In[3]:= `Clear["Global`*"]`

(*Define the matrices PHR and PWF as in the type up, Pmid is the same as Lemma 6*)

In[4]:= $$PHR = \left\{\left\{-\frac{1-x^2}{n}+\frac{1}{2}\left(2-x^2\right), \frac{x^2}{4}+\frac{1-x^2}{2\,n}, \frac{x^2}{4}+\frac{1-x^2}{2\,n}\right\}, \left\{1-\frac{1-x}{n}, \frac{1-x}{2\,n}, \frac{1-x}{2\,n}\right\}, \{0, 0, 1\}\right\};$$

$$PWF = \left\{\left\{1-1/n, 1\big/\left(2\,n\right), 1\big/\left(2\,n\right)\right\}, \left\{1-1/n, 1\big/\left(2\,n\right), 1\big/\left(2\,n\right)\right\}, \{0, 0, 1\}\right\};$$

$$Pmid = \frac{l}{n^s} * \left(1-\frac{x^2}{4}\right) * PHR + \left(1-\frac{l}{n^s}\right) * PWF;$$

(*the row sum of Pmid is not 1*)

In[5]:= `Simplify[Total[Pmid, {2}]]`

Out[5]= $$\left\{1-\frac{1}{4}\,l\,n^{-s}\,x^2, 1-\frac{1}{4}\,l\,n^{-s}\,x^2, 1-\frac{1}{4}\,l\,n^{-s}\,x^2\right\}$$

(*in order to apply Mohle's lemma later we need a stochastic matrix*)

In[6]:= $$Pmidfinal = \frac{1}{1-\frac{l\,x^2}{4*n^s}} * Pmid;$$

In[7]:= `Simplify[Total[Pmidfinal, {2}]]`

Out[7]= {1, 1, 1}

(*the matrix A is defined as:
A=Limit[Pmidfinal,n→Infinity]*)
(* for simplicity evaluate A for different values of s and infer its true value*)

In[8]:= `For[s = 0.1, s ≤ 1, s = s + 0.1, Print[Limit[Pmidfinal, n → Infinity]]]`

```
{{1., 0., 0.}, {1., 0., 0.}, {0, 0, 1.}}
{{1., 0., 0.}, {1., 0., 0.}, {0, 0, 1.}}
{{1., 0., 0.}, {1., 0., 0.}, {0, 0, 1.}}
{{1., 0., 0.}, {1., 0., 0.}, {0, 0, 1.}}
{{1., 0., 0.}, {1., 0., 0.}, {0, 0, 1.}}
{{1., 0., 0.}, {1., 0., 0.}, {0, 0, 1.}}
{{1., 0., 0.}, {1., 0., 0.}, {0, 0, 1.}}
{{1., 0., 0.}, {1., 0., 0.}, {0, 0, 1.}}
{{1., 0., 0.}, {1., 0., 0.}, {0, 0, 1.}}
{{1., 0., 0.}, {1., 0., 0.}, {0, 0, 1.}}
```

In[9]:= `Clear[n, s, t, l, x]`

In[10]:= `A = {{1, 0, 0}, {1, 0, 0}, {0, 0, 1}};`

```
(* the matrix B is defined as Bs=Limit[n^s*(Pmidfinal-A),n→Infinity]*)

(*Similarly to the matrix A we define B for theta in (0,1) and theta=1*)

For[s = 0.1, s ≤ 1, s = s + 0.1, Print[Limit[n^s * (Pmidfinal - A), n → Infinity]]];
```

$\{\{0.125\, l\, x^2(-4.+x^2), -0.0625\, l\, x^2(-4.+x^2), -0.0625\, l\, x^2(-4.+x^2)\}, \{0., 0., 0.\}, \{0, 0, 0.\}\}$

$\{\{0.125\, l\, x^2(-4.+x^2), -0.0625\, l\, x^2(-4.+x^2), -0.0625\, l\, x^2(-4.+x^2)\}, \{0., 0., 0.\}, \{0, 0, 0.\}\}$

$\{\{0.125\, l\, x^2(-4.+x^2), -0.0625\, l\, x^2(-4.+x^2), -0.0625\, l\, x^2(-4.+x^2)\}, \{0., 0., 0.\}, \{0, 0, 0.\}\}$

$\{\{0.125\, l\, x^2(-4.+x^2), -0.0625\, l\, x^2(-4.+x^2), -0.0625\, l\, x^2(-4.+x^2)\}, \{0., 0., 0.\}, \{0, 0, 0.\}\}$

$\{\{0.125\, l\, x^2(-4.+x^2), -0.0625\, l\, x^2(-4.+x^2), -0.0625\, l\, x^2(-4.+x^2)\}, \{0., 0., 0.\}, \{0, 0, 0.\}\}$

$\{\{0.125\, l\, x^2(-4.+x^2), -0.0625\, l\, x^2(-4.+x^2), -0.0625\, l\, x^2(-4.+x^2)\}, \{0., 0., 0.\}, \{0, 0, 0.\}\}$

