Higher Order Session Types

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1 Calculus

1.1 Syntax

We assume the countable sets:

$$\begin{array}{llll} S = \{s, s_1, \ldots\} & \text{Session names} & \overline{S} = \{\overline{s} \mid s \in S\} & \text{Dual session names} \\ \mathcal{V} = \{x, y, z, \ldots\} & \text{Variable names} & \mathcal{PV} = \{X, Y, Z, \ldots\} & \text{Process Variable names} \end{array}$$

with the set of names $\mathcal{N} = S \cup \overline{S}$ and let $k \in \mathcal{N} \cup \mathcal{V}$.

We can now define the syntax of processes:

$$\begin{array}{lllllll} P & ::= & k!\langle (x)P_1\rangle; P_2 & \text{Out} & \mid & k?(X); P & \text{In} \\ & \mid & s \oplus l; P & \text{Sel} & \mid & s\&\{l_i : P_i\}_{i\in I} & \text{Bra} \\ & \mid & P_1 \mid P_2 & \text{Par} & \mid & (\nu\ s)P & \text{New} \\ & \mid & X\langle k\rangle & \text{Appl} & \mid & \mathbf{0} & \text{Inact} \end{array}$$

1.2 Reduction

Structural Congruence

$$P \mid \mathbf{0} \equiv P \qquad P_1 \mid P_2 \equiv P_2 \mid P_1 \qquad P_1 \mid (P_2 \mid P_3) \qquad (P_1 \mid P_2) \mid P_3 \qquad (\nu\ s)\mathbf{0} \equiv \mathbf{0}$$
$$s \notin \mathtt{fn}(P_1) \Rightarrow P_1 \mid (\nu\ s)P_2 \equiv (\nu\ s)(P_1 \mid P_2)$$

Process Variable Substitution

$$\begin{array}{rcl} (s!\langle (y)P_1\rangle; P_2)\{(x)Q/X\} & = & s!\langle (y)P_1\{(x)Q/X\}\rangle; (P_2\{(x)Q/X\}) \\ (s?(Y); P)\{(x)Q/X\} & = & s?(Y); (P\{(x)Q/X\}) \\ (s \oplus l; P)\{(x)Q/X\} & = & s \oplus l; (P\{(x)Q/X\}) \\ (s\&\{l_i : P_i\}_{i\in I})\{(x)Q/X\} & = & s\&\{l_i : P_i\{(x)Q/X\}\}_{i\in I} \\ (P_1 \mid P_2)\{(x)Q/X\} & = & P_1\{(x)Q/X\} \mid P_2\{(x)Q/X\} \\ ((\nu\ s)P)\{(x)Q/X\} & = & (\nu\ s)(P\{(x)Q/X\}) \\ \mathbf{0}\{(x)Q/X\} & = & \mathbf{0} \\ X\langle k\rangle\{(x)Q/X\} & = & Q\{k/x\} \end{array}$$

$$\begin{array}{rcl} & & s!\langle (x)P\rangle; P_1 \mid s?(X); P_2 \longrightarrow P_1 \mid P_2\{(x)P/X\} \\ & & s \oplus l_k; P \mid s\&\{l_i : P_i\}_{i\in I} \longrightarrow P \mid P_k \qquad k \in I \\ P_1 \longrightarrow P_1' & \Rightarrow & P_1 \mid P_2 \longrightarrow P_1' \mid P_2 \\ P \longrightarrow P' & \Rightarrow & (\nu\ s)P \longrightarrow (\nu\ s)P' \\ P \equiv\longrightarrow\equiv P' & \Rightarrow & P \longrightarrow P' \end{array}$$

2 Types

2.1 Session Types

$$
\begin{array}{lcl}
H & ::= & T \multimap \diamond \ \mid\ T \rightarrow \diamond \\
T & ::= & \mathsf{end} \ \mid\ \mu \mathsf{t}.T \ \mid\ \mathsf{t} \ \mid\ !\langle H\rangle;T \ \mid\ ?(H);T \ \mid\ \oplus\{l_i : T_i\}_{i\in I} \ \mid\ \&\{l_i : T_i\}_{i\in I}
\end{array}
$$

We let $\multimap\!\!\bullet \in \{\multimap, \rightarrow\}$.

2.2 Subtyping

Definition 2.1. *Let $\multimap < \rightarrow$ to define $\leq\, = <^*$.*

Definition 2.2 (Session Subtyping). *Let $\mathcal{T}$ to be the set of all session types. Define the monotone function $F : \mathcal{T} \longrightarrow \mathcal{T}$:*

$$
\begin{array}{rl}
F(R) = & \{\mathsf{end}, \mathsf{end}\} \\
& \cup\ \{(T_1, T_2)\ \mid\ T_1 \multimap\!\!\bullet \diamond\ R\ T_2 \multimap\!\!\bullet \diamond\} \cup \{(T_1, T_2)\ \mid\ T_1 \multimap \diamond\ R\ T_2 \rightarrow \diamond\} \\
& \cup\ \{(T_1, T_2)\ \mid\ !\langle H_1\rangle;T_1\ R\ !\langle H_2\rangle;T_2, H_2\ R\ H_1\} \\
& \cup\ \{(T_1, T_2)\ \mid\ ?(H_1);T_1\ R\ ?(H_2);T_2, H_1\ R\ H_2\} \\
& \cup\ \{(T_k, T'_k)\ \mid\ \oplus\{l_i : T_i\}_{i\in I}\ R\ \oplus\{l_i : T'_i\}_{i\in I}, k \in I\} \\
& \cup\ \{(T_k, T'_k)\ \mid\ \&\{l_i : T_i\}_{i\in I}\ R\ \&\{l_i : T'_i\}_{i\in I}, k \in I\} \\
& \cup\ \{(T_1, \mu\mathsf{t}.T_2)\ \mid\ T_1\{\mu\mathsf{t}.T_1/t\}\ R\ \mu\mathsf{t}.T_2\} \cup \{(\mu\mathsf{t}.T_1, T_2)\ \mid\ \mu\mathsf{t}.T_1\ R\ T_2\{\mu\mathsf{t}.T_2/t\}\} \\
& \cup\ \{(T_1, T_2)\ \mid\ \mu\mathsf{t}.T_1\ R\ \mu\mathsf{t}.T_2\}
\end{array}
$$

$\leq = \nu R.F(R)$.

