Nonlinear Optimal Tracking With Incomplete State Information Using Finite-Horizon State Dependent Riccati Equation (SDRE)

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Nonlinear Optimal Tracking With Incomplete State Information Using Finite-Horizon State Dependent Riccati Equation (SDRE)

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Abstract— **In this paper, an online technique for finite-horizon nonlinear stochastic tracking problems is presented. The idea of the proposed technique is to integrate the Kalman filter algorithm and the State Dependent Riccati Equation (SDRE) technique. Unlike the ordinary methods which deal with the linearized system, this technique will estimate the unmeasured states of the nonlinear system directly, and this will make the proposed technique effective for wide range of operating points. Numerical example is given to illustrate the effectiveness of the proposed technique.**

I. INTRODUCTION

Kalman filter is an effective minimum variance linear state estimator that estimates the unmeasured system states corrupted with white process and measurement noise. The standard Kalman filter is limited only to linear systems. Most real-world systems are nonlinear, in which case standard Kalman filters are not applicable [1]. Therefore, it becomes necessary to use some other nonlinear filter techniques. The extended Kalman filter (EKF) is the most widely applied state estimation algorithm for nonlinear systems. The EKF is used to estimate the unmeasured states of nonlinear systems. The EKF relies on linearization of the nonlinear system using Taylor series expansion near the operating point [2]. In linearization, we assume that the range of operation is small. Consequently, the EKF will only be effective in the small neighborhood of the operating points, and the accuracy of this technique will decrease for large operating range of nonlinear systems.

There exist many nonlinear control design techniques, each has benefits and flaws. Selecting the suitable control technique for nonlinear system usually requires consideration of different factors, e.g. performance, optimality, and cost. One of the highly promising and rapidly developing techniques for nonlinear optimal controllers is the State Dependent Riccati Equation (SDRE) technique. The SDRE has become a very attractive tool for the systematic design of nonlinear controllers, and it is very effective algorithm for designing the nonlinear feedback control by allowing nonlinearities in the system states [3]. Moreover, The SDRE offers a great design flexibility by tuning the state dependent coefficient matrices [4].

Inspired by the great potential of the SDRE for infinite horizon optimal control of nonlinear systems [5], this paper offers a new technique for tracking of finite-horizon nonlinear stochastic systems. This is accomplished by integrating the Kalman filter with the finite-horizon SDRE technique. Using the change of variable [6] to convert the differential Riccati equation (DRE) to a linear differential Lyapunov equation [7], [8]. The tracking problem is solved in real time [9].

The structure of the paper is as follows: Section II presents a brief overview of standard Kalman filter. In Section III the finite-horizon tracking problem using SDRE is presented. Section IV presents the continuous time tracking for optimal nonlinear stochastic systems. A numerical example is given in Section V. Finally, conclusions of the paper are in Section VI.

II. STANDARD KALMAN FILTER

The Kalman filter was developed by Rudolf E. Kalman in 1960 [10]. The Kalman filter can be used to estimate the states of continuous-time or discrete-time linear systems. Here, a brief overview of continuous-time Kalman filter for linear systems, which is needed in late sections, is given.

Consider the linear, continuous-time, stochastic system with dynamic model:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}(t)\mathbf{x}(t)+\mathbf{B}(t)\mathbf{u}(t)+\mathbf{B}_w(t)\mathbf{w}(t), \qquad (1)\\ \mathbf{y}(t) &= \mathbf{C}(t)\mathbf{x}(t)+\mathbf{v}(t), \qquad (2)\end{aligned}$$

where, $\mathbf{w}(t)$ and $\mathbf{v}(t)$ are process, and measurement (white, Gaussian) random noises with zero mean (i.e., $\bar{\mathbf{w}}(t)=\bar{\mathbf{v}}(t)=0$) and covariances $\mathbf{Q}_w(t)$ and $\mathbf{R}_v(t)$, respectively, and assumed to be *uncorrelated*. The estimated state $\hat{\mathbf{x}}(t)$ is given by:

$$\dot{\hat{\mathbf{x}}}(t)=\mathbf{A}(t)\hat{\mathbf{x}}(t)+\mathbf{B}(t)\mathbf{u}(t)+\mathbf{K}_e(t)\left[\mathbf{y}(t)-\mathbf{C}(t)\hat{\mathbf{x}}(t)\right], \qquad (3)$$

which can be rewritten as

$$\dot{\hat{\mathbf{x}}}(t)=\left[\mathbf{A}(t)-\mathbf{K}_e(t)\mathbf{C}(t)\right]\hat{\mathbf{x}}(t)+\mathbf{B}(t)\mathbf{u}(t)+\mathbf{K}_e(t)\mathbf{y}(t), \qquad (4)$$

where, $\mathbf{K}_e(t)$ is the estimator gain, $\hat{\mathbf{x}}(t)$ is the *state estimate* with initial value

$$\mathscr{E}\{\mathbf{x}(t=0)\}=\bar{\mathbf{x}}(t_0)=\hat{\mathbf{x}}(t_0). \qquad (5)$$

where, $\mathscr{E}$ stands for *expected, average or mean value* and considered intuitively equal to the *estimate*.
Let us define the error $\mathbf{e}(t)$ between the true or actual state $\mathbf{x}(t)$ and the state estimate $\hat{\mathbf{x}}(t)$ as

$$\mathbf{e}(t)=\mathbf{x}(t)-\hat{\mathbf{x}}(t); \qquad \dot{\mathbf{e}}(t)=\dot{\mathbf{x}}(t)-\dot{\hat{\mathbf{x}}}(t). \qquad (6)$$

