An Invariant of Spatial Graphs*

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INTRODUCTION

Some useful invariants for links have appeared in the last few years, e.g., the Jones polynomial, the 2-variable Jones polynomial, the Kauffman polynomial, etc. These invariants are not defined for spatial graphs. The Alexander ideals and the Alexander polynomials of spatial graphs [5–7] are determined by the fundamental groups of the complements of spatial graphs. Therefore they are neighborhood equivalence class invariants of spatial graphs. Thus the two spatial graphs shown in Figure 7 cannot be distinguished by them.

In this paper, we will introduce a 1-variable Laurent polynomial invariant for nondirected spatial graphs. It is a simple and useful invariant. We will define two types of spatial graphs: one is a spatial graph with flat vertices and the other is a spatial graph with pliable vertices. Our polynomial is an invariant for flat vertex graphs. Also, it is an invariant for pliable vertex graphs whose maximum degrees are less than 4.

The restriction of our invariant to 2-regular graphs is an invariant of links. We will show that it is a specialization of Kauffman's Dubrovnik polynomial; moreover, it is the Jones polynomial of the $(2,0)$-cabling of a knot.

1. SPATIAL GRAPHS, DIAGRAMS AND REIDEMEISTER MOVES

Throughout this paper we work in the piecewise-linear category. Let $G = (V, E)$ be a graph embedded in $\mathbb{R}^3$, we say G is a *spatial graph*. If for each vertex v of G, there exist a neighborhood B_v of v and a small flat plane P_v such that $G \cap B_v \subset P_v$, then we say that G is a *flat vertex graph*. For two spatial graphs G and G', if there exists an isotopy $h_t:\mathbb{R}^3 \to \mathbb{R}^3$ $t \in [0, 1]$ such that $h_0 = \mathrm{id}$

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and $h_1(G) = G'$, then we say that G and G' are *ambient isotopic as pliable vertex graphs (pliably isotopic)*. For two flat vertex graphs G, G', if there exists an isotopy $h_t: \mathbb{R}^3 \to \mathbb{R}^3$ $t \in [0, 1]$ such that $h_0 = \text{id}$, $h_1(G) = G'$, and $h_t(G)$ are flat vertex graphs for each $t \in [0, 1]$, then we say that G and G' are *ambient isotopic as flat vertex graphs (flatly isotopic)*.

Let $G \subset \mathbb{R}^3$ be a spatial graph. We say that a projection $p: \mathbb{R}^3 \to \mathbb{R}^2$ is a regular projection corresponding to G if each multi point of $p(G)$ is a double point of two transversal edges. Then we say the image $p(G)$ with information about the over crossings at all crossings of $p(G)$ is a *diagram* of G.

We shall define fundamental moves of diagrams, called Reidemeister moves, as in Figure 1.

It is easy to see that (0) is generated by (I), (I) is included in (VI), (II) is included in (IV), and that (V) is generated by (II), (III), and (VI). We say the deformation generated by (I) $\sim$ (VI) is *pliable deformation*, one generated by (I) $\sim$ (V) is a *flat deformation*, and we say one generated by (0) and (II)–(IV) is a *regular deformation*.

The next basic lemma is shown by the same argument as in Kauffman's recent paper [4].

Lemma 1. Let G_1 and G_2 be two spatial graphs. G_1 is pliably isotopic (respectively flatly isotopic) to G_2 if and only if a diagram of G_1 is deformable to a diagram of G_2 by pliable deformation (respectively, flat deformation).

2. AN INVARIANT OF SPATIAL GRAPHS

Let $G = (V, E)$ be a graph, where V is the vertex set and E is the edge set of G. Let $\mu(G)$ and $\beta(G)$ be the number of connected components of G and the first Betti number of G, respectively. Put $f(G) = x^{\mu(G)}y^{\beta(G)}$ and define a 2-variable Laurent polynomial by

$$h(G) = h(G)(x,y) = \sum_{F \subset E} (-x)^{-|F|} f(G - F),$$

where F ranges over the family of subsets of E, $|F|$ is the number of elements of F, $G - F = (V, E - F)$, and x and y are indeterminates. In particular, define $h(\phi) = 1$.

Of course, $h(G)$ is an invariant of graph G. And it is a specialization of Negami's polynomial [8] of G. It has the following properties:

Proposition 1. Let e be a nonloop edge of a graph G. Then $h(G) = h(G/e) - 1/x\, h(G - e)$, where G/e is the graph obtained from G by contracting e to a point and $G - e = G - \{e\}$.

