# MATH7501: Exercise 3 Solutions

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## 1 Question 1 (6 MARKS)

### 1.1 Expression for $P(Y \leq y)$

Distribution functon of

$$\begin{aligned} X &: F_x(x) = P(X \leq x) \\ Y &: F_y(y) = P(Y \leq y) \end{aligned}$$

Then

$$\begin{aligned} P(Y \leq y) &= P(aX + b \leq y) \\ &= P(aX \leq y - b) \end{aligned}$$

**if ($a > 0$)**

$$\begin{aligned} P(Y \leq y) &= P\left(X \leq \frac{y-b}{a}\right) \\ &= \boxed{F_x\left(\frac{y-b}{a}\right)} \end{aligned}$$

**if ($a < 0$)**

$$\begin{aligned} P(Y \leq y) &= P(X \geq \frac{y-b}{a}) \\ &= 1 - P(X \leq \frac{y-b}{a}) \\ &= \boxed{1 - F_x\left(\frac{y-b}{a}\right)} \end{aligned}$$

**Note**

$$P\left(X \leq \frac{y-b}{a}\right) = P\left(X < \frac{y-b}{a}\right)$$

because X is continuous

### 1.2 Distribution Function of Y

In general if $U \sim U(c,d)$ the distribution function is given by equation

$$F_U(x) = \begin{cases} 0 & \text{if } x \leq c \\ \frac{x-c}{d-c} & \text{if } c \leq x \leq d \\ 1 & \text{if } x > d \end{cases}$$

Here $X = U \sim (0,1)$ so we obtain

$$F_X(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$$

Then since $a > 0$

$$F_Y(y) = P(Y \leq y) = F_X\left(\frac{y-b}{a}\right)$$

$$F_Y(y) = \begin{cases} 0 & \text{if } \frac{y-b}{a} \leq 0 \\ \\ \frac{y-b}{a} & \text{if } 0 \leq \frac{y-b}{a} \leq 1 \\ \\ 1 & \text{if } \frac{y-b}{a} > 1 \end{cases}$$

This can then rewritten so

$$F_Y(y) = \begin{cases} 0 & \text{if } y \leq b \\ \\ \frac{y-b}{a} & \text{if } b \leq y \leq a+b \\ \\ 1 & \text{if } y > a+b \end{cases}$$

Then $Y \sim U(a,b)$ so Y has a **uniform distribution**

## 2 Question 2 (4 MARKS)

### 2.1 K in terms of $\alpha$

$$f(x) = \begin{cases} \frac{K}{x^\alpha} & \text{if } x \geq 1 \\ \\ 0 & \text{Otherwise} \end{cases}$$

Probability Density Function of X. For this to be a valid pdf we require

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$= \int_{-\infty}^{1} f(x)dx + \int_{1}^{\infty} f(x)dx$$

$$= \int_{-\infty}^{1} 0 + \int_{1}^{\infty} Kx^{-\alpha}dx$$

$$= K\int_{1}^{\infty} x^{-\alpha}dx$$

$$= \frac{K}{1-\alpha}\left[x^{1-\alpha}\right]_{-\infty}^{1}$$

$$= \frac{K}{1-\alpha}\left[0 - 1^{1-\alpha}\right]$$

$$= \frac{K}{\alpha - 1}$$

$$\implies \boxed{K = \alpha - 1}$$

**Note**: $\alpha > 1 \implies 1-\alpha < 0$ and $x^{1-\alpha} = 0$ as $x \to 0$

## 2.2 State the values of r where the $r^{th}$ moment $E(X^r)$ exists

Consider

$$E(X^r) = \int_{-\infty}^{\infty} x^r f(x)dx$$

$$= \int_{1}^{\infty} x^r Kx^{-\alpha}dx$$

$$= \boxed{K\int_{1}^{\infty} x^{r-\alpha}dx}$$

so the integral for the $r^{th}$ moment is $E(X^r) = K\int_{1}^{\infty} x^{r-\alpha}dx$ . This integral only converges absolutely if $r - \alpha < -1$ i.e. $\boxed{r < \alpha - 1}$

**Note** $r - \alpha - 1 < 0$ to ensure that $x^{r-\alpha+1} \to 0$ as $x \to \infty$

# 3 Question 3 (7 MARKS)

Let $X$ be the *Number of battery replacements in 10 hours* and $C$ be the event that "The batteries are counterfeit" (i.e. normal, not longlife) Given

$$C : X \sim Poi(3) \text{ (3 batteries per 10 hour use)}$$
$$C^c : X \sim Poi(1) \text{(1 battery per 10 hour use)}$$

Where $C^c$ is the complement of C long life batteries $\boldsymbol{P(C) = 0.05 \wedge P(C^c = 0.95)}$ if $X \sim Poi(\mu)$ the

$$P(X = k) = \frac{\mu^k e^{-\mu}}{k!}$$

So consider

$$P(C|X = 3) = \frac{P(X = 3|C)P(C)}{P(X = 3)}$$

$$: \text{(By Baye's Theorem)}$$

$$= \frac{P(X = 3|C)P(C)}{P(X = 3|C)P(C + P(X = 3|C^c)P(C^c)}$$

$$: \text{(By Total Law of Probability)}$$

$$= \frac{\left(\frac{3^3 e^{-3}}{3!}\right)(0.05)}{\left(\frac{3^3 e^{-3}}{3!}\right)(0.05) + \left(\frac{1^3 e^{-1}}{3!}\right)(0.95)}$$

$$= \boxed{0.163 \text{ to 3 sf}}$$

## 3.1 Is the poisson distibution suitable

No its not suitable. The poisson process requires the independence from overlapping time intervals.