2.3 Duality

Definition 2.3 (Session Duality). *Define the monotone function $F : \mathcal{T} \longrightarrow \mathcal{T}$:*

$$
\begin{array}{rl}
F(R) = & \{\mathsf{end}, \mathsf{end}\} \\
& \cup\ \{(T_1, T_2)\ \mid\ ?(H_1);T_1\ R\ !\langle H_2\rangle;T_2, H_1\ \leq\ H_2\} \\
& \cup\ \{(T_1, T_2)\ \mid\ !\langle H_1\rangle;T_1\ R\ ?(H_2);T_2, H_2\ \leq\ H_1\} \\
& \cup\ \{(T_k, T'_k)\ \mid\ \oplus\{l_i : T_i\}_{i\in I}\ R\ \&\{l_i : T'_i\}_{i\in I}, k \in I\} \\
& \cup\ \{(T'_k, T_k)\ \mid\ \&\{l_i : T_i\}_{i\in I}\ R\ \oplus\{l_i : T'_i\}_{i\in I}, k \in I\} \\
& \cup\ \{(T_1, \mu\mathsf{t}.T_2)\ \mid\ T_1\{\mu\mathsf{t}.T_1/t\}\ R\ \mu\mathsf{t}.T_2\} \cup \{(\mu\mathsf{t}.T_1, T_2)\ \mid\ \mu\mathsf{t}.T_1\ R\ T_2\{\mu\mathsf{t}.T_2/t\}\} \\
& \cup\ \{(T_1, T_2)\ \mid\ \mu\mathsf{t}.T_1\ R\ \mu\mathsf{t}.T_2\}
\end{array}
$$

$\mathsf{dual} = \nu R.F(R)$.

2.4 Typing System

$$\Gamma\ ::=\ \Gamma \cdot X : H\ \mid\ \emptyset$$

$$\Delta\ ::=\ \Delta \cdot k : T\ \mid\ \Delta \cdot X\ \mid\ \emptyset$$

$$\Gamma \vdash P \triangleright \Delta$$

$$\Gamma \vdash \mathbf{0} \triangleright \emptyset \quad \text{[Inact]} \qquad \frac{\Gamma \vdash P \triangleright \Delta \quad k \notin \mathtt{dom}(\Delta)}{\Gamma \vdash P \triangleright \Delta \cdot k : \mathtt{end}} \quad \text{[Comp]}$$

$$\Gamma \cdot X : T \multimap \diamond \vdash X\langle k \rangle \triangleright k : T \cdot X \quad \text{[LAppl]} \qquad \Gamma \cdot X : T \rightarrow \diamond \vdash X\langle k \rangle \triangleright k : T \quad \text{[SAppl]}$$

$$\frac{\Gamma \vdash P \triangleright \Delta \cdot s : T' \quad T' \leq T}{\Gamma \vdash P \triangleright \Delta \cdot s : T} \quad \text{[Subs]}$$

$$\frac{\Gamma \vdash Q \triangleright \Delta \cdot x : T' \quad \Gamma \vdash P \triangleright k : T \quad k \notin \mathtt{dom}(\Delta)}{\Gamma \vdash k!\langle (x)Q \rangle; P \triangleright \Delta \cdot k : !\langle T' \rightarrow \diamond \rangle; T} \quad \text{[SOut]}$$

$$\frac{\Gamma \vdash Q \triangleright \Delta_1 \cdot x : T' \quad \Gamma \vdash P \triangleright \Delta_2 \cdot k : T \quad \Delta_1 \neq \emptyset \quad \mathtt{dom}(\Delta_1) \cap \mathtt{dom}(\Delta_2 \cdot k : T) = \emptyset}{\Gamma \vdash k!\langle (x)Q \rangle; P \triangleright \Delta_1 \cdot \Delta_2 \cdot k : !\langle T' \multimap \diamond \rangle; T} \quad \text{[LOut]}$$

$$\frac{\Gamma \cdot X : T' \rightarrow \diamond \vdash P \triangleright \Delta \cdot k : T \quad X \notin \Delta}{\Gamma \vdash k?(X); P \triangleright \Delta \cdot k : ?(T' \rightarrow \diamond); T} \quad \text{[SIn]}$$

$$\frac{\Gamma \cdot X : T' \multimap \diamond \vdash P \triangleright \Delta \cdot k : T \cdot X}{\Gamma \vdash k?(X); P \triangleright \Delta \cdot k : ?(T' \multimap \diamond); T} \quad \text{[LIn]}$$

$$\frac{\Gamma \vdash P \triangleright \Delta \cdot s : T_1 \cdot \overline{s} : T_2 \quad T_1 \ \mathsf{dual}\ T_2}{\Gamma \vdash (\nu\ s)P \triangleright \Delta} \quad \text{[Res]} \qquad \frac{\begin{array}{c}\Gamma \vdash P_1 \triangleright \Delta_1 \quad \Gamma \vdash P_2 \triangleright \Delta_2 \\ \mathtt{dom}(\Delta_1) \cap \mathtt{dom}(\Delta_2) = \emptyset\end{array}}{\Gamma \vdash P_1 \mid P_2 \triangleright \Delta_1 \cdot \Delta_2} \quad \text{[Par]}$$

$$\frac{\Gamma \vdash P \triangleright \Delta \cdot k : T}{\Gamma \vdash P \triangleright \Delta \cdot k : \oplus\{l : T\}} \quad \text{[Sel]} \qquad \frac{\forall i \in I, \Gamma \vdash P_i \triangleright \Delta \cdot k : T_i}{\Gamma \vdash s\&\{l_i : P_i\}_{i \in I} \triangleright \Delta \cdot k : \&\{l_i : T_i\}_{i \in I}} \quad \text{[Bra]}$$

2.5 Examples

Example 2.1.

$$P = s?(X); (X\langle s_1 \rangle \mid X\langle s_2 \rangle)$$

is untypable under environment $\Gamma = X : T \multimap \diamond$, since:

$$\Gamma \vdash X\langle s_1 \rangle \triangleright s_1 : T \cdot X \quad \Gamma \vdash X\langle s_2 \rangle \triangleright s_2 : T \cdot X$$

We cannot apply rule [Par] to get:

$$\Gamma \vdash X\langle s_1 \rangle \mid X\langle s_2 \rangle \triangleright s_1 : T \cdot s_2 : T$$

because $\mathtt{dom}(s_1 : T \cdot X) \cup \mathtt{dom}(s_2 : T \cdot X) = X$.

