscope to the motor position and ground.

$$\Delta\theta = -14.8\,\mathrm{V} \tag{6}$$

$$\Delta t = 1.39\,\mathrm{s} \tag{7}$$

The time Δt is the period of one complete rotation of the output axis; therefore, its rotational speed in rpm is:

$$\omega_{\text{out}} = \frac{60\,\mathrm{s}}{\Delta t} = 43.165\,\mathrm{rpm} \tag{8}$$

$$k_\mu = \frac{\omega_{\text{out}}}{\omega} \tag{9}$$

From equation 4 and equation 9, we get:

$$\omega = 1553.94\,\mathrm{rpm} \tag{10}$$

From equation 3:

$$V_{\text{tacho,ss}} = kt \cdot \omega \tag{11}$$

Finally, from our previous equations we find that

$$k_0 = \frac{\Delta\theta}{k_\mu \cdot \omega \cdot \Delta t} = 0.24667 \tag{12}$$

$$k_m = 155.39 \tag{13}$$

$$k_t = 9.2667 * 10^{-4} \tag{14}$$

1.2 State equations

Consider U the input of our system. The transfer functions of the two states variables(Vtacho and Output) are given from.

for x1 state variable

$$\frac{OUT(s)}{U(s)} = -\frac{k_\mu k_o k_m}{s(T_m s + 1)}.$$

$$\frac{Vtacho(s)}{U(s)} = \frac{k_m k_t}{T_m s + 1}.$$

$$\frac{OUT(s)}{Vtacho(s)} = \frac{-k_\mu ko}{skt}$$

Therefore for x1=output and x2=Vtacho taking the inverse Laplace Transform for the (3) equation we get

$$\frac{dx_1}{dt} = \frac{-k_\mu k_o}{k_t} x2$$

for the x2 state variable

$$sVtacho(s) = \frac{sk_m k_t}{T_m s + 1} U(s)$$

$$sVtacho(s) = k_m k_t \frac{s + \frac{1}{T_m} - \frac{1}{T_m}}{T_m s + 1} U(s)$$

$$sVtacho(s) = \frac{k_m k_t}{T_m} U(s) - \frac{1}{T_m} \frac{k_m k_t}{T_m(s + \frac{1}{T_m})} U(s)$$

eventually from (2)

$$sVtacho(s) = \frac{k_m k_t}{T_m} U(s) - \frac{1}{T_m} Vtacho(s)$$

Taking the inverse Laplace Transform

$$\frac{dx_2}{dt} = -\frac{1}{T_m} x_2 + \frac{k_m k_t}{T_m} u$$

So our system with the state equations can be described by

$$\frac{dx_1}{dt} = \frac{-k_\mu k_o}{k_t} x_2$$

$$\frac{dx_2}{dt} = -\frac{1}{T_m} x_2 + \frac{k_m k_t}{T_m} u$$

The system can be writen in a matrix form:

$$\dot{x} = Ax + Bu, \quad y = Cx$$

where:

$$A = \begin{bmatrix} 0 & \frac{-k_\mu k_o}{k_t} \\ 0 & \frac{-1}{T_m} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{k_m k_t}{T_m} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

2 Linear feedback controller

2.1 Choosing the gains of the controller

First of all we need to check the controllabity of the system The controllability array M is:

$$M = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & -\frac{k_m k_t k_\mu ko}{T_m k_t} \\ \frac{k_m k_t}{T_m} & -\frac{k_m k_t}{T_m^2} \end{bmatrix}$$

and we can conclude that the system can be controlled because

$$det(M) = \frac{(k_m k_t)^2}{(T_m)^2} \frac{k_\mu k_o}{k_t}$$

the result is not zero so rank(M)=2 and our system is controllable.Since we use a linear states feedback controller $u = -k_1 x1 - k_2 x2 + k_r r$

$$\dot{x} = \begin{bmatrix} 0 & -\frac{k_\mu k_o}{kt} \\ -\frac{k_m k_t k_1}{Tm} & -\frac{1+k_m k_t k_2}{T_m} \end{bmatrix} x + \begin{bmatrix} 0 \\ k_r \end{bmatrix} r$$

In order to calculate k1,k2 we need to evaluate the characteristic polynomial of the system by calculating

$$det(sI - \tilde{A})$$

, where

$$\tilde{A}$$

is the new matrix we calculated after using u=-k1x1-k2x2+krr.

$$\begin{aligned} \det(sI - \tilde{A}) &= \left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & -\frac{k_\mu k0}{k_t} \\ -\frac{k_m k_t k_1}{T_m} & -\frac{1+k_m k_t k_2}{T_m} \end{bmatrix} \right| \\ &= s^2 + \frac{1}{T_m}(1 + k_2 k_t k_m))s + \frac{-k_m k_o k_\mu k_1}{T_m} \end{aligned}$$

