MATH7501: Exercise 5 Solutions

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1 Question 1 (6 MARKS)

1.1 Find pdf of Y

$Y = g(X)$ and g is a strictly monotone function. Then we may use the formula

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy}(g^{-1}(y)) \right|$$

Let $X = 1/Y \implies Y = 1/X \implies X \sim \Gamma(\alpha, \lambda)$

pdf of X is given by

$$f_X(x) = \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}$$

Here $g(x) = 1/x$ and X takes values in $\mathbb{R}^+$ so we can say that g(x) is **monotonic decreasing** in $\mathbb{R}^+$

$Y = 1/X$ so Y can only take values greater that zero. so for $f_Y(y) = 0$ for $y \leq 0$ then for $y > 0$ apply the transformation formula.

$g(x) = \frac{1}{x} \implies g^{-1}(y) = \frac{1}{y}$ and $\frac{dx}{dy} = -\frac{1}{y^2}$ for $y > 0$ we then then have

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy}(g^{-1}(y)) \right|$$

$$= \frac{\lambda^\alpha (\frac{1}{y})^{\alpha-1} e^{-\lambda(\frac{1}{y})}}{\Gamma(\alpha)} \cdot \left| -\frac{1}{y^2} \right|$$

$$= \frac{\lambda^\alpha y^{-(\alpha+1)} e^{-\frac{\lambda}{y}}}{\Gamma(\alpha)}$$

so combining this together

$$\boxed{f_Y(y) = \begin{cases} 0 & \text{if } y \leq 0 \\ \\ \frac{\lambda^\alpha y^{-(\alpha+1)} e^{-\frac{\lambda}{y}}}{\Gamma(\alpha)} & \text{Otherwise i.e. } y > 0 \end{cases}}$$

2 Question 2 (4 MARKS)

Proof.

$$E(\Phi(X_1, X_2)) = \sum_{x1} \sum_{x2} \Phi(X_1, X_2) p(x_1, x_2)$$

$$= \sum_{x1} \sum_{x2} x_1 p(x_1, x_2)$$

$$= \sum x_{x1} P(X = x_1) \text{ as } \sum_{x2} P(X_1 = x_1, X_2 = x_2) = P(X = x_1)$$

$$= E(X_1) \text{ by defiintion}$$

Quod Erat Demonstrandum

3 Question 3 (10 MARKS)

3.1 Find joint pmf table

X, Y take values 0, 1 and 2

$X \sim Bin(2, 1/2)$ and Given $X = x$ we have $Y \sim Bin(x, 1/2)$

Then we can construct table using $P(X = x, Y = y) = P(X = x \cap Y = y)$ which is the same as $P(Y = y|X = x)P(X = x)$

We note that $P(X = x, Y = y) = 0$ if $y \geq x$ which is obvious because we can't toss a coin Y more times we tossed the coin X.

We can trivially find the probabilities of the coin tosss of X so we have that

$$P(X = 0) = \frac{1}{4}$$
$$P(X = 1) = \frac{1}{2}$$
$$P(X = 2) = \frac{1}{4}$$

So now we find the joint probabilities

$$P(X = 0, Y = 0) = P(Y = 0|X = 0)P(X = 1) = 1 \times \frac{1}{4} = \frac{1}{4}$$

$$P(X = 1, Y = 0) = P(Y = 0|X = 1)P(X = 1) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

$$P(X = 1, Y = 1) = P(Y = 1|X = 1)P(X = 1) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

$$P(X = 2, Y = 0) = P(Y = 0|X = 2)P(X = 2) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$$

$$P(X = 2, Y = 1) = P(Y = 1|X = 2)P(X = 2) = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$$

$$P(X = 2, Y = 2) = P(Y = 2|X = 2)P(X = 2) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$$

So we obtain the following table that expresses the joint pmf

$P(X = x, Y = y)$	$\boldsymbol{y = 0}$	$\boldsymbol{y = 1}$	$\boldsymbol{y = 2}$
$\boldsymbol{x = 0}$	1/4	0	0
$\boldsymbol{x = 1}$	1/4	1/4	0
$\boldsymbol{x = 2}$	1/16	1/8	1/16

3.2 Marginal pmfs

The marginal pmfs are obtained by summing over rows and columns respectively

	$\boldsymbol{k = 0}$	$\boldsymbol{k = 1}$	$\boldsymbol{k = 2}$
$\boldsymbol{P(X = k)}$	1/4	1/2	1/4
$\boldsymbol{P(Y = k)}$	9/16	6/16	1/16

3.3 Are X and Y independent

For independence require that $\forall x, y : P(X = x, Y = y) = P(X = x)P(Y = y)$ where $P(X = x)$ and $P(Y = y)$ are from the marginal pmfs

So in this case take the easy example of $P(X = 0, Y = 1) = 0$ we have that

$$P(X = 0)P(Y = 1) = \frac{1}{4} \times \frac{6}{16} = \frac{6}{64} \neq 0$$

Then X are Y are **not independent** as $P(X = 0, Y = 1) \neq P(X = 0)P(Y = 1)$