4. Übung zur Physik III

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A1)

$T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$

$= \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$ ✓

$V = \frac{1}{2} c_1 x_1^2 + \frac{1}{2} c_2 (x_2 - x_1)^2 + \frac{1}{2}(c_3 x_2^2)$ ✓

$\mathcal{L} = T - V$

$\frac{\partial \mathcal{L}}{\partial \dot{x}} \begin{cases} \frac{\partial \mathcal{L}}{\partial \dot{x}_1} = m_1 \dot{x}_1 \\ \frac{\partial \mathcal{L}}{\partial \dot{x}_2} = m_2 \dot{x}_2 \end{cases}$

$\frac{\partial \mathcal{L}}{\partial x} \begin{cases} \frac{\partial \mathcal{L}}{\partial x_1} = -\frac{1}{2} c_1 \cdot 2x_1 - c_2 x_1 + c_2 x_2 \\ \frac{\partial \mathcal{L}}{\partial x_2} = -c_2 x_2 + c_2 x_1 - c_3 x_2 \end{cases}$

I $m_1 \ddot{x}_1 + x_1 (c_1 + c_2) \pm c_2 x_2 = 0$

II $m_2 \ddot{x}_2 + x_2 (c_2 + c_3) \pm c_2 x_1 = 0$

Euler-Lagrange: $0 = \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}$

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Aufgabe 2: System 1: Masse m $z_1 = 0$ $\quad l = r + z_2$ (✓)

System 2: Masse $M(x_2, y_2, z_2)$ $\quad r = l - z$

$V = m \cdot g z$

$T = \frac{1}{2} m v_1^2 + \frac{1}{2} M \dot{z}_2^2 \qquad$ mit $v^2 = \dot{r}^2 + r^2 \cdot \dot{\varphi}^2$; $\frac{d}{dt} \dot{z}^2 = \dot{r}^2$

$\mathcal{L} = T - V = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\varphi}^2) + M \cdot g (l - r) + \frac{1}{2} M \dot{r}^2$ ✓

$$\frac{\partial \mathcal{L}}{\partial q_k} \begin{cases} \frac{\partial \mathcal{L}}{\partial r} = m r \dot{\varphi}^2 - Mg \\ \frac{\partial \mathcal{L}}{\partial \varphi} = 0 \end{cases}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_k} \begin{cases} \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r} + M\dot{r} & \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = m\ddot{r} + M\ddot{r} \\ \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = m r^2 \dot{\varphi} & \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right) = m(r^2 \ddot{\varphi}) + m(2 r \dot{r} \dot{\varphi}) \end{cases}$$

mit $\mathcal{L}$ und $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) - \left(\frac{\partial \mathcal{L}}{\partial q_k}\right) = 0$ ergibt sich

I $\quad m\ddot{r} + M\ddot{r} - m r \dot{\varphi}^2 + M \cdot g = 0$ ✓

II $\quad r\ddot{\varphi} + 2\dot{\varphi}\dot{r} = 0$

φ ist zyklisch denn $\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = 0 \quad | \int dt$

$$\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \underbrace{m r^2 \dot{\varphi}}_{L} = \text{const.}$$

$\Rightarrow$ Drehimpuls L ist erhalten ✓

$\mathcal{L}$ hängt nicht explizit von der Zeit ab $\Rightarrow$ H = konst.

$\Rightarrow$ Energieerhaltung (✓) $\qquad$ S. Vorlesung

c) Geg.: $\dot{\varphi} < 0$ $\Rightarrow$ $\dot{r} > 0$

$\dot{r} > 0 \quad | \cdot 2\dot{\varphi}$

$2\dot{\varphi} > 0 \quad |$ mit II $\qquad$ 3/5

anderer Ansatz: $\quad -r\ddot{\varphi} > 0 \xRightarrow{(r>0)} \boxed{\ddot{\varphi} < 0}$

↳ Betrachte I: $(M+m)\ddot{r} = m r \dot{\varphi}^2 - Mg \Rightarrow \ddot{r} > 0 \Leftrightarrow m r \dot{\varphi}^2 > Mg$ ✓

