Galerkin (1)

$$\sum_i A_{ji}\, a_i = b_j \qquad b_j = -(1, w_j)$$
$$A_{ji} = (\nabla^2 w_i, w_j)$$
$$(f,g) \equiv \int_0^1\int_0^\pi f(x,\varphi)\, g(x,\varphi)\, x\, dx\, d\varphi$$

$$w_{mn}(x,\varphi) = x^{2m+1}(1-x)^n \sin(2m+1)\varphi \qquad \begin{matrix} m \geq 0 \\ n \geq 1 \end{matrix}$$

ortogonalnost w

$$\int_0^1\int_0^\pi w_{mn}(x,\varphi)\, w_{m'n'}(x,\varphi)\, x\, dx\, d\varphi \qquad \forall\, m, m' \in \mathbb{Z}$$

$$= \frac{\Gamma(4+2m+2m')\,\Gamma(1+n+n')\left\{(1+m+m')\overbrace{\sin 2(m-m')\pi}^{=0} + (m'-m)\overbrace{\sin 2(m+m')\pi}^{=0}\right\}}{4\left[m(1+m) - m'(1+m')\right]\Gamma(5+2m+2m'+n+n')}$$

MATHEMATICA

$\neq 0$ samo za $m \neq m'$, ko pa števec $=0$ in imenovalec $=0$:

$x \equiv m - m'$

$$I_1 \equiv \frac{(1+m+m')\sin 2(m-m')\pi}{m(1+m) - m'(1+m')} = \frac{(1+m+m')\sin 2x\pi}{(m-m') + (m+m')(m-m')} = \frac{(1+m+m')\sin 2x\pi}{x + (m+m')x}$$

$$\lim_{x\to 0} I_1 \underset{\text{L'Hospital}}{=} \frac{\cancel{(1+m+m')}\, 2\pi \cos 2x\pi}{\cancel{1+m+m'}} = \underline{\underline{2\pi}}$$

$$I_2 \equiv \frac{(m'-m)\sin 2(m+m')\pi}{x + (m+m')x} = \frac{x \sin 2(m+m')\pi}{x(1+m+m')}$$

$$\lim_{x\to 0} I_2 = \frac{\sin 2(m+m')\pi}{1+m+m'} \equiv 0, \text{ ker števec} \equiv 0, \text{ imen.} \neq 0$$

torej

$$= 2\pi\, \frac{1}{4}\, \frac{\Gamma(4+4m)\,\Gamma(1+n+n')}{\Gamma(5+4m+n+n')}\, \delta_{mm'} = \underline{\underline{\frac{\pi}{2}\, \delta_{mm'}\, B(4+4m, 1+n+n')}}$$

$$b_j = -\int_0^1\int_0^\pi w_{m'n'}(x,\varphi)\, x\, dx\, d\varphi$$

$$\underset{\substack{\uparrow \\ \text{MATH.}}}{=} -\frac{2\cos^2(m'\pi)\,\Gamma(3+2m')\,\Gamma(1+n')}{(1+2m')\,\Gamma(4+2m'+n')} = \boxed{-\frac{2}{2m'+1}\,B(2m'+3,\,n'+1)}$$

$$(= b_{m'n'})$$

in $\dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = B(p,q)$

$$\nabla^2 f = \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial f}{\partial x}\right) + \frac{1}{x^2}\frac{\partial^2 f}{\partial \varphi^2}$$

$$A_{ji} = (\nabla^2 w_i, w_j)$$

$$\| \qquad\qquad\qquad\qquad$$

$$A_{(m'n')(mn)} = \int_0^1\int_0^\pi \nabla^2 w_{mn}(x,\varphi)\cdot w_{m'n'}(x,\varphi)\, x\, dx\, d\varphi$$

$$= \frac{n}{4}\Gamma(n+n'-1)\left\{\frac{(2+4m+n)\,\Gamma(4+2m+2m')}{\Gamma(3+2m+2m'+n+n')} - \frac{(3+4m)\,\Gamma(3+2m+2m')}{\Gamma(2+2m+2m'+n+n')}\right\}$$

$$\cdot\left\{\underbrace{\frac{\sin 2(m-m')\pi}{m-m'} - \frac{\sin 2(m+m')\pi}{1+m+m'}}_{=0 \text{ za } m\neq m'}\right\}$$

ko $m=m'$, spet po l'Hospitalu:

$$x = m-m';\quad \lim_{x\to 0}\frac{\sin 2x\pi}{x} = \left.\frac{2\pi\cos 2x\pi}{1}\right|_{x=0} = 2\pi$$

(drugi člen itak $\equiv 0$)

$$= \frac{n}{4}\Gamma(n+n'-1)\left\{\frac{(2+4m+n)\,\Gamma(4+4m)}{\Gamma(3+4m+n+n')} - \frac{(3+4m)\,\Gamma(3+4m)}{\Gamma(2+4m+n+n')}\right\} 2\pi\,\delta_{mm'}$$

$$\Gamma(4+4m) = (3+4m)\Gamma(3+4m)\,, \quad \Gamma(3+4m+n+n') = (2+4m+n+n')\cdot\Gamma(2+4m+n+n')$$

$$\left(\Gamma(z+1) = z\,\Gamma(z)\right)$$

$$= \frac{n}{4}\Gamma(n+n'-1)\left\{\frac{(2+4m+n)(3+4m)\Gamma(3+4m)}{(2+4m+n+n')\Gamma(2+4m+n+n')} - \frac{(3+4m)\Gamma(3+4m)}{\Gamma(2+4m+n+n')}\right\}2\pi\,\delta_{mm'}$$