Since we have a second degree system we need both coefficients of the characteristic polynomial must be positive in order to have stability from Routh-Hurwitz criteria.So we choose

$$k2 > 0$$

and

$$k_1 < 0$$

. Our characteristic polynomial can be written in a form,

$$s^2 + 2zw + w^2$$

In order to achieve a fast response time and no elevation we need z =1 so we wont get elevation in the output.Also the recovery time of the system can be roughly calculated from

$$ts = \frac{4}{2zw}$$

So from equalizing (10) and (11) while z=1 we get

$$w = \frac{1}{2T_m}(1 + k2ktkm))$$

and

$$w = \sqrt{\frac{-kmktkok_\mu k1}{Tm}}$$

So for example if we want a recovery time ts=0.95s with no elevation we get w=8rad/s and making a substitution in our previous equations we get approximately

$$k1 = -10$$

and

$$k2 = 27.17$$

Moreover we need to send the position to the desired position r=5 so kr can be calculated from

$$kr = -\frac{1}{C(A - Bk)^{-1}B}$$

where as k=[k1 k2], and kr=-10.

Here is a simulation made in simulink that confirms the following calculations,

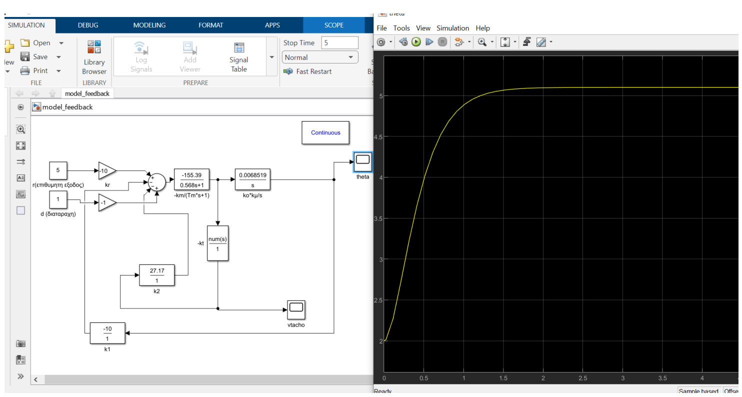


Figure 1: Simulink position diagramm with u=10x1-27.17x2-10r

2.2 Laboratory results

2.2.1 Reaching the desired position

Since we calculated the gains of the controller we can continue to the actuall system simulations.

1*Note. The arduino controller u can only take values till u=10 because the arduino controller can read voltage values untill 5V so in order to design our controller we need to choose smaller values for k1,k2,kr

2*Note. Although in theorytical analysis we found that $k1 < 0$ the gains of the controller are getting positive values because pin9 and pin6 of the uno board multiply by the values with -1

1)following results for k=1.5,k2=6.2845,kr=1.5

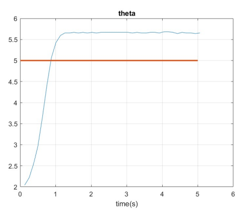


Figure 2: Position diagramm

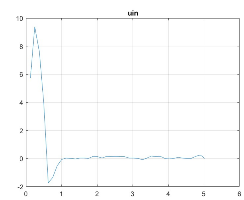


Figure 3: Controller diagram

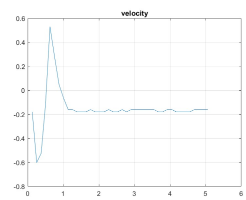


Figure 4: Vtacho diagramm

2)following results for k=1,k2=3.8584,kr=1

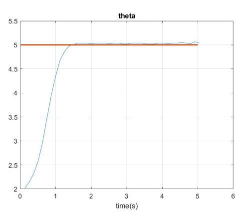


Figure 5: Position diagramm

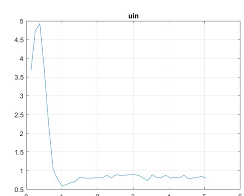


Figure 6: Controller diagram

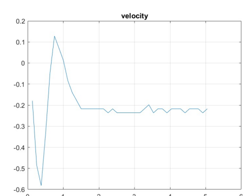


Figure 7: Vtacho diagramm

2.2.2 Error in steady state

We observe that for lower gains of our controller the system seems to have a smaller error in the steady state ,moreover position comes closer to r=5.This error is more clear in the first set of gains(k1=1.5),that is caused by the disturbances,noise and the friction in the belt of the motor system.

2.2.3 Pulling down the magnetic brake

After lowering the magnetic brake we get the following diagrams for the corresponding values of the k parameters we calculated above.