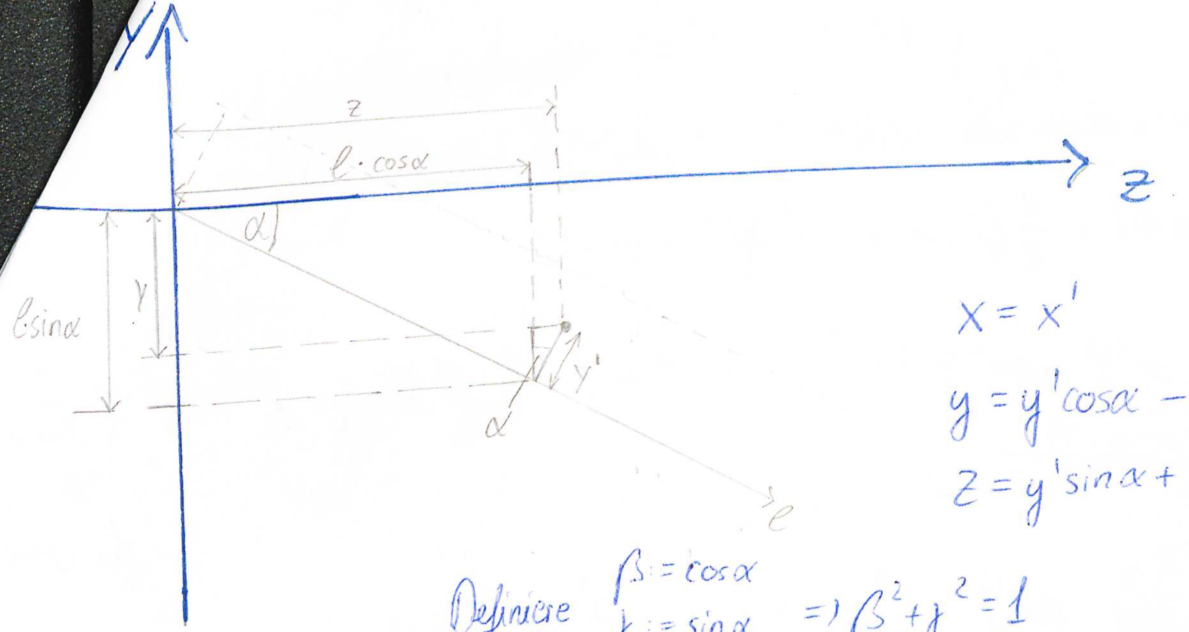
$L > Mg$

$\ddot{r} < 0 \Leftrightarrow \boxed{L < Mg}$

d) $\dot{\varphi} = 0 \overset{\mathrm{II}}{\Rightarrow} r\ddot{\varphi} = 0$

$\mathrm{I} \Rightarrow (m+M)\ddot{r} = -Mg$

$\Leftrightarrow \ddot{r} = \frac{-Mg}{m+M}$



$x = x'$

$y = y'\cos\alpha - l\sin\alpha$

$z = y'\sin\alpha + l\cos\alpha$

Definiere $\begin{cases} \beta := \cos\alpha \\ \gamma := \sin\alpha \end{cases} \Rightarrow \beta^2 + \gamma^2 = 1$

- $$\vec{v} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} a(\dot{\theta} - \cos(\theta)\cdot\dot{\theta} \\ a\beta(-\sin\theta\,\dot{\theta}) - \dot{l}\gamma \\ a\gamma(-\sin\theta\,\dot{\theta}) + \dot{l}\beta \end{pmatrix} \Rightarrow \underset{\beta^2+\gamma^2=1}{\vec{v}^2} = \dot{l}^2 + a^2(\dot{\theta}^2(1-\cos\theta)^2 + \sin(\theta)\dot{\theta}^2)$$

$$= \dot{l}^2 + a^2\dot{\theta}^2(1 - 2\cos\theta + 1)$$

$$= \dot{l}^2 + 2a^2\dot{\theta}^2(1-\cos\theta)$$

$\Rightarrow \mathcal{L}$ $\quad T = \frac{1}{2}m\vec{v}^2, \quad V = mgy = mg(\beta a(1+\cos\theta) - l\gamma)$