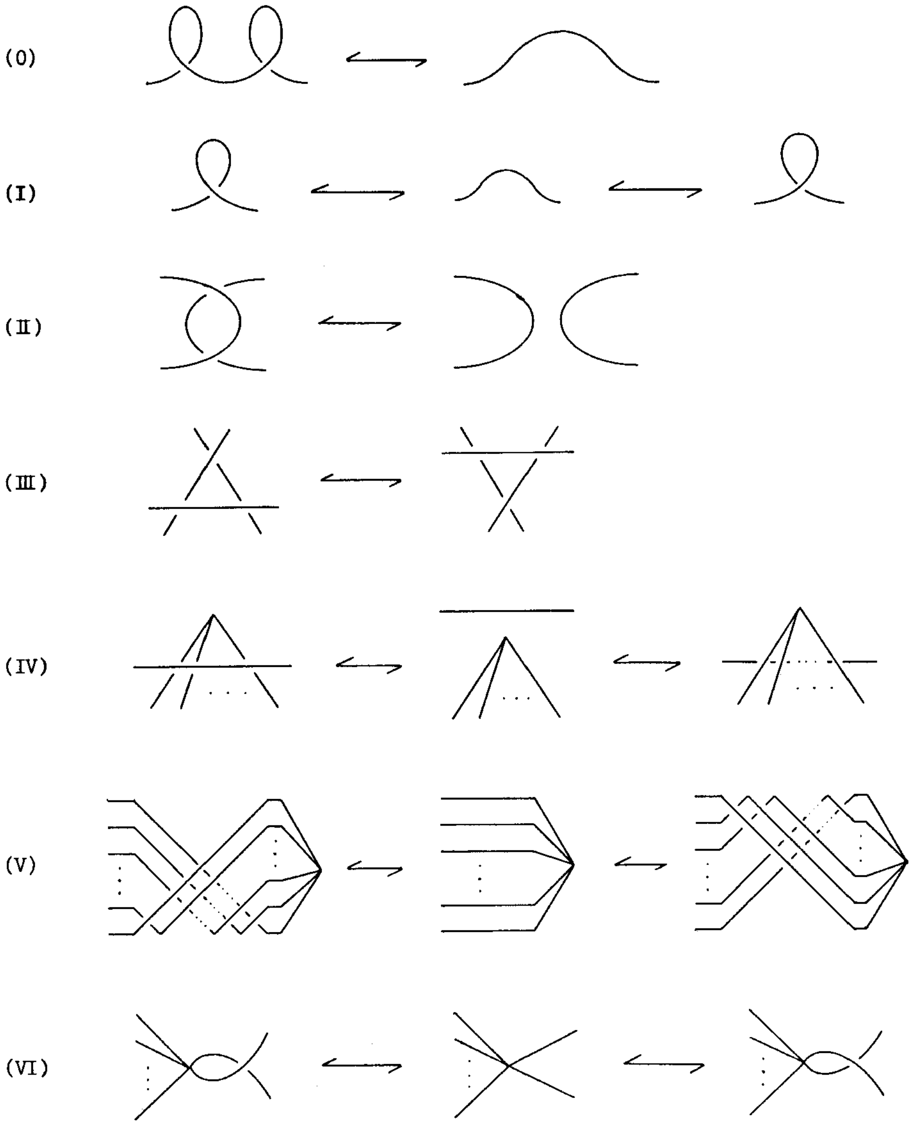


FIGURE 1

Proof.

$$h(G) = \sum_{e \notin F \subset E} (-x)^{-|F|} f(G - F) + \sum_{e \in F \subset E} (-x)^{-|F|} f(G - F)$$

$$= \sum_{F \subset E - e} (-x)^{-|F|} f(G/e - F) - 1/x$$

$$\cdot \sum_{F' \subset E - e} (-x)^{-|F'|} f(G - e - F)$$

$$= h(G/e) - 1/x\, h(G - e) . \quad \blacksquare$$

For two graphs G_1 and G_2, $G_1 \coprod G_2$ denotes the disjoint union of G_1 and G_2, and $G_1 \bigvee G_2$ denotes a wedge at a vertex of G_1 and G_2, i.e., $G_1 \bigvee G_2 = G_1 \cup G_2$ and $G_1 \cap G_2 = \{\text{a vertex}\}$. The symbol $\bigvee$ is quoted from [9]. We say that a graph G has a cut edge e if $G - e$ has more connected components than G. Then the following proposition holds:

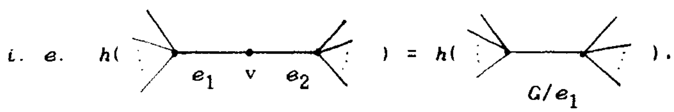
Proposition 2.

