2. Übungszettel zur Physik III

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| A1 | A2 | A3 | A4 | Σ |
|---|---|---|---|---|
| 1,5 | 5 | 4½ | 4,5 | 15/ |

Hausaufgabe 1: Schwingungen an zwei Federn

$$\Delta\ddot{x} + \frac{2k}{m}\cdot\Delta l \cdot \sin(\alpha) = 0$$

und die Gewichtskraft? Außerdem $\ddot{x}$, nicht nur $\Delta\ddot{x}$

$$\Leftrightarrow \Delta\ddot{x} + \frac{2k}{m}(l - l_0)\cdot \sin(\alpha - \alpha_0) = 0$$

Hier ist die Länge der Federn ohne Masse unnötig

Es gilt: $l = \sqrt{a^2 + (x_0 + \Delta x)^2}$ ✓

$$\sin\alpha = \frac{x_0 + \Delta x}{l} = \sin(\Delta\alpha + \alpha_0)$$ ✓

$$\Rightarrow \Delta\ddot{x} + 2k/m\left(\sqrt{a^2 + (x_0 + \Delta x)^2} - l_0\right)\cdot\left(\frac{x_0 + \Delta x}{\sqrt{a^2 + (x_0 - \Delta x)^2}}\right) = 0$$

$$\Leftrightarrow \Delta\ddot{x} + \frac{2k}{m}\left(x_0 + \Delta x - \frac{l_0(x_0 + \Delta x)}{\sqrt{a^2 + (x_0 - \Delta x)^2}}\right)$$ (✓)

$$T\left(\frac{x_0 + \Delta x}{\sqrt{a^2 + (x_0 + \Delta x)^2}}, x_0\right) = \frac{x_0}{\sqrt{a^2 + x_0^2}} + \frac{a^2\cdot\Delta x}{\sqrt{a^2 + x_0^2}^3} + h.o.t$$

$$\Rightarrow \Delta\ddot{x} + \frac{2k}{m}\left(x_0 + \Delta x - l_0\left(\frac{x_0}{\sqrt{a^2 + x_0^2}} + \frac{a^2\cdot\Delta x}{\sqrt{a^2 + x_0^2}^3}\right)\right)$$

$$\Leftrightarrow \Delta\ddot{x} + \frac{2k}{m}\left[\Delta x\left(1 - \frac{l_0 a^2}{\sqrt{a^2 + x_0^2}^3}\right) + x_0 - \frac{l_0 x_0}{\sqrt{a^2 + x_0^2}^3}\right]$$ (✓)

$$\beta := \frac{2k}{m}\left(1 - \frac{l_0 a^2}{\sqrt{a^2 + x_0^2}^3}\right) \qquad \gamma := \left(x_0 - \frac{l_0 x_0}{\sqrt{a^2 + x_0^2}}\right)\frac{2k}{m}$$

$$\Rightarrow \Delta\ddot{x} + [\Delta x\cdot\beta - \gamma] = 0 \Leftrightarrow \Delta\ddot{x} + \Delta x\cdot\underbrace{\beta}_{\omega_0^2} = \gamma$$

$$\omega_0 = \frac{2\pi}{T} \Leftrightarrow T = \frac{2\pi}{\omega_0} = \frac{2\pi}{\sqrt{\beta}} = \frac{2\pi}{\sqrt{\frac{2k}{m}\left(1 - \frac{l_0\cdot a^2}{\sqrt{a^2 + x_0^2}^3}\right)}}$$

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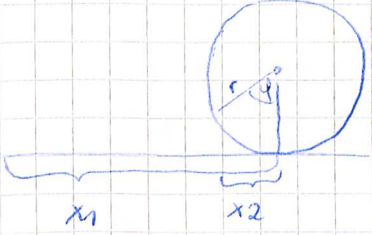
Aufgabe 2)



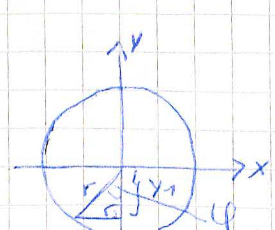

$\Rightarrow x_1 - x_2 = x$

$x_1 = r\cdot\varphi$ (Umfang)


$\Rightarrow \sin\varphi = \frac{x_2}{r} \Leftrightarrow x_2 = \sin\varphi\cdot r$