1)following results for k=1.5,k2=6.2845,kr=1.5

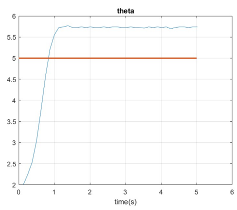


Figure 8: Position diagramm

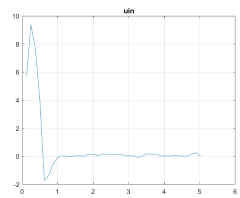


Figure 9: Controller diagram

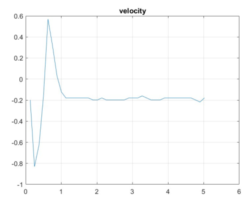


Figure 10: Vtacho diagramm

2)following results for k=1,k2=3.8584,kr=1

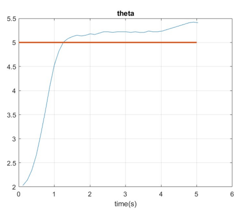


Figure 11: Position diagramm

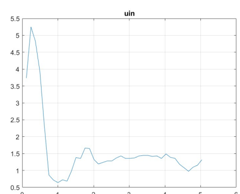


Figure 12: Controller diagram

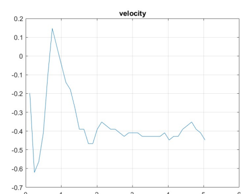


Figure 13: Vtacho diagramm

2.2.4 Changing thetaref

Our goal in this problem is to observe the outcomes of the system for r=5+2sin(w*t),for the diferent values of w.

*Note for the following diagramms we choose k1=1.2,k2=4.888,kr=1.2

1)for w=0.5

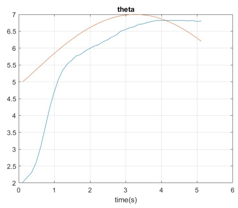


Figure 14: Position diagramm

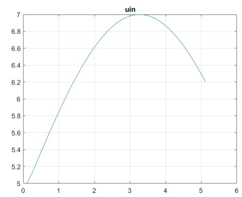


Figure 15: Controller diagram

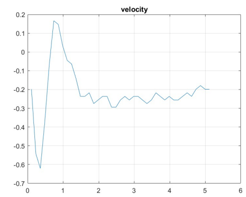


Figure 16: Vtacho diagramm

2)for w=2

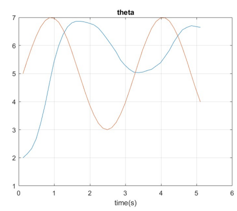


Figure 17: Position diagramm

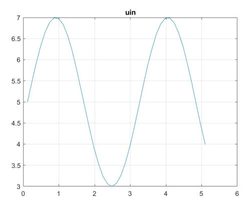


Figure 18: Controller diagram

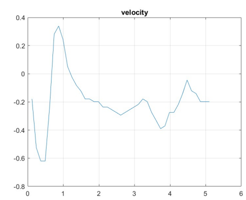


Figure 19: Vtacho diagramm

3)for w=5

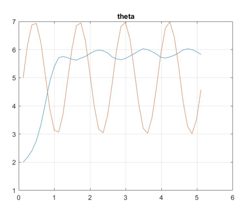


Figure 20: Position diagramm

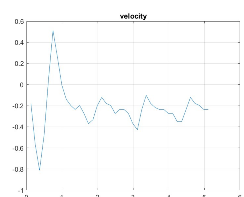


Figure 21: Controller diagram

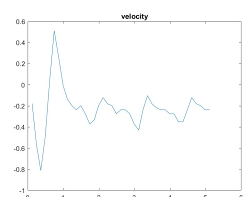


Figure 22: Vtacho diagramm

Through the diagrams we observe that the output follows the changes of the curvature for the different angular frequency values of the desired output, however it does not follow the amplitude changes because as the angular frequency increases the amplitude of the oscillations in the output is smaller than that of the desired.This again is caused due to the disturbances of the system.

3 Dynamic state feedback controller

3.1 Theorytical analysis

In order to achieve r=5 regardless the disturbance we must use a dynamic state feedback controller in a form of

u=-k1x1-k2x2-kiz where

$$\frac{dz}{dt} = y - r$$

and y=Cx ,while C=[1,0] by substituting in our initial system we have new state equations,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{k_\mu k_0}{k_t} & 0 \\ -k1\frac{k_m k_t}{T_m} & -\frac{1}{T_m} - k2\frac{k_m k_t}{T_m} & -ki\frac{k_m k_t}{T_m} \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} r.$$

In order to calculate the new characteristic polynomial

$$\det(sI - \tilde{A}) = \left| \begin{bmatrix} s & \frac{k_\mu k_0}{k_T} & 0 \\ k1\frac{k_m k_t}{T_m} & s + \frac{1}{T_m} + k2\frac{k_m k_t}{T_m} & ki\frac{k_m k_t}{T_m} \\ -1 & 0 & s \end{bmatrix} \right| =$$

$$s^3 + \frac{1}{T_m}(1 + k_t k_m k_2)s^2 - \frac{k_\mu k_0 k_m}{T_m} k_1 s - \frac{k_\mu k_0 k_m}{T_m} k_i$$