(1) $h(G_1 \coprod G_2) = h(G_1)h(G_2)$,
(2) $h(G_1 \bigvee G_2) = 1/x\, h(G_1)h(G_2)$, and
(3) If G has a cut edge then $h(G) = 0$.

Proof. (1) and (2) are trivial.

(3) Let e be the cut edge of G. Then $G - e = G_1 \coprod G_2$ and $G/e = G_1 \bigvee G_2$ for some graphs G_1 and G_2. By Proposition 1, $h(G) = h(G_1 \bigvee G_2) - 1/x\, h(G_1 \coprod G_2) = 0$. ∎

Theorem 1. Let v be a vertex of a graph $G = (V, E)$, which is a terminal point of just two nonloop edges e_1 and e_2. Then $h(G) = h(G/e_1)$, i.e.,



Proof. The graph $G - e_1$ has a cut edge e_2, so $h(G - e_1) = 0$. Therefore from the previous propositions,

$$h(G/e_1) = h(G) - 1/x\, h(G - e_1) = h(G). \quad ∎$$

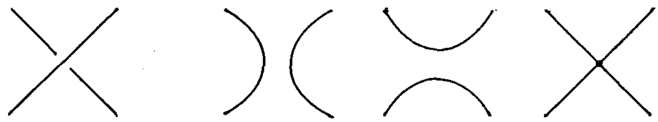
Corollary 1. $h(G)$ is a topological invariant of a graph G, i.e., if G is homeomorphic to a graph G', then $h(G) = h(G')$.

Now we will define an invariant of spatial graphs. Let g be a diagram of a graph. For a crossing c of g, we define s_+, s_-, and s_0, called the spin of $+1$, -1, and 0, as shown in Figure 2. Let S be the plane graph obtained from g by replacing each crossing with a spin. S is called a state on g and $\mathcal{S}(g)$ denotes the set of states on g. Put $\{g \mid S\} = A^{p-q}$, where p and q are the numbers of crossings with spin of $+1$ and spin of -1 in S, respectively, and A is an indeterminate. We define a 1-variable Laurent polynomial $R\{g\}(A)$ as follows:

$$R(g) = R(g)(A) = \sum_{S \in \mathcal{S}(g)} \{g \mid S\} H(S),$$

where $H(S) = h(S)(-1, -A - 2 - A^{-1})$. In particular, define that $R(\phi) = 1$.

The next proposition is derived from the definition of $R(g)$ and previous propositions.



c $\quad s_+ \quad s_- \quad s_0$

FIGURE 2

Proposition 3.

(1) $R(\quad) = A\,R(\quad) + A^{-1}\,R(\quad) + R(\quad)$,

(2) $R(\quad) = R(\quad) + R(\quad)$,

(3) $R(g_1 \perp\!\!\!\perp g_2) = R(g_1)R(g_2)$,

(4) $R(g_1 \vee g_2) = -R(g_1)R(g_2)$,

(5) If g has a cut edge then $R(g) = 0$.

Remark. The figures in an equation represent diagrams that differ only as indicated in the figures.

The next proposition is very useful for the proof of invariance of $R(g)$ and its calculation.

Proposition 4.

(1) $R(\quad) = \sigma$, where $\sigma = A + 1 + A^{-1}$,

(2) $R(\quad) = -\sigma\,R(\quad)$,

(3) $R(\quad) = -A\,R(\quad) - (A^2+A)\,R(\quad)$,

(4) $R(\quad) = -A^{-1}\,R(\quad) - (A^{-2}+A^{-1})\,R(\quad)$,

(5) $R(\quad) = -A\,R(\quad)$, $\quad R(\quad) = -A^{-1}\,R(\quad)$,

(6) $R(\quad) = A^2\,R(\quad)$, $\quad R(\quad) = A^{-2}\,R(\quad)$.

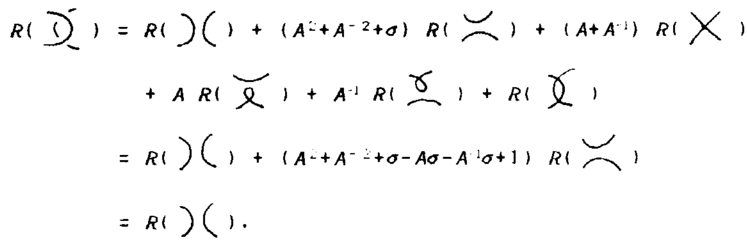
Proof. (1): $h(\quad)(x,y) = xy-1$, so $R(\quad) = H(\quad) = h(\quad)(-1, -A-2-A^{-1}) = A+1+A^{-1}$.