Substituting from (1) and (4) in (6)

$$\dot{\mathbf{e}}(t) = \mathbf{A}(t)\mathbf{e}(t)+\mathbf{K}_e(t)\mathbf{C}(t)\mathbf{e}(t)+\mathbf{B}_w(t)\mathbf{w}(t)-\mathbf{K}_e(t)\mathbf{v}(t), \qquad (7)$$

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which can be rewritten as

$$\dot{\mathbf{e}}(t) \quad = \quad [\mathbf{A}(t)-\mathbf{K}_e(t)\mathbf{C}(t)]\,\mathbf{e}(t)+\mathbf{B}_{wk}(t)\mathbf{z}_{wk}(t), \quad (8)$$

where

$$\mathbf{B}_{wk}(t)=[\mathbf{B}_w(t) \quad -\mathbf{K}_e(t)]; \qquad \mathbf{z}_{wk}=[\mathbf{w}(t) \quad \mathbf{v}(t)]'. \quad (9)$$

Using the results from [11] on the propagation of state vector

$$\dot{\mathbf{x}}(t)=\mathbf{A}(t)\mathbf{x}(t)+\mathbf{B}_w(t)\mathbf{w}(t), \quad (10)$$

and the corresponding state estimate error covariance $\mathbf{P}_e(t)$ which can be calculated from

$$\dot{\mathbf{P}}_e(t)=\mathbf{A}(t)\mathbf{P}_e(t)+\mathbf{P}_e(t)\mathbf{A}'(t)+\mathbf{B}_w(t)\mathbf{Q}_w(t)\mathbf{B}'_w(t). \quad (11)$$

Now, using the result (11) for the error dynamics (8)

$$\begin{aligned}\dot{\mathbf{P}}_e(t)&=[\mathbf{A}(t)-\mathbf{K}_e(t)\mathbf{C}(t)]\,\mathbf{P}_e(t)+\mathbf{P}_e(t)\,[\mathbf{A}(t)-\mathbf{K}_e(t)\mathbf{C}(t)]'\\ &\quad+[\mathbf{B}_w(t) \quad -\mathbf{K}_e(t)]\begin{bmatrix}\mathbf{Q}_w(t) & \\ & \mathbf{R}_v(t)\end{bmatrix}[\mathbf{B}_w(t) \quad -\mathbf{K}_e(t)]';\\ \dot{\mathbf{P}}_e(t)&=[\mathbf{A}(t)-\mathbf{K}_e(t)\mathbf{C}(t)]\,\mathbf{P}_e(t)+\mathbf{P}_e(t)\,[\mathbf{A}(t)-\mathbf{K}_e(t)\mathbf{C}(t)]'\\ &\quad+[\mathbf{B}_w(t)\mathbf{Q}_w(t)\mathbf{B}'_w(t)+\mathbf{K}_e(t)\mathbf{R}_v(t)\mathbf{K}'_e(t), \quad (12)\end{aligned}$$

where, $\mathbf{P}_e=\mathbf{P}_e(t)=\mathscr{E}\{[\mathbf{x}(t)-\hat{\mathbf{x}}(t)]\,[\mathbf{x}(t)-\hat{\mathbf{x}}(t)]'\}$ is to be solved in *forward* direction with initial condition

$$\mathbf{P}_{e0}=\mathbf{P}_e(t_0)=\mathscr{E}\{[\mathbf{x}(t_0)-\hat{\mathbf{x}}(t_0)]\,[\mathbf{x}(t_0)-\hat{\mathbf{x}}(t_0)]'\}. \quad (13)$$

We have the condition on $\mathbf{K}_e(t)$ for minimum error variance as

$$\mathbf{K}_e(t)=\mathbf{P}_e(t)\mathbf{C}'(t)\mathbf{R}_v^{-1}(t). \quad (14)$$

Using the optimal Kalman gain (14) in the covariance relation (12), we get

$$\begin{aligned}\dot{\mathbf{P}}_e(t)=\mathbf{A}(t)\mathbf{P}_e(t)+\mathbf{P}_e(t)\mathbf{A}'(t)-\mathbf{P}_e(t)\mathbf{C}'(t)\mathbf{R}_v^{-1}(t)\mathbf{C}(t)\mathbf{P}_e(t)\\ +\mathbf{B}_w(t)\mathbf{Q}_w(t)\mathbf{B}'_w(t), \quad (15)\end{aligned}$$

with initial condition $\mathbf{P}_e(t=0)=\mathbf{P}_{e0}$. This is called the matrix continuous, differential Riccati equation (CDRE) arising in optimal state estimation.

Fig. 1 shows a structure of the standard linear continuous-time Kalman filter.