It is though typable under environment $\Gamma = X : T \rightarrow \diamond$, since:

$$\cfrac{\cfrac{\Gamma \vdash X\langle s_1 \rangle \triangleright s_1 : T \quad \Gamma \vdash X\langle s_2 \rangle \triangleright s_2 : T \quad \mathtt{dom}(s_1 : T) \cup \mathtt{dom}(s_2 : T) = \emptyset}{\Gamma \vdash X\langle s_1 \rangle \mid X\langle s_2 \rangle \triangleright s_1 : T \cdot s_2 : T}}{\vdash s?(X); (X\langle s_1 \rangle \mid X\langle s_2 \rangle) \triangleright s : ?(T \rightarrow \diamond); \mathtt{end} \cdot s_1 : T \cdot s_2 : T}$$

Now let

$$Q_1 = \overline{s}!\langle (x)x!\langle \mathbf{0}\rangle;\mathbf{0}\rangle;\mathbf{0}$$
$$Q_2 = \overline{s}!\langle (x)x!\langle \mathbf{0}\rangle; s'!\langle \mathbf{0}\rangle;\mathbf{0}\rangle;\mathbf{0}$$

Process $(\nu\ s)(Q_1 \mid P)$ is typable, whereas $(\nu\ s)(Q_2 \mid P)$ is not. This is due to the fact that abstraction $(x)x!\langle \mathbf{0}\rangle; s'!\langle \mathbf{0}\rangle;\mathbf{0}$ contains linear session s' and should not be duplicated:

$$P \mid Q_2 \longrightarrow s_1!\langle \mathbf{0}\rangle; s'!\langle \mathbf{0}\rangle;\mathbf{0} \mid s_2!\langle \mathbf{0}\rangle; s'!\langle \mathbf{0}\rangle;\mathbf{0}$$

The last process should not be typable because name s' is appeared twice.

The type system avoids the above situation on rule [Out] and the duality relation:

$$\frac{\vdash x!\langle \mathbf{0}\rangle; s'!\langle \mathbf{0}\rangle;\mathbf{0} \triangleright x : T \cdot s' : T' \quad \vdash \mathbf{0} \triangleright \overline{s} : \mathsf{end}}{\vdash Q_2 \triangleright s' : T' \cdot \overline{s} : !\langle T \multimap \diamond\rangle;\mathsf{end}}$$

We then apply rule [Par] to get:

$$\vdash P \mid Q_2 \triangleright s' : T' \cdot \overline{s} : !\langle T \multimap \diamond\rangle;\mathsf{end} \cdot s : ?(T \to \diamond);\mathbf{0} s_1 : T \cdot s_2 : T$$

On this typing node, rule [Res] is not applicable since $!\langle T \multimap \diamond\rangle;\mathsf{end}$ is not dual with $?(T \to \diamond);\mathbf{0}$.

3 Extensions

$$\begin{array}{rcll}
P & ::= & \ldots & \\
& \mid & k!\langle k'\rangle;P & \text{Name passing} \\
& \mid & k?(x);P & \text{Receive name} \\
& \mid & k!\langle \tilde{k}\rangle;P & \text{Polyadic send} \\
& \mid & k?(\tilde{x});P & \text{Polyadic receive} \\
& \mid & k!\langle (X)P_1\rangle;P_2 & \text{Process Abstraction} \\
& \mid & X\langle (x)P\rangle & \text{Process Application} \\
& \mid & k!\langle (\tilde{x})P_1\rangle;P_2 & \text{Polyadic Abstraction} \\
& \mid & X\langle \tilde{k}\rangle & \text{Polyadic Application}
\end{array}$$

give semantics

3.1 Encoding

$$\begin{array}{rcl}
[\![k!\langle k'\rangle;P]\!] & ::= & k!\langle (z)z?(X);X\langle k'\rangle\rangle;[\![P]\!] \\
[\![k?(x);P]\!] & ::= & k?(X);(\nu\ s)(X\langle s\rangle \mid \overline{s}!\langle (x)[\![P]\!]\rangle;\mathbf{0}) \\
[\![k!\langle k'\cdot\tilde{k}\rangle;P]\!] & ::= & [\![k!\langle k'\rangle;k!\langle \tilde{k}\rangle;P]\!] \\
[\![k?(x\cdot\tilde{x});P]\!] & ::= & [\![k?(x);k?(\tilde{x});P]\!] \\
[\![k!\langle (X)Q\rangle;P]\!] & ::= & k!\langle (z)z?(X);[\![Q]\!]\rangle;[\![P]\!] \\
[\![X\langle (x)P\rangle]\!] & ::= & (\nu\ s)(X\langle s\rangle \mid \overline{s}!\langle (x)[\![P]\!]\rangle;\mathbf{0}) \\
[\![k!\langle (\tilde{x})P_1\rangle;P_2]\!] & ::= & k!\langle (z)[\![z?(\tilde{x});P_1]\!]\rangle;[\![P_2]\!] \\
[\![X\langle \tilde{k}\rangle]\!] & ::= & (\nu\ s)(X\langle s\rangle \mid [\![\overline{s}!\langle \tilde{k}\rangle;\mathbf{0}]\!]) \\
& \ldots & \text{the rest isomorphic}
\end{array}$$

3.2 Operational Correspondence

$$\begin{array}{rcl} s!\langle k'\rangle; P_1 \mid s?(x); P_2 & \longrightarrow & P_1 \mid P_2\{k'/x\} \\ s!\langle (Y)P\rangle; P_1 \mid s?(X); X\langle (x)P_2\rangle & \longrightarrow & P_1 \mid P\{(x)P_2/Y\} \\ s!\langle (\tilde{x})P_1\rangle; P_2 \mid s?(X); X\langle \tilde{k}\rangle & \longrightarrow & P_2 \mid P_1\{\tilde{k}/\tilde{x}\} \end{array}$$

$$\begin{array}{rcl} [\![s!\langle k'\rangle; P_1 \mid s?(x); P_2]\!] & ::= & s!\langle (z)z?(X); X\langle k'\rangle\rangle; [\![P_1]\!] \mid s?(X); (\nu\ s')(X\langle s'\rangle \mid \overline{s'}!\langle (x)[\![P_2]\!]\rangle; \mathbf{0}) \\ & \longrightarrow & [\![P_1]\!] \mid (\nu\ s')(s?(X); X\langle k'\rangle \mid \overline{s'}!\langle (x)[\![P_2]\!]\rangle; \mathbf{0}) \\ & \longrightarrow & [\![P_1]\!] \mid [\![P_2]\!]\{k'/x\} \\ \\ [\![s!\langle (Y)P\rangle; P_1 \mid s?(X); X\langle (x)P_2\rangle]\!] & ::= & s!\langle (z)z?(Y); [\![P]\!]\rangle; [\![P_1]\!] \mid s?(X); (\nu\ s')(X\langle s'\rangle \mid \overline{s'}!\langle (x)[\![P_2]\!]\rangle; \mathbf{0}) \\ & \longrightarrow & [\![P_1]\!] \mid (\nu\ s')(s'?(Y); [\![P]\!] \mid \overline{s'}!\langle (x)[\![P_2]\!]\rangle; \mathbf{0}) \\ & \longrightarrow & [\![P_1]\!] \mid [\![P]\!]\{(x)[\![P_2]\!]/Y\} \\ \\ [\![s!\langle (\tilde{x})P_1\rangle; P_2 \mid s?(X); X\langle \tilde{k}\rangle]\!] & ::= & s!\langle (z)[\![z?(\tilde{x}); P_1]\!]\rangle; [\![P_2]\!] \mid s?(X); (\nu\ s')(X\langle s'\rangle \mid [\![\overline{s}!\langle \tilde{k}\rangle; \mathbf{0}]\!]) \\ & \longrightarrow & [\![P_2]\!] \mid (\nu\ s')([\![s'?(\tilde{x}); P_1]\!] \mid [\![\overline{s}!\langle \tilde{k}\rangle; \mathbf{0}]\!]) \\ & \longrightarrow^* & [\![P_2]\!] \mid [\![P_1]\!]\{\tilde{k}/\tilde{x}\} \end{array}$$

3.3 Encode Processes to non Linear Abstractions

Processes with free sessions can only be used as linear abstractions. As we have seen in Example 2.1 a process:

$$s!\langle ()P\rangle; P_1 \mid s?(X); (X\langle\rangle \mid X\langle\rangle)$$

with $\mathtt{fs}(P) \neq \emptyset$ is not typable since abstraction $()P$ can only be used in a linear way.

