Assignment-2

1) Find the absolute minimum value of the function $f(x,y) = x^2+y^2$ subject to the constraint $x^2+2y^2=1$.

Ans → $f(x,y) = x^2+y^2$ $\qquad g(x,y) = x^2+2y^2=1$

$\Rightarrow x^2+2y^2-1=0$

$L(x,y,\lambda) = x^2+y^2-\lambda(x^2+2y^2-1)$

$\frac{dL}{dx} = 2x-2\lambda x=0 \Rightarrow x(1-\lambda)=0$ So, $x=0, \lambda=1$ – eqn (i)

$\frac{dL}{dy} = 2y-4\lambda y=0 \Rightarrow y(1-2\lambda)=0$ so, $y=0, \lambda=1/2$ – eqn (ii)

$\frac{dL}{d\lambda} = x^2+2y^2-1=0$ → eqn (iii)

for eqn (iii),

Case a → $x=0$

$2y^2-1=0$

$\Rightarrow y^2 = 1/2$

$\Rightarrow y=\pm 1/\sqrt{2}$

Case b → $y=0$

$x^2-1=0$

$\Rightarrow x^2=1$

$\Rightarrow x=\pm 1$

Now applying x & y value to $f(x,y)$ →

$f(0,\pm 1/\sqrt{2}) = x^2+y^2$

$=(0)^2+(1/\sqrt{2})^2$

$= 1/2$

$f(1,0) = x^2+y^2$

$=(1)^2+(0)^2$

$=1$

∴ The absolute minimum value at points $(0,\pm 1/\sqrt{2})$ is $1/2$.

2) The firm manufactures a commodity at too different factories. The total cost of manufacturing depends on the quantities, p & q supplied by each factor, ...

Ans → Cost function, $C(p,q) = 2pq^2+pq+22q+500$

Constraint $= p+q=200 \Rightarrow p=200-q$

& $L(p,q,\lambda) = 2pq^2+pq+22q+500-\lambda(p+q-200)$

$$\frac{dL}{dp} = 2q^2 + q - \lambda = 0 \quad (i)$$

$$\frac{dL}{dq} = 4pq + p + 22 - \lambda = 0 \quad (ii)$$

$$\frac{dL}{d\lambda} = -(p+q-200) = 0 \Rightarrow p+q = 200 \quad (iii)$$

Equating λ in eqn (i) & (ii), we get

$$2q^2 + q = 4pq + p + 22 \quad (iv)$$

Substitute constraint to eqn (iv) →

$$2q^2 + q = 4(200-q)q + (200-q) + 22 \quad (\because p = 200-q)$$

$$\Rightarrow 2q^2 + q = 800q - 4q^2 + 200 - q + 22$$

$$\Rightarrow 6q^2 - 798q - 222 = 0$$

$$\Rightarrow q^2 - 133q - 37 = 0$$

By quadratic formula,

$$q = \frac{133 \pm \sqrt{133^2 + 4 \cdot 37}}{2}$$

$$= \frac{133 \pm \sqrt{17837}}{2}$$

$$= \frac{133 + 133.5}{2} \text{ or } \frac{133 - 133.5}{2}$$

$$= \frac{266.5}{2} \text{ or } \frac{-0.5}{2}$$

$$= 133.25 \text{ or } -0.25 \quad \text{(only +ve to be taken)}$$

Then, $p = 200 - q$

$$= 200 - 133.25 = 66.75$$

So, $p = 66.75$ & $q = 133.25$ are used to minimize the production cost subject to the total production constraint.

3) Find the maximum possible weight of the minimum spanning tree of a graph with four vertices and six edges having weights 6, 7, 4, 8, 11, 9.

Ans → MST of a graph with n vertices has exactly n-1 edges

So, here there will be 3 edges.

For maximum possible weight of the MST, we must choose the heaviest possible combination of 3 edges.

From the weights given (6, 7, 4, 8, 11, 9), heaviest edges → 11, 9, 8

So, the total weight of MST = 11+9+8 = $\underline{\underline{28}}$.

4) A manufacturer produces two types of products x & y. The profit is given by $2x+3y$. What will be the maximum profits if the constraints are $x+3y \leq 40$, $3x+y \leq 24$, $x+y \leq 10$, $x > 0$, $y > 0$. Use graphical method.