III. NONLINEAR FINITE-HORIZON TRACKING USING SDRE

A. Problem Formulation

The nonlinear system considered in this paper is assumed to be in the form

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x})+\mathbf{g}(\mathbf{x})\mathbf{u}(t), \quad (16)\\ \mathbf{y}(t) &= \mathbf{h}(\mathbf{x}). \quad (17)\end{aligned}$$

That nonlinear system can be expressed in a state-dependent like linear form, as

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}(\mathbf{x})\mathbf{x}(t)+\mathbf{B}(\mathbf{x})\mathbf{u}(t), \quad (18)\\ \mathbf{y}(t) &= \mathbf{C}(\mathbf{x})\mathbf{x}(t), \quad (19)\end{aligned}$$

where $\mathbf{f}(\mathbf{x})=\mathbf{A}(\mathbf{x})\mathbf{x}(t)$, $\mathbf{B}(\mathbf{x})=\mathbf{g}(\mathbf{x})$, $\mathbf{h}(\mathbf{x})=\mathbf{C}(\mathbf{x})\mathbf{x}(t)$.

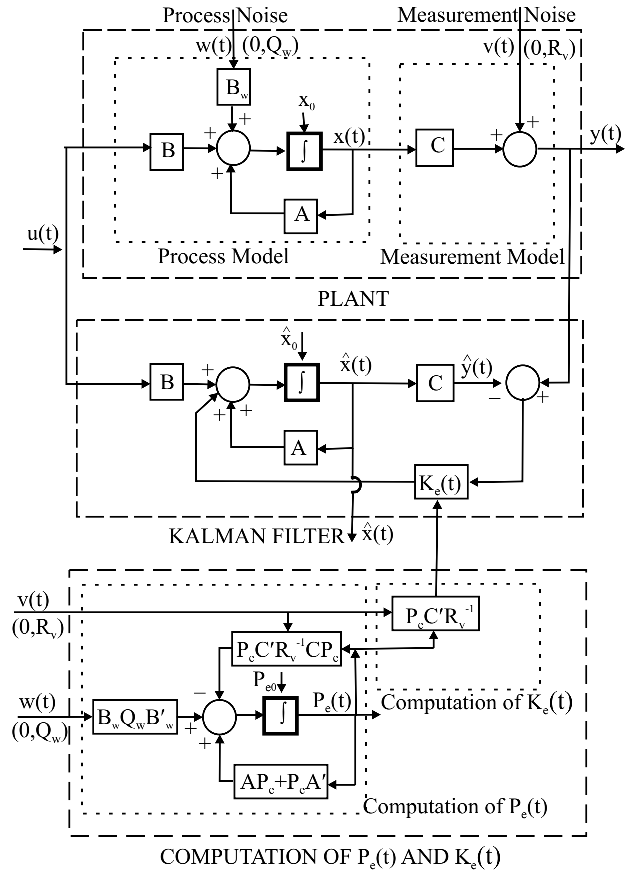

Fig. 1. Standard Linear Continuous-Time Kalman Filter

Let $\mathbf{z}(t)$ be the desired output. The goal is to find a state feedback control law that minimizes a cost function given by [12]:

$$\begin{aligned}\mathbf{J}(\mathbf{x},\mathbf{u})=\frac{1}{2}\mathbf{e}'(t_f)\mathbf{F}\mathbf{e}(t_f)\\ +\frac{1}{2}\int_{t_0}^{t_f}\left[\mathbf{e}'(t)\mathbf{Q}(\mathbf{x})\mathbf{e}(t)+\mathbf{u}'(\mathbf{x})\mathbf{R}(\mathbf{x})\mathbf{u}(\mathbf{x})\right]dt, \quad (20)\end{aligned}$$

where $\mathbf{e}(t)=\mathbf{z}(t)-\mathbf{y}(t)$, $\mathbf{Q}(\mathbf{x})$ and $\mathbf{F}$ are symmetric positive semi-definite matrices, and $\mathbf{R}(\mathbf{x})$ is a symmetric positive definite matrix. Moreover, $\mathbf{x}'(t)\mathbf{Q}(\mathbf{x})\mathbf{x}(t)$ is a measure of control accuracy and $\mathbf{u}'(\mathbf{x})\mathbf{R}(\mathbf{x})\mathbf{u}(\mathbf{x})$ is a measure of control effort [3].

B. Solution for Finite-Horizon SDRE Tracking

To minimize the above cost function (20), a feedback control law can be given as

$$\mathbf{u}(\mathbf{x})=-\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}'(\mathbf{x})[\mathbf{P}_c(\mathbf{x})\mathbf{x}(t)-\mathbf{g}(\mathbf{x})], \quad (21)$$

where $\mathbf{P}_c(\mathbf{x})$ is a symmetric, positive-definite solution of the state-dependent Differential Riccati Equation (SDDRE) of the form

$$\begin{aligned}-\dot{\mathbf{P}}_\mathbf{c}(\mathbf{x})=\mathbf{P}_\mathbf{c}(\mathbf{x})\mathbf{A}(\mathbf{x})+\mathbf{A}'(\mathbf{x})\mathbf{P}_\mathbf{c}(\mathbf{x})\\ -\mathbf{P}_\mathbf{c}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}\mathbf{B}'(\mathbf{x})\mathbf{P}_\mathbf{c}(\mathbf{x})+\mathbf{C}'(\mathbf{x})\mathbf{Q}(\mathbf{x})\mathbf{C}(\mathbf{x}), \quad (22)\end{aligned}$$

with the final condition

$$\mathbf{P}_c(\mathbf{x},t_f)=\mathbf{C}'(t_f)\mathbf{F}\mathbf{C}(t_f). \tag{23}$$