It is convenient to have an encoding from a process to an abstraction with no free names, that can be used a shared value:

$$\mathcal{A}[\![P]\!] \ ::= \ ((\!|\mathtt{fn}(P)|\!)^v)\mathcal{A}[\![P]\!]^{\emptyset}$$

where

Function $(\!| \cdot |\!)^s : 2^{\mathcal{N}} \longrightarrow \mathcal{N}^{\omega}$ orders lexicographically a set of names, function $(\!| \cdot |\!)^v : 2^{\mathcal{N}} \longrightarrow \mathcal{V}^{\omega}$ maps a set of names to variables:

$$\begin{array}{rcl} (\!|\{s_i\}_{i\in I}|\!)^v & = & (\!|(\!|\{s_i\}_{i\in I}|\!)^s|\!)^{s\rightarrow v} \\ (\!|s \cdot \tilde{s}|\!)^{s\rightarrow v} & = & x_s \cdot (\!|\tilde{s}|\!)^{s\rightarrow v} \\ (\!|s|\!)^{s\rightarrow v} & = & x_s \end{array}$$

$$
\begin{array}{rcl}
\mathcal{A}\llbracket s!\langle (x)P' \rangle; P \rrbracket^{\sigma} & ::= & \begin{cases} x_s!\langle ((\llparenthesis x \rrparenthesis^{v} P)\mathcal{A}\llbracket P' \rrbracket^{\emptyset} \rangle; \mathcal{A}\llbracket P \rrbracket^{\sigma} \;\; s \notin \sigma \\ \;\; s!\langle ((\llparenthesis x \rrparenthesis^{v} P)\mathcal{A}\llbracket P' \rrbracket^{\emptyset} \rangle; \mathcal{A}\llbracket P \rrbracket^{\sigma} \;\; s \in \sigma \end{cases} \\
\mathcal{A}\llbracket s?(X); P \rrbracket^{\sigma} & ::= & \begin{cases} x_s?(X); \mathcal{A}\llbracket P \rrbracket^{\sigma} \;\; s \notin \sigma \\ \;\; s?(X); \mathcal{A}\llbracket P \rrbracket^{\sigma} \;\; s \in \sigma \end{cases} \\
\mathcal{A}\llbracket s \oplus l; P \rrbracket^{\sigma} & ::= & \begin{cases} x_s \oplus l; \mathcal{A}\llbracket P \rrbracket^{\sigma} \;\; s \notin \sigma \\ \;\; s \oplus l; \mathcal{A}\llbracket P \rrbracket^{\sigma} \;\; s \in \sigma \end{cases} \\
\mathcal{A}\llbracket s\&\{l_i : P_i\}_{i \in I} \rrbracket^{\sigma} & ::= & \begin{cases} x_s\&\{l_i : \mathcal{A}\llbracket P_i \rrbracket^{\sigma}\}_{i \in I} \;\; s \notin \sigma \\ \;\; s\&\{l_i : \mathcal{A}\llbracket P_i \rrbracket^{\sigma}\}_{i \in I} \;\; s \in \sigma \end{cases} \\
\mathcal{A}\llbracket P_1 \mid P_2 \rrbracket^{\sigma} & ::= & \mathcal{A}\llbracket P_1 \rrbracket^{\sigma} \mid \mathcal{A}\llbracket P_2 \rrbracket^{\sigma} \qquad\qquad\qquad\qquad s \notin \sigma \\
\mathcal{A}\llbracket (\nu\ s)P \rrbracket^{\sigma} & ::= & (\nu\ s)\mathcal{A}\llbracket P \rrbracket^{\sigma \cdot s} \\
\mathcal{A}\llbracket \mathbf{0} \rrbracket^{\sigma} & ::= & \mathbf{0} \\
\mathcal{A}\llbracket X\langle s \rangle \rrbracket^{\sigma} & ::= & \begin{cases} X\langle x_s \rangle \;\; s \notin \sigma \\ \;\; X\langle s \rangle \;\; s \in \sigma \end{cases}
\end{array}
$$

3.4 Encode Recursion

$$
\begin{array}{rcl}
P & ::= & \dots \\
 & \mid & \mu r.P \\
 & \mid & r
\end{array}
$$

$$
\begin{array}{rcl}
\llbracket \mu r.P \rrbracket & = & (\nu\ s)(s?(X); \llbracket P \rrbracket \mid \overline{s}!\langle (z \cdot \llparenthesis \mathtt{fn}(P) \rrparenthesis^{v}) z?(X); \mathcal{A}\llbracket P \rrbracket^{\emptyset} \rangle; \mathbf{0}) \\
\llbracket r \rrbracket & = & (\nu\ s)(X\langle s \cdot \llparenthesis \mathtt{fn}(P) \rrparenthesis^{s} \rangle \mid \overline{s}!\langle (z \cdot \llparenthesis \mathtt{fn}(P) \rrparenthesis^{v}) X\langle z \cdot \llparenthesis \mathtt{fn}(P) \rrparenthesis^{v} \rangle \rangle; \mathbf{0})
\end{array}
$$

3.5 Operational Correspondence for Recursion

todo

3.6 Typing

We type the encodings:

1. $s!\langle k \rangle; P$

$$
\dfrac{\Gamma \vdash \llbracket P \rrbracket \triangleright \Delta \cdot s : T \qquad \dfrac{\dfrac{\Gamma \cdot X : T' \multimap \diamond \vdash X\langle k \rangle \triangleright k : T' \cdot X}{\Gamma \cdot X : T' \multimap \diamond \vdash X\langle k \rangle \triangleright k : T' \cdot X \cdot z : \mathsf{end}}}{\Gamma \vdash z?(X); X\langle k \rangle \triangleright z : ?(T' \multimap \diamond); \mathsf{end}}}{\Gamma \vdash s!\langle (z) z?(X); X\langle k \rangle \rangle; \llbracket P \rrbracket \triangleright s : \Delta \cdot s : !\langle ?(T' \multimap \diamond); \mathsf{end} \multimap \diamond \rangle; T}
$$

