MATH7501: Exercise 2 Solutions

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1 Question 1 (4 MARKS)

1.1 Number of sixes obtained in three successive throws of a fair die

Yes, $\boldsymbol{n = 3}$ and $\boldsymbol{p = \frac{1}{6}}$

1.2 Number of aces dealt in a hand of four cards from a standard pack

No, trials are not independent (success probability at each stage depends on what has already happened). Can say this is modelled by a *hypergeometric distribution*

1.3 The number of students in a class of 40 whose birthday falls on a Sunday this year

Yes (providing there are no siblings in the class) $\boldsymbol{n = 40}$ and $\boldsymbol{p = \frac{1}{7}}$
(Though this question is debatable if its able to be modelled by binomial distribution and the probability is debabtable. The answers say $p = \frac{1}{7}$)

1.4 The number of throws of a fair coin until the first head obtained

No, this is modelled by a *geometric distribution*

2 Question 2 (4 MARKS)

Let X be the number of correct answers.Then X can be modelled by $X \sim Bin(10, \frac{1}{4})$

2.1 $\mathbf{P(X \geq 8)}$

We require here

$$P(X = 8) + P(X = 9) + P(X = 10)$$

In general for $Y \sim Bin(n, p)$:

$$P(Y = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

So what we have is then

$$\binom{10}{8}\left(\frac{1}{4}\right)^8\left(\frac{3}{4}\right)^2+\binom{10}{9}\left(\frac{1}{4}\right)^9\left(\frac{3}{4}\right)^1+\binom{10}{10}\left(\frac{1}{4}\right)^{10}\left(\frac{3}{4}\right)^0$$

Which simplifies to $\boldsymbol{P(X \geq 8) = 0.000416}$ *to 3sf*

2.2 Probability that the last of the ten answers given is the eighth one that is correct

Required probability that

$$P(7 \text{ out of 9 is correct and } 10^{th} \text{ is correct}) = P(10^{th} \text{ is correct } |(7 \text{ out of 9 is correct }))$$

Now we use the fact that

$$\begin{aligned} P(A \cap B) &= P(A)P(B|A) \text{ (conditional probability)} \\ &= P(A)P(B) \text{ in this case as events are } \textbf{independent} \end{aligned}$$

So

$$= \binom{9}{7}\left(\frac{1}{4}\right)^7\left(\frac{3}{4}\right)^2 \times \left(\frac{1}{4}\right)$$

$$= \mathbf{0.000309} \textit{ to 3sf}$$

3 Question 3 (5 MARKS)

Let X be the number of breakdowns in a year. Then $X \sim Geo(0.8)$ Then in this case

$$E(X) = \frac{1}{p} = \frac{1}{0.8} = 1.25$$

$$Var(X) = \frac{1-p}{p^2} = \frac{0.2}{0.8^2} = 0.3125$$

3.1 Find the expected value and the variance of the total cost of repairs in a year.

Let Y be the cost of repairs then $Y = 150X$. Now recall these two facts for when $Y = aX + b$

$$\boxed{E(Y) = aE(X) + b} \tag{3.1.1}$$

$$\boxed{Var(Y) = a^2Var(X)} \tag{3.1.2}$$

So lets use these in the case of where $Y = 150X$

$$E(Y) = E(150X) = 150 \times 1.25 = \textbf{£187.50} \textit{ (by 3.1.1)}$$

$$Var(Y) = Var(150X) = 150^2 \times 0.3125 = \textbf{£7031.25} \textit{ (by 3.1.2)}$$

3.2 Expected Cost and Variance under the Insurance Policy

Under this insurance policy the repair cost is now given by

$$\begin{aligned} Y &= (150 - 100)X + 60 \\ &= 50X + 60 \end{aligned}$$

So

$$E(Y) = E(50X + 60) = 50 \times 1.25 + 60 = \textbf{£122.50} \ \textit{(by 3.1.1)}$$

$$Var(Y) = Var(50X + 60) = 50^2 \times 0.3125 = \textbf{£781.25} \ \textit{(by 3.1.2)}$$

4 Question 4 (7 MARKS)

$X \sim Bin(n,p)$ and we know that

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \text{ where } k = 0, 1, .., n$$

Then the **pgf** is defined by

$$\Pi_X(z) = \sum_k z^k P(X = k)$$

There is no ambiguity in variable so $\Pi_X(z) = \Pi(z)$
So now let us find $\Pi^{'}(z)$ and $\Pi^{''}(z)$

$$\begin{aligned} \Pi^{'}(z) &= n(pz + (1-p))^{n-1} p \\ &= np(pz + (1-p))^{n-1} \ (\textit{chain rule wrt } z) \end{aligned}$$

$$\begin{aligned} \Pi^{''}(z) &= n(n-1)p(pz + (1-p))^{n-2} p \\ &= n(n-1)p^2(pz + (1-p))^{n-2} \ (\textit{chain rule wrt } z) \end{aligned}$$

Then recall three facts

$$\boxed{E(X) = \Pi^{'}(1)} \tag{4.0.1}$$

$$\boxed{E(X(X-1)) = \Pi^{''}(1)} \tag{4.0.2}$$

$$\boxed{Var(X) = E(X(X-1)) - \mu(\mu - 1) \text{ where } \mu = E(X)} \tag{4.0.3}$$

Now;

$$\begin{aligned} E(X) &= \Pi^{'}(1) \ \textit{(by 4.0.1)} \\ &= np((1)p + (1-p))^{n-1} \\ &= np(p + 1 - p)^{n-1} \\ &= np(1) \\ &= \boldsymbol{np} \end{aligned}$$

$$
\begin{aligned}
Var(X) &= E(X(X-1)) - \mu(\mu - 1) \text{ \textit{(by 4.0.3)}} \\
&= \Pi''(1) - np(np-1) \\
&= n(n-1)p^2(p(1) + (1-p))^{n-2} - np(np-1) \\
&= np^2(n-1) - np(np-1) \\
&= np(p(n-1) - (np-1)) \\
&= np(np - p - np + 1) \\
&= \boldsymbol{np(1-p)}
\end{aligned}
$$