The resulting SDRE-controlled trajectory becomes the solution of the state-dependent closed-loop dynamics

$$\dot{\mathbf{x}}(t)=[\mathbf{A}(\mathbf{x})-\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}'(\mathbf{x})\mathbf{P}_c(\mathbf{x})]\mathbf{x}(t) +\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}'(\mathbf{x})\mathbf{g}(\mathbf{x}), \tag{24}$$

where $\mathbf{g}(\mathbf{x})$ is a solution of the state-dependent non-homogeneous vector differential equation

$$\dot{\mathbf{g}}(\mathbf{x})=-[\mathbf{A}(\mathbf{x})-\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}'(\mathbf{x})\mathbf{P}(\mathbf{x})]'\mathbf{g}(\mathbf{x}) -\mathbf{C}'(\mathbf{x})\mathbf{Q}(\mathbf{x})\mathbf{z}(\mathbf{x}), \tag{25}$$

with the final condition

$$\mathbf{g}(\mathbf{x},t_f)=\mathbf{C}'(t_f)\mathbf{F}\mathbf{z}(t_f). \tag{26}$$

As the SDRE function of $(\mathbf{x},t)$, we do not know the value of the states ahead of present time step. Consequently, the state dependent coefficients cannot be calculated to solve (22) with the final condition (23) by backward integration from t_f to t_0. To overcome this problem, an approximate analytical approach is used [6], [7], [8], [9], which converts the original nonlinear Ricatti equation to a differential Lyapunov equation. At each time step, the Lyapunov equation can be solved in closed form. In order to solve the DRE (22), and the non-homogeneous differential equation (25) one can follow the following steps:

- Solve Algebraic Riccati Equation (ARE) to calculate the steady state value $\mathbf{P_{ss}}(\mathbf{x})$

$$\mathbf{P_{ss}}(\mathbf{x})\mathbf{A}(\mathbf{x})+\mathbf{A}'(\mathbf{x})\mathbf{P_{ss}}(\mathbf{x})- \mathbf{P_{ss}}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}'(\mathbf{x})\mathbf{P_{ss}}(\mathbf{x})+\mathbf{Q}(\mathbf{x})=0. \tag{27}$$

- Use change of variables technique and assume that $\mathbf{K}(\mathbf{x},t)=[\mathbf{P_c}(\mathbf{x},t)-\mathbf{P_{ss}}(\mathbf{x})]^{-1}$.
- Calculate the value of $\mathbf{A_{cl}}(\mathbf{x})$ as $\mathbf{A_{cl}}(\mathbf{x})=\mathbf{A}(\mathbf{x})-\mathbf{B}(\mathbf{x})\mathbf{R^{-1}}\mathbf{B}'(\mathbf{x})\mathbf{P_{ss}}(\mathbf{x})$.
- Solve the algebraic Lyapunov equation [13]

$$\mathbf{A_{cl}}\mathbf{D}+\mathbf{D}\mathbf{A}'_{\mathbf{cl}}-\mathbf{B}\mathbf{R}^{-1}\mathbf{B}'=\mathbf{0}. \tag{28}$$

- Solve the differential Lyapunov equation

$$\dot{\mathbf{K}}(\mathbf{x},\mathbf{t})=\mathbf{K}(\mathbf{x},\mathbf{t})\mathbf{A}'_{\mathbf{cl}}(\mathbf{x})+\mathbf{A_{cl}}(\mathbf{x})\mathbf{K}(\mathbf{x},\mathbf{t}) -\mathbf{B}(\mathbf{x})\mathbf{R^{-1}}(\mathbf{x})\mathbf{B}'(\mathbf{x}). \tag{29}$$

  The solution of (29) is given by [14]

$$\mathbf{K}(\mathbf{t})=\mathbf{e}^{\mathbf{A_{cl}}(\mathbf{t}-\mathbf{t_f})}(\mathbf{K}(\mathbf{x},\mathbf{t_f})-\mathbf{D})\mathbf{e}^{\mathbf{A_{cl}}'(\mathbf{t}-\mathbf{t_f})}+\mathbf{D}. \tag{30}$$

- Calculate the value of $\mathbf{P_c}(\mathbf{x},t)$ from the equation

$$\mathbf{P_c}(\mathbf{x},\mathbf{t})=\mathbf{K^{-1}}(\mathbf{x},\mathbf{t})+\mathbf{P_{ss}}(\mathbf{x}). \tag{31}$$

- Calculate the steady state value $\mathbf{g_{ss}}(\mathbf{x})$ from the equation

$$\mathbf{g_{ss}}(\mathbf{x})=[\mathbf{A}(\mathbf{x})-\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}'(\mathbf{x})\mathbf{P_{ss}}(\mathbf{x})]'^{-1} \mathbf{C}'(\mathbf{x})\mathbf{Q}(\mathbf{x})\mathbf{z}(\mathbf{x}). \tag{32}$$

- Use change of variables technique and assume that $\mathbf{K_g}(\mathbf{x},t)=[\mathbf{g}(\mathbf{x},t)-\mathbf{g_{ss}}(\mathbf{x})]$.