2. $s?(x); P$ with $\Gamma' = \Gamma \cdot X : ?(T' \multimap \diamond); \mathsf{end}$

$$
\dfrac{\dfrac{\dfrac{\Gamma' \vdash X\langle s' \rangle \triangleright s' : ?(T' \multimap \diamond); \mathsf{end} \cdot X \qquad \dfrac{\dfrac{\Gamma' \vdash \mathbf{0} \triangleright \emptyset}{\Gamma' \vdash \mathbf{0} \triangleright \overline{s'} : \mathsf{end}} \qquad \Gamma' \vdash \llbracket P \rrbracket \triangleright \Delta \cdot x : T' \cdot s : T}{\Gamma' \vdash \overline{s'}!\langle (x) \llbracket P \rrbracket \rangle; \mathbf{0} \triangleright \Delta \cdot \overline{s'} : !\langle T' \multimap \diamond \rangle; \mathsf{end} \cdot s : T}}{\Gamma' \vdash X\langle s' \rangle \mid \overline{s'}!\langle (x) \llbracket P \rrbracket \rangle; \mathbf{0} \triangleright \Delta \cdot s' : ?(T' \multimap \diamond); \mathsf{end} \cdot \overline{s'} : !\langle T' \multimap \diamond \rangle; \mathsf{end} \cdot s : T \cdot X}}{\Gamma' \vdash (\nu\ s')(X\langle s' \rangle \mid \overline{s'}!\langle (x) \llbracket P \rrbracket \rangle; \mathbf{0}) \triangleright \Delta \cdot s : T \cdot X}}{\Gamma \vdash s?(X); (\nu\ s')(X\langle s' \rangle \mid \overline{s'}!\langle (x) \llbracket P \rrbracket \rangle; \mathbf{0}) \triangleright \Delta \cdot s : ?(?(T' \multimap \diamond); \mathsf{end} \multimap \diamond); T}
$$

3. $s!\langle (Y)P_2\rangle; P_1$

$$\dfrac{\Gamma \vdash \llbracket P_1 \rrbracket \triangleright \Delta_1 \cdot s : T \qquad \dfrac{\dfrac{\Gamma \cdot Y : T' \multimap \diamond \vdash \llbracket P_2 \rrbracket \triangleright \Delta_2}{\Gamma \cdot Y : T' \multimap \diamond \vdash \llbracket P_2 \rrbracket \triangleright \Delta_2 \cdot z : \mathsf{end}}}{\Gamma \vdash z?(Y); \llbracket P_2 \rrbracket \triangleright \Delta_2 \backslash Y \cdot z : ?(T' \multimap \diamond); \mathsf{end}}}{\Gamma \vdash s!\langle (z)z?(Y); \llbracket P_2 \rrbracket\rangle; \llbracket P_1 \rrbracket \triangleright \Delta_1 \cdot \Delta_2 \backslash Y \cdot z : !\langle ?(T' \multimap \diamond); \mathsf{end} \multimap \diamond\rangle; T}$$

4. $X\langle (x)P\rangle$

$$\dfrac{\dfrac{\Gamma \cdot X : ?(T' \multimap \diamond); \mathbf{0} \multimap \diamond \vdash X\langle s\rangle \triangleright \Delta_1 \cdot s : ?(T' \multimap \diamond); \mathbf{0} \qquad \dfrac{\Gamma' \vdash \llbracket P \rrbracket \triangleright \Delta_2 \cdot x : T' \qquad \dfrac{\Gamma' \vdash \mathbf{0} \triangleright \emptyset}{\Gamma' \vdash \mathbf{0} \triangleright \overline{s'} : \mathsf{end}}}{\Gamma' \vdash \overline{s}!\langle (x)P\rangle; \mathbf{0} \triangleright \Delta_2 \cdot \overline{s} : !\langle T' \multimap \diamond\rangle; \mathsf{end}}}{\Gamma \cdot X : ?(T' \multimap \diamond); \mathbf{0} \multimap \diamond \vdash X\langle s\rangle \mid \overline{s}!\langle (x)P\rangle; \mathbf{0} \triangleright \Delta_1 \cdot \Delta_2 \cdot s : ?(T' \multimap \diamond); \mathbf{0} \cdot \overline{s} : !\langle T' \multimap \diamond\rangle; \mathsf{end}}}{\Gamma \cdot X : ?(T' \multimap \diamond); \mathbf{0} \multimap \diamond \vdash (\nu\ s)(X\langle s\rangle \mid \overline{s}!\langle (x)P\rangle; \mathbf{0}) \triangleright \Delta_1 \cdot \Delta_2}$$

5. $\mu r.P$

$$\dfrac{\Gamma \cdot X : ?(T' \to \diamond); \mathsf{end} \to \diamond \vdash \llbracket P \rrbracket \triangleright \Delta \cdot s : T}{\Gamma \vdash s?(X); \llbracket P \rrbracket \triangleright \Delta \cdot s : ?(?(T' \to \diamond); \mathsf{end} \to \diamond); T}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \cdot X : T' \to \diamond \vdash \mathcal{A}\llbracket P \rrbracket^{\emptyset} \triangleright z : \mathsf{end} \cdot \tilde{y} : \tilde{T}}{\Gamma \vdash z?(X); \mathcal{A}\llbracket P \rrbracket^{\emptyset} \triangleright z : ?(T' \to \diamond); \mathsf{end} \cdot \tilde{y} : \tilde{T}} \qquad \dfrac{\Gamma \vdash \mathbf{0} \triangleright \emptyset}{\Gamma \vdash \mathbf{0} \triangleright \overline{s} : \mathsf{end}}}{\Gamma \vdash \overline{s}!\langle (z\tilde{y})z?(X); \mathcal{A}\llbracket P \rrbracket^{s}\rangle; \mathbf{0} \triangleright \overline{s} : !\langle ?(T' \to \diamond); \mathsf{end} \to \diamond\rangle; \mathsf{end}}}{\Gamma \vdash s?(X); \llbracket P \rrbracket \mid \overline{s}!\langle (z\tilde{y})z?(X); \mathcal{A}\llbracket P \rrbracket^{s}\rangle; \mathbf{0} \triangleright \Delta \cdot s : ?(?(T' \to \diamond); \mathsf{end} \to \diamond); T \cdot \overline{s} : !\langle ?(T' \to \diamond); \mathsf{end} \to \diamond\rangle; \mathsf{end}}}{\Gamma \vdash (\nu\ s)(s?(X); \llbracket P \rrbracket \mid \overline{s}!\langle (z\tilde{y})z?(X); \mathcal{A}\llbracket P \rrbracket^{s}\rangle; \mathbf{0}) \triangleright \Delta}$$