In order to ensure stability to the system the polynomial must fullfill the Routh-Hurwitz criteria

$$\begin{array}{ccc} s^3 & 1 & -\frac{k_\mu k_o k_m}{T_m} k_1 \\ s^2 & \frac{1}{T_m}(1 + k_t k_m k_2) & -\frac{k_\mu k_o k_m}{T_m} k_i \\ s^1 & a & 0 \\ s^0 & b & 0 \end{array}$$

where

$$a = \frac{k_\mu k_0 k_m}{T^2 m} \frac{k_1(1 + k_t k_m k_2) - k_i}{\frac{1}{T_m}(1 + k_t k_m k_2)},$$

and

$$b = -k_i \frac{k_\mu k_o k_m}{T_m}.$$

To avoid having roots in the right half-plane, there should be no sign changes in the elements of the first column of the Routh matrix. So $1 > 0$,also $a > 0$ and $b > 0$ and

$$k2 > -\frac{1}{k_t k_m}$$

,

$$ki < 0.$$

$$k1 > \frac{k_i}{\frac{1}{T_m} + \frac{k_m k_t}{T_m}}$$

. Lets assume we want to send the poles of the system in a desired polynomial such as,

$$(s+20)(s+2)(s+2) = s^3 + 24s^2 + 84s + 80$$

so that we wont have elevation and we would achieve a fast ts .By equalizing the characteristic polynomial with the desired polynomial we can choose k1,k2,ki such as

$$k2 = 101.67$$

$$k1 = -44.84$$

$$ki = -42.7039$$

. We actually choose a pole further from the other 2 which is faster and simplify the system so the characteristic polynomial can approximately written in

$$(20)(s+2)(s+2) = (s^2 + 4s + 4)20$$

And we multiply by 20 so as we can have the same dc gain. The following simulation in simulink confirms the choise we made because as we see regardless the disturbance the output reaches our desired position.