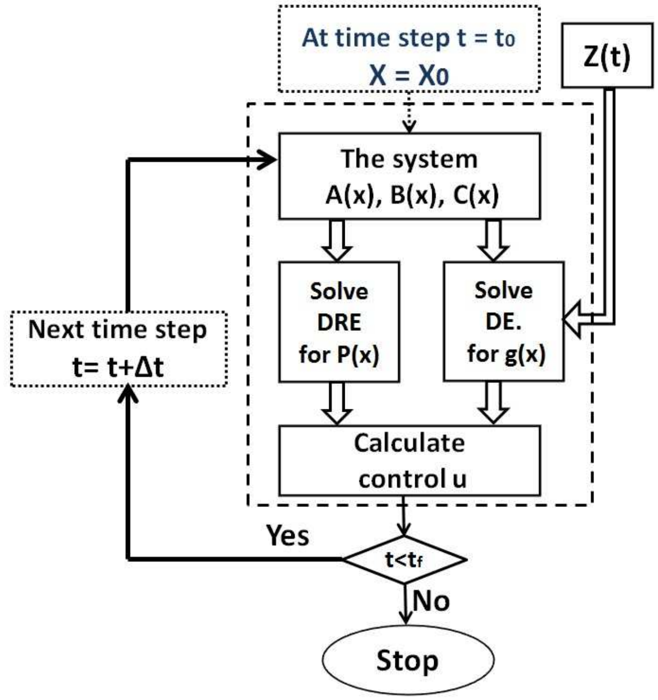

Fig. 2. Overview of The Process of Finite-Horizon SDDRE Tracking

- Solve the differential equation

$$\mathbf{K_g}(t)=\mathbf{e}^{-(\mathbf{A}-\mathbf{B}\mathbf{R^{-1}}\mathbf{B}'\mathbf{P})'(\mathbf{t}-\mathbf{t_f})}[\mathbf{g}(\mathbf{x},t_f)-\mathbf{g_{ss}}(\mathbf{x})]. \tag{33}$$

- Calculate the value of $\mathbf{g}(\mathbf{x},t)$ from the equation

$$\mathbf{g}(\mathbf{x},t)=\mathbf{K_g}(\mathbf{x},t)+\mathbf{g_{ss}}(\mathbf{x}). \tag{34}$$

- Calculate the value of the optimal control $\mathbf{u}(\mathbf{x},t)$ as

$$\mathbf{u}(\mathbf{x},t)=-\mathbf{R^{-1}}(\mathbf{x})\mathbf{B}'(\mathbf{x})[\mathbf{P}_c(\mathbf{x},t)\mathbf{x}(t)-\mathbf{g}(\mathbf{x},t)]. \tag{35}$$

Fig. 2 summarized the overview of the process of finite-horizon SDDRE tracking technique

Note : It is easily seen that this technique with finite-horizon SDDRE can be used for linear systems and the resulting SDDRE becomes the standard DRE [12].

IV. THE NONLINEAR CONTINUOUS TIME TRACKING WITH INCOMPLETE STATE INFORMATION

A. Optimal Estimation

let us reproduce the nonlinear system with noises in state dependent form

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{x})\mathbf{x}(t)+\mathbf{B}(\mathbf{x})\mathbf{u}(t)+\mathbf{B}_w(t)\mathbf{w}(t), \tag{36}$$

$$\mathbf{y}(t) = \mathbf{C}(\mathbf{x})\mathbf{x}(t)+\mathbf{v}(t). \tag{37}$$

In order to find the best estimate $\hat{\mathbf{x}}(t)$ and the corresponding covariance matrix $\mathbf{P}_e(\hat{\mathbf{x}},t)$, we use the results of Section II. At each time step, the estimate equations are

$$\dot{\hat{\mathbf{x}}}(t)=\mathbf{A}(\hat{\mathbf{x}})\hat{\mathbf{x}}(t)+\mathbf{B}(\hat{\mathbf{x}})\mathbf{u}(t)+\mathbf{K}_e(\hat{\mathbf{x}},t)[\mathbf{y}(t)-\mathbf{C}(\hat{\mathbf{x}})\hat{\mathbf{x}}(t)]; \quad \hat{\mathbf{x}}(t_0)=\bar{\mathbf{x}}(t_0), \tag{38}$$