6. $\llbracket r \rrbracket$

4 Observational Semantics

4.1 Labelled Transition Semantics

$$\begin{array}{lcl}
\lambda & ::= & \tau \ \mid \ s!\langle (x)P\rangle \ \mid \ s?\langle (x)P\rangle \ \mid \ s \oplus l \ \mid \ s\& l \ \mid \ o \\
o & ::= & (\nu\ s)s!\langle (x)P\rangle \ \mid \ (\nu\ s)o
\end{array}$$

$$\begin{array}{c}
\mathtt{fn}(s \oplus l) = \mathtt{fn}(s\& l) = \{s\} \qquad \mathtt{fn}(\tau) = \emptyset \\
\mathtt{fn}(s!\langle (x)P\rangle) = \mathtt{fn}(s!\langle (x)P\rangle) = \{s\} \cup \mathtt{fn}((x)P) \\
\mathtt{bn}(\tau) = \mathtt{bn}(s \oplus l) = \mathtt{bn}(s\& l) = \mathtt{bn}(s?\langle (x)P\rangle) = \emptyset \\
\mathtt{bn}((\nu\ \tilde{s})s!\langle (x)P\rangle) = \tilde{s}
\end{array}$$

$$s \oplus l \asymp s\& l \qquad (\nu\ \tilde{s})s!\langle (x)P\rangle \asymp s?\langle (x)P\rangle$$

$$s!\langle (x)Q\rangle;P \xrightarrow{s!\langle (x)Q\rangle} P \qquad s?(X);P \xrightarrow{s?\langle (x)Q\rangle} P\{(x)Q/X\}$$

$$s\oplus l;P \xrightarrow{s\oplus l} P \qquad s\&\{l_i : P_i\}_{i\in I} \xrightarrow{s\& l_k} P_k \quad k\in I$$

$$\frac{P \xrightarrow{\lambda} P' \quad s\notin \mathtt{fn}(\lambda)}{(\nu\ s)P \xrightarrow{\lambda} (\nu\ s)P'} \qquad \frac{P \xrightarrow{(\nu\ \tilde{s})s!\langle (x)Q\rangle} P' \quad s'\in \mathtt{fn}((x)Q)}{(\nu\ s')P \xrightarrow{(\nu\ s'\cdot\tilde{s})s!\langle (x)Q\rangle} P'}$$

$$\frac{P \xrightarrow{\lambda} P' \quad \mathtt{bn}(\lambda)\cap\mathtt{fn}(Q)=\emptyset}{P\mid Q \xrightarrow{\lambda} P'\mid Q} \qquad \frac{Q \xrightarrow{\lambda} Q' \quad \mathtt{bn}(\lambda)\cap\mathtt{fn}(P)=\emptyset}{P\mid Q \xrightarrow{\lambda} P\mid Q'}$$

$$\frac{P \xrightarrow{\lambda_1} P' \qquad Q \xrightarrow{\lambda_2} Q'}{P\mid Q \xrightarrow{\tau} (\nu\ \mathtt{bn}(\lambda_1)\cup\mathtt{bn}(\lambda_2))(P'\mid Q')} \qquad \frac{P \equiv_\alpha P'' \quad P'' \xrightarrow{\lambda} P'}{P \xrightarrow{\lambda} P'}$$

4.2 LTS for Types

Define $(\Gamma,\Delta) \xrightarrow{\lambda} (\Gamma',\Delta')$

Define $\Gamma \vdash P \triangleright \Delta \xrightarrow{\lambda} P \triangleright \Delta'$

4.3 Bisimulation

Definition 4.1 (Barbed congruence). Let relation $\mathcal{R}$ such that $\Gamma \vdash P_1 \triangleright \Delta \mathcal{R} Q_1 \triangleright \Delta$. $\mathcal{R}$ is a barbed congruence if whenever:

- $\forall (\nu\ \tilde{s})s!\langle (x)P\rangle$ such that $\Gamma \vdash P_1 \triangleright \Delta \xrightarrow{(\nu\ \tilde{s})s!\langle (x)P\rangle} P_2 \triangleright \Delta', \exists Q_2$ such that $\Gamma \vdash Q_1 \triangleright \Delta \xrightarrow{(\nu\ \tilde{s})s!\langle (x)P\rangle} Q_2 \triangleright \Delta'$ and $\forall C, s'$ such that $\Gamma \vdash (\nu\ \tilde{s})(P_2 \mid C[P\{s'/x\}]) \triangleright \Delta''$ and $\Gamma \vdash (\nu\ \tilde{s})(Q_2 \mid C[P\{s'/x\}]) \triangleright \Delta''$ then $\Gamma \vdash (\nu\ \tilde{s})(P_2 \mid C[P\{s'/x\}]) \triangleright \Delta'' \mathcal{R} (\nu\ \tilde{s})(Q_2 \mid C[P\{s'/x\}]) \triangleright \Delta''$.
- $\forall \lambda \neq (\nu\ \tilde{s})s!\langle (x)P\rangle$ such that $\Gamma \vdash P_1 \triangleright \Delta \xrightarrow{\lambda} P_2 \triangleright \Delta', \exists Q_2$ such that $\Gamma \vdash Q_1 \triangleright \Delta \xrightarrow{\lambda} Q_2 \triangleright \Delta'$ and $\Gamma \vdash P_2 \triangleright \Delta' \mathcal{R} Q_2 \triangleright \Delta'$.
- The symmetric cases of 1 and 2.

The largest barbed congruence is denoted by $\approx^c$.

Definition 4.2 (Bisimulation). Let relation $\mathcal{R}$ such that $\Gamma \vdash P_1 \triangleright \Delta \mathcal{R} Q_1 \triangleright \Delta$. $\mathcal{R}$ is a bisimulation if whenever:

- $\forall (\nu\ \tilde{s})s!\langle (x)P\rangle$ such that $\Gamma \vdash P_1 \triangleright \Delta \xrightarrow{(\nu\ \tilde{s})s!\langle (x)P\rangle} P_2 \triangleright \Delta', \exists Q_2$ such that $\Gamma \vdash Q_1 \triangleright \Delta \xrightarrow{(\nu\ \tilde{s})s!\langle (x)P\rangle} Q_2 \triangleright \Delta'$ and s' such that $\Gamma \vdash (\nu\ \tilde{s})(P_2 \mid P\{s'/x\}) \triangleright \Delta''$ and $\Gamma \vdash (\nu\ \tilde{s})(Q_2 \mid P\{s'/x\}) \triangleright \Delta''$ then $\Gamma \vdash (\nu\ \tilde{s})(P_2 \mid P\{s'/x\}) \triangleright \Delta'' \mathcal{R} (\nu\ \tilde{s})(Q_2 \mid P\{s'/x\}) \triangleright \Delta''$.
- $\forall \lambda \neq (\nu\ \tilde{s})s!\langle (x)P\rangle$ such that $\Gamma \vdash P_1 \triangleright \Delta \xrightarrow{\lambda} P_2 \triangleright \Delta', \exists Q_2$ such that $\Gamma \vdash Q_1 \triangleright \Delta \xrightarrow{\lambda} Q_2 \triangleright \Delta'$ and $\Gamma \vdash P_2 \triangleright \Delta' \mathcal{R} Q_2 \triangleright \Delta'$.
- The symmetric cases of 1 and 2.