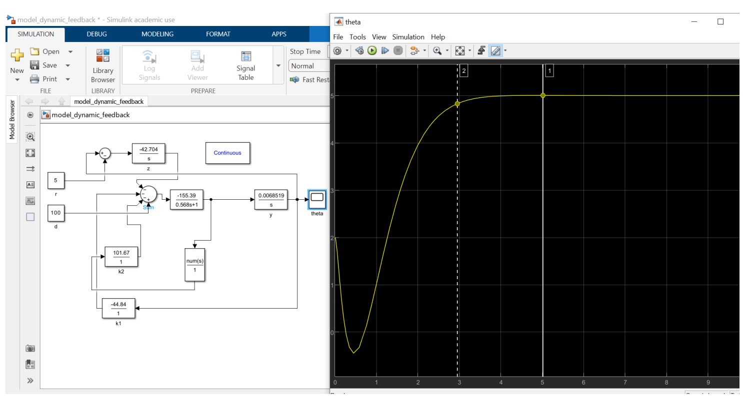


Figure 23: Simulink position diagramm with u=44.84x1-101.67x2+42.7039z

3.2 Laboratory results

3.2.1 Without magnetic brake

1)following results for poles in p1=p2=p3=1.2

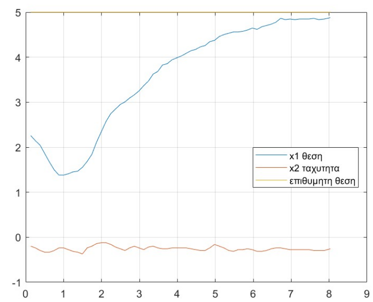


Figure 24: Position and velocity diagram for u=2.30x1+7.25x2+0.92z

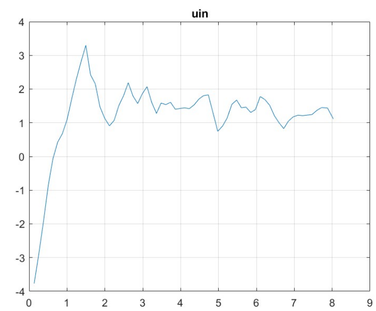


Figure 25: Controller diagram

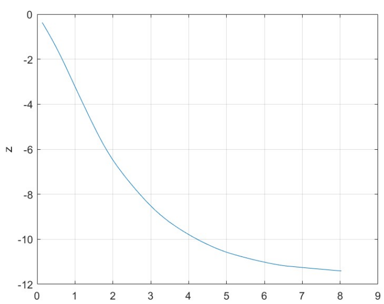


Figure 26: Diagram for z state variable

2)following results for poles in p1=p2=p3=1.5

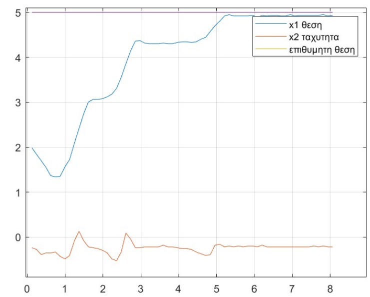


Figure 27: Position and velocity diagram for u=3.6x1+10.8x2+1.8005z

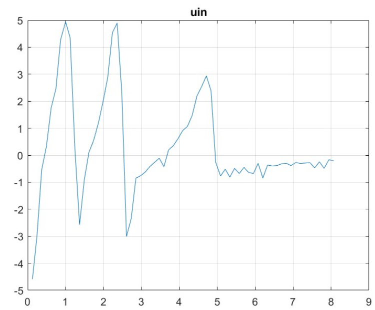


Figure 28: Controller diagram

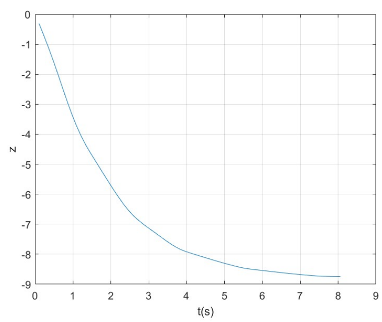


Figure 29: Diagram for z state variable

3)following results for poles in p1=p2=p3=1.7

We can observe that for higher values for the poles of the system we can achieve better recovery time.