$\Rightarrow \mathcal{L} = T - V = \frac{1}{2}m(\dot{l}^2 + 2a^2\dot{\theta}^2(1-\cos\theta)) - mg(\beta a(1+\cos\theta) - l\gamma)$

- $$\frac{\partial\mathcal{L}}{\partial\theta} = \frac{1}{2}ma^2\dot{\theta}^2\,2\sin\theta + mg\beta a\sin\theta$$

$$\frac{\partial\mathcal{L}}{\partial l} = mg\gamma$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = 2ma^2\dot{\theta}(1-\cos\theta)$$

$$\frac{\partial\mathcal{L}}{\partial\dot{l}} = m\dot{l} \qquad \Rightarrow \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_k} = \begin{cases} \theta: & 2ma^2\ddot{\theta}(1-\cos\theta) + 2ma^2\dot{\theta}^2\sin\theta \\ l: & m\ddot{l} \end{cases}$$

$$\Rightarrow \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_k} - \frac{\partial\mathcal{L}}{\partial q_k} = 0 \begin{cases} 2a^2\ddot{\theta}(1-\cos\theta) + a^2\dot{\theta}^2\sin\theta - a\beta g\sin(\theta) = 0 & \mathrm{I} \\ m\ddot{l} - mg\gamma = 0 & \mathrm{II} \end{cases}$$

b) $u := \cos\frac{\Theta}{2}$

$$(*)\begin{cases}\cos\Theta = \cos^2\frac{\Theta}{2} - \sin^2\frac{\Theta}{2}\\ \sin\Theta = 2\sin\frac{\Theta}{2}\cos\frac{\Theta}{2}\end{cases}$$

I: $\frac{1}{2}\ddot\Theta(1-\cos\Theta) + \frac{1}{4}\dot\Theta^2\sin\Theta - \frac{1}{4}\frac{g\beta}{a}\sin\Theta = 0$

$$(*) \hookrightarrow \frac{1}{2}\ddot\Theta\Big(\underbrace{1-\underbrace{\cos^2\tfrac{\Theta}{2}}_{=1-\sin^2\frac{\Theta}{2}} + \sin^2\tfrac{\Theta}{2}}_{=2\sin^2\frac{\Theta}{2}}\Big) + \frac{1}{4}\dot\Theta^2\, 2\sin\frac{\Theta}{2}\cos\frac{\Theta}{2} - \frac{1}{4}\frac{g\beta}{a}2\sin\frac{\Theta}{2}\cos\frac{\Theta}{2} = 0 \quad \Big| \cdot \frac{1}{2\sin\frac{\Theta}{2}}$$

für $\Theta \neq 0$

$(=)$ $\cancel{\frac{1}{2}\ddot\Theta}$

$$\underbrace{\frac{1}{2}\ddot\Theta\sin\frac{\Theta}{2} + \frac{1}{4}\dot\Theta^2\cos\frac{\Theta}{2}}_{-\ddot u} - \frac{1}{4}\frac{g\beta}{a}\underbrace{\cos\frac{\Theta}{2}}_{u} = 0$$

Jetzt vergleiche mit

$$\frac{d}{dt}(\dot u) = \frac{d}{dt}\left(\frac{\partial}{\partial\Theta}u(\Theta)\frac{d\Theta}{dt}\right) = \frac{d}{dt}\left(-\frac{1}{2}\sin\frac{\Theta}{2}\dot\Theta\right)$$

$$\ddot u = \frac{\partial}{\partial\Theta}\dot\Theta + \frac{\partial}{\partial\dot\Theta}\ddot\Theta = -\frac{1}{2}\sin\frac{\Theta}{2}\ddot\Theta + \left(-\frac{1}{4}\cos\frac{\Theta}{2}\dot\Theta^2\right)$$

Oh, das sieht gleich aus...

$$\Rightarrow +\ddot u + \underbrace{\frac{1}{4}\frac{g\beta}{a}}_{\omega^2}u = 0$$ → harmon. Oszillator