$\Rightarrow x = r\cdot\varphi - \sin\varphi\cdot r$

$\Leftrightarrow x = r(\varphi - \sin\varphi)$ ✓




$\Rightarrow \cos(\varphi) = \frac{-y_1}{r} \Rightarrow y_1 = -r\cos(\varphi)$

$\Rightarrow y = r + y_1 = r - r\cdot\cos(\varphi)$

$\Leftrightarrow r(1 - \cos\varphi)$ ✓

$$\Rightarrow \vec{x}(\varphi) = \begin{pmatrix} r \cdot \varphi - \sin\varphi \cdot r \\ r - r \cdot \cos(\varphi) \end{pmatrix} = r \begin{pmatrix} \varphi - \sin\varphi \\ 1 - \cos\varphi \end{pmatrix}$$ ✓

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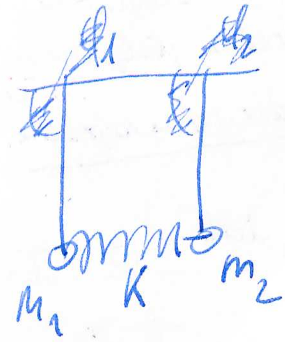
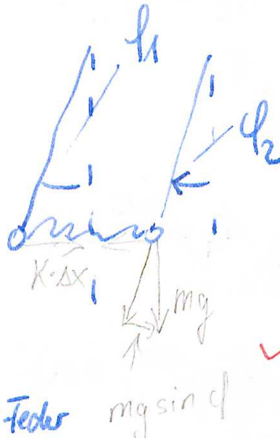
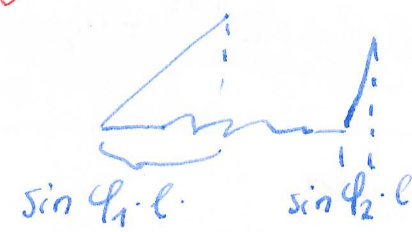
a)





einzelnes Fadenpendel ohne Feder



$$m l \ddot{\varphi} + m g \sin \varphi = 0$$

Jetzt mit Feder für Masse $m_1$

$$m_1 l \ddot{\varphi} + m_1 g \sin \varphi_1 + K \cdot \Delta x = 0$$

$\Delta x$: Auslenkung der Feder $\Delta x = l \cdot (\sin \varphi_1 - \sin \varphi_2)$

$$\rightarrow m_1 l \ddot{\varphi}_1 + m_1 g \varphi_1 + K \cdot l (\varphi_1 - \varphi_2) = 0 \quad \mathrm{I}$$

für Masse 2 äquiv.

$\sin \varphi \approx \varphi$

$$m_2 l \ddot{\varphi}_2 + m_2 g \varphi_2 - Kl(\varphi_1 - \varphi_2) = 0 \quad \mathrm{II}$$

b)

I + II

Das wäre gut gewesen

$$\varphi_1 = \frac{1}{a_{11}} \cdot (\psi_1 - a_{12} \varphi_2)$$

$$\psi_2 = \frac{a_{21}}{a_{11}} (\psi_1 - a_{12} \varphi_2) + a_{22} \varphi_2$$

$$\psi_2 - \frac{a_{21}}{a_{11}} \psi_1 = \varphi_2 \cdot \left(a_{22} - \frac{a_{12} a_{21}}{a_{11}}\right)$$

$$\varphi_2 = \frac{1}{a_{22} - \frac{a_{12} a_{21}}{a_{11}}} \cdot \left(\psi_2 - \frac{a_{21}}{a_{11}} \psi_1\right)$$

Ansatz: $\varphi_1 = A_1 e^{i\omega t}$ $\quad \varphi_2 = A_2 e^{i\omega t}$

das hier scheint nicht sonderlich zielführend zu sein, besser weiter auf nächster Seite.

