Homework #3

$\boxed{Q.1}$ Problem statement: $f(\cdot)$ and $g(\cdot)$ are convex. Prove that $h(x) = \max\{f(x), g(x)\}$ is convex.

Solution Δ: $f(x)$ is convex, $\Rightarrow$

$$f([1-\alpha]x + \alpha y) \leq [1-\alpha]f(x) + \alpha f(y)$$

$\forall x, y,\ \alpha \in [0,1]$ by definition

Similarly, $g([1-\alpha]x + \alpha y) \leq [1-\alpha]g(x) + \alpha g(y), \forall x, y, \alpha \in [0,1]$

$h(x) = \max\{f(x), g(x)\}$, consider $z = [1-\alpha]x +$ $+\alpha y$, for arbitrary x, y and $\alpha \in [0,1]$

$$h(z) = \begin{cases} f(z), & \text{if } f(z) \geq g(z) \\ g(z), & \text{if } f(z) < g(z) \end{cases}$$

$$\left.\begin{aligned} f(z) &\leq [1-\alpha] f(x) + \alpha f(y) \\ g(z) &\leq [1-\alpha] g(x) + \alpha g(y) \end{aligned}\right\}, 1-\alpha \geq 0, \alpha \geq 0, \Rightarrow$$

$$f(z) \leq (1-\alpha) \underbrace{\max\{f(x), g(x)\}}_{h(x)} + \alpha \underbrace{\max\{f(y), g(y)\}}_{h(y)}$$

because $h(x) \geq f(x)$ and $h(y) \geq g(y)$

Similarly, $g(z) \leq (1-\alpha) h(x) + \alpha h(y)$.

Thus, $h([1-\alpha]x + \alpha y) = \begin{cases} f(z), & \text{if } f(z) \geq g(z) \\ g(z), & \text{if } f(z) < g(z) \end{cases} \leq$

$\leq (1-\alpha) h(x) + \alpha h(y), \forall x, y,\ \alpha \in [0,1]$

$\Rightarrow h(\cdot)$ is convex by definition $\blacksquare$

$\boxed{Q.3}$ Problem statement: Solve $\min\limits_{z \in \mathbb{R}} \frac{1}{m} \sum\limits_{i=1}^{m} (z - x_i)^2$

Solution: $f(z) = \frac{1}{m} \sum\limits_{i=1}^{m} (z - x_i)^2$

$$f'(z) = \frac{1}{m} \sum_{i=1}^{m} ((z - x_i)^2)'_z = \frac{1}{m} \sum_{i=1}^{m} 2(z - x_i) =$$

$$= 2\frac{1}{m}\left[\sum_{i=1}^{m} z - \sum_{i=1}^{m} x_i\right] = 2z - 2\frac{1}{m}\sum_{i=1}^{m} x_i$$

$f'(z) = 0$, $\Rightarrow$ $z = \frac{1}{m}\sum\limits_{i=1}^{m} x_i$ is an extremum of $f(z)$. $f''(z) = 2 > 0$, $\Rightarrow z = \frac{1}{m}\sum\limits_{i=1}^{m} x_i$ is the minimum

Answer: $z^* = \frac{1}{m}\sum\limits_{i=1}^{m} x_i$, $f^* = \frac{1}{m}\sum\limits_{i=1}^{m} (z^* - x_i)^2$

$\boxed{Q.4}$ Problem statement: Solve $\min\limits_{a,b} \frac{1}{m} \sum\limits_{i}^{m} (ax_i + b - y_i)^2$