The largest barbed congruence is denoted by $\approx$.

Theorem 4.1. – $\approx^c$ *is a congruence.*
- $\cong$ *implies* $\approx^c$
- $\approx^c = \approx$

5 Encode the λ-calculus

$$
\begin{aligned}
[\![x]\!] &\stackrel{\texttt{def}}{=} (z)(X\langle z\rangle)\\
[\![\lambda x.M]\!]^w &\stackrel{\texttt{def}}{=} (z)z?(X);[\![M]\!]\langle w\rangle\\
[\![MN]\!] &\stackrel{\texttt{def}}{=} (z)(\nu\ s)([\![M]\!]^z\langle s\rangle \mid \overline{s}!\langle[\![N]\!]\rangle;\mathbf{0})\\
[\![M]\!]\langle z\rangle &\stackrel{\texttt{def}}{=} (\nu\ s)(s?(X);X\langle z\rangle \mid \overline{s}!\langle[\![M]\!]\rangle;\mathbf{0})\\
\lambda[M] &\stackrel{\texttt{def}}{=} [\![M]\!]\langle -\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\lambda[(\lambda x.(\lambda x.x)x)y]\\
\stackrel{\texttt{def}}{=}\ & [\![(\lambda x.(\lambda x.x)x)y]\!]\langle w\rangle\\
\stackrel{\texttt{def}}{=}\ & (\nu\ s_1)(s_1?(X);X\langle w\rangle \mid \overline{s_1}!\langle[\![(\lambda x.(\lambda x.x)x)y]\!]\rangle;\mathbf{0})\\
\stackrel{\texttt{def}}{=}\ & (\nu\ s_1)(s_1?(X);X\langle w\rangle \mid \overline{s_1}!\langle(z_1)(\nu\ s_2)([\![(\lambda x.(\lambda x.x)x]\!]^{z_1}\langle s_2\rangle \mid \overline{s_2}!\langle[\![y]\!]\rangle;\mathbf{0}))\rangle;\mathbf{0})\\
\stackrel{\texttt{def}}{=}\ & (\nu\ s_1)(s_1?(X);X\langle w\rangle \mid \overline{s_1}!\langle(z_1)(\nu\ s_2)((\nu\ s_3)(s_3?(X);X\langle s_2\rangle \mid \overline{s_3}!\langle[\![(\lambda x.(\lambda x.x)x]\!]^{z_1}\rangle;\mathbf{0}) \mid \overline{s_2}!\langle[\![y]\!]\rangle;\mathbf{0}))\rangle;\mathbf{0})\\
\stackrel{\texttt{def}}{=}\ & (\nu\ s_1)(s_1?(X);X\langle w\rangle \mid \overline{s_1}!\langle(z_1)(\nu\ s_2)((\nu\ s_3)(s_3?(X);X\langle s_2\rangle \mid \overline{s_3}!\langle(z_2)z_2?(X);[\![(\lambda x.x)x]\!]\langle z_1\rangle\rangle;\mathbf{0}) \mid \overline{s_2}!\langle[\![y]\!]\rangle;\mathbf{0}))\rangle;\mathbf{0})\\
\stackrel{\texttt{def}}{=}\ & (\nu\ s_1)(s_1?(X);X\langle w\rangle \mid \overline{s_1}!\langle(z_1)(\nu\ s_2)((\nu\ s_3)(s_3?(X);X\langle s_2\rangle \mid\\
& \overline{s_3}!\langle(z_2)z_2?(X);(\nu\ s_4)(s_4?(X);X\langle z_1\rangle \mid \overline{s_4}!\langle[\![(\lambda x.x)x]\!]\rangle;\mathbf{0})\rangle;\mathbf{0}) \mid \overline{s_2}!\langle[\![y]\!]\rangle;\mathbf{0}))\rangle;\mathbf{0})\\
\stackrel{\texttt{def}}{=}\ & (\nu\ s_1)(s_1?(X);X\langle w\rangle \mid \overline{s_1}!\langle(z_1)(\nu\ s_2)((\nu\ s_3)(s_3?(X);X\langle s_2\rangle \mid\\
& \overline{s_3}!\langle(z_2)z_2?(X);(\nu\ s_4)(s_4?(X);X\langle z_1\rangle \mid \overline{s_4}!\langle(z_3)(\nu\ s_5)([\![(\lambda x.x)]\!]^{z_3}\langle s_5\rangle \mid \overline{s_5}!\langle[\![x]\!]\rangle;\mathbf{0}))\rangle;\mathbf{0})\rangle;\mathbf{0}) \mid \overline{s_2}!\langle[\![y]\!]\rangle;\mathbf{0}))\rangle;\mathbf{0})\\
\stackrel{\texttt{def}}{=}\ & (\nu\ s_1)(s_1?(X);X\langle w\rangle \mid \overline{s_1}!\langle(z_1)(\nu\ s_2)((\nu\ s_3)(s_3?(X);X\langle s_2\rangle \mid\\
& \overline{s_3}!\langle(z_2)z_2?(X);(\nu\ s_4)(s_4?(X);X\langle z_1\rangle \mid \overline{s_4}!\langle(z_3)(\nu\ s_5)((\nu\ s_6)(s_6?(X);X\langle s_5\rangle \mid \overline{s_6}!\langle[\![(\lambda x.x)]\!]^{z_3}\rangle;\mathbf{0}) \mid\\
& \overline{s_5}!\langle[\![x]\!]\rangle;\mathbf{0}))\rangle;\mathbf{0})\rangle;\mathbf{0}) \mid \overline{s_2}!\langle[\![y]\!]\rangle;\mathbf{0}))\rangle;\mathbf{0})\\
\stackrel{\texttt{def}}{=}\ & (\nu\ s_1)(s_1?(X);X\langle w\rangle \mid \overline{s_1}!\langle(z_1)(\nu\ s_2)((\nu\ s_3)(s_3?(X);X\langle s_2\rangle \mid\\
& \overline{s_3}!\langle(z_2)z_2?(X);(\nu\ s_4)(s_4?(X);X\langle z_1\rangle \mid \overline{s_4}!\langle(z_3)(\nu\ s_5)((\nu\ s_6)(s_6?(X);X\langle s_5\rangle \mid\\
& \overline{s_6}!\langle(z_4)z_4?(X);[\![x]\!]\langle z_3\rangle\rangle;\mathbf{0}) \mid\\
& \overline{s_5}!\langle[\![x]\!]\rangle;\mathbf{0}))\rangle;\mathbf{0})\rangle;\mathbf{0}) \mid \overline{s_2}!\langle[\![y]\!]\rangle;\mathbf{0}))\rangle;\mathbf{0})\\
\stackrel{\texttt{def}}{=}\ & (\nu\ s_1)(s_1?(X);X\langle w\rangle \mid \overline{s_1}!\langle(z_1)(\nu\ s_2)((\nu\ s_3)(s_3?(X);X\langle s_2\rangle \mid\\
& \overline{s_3}!\langle(z_2)z_2?(X);(\nu\ s_4)(s_4?(X);X\langle z_1\rangle \mid \overline{s_4}!\langle(z_3)(\nu\ s_5)((\nu\ s_6)(s_6?(X);X\langle s_5\rangle \mid\\
& \overline{s_6}!\langle(z_4)z_4?(X);(\nu\ s_7)(s_7?(X);X\langle z_3\rangle \mid \overline{s_7}!\langle[\![x]\!]\rangle;\mathbf{0})\rangle;\mathbf{0}) \mid\\
& \overline{s_5}!\langle[\![x]\!]\rangle;\mathbf{0}))\rangle;\mathbf{0})\rangle;\mathbf{0}) \mid \overline{s_2}!\langle[\![y]\!]\rangle;\mathbf{0}))\rangle;\mathbf{0})\\
\stackrel{\texttt{def}}{=}\ & (\nu\ s_1)(s_1?(X);X\langle w\rangle \mid \overline{s_1}!\langle(z_1)(\nu\ s_2)((\nu\ s_3)(s_3?(X);X\langle s_2\rangle \mid\\
& \overline{s_3}!\langle(z_2)z_2?(X);(\nu\ s_4)(s_4?(X);X\langle z_1\rangle \mid \overline{s_4}!\langle(z_3)(\nu\ s_5)((\nu\ s_6)(s_6?(X);X\langle s_5\rangle \mid\\
& \overline{s_6}!\langle(z_4)z_4?(X);(\nu\ s_7)(s_7?(X);X\langle z_3\rangle \mid \overline{s_7}!\langle(z_5)(X\langle z_5\rangle)\rangle;\mathbf{0})\rangle;\mathbf{0}) \mid\\
& \overline{s_5}!\langle(z_6)(X\langle z_6\rangle)\rangle;\mathbf{0}))\rangle;\mathbf{0})\rangle;\mathbf{0}) \mid \overline{s_2}!\langle(z_7)(Y\langle z_7\rangle)\rangle;\mathbf{0}))\rangle;\mathbf{0})
\end{aligned}
$$

