MATH7501 Exercise sheet 4 — to be done by Friday 9th February

THIS SET OF EXERCISES IS THE SECOND ASSESSMENT

To assist the marking of this assessment, please write your name CLEARLY at the top of each sheet of paper, and indicate clearly (e.g. by underlining) the final answer to each part of a question.

All solutions must be your own work.

1. The lifetimes of lightbulbs are exponentially distributed with mean 500 hours; lifetimes of different bulbs are independent.
   (a) What proportion of bulbs have lifetimes in excess of 50 hours?
   (b) Bulbs are packaged in boxes of 10. The manufacturer guarantees that in each box, there will be at least 9 bulbs with lifetimes in excess of 50 hours. What percentage of boxes will meet the guarantee?

5 marks

2. Mrs Smith walks to work every day. If she takes the most direct route, her journey time is 10 minutes. However, this route follows a busy main road and, if it has been raining, she often gets splashed by traffic driving through puddles. Before she leaves for work therefore, she checks the weather: if the sky is no more than 40% covered by cloud, she takes the most direct route, but otherwise she takes a longer route where she is less likely to get splashed. This longer route takes her 15 minutes.

   Suppose that the proportion of sky covered by cloud follows a beta distribution with parameters $\alpha = 1/2$, $\beta = 1$. What is Mrs Smith's expected journey time? Compare this with her journey time if the cloud cover is equal to its expected value. **7 marks**

3. Let X be a continuous random variable taking non-negative values. The SURVIVOR FUNCTION of X is defined as $S(x) = 1 - F(x) = \mathrm{P}(X > x)$. If X has finite expected value $\mathrm{E}(X)$, use integration by parts to show that

$$\int_0^\infty S(x)dx = \mathrm{E}(X).$$

   (**Hint:** write the integral as $\int 1 \times S(x)dx$. You may assume that $\lim_{x\to\infty} xP(X > x) = 0$ when X has finite expectation).

   Use this result to verify the mean of the exponential distribution. **3 marks**

4. (a) Let X be a continuous random variable with moment generating function $M_X(\cdot)$. Suppose that $Y = aX + b$ for constants a and b. Show that the moment generating function of Y, $M_Y(t)$ say, can be written as $M_Y(t) = e^{bt}M_X(at)$.
   (b) Suppose that X has a gamma distribution with parameters α and λ, and that $Y = 3X$. Use the expression given in the lecture notes for the MGF of a gamma distribution, coupled with the result from part (a), to show that Y also has a gamma distribution. State the parameters of this distribution.

5 marks