Ja ü

$\Rightarrow$ einsetzen in I, II:

$$-m_1 l A_1 \omega^2 \cancel{e^{i\omega t}} + m_1 g A_1 \cancel{e^{i\omega t}} + Kl(A_1 - A_2)\cancel{e^{i\omega t}} = 0 \quad \mathrm{Ia}$$

$$-m_2 l A_2 \omega^2 \cancel{e^{i\omega t}} + m_2 g A_2 \cancel{e^{i\omega t}} \mp Kl(A_1 - A_2)\cancel{e^{i\omega t}} = 0 \quad \mathrm{IIa}$$

Gleichungssystem

$$\underbrace{\begin{pmatrix} m_1(g - l\omega^2) + Kl & -Kl \\ -Kl & m_2(g - l\omega^2) + Kl \end{pmatrix}}_{A} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

$$\Leftrightarrow \det A \stackrel{!}{=} 0$$

$$K^2 l^2 + (m_1(g - l\omega^2) + Kl) \cdot (m_2(g - l\omega^2) + Kl) = 0$$

$$2K^2 l^2 + m_1 m_2 (g - l\omega^2)^2 + m_1 g K l - m_1 l \omega^2 K l$$
$$+ m_2 g K l - m_2 l \omega^2 K l = 0$$

$$2K^2 l^2 + gKl(m_1 + m_2) + \omega^2 \cdot (-Kl^2(m_1 + m_2) - 2gl m_1 m_2) + \omega^4 \cdot l^2 m_1 m_2 =$$
$$+ m_1 m_2 g^2 \qquad \Big| \cdot \frac{1}{l^2 m_1 m_2}$$

$$2\frac{K^2}{m_1 m_2} + gK\frac{m_1 + m_2}{l m_1 m_2} + \frac{g^2}{l^2} + \omega^2 \cdot \left(\frac{-K(m_1 + m_2)}{m_1 m_2} - \frac{2g}{l}\right) + \omega^4 = 0$$

$M := \frac{m_1 m_2}{m_1 + m_2}$

$m = m_1 + m_2$

$$\omega_{1/2}^2 = \frac{K}{2M} + \frac{g}{l} \pm \sqrt{\left(\frac{K}{2M} + \frac{g}{l}\right)^2 - 2\frac{K^2}{Mm} - gK\frac{1}{Ml} - \frac{g^2}{l^2}}$$

$$= \frac{K}{2M} + \frac{g}{l} \pm \sqrt{\frac{K^2}{4M^2} - 2\frac{K^2}{Mm}}$$

$\mathrm{I}+\mathrm{II}: \ l(m_1\ddot{\varphi}_1+m_2\ddot{\varphi}_2)+g(m_1\varphi_1+m_2\varphi_2)=0$

$a_{11}=m_1 \quad a_{12}=m_2 \qquad \psi_1=m_1\varphi_1+m_2\varphi_2$

Mit Normierung $\hat{=}$ wieder schöner

$\Rightarrow l\ddot{\psi}_1+g\psi_1=0$ $\qquad$ Ib

~~$\mathrm{I}-\mathrm{II}: l(m_1\ddot{\varphi}_1-m_2\ddot{\varphi}_2)+g(m_1\varphi_1-m_2\varphi_2)+2Kl(\varphi_1-\varphi_2)=0 \quad |\cdot\frac{1}{m_1}\cdot\frac{1}{m_2}$~~

~~$l\left(\frac{\ddot{\varphi}_1}{m_2}-\frac{\ddot{\varphi}_2}{m_1}\right)+g\left(\frac{\varphi_1}{m_2}-\frac{\varphi_2}{m_1}\right)+2Kl\left(\frac{\varphi_1}{m_1m_2}-\frac{\varphi_2}{m_1m_2}\right)=0$~~

$\mathrm{I}\cdot m_2-\mathrm{II}\cdot m_1:$

$m_1m_2l(\ddot{\varphi}_1-\ddot{\varphi}_2)+gm_1m_2(\varphi_1-\varphi_2)+Kl(m_1+m_2)(\varphi_1-\varphi_2)$

$\rightarrow a_{21}=1 \quad , \quad a_{22}=-1 \qquad \psi_2=\varphi_1-\varphi_2$

$\Rightarrow m_1m_2l\ddot{\psi}_2+\psi_2\cdot(gm_1m_2+Kl(m_1+m_2))=0$ $\qquad$ IIb

bzw. $\ddot{\psi}_2+\psi_2\cdot\left(\frac{g}{l}+K\frac{m_1+m_2}{m_1m_2}\right)=0$

Ansatz: $\psi_1=A_1e^{i\omega t} \qquad \psi_2=A_2e^{i\omega t}$

$\overset{\mathrm{Ib}}{\Rightarrow} -A_1\omega^2e^{i\omega t}+\frac{g}{l}A_1e^{i\omega t}=0$

