MATH7501: Exercise 1 Solutions

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1 Question 1 (6 MARKS)

1.1 Find $E(X)$

The possible outcomes are $\{TT, TH, HT, HH\}$ X is the number of heads so here we have $\{0, 1, 1, 2\}$ as the values of X following the order of the outcomes in the previous set
The probability mass function (pmf) is then given by:

k	0	1	2
$P(X = k)$	1/4	1/2	1/4

Then

$$\begin{aligned} E(X) &= \sum_k kP(X = k) \\ &= 0 \cdot P(X = 0) + 1 \cdot P(X = 1) + 2 \cdot P(X = 2) \\ &= (0 \cdot 1/4) + (1 \cdot 1/2) + (2 \cdot 1/4) \\ &= 1 \end{aligned}$$

So we have that $E(X) = 1$

1.2 Find $E(1/1 + X)$

Following a similar process:

$$\begin{aligned} E(1/1 + X) &= \sum_k \frac{1}{1 + k} P(X = k) \\ &= 1 \cdot P(X = 0) + 1/2 \cdot P(X = 1) + 1/3 \cdot P(X = 2) \\ &= (1 \cdot 1/4) + (1/2 \cdot 1/2) + (1/3 \cdot 1/4) \\ &= \frac{7}{12} \end{aligned}$$

1.3 Verify that $E(\frac{1}{1+X}) \neq \frac{1}{1+E(X)}$

We just got $E(\frac{1}{1+X})$ in **section 1.2** $= \frac{7}{12}$ and in **section 1.1** $E(X) = 1$ so:

$$\begin{aligned} \frac{1}{1+E(X)} &= \frac{1}{1+1} \\ &= \frac{1}{2} \\ &\neq \frac{7}{12} \\ &\neq E(\frac{1}{1+X}) \\ &\text{As required} \end{aligned}$$

1.4 Find $Var(1/1+X)$

To find this compute :

$$E((1/1+X)^2) - (E(1/1+X))^2 \tag{1.4.1}$$

First Compute

$$\begin{aligned} E((1/1+X)^2) &= (1 \cdot (1/4)^2) + ((1/2)^2 \cdot (1/2)) + ((1/3)^2 \cdot (1/4)) \\ &= \frac{29}{72} \end{aligned}$$

So back to **Equation 1.4.1**

$$\begin{aligned} E((1/1+X)^2) - (E(1/1+X))^2 &= \frac{29}{72} - (\frac{7}{12})^2 \\ &= \frac{1}{16} \end{aligned}$$

2 Question 2 (5 MARKS)

2.1 Find possible values of X and corresponding pmf

X is our winnings with Red worth -1, White worth 0 , Blue worth +2. We have 5 Red (R), 3 Blue (B) and 2 White (W).
Then we have the possible events ***f RR, RW, WR, WW, RB, BR, WB, BW, BBg*** Then values of X are *f* 2, 1, 1, 0, 1, 1, 2, 2, 4*g* Let us now find the probability mass function. It is given by:
$P(X = -2) = P(\{RR\}) = (\frac{5}{10}) \times (\frac{4}{9}) = \frac{20}{90}$

$P(X = -1) = P(\{RW, WR\}) = 2 \times (\frac{5}{10}) \times (\frac{2}{9}) = \frac{20}{90}$

$P(X = 0) = P(\{WW\}) = (\frac{2}{10}) \times (\frac{1}{9}) = \frac{2}{90}$

$P(X = 1) = P(\{RB, BR\}) = 2 \times (\frac{5}{10}) \times (\frac{3}{9}) = \frac{30}{90}$

$P(X = 2) = P(\{WB, BW\}) = 2 \times (\frac{2}{10}) \times (\frac{3}{9}) = \frac{12}{90}$

$P(X=4) = P(\{BB\}) = (\frac{3}{10}) \times (\frac{2}{9}) = \frac{6}{90}$

2.2 Expected Profit

The Expected Profit can be found by calculating $E(X)$

$$
\begin{aligned}
E(X) &= \sum_k kP(X=k) \\
&= (-2 \cdot P(X=-2)) + ((-1) \cdot P(X=-1)) + ... + (4 \cdot P(X=4)) \\
&= 0.2
\end{aligned}
$$

So in terms of the context we can say that the Expected Profit is £0.20

2.3 Probability that you lose a pound given that you make a loss

Recall: $P(A|B) = \frac{P(A \cap B)}{P(B)}$ conditional probability

$$
\begin{aligned}
P(X < -1 | X < 0) &= \frac{P((X < -1) \cap (X < 0))}{P(X < 0)} \\
&= \frac{P(X < -1)}{P(X < 0)} \\
&= \frac{P(X = -2)}{P(X = -2) + P(X = -1)} \\
&= \frac{20/90}{20/90 + 20/90} \\
&= \frac{1}{2}
\end{aligned}
$$

3 Question 3 (4 MARKS)

3.1 Calculate probability that at most a proportion $\alpha = k/n$ of the organisms survive

$$P(X = r) = \frac{2(r+1)}{(n+1)(n+2)} \text{where } r = 0, 1, ..., n$$

"At most k out of n organisms survive" and the corresponding probability:

$$= \frac{2}{(n+1)(n+2)} \cdot \frac{(k+1)(k+2)}{2}$$

$$= \frac{(k+1)(k+2)}{(n+1)(n+2)}$$

3.2 Deduce that for large n this probability is approximately α^2

$$P(X \leq k) = \frac{(\frac{k}{n} + \frac{1}{n})(\frac{k}{n} + \frac{2}{n})}{(1 + \frac{1}{n})(1 + \frac{2}{n})}$$

$$= \frac{(\alpha + \frac{1}{n})(\alpha + \frac{2}{n})}{(1 + \frac{1}{n})(1 + \frac{2}{n})}$$

We can now find $\lim_{n\to\infty}$ which would be

$$\frac{(\alpha + 0)(\alpha + 0)}{(1 + 0)(1 + 0)} = \alpha^2$$

As required

3.3 Find the smallest value of n for which the probability of there being at least one survivor among the n organisms is at least 0:95.

The required probability for this final part is
i.e.

$$\begin{aligned} P(X \geq 1) \geq 0.95 &\iff P(X = 1) + ... + P(X = n) \geq 0.95 \\ &\iff P(X < 1) \leq 0.05 \\ &\iff P(X = 0) \leq 0.05 \\ &\iff \frac{2}{(n+1)(n+2)} \leq 0.05 \\ &\iff (n+1)(n+2) \geq 40 \end{aligned}$$

Now the smallest value of n that satisifies this would be $\boldsymbol{n}$ **= 5**. This is because $n = 5$ is the first value of n s.t. $(n + 1)(n + 2) \geq 40$ $[(5)(6) \leq 40$ but $(6)(7) \geq 40]$

So the answer is $\boldsymbol{n}$ **= 5**