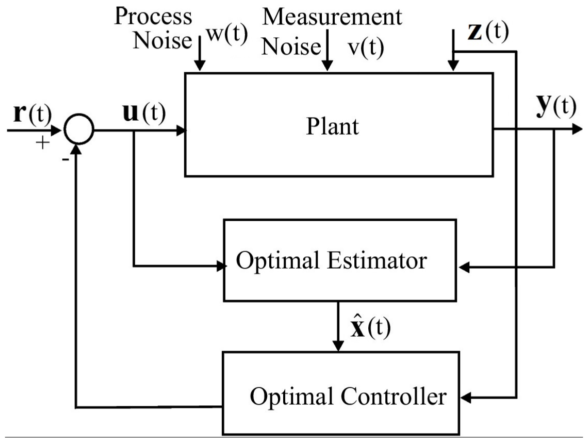

Fig. 3. Summary of Continuous-Time Nonlinear Tracking

where, $\mathbf{K}_e(\hat{\mathbf{x}},t)$, the optimal Kalman estimator gain, is obtained as

$$\mathbf{K}_e(\hat{\mathbf{x}},t) \quad = \quad \mathbf{P}_e(\hat{\mathbf{x}},t)\mathbf{C}'(\hat{\mathbf{x}})\mathbf{R}_v^{-1}(t), \qquad (39)$$

and $\mathbf{P}_e(\hat{\mathbf{x}},t)$ is the solution of the matrix differential Riccati equation

$$\begin{aligned}\dot{\mathbf{P}}_e(\hat{\mathbf{x}},t) = \mathbf{A}(\hat{\mathbf{x}})\mathbf{P}_e(\hat{\mathbf{x}},t) + \mathbf{P}_e(\hat{\mathbf{x}},t)\mathbf{A}'(\hat{\mathbf{x}}) + \mathbf{B}_w(t)\mathbf{Q}_w(t)\mathbf{B}_w'(t) \\ -\mathbf{P}_e(\hat{\mathbf{x}},t)\mathbf{C}'(\hat{\mathbf{x}})\mathbf{R}_v^{-1}(t)\mathbf{C}(\hat{\mathbf{x}})\mathbf{P}_e(\hat{\mathbf{x}},t), \quad (40)\end{aligned}$$

is to be solved in *forward* direction with initial condition $\mathbf{P}_e(\hat{\mathbf{x}},t_0) = \mathbf{P}_{e0}$.

The minimization of J is equivalent to minimization of

$$\begin{aligned}J_a(\mathbf{x},u) = \mathscr{E}\Bigg\{\frac{1}{2}\hat{\mathbf{e}}'(t_f)\mathbf{F}\hat{\mathbf{e}}(t_f) + \\ \frac{1}{2}\int_{t_0}^{t_f}[\hat{\mathbf{e}}'(t)\mathbf{Q}(\hat{\mathbf{x}})\hat{\mathbf{e}}(t) + \mathbf{u}'(\hat{\mathbf{x}},t)\mathbf{R}(\hat{\mathbf{x}})\mathbf{u}(\hat{\mathbf{x}},t)dt\Bigg\}, \quad (41)\end{aligned}$$

where $\hat{\mathbf{e}}(t) = \mathbf{z}(t) - \mathbf{C}(\hat{\mathbf{x}})\hat{\mathbf{x}}(t)$.

B. Optimal Control

At each time step, using the results of nonlinear tracking obtained in Section III except that the state is now the optimal estimate $\hat{\mathbf{x}}(t)$

$$\mathbf{u}(\hat{\mathbf{x}},t) = -\mathbf{R}^{-1}(\hat{\mathbf{x}})\mathbf{B}'(\hat{\mathbf{x}})[\mathbf{P}_c(\hat{\mathbf{x}},t)\hat{\mathbf{x}}(t) - \mathbf{g}(\hat{\mathbf{x}},t)]. \qquad (42)$$

where, $\mathbf{P}_c(\hat{\mathbf{x}},t)$ and $\mathbf{g}(\hat{\mathbf{x}},t)$ are the approximate solutions of the SDRE, (31) and (34) respectively.

The summary of the nonlinear tracking problem is shown in Fig. 3. Here, we see that the original plant is subjected to process noise $\mathbf{w}(t)$ and measurement noise $\mathbf{v}(t)$.

The entire algorithm of combined estimation and control leading to nonlinear tracking problem is shown in Table I where we introduced the command $\mathbf{r}(t)$ for a more general treatment [15].

V. ILLUSTRATIVE EXAMPLE

For numerical simulation and analysis, the developed estimation and optimal tracking technique is implemented for noise cancellation for inverted pendulum controlled by DC motor, as shown in Fig. 4

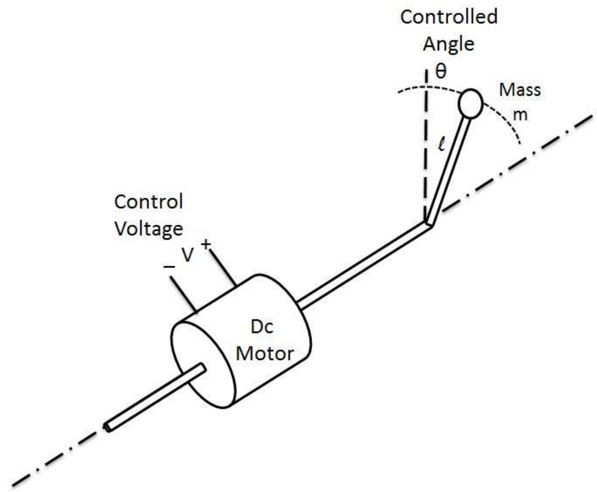

Fig. 4. Inverted Pendulum Controlled by DC Motor

The dynamic equations for system under concern are

$$V(t) \quad = \quad L\frac{di(t)}{dt} + Ri(t) + k_b\frac{d\theta(t)}{dt}, \qquad (43)$$

$$ml^2\frac{d^2\theta(t)}{dt^2} \quad = \quad -mglsin(\theta(t)) - k_m i(t), \qquad (44)$$

where, V is the control voltage, L is the motor inductance,i is the current through the motor winding, R the motor winding resistance,k_b the motor's back electro magnetic force constant, θ the angle of pendulum, m the mass of pendulum, l the length of rod, g the gravitational constant, and k_m the damping (friction) constant.

