In[∘]:= `Clear["Global`*"]`

- **Define the matrices PHR, PWF and the mixed matrix**

In[77]:=

$$\mathrm{PHR} = \Big\{\Big\{(-1+x)^2\,(1+2\,x)+\frac{(-1+x)^2\,(-24-350\,x+239\,x^2)}{4\,n},$$

$$-2\,(-1+x)\,x^2+\frac{16-60\,x-18\,x^2+121\,x^3-59\,x^4}{4\,n},\ \frac{x^4}{4}+\frac{x^2-x^4}{8\,n},$$

$$-\frac{1}{4}\,(-1+x)\,x^2+\frac{4-15\,x-x^2+23\,x^3-11\,x^4}{8\,n},\ -\frac{1}{8}\,(-2+x)\,x^3+\frac{7\,(-1+x)\,x^3}{8\,n},$$

$$-\frac{1}{4}\,(-1+x)\,x^2+\frac{4-15\,x-x^2+23\,x^3-11\,x^4}{8\,n},\ -\frac{1}{8}\,(-2+x)\,x^3+\frac{7\,(-1+x)\,x^3}{8\,n},\ \frac{x^4}{32}+\frac{x^2\,(2-3\,x+x^2)}{16\,n},$$

$$-\frac{(-1+x)\,x^3}{16\,n}+\frac{x^4}{32},\ -\frac{1}{16}\,x^2\,(-8+x^2)+\frac{8+754\,x-1794\,x^2+1383\,x^3-351\,x^4}{8\,n}\Big\},$$

$$\Big\{\frac{1}{2}-\frac{3}{n}-\frac{13\,x}{2\,n}-\frac{3\,x^2}{2}+\frac{22\,x^2}{n}+x^3-\frac{25\,x^3}{2\,n},\ \frac{1}{2}+\frac{1}{2\,n}-\frac{25\,x}{2\,n}+\frac{77\,x^2}{4\,n}-\frac{x^3}{2}-\frac{29\,x^3}{4\,n},$$

$$\frac{x^2}{2}-\frac{3\,x^2}{4\,n}-\frac{x^3}{4}+\frac{3\,x^3}{4\,n},\ \frac{1}{2\,n}-\frac{x}{n}+\frac{x^2}{8}+\frac{3\,x^2}{16\,n}-\frac{x^3}{8}+\frac{5\,x^3}{16\,n},\ \frac{1}{2\,n}-\frac{x}{n}+\frac{x^2}{8}+\frac{x^2}{2\,n},$$

$$\frac{1}{2\,n}-\frac{x}{n}+\frac{x^2}{8}+\frac{3\,x^2}{16\,n}-\frac{x^3}{8}+\frac{5\,x^3}{16\,n},\ \frac{1}{2\,n}-\frac{x}{n}+\frac{x^2}{8}+\frac{5\,x^2}{8\,n}-\frac{x^3}{8\,n},\ \frac{x^2}{16\,n}+\frac{x^3}{32}-\frac{x^3}{16\,n},$$

$$\frac{3\,x^2}{32\,n}+\frac{x^3}{32}-\frac{3\,x^3}{32\,n},\ \frac{1}{2\,n}+\frac{23\,x}{n}+\frac{x^2}{2}-\frac{1349\,x^2}{32\,n}-\frac{x^3}{16}+\frac{597\,x^3}{32\,n}\Big\},$$

$$\Big\{\frac{1}{4}-\frac{3}{2\,n}+\frac{x}{2\,n}-\frac{x^2}{4}+\frac{x^2}{n},\ \frac{1}{2}-\frac{1}{2\,n}-\frac{x}{2\,n}-\frac{x^2}{2}+\frac{x^2}{n},\ \frac{1}{4}+\frac{1}{4\,n}-\frac{x}{4\,n},\ \frac{1}{8\,n}-\frac{x}{8\,n},\ \frac{1}{4\,n}-\frac{x}{4\,n}+\frac{x^2}{8},$$

$$\frac{1}{8\,n}-\frac{x}{8\,n},\ \frac{1}{4\,n}-\frac{x}{4\,n}+\frac{x^2}{8},\ \frac{x^2}{32},\ \frac{1}{16\,n}-\frac{x}{16\,n}+\frac{x^2}{32},\ \frac{15}{16\,n}+\frac{17\,x}{16\,n}+\frac{7\,x^2}{16}-\frac{2\,x^2}{n}\Big\},$$

