Anhang

Nr. 3 $X \hat{=}$ Summe der Augenzahl

a) $P(X=9) = P(W_{rot} + W_{Blau} = 9)$

$= P(W_{rot} = 3 \wedge W_{Blau} = 6) + P(W_r = 4 \wedge W_{Blau} = 5)$

$+ P(W_{rot} = 5 \wedge W_{Blau} = 4) + P(W_{rot} = 6 \wedge W_{Blau} = 3)$

unabhängige Ereignisse $= \left(\frac{1}{6} \cdot \frac{1}{6}\right) \cdot 4 = \frac{4}{36} = \frac{1}{9}$

b) $P(X \geq 9) = P(W_{rot} + W_{Blau} \geq 9)$

$= P(W_{rot} + W_{Blau} = 9) + P(W_{rot} + W_{Blau} = 10) + P(W_{rot} + W_{Blau} = 11)$

$+ P(W_{rot} + W_{Blau} = 12)$

$= \frac{1}{9} + 3 \cdot \frac{1}{36} + 2 \cdot \frac{1}{36} + \frac{1}{36} = \frac{1}{9} + \frac{6}{36} = 5/18$

c) $P(W_{rot} = 4 \wedge W_{Blau} = 5) + P(W_{Blau} = 4 + W_{rot} = 5)$

$= \frac{1}{6} \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{1}{6} = \frac{2}{36} = \frac{1}{18}$

d) $P(W_{rot} = 4 \wedge W_{Blau} = 5) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$

e) $P(W_{rot} + W_{Blau} = 9 \mid W_{rot} = 4)$

$= \frac{P(W_{rot} + W_{Blau} = 9 \wedge W_{rot} = 4)}{P(W_{rot} = 4)} = \frac{P(W_{Blau} = 5 \wedge W_{rot} = 4)}{P(W_{rot} = 4)} = \frac{\frac{1}{6} \cdot \frac{1}{6}}{\frac{1}{6}} = \frac{1}{6}$

f) $P(W_{rot} + W_{Blau} \geq 9 \mid W_{rot} = 4)$

$= \frac{P(W_{rot} + W_{Blau} \geq 9 \wedge W_{rot} = 4)}{P(W_{rot} = 4)} = \frac{P(5,4) + P(6,4)}{P(W_{rot} = 4)} = \frac{2 \cdot \frac{1}{36}}{\frac{1}{6}} = \frac{2}{6} = \frac{1}{3}$

~~$P(W_{rot} = 4, W_{rot} = 5$~~

g) $P(W_{rot} = 4 \wedge W_{Blau} = 5 \mid W_{rot} = 4)$

$= \frac{P(W_{rot} = 4 \wedge W_{Blau} = 5 \wedge W_{rot} = 4)}{P(W_{rot} = 4)} = \frac{\frac{1}{6} \cdot \frac{1}{6}}{1/6} = \frac{1}{6}$

$$g(x) := \int_{-\infty}^{\infty} k\, e^{-\frac{1}{2}\frac{1}{\sigma_x^2\sigma_y^2-cov^2}\left(\sigma_y^2(\underbrace{x-\mu_x}_{\tilde{x}})^2-2cov(\underbrace{y-\mu_y}_{\tilde{y}})(\underbrace{x-\mu_x}_{\tilde{x}})+\sigma_x^2(\underbrace{y-\mu_y}_{\tilde{y}})^2\right)}\,dy$$

$$k = \frac{1}{\sqrt{4\pi^2(\sigma_y^2\sigma_x^2-cov^2)}}$$

$$c = \frac{1}{2}\frac{1}{\sigma_x^2\sigma_y^2-cov^2}$$

$$= k\int_{-\infty}^{\infty} e^{-\underbrace{\frac{1}{2}\frac{\sigma_y^2}{\sigma_x^2\sigma_y^2-cov^2}}_{c}\tilde{x}^2}\, e^{-\frac{1}{2}\frac{1}{\sigma_x^2\sigma_y^2-cov^2}\left(\sigma_x^2\tilde{y}^2-2cov\,\tilde{y}\tilde{x}\right)}\,d\tilde{y}$$

$$= k\, e^{-c\tilde{x}\sigma_y^2}\int_{-\infty}^{\infty} e^{-c(\sigma_x^2\tilde{y}^2-2cov\,\tilde{y}\tilde{x})}\,d\tilde{y}$$