Others are easy calculation using Proposition 3. ∎

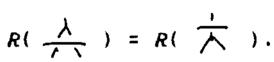
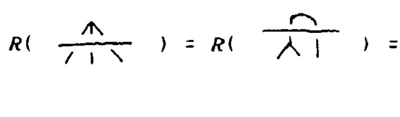
Theorem 2. $R(g)$ is a regular deformation invariant of a diagram g.

Proof. We shall show that $R(g)$ does not change under the Reidemeister moves (0), (II) $\sim$ (IV).

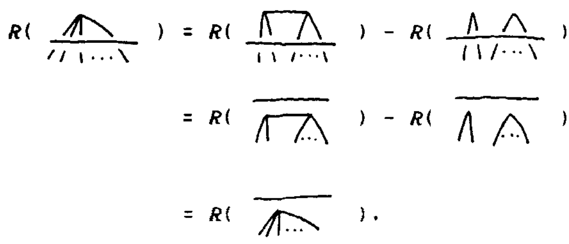
(0) It is derived from Proposition 4(6).

(II) 

(IV) Let v be the moving vertex in the figure of Reidemeister move (IV). Our proof is an induction on the degree of v. If degree(v) $= 1$, then such diagrams have cut edges, so both of their polynomials are zero. If degree(v) $= 2$, then it is shown in the case of (II) of this proof. If degree(v) $= 3$,

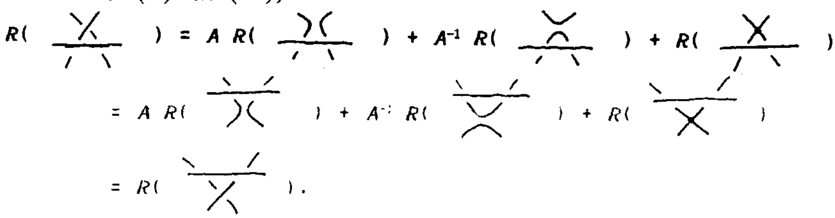
So  And 

If degree(v) > 3, from Proposition 3(2) and the hypothesis of the induction,



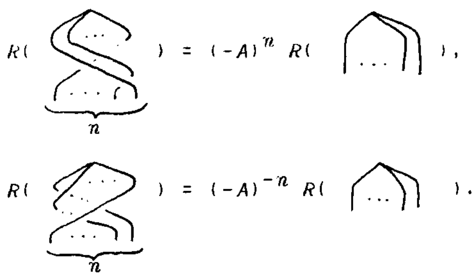
The other equation is shown similarly.

(III) From the definition of $R(g)$ and its invariance under the Reidemeister moves (II) and (IV),

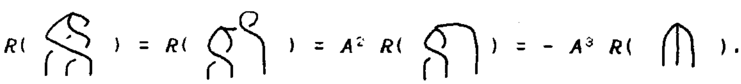


This completes the proof. ∎

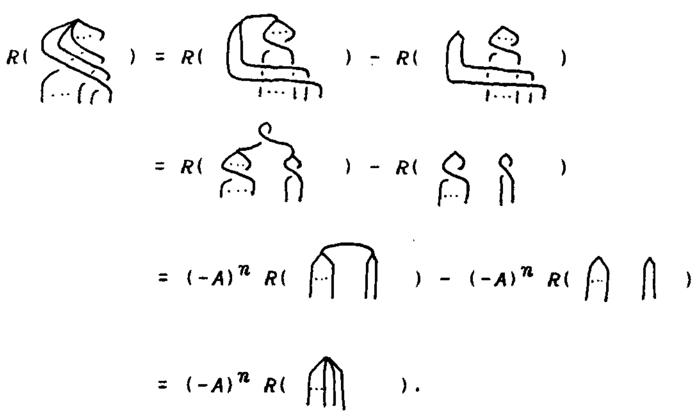
Proposition 5.



Proof. Our proof is an induction on n. If $n = 1$ then such diagrams have a cut edge, so both of their polynomials are zero. If $n = 2$, then it is shown in Proposition 4(6). If $n = 3$,



If $n > 3$, then from the hypothesis of the induction,



The other equation is shown similarly. This completes the proof. ∎

The above proposition implies the next theorem.

Theorem 3. $R(g)$ is a flat deformation invariant of a diagram g up to multiplying $(-A)^n$ for some integer n.

