MATH7501 Exercise sheet 7 — to be done by Friday 9th March

1. Write down the moment generating function (MGF) of a Bernoulli random variable with parameter p. Use this to deduce the MGF of a Binomial (n, p) random variable, and verify that the correct mean and variance of the Binomial distribution can be obtained from this MGF.

   Hint: remember that a Binomial (n, p) random variable can be regarded as a sum of n independent Bernoullis, each with parameter p.

   5 marks

2. Let $X_1, \ldots, X_n$ be mutually independent random variables, such that $X_i \sim \Gamma(\alpha_i, \lambda)$ (note that λ is the same for each variable). Using moment generating functions, show that $S_n = X_1 + \ldots + X_n$ has a gamma distribution with index $\sum_{i=1}^n \alpha_i$ and scale parameter λ (**NB** the formula for the MGF of a gamma distribution is in your notes).

   Use the Central Limit Theorem to deduce that, if X has a gamma distribution with index n and parameter λ, where n is large, then

   $$\mathrm{P}(n < \lambda X < n + \sqrt{n}) \simeq 0.34 \ .$$

   5 marks

3. (a) Show that the moment generating function of the $Poi(\mu)$ distribution is $M(t) = \exp\left[\mu\left(e^t - 1\right)\right]$.

   (b) Suppose that $X_1, \ldots, X_n$ are independent, identically distributed random variables, each distributed as $Poi(\mu)$. Let $Z_n = (n\mu)^{-1/2} \sum_{i=1}^n (X_i - \mu)$ and let $M_n(\cdot)$ denote the moment generating function of Z_n. Show that

   $$M_n(t) = \exp\left[-t\,(n\mu)^{1/2} + n\mu\left(e^{t(n\mu)^{-1/2}} - 1\right)\right] \ ,$$

   and deduce that $\lim_{n\to\infty} \log M_n(t) = t^2/2$. What is the limiting distribution of Z_n as $n \to \infty$?

   Hint: note that Z_n can be written in the form $aS_n + b$ for some values a and b, where $S_n = \sum_{i=1}^n X_i$. Thus the MGF of Z_n is $M_n(t) = E(e^{t(aS_n + b_n)})$.

   10 marks