NR:
$$-c\sigma_x^2\left(\tilde{y}^2-\frac{2cov}{\sigma_x^2}\tilde{y}\tilde{x}\right)$$
$$= -c\sigma_x^2\left(\tilde{y}^2-\frac{2cov}{\sigma_x^2}\tilde{y}+\frac{cov^2}{\sigma_x^4}\tilde{x}^2-\frac{cov^2}{\sigma_x^4}\tilde{x}^2\right)$$
$$= -c\sigma_x^2\left(\tilde{y}-\frac{cov}{\sigma_x^2}\tilde{x}\right)^2+\frac{c\,cov^2}{\sigma_x^2}\tilde{x}^2$$

$h(y)$
$\rightarrow$ vertausche x mit y und σ_x mit σ_y

$$= k\, e^{-c\left(\tilde{x}\sigma_y^2-\frac{cov^2}{\sigma_x^2}\tilde{x}^2\right)}\int e^{-c\sigma_x^2(\underbrace{\tilde{y}-\frac{cov}{\sigma_x^2}\tilde{x}}_{y'})^2}\,d\tilde{y}$$

$$= k\, e^{-c\left(\tilde{x}\sigma_y^2-\frac{cov^2}{\sigma_x^2}\tilde{x}^2\right)}\int_{-\infty}^{\infty} e^{-c\sigma_x^2 y'^2}\,dy'$$

$$= k\, e^{-c\left(\tilde{x}\sigma_y^2-\frac{cov^2}{\sigma_x^2}\tilde{x}\right)}\sqrt{\frac{c\sigma_x^2}{\pi}} = k\, e^{-\frac{1}{2}\frac{1}{\sigma_x^2\sigma_y^2-cov^2}\left((x-\mu_x)\sigma_y^2-\frac{cov^2}{\sigma_x^2}(x-\mu_x)^2\right)}\cdot\sqrt{\frac{1}{2\pi}\frac{\sigma_x^2}{\sigma_y^2\sigma_x^2-cov^2}}$$

Nr. 1 $\int_0^\infty f(v)\,dv = N\left(\frac{m}{2\pi k_B T}\right)^{3/2} \cdot 4\pi \int_0^\infty e^{-\frac{mv^2}{2k_BT}} \cdot v^2\,dv \overset{!}{=} 1$

$a x^2 = u^2$

$\sqrt{a}\cdot x = u$

$\frac{du}{dx} = \sqrt{a}$

$$\Big| \int_{-\infty}^{\infty} e^{-a x^2} x^2\,dx = \int_0^\infty e^{-ax^2} \cdot x \cdot x\,dx$$

$$= \left[\frac{-1}{2a} e^{-ax^2} \cdot x\right]_0^\infty + \int_0^\infty \frac{1}{2a} e^{-ax^2}\,dx$$

$$= \int_0^\infty \frac{1}{2a} \cdot \frac{1}{\sqrt{a}}\, e^{-u^2}\,du \qquad \leftarrow \text{"halbes Gaußintegral"}$$

$$= \frac{\sqrt{\pi}}{2a\sqrt{a}} \cdot \frac{1}{2} \quad , \operatorname{Re}(a) > 0$$

$$= N \cdot \left(\frac{m}{2\pi k_B T}\right)^{3/2} \cdot 4\pi \cdot \frac{\sqrt{\pi}}{2}\,\frac{1}{2}\left(\frac{m}{2k_BT}\right)^{-3/2}$$

$$= N \cdot 1 \overset{!}{=} 1 \quad \Rightarrow \quad \boxed{N = 1}$$

c) $\frac{d}{dv} f(v) \overset{!}{=} 0$

$$\frac{1}{2} \cdot \left(\frac{m}{2\pi k_B T}\right)^{3/2} \cdot 4\pi \;\frac{d}{dv}\left(e^{-\frac{mv^2}{2k_BT}} v^2\right)$$

