MATH7501: Exercise 6 Solutions

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1 Question 1 (6 MARKS)

1.1 Are X_1 and X_2 independent?

If X_1 and X_2 are independent then their jpint probability funciton must factorise into oao product of the form $[f(x_1, x_2)]f_{X_1}(x_1)f_{X_2}(x_2)]$. Clearly this is not possible here so X_1 and X_2 are not independent

1.2 Find Covariance between X_1 and X_2 how does it relate to the previous answer?

NOTE:

$$Cov(X_1, X_2) = E(X_1, X_2) - \mu_1\mu_2$$

Where $\mu_1 = E(X_1)$ and $\mu_2 = E(X_2)$

So now we find the Integral

$$E(X_1) = \int_{x_2=-\infty}^{\infty} \int_{x_1=-\infty}^{\infty} x_1 f(x_1, x_2) dx_1 dx_2$$

$$= \int_{x_2=0}^{1} \int_{x_1=0}^{1} x_1 f(x_1, x_2) dx_1 dx_2$$

$$= \int_{x_2=0}^{1} \int_{x_1=0}^{1} x_1(x_1 + x_2) dx_1 dx_2$$

$$= \int_{x_2=0}^{1} \int_{x_1=0}^{1} x_1^2 + x_1 x_2 dx_1 dx_2$$

$$= \int_{x_2=0}^{1} \left[\frac{1}{3}x_1^3 + \frac{1}{2}x_1^2 x_2\right]_{x_1=0}^{1} dx_2$$

$$= \int_{x_2=0}^{1} \frac{1}{3} + \frac{1}{2}x_2 dx_2$$

$$= \left[\frac{1}{3}x_2 + \frac{1}{4}x_2^2\right]_{x_2=0}^{1} dx_2$$

$$= \frac{1}{3} + \frac{1}{4} = \mathbf{\frac{7}{12}}$$

It is also trivially true that $\mu_2 = E(X_2) = \frac{7}{12}$ By symmetry

Then we find

$$E(X_1, X_2) = \int_{X_2=0}^{1} \int_{X_1=0}^{1} x_1 x_2 (x_1, x_2) dx_1 x_2$$

Which when we perform the computation gives us the answer $\frac{1}{3}$

Finally we can do

$$Cov(X_1, X_2) = E(X_1, X_2) - \mu_1\mu_2$$

$$= \frac{1}{3} - \left(\frac{7}{12}\right)^2$$

$$= \boxed{-\frac{\mathbf{1}}{\mathbf{144}}}$$

2 Question 2 (4 MARKS)

2.1 Conditional distribution of Y and its parameters

From the exercise $X \sim Poi(\mu)$ and this is where $E(X) = \mu$. Given that $X = r$ The random variable Y is then distributed as $Y \sim NB(r, p)$ (Bernoulli Trials to obtain r successes where probability of success p is independent from trial to trial). NOTE that the "independently of each treatment occassion" tells us that even if the same animal goes for multiple treatments nothing will change.

$\boxed{\text{Y has a NEGATIVE BINOMIAL distribution with parameters r,p}}$

2.2 Expected value of Y

Given that $X = r$, we have that $E(Y) = \frac{r}{p}$ i.e. $E_{Y|X}(Y|X) = \frac{X}{p}$ Where $\frac{X}{p}$ takes the value $\frac{k}{p}$ when $X = k$ Then using the "Iterated Expectation Law"

$$E_Y(Y) = E_X[E_{Y|X}(Y|X)]$$

$$= E_X\left(\frac{X}{P}\right)$$

$$= \boxed{\frac{\boldsymbol{\mu}}{\boldsymbol{p}}}$$

3 Question 3 (10 MARKS)

3.1 Unconditional Expectation of X_2

From quesiton $X_1 \sim N(0,1)$ so $E_{X_1}(X_{X_1}) = 0$

Given that $X_1 = x_1$ we have that $X_2 \sim N(\alpha x_1, \tau^2)$

So we have that $E_{(X_2|X_1)}(X_2|X_1) = \alpha X_1$ which we can use the Iterated Expectation Law on to get

$$\begin{aligned}
E_{(X_2|X_1)}(X_2|X_1) &= E_{X_1}(\alpha X_1) \\
&= \alpha E_{X_1}(X_1) \\
&= \alpha \cdot 0 \\
&= 0
\end{aligned}$$

$\boxed{\boldsymbol{E(X_2 = 0)}}$

3.2 Find expressions for marginal density of X_1 and for the Conditional density of X_2 given that $X_1 = x_1$. Hence find the joint density of X_1 and X_2. Show that this joint density can be written in the form given in the sheet.

NOTE THE FORM REQUIRED IN THE SHEET IS GIVEN AS

$$f(x_1, x_2) = \frac{1}{2\pi\sigma\sqrt{1-\rho^2}} exp\left[-\frac{1}{2(1-\rho^2)}\left(x_1^2 - \frac{2\rho x_1 x_2}{\sigma} + \frac{x_2^2}{\sigma^2}\right)\right]$$

FROM NOTES: If $X \sim N(\mu, \sigma^2)$ then density function of X is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Then the marginal desnity of X_1 is given by

$$f_{X_1}(x_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_1)^2}{2}}$$

Where $x_1 \in \mathbb{R}$. Also conditional on $X_1 = x_1$ the conditional density of x_2 is given by

$$f_{X_2|X_1}(x_2|x_1) = \frac{1}{\sqrt{2\pi\tau^2}} e^{-\frac{(x_2-\alpha x_1)^2}{2\sigma^2}}$$

Where $x_2 \in \mathbb{R}$. Then given that $X_1 = x_1$ we have that $X_2 \sim N(\alpha x_1, \tau^2)$

The joint density function of X_1 and X_2 given by

$$f(x_1, x_2) = f_{X_1}(X_1) \cdot f_{X_2|X_1}(x_2|x_1)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_1)^2}{2}} \cdot \frac{1}{\sqrt{2\pi\tau^2}} e^{-\frac{(x_2 - \alpha x_1)^2}{2\tau^2}}$$

$$= \frac{1}{2\pi\tau} exp\left(-\frac{1}{2}\left(x_1^2 + \frac{(x_2 - \alpha x_1)^2}{\tau^2}\right)\right)$$

Then keeping the form required in mind we can rearrange to

$$f(x_1, x_2) = \frac{1}{2\pi\tau} exp\left(-\frac{1}{2}\left(1 + \frac{\alpha^2}{\tau^2}\right) x_1^2 - \frac{2\alpha x_1 x_2}{\tau^2} + \frac{x_2^2}{\tau^2}\right)$$

Equating the coefficients of x_1^2 and x_2^2 respectively

First x_1^2

$$x_1^2 = \left(1 + \frac{\alpha^2}{\tau^2}\right) = \frac{1}{1 - \rho^2}$$

$$\implies (1 - \rho^2) = \frac{\tau^2}{\alpha^2 + \tau^2}$$

Then x_2^2

$$x_2^2 = \frac{1}{(1 - \rho)^2 \sigma^2} = \frac{1}{\tau^2}$$

$$\implies \sigma^2 = \frac{\tau^2}{1 - \rho^2}$$

This leads to

$$e^2 = \frac{\alpha^2}{\alpha^2 + \tau^2}$$

$$\sigma^2 = \alpha^2 + \tau^2$$

We can verify that these values make all corresponding coefficients equal and conclude that

$$\rho = \frac{\alpha}{\sqrt{\alpha^2 + \tau^2}}$$

$$\tau = \sqrt{\alpha^2 + \tau^2}$$

In the form asked by in the question