Let G_1 and G_2 be spatial graphs whose maximum degrees are less than 4, where the *maximum degree* of a graph $G = (V, E)$ is $\max\{\text{degree}(v) \mid v \in V\}$. Then, G_1 is pliably isotopic to G_2 if and only if G_1 is flatly isotopic to G_2. So, we get the next theorem.

Theorem 4. If g is a diagram of a graph whose maximum degree is less than 4, then $R(g)$ is a pliable deformation invariant of g up to multiplying $(-A)^n$ for some integer n.

For a spatial graph G, define $\overline{R}(g) = (-A)^{-m}R(g)$, where g is a diagram of G and m is the lowest degree of $R(g)$. By Theorem 3, $\overline{R}(G)$ is a flat isotopy invariant; moreover, by Theorem 4, if G is a graph whose maximum degree is less than 4, then $\overline{R}(G)$ is a pliable isotopy invariant of G.

3. CONNECTED SUM OF GRAPHS

For a positive integer n and graphs G and G', let v (respectively v') be a vertex of G (respectively G') of degree n, and $e_1, \ldots, e_n$ (respectively $e'_1, \ldots, e'_n$) be the edges adjacent to v (respectively v'). Then we construct a graph $G\#_n G' = (V \cup V' \cup \{v_1, \ldots, v_n\} \setminus \{v, v'\}, E \cup E')$ by removing v and v' from $G \coprod G'$ and adding n vertices $v_1, \ldots, v_n$ and changing the end point v and v' of e_i and e'_i to v_i for $i = 1, \ldots, n$. We say $G\#_n G'$ is a *connected sum* of G and G' of order n. See Figure 3.

For a positive integer n and spatial graphs G, G', let v (respectively v') be a vertex of G (respectively G') of degree n. Assume that G is in the upper half-space $\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 \mid z \geq 0\}$ except for some small neighborhood of v, and G' is in the lower half-space $\mathbb{R}^3_- = \{(x, y, z) \in \mathbb{R}^3 \mid z \leq 0\}$, except for some small neighborhood of v', and $G \cap \mathbb{R}^2_0 = G' \cap \mathbb{R}^2_0 = \{n \text{ points}\}$, where $\mathbb{R}^2_0$ is

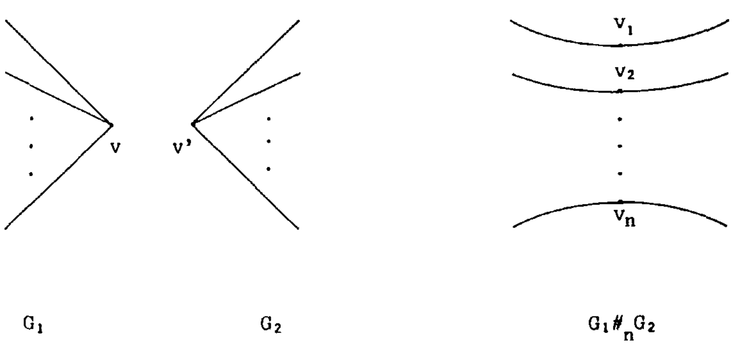


FIGURE 3

the boundary plane of those half-spaces. Then the spatial graph $(G \cap \mathbb{R}^3_+) \cup (G' \cap \mathbb{R}^3_-) \subset \mathbb{R}^3$ is called a *connected sum* of G, and G' is of order n and it is denoted by $G\#_n G'$.

For a positive integer n and diagrams $g, g' \subset \mathbb{R}^2$, v (respectively v') be a vertex of g (respectively g') of degree n. Assume that g is in the upper half-plane $\mathbb{R}^2_+$ except for some small neighborhood of v, and g' is in the lower half-plane $\mathbb{R}^2_-$ except for some small neighborhood of v', and $g \cap \mathbb{R}_0 = g' \cap \mathbb{R}_0 = \{n \text{ points}\}$, where $\mathbb{R}_0$ is the boundary line of those half-planes. Then the diagram $(g \cap \mathbb{R}^2_+) \cup (g' \cap \mathbb{R}^2_-)$ is called a *connected sum* of g and g', and it is denoted by $g\#_n g'$.

Let Θ_n be the graph that consists of two vertices and n edges, none of which is a loop. We say Θ_n the *θ_n-graph*. See Figure 4.

Proposition 6.

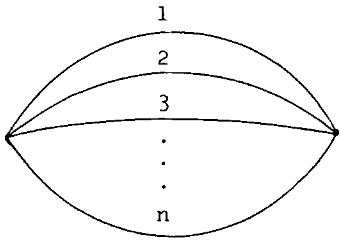
(1) $h(G\#_2 G') = h(G)h(G')/h(\Theta_2)$,
(2) $h(G\#_3 G') = h(G)h(G')/h(\Theta_3)$.