Solution: $f(a,b) = \frac{1}{m} \sum\limits_{i}^{m} (ax_i + b - y_i)^2$

$$\frac{\partial f}{\partial a} = \frac{1}{m} \sum_{i}^{m} \frac{\partial}{\partial a}((ax_i + b - y_i)^2) = \frac{1}{m} \sum_{i=1}^{m} 2(ax_i + b - y_i) \cdot x_i$$

$$\frac{\partial f}{\partial b} = \frac{1}{m} \sum_{i}^{m} \frac{\partial}{\partial b}((ax_i + b - y_i)^2) = \frac{1}{m} \sum_{i} 2(ax_i + b - y_i)$$

$$\nabla f = 0: \begin{cases} \sum\limits_{i=1}^{m} (ax_i + b - y_i) x_i = 0 \\ \sum\limits_{i=1}^{m} (ax_i + b - y_i) = 0 \end{cases}$$

$$\sum_{i=1}^{m} (ax_i + b - y_i) = a\sum_{i=1}^{m} x_i + mb - \sum_{i=1}^{m} y_i = 0$$

I) $\sum\limits_{i=1}^{m} x_i = 0$, $\Rightarrow mb = \sum\limits_{i=1}^{m} y_i$, $b = \frac{\sum_{i=1}^{m} y_i}{m}$

$$\sum_{i=1}^{m} (ax_i + b - y_i) x_i = a\sum_{i=1}^{m} x_i^2 + \cancel{b\sum_{i=1}^{m} x_i} - \sum_{i=1}^{m} y_i x_i = 0,$$

$a = \frac{\sum_{i=1}^{m} x_i y_i}{\sum_{i=1}^{m} x_i^2}$ if $\sum\limits_{i=1}^{m} x_i^2 \neq 0$. $\sum\limits_{i=1}^{m} x_i^2 = 0$ has no sense because of that means $x_i = 0\ \forall i = \overline{1,m}$

II) $\sum\limits_{i=1}^{m} x_i \neq 0$, $\Rightarrow$ $a = \frac{\sum_{i=1}^{m} y_i - mb}{\sum_{i=1}^{m} x_i}$

$$\sum_{i=1}^{m}\left[\left(\frac{\sum_{j=1}^{m} y_j}{\sum_{j=1}^{m} x_j} - \frac{mb}{\sum_{j=1}^{m} x_j}\right)x_i + b - y_i\right] x_i =$$

$$= \sum_{i=1}^{m} \frac{\sum_{j=1}^{m} y_j}{\sum_{j=1}^{m} x_j} x_i^2 - mb\sum_{i=1}^{m}\frac{x_i^2}{\sum_{j=1}^{m} x_j} + b\sum_{i=1}^{m} x_i - \sum_{i=1}^{m} x_i y_i = 0$$

$$b = \frac{\sum_{i=1}^{m} x_i y_i - \frac{\sum_{j=1}^{m} y_j}{\sum_{j=1}^{m} x_j}\sum_{i=1}^{m} x_i^2}{\sum_{i=1}^{m} x_i - m\sum_{i=1}^{m}\frac{x_i^2}{\sum_{j=1}^{m} x_j}}, \quad \frac{\sum_{j=1}^{m} x_j}{m} = \bar{x}, \frac{\sum_{j=1}^{m} y_j}{m} = \bar{y}, \Rightarrow$$

$$b = \frac{\sum_{i=1}^{m} x_i y_i - \frac{\bar{y}}{\bar{x}}\sum_{i=1}^{m} x_i^2}{\sum x_i - \frac{\sum_{i=1}^{m} x_i^2}{\bar{x}}} = \frac{\bar{x}\sum_{i=1}^{m} x_i y_i - \bar{y}\sum_{i=1}^{m} x_i^2}{\bar{x}\sum_{i=1}^{m} x_i - \sum_{i=1}^{m} x_i^2} = \frac{\bar{x}\sum_{i=1}^{m} x_i y_i - \bar{y}\sum_{i=1}^{m} x_i^2}{m(\bar{x})^2 - \sum_{i=1}^{m} x_i^2}$$

$$a = \frac{m \cdot \bar{y} - m \cdot \dfrac{\bar{x}\sum_{i=1}^{m} x_i y_i - \bar{y}\sum_{i=1}^{m} x_i^2}{\bar{x}\sum_{i=1}^{m} x_i - \sum_{i=1}^{m} x_i^2}}{m\bar{x}} \Leftrightarrow$$