Ans → Maximise $P = 2x+3y$

Constraints → $x+3y \leq 40$

$3x+y \leq 24$

$x+y \leq 10$

$x, y > 0$

Intercepts for each line →

$x+3y = 40$ ①

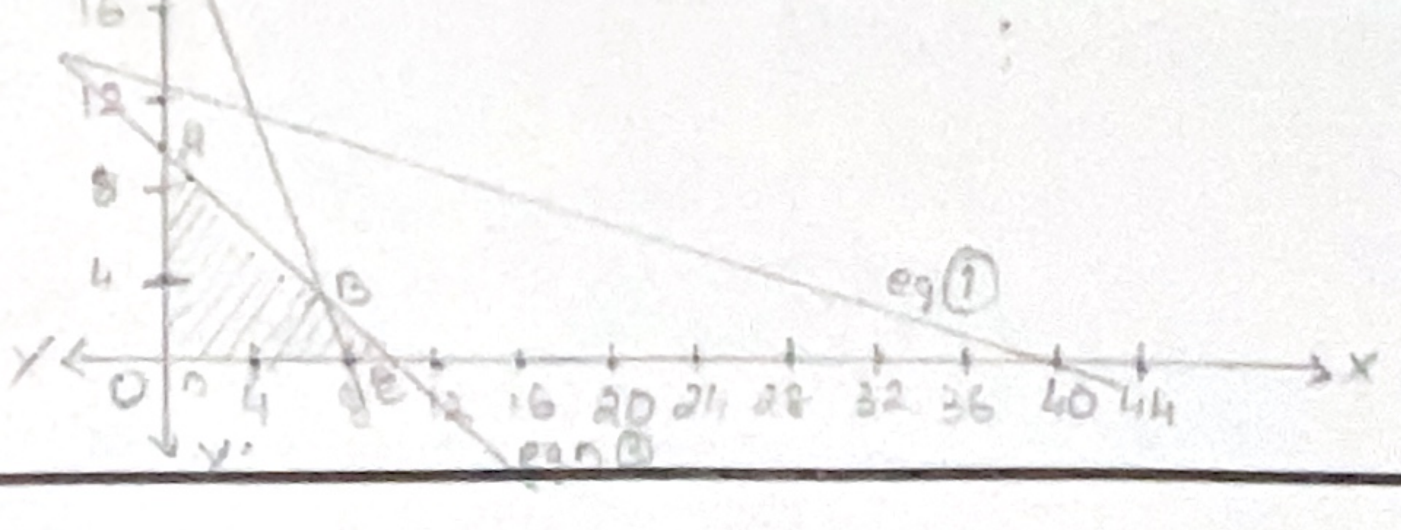
x	0	40
y	$\frac{40}{3} = 13.33$	0

$3x+y = 24$ ②

x	0	8
y	24	0

$x+y = 10$ ③

x	0	10
y	10	0



Feasible region from the graph,

A(0,10), B(from intersection of eqn ② & ③),

C(8,0), D(0,0)

Solving for B →

$3x+y=24$ & $x+y=10$

$3x+y-(x+y)=24-10$

$\Rightarrow 3x-x+y-y=14$

$\Rightarrow 2x=14$

$\Rightarrow x=7$

$\Rightarrow y=3$ (from equating $x+y=10$)

So, B is (7,3).

Now, evaluating profit at each corner point.

$P(0,10) = 2(0)+3(10)=30$

$P(7,3) = 2(7)+3(3)=23$

$P(8,0) = 2(8)+3(0)=16$

∴ Maximum profit is 30 at ($x=0, y=10$).

5) Use graphical method to solve the following LPP. Max $Z=3x+2y$. Subject to $5x+y\geq 10$, $x+y\geq 6$, $x+4xy\geq 12$, $x,y\geq 0$.

Ans → Max $Z=3x+2y$

Constraints → $5x+y\geq 10$

$x+y\geq 6$

$x+4xy\geq 12$

$x,y\geq 0$

Here, the third constant ($x+4xy\geq 12$) is non linear. So the given problem can't be solved using LPP through graphical method.

6) Use the graphical method to solve the following LPP. Max $Z=x+y$ subject to $x+y\leq 10$, $x+y\geq 3$, $x,y\geq 0$.