Proof. In this proof let $n = 2$ or 3. Let v, v' be the vertices of G, G' that are removed when constructing the connected sum $G\#_n G'$ and $e_1, \dots, e_n$, $e'_1, \dots, e'_n$ be the edges of G, G' that are adjacent to v, v', respectively. It suffices to assume that G and G' are connected and have no cut edges. Our proof is by induction on the number $k = |E \cup E'| - 2n$, where E and E' are the edge sets of G and G', respectively.

If $k = 0$, G, G' and $G\#_n G'$ are homeomorphic to the θ_n-graph Θ_n; hence the equalities hold. If $k > 0$, let e be an edge that is distinct from $e_1, \dots, e_n, e'_1, \dots, e'_n$. It suffices to assume that e is an edge of G. We shall prove the equations in the next two cases.

If e is a loop, from Proposition 2 and the hypothesis of the induction,

$$\begin{aligned} h(G\#_n G') &= (y - 1/x)h((G - e)\#_n G') \\ &= (y - 1/x)h(G - e)h(G')/h(\Theta_n) \\ &= h(G)R(G')/h(\Theta_n). \end{aligned}$$



θ_n-graph

FIGURE 4

If e is not a loop, by Proposition 1 and the hypothesis of the induction,

$$\begin{aligned} h(G\#_n G') &= h((G/e)\#_n G') - 1/x\, h((G - e)\#_n G') \\ &= h(G/e)h(G')/h(\Theta_n) - 1/x\, h(G - e)h(G')/h(\Theta_n) \\ &= (h(G/e) - 1/x\, h(G - e))h(G')/h(\Theta_n) \\ &= h(G)h(G')/h(\Theta_n)\,. \end{aligned}$$

This completes the proof. ∎

This proposition implies the next theorem.

Theorem 5.

(1) $R(g\#_2 g') = R(g)R(g')/\sigma$,
(2) $R(g\#_3 g') = R(g)R(g')/(\sigma - \sigma^2)$, where $\sigma = A + 1 + A^{-1}$.

Proof. In this proof let $n = 2$ or 3. By Proposition 6,

$$\begin{aligned} R(g\#_n g') &= \sum_{\substack{S\in\zeta(g)\\ S'\in\zeta(g')}} \{g\#_n g' \mid S\#_n S'\}H(S\#_n S') \\ &= \sum_{\substack{S\in\zeta(g)\\ S'\in\zeta(g')}} \{g \mid S\}H(S)\{g' \mid S'\}H(S')/H(\Theta_n) \\ &= \sum_{S\in\zeta(g)} \{g \mid S\}H(S) \sum_{S'\in\zeta(g')} \{g' \mid S'\}H(S')/H(\Theta_n) \\ &= R(g)R(g')/H(\Theta_n)\,, \end{aligned}$$

And $H(\Theta_2) = \sigma, H(\Theta_3) = \sigma - \sigma^2$. Hence we complete the proof. ∎

4. TWISTING NUMBER AND THE INVARIANT OF θ_n-CURVES

Let k be a knot diagram, i.e., a diagram of a 2-regular 1-component graph. We define the *twisting number* $t(k)$ as follows: We fix an orientation on k, and put $t(k) = \Sigma_c\, \mathrm{sign}(c)$, where c ranges over all the crossings of k and sign(⤱) $= +1$, sign(⤲) $= -1$. $t(k)$ does not depend on the choice of the orientation of k.

Let $\Theta_n = (\{u, v\}, \{e_1, \ldots, e_n\})$ be a spatial θ_n-graph. We say Θ_n is a *θ_n-curve*. In particular, we say Θ_3 is a *θ-curve*. Let C_{ij} be the cycle $u \cdot e_i \cdot v \cdot e_j \cdot u$ of Θ_n $(i \neq j)$. Let Θ_n be a diagram of Θ_n and c_{ij} be the subdiagram of θ_n corresponding to C_{ij}. Then, we define the *twisting number* of θ_n by

$$t(\theta_n) = \sum_{i<j} t(c_{ij})/(n - 1)\,.$$

More generally, let $\Xi = \Theta_{n_1} \cup \cdots \cup \Theta_{n_s}$ be a link of some θ_n-curves, *and* ξ be a diagram of Ξ, and θ_{n_i} be the subdiagram of ξ corresponding to θ_{n_i}. We define the *twisting number* $t(\xi)$ of ξ by $t(\xi) = \sum_{i=1}^{s} t(\theta_{n_i})$. It is easy to see that $t(\xi') = -t(\xi)$, where ξ' is the mirror image of ξ.