$\Rightarrow \omega_{1/2}=\pm\sqrt{g/l} \quad \Rightarrow \psi_1=A_{11}e^{i\sqrt{g/l}\,t}+A_{12}e^{-i\sqrt{g/l}\,t}$

$\overset{\mathrm{IIb}}{\Rightarrow}$

$\omega_{3/4}=\pm\sqrt{\frac{g}{l}+K\frac{m_1+m_2}{m_1m_2}} \quad \Rightarrow \psi_2=A_{21}e^{i\sqrt{\frac{g}{l}+K\dots}}+A_{22}e^{-i\sqrt{\dots}}$

c) $t=0$: • $\psi_1=m_1\cdot 0+m_2\cdot\hat{\varphi}=A_{11}e^0+A_{12}\cdot e^0$ $\quad$ gut

Anfangsbed. sind für $\varphi$ und nicht für $\psi$ gegeben

$\Rightarrow A_{11}=m_2\hat{\varphi}-A_{12}$

$\dot{\psi}_1(t=0)=m_1\dot{\varphi}_1(t=0)+m_2\dot{\varphi}_2(t=0)=0$

$\Leftrightarrow 0=A_{11}i\omega+A_{12}(-i)\omega \quad |\cdot i \quad |\cdot\frac{1}{\omega}$

$\omega=\sqrt{g/l}$

$\Leftrightarrow A_{11}=A_{12} \quad \Rightarrow A_{11}=A_{12}=\frac{1}{2}m_2\hat{\varphi}$

• $\psi_2(t=0)=-\hat{\varphi}=A_{21}+A_{22}$

$\dot{\psi}_2(t=0)=0 \quad \Rightarrow A_{21}=A_{22}$

$\Rightarrow A_{21}=A_{22}=-\frac{1}{2}\hat{\varphi}$

„Zurückkoppeln" = Verkuppeln ? :)

$\Psi_1 = m_1 \varphi_1 + m_2 \varphi_2 \quad \Rightarrow \Psi_1 = m_1 \varphi_1 + m_2 \varphi_1 - m_2 \Psi_2$

$\Psi_2 = \varphi_1 - \varphi_2 \quad \rightarrow \varphi_2 = \varphi_1 - \Psi_2$

$\Rightarrow \varphi_1 = \dfrac{\Psi_1 + m_2 \Psi_2}{m_1 + m_2}$ $(\checkmark)$

$\Rightarrow \varphi_2 = \dfrac{\Psi_1 - m_1 \Psi_2}{m_1 + m_2}$ $(\checkmark)$

und ab hier hättet ihr mit den AB arbeiten können

$$\rightarrow \varphi_1 = \frac{1}{m_1+m_2} \cdot \left[\frac{1}{2} m_2 \hat{\varphi} \left(e^{i\sqrt{\frac{g}{l}}t} + e^{-i\sqrt{\frac{g}{l}}t}\right) - \frac{1}{2} m_2 \hat{\varphi} \left(e^{+i\omega_3 t} + e^{-i\omega_3 t}\right)\right]$$

$$\varphi_2 = \frac{1}{m_1+m_2} \left[\frac{1}{2} m_2 \hat{\varphi} \left(e^{i\sqrt{\frac{g}{l}}t} + e^{-i\sqrt{\frac{g}{l}}t}\right) + \frac{1}{2} m_1 \hat{\varphi} \left(e^{+i\omega_3 t} + e^{-i\omega_3 t}\right)\right]$$

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$$U = \frac{1}{2} k (x_1 - x_2)^2$$ ✓

$$F = m\ddot{x} = -\nabla U$$

$$m_1 \ddot{x}_1 = -k(x_1 - x_2) \quad \text{I}$$ ✓

$$m_2 \ddot{x}_2 = k(x_1 - x_2) \quad \text{II}$$ ✓

schön

b)