$$
\begin{aligned}
& (\nu\ s_1)(s_1?(X);X\langle w\rangle \mid \overline{s_1}!\langle (z_1)(\nu\ s_2)((\nu\ s_3)(s_3?(X);X\langle s_2\rangle \mid \\
& \overline{s_3}!\langle (z_2)z_2?(X);(\nu\ s_4)(s_4?(X);X\langle z_1\rangle \mid \overline{s_4}!\langle (z_3)(\nu\ s_5)((\nu\ s_6)(s_6?(X);X\langle s_5\rangle \mid \\
& \overline{s_6}!\langle (z_4)z_4?(X);(\nu\ s_7)(s_7?(X);X\langle z_3\rangle \mid \overline{s_7}!\langle (z_5)(X\langle z_5\rangle)\rangle;\mathbf{0})\rangle;\mathbf{0}) \mid \\
& \overline{s_5}!\langle (z_6)(X\langle z_6\rangle)\rangle;\mathbf{0})\rangle;\mathbf{0})\rangle;\mathbf{0}) \mid \overline{s_2}!\langle (z_7)(Y\langle z_7\rangle)\rangle;\mathbf{0})\rangle;\mathbf{0}) \\
\longrightarrow\ & (\nu\ s_2)((\nu\ s_3)(s_3?(X);X\langle s_2\rangle \mid \\
& \overline{s_3}!\langle (z_2)z_2?(X);(\nu\ s_4)(s_4?(X);X\langle w\rangle \mid \overline{s_4}!\langle (z_3)(\nu\ s_5)((\nu\ s_6)(s_6?(X);X\langle s_5\rangle \mid \\
& \overline{s_6}!\langle (z_4)z_4?(X);(\nu\ s_7)(s_7?(X);X\langle z_3\rangle \mid \overline{s_7}!\langle (z_5)(X\langle z_5\rangle)\rangle;\mathbf{0})\rangle;\mathbf{0}) \mid \\
& \overline{s_5}!\langle (z_6)(X\langle z_6\rangle)\rangle;\mathbf{0})\rangle;\mathbf{0})\rangle;\mathbf{0}) \mid \overline{s_2}!\langle (z_7)(Y\langle z_7\rangle)\rangle;\mathbf{0}) \\
\longrightarrow\ & (\nu\ s_4)(s_4?(X);X\langle w\rangle \mid \overline{s_4}!\langle (z_3)(\nu\ s_5)((\nu\ s_6)(s_6?(X);X\langle s_5\rangle \mid \\
& \overline{s_6}!\langle (z_4)z_4?(X);(\nu\ s_7)(s_7?(X);X\langle z_3\rangle \mid \overline{s_7}!\langle (z_5)(X\langle z_5\rangle)\rangle;\mathbf{0})\rangle;\mathbf{0}) \mid \\
& \overline{s_5}!\langle (z_6)(Y\langle z_6\rangle)\rangle;\mathbf{0})\rangle;\mathbf{0}) \\
\longrightarrow\ & (\nu\ s_5)((\nu\ s_6)(s_6?(X);X\langle s_5\rangle \mid \\
& \overline{s_6}!\langle (z_4)z_4?(X);(\nu\ s_7)(s_7?(X);X\langle w\rangle \mid \overline{s_7}!\langle (z_5)(X\langle z_5\rangle)\rangle;\mathbf{0})\rangle;\mathbf{0}) \mid \\
& \overline{s_5}!\langle (z_6)(Y\langle z_6\rangle)\rangle;\mathbf{0}) \\
\longrightarrow\ & (\nu\ s_5)(s_5?(X);(\nu\ s_7)(s_7?(X);X\langle w\rangle \mid \overline{s_7}!\langle (z_5)(X\langle z_5\rangle)\rangle;\mathbf{0}) \mid \\
& \overline{s_5}!\langle (z_6)(Y\langle z_6\rangle)\rangle;\mathbf{0}) \\
\longrightarrow\ & (\nu\ s_7)(s_7?(X);X\langle w\rangle \mid \overline{s_7}!\langle (z_5)(Y\langle z_5\rangle)\rangle;\mathbf{0}) \\
\longrightarrow\ & Y\langle w\rangle
\end{aligned}
$$