1. Linear expression of HES

$$\Delta Q + \Delta t A \left(\frac{\partial Q}{\partial x} \right) + \Delta t B \left(\frac{\partial Q}{\partial y} \right) = 0 \tag{1}$$

$$Q = (\phi, u, v)^T, \qquad A = \begin{pmatrix} & 1 & \\ 1 & & \\ & & \end{pmatrix}, \qquad B = \begin{pmatrix} & & 1 \\ & & \\ 1 & & \end{pmatrix}. \tag{2}$$

1.1 First-order discretization

$$\Delta t \left(\frac{\partial E}{\partial x} \right) = \begin{pmatrix} \sum_{i-1}^{i+1} (a_k \phi_k + b_k u_k) \\ \sum_{i-1}^{i+1} (c_k \phi_k + d_k u_k) \\ 0 \end{pmatrix}, \qquad \Delta t \left(\frac{\partial F}{\partial x} \right) = \begin{pmatrix} \sum_{j-1}^{j+1} (e_k \phi_k + f_k v_k) \\ 0 \\ \sum_{j-1}^{j+1} (g_k \phi_k + h_k v_k) \end{pmatrix}. \tag{3}$$

$$a_{i-1} = -\frac{\Delta t}{2\Delta x} \qquad a_i = \frac{\Delta t}{\Delta x} \qquad a_{i+1} = -\frac{\Delta t}{2\Delta x} \tag{4}$$

$$b_{i-1} = -\frac{\Delta t}{2\Delta x} \qquad b_i = 0 \qquad b_{i+1} = \frac{\Delta t}{2\Delta x} \tag{5}$$

$$c_{i-1} = -\frac{\Delta t}{2\Delta x} \qquad c_i = 0 \qquad c_{i+1} = \frac{\Delta t}{2\Delta x} \tag{6}$$

$$d_{i-1} = -\frac{\Delta t}{2\Delta x} \qquad d_i = \frac{\Delta t}{\Delta x} \qquad d_{i+1} = -\frac{\Delta t}{2\Delta x} \tag{7}$$

$$e_{j-1} = -\frac{\Delta t}{2\Delta y} \qquad e_j = \frac{\Delta t}{\Delta y} \qquad e_{j+1} = -\frac{\Delta t}{2\Delta y} \tag{8}$$

$$f_{j-1} = -\frac{\Delta t}{2\Delta y} \qquad f_j = 0 \qquad f_{j+1} = \frac{\Delta t}{2\Delta y} \tag{9}$$

$$g_{j-1} = -\frac{\Delta t}{2\Delta y} \qquad g_j = 0 \qquad g_{j+1} = \frac{\Delta t}{2\Delta y} \tag{10}$$

$$h_{j-1} = -\frac{\Delta t}{2\Delta y} \qquad h_j = \frac{\Delta t}{\Delta y} \qquad h_{j+1} = -\frac{\Delta t}{2\Delta y} \tag{11}$$

1.2 Second-order discretization

$$\Delta t\left(\frac{\partial E}{\partial x}\right)=\begin{pmatrix}\sum_{i-2}^{i+2}\left(a_k\phi_k+b_ku_k\right)\\ \sum_{i-2}^{i+2}\left(c_k\phi_k+d_ku_k\right)\\ 0\end{pmatrix},\qquad \Delta t\left(\frac{\partial F}{\partial x}\right)=\begin{pmatrix}\sum_{j-2}^{j+2}\left(e_k\phi_k+f_kv_k\right)\\ 0\\ \sum_{j-2}^{j+2}\left(g_k\phi_k+h_kv_k\right)\end{pmatrix}. \tag{12}$$

$$\begin{aligned}
a_{i-2} &= \frac{\Delta t}{4\Delta x}s_{\phi1,i-1}\\
a_{i-1} &= -\frac{\Delta t}{4\Delta x}\left(2+s_{\phi1,i-1}-s_{\phi2,i-1}\right)\\
a_{i} &= \frac{\Delta t}{4\Delta x}\left(4-s_{\phi1,i+1}-s_{\phi2,i-1}\right)\\
a_{i+1} &= -\frac{\Delta t}{4\Delta x}\left(2-s_{\phi1,i+1}+s_{\phi2,i+1}\right)\\
a_{i+2} &= \frac{\Delta t}{4\Delta x}s_{\phi2,i+1}
\end{aligned} \tag{13}$$

$$\tag{14}$$

$$\begin{aligned}
b_{i-2} &= \frac{\Delta t}{4\Delta x}s_{u1,i-1}\\
b_{i-1} &= -\frac{\Delta t}{4\Delta x}\left(2+s_{u1,i-1}-s_{u2,i-1}+2s_{u1,i}\right)\\
b_{i} &= \frac{\Delta t}{4\Delta x}\left(-s_{u2,i-1}+2s_{u1,i}-2s_{u2,i}+s_{u1,i+1}\right)\\
b_{i+1} &= \frac{\Delta t}{4\Delta x}\left(2-s_{u1,i+1}+s_{u2,i+1}+2s_{u2,i}\right)\\
b_{i+2} &= -\frac{\Delta t}{4\Delta x}s_{u2,i+1}
\end{aligned} \tag{15}$$