$$= \frac{1}{2}\left(\frac{m}{2\pi k_B T}\right)^{3/2} 4\pi \;\underbrace{\left(\frac{-2m}{2k_BT} \cdot v^3 e^{-\frac{mv^2}{2k_BT}} + 2v\, e^{-\frac{mv^2}{2k_BT}}\right)}_{\overset{!}{=}0} \overset{!}{=} 0$$

$$\Leftrightarrow \left(\frac{-m}{k_BT} \cdot v^3 + 2v\right) = 0$$

$$\Leftrightarrow v = 0 \quad \vee \quad v^2 = \frac{2k_BT}{m}$$

$$\Rightarrow \boxed{v_m = \sqrt{\frac{2k_BT}{m}}}$$

$$\left[\frac{J}{K} \cdot \frac{K}{kg} = \frac{J}{kg} = \frac{kg\, m^2}{s^2}\frac{1}{kg} = \frac{m^2}{s^2}\right]$$

b) $\int_0^\infty f(v)\, v\, dv = 1 \cdot \left(\frac{m}{2\pi k_BT}\right)^{3/2} 4\pi \int_0^\infty e^{-\frac{mv^2}{2k_BT}} v^3\,dv$

$$= 4\pi \left(\frac{1}{\pi \cdot v_m^2}\right)^{3/2} \int_0^\infty e^{-v^2/v_m^2}\, v \cdot v^2\,dv$$

$$= 4\pi \left(\frac{1}{\pi v_m^2}\right)^{3/2}\left(\left[\frac{-v_m^2}{2} e^{-v^2/v_m^2} \cdot v^2\right]_0^\infty + \int_0^\infty v_m^2\, e^{-v^2/v_m^2} \cdot v\,dv\right)$$

$$= 4\pi \left(\frac{1}{\pi v_m^2}\right)^{3/2}\left[\left(v_m^2 \cdot \left(\frac{-v_m^2}{2} e^{-v^2/v_m^2}\right)\right)\right]_0^\infty$$

$$= 4\pi \left(\frac{1}{\pi v_m^2}\right)^{3/2} \frac{v_m^4}{2}$$

$$\boxed{= \frac{2}{\sqrt{\pi}}\, v_m}$$

c) $\int_0^{\tilde{x}=x_{0,5}} f(v) \cdot dx \overset{!}{=} 0,5$

$\Leftrightarrow \left(\frac{1}{\pi v_m^2}\right)^{3/2} \cdot 4\pi \cdot \int_0^{\tilde{x}} e^{-v^2/v_m^2} v^2 dv \overset{!}{=} 0,5$

$\Leftrightarrow \int_0^{\tilde{x}} e^{-v^2/v_m^2} v^2 dv = \frac{1}{8\pi} (\pi \cdot v_m^2)^{3/2}$ | Subst $\frac{v^2}{v_m^2} = u^2$, $v = u \cdot v_m$, $\frac{du}{dv} = \frac{1}{v_m}$

$\Leftrightarrow \int_0^{\tilde{x}/v_m} e^{-u^2} u^2 \cdot v_m^2 \cdot v_m \, du = \frac{1}{8\pi} \sqrt[3]{\pi} \cdot v_m^3$

$\Leftrightarrow \int_0^{\tilde{x}/v_m} e^{-u^2} u^2 \, du = \frac{\sqrt[3]{\pi}}{8\pi} = \frac{\sqrt{\pi}}{8}$

und das ist numerisch gut lösbar!

