MATH7501 Exercise sheet 6 — to be done by Friday 3rd March

1. Let X_1 and X_2 be the proportions of time that two employees spend at their desks on a particular day, and suppose that the joint density of X_1 and X_2 is given by

$$f(x_1, x_2) = \begin{cases} x_1 + x_2 & 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) *Without carrying out any calculations or mathematical manipulation*, state whether or not X_1 and X_2 are independent. Explain your reasoning clearly.

   (b) Find the covariance between X_1 and X_2. How does the result relate to your answer for part (a)?

6 marks

2. In an outbreak of disease among cattle, the number of infected animals has a Poisson distribution with mean μ. Each infected animal is treated with antibiotics; however, the treatment is not always successful. If the treatment is unsuccessful then it is repeated indefinitely until the animal is cured. Suppose that the probability of being cured is p, independently on each treatment occasion and independently between animals. Let X denote the number of animals infected; and let Y denote the total number of treatments administered before all of the infected animals are cured.

   (a) Suppose that $X = r$. Given this information, name the conditional distribution of Y and give the values of its parameters.

   (b) Find the expected value of Y.

4 marks

3. The random variable X_1 is distributed as $N(0, 1)$. Conditional on $X_1 = x_1$, the random variable X_2 is distributed as $N(\alpha x_1, \tau^2)$.

   (a) What is the (unconditional) expectation of X_2?

   (b) Write down expressions for the marginal density of X_1 and for the conditional density of X_2 given $X_1 = x_1$. Hence find the joint density of X_1 and X_2. Show that this joint density can be written in the form

$$f(x_1, x_2) = \frac{1}{2\pi\sigma\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(x_1^2 - \frac{2\rho x_1 x_2}{\sigma} + \frac{x_2^2}{\sigma^2}\right)\right] ,$$

   and give expressions for ρ and σ^2 in terms of α and τ^2.

   (c) *Without carrying out any further calculations*, name the marginal distribution of X_2 and give the values of its parameters in terms of α and τ^2. What, if any, restrictions are required on α and τ^2 to ensure that X_1 and X_2 have the same marginal distributions?

10 marks