$\{\{0.125\, l\, x^2(-4.+x^2), -0.0625\, l\, x^2(-4.+x^2), -0.0625\, l\, x^2(-4.+x^2)\}, \{0., 0., 0.\}, \{0, 0, 0.\}\}$

$\{\{0.125\, l\, x^2(-4.+x^2), -0.0625\, l\, x^2(-4.+x^2), -0.0625\, l\, x^2(-4.+x^2)\}, \{0., 0., 0.\}, \{0, 0, 0.\}\}$

$\{\{0.125\, l\, x^2(-4.+x^2), -0.0625\, l\, x^2(-4.+x^2), -0.0625\, l\, x^2(-4.+x^2)\}, \{0., 0., 0.\}, \{0, 0, 0.\}\}$

$\{\{-1.-0.5\, l\, x^2+0.125\, l\, x^4, 0.5+0.25\, l\, x^2-0.0625\, l\, x^4, 0.5+0.25\, l\, x^2-0.0625\, l\, x^4\},$
$\{-1., 0.5, 0.5\}, \{0, 0, 0.\}\}$

In[12]:= `Clear[n, s, t, l, x]`

In[13]:= $B1 = \left\{\left\{-1+\frac{1}{8}\, l\, x^2(-4+x^2), \frac{1}{16}\left(8-l\, x^2(-4+x^2)\right), \frac{1}{16}\left(8-l\, x^2(-4+x^2)\right)\right\}, \left\{-1, \frac{1}{2}, \frac{1}{2}\right\}, \{0, 0, 0\}\right\};$

In[14]:= `TableForm[B1]`

Out[14]//TableForm=

$-1+\frac{1}{8}\, l\, x^2(-4+x^2)$	$\frac{1}{16}\left(8-l\, x^2(-4+x^2)\right)$	$\frac{1}{16}\left(8-l\, x^2(-4+x^2)\right)$
-1	$\frac{1}{2}$	$\frac{1}{2}$
0	0	0

In[15]:= $Bs = \left\{\left\{\frac{1}{8}\, l\, x^2(-4+x^2), \frac{-1}{16}\, l\, x^2(-4+x^2), \frac{-1}{16}\, l\, x^2(-4+x^2)\right\}, \{0, 0, 0\}, \{0, 0, 0\}\right\};$

In[16]:= `TableForm[Bs]`

Out[16]//TableForm=

$\frac{1}{8}\, l\, x^2(-4+x^2)$	$-\frac{1}{16}\, l\, x^2(-4+x^2)$	$-\frac{1}{16}\, l\, x^2(-4+x^2)$
0	0	0
0	0	0

```
(*define the matrices G for different values of theta*)
```

In[17]:=
```
G1 = A.B1.A;
Gs = A.Bs.A;
```

In[19]:=
```
TableForm[G1]
```
Out[19]//TableForm=

$$\begin{array}{lll} -1+\frac{1}{8}\,l\,x^2\left(-4+x^2\right)+\frac{1}{16}\left(8-l\,x^2\left(-4+x^2\right)\right) & 0 & \frac{1}{16}\left(8-l\,x^2\left(-4+x^2\right)\right) \\ -1+\frac{1}{8}\,l\,x^2\left(-4+x^2\right)+\frac{1}{16}\left(8-l\,x^2\left(-4+x^2\right)\right) & 0 & \frac{1}{16}\left(8-l\,x^2\left(-4+x^2\right)\right) \\ 0 & 0 & 0 \end{array}$$

In[20]:=
```
TableForm[Gs]
```
Out[20]//TableForm=

$$\begin{array}{lll} \frac{1}{16}\,l\,x^2\left(-4+x^2\right) & 0 & -\frac{1}{16}\,l\,x^2\left(-4+x^2\right) \\ \frac{1}{16}\,l\,x^2\left(-4+x^2\right) & 0 & -\frac{1}{16}\,l\,x^2\left(-4+x^2\right) \\ 0 & 0 & 0 \end{array}$$

```
(*and the matrix exponentials*)
```

In[22]:=
```
expo1 = MatrixExp[G1 * t];
expos = MatrixExp[Gs * t];
```

```
(*so the final matrix we work with is the one
 in \eqref{E_mixed_term_lim} and they are defined as *)
```

In[24]:=
```
final1 = A.expo1;
finals = A.expos;
```

```
(*thus for theta=1 we have that *)
```

In[26]:=
```
{1, 0, 0}.final1.{{1}, {1}, {0}}
```
Out[26]=

$$\left\{e^{\frac{1}{16}\,t\left(-8-4\,l\,x^2+l\,x^4\right)}\right\}$$

```
(*and for theta in (0,1) we have that *)
```

In[27]:=
```
{1, 0, 0}.finals.{{1}, {1}, {0}}
```
Out[27]=

$$\left\{e^{\frac{1}{16}\,l\,t\,x^2\left(-4+x^2\right)}\right\}$$

```
(*note, the above results are multiplied by a factor of exp(-l*t*x^2/4) *)
```