Ans → Max $Z=x+y$

Constraints → $x+y\leq 10$, $x+y\geq 3$, $x,y\geq 0$

Regd No: ____________

Name: ____________

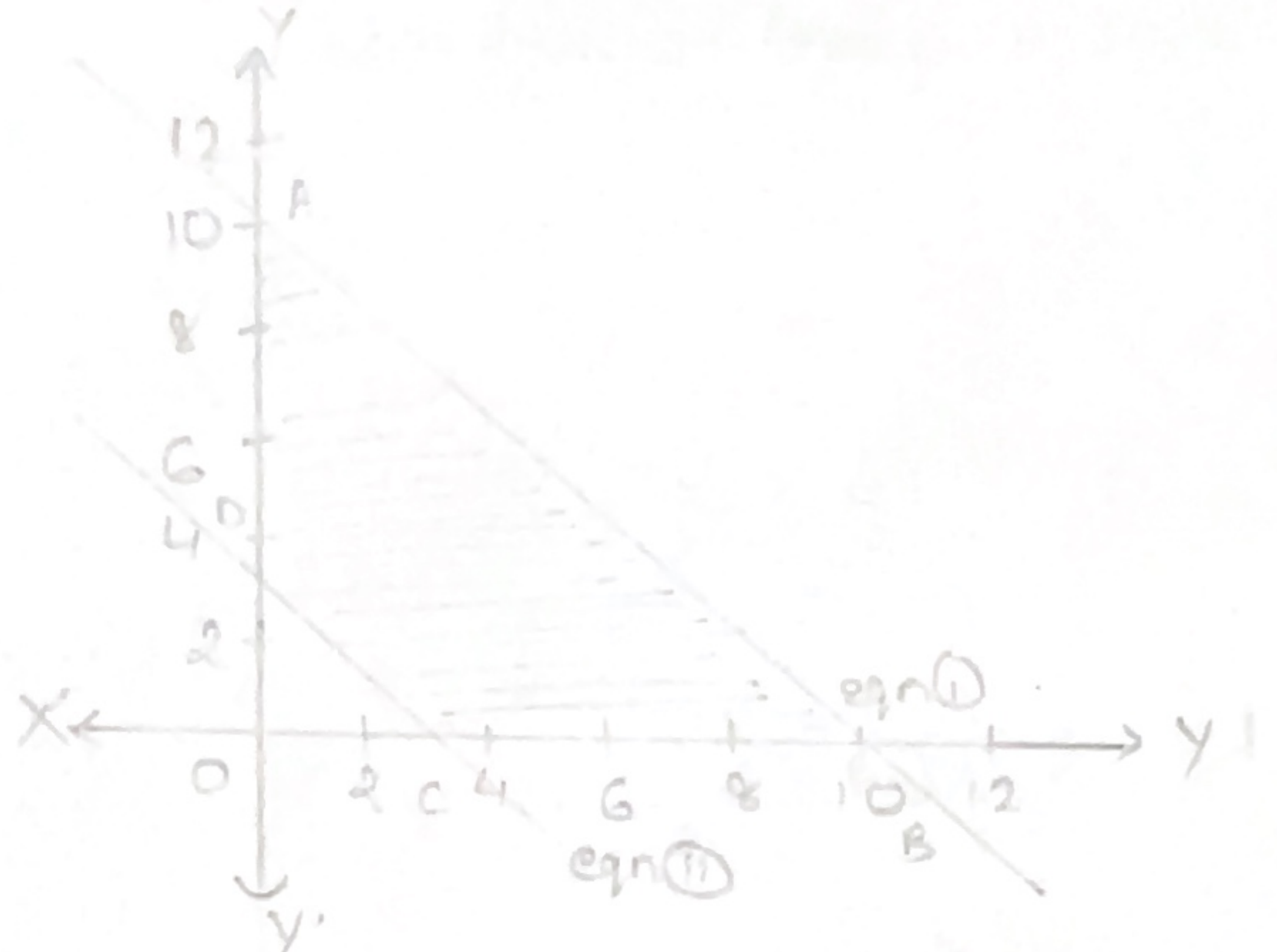
Intercepts for each line →

$x+y=10$ (i)

x	0	10
y	10	0

$x+y=3$ (ii)

x	0	3
y	3	0



Feasible corner points →

$A(0,10)$, $B(10,0)$, $C(3,0)$, $D(0,3)$

Maximizing at each point →

$Z(0,10) = 0+10 = 10$

$Z(10,0) = 10+0 = 10$

$Z(3,0) = 3+0 = 3$

$Z(0,3) = 0+3 = 3$

∴ Maximum $Z=10$ at either $(0,10)$ or $(10,0)$.

7) Use simplex methods to solve the following LPP. Max $Z = 3x+2y$. Subject to $x+3y \leq 40$, $3x+y \leq 24$, $x+y \leq 10$, $x>0$, $y>0$

Ans → Max $Z = 3x+2y$

Constraints → $x+3y \leq 40$, $3x+y \leq 24$, $x+y \leq 10$, $x>0, y>0$

Add slack variables → $Z = 3x+2y+0S_1+0S_2+0S_3$

$x+3y+S_1=40$

$3x+y+S_2=24$

$x+y+S_3=10$

Initial table →

	C_j	3	2	0	0	0		
C_{Bi}	Bv	x (key column)	y	S_1	S_2	S_3	Sol^n	Ratio
0	S_1	1	3	1	0	0	40	$40/1 = 40$
0	S_2	$\boxed{3}$	1	0	1	0	24	$24/3 = 8$ → min
0	S_3	1 (key elem)	1	0	0	1	10	$10/1 = 10$
	Z_j	0	0	0	0	0		
	$C_j - Z_j$	$\boxed{3}$ → max	2	0	0	0		

For above table, key element = 3

For Bv, $S_2^{(x)}$ (new val) = $\frac{\text{~~updated~~ old value}}{\text{key elem}}$ $\quad \frac{3}{3}=1,\ \frac{1}{3},\ 0,\ \frac{1}{3},\ 0$

S_1 = new value = old val − $\frac{\text{corresponding keycol} \times \text{corresponding keyrow}}{\text{key elem}}$

$1-\frac{1\times3}{3}=0,\quad 3-\frac{1\times1}{3}=\frac{8}{3},\quad 1-\frac{1\times0}{3}=1,\quad 0-\frac{1\times1}{3}=-\frac{1}{3},\quad 0-\frac{1\times0}{3}=0$