$$\Big\{1-\frac{6}{n}-\frac{13\,x}{n}-\frac{5\,x^2}{2}+\frac{83\,x^2}{2\,n}+\frac{3\,x^3}{2}-\frac{45\,x^3}{2\,n},\ \frac{4}{n}-\frac{8\,x}{n}+2\,x^2-\frac{x^2}{n}-2\,x^3+\frac{5\,x^3}{n},$$

$$\frac{x^2}{n}+\frac{x^3}{2}-\frac{x^3}{n},\ \frac{1}{2\,n}-\frac{x}{n}+\frac{x^2}{4}-\frac{x^3}{4}+\frac{x^3}{2\,n},\ 0,\ \frac{1}{2\,n}-\frac{x}{2\,n}-\frac{x^2}{4\,n}+\frac{x^3}{4\,n},$$

$$\frac{x^2}{4\,n}+\frac{x^3}{4}-\frac{x^3}{4\,n},\ \frac{x^2}{4\,n}-\frac{x^3}{4\,n},\ 0,\ \frac{1}{n}+\frac{45\,x}{2\,n}+\frac{x^2}{4}-\frac{167\,x^2}{4\,n}+\frac{73\,x^3}{4\,n}\Big\},$$

$$\left\{0,\ 1-\frac{3}{n}-x^2+\frac{3x^2}{n},\ \frac{1}{n}-\frac{x}{n}+\frac{x^2}{2},\ 0,\ \frac{1}{2n}-\frac{x^2}{2n},\ 0,\ \frac{1}{n}-\frac{x}{n}+\frac{x^2}{4},\ 0,\ 0,\ \frac{1}{2n}+\frac{2x}{n}+\frac{x^2}{4}-\frac{5x^2}{2n}\right\},$$

$$\left\{1-\frac{6}{n}-\frac{13x}{n}-\frac{5x^2}{2}+\frac{83x^2}{2n}+\frac{3x^3}{2}-\frac{45x^3}{2n},\ \frac{4}{n}-\frac{8x}{n}+2x^2-\frac{x^2}{n}-2x^3+\frac{5x^3}{n},\right.$$

$$\frac{x^2}{n}+\frac{x^3}{2}-\frac{x^3}{n},\ \frac{1}{2n}-\frac{x}{n}+\frac{x^2}{4}-\frac{x^3}{4}+\frac{x^3}{2n},\ \frac{x^2}{4n}+\frac{x^3}{4}-\frac{x^3}{4n},$$

$$\left.\frac{1}{2n}-\frac{x}{2n}-\frac{x^2}{4n}+\frac{x^3}{4n},\ 0,\ \frac{x^2}{4n}-\frac{x^3}{4n},\ 0,\ \frac{1}{n}+\frac{45x}{2n}+\frac{x^2}{4}-\frac{167x^2}{4n}+\frac{73x^3}{4n}\right\},$$

$$\left\{0,\ 1-\frac{3}{n}-x^2+\frac{3x^2}{n},\ \frac{1}{n}-\frac{x}{n}+\frac{x^2}{2},\ 0,\ \frac{1}{n}-\frac{x}{n}+\frac{x^2}{4},\ 0,\ \frac{1}{2n}-\frac{x^2}{2n},\ 0,\ 0,\ \frac{1}{2n}+\frac{2x}{n}+\frac{x^2}{4}-\frac{5x^2}{2n}\right\},$$

$$\left\{1-x^2+\frac{-6+2x+4x^2}{n},\ \frac{4-4x}{n},\ x^2,\ \frac{1-x^2}{2n},\ 0,\ \frac{1-x^2}{2n},\ 0,\ 0,\ 0,\ \frac{1+2x-3x^2}{n}\right\},$$

$$\left\{0,\ 0,\ 1+\frac{-1+x}{n},\ 0,\ 0,\ 0,\ 0,\ 0,\ \frac{1-x}{4n},\ \frac{3-3x}{4n}\right\},\ \{0,0,0,0,0,0,0,0,0,1\}\Big\};$$