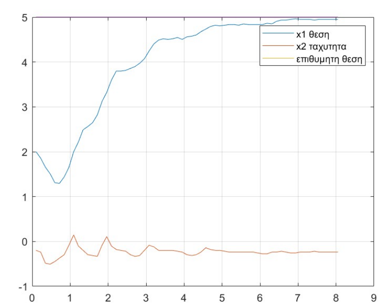


Figure 30: Position and velocity diagram for u=4.62x1+13.17x2+2.62z

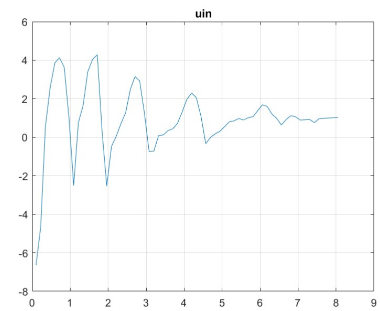


Figure 31: Controller diagram

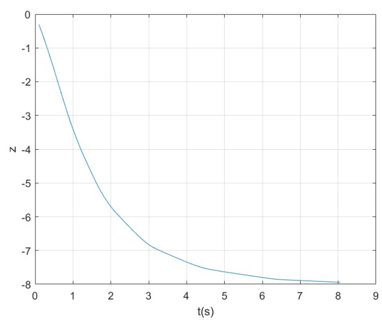


Figure 32: Diagram for z state variable

3.2.2 With magnetic brake

1)following results for poles in p1=p2=p3=1.2

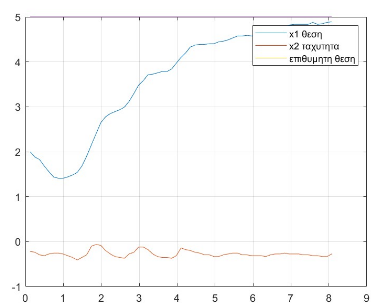


Figure 33: Position and velocity diagram for u=2.30x1+7.25x2+0.92z

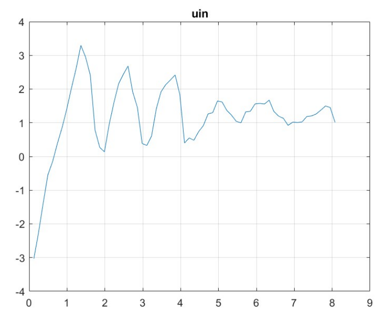


Figure 34: Controller diagram

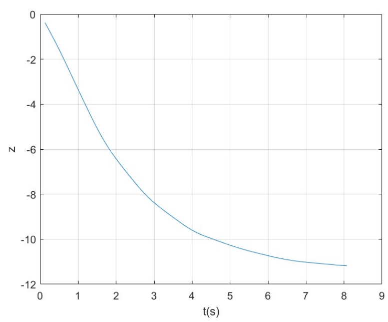


Figure 35: Diagram for z state variable

2)following results for poles in p1=p2=p3=1.5

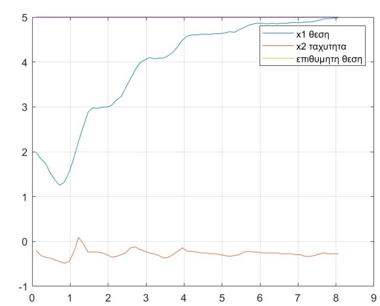


Figure 36: Position and velocity diagram for u=3.6x1+10.8x2+1.8005z

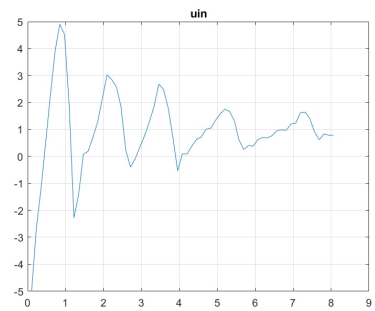


Figure 37: Controller diagram

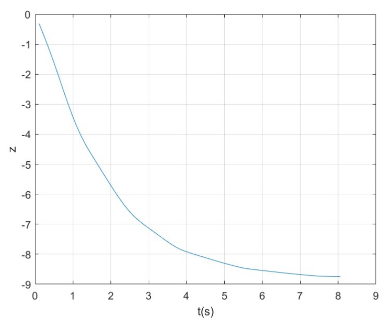


Figure 38: Diagram for z state variable

3)following results for poles in p1=p2=p3=1.7

Again we can observe that for higher values of the poles the system can achieve better recovery time Although the system has increaced its disturbance we can still reach the desired position .

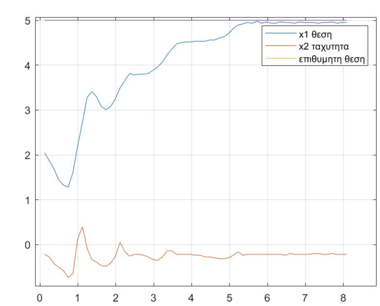


Figure 39: Position and velocity diagram for u=4.62x1+13.17x2+2.62z

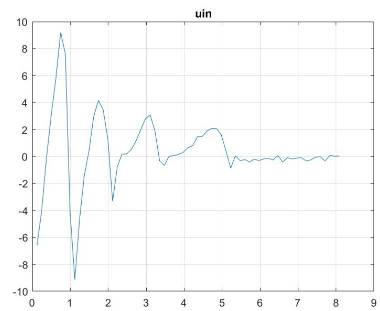


Figure 40: Controller diagram

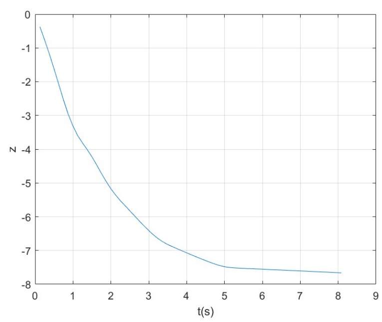


Figure 41: Diagram for z state variable

4 Linear output feedback controller

4.1 Construction of the observer

In order to construct the observer we must assure that the system is observable

$$W = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{k_\mu ko}{kt} \end{pmatrix}$$

$$det(M) = -\frac{k_\mu ko}{kt}$$

is not 0 so rank(M)=2 and the system is observable

We define an observer in a form of

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

Lets assume $\overline{x}$ the error between the state variable and the observer such as

$$\overline{x} = x - \hat{x}$$

Given that $\dot{x} = Ax + Bu$ by subtituting we have

$$\dot{\overline{x}} = (A - LC)\overline{x}$$

That means that we have to choose the gains of the column array L so as we have stable system in the error equation and faster observation time. For instance let ,

$$p_d(s) = s^2 + 40s + 400$$

be the desired characteristic polynomial and

$$L = \begin{pmatrix} l1 \\ l2 \end{pmatrix}$$

The characteristic polynomial of the system is given from

$$det(sI - A) = |\begin{bmatrix} s & \frac{k_\mu k_o}{k_t} \\ 0 & s + \frac{1}{T_m} \end{bmatrix}|$$

$$p_A = s^2 + \frac{1}{T_m}s$$

$$\alpha_1 = \frac{1}{T_m}$$

$$\alpha_2 = 0$$

We will calculate the values l1,l2 from $L = W^{-1}\overline{W}\begin{bmatrix} p_1 - a_1 \\ p_2 - a_2 \end{bmatrix}$

, where $\overline{W}$ is the observability matrix of the normal observable form since the system is already observable.

$$\overline{W} = \begin{bmatrix} 1 & 0 \\ \frac{1}{T_m} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{T_m} & 1 \end{bmatrix}$$

$$L = -\frac{k_t}{k_\mu k_o}\begin{bmatrix} -\frac{k_\mu ko}{k_t} & 0 \\ -\frac{1}{T_m} & 1 \end{bmatrix}\begin{bmatrix} p_1 - \alpha_1 \\ p_2 - \alpha_2 \end{bmatrix}$$

$$L = -\frac{k_t}{k_\mu k_o}\begin{bmatrix} -\frac{k_\mu k_o}{k_t}(p_1 - \alpha_1) \\ p_2 - \alpha_2 - \frac{1}{T_m}(p_1 - \alpha_1) \end{bmatrix}$$

Having already calculated the parameters from above we can finally define l1,l2

$$L = \begin{bmatrix} 38.23 \\ -44.91 \end{bmatrix}$$

The following simulation is made with simulink justifying the gains of the L column array. The green and the blue lines represent the velocity and the position of the system respectively,while the red and the yellow represent the velocity and position of the observer.