$S_2 \rightarrow 1-\frac{1\times3}{3}=0,\quad 1-\frac{1\times1}{3}=\frac{2}{3},\quad 0-\frac{1\times0}{3}=0,\quad 0-\frac{1\times1}{3}=-\frac{1}{3},\quad 1-\frac{1\times0}{3}=1$

Iteration 1 table →

	C_j	3	2	0	0	0		
C_{Bi}	Bv	x	y	S_1	S_2	S_3	Soln	Ratio
0	S_1	0	8/3	1	−1/3	0	$40-\frac{1\times24}{3}=32$	$32/\frac{8}{3}=12$
3	x	1	1/3	0	1/3	0	$24/3=8$	$8/\frac{1}{3}=24$
0	S_3	0	2/3	0	−1/3	1	$10-\frac{1\times24}{3}=2$	$2/\frac{2}{3}=3$ → min
	Z_j	3	1	0	1	0		
	C_j-Z_j	0	1 → max	0	−1	0		

key row, key column, key elem

For next table updated value,

(y) $S_3 \Rightarrow 0/\frac{2}{3}=0,\quad \frac{2/3}{2/3}=1,\quad 0,\quad \frac{-1/3}{2/3}=-\frac{1}{2},\quad \frac{1}{2/3}=\frac{3}{2}$

$S_1 \rightarrow 0-\frac{0\times 8/3}{2/3}=0,\quad \frac{8}{3}-\frac{2/3\times 8/3}{2/3}=0,\quad 1-\frac{0\times 8/3}{2/3}=1,\quad -\frac{1}{3}-\frac{(-1/3)\times 8/3}{2/3}=1,\quad 0-\frac{1\times 8/3}{2/3}=-4$

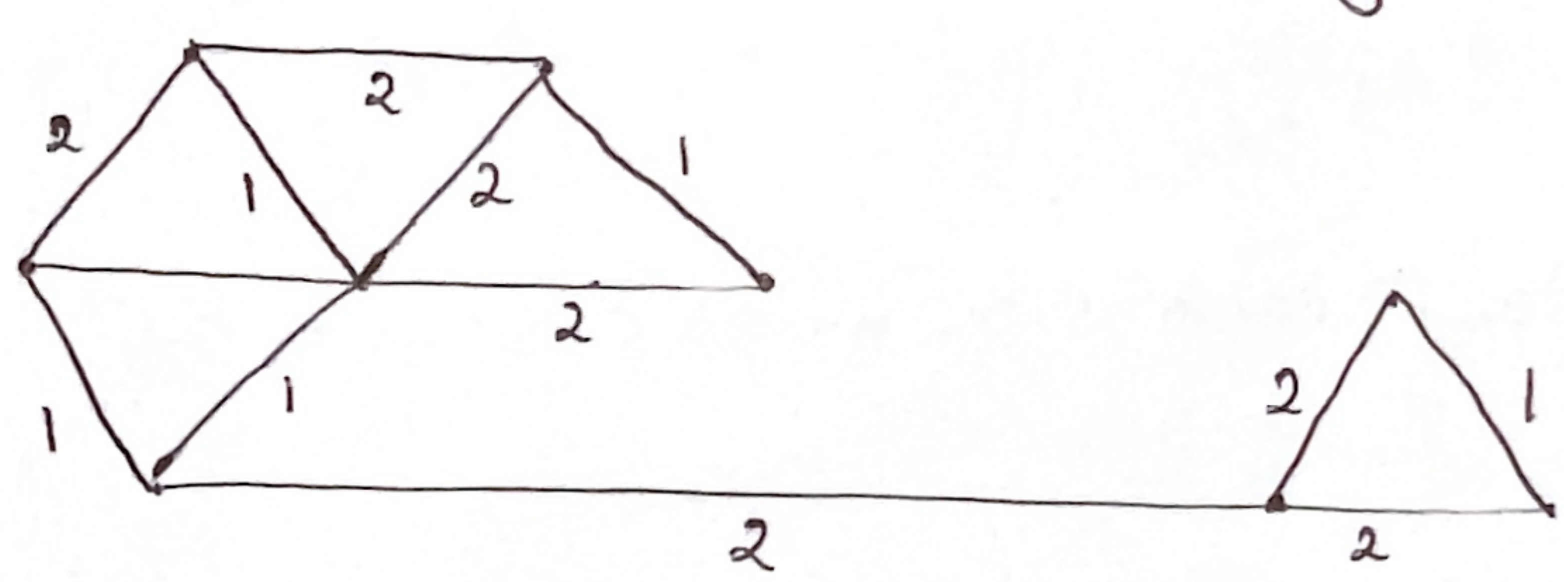
$x \rightarrow 1-\frac{0\times 1/3}{2/3}=1,\quad \frac{1}{3}-\frac{2/3\times 1/3}{2/3}=0,\quad 0-\frac{0\times 1/3}{2/3}=0,\quad \frac{1}{3}-\frac{1/3\times(-1/3)}{2/3}=\frac{1}{2},\quad 0-\frac{1\times 1/3}{2/3}=-\frac{1}{2}$