→ $\omega = \frac{1}{2}\sqrt{\frac{g\beta}{a}}$

$$u(t) = A\cdot e^{i\omega t} + Be^{-i\omega t} = \cos\frac{\Theta(t)}{2}$$

$$\underline{\underline{\Theta = 2\cdot\arccos\left(A\cdot e^{i\omega t} + Be^{-i\omega t}\right)}}$$ ✓

II: $\ddot\ell = \gamma g \rightarrow \underline{\underline{\ell(t) = \frac{1}{2}\gamma g t^2 + v_0 t + \ell_0}}$ $(\gamma = \sin\alpha)$ ✓

Super!

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a) Weg und Geschwindigkeit. & t

„Neu" ist die Geschwindigkeitsabhängigkeit. Das geht am Ende auf die Lorentz-Kraft zurück!

(✓)

b) $\mathcal{L} = \underbrace{\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)}_{T} - \underbrace{\left(Q\varphi(x,y,z,t) - Q\sum_j \dot{x}_j A_j(x,y,z,t)\right)}_{V=U}$ ✓

c) $$\frac{\partial U}{\partial q_k} = Q\cdot\varphi'(q_k,t) - Q\sum_j \dot{q}_j A_j'(q_k,t) = Q\frac{\partial\varphi}{\partial q_k} - Q\sum_j \dot{q}_j \frac{\partial}{\partial q_k}A_j(q_k,t)$$

$$\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_k}\right) = \cancel{\frac{d}{dt}\left(-Q\sum_j A_k(q_k,t)\right)}$$

$$\frac{d}{dt}\left(-Q\cdot A_k(q_l,t)\right) = -Q\cdot\sum_{l=1}^{3}\frac{\partial A_k(q_l,t)}{\partial q_l}\frac{dq_l}{dt} + \frac{\partial A}{\partial t}\cancel{\frac{dt}{dt}}$$

$$\Rightarrow Q_k = +Q\left(-\frac{\partial\varphi}{\partial q_k} + \overbrace{\sum_j \dot{q}_j\frac{\partial}{\partial q_k}A_j(q_k,t) - \left(\sum_{l=1}^{3}\frac{\partial A_k}{\partial q_l}\frac{dq_l}{dt}\right)}^{I} - \frac{\partial A_k}{\partial t}\right)$$ ✓

sehr gut

d) $\vec{B} = \vec{\nabla}\times\vec{A}$ $\quad \vec{E} = -\vec{\nabla}\varphi - \frac{\partial\vec{A}}{\partial t} = -\left(\frac{\partial\varphi}{\partial q_k} + \frac{\partial A_k}{\partial t}\right)$

$$I: = \dot{x}\frac{\partial}{\partial q_k}A_x + \dot{y}\frac{\partial}{\partial q_k}A_y + \dot{z}\frac{\partial}{\partial q_k}A_z - \frac{\partial A_k}{\partial x}\underbrace{\frac{dx}{dt}}_{\dot{x}} - \frac{\partial A_k}{\partial y}\dot{y} - \frac{\partial A_k}{\partial z}\dot{z}$$

$k=x$: $$\cancel{\dot{x}\frac{\partial}{\partial x}A_x - \frac{\partial A_x}{\partial x}\dot{x}} + \dot{y}\frac{\partial}{\partial x}A_y - \frac{\partial A_x}{\partial y}\dot{y} + \dot{z}\frac{\partial}{\partial x}A_z - \frac{\partial A_x}{\partial z}\dot{z}$$

$$= \dot{y}\cdot\left(\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right) + \dot{z}\left(\frac{\partial}{\partial x}A_z - \frac{\partial}{\partial z}A_x\right)$$

das entspricht gerade der x-Komponente von $\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} \times (\vec{\nabla} \times \vec{A}) = \dot{\vec{q}} \times \vec{B}$

$= \vec{v} \times \vec{B}$

$k=y$ und $k=z$ erfolgen analog (Symmetrie und so...)

$\Rightarrow \vec{Q}_k = Q(\vec{E} + \dot{\vec{q}} \times \vec{B}) = \vec{F}$ ☒

✓

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