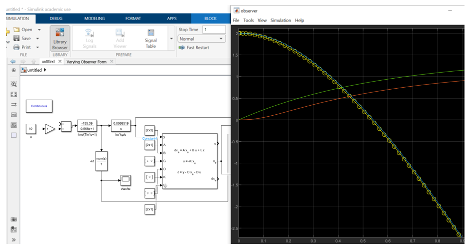


Figure 42:

4.2 Linear output feedback controller

In this section we will study the construction of a linear output feedback controller that uses the measurments of the observer. Since we already see from previous equation that the system is both controllable and observable we can use $u = -k\hat{x} + kr * r$ as a controller. We know that

$$\frac{dx}{dt} = Ax + Bu$$

and

$$\frac{d\overline{x}}{dt} = (A - LC)\overline{x}$$

By substituting u controller we have

$$\dot{x} = Ax - Bk\hat{x} + Bk_r * r$$

$$\hat{x} = x - \overline{x}$$

$$\dot{x} = (A - Bk)x + Bk\overline{x} + Bk_r * r$$

We have an equivalent system in form of

$$\begin{bmatrix} \dot{x} \\ \dot{\overline{x}} \end{bmatrix} = \begin{bmatrix} A - Bk & Bk \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ \overline{x} \end{bmatrix} + \begin{bmatrix} Bk_r \\ 0 \end{bmatrix} r$$

The new characteristic polynomial is given from

$$p(s) = det(sI - A + LC)det(sI - A + Bk)$$

The characteristic polynomial of the controller

$$det(sI - A + Bk) = s^2 + s(\frac{k_m k_t}{T_m} k_2 + \frac{1}{T_m}) - \frac{k_m k_\mu k_o}{T_m} k_1$$

And for the observer

$$A - LC = \begin{bmatrix} -l_1 & -\frac{k_\mu k_o}{k_t} \\ -l_2 & -\frac{1}{T_m} \end{bmatrix}$$

$$det(sI - A + LC) = \begin{bmatrix} s + l_1 & \frac{k_\mu k_o}{kt} \\ l_2 & s + \frac{1}{T_m} \end{bmatrix} = s^2 + s(l_1 + \frac{1}{T_m}) + \frac{l1}{T_m} - l_2 \frac{k_\mu k_o}{k_t}$$

The total polynomial is the multiplication of them ,

$$p(s) = (s^2 + s(l_1 + \frac{1}{T_m}) + \frac{l1}{T_m} - l_2\frac{k_\mu k_o}{k_t}) * (s^2 + s(\frac{k_m k_t}{Tm}k_2 + \frac{1}{T_m}) - \frac{k_m k_\mu k_o}{T_m}k_1)$$

We can choose the poles of the observer to be faster than the ones of the controller . So lets say we want a characteristic polynomial for the observer to be $p_{obs} = s^2 + 40s + 400$ We will have $l_1 = 38.24$ and $l_2 = -45$.Also we can choose the poles of the controller to be slower (poles of the observer at least 2 times faster), so lets assume we want both of them to be at -3. We would have a characteristic polynomial $p_{obs} = s^2 + 6s + 9$ and $k_1 = -4.8$,$k_2 = 16.72$. The k_r gain can be calculated through

$$k_r = -\frac{1}{C(A - Bk)^{-1}B} = -4.8$$

The characteristic polynomial of the system is

$$p_d(s) = (s^2 + 40s + 400)(s^2 + 6s + 9) = s^4 + 46s^3 + 649s^2 + 2760s + 3600$$

The following simulation is made with simulink and justifies the choises we made for the gains l1,l2,k1,k2 and kr.

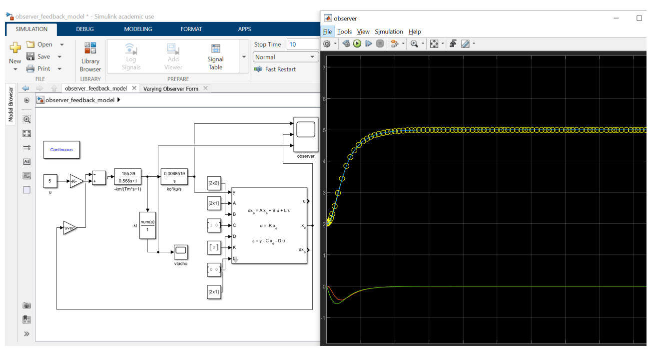


Figure 43: The green and the blue lines represent the velocity and the position of the system respectively,while the red and the yellow represent the velocity and position of the observer.The observer follows the values of the state variables and as we see we have no elevation and a recovery time $t_s = 1.652s$