$B(n+n'-1,\,3+4m)$

$$= \frac{n\pi}{2}\delta_{mm'}\,B(n+n'-1,3+4m)\left\{\frac{2+4m+n}{2+4m+n+n'} - 1\right\}(3+4m)$$

$$\frac{2+4m+n}{2+4m+n+n'} - 1 = -\frac{n'}{2+4m+n+n'}$$

$$= -\frac{\pi}{2}nn'\frac{3+4m}{2+4m+n+n'}B(n+n'-1,3+4m)\cdot\delta_{mm'}$$

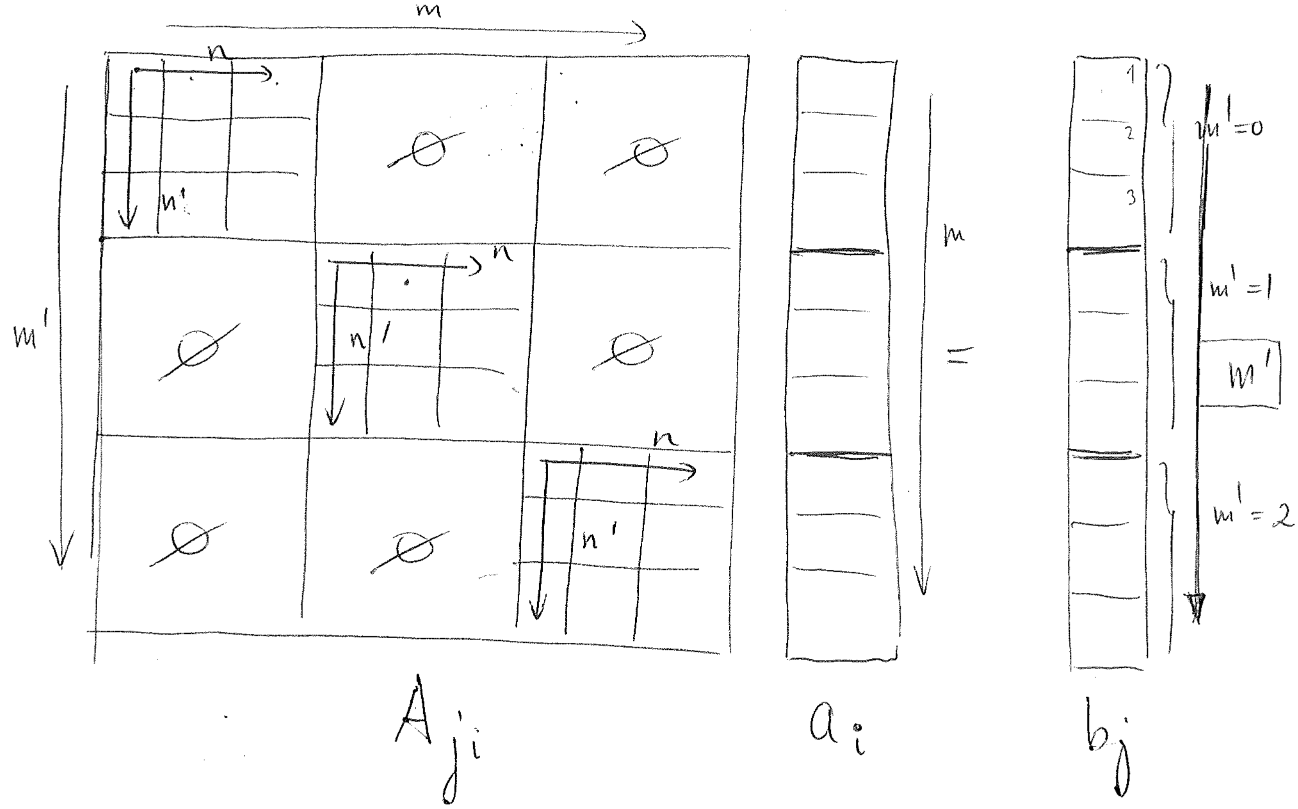
$$\boxed{A_{(m'n')(mn)} = -\delta_{mm'}\frac{\pi}{2}nn'\frac{3+4m}{2+4m+n+n'}B(n+n'-1,3+4m)}$$

$m, m' \geq 0$

$n, n' \geq 1$

$\in \mathbb{Z}$

structure of matrix A for $M = m_{max} = 2$, $N = n_{max} = 3$



$\underline{\underline{A}}\,\underline{a} = \underline{b}$

ko dobimo $a_i (= a_{mn})$,

$$u(x,\varphi) = \sum_{mn} a_{mn} W_{mn}(x,\varphi) \quad , \quad C = \frac{32}{\pi^2}\int_0^1\!\!\int_0^\pi u(x,\varphi)\, x\, dx\, d\varphi$$

z Mathematico dobim npr.

$M_{max} = 0$, $N_{max} = 1$ $(1\times 1\ !)$ $C = 0.720506$

$M_{max} = 1$, $N_{max} = 2$ (4×4) $C = 0.748738$

2 3 (9×9) $C = 0.754326$

3 3 $\underbrace{(12\times 12)}_{= \dim A}$ $C = 0.755854$

⋮

naprej gre počasi $\longrightarrow$ rajši C++

(naj gledajo tudi konvergenco v m, n)

Trivialno, a pomembno:

<u>Inverza matrike se "ne računa" (oz. ga ni treba):</u>

$$c = \ldots \sum_{ij} b_i A^{-1}_{ij} b_j = \ldots b^T(A^{-1}b)$$

$$c = b^T(A^{-1}b) = b^T x$$

$$x = A^{-1}b$$

$$Ax = AA^{-1}b = Ib = b$$

$\rightsquigarrow$ rešimo sistem $Ax=b$ po x

in damo x v