	C_j	3	2	0	0	0	
C_{B_i}	B_v	x	y	S_1	S_2	S_3	Sol
0	S_1	0	0	1	1	-4	24
3	x	1	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	7
2	y	0	1	0	$-\frac{1}{2}$	$\frac{3}{2}$	3
	Z_j	3	2	0	$\frac{5}{2}$	$\frac{3}{2}$	27
	$C_j - Z_j$	0	0	0	$-\frac{5}{2}$	$-\frac{3}{2}$	

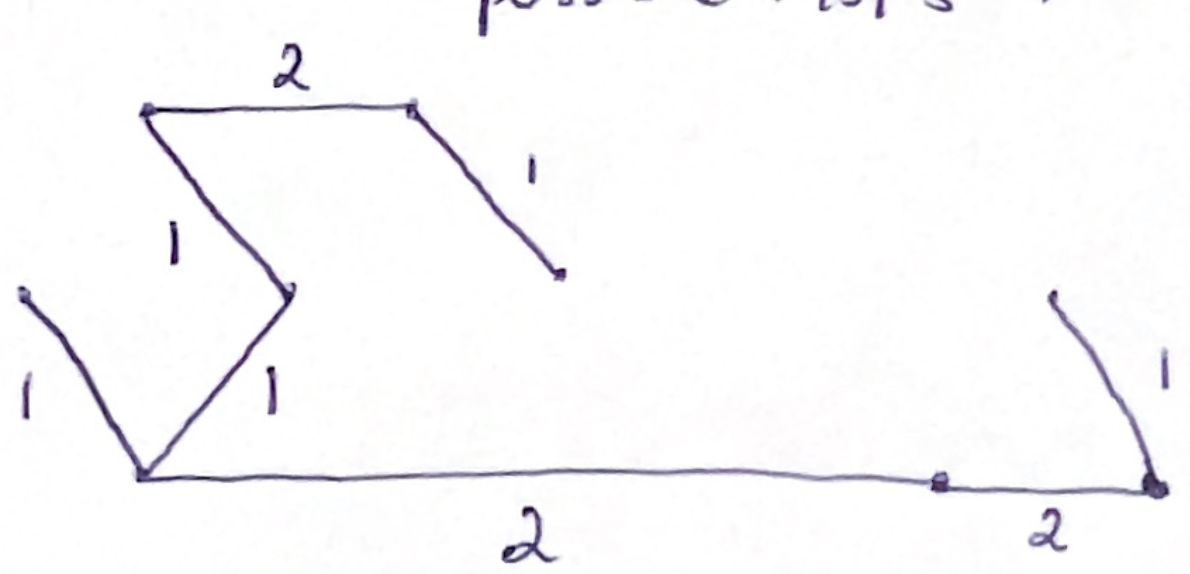
We have reached optimal solution for $C_j - Z_j$.

So max $Z_j = 3x + 2y = \underline{\underline{27}}$.

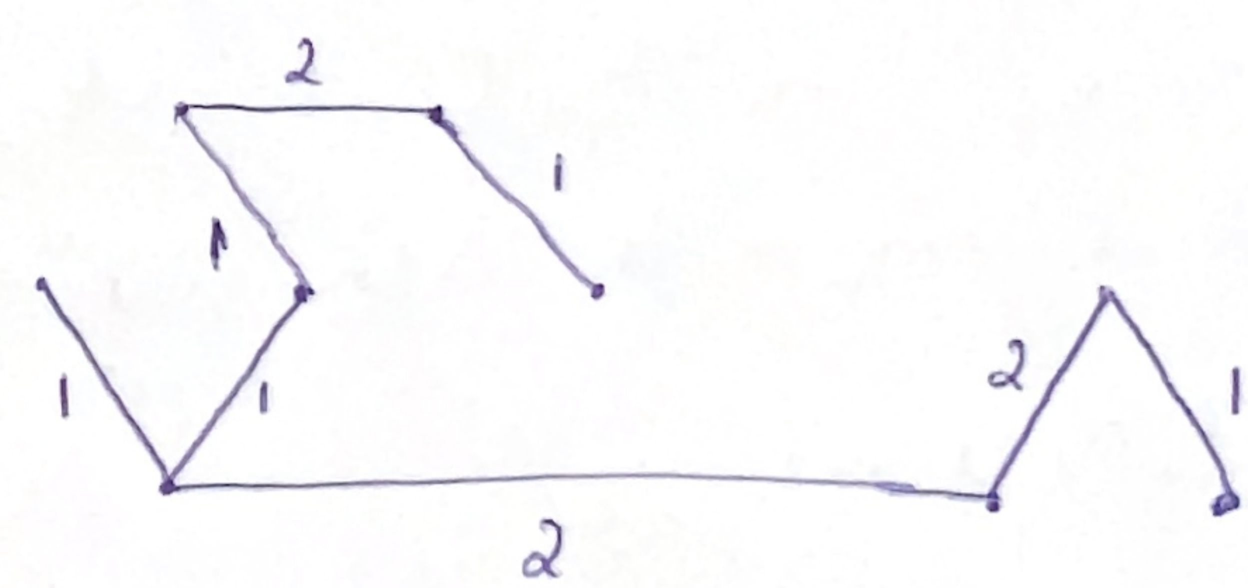
8) The number of distinct minimum spanning trees for the weighted graph below is $\underline{6}$.