4.3 Laboratory results

4.3.1 Observer

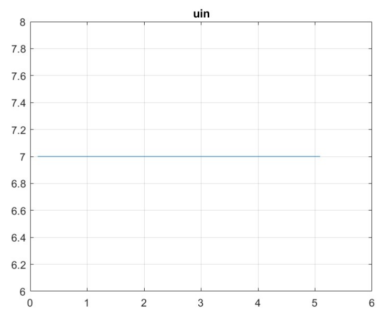


Figure 44: Control diagram

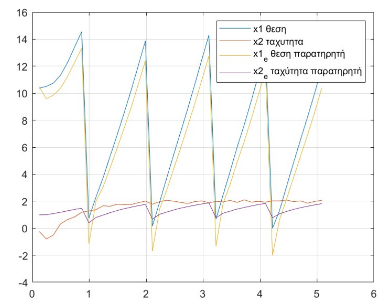


Figure 45: State variables diagram for observer and initial system

We choose our input in u=7Volt and $L = [18.23, -9.18]^T$.Both position estimation and velocity estimation works fine although the velocity estimation is a bit slower.We could resolve this either by increasing the gains of the L vector or its possible to have some modeling errors. We choose our input in u=7Volt

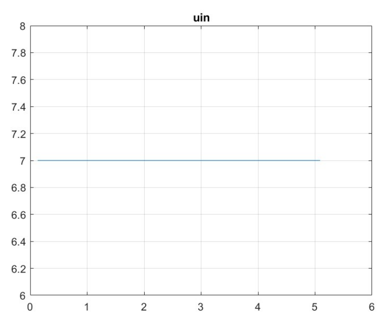


Figure 46: Control diagram

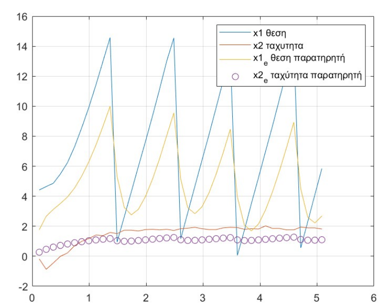


Figure 47: State variables diagram for observer and initial system

and $L = [8.23, -0.7]^T$.Both position estimation and the velocity estimation are slower.This occurs due to the fact that we choose lower gains for the L vector , so the poles of the characteristic polynomial are closer to the imaginary axis witch leads to higher recovery time.

4.3.2 Linear output feedback controller

Our controller has a form $u = -K\hat{x} - k_r r$ with gains as calculated during the 2nd lab exercise, which are k1 = 1, k2 = 3.88, and kr = k1. We choose initial state for theta variable x1(0)=2 and desired position r=5. We choose $L =$

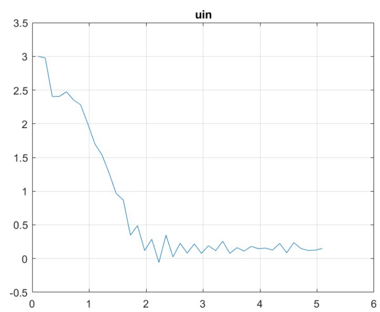


Figure 48: Control diagram

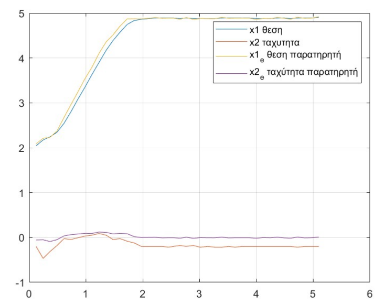


Figure 49: State variables diagram for observer and initial system

$[18.23, -9.18]^T$.Both position estimation and velocity does not have significant deviation from both state variables.

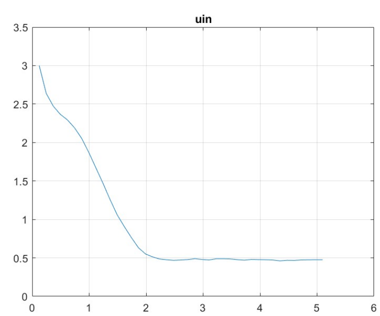


Figure 50: Control diagram

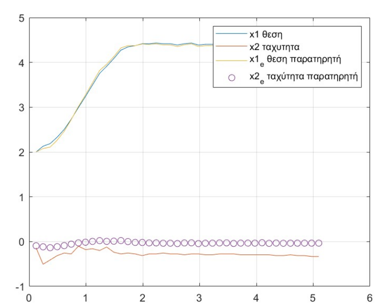


Figure 51: State variables diagram for observer and initial system

We choose $L = [10.23, -2.44]^T$.Both position estimation and velocity does not have significant deviation from both state variables.The system appears to be a bit slower in contrast to the previous choise for the L vector.