$$\Leftrightarrow \frac{\bar{y}}{\bar{x}} - \frac{\sum_{i=1}^{m} x_i y_i - \frac{\bar{y}}{\bar{x}}\sum_{i=1}^{m} x_i^2}{\bar{x}\sum_{i=1}^{m} x_i - \sum_{i=1}^{m} x_i^2} \Leftrightarrow$$

$$\Leftrightarrow \frac{\bar{x}\bar{y}\sum_{i=1}^{m} x_i - \cancel{\bar{y}\sum_{i=1}^{m} x_i^2} - \bar{x}\sum_{i=1}^{m} x_i y_i + \cancel{\bar{y}\sum_{i=1}^{m} x_i^2}}{\bar{x}^2\sum_{i=1}^{m} x_i - \bar{x}\sum_{i=1}^{m} x_i^2} \Leftrightarrow$$

$$\Leftrightarrow \frac{\bar{y}\sum_{i=1}^{m} x_i - \sum_{i=1}^{m} x_i y_i}{\bar{x}\sum_{i=1}^{m} x_i - \sum_{i=1}^{m} x_i^2} = \frac{m\bar{x}\bar{y} - \sum_{i=1}^{m} x_i y_i}{m(\bar{x})^2 - \sum_{i=1}^{m} x_i^2}$$

Thus, $a = \frac{m\bar{x}\bar{y} - \sum_{i=1}^{m} x_i y_i}{m(\bar{x})^2 - \sum_{i=1}^{m} x_i^2}$ and $b = \frac{\bar{x}\sum_{i=1}^{m} x_i y_i - \bar{y}\sum_{i=1}^{m} x_i^2}{m(\bar{x})^2 - \sum_{i=1}^{m} x_i^2}$

is the extremum (if $\bar{x} = 0$, i.e., $\sum_{i=1}^{m} x_i = 0$ $a = \frac{\sum_{i=1}^{m} x_i y_i}{\sum_{i=1}^{m} x_i^2}$, $b = \bar{y}$, therefore (I) is in the general case (II))

$$\frac{\partial^2 f}{\partial a^2} = \frac{2}{m}\sum_{i=1}^{m} \frac{\partial}{\partial a} x_i(ax_i + b - y_i) = \frac{2}{m}\sum_{i=1}^{m} x_i^2$$

$$\frac{\partial^2 f}{(\partial b)^2} = \frac{2}{m}\sum_{i=1}^{m} \underbrace{\frac{\partial}{\partial b}(ax_i + b - y_i)}_{=1} = \frac{2}{m}\cdot m = 2$$

$$\frac{\partial^2 f}{\partial a \partial b} = \frac{\partial^2 f}{\partial a \partial b} = \frac{2}{m}\sum_{i=1}^{m} \frac{\partial}{\partial a}(ax_i + b - y_i) = \frac{2}{m}\cdot\sum_{i=1}^{m} x_i = 2\bar{x}$$

$$\nabla^2 f(a,b) = \begin{pmatrix} \frac{2}{m}\sum_{i=1}^{m} x_i^2 & 2\bar{x} \\ 2\bar{x} & 2 \end{pmatrix} = 2\begin{pmatrix} \sum_{i=1}^{m} x_i^2 & \bar{x} \\ \bar{x} & 1 \end{pmatrix}$$

$$\sum_{i=1}^{m} x_i^2 \geq 0$$

$$\begin{vmatrix} \sum_{i=1}^{m} x_i^2 & \bar{x} \\ \bar{x} & 1 \end{vmatrix} = \sum_{i=1}^{m} x_i^2 - (\bar{x})^2 \geq 0$$

Answer: $\boxed{a = \frac{m\bar{x}\bar{y} - \sum_{i=1}^{m} x_i y_i}{m(\bar{x})^2 - \sum_{i=1}^{m} x_i^2} \quad b = \frac{\bar{x}\sum_{i=1}^{m} x_i y_i - \bar{y}\sum_{i=1}^{m} x_i}{m(\bar{x})^2 - \sum_{i=1}^{m} x_i^2}}$