$\rightarrow$ Numerisch:

$v_{0,5} = 1,087652 \cdot v_m$ //

d) $\frac{1}{2} f(v_m) = \frac{1}{2} f\left(\sqrt{\frac{2k_BT}{m}}\right)$

$$= \frac{1}{2} \cdot \left(\frac{1}{\pi v_m^2}\right)^{3/2} 4\pi \cdot e^{-v_m^2/v_m^2} \cdot v_m^2$$

$$= \frac{1}{2 \cdot e} \cdot \frac{1}{\sqrt{\pi}^3} \frac{1}{v_m^3} \cdot 4\pi \cdot v_m^2$$

$$= \frac{2}{e} \cdot \frac{1}{v_m} \cdot \frac{1}{\sqrt{\pi}}$$

$$f(v') \stackrel{!}{=} \frac{2}{e} \cdot \frac{1}{v_m} \frac{1}{\sqrt{\pi}}$$

$$\Leftrightarrow \frac{1}{\sqrt{\pi}^3 \cdot v_m^3} \cdot 4\pi \cdot e^{-v'^2/v_m^2} \cdot v'^2 \stackrel{!}{=} \frac{2}{e} \frac{1}{v_m} \frac{1}{\sqrt{\pi}}$$

$$\Leftrightarrow e^{-v'^2/v_m^2} v'^2 = \frac{1}{2e} \cdot v_m^2 \qquad | \ln$$

$$\Leftrightarrow \ln(e^{-v'^2/v_m^2}) + \ln(v'^2) = \ln\left(\frac{v_m^2}{2e}\right)$$

$$\Leftrightarrow -v'^2/v_m^2 + \ln(v'^2) = \ln(v_m^2/2e)$$

$$\Leftrightarrow -v'^2/v_m^2 + \ln(v'^2) = \ln(v_m^2) - \ln(2e) \qquad | \frac{v'}{v_m} = a$$

$$\Leftrightarrow -a^2 + \cancel{\ln(v_m^2)} + \ln(a^2) = \cancel{\ln(v_m^2)} - \ln(2e)$$

$$\Leftrightarrow a^2 - \ln(a^2) - \ln(2e) = 0$$

$$\stackrel{WA}{\Rightarrow} a_1 \approx 1{,}6366$$

$$a_2 \approx 0{,}4816$$

$$\rightarrow v_{FWHM} = (a_1 - a_2) v_m = 1{,}155\, v_m$$

e) $\sigma_v = \sqrt{\langle v^2\rangle - \langle v\rangle^2}$

$$\langle v^2\rangle = \int_0^\infty f(v)\cdot v^2\,dv$$

$$= \int_0^\infty \frac{1}{\sqrt[3]{\pi}\,v_m^3}\cdot 4\pi\cdot e^{-v^2/v_m^2}\cdot v^4\cdot dv$$

$$= \frac{1\cdot 4\pi}{\sqrt[3]{\pi}\,v_m^3}\int_0^\infty e^{-v^2/v_m^2}\,v\cdot v^3\,dv$$

$$= \frac{4\pi}{\sqrt[3]{\pi}\,v_m^3}\left(\cancelto{0}{\left[\frac{-v_m^2}{2}\,e^{-v^2/v_m^2}\cdot v^3\right]_0^\infty} + \int_0^\infty \frac{v_m^2}{2}\cdot e^{-v^2/v_m^2}\cdot 3v^2\,dv\right.$$

$$= \frac{6\pi}{\sqrt[3]{\pi}\,v_m}\int_0^\infty e^{-v^2/v_m^2}\,v^2\,dv$$

$$= \frac{6\pi}{\sqrt[3]{\pi}\,v_m}\left(\cancelto{0}{\left[\frac{-v_m^2}{2}\,e^{-v^2/v_m^2}\cdot v\right]_0^\infty} + \int_0^\infty \frac{+v_m^2}{2}\cdot e^{-v^2/v_m^2}\,dv\right)$$

$$\int_0^\infty e^{-ax^2} = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

$$= \frac{3\pi v_m}{\sqrt[3]{\pi}}\cdot\frac{1}{2}\cdot\sqrt{\pi}\cdot\sqrt{v_m^2} = \frac{3}{2}\cdot v_m^2 \quad //$$

$$\rightarrow \sigma_v = \sqrt{\frac{3}{2}v_m^2 - \left(\frac{2}{\sqrt{\pi}}v_m\right)^2}$$

$$= \sqrt{v_m^2\left(\frac{3}{2} - \frac{4}{\pi}\right)} = v_m\cdot\sqrt{\frac{3}{2} - \frac{4}{\pi}}$$