

MATH7501 Exercise sheet 1 — to be done by Friday 19th January

1. Let X be the number of heads on two throws of a fair coin. Find $E(X)$ and $E(1/(1 + X))$. Verify that $E(1/(1 + X)) \neq 1/(1 + E(X))$. Find $\text{Var}(1/(1 + X))$. **6 marks**
2. A sample of two balls is selected at random and without replacement from an urn containing five red, three blue and two white balls. You win two pounds for each blue ball chosen and lose one pound for each red ball. Let X be your winnings. Write down the possible values of X and the corresponding probability mass function. Find your expected profit. What is the probability that you lose more than a pound, given that you do make a loss? **5 marks**
3. In a biochemical experiment, n organisms are placed in a nutrient medium and X , the number of organisms that survive for a given period, is recorded. If

$$P(X = r) = 2(r + 1)/[(n + 1)(n + 2)],$$

for $r = 0, \dots, n$ and is zero otherwise, calculate the probability that at most a proportion $\alpha = k/n$ of the organisms survive. Deduce that for large n , this probability is approximately α^2 . Find the smallest value of n for which the probability of there being at least one survivor among the n organisms is at least 0.95. **4 marks**

4. Meteorologists are required to issue daily forecasts of the probability of rainfall in a certain location. In an attempt to make them more accountable, or to save money, it is proposed to link their salary to their forecast performance. Performance is measured by scoring each forecast probability \hat{p} against the subsequent outcome X (taking the value 1 if it rains and 0 otherwise). It is proposed to define a the forecast score as $S = X \ln \hat{p} + (1 - X) \ln(1 - \hat{p})$.
 - (a) If the true probability of rain is p , find the forecast value \hat{p} that maximises the expected score. If forecasters receive salary bonuses when they score highly according to this rule, how should they act so as to maximise their salaries in the long run?
 - (b) In the West of Ireland, rain has occurred on 69% of all days in the last 50 years. In New South Wales, Australia, rain has occurred on 24% of all days. Would it be appropriate to use this scoring rule to compare the performance of Irish and Australian forecasters?

5 marks