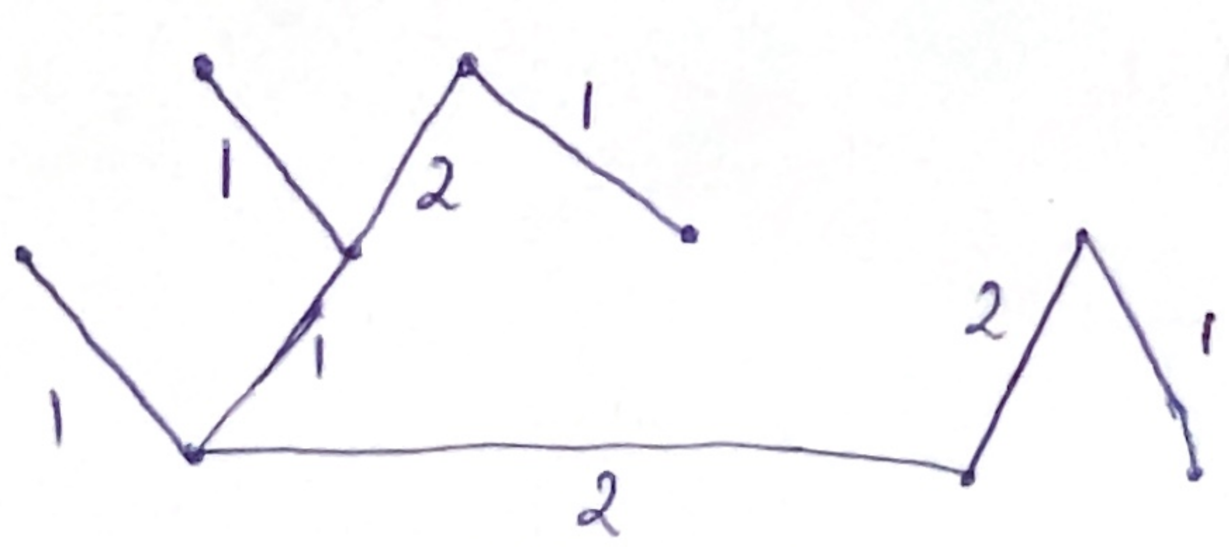
Ans → All the possible MST's →



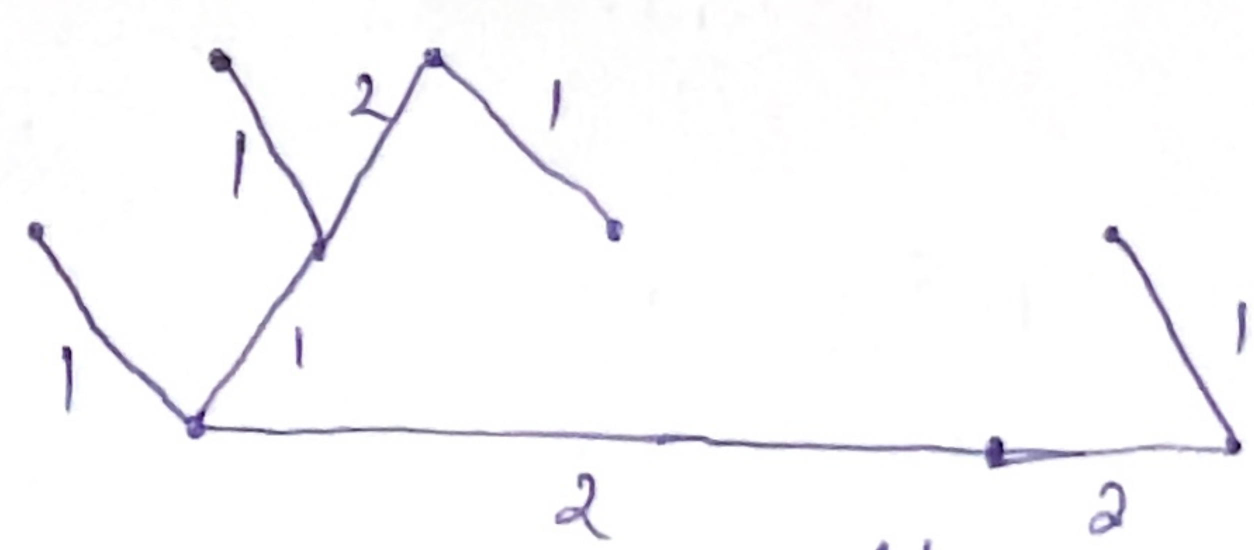
Weight = 11



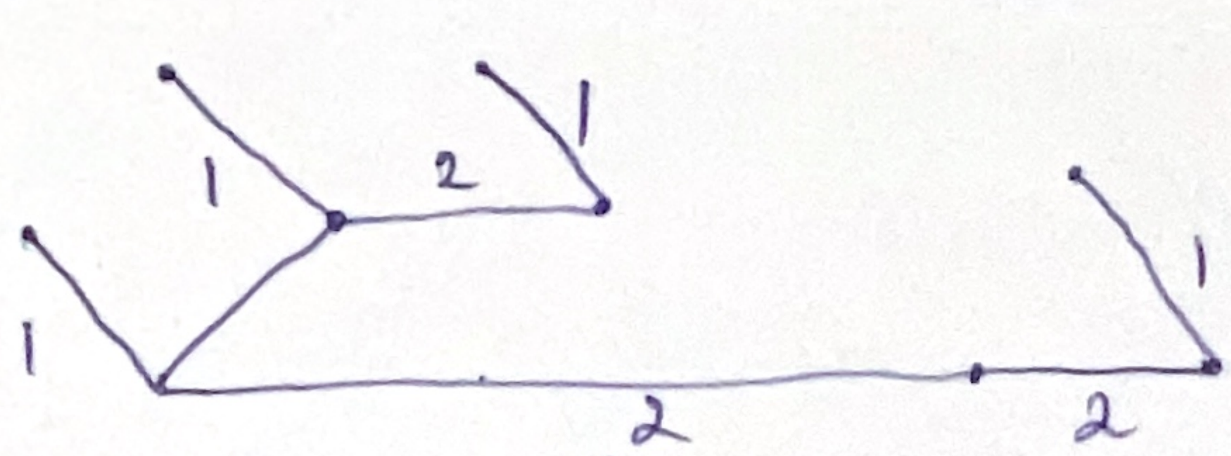
Weight = 11



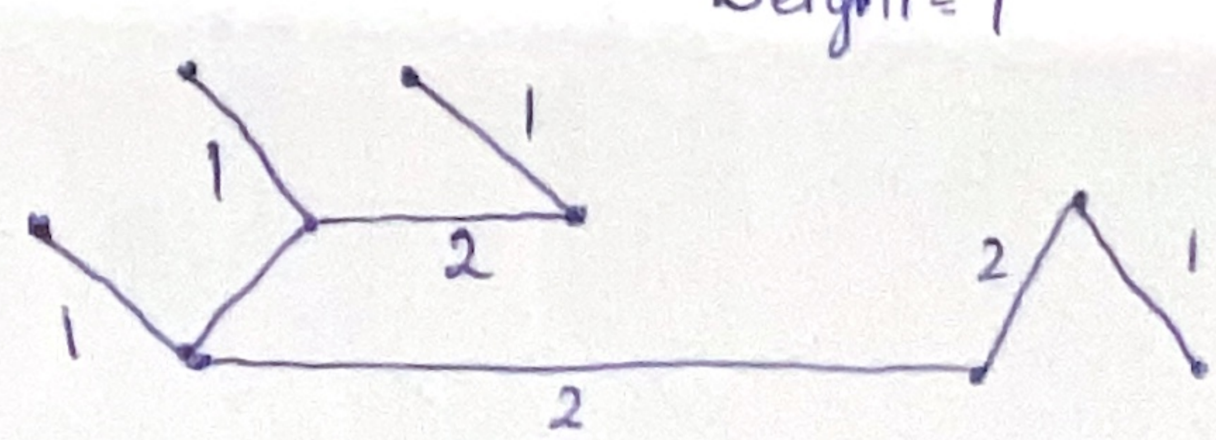
Weight = 11



