MATH7501: Exercise 7 Solutions

Dinesh Kalamegam

March 9, 2018

1 Question 1 (5 MARKS)

1.1 MGF of Bernoulli Distribution with parameter p

Let $X_2 \sim Ber(p)$ MGF can be found directly. $M_x(.)$ is given by

$$\begin{aligned} M_x(t) &= \Sigma_k e^{tk} P(X = k) \\ &= e^{t \cdot 0}(1-p) + e^{t \cdot 1}(p) \\ &\boxed{\implies M_x(t) = E(e^{tX}) = 1 - p + pe^t} \end{aligned}$$

1.2 Deduce MGF of Binomail Distribution

Let $Y \sim (n, p)$ Then $Y = X_1 + ... + X_n$ Where $X_1, ..., X_n$ are independent Bernoulli random variables each with parameter p $X_i \sim Ber(p)$ for $i = 1, ..., n$

Then from result in notes MGF of Y given by

$$\begin{aligned} M_Y(t) &= M_{X_1}(t) + ... + M_{X_n} \\ &= (1 - p + pe^t) + ... + (1 - p + pe^t) \\ &= \boxed{(1 - p + pe^t)^n} \end{aligned}$$

1.3 Verify mean and variannce of the binomial distribution

First derivative of the MGF given by

$$\begin{aligned} M_Y'(t) &= \frac{d}{dt} M_y(t) \\ &= n(1 - p + pe^t)^{n-1} \times npe^t \end{aligned}$$

Second derivative of the MGF given by

$$\begin{aligned}M''_Y(t) &= \frac{d^2}{dt^2}M_y(t)\\ &= n(n-1)p^2e^t(1-p+pe^t)^{n-2} + npe^t(1-p+pe^t)^{n-1}\end{aligned}$$

Hence

$$\begin{aligned}\boxed{E(Y) = M'_Y(0) = np^0(1-p+pe^0) = np}\\ E(Y^2) = M''_Y(0) = n(n-1)p^2 + np\end{aligned}$$

$$\boxed{Var(Y) = n(n-1)p^2 + np - (np)^2 = np(1-p)}$$

And this is as required

2 Question 2 (5 MARKS)

2.1 Show that S_n has a gamma distribution

Here $X_i \sim \Gamma(\alpha_i, \lambda)$ MGF of X_i are given by

$$M_{X_i}(t) = E(e^{tX_i}) = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha_i}$$

Then since $S_n = X_1 + ... + X_n$ where X_i are independent $(i = 1, ..., n)$

$$\begin{aligned}M_{S_n}(t) &= M_{X_1}(t) + ... + M_{X_n}(t)\\ &= \left(\frac{\lambda}{\lambda - t}\right)^{\alpha_1} ... \left(\frac{\lambda}{\lambda - t}\right)^{\alpha_n}\\ &= \left(\frac{\lambda}{\lambda - t}\right)^{\alpha_1 + ... \alpha_n}\end{aligned}$$

Has same form of MGF of a gamma distribution with parameters λ , $\alpha_1 + ... + \alpha_n$

i.e. $S_n \sim \Gamma\left(\sum_{i=1}^{n} \alpha_i, \lambda\right)$

2.2 Use Central Limit Theorem to deduce result in sheet

Consider $X \sim \Gamma(\lambda, n)$ using the above cna express X as $X = X_1 + ... + X_n$ where $X_1 \sim \Gamma(1, \lambda)$ then substitute $\alpha_i = 1$ for $i = 1, ...n$ above

By Central Limit Theorem if $X_1, ...X_n$ independent identically distributed random variables with common mean μ and common variance of σ^2 Then

$$\bar{X} = \frac{X_1 + ... + X_n}{n}$$

approximates to a normal distribution with mean μ and variance $\frac{\sigma^2}{n}$ as n gets larger

so when is large $\frac{X_1+...+X_n}{n} = N\left(\mu, \frac{\sigma^2}{n}\right)$

Then $E(n\bar{X}) = nE(\bar{X}) = n\mu$ and $Var(n\bar{X}) = n^2 Var(\bar{X}) = n^2(\sigma^2/n) = n\sigma^2$ for n large

So for large n: $n\bar{X} = X_1 + ... + X_n \sim N(n\mu, n\sigma^2)$ in this case we have that $X_i \sim \Gamma(1, \lambda)$ so that $E(X_i) = \frac{1}{\lambda}$ and $Var(X_i) = \frac{1}{\lambda^2}$ so we may set $\mu = \frac{1}{\lambda}$ and $\sigma^2 = \frac{1}{\lambda^2}$ as above

This leads to $X = X_1 + ... + X_n \sim N(n(1/\lambda), n(1/\lambda^2))$ for large n

Then

$$\begin{aligned} P(n < \lambda X < n + \sqrt{n}) &= P\left(\frac{n}{\lambda} < X < \frac{n+\sqrt{n}}{\lambda}\right) \\ &= P\left(0 < X - n/\lambda < \sqrt{n}/\lambda\right) \\ &= P\left(0 < \frac{X - n/\lambda}{\sqrt{n}/\lambda} < 1\right) \end{aligned}$$

So we have that $\frac{X-n/\lambda}{\sqrt{n}/\lambda} \sim N(0, 1)$

So for large n:

$$\begin{aligned} P(n < \lambda X < n + \sqrt{n}) &= P(0 < Z < 1) \\ &= P(Z < 1) - P(Z < 0) \\ &= 0.8413 - 0.5 \text{ get values from table} \\ &= 0.3413 \end{aligned}$$

Which is approximately $\boxed{0.34}$ as required

3 Question 3 (10 MARKS)

3.1 MGF of Poisson Distribution

Suppose $X \sim Poi(\mu) \implies P(x = k) = \frac{e^{-\mu}\mu^k}{k!}$ then the MGF of X is given by

$$\begin{aligned}
M_X(t) &= E(e^{tX}) \\
&= \sum_k e^{tk} P(X = k) \\
&= \sum_{k=0}^{\infty} e^{tk} \frac{e^{-\mu}\mu^k}{k!} \\
&= e^{-\mu} \sum_{k=0}^{\infty} - \frac{(\mu e^t)^k}{k!} \\
&= e^{-\mu} e^{\mu e^t} \\
&= e^{\mu(e^t - 1)} \\
&= exp(\mu(e^t - 1))
\end{aligned}$$

As required