Now we define that $S(\xi) = (-A)^{-2t(\xi)}R(\xi)$.

Theorem 6. $S(\xi)$ is a flat deformation invariant of ξ.

Proof. It is easy to see that the twisting number is a regular deformation invariant. So, $S(\xi)$ does not change under the regular deformation. We shall show the invariance of $R(\xi)$ under the Reidemeister move (V). Put

$\xi =$  n, $\xi' =$  .

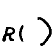
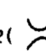
Then $t(\xi') = t(\xi) + n/2$. By Proposition 5, $R(\xi') = A^n R(\xi)$. So, $R(\xi') = (-A)^{-2(t(\xi) + n/2)}(-A)^n R(\xi) = (-A)^{-2t(\xi)}R(\xi) = S(\xi)$. The other equation is shown similarly. ∎

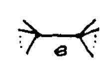
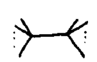
So, we define $S(\Xi) = S(\xi)$. Then $S(\Xi)$ is a flat isotopy invariant of Ξ. Theorem 4 and the above theorem imply the next.

Theorem 7. Let Ξ be a link of some θ-curves and knots. Then $S(\Xi)$ is a pliable isotopy invariant of Ξ.

5. RECURSIVE FORMULA OF $R(g)$ AND INVARIANTS OF LINKS

From the definition of $R(g)$ and the previous propositions we get the following formulas:

(1) $R($  $) = A\,R($  $) + A^{-1}R($  $) + R($  $)$.

(2) $R($  $) = R($  $) + R($  $)$,

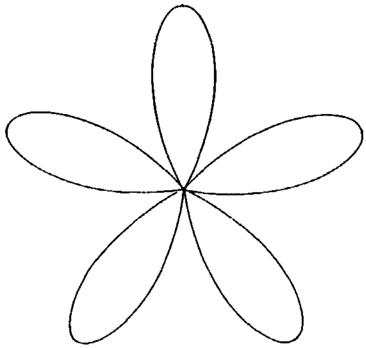
where e is a nonloop edge.

(3) $R(g_1 \amalg g_2) = R(g_1)R(g_2)$.

(4) $R(B_n) = -(-\sigma)^n$, where B_n is the n-leafed bouquet (see Figure 5) and $\sigma = A + 1 + A^{-1}$.

(5) $R(\phi) = 1$, $R(\cdot) = R(B_0) = -1$, $R(\bigcirc) = R(B_1) = \sigma$.

We can adopt the above formulas for the definition of $R(g)$. In fact, for any diagram g, we can resolve $R(g)$ to a summation of the invariant of some disjoint union of some bouquets with some coefficients by using (1) and (2) of the above formulas.



B_5

FIGURE 5

Kauffman discovered a regular deformation invariant $D(l)(\alpha, z)$ of a link diagram l (i.e., a diagram of 2-regular graph) [2, 3]. It is a specialization of the Kauffman polynomial usually denoted by F. That is called the Dubrovnik polynomial and is defined by the following recursive definition.

$$D(\ \ ) - D(\ \ ) = z\,\{D(\ \ ) - D(\ \ )\}$$

$$D(\ \ ) = a\,D(\ \ )$$

$$D(\ \ ) = a^{-1}\,D(\ \ )$$

$$D(\phi) = 1, \quad D(\ \ ) = (a-a^{-1})/z+1.$$

The above definition is different from the original one. The original one defines that $D(\ \ ) = 1$.

The next equation (1′) is derived from (1) of the recursive definition of $R(g)$.

$$(1')\qquad R(\ \ ) = A^{-1}\,R(\ \ ) + A\,R(\ \ ) + R(\ \ ).$$

By (1) and (1′),

$$R(\ \ ) - R(\ \ ) = (A-A^{-1})\,\{R(\ \ ) - R(\ \ )\}.$$

Moreover,

$$R(\ \ ) = A^{2}\,R(\ \ ),\quad R(\ \ ) = A^{-2}\,R(\ \ )\ \text{and}\quad R(\ \ ) = A+1+A^{-1}.$$

So $R(l)(A)$ satisfies the defining formulas of $D(l)(A^2, A - A^{-1})$. Now we get the next theorem.

Theorem 8. Let l be a link diagram. Then $R(l)(A) = D(l)(A^2, A - A^{-1})$.