$$
\begin{aligned}
c_{i-2} &= \frac{\Delta t}{4\Delta x} s_{\phi 1,i-1} \\
c_{i-1} &= -\frac{\Delta t}{4\Delta x} \left(2 + s_{\phi 1,i-1} - s_{\phi 2,i-1} + 2 s_{\phi 1,i}\right) \\
c_{i} &= \frac{\Delta t}{4\Delta x} \left(-s_{\phi 2,i-1} + 2 s_{\phi 1,i} - 2 s_{\phi 2,i} + s_{\phi 1,i+1}\right) \\
c_{i+1} &= \frac{\Delta t}{4\Delta x} \left(2 - s_{\phi 1,i+1} + s_{\phi 2,i+1} + 2 s_{\phi 2,i}\right) \\
c_{i+2} &= -\frac{\Delta t}{4\Delta x} s_{\phi 2,i+1}
\end{aligned}
\tag{16}
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\tag{17}
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\begin{aligned}
d_{i-2} &= \frac{\Delta t}{4\Delta x} s_{u1,i-1} \\
d_{i-1} &= -\frac{\Delta t}{4\Delta x} \left(2 + s_{u1,i-1} - s_{u2,i-1}\right) \\
d_{i} &= \frac{\Delta t}{4\Delta x} \left(4 - s_{u1,i+1} - s_{u2,i-1}\right) \\
d_{i+1} &= -\frac{\Delta t}{4\Delta x} \left(2 - s_{u1,i+1} + s_{u2,i+1}\right) \\
d_{i+2} &= \frac{\Delta t}{4\Delta x} s_{u2,i+1}
\end{aligned}
\tag{18}
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\tag{19}
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\begin{aligned}
e_{j-2} &= \frac{\Delta t}{4\Delta y} t_{\phi 1,j-1} \\
e_{j-1} &= -\frac{\Delta t}{4\Delta y} \left(2 + t_{\phi 1,j-1} - t_{\phi 2,j-1}\right) \\
e_{j} &= \frac{\Delta t}{4\Delta y} \left(4 - t_{\phi 1,j+1} - t_{\phi 2,j-1}\right) \\
e_{j+1} &= -\frac{\Delta t}{4\Delta y} \left(2 - t_{\phi 1,j+1} + t_{\phi 2,j+1}\right) \\
e_{j+2} &= \frac{\Delta t}{4\Delta y} t_{\phi 2,j+1}
\end{aligned}
\tag{20}
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\tag{21}
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\begin{aligned}
f_{j-2} &= \frac{\Delta t}{4\Delta y} t_{v1,j-1} \\
f_{j-1} &= -\frac{\Delta t}{4\Delta y} \left(2 + t_{v1,j-1} - t_{v2,j-1} + 2t_{v1,j}\right) \\
f_{j} &= \frac{\Delta t}{4\Delta y} \left(-t_{v2,j-1} + 2t_{v1,j} - 2t_{v2,j} + t_{v1,j+1}\right) \\
f_{j+1} &= \frac{\Delta t}{4\Delta y} \left(2 - t_{v1,j+1} + t_{v2,j+1} + 2t_{v2,j}\right) \\
f_{j+2} &= -\frac{\Delta t}{4\Delta y} t_{v2,j+1}
\end{aligned}
\tag{22}
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\tag{23}
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\begin{aligned}
g_{j-2} &= \frac{\Delta t}{4\Delta y} t_{\phi 1,j-1} \\
g_{j-1} &= -\frac{\Delta t}{4\Delta y} \left(2 + t_{\phi 1,j-1} - t_{\phi 2,j-1} + 2t_{\phi 1,j}\right) \\
g_{j} &= \frac{\Delta t}{4\Delta y} \left(-t_{\phi 2,j-1} + 2t_{\phi 1,j} - 2t_{\phi 2,j} + t_{\phi 1,j+1}\right) \\
g_{j+1} &= \frac{\Delta t}{4\Delta y} \left(2 - t_{\phi 1,j+1} + t_{\phi 2,j+1} + 2t_{\phi 2,j}\right) \\
g_{j+2} &= -\frac{\Delta t}{4\Delta y} t_{\phi 2,j+1}
\end{aligned}
\tag{24}
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\tag{25}
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\begin{aligned}
h_{j-2} &= \frac{\Delta t}{4\Delta y} t_{v1,j-1} \\
h_{j-1} &= -\frac{\Delta t}{4\Delta y} \left(2 + t_{v1,j-1} - t_{v2,j-1}\right) \\
h_{j} &= \frac{\Delta t}{4\Delta y} \left(4 - t_{v1,j+1} - t_{v2,j-1}\right) \\
h_{j+1} &= -\frac{\Delta t}{4\Delta y} \left(2 - t_{v1,j+1} + t_{v2,j+1}\right) \\
h_{j+2} &= \frac{\Delta t}{4\Delta y} t_{v2,j+1}
\end{aligned}
\tag{26}
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