Weight = 1

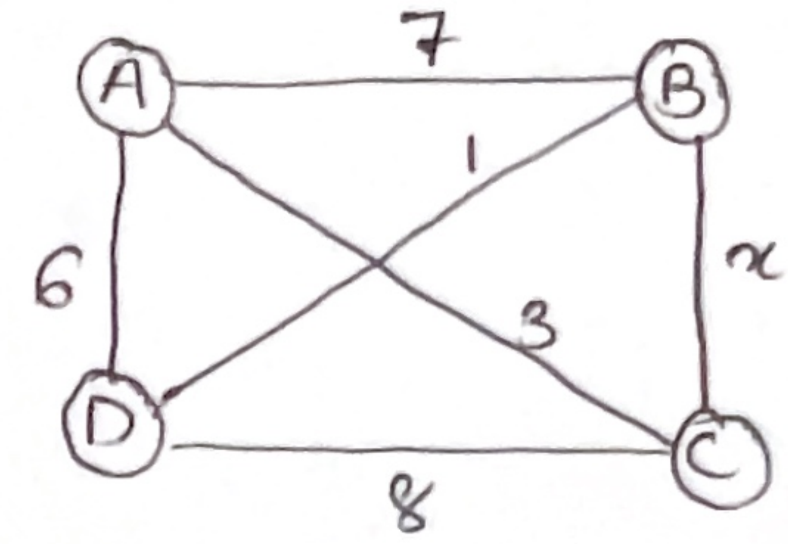


Weight = 11



Weight = 11

9) The maximum value of x such that edge between the nodes B & C is included in every minimum spanning tree of the given graph is $\underline{5}$.