It is shown in [10] that $-(t^{1/2} + t^{-1/2})V_{K^{(2)}}(t) = D(\tilde{K})(t^{-2}, t^{-1} - t) + 1$, where $V_{K^{(2)}}(t)$ is the Jones polynomial [1] of the $(2,0)$-cabling $K^{(2)}$ of a knot K and $\tilde{K}$ is a diagram of K such that $t(\tilde{K}) = 0$.

Corollary 2. $-(t^{1/2} + t^{-1/2})V_{K^{p(2)}}(t) = R(\tilde{K})(t^{-1}) + 1$.

6. APPLICATIONS

Let g_1 and g_2 be the diagrams shown in Figure 6. Then $R(g_1) = 0$ and $R(g_2) = -A^5 - A^4 - A^3 - A^2 + A^{-1} + A^{-2} + A^{-3} + A^{-4}$. Therefore, the two spatial graphs G_1 and G_2 presented by g_1 and g_2 are not pliably isotopic. Note that $(\mathbb{R}^3, G_1)$ and $(\mathbb{R}^3, G_2)$ are also neighborhood equivalent (after S. Suzuki [9]), i.e., $(\mathbb{R}^3, N(G_1)) \cong (\mathbb{R}^3, N(G_2))$.

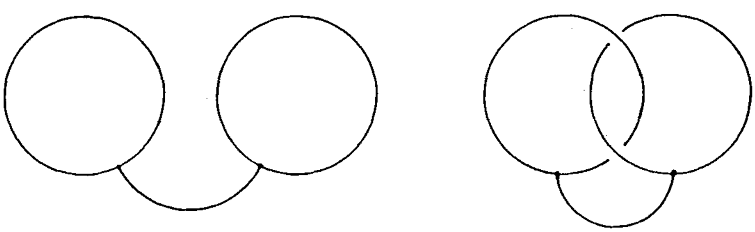
Proposition 6. Let g' be the mirror image of a diagram g. Then $R(g')(A) = R(g)(A^{-1})$.

This proposition implies the following theorems:

Theorem 9. Let G be a spatial graph. If G is amphicheiral (i.e., isotopic to the mirror image of itself) as a flat vertex graph, then $\overline{R}(G)(A) = (-A)^{-d}\overline{R}(G)(A^{-1})$, where d is the degree of $\overline{R}(G)(A)$.

Theorem 10. Let G be a spatial graph whose maximum degree is less than 4. If G is amphicheiral as a pliable vertex graph, then $\overline{R}(G)(A) = (-A)^{-d}\overline{R}(G)(A^{-1})$, where d is the degree of $\overline{R}(G)(A)$.

Theorem 11. Let Θ be an amphicheiral θ-curve. Then $S(\Theta)(A^{-1}) = S(\Theta)(A)$.



θ_1 θ_2

FIGURE 6

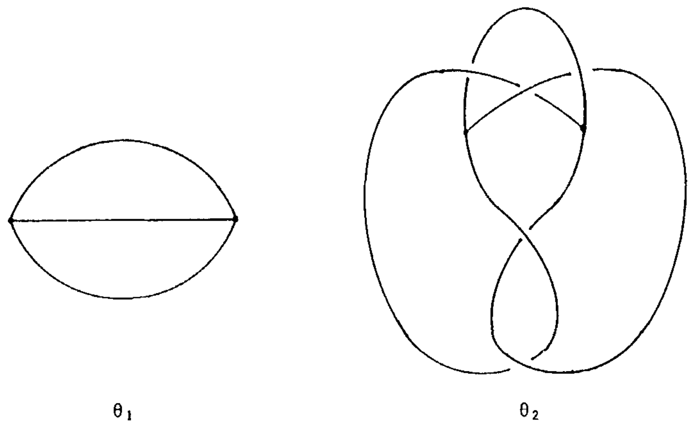


FIGURE 7

Let θ_1 and θ_2 be the diagram shown in Figure 7. Let Θ_1 and Θ_2 be the spatial graphs presented by θ_1 and θ_2. Then $t(\theta_1) = 0$, $R(\theta_1) = -A^2 - A - 2 - A^{-1} - A^{-2}$, $t(\theta_2) = -3/2$, $R(\theta_2) = A^9 - A^8 - 2A^7 + A^6 - A^5 + 2A^3 + A^2 + 2A + A^{-1} - A^{-3} + A^{-4} + A^{-5} - A^{-6} + A^{-7} + A^{-8}$. Therefore Θ_2 is not pliably isotopic to trivial θ-curve Θ_1. Moreover, by Theorem 9, Θ_2 is not amphicheiral as a pliable vertex graph. Note that each of the three cycles of Θ_2 is a trivial knot.

Those spatial graphs shown in Figures 6 and 7 are presented in [5,6].

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