In[78]:=

$$\text{PWF}=\left\{\left\{1-\frac{6}{n},\ \frac{4}{n},\ 0,\ \frac{1}{2n},\ 0,\ \frac{1}{2n},\ 0,\ 0,\ 0,\ \frac{1}{n}\right\},\right.$$

$$\left\{\frac{1}{2}-\frac{3}{n},\ \frac{1}{2}+\frac{1}{2n},\ 0,\ \frac{1}{2n},\ \frac{1}{2n},\ \frac{1}{2n},\ \frac{1}{2n},\ 0,\ 0,\ \frac{1}{2n}\right\},\ \left\{\frac{1}{4}-\frac{3}{2n},\ \frac{1}{2}-\frac{1}{2n},\ \frac{1}{4}+\frac{1}{4n},\right.$$

$$\left.\frac{1}{8n},\ \frac{1}{4n},\ \frac{1}{8n},\ \frac{1}{4n},\ 0,\ \frac{1}{16n},\ \frac{15}{16n}\right\},\ \left\{1-\frac{6}{n},\ \frac{4}{n},\ 0,\ \frac{1}{2n},\ 0,\ \frac{1}{2n},\ 0,\ 0,\ 0,\ \frac{1}{n}\right\},$$

$$\left\{0,\ 1-\frac{3}{n},\ \frac{1}{n},\ 0,\ \frac{1}{n},\ 0,\ \frac{1}{2n},\ 0,\ 0,\ \frac{1}{2n}\right\},\ \left\{1-\frac{6}{n},\ \frac{4}{n},\ 0,\ \frac{1}{2n},\ 0,\ \frac{1}{2n},\ 0,\ 0,\ 0,\ \frac{1}{n}\right\},$$

$$\left\{0,\ 1-\frac{3}{n},\ \frac{1}{n},\ 0,\ \frac{1}{2n},\ 0,\ \frac{1}{n},\ 0,\ 0,\ \frac{1}{2n}\right\},\ \left\{1-\frac{6}{n},\ \frac{4}{n},\ 0,\ \frac{1}{2n},\ 0,\ \frac{1}{2n},\ 0,\ 0,\ 0,\ \frac{1}{n}\right\},$$

$$\left\{0,\ 0,\ 1-\frac{1}{n},\ 0,\ 0,\ 0,\ 0,\ 0,\ \frac{1}{4n},\ \frac{3}{4n}\right\},\ \{0,0,0,0,0,0,0,0,0,1\}\Big\};$$

In[79]:=

$$\text{PMIX}=\left(1-\frac{l}{n^s}\right)*\text{PWF}+\frac{l}{n^s}*\text{PHR};$$

- **O(N) timescale**
- Now define the matrices needed for the application of Mohle’s Lemma:

```
In[ ]:= AWF = Limit[PWF, n → Infinity];
AHR = Limit[PHR, n → Infinity];
Amix = AWF;
Pmix = Limit[MatrixPower[Amix, k], k → Infinity];
BWF = Limit[n * (PWF - AWF), n → Infinity];
BHR = Limit[n * (PHR - AHR), n → Infinity];
Bmix = Limit[n * (PMIX - Amix), n → Infinity];
Bmix = l * (AHR - AWF) + BWF;
Gmix = Pmix.Bmix.Pmix;
```

- Apply Mohle’s Lemma in the O(N) timescale

```
In[ ]:= finalmix = Pmix.MatrixExp[t * Gmix];
{{1, 0, 0, 0, 0, 0, 0, 0, 0, 0}}.finalmix.{{1}, {1}, {1}, {1}, {1}, {1}, {1}, {1}, {1}, {0}}
```

Out[ ]=

$$\left\{\left\{e^{\frac{1}{16} t \left(-16-8 l x^2+l x^4\right)}\right\}\right\}$$

- **O(N^\theta) timescale**
- Define the matrices needed for the application of Mohle’s Lemma:

```
In[ ]:= AWF = Limit[PWF, n → Infinity];
AHR = Limit[PHR, n → Infinity];
Amix = AWF;
Pmix = Limit[MatrixPower[Amix, k], k → Infinity];
Bmix = l * (AHR - AWF);
BWF = Limit[n * (PWF - AWF), n → Infinity];
BHR = Limit[n * (PHR - AHR), n → Infinity];
```

- Apply Mohle’s Lemma in the O(N^\theta) timescale

```
In[ ]:= Gmix = Pmix.Bmix.Pmix;
finalmix = Pmix.MatrixExp[t * Gmix];
{{1, 0, 0, 0, 0, 0, 0, 0, 0, 0}}.finalmix.{{1}, {1}, {1}, {1}, {1}, {1}, {1}, {1}, {1}, {0}}
```

Out[ ]=

$$\left\{\left\{e^{\frac{1}{16} l t x^2 \left(-8+x^2\right)}\right\}\right\}$$