The system nonlinear state equations can be written in the state dependent form

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{(g/l)sin(x_1)}{x_1} & 0 & \frac{k_m}{ml^2} \\ 0 & -\frac{k_b}{L} & \frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} u, \quad (45)$$

where: $\theta = x_1$, $\dot{\theta} = x_2$, $i = x_3$, $V = u$.

Let the selected weighted matrices are

$$\mathbf{Q} = diag(100,0,0), \mathbf{R} = 0.07, \mathbf{F} = diag(1,1,1). \qquad (46)$$

The covariances of the noises have been taken as

$$\mathbf{Q_w} = diag(0.2,0.2,0.2), \mathbf{R_v} = 10. \qquad (47)$$

The simulations are performed for final time of 8 seconds and the resulting angle trajectories are shown in Fig. 5, where the dash-dot line denotes *the reference* angle trajectory, the dashed line denotes the *actual* (without noise) angle trajectory, and the solid line denotes *the estimated* (with noise) angle trajectory. The optimal control voltage is shown in Fig. 6, where the solid line denotes the estimated optimal control and the doted line denotes the actual optimal control signal.

Comparing these trajectories in Fig. 5, it's clear that the propose algorithm gives very good results as the estimated

TABLE I

PROCEDURE SUMMARY OF CONTINUOUS-TIME TACKING PROBLEM: INCOMPLETE STATE INFORMATION

Statement of the Problem	
Given the process as $\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{x})\mathbf{x}(t) + \mathbf{B}(\mathbf{x})\mathbf{u}(t) + \mathbf{B}_w(t)\mathbf{w}(t)$, the observation of the state as $\mathbf{y}(t) = \mathbf{C}(\mathbf{x})\mathbf{x}(t) + \mathbf{v}(t), \mathbf{e}(t) = \mathbf{z}(t) - \mathbf{y}(t)$, the performance measure for estimate as $J(t) = \text{trace}\,[var[\tilde{\mathbf{x}}(t)]] + \text{trace}[\mathbf{P}_e(t)]$, the conditions as $\mathscr{E}[\mathbf{w}(t)] = 0, \quad \text{COV}[\mathbf{w}(t), \mathbf{w}(\tau)] = \mathbf{Q}_w(t)\delta(t-\tau), \quad \mathscr{E}[\mathbf{x}(t_0)] = \bar{\mathbf{x}}_0$, $\text{VAR}[\mathbf{x}(t_0)] = \mathbf{P}_0; \quad \text{COV}[\mathbf{x}(t), \mathbf{w}(\tau)] = 0 \quad$ for all $\tau > t$, $E[\mathbf{v}(t)] = 0, \quad COV[\mathbf{v}(t), \mathbf{v}(\tau)] = \mathbf{R}_v(t)\delta(t-\tau)$, $COV[\mathbf{v}(t), \mathbf{w}(\tau)] = 0, \quad COV[\mathbf{v}(t), \mathbf{x}(\tau)] = 0$ for all t, τ, and the performance measure for control as $\hat{J}(\mathbf{x}_0, t_0) = \mathscr{E}\left\{\frac{1}{2}\mathbf{e}'(t_f)\mathbf{F}\mathbf{e}(t_f) + \frac{1}{2}\int_{t_0}^{t_f}[\mathbf{e}'(t)\mathbf{Q}(\mathbf{x})\mathbf{e}(t) + \mathbf{u}'(\mathbf{x})\mathbf{R}(\mathbf{x})\mathbf{u}(\mathbf{x})]\right\} dt$ find the optimal estimator and controller	
Solution of the Problem I: Optimal Estimator/Kalman Filter	
Step 1	At each time step, solve the matrix differential Riccati equation $\dot{\mathbf{P}}_e(\hat{\mathbf{x}}, t) = \mathbf{A}(\hat{\mathbf{x}})\mathbf{P}_e(\hat{\mathbf{x}}, t) + \mathbf{P}_e(\hat{\mathbf{x}}, t)\mathbf{A}'(\hat{\mathbf{x}}) + \mathbf{B}_w(t)\mathbf{Q}_w(t)\mathbf{B}'_w(t) -$ $\mathbf{P}_e(\hat{\mathbf{x}}, t)\mathbf{C}'(\hat{\mathbf{x}})\mathbf{R}_v^{-1}(t)\mathbf{C}(\hat{\mathbf{x}})\mathbf{P}_e(\hat{\mathbf{x}}, t); \qquad \mathbf{P}_e(\hat{\mathbf{x}}, t_0) = \mathbf{P}_{e0}$.
Step 2	Using $\mathbf{P}_e(\hat{\mathbf{x}}, t)$ from Step 1, obtain the optimal estimator (filter) gain as $\mathbf{K}_e(\hat{\mathbf{x}}, t) = \mathbf{P}_e(\hat{\mathbf{x}}, t)\mathbf{C}'(\hat{\mathbf{x}})\mathbf{R}_v^{-1}(t)$.
Step 3	Using $\mathbf{K}_e(\hat{\mathbf{x}}, t)$ from Step 2, solve the optimal state estimate $\hat{\mathbf{x}}(t)$ from $\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}(\hat{\mathbf{x}})\hat{\mathbf{x}}(t) + \mathbf{B}(\hat{\mathbf{x}})\mathbf{u}(t) + \mathbf{K}_e(\hat{\mathbf{x}}, t)[\mathbf{y}(t) - \mathbf{C}(\hat{\mathbf{x}})\hat{\mathbf{x}}(t)]; \quad \hat{\mathbf{x}}(t_0) = \bar{\mathbf{x}}_0$.
Solution of the Problem II: Optimal Controller	
Step 4	At each time step, calculate the value of $\mathbf{P}_c(\hat{\mathbf{x}}, t)$ from the equation $\mathbf{P_c}(\hat{\mathbf{x}}, \mathbf{t}) = \mathbf{K}^{-1}(\hat{\mathbf{x}}, \mathbf{t}) + \mathbf{P_{ss}}(\hat{\mathbf{x}})$, with $K(\hat{\mathbf{x}}, t)$ is the solution differential Lyapunov equation (29) .
Step 5	Calculate the value of $\mathbf{g}(\mathbf{x}, t)$ from the equation $\mathbf{g}(\hat{\mathbf{x}}, t) = \mathbf{K_g}(\hat{\mathbf{x}}, t) + \mathbf{g_{ss}}(\hat{\mathbf{x}})$ with $K_g(\hat{\mathbf{x}}, t)$ is the solution differential equation (33) .
Step 6	Obtain the closed-loop optimal control as $\mathbf{u}(\hat{\mathbf{x}}, t) = -\mathbf{R^{-1}}(\hat{\mathbf{x}})\mathbf{B}'(\hat{\mathbf{x}})[\mathbf{P}_c(\hat{\mathbf{x}}, t)\hat{\mathbf{x}}(t) - \mathbf{g}(\hat{\mathbf{x}}, t)]$.