$\text{I} + \text{II}$

$$m_1 \ddot{x}_1 + m_2 \ddot{x}_2 = 0$$ ✓

$$\ddot{\mathcal{X}}_1 = 0$$ ✓

$$\mathcal{X}_1 = m_1 x_1 + m_2 x_2$$ ✓

$$\mathcal{X}_2 = a_{21} x_1 + a_{22} x_2$$

$$-\text{I} \cdot m_2 + \text{II} \cdot m_1$$

$$\cancel{m_1 m_2} (\ddot{x}_2 - \ddot{x}_1) = k (x_1 - x_2) \cdot \frac{(m_2 + m_1)}{m_1 m_2}$$

$$\ddot{\mathcal{X}}_2 = -k \cdot \frac{(m_2 + m_1)}{m_1 m_2} \mathcal{X}_2$$ ✓

$$\mathcal{X}_2 = x_2 - x_1$$ ✓

schön

c) $\frac{\partial^2 \vec{x}}{\partial t^2} = M \vec{x} \qquad \vec{x} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\ddot{x}_1 = -\frac{k}{m_1}(x_1 - x_2) \qquad \ddot{x}_2 = \frac{k}{m_2}(x_1 - x_2)$$

$$\Rightarrow \ddot{\vec{x}} = \underbrace{\begin{pmatrix} -\frac{k}{m_1} & \frac{k}{m_1} \\ \frac{k}{m_2} & -\frac{k}{m_2} \end{pmatrix}}_{M} \vec{x}$$ ✓

$$\det (M - \lambda E) \stackrel{!}{=} 0$$ ✓

$$= \left| \begin{pmatrix} -\frac{k}{m_1} - \lambda & \frac{k}{m_1} \\ \frac{k}{m_2} & -\frac{k}{m_2} - \lambda \end{pmatrix} \right| = \frac{k^2}{m_1 m_2} + \lambda \frac{k}{m_1} + \lambda \frac{k}{m_2} + \lambda^2 - \frac{k^2}{m_1 m_2}$$

$$= \lambda \left( \lambda + \left( \frac{k}{m_1} + \frac{k}{m_2} \right) \right) \stackrel{!}{=} 0$$

$$\Rightarrow \lambda = 0 \qquad \Rightarrow \lambda = -k \left( \frac{1}{m_1} + \frac{1}{m_2} \right)$$ ✓

$\lambda = i\omega$

$\lambda = \omega^2 \Rightarrow \omega_{1/2} = 0$ $\quad -0{,}5$

$\omega = \pm\sqrt{-k\left(\frac{1}{m_1}+\frac{1}{m_2}\right)} \Rightarrow \lambda = -k\left(\frac{1}{m_1}+\frac{1}{m_2}\right)$

Einsetzen in $(M-\lambda E)\vec{u}$

$$= \begin{pmatrix} -\cancel{\frac{k}{m_1}} + \frac{k}{m_1} + \cancel{\frac{k}{m_2}} & \frac{k}{m_1} \\ \frac{k}{m_2} & -\cancel{\frac{k}{m_2}} + \frac{k}{m_1} + \cancel{\frac{k}{m_2}} \end{pmatrix}\vec{u}$$ $\Rightarrow \omega = \pm\sqrt{k\left(\frac{1}{m_1}+\frac{1}{m_2}\right)}$

$$= \begin{pmatrix} k/m_2 & k/m_1 \\ k/m_2 & k/m_1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \overset{!}{=} 0$$

$$u_1 \frac{k}{m_2} + u_2 \frac{k}{m_1} = 0$$

$$u_1 \frac{k}{m_2} + u_2 \frac{k}{m_1} = 0$$

$\begin{pmatrix} -\frac{m_2}{m_1} \\ 1 \end{pmatrix}$ EV

$u_1 = -\frac{m_2}{m_1} u_2 \Rightarrow \vec{u} = a\begin{pmatrix} -\frac{m_2}{m_1} u_2 \\ u_2 \end{pmatrix}$ Eigenvektor $\quad (\checkmark)$

$\lambda = 0$

$$(M-\lambda E)\vec{u} = k\begin{pmatrix} -\frac{1}{m_1} & \frac{1}{m_1} \\ \frac{1}{m_2} & -\frac{1}{m_2} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \overset{!}{=} \vec{0}$$

$$-\frac{k}{m_1} u_1 + \frac{k}{m_1} u_2 = 0 \Leftrightarrow -u_1 + u_2$$

$$\frac{k}{m_2} m + \frac{k}{m_2} u_2 = 0 \Leftrightarrow u_1 = u_2$$

$\Rightarrow \vec{u}_0 = a\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ Eigenvektor $\quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ EV

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