Ans→ Let's find MST of the graph without edge BC.

Total weight = AD+AC+BD
= 6+1+3
= 10

~~To find make the~~

To make BC included in every MST x should be such a max value that when added with AC & BD it should be less than 10.

So, $AC+BD+BC<10$

$\Rightarrow 1+3+x<10$

$\Rightarrow x<6$

So, the highest value of x will be $\underline{\underline{5}}$.

10) Suppose there are two bowls of cookies. Bowl 1 contains 30 vanilla cookies & 10 chocolate cookies. Bowl 2 contains 20 each. ....

Ans→ Given,

Bowl 1: 30 vanilla, 10 chocolate → total 40

Bowl 2: 20 vanilla, 20 chocolate → total 40

$P(B_1) = P(B_2) = 1/2$

By Bayes' Theorem,

$$P(B_1 | Vanilla) = \frac{P(Vanilla | B_1) \cdot P(B_1)}{P(Vanilla)}$$

Now computing each term,

$P(vanilla | B_1) = 30/40 = 3/4$

$P(vanilla | B_2) = 20/40 = 1/2$

$P(Vanilla) = P(B_1) \cdot P(Vanilla | B_1) + P(B_2) \cdot P(Vanilla | B_2)$

$= 1/2 \cdot 3/4 + 1/2 \cdot 1/2 = 5/8$

Now, $P(B_1 | \text{Vanilla}) = \dfrac{3/4 \cdot 1/2}{5/8} = \dfrac{3}{8} \div \dfrac{5}{8} = 3/5$ (Ans).

$\therefore$ Probability of vanilla cookie coming from Bowl 1 is $3/5$.

11) Three prisoners, A, B, and C, are in separate cells and sentence to death. The governor has selected one of them at random to be pardoned. ....

Ans → Given,

One of them to be pardoned $= P(A) = P(B) = P(C) = 1/3$

Warden says "B will be executed".

Now, $P(W_B | P_A) = 1/2$ (Warden flips coin between B & C)

$P(W_B | P_B) = 0$ (Warden must say C, never B)

$P(W_B | P_C) = 1$ (Warden must say B)

$$P(W_B) = P(P_A) \cdot P(W_B|P_A) + P(P_B) \cdot P(W_B|P_B) + P(P_C) \cdot P(W_B|P_C)$$
$$= 1/3 \cdot 1/2 + 1/3 \cdot 0 + 1/3 \cdot 1$$
$$= \frac{1}{6} + \frac{1}{3} = \frac{3}{6} = 1/2$$

Probability that A is pardoned $= P(P_A | W_B) = \dfrac{P(P_A) \cdot P(W_B | P_A)}{P(W_B)} = \dfrac{1/3 \cdot 1/2}{1/2} = 1/3$

Probability that ~~B~~ C is pardoned $= P(P_C | W_B) = \dfrac{P(P_C) \cdot P(W_B | P_C)}{P(W_B)} = \dfrac{1/3 \cdot 1}{1/2} = 2/3$

This means, after hearing "B will be executed", A learns that C is now more likely to be pardoned than himself.

12) If it's sunny today, there is a 70% chance it will be sunny tomorrow and 30% chance it will rain. If it's rainy today, there is a 60% chance it will be sunny tomorrow ....

Ans → Transition Probabilities =

$$\begin{array}{c c} \text{From / To} & \begin{array}{cc} \text{Sunny (S)} & \text{Rainy (R)} \end{array} \\ \begin{array}{c} \text{Sunny} \\ \text{Rain} \end{array} & \begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix} \end{array}$$

Name: ____________________ Regd No: ____________________

Let, initial state vector = $v_0 = [1 \quad 0]$

After one day, $v_1 = v_0 \cdot P$

$$= [1 \quad 0] \begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix} = [0.7 \quad 0.3]$$

After two days, $v_2 = v_1 \cdot P$

$$= [0.7 \quad 0.3] \begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix} = [0.67 \quad 0.33]$$

∴ Probability it will be sunny after two days, given today is sunny is 67%.

13) A company has two states:

Good (G): The company is profitable and

Bad (B): The company is making a loss ....

Ans → Transition Matrix, $P = \begin{array}{c c} & \begin{array}{cc} G & B \end{array} \\ \begin{array}{c} G \\ B \end{array} & \begin{bmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{bmatrix} \end{array}$

Initial state vector, $v_0 = [1 \quad 0]$

After 1 day, $v_1 = v_0 \cdot P$

$$= [1 \quad 0] \begin{bmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{bmatrix} = [0.8 \quad 0.2]$$

After 2 days, $v_2 = v_1 \cdot P$

$$= [0.8 \quad 0.2] \begin{bmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{bmatrix} = [0.74 \quad 0.26]$$

∴ Probability the company is in Good state after two days (starting in Good) is 74%.