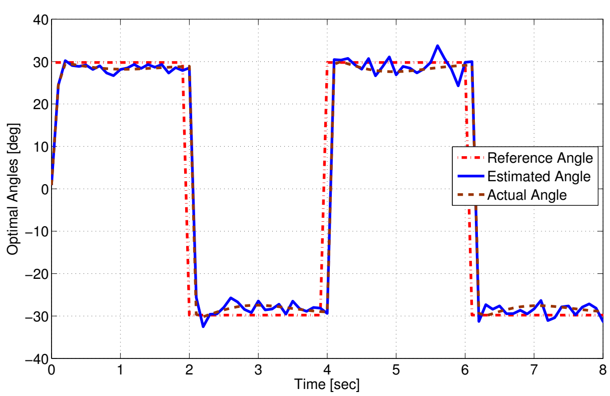

Fig. 5. Angle Trajectories for Inverted Pendulum Controlled by DC Motor

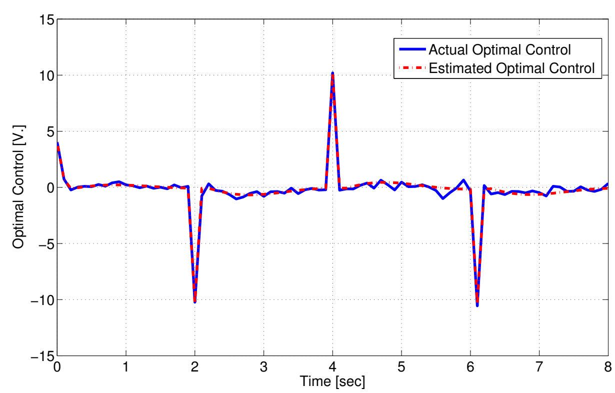

Fig. 6. Optimal Control for Inverted Pendulum Controlled by DC Motor

optimal output is making a very good tracking to the reference angle, and also the estimated trajectory is very close to the actual output trajectory.

From these results, it can be seen that the developed algorithm is able to solve the SDDRE finite-horizon nonlinear tracking stochastic problem.

Note : The regulator problem for a nonlinear stochastic system using finite-horizon SDRE was investigated [16].

VI. CONCLUSIONS

The paper proposed a new optimal tracking technique used for finite-horizon nonlinear stochastic systems. The idea of the proposed technique is to integrate the Kalman filter algorithm and the SDDRE technique. Kalman filter is used to estimate the unmeasured states which are corrupted with noises in the nonlinear model. This technique is effective for wide range of operating points. Numerical example is given to illustrate the effectiveness of the proposed technique.

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