Summary of Continuous Distributions for a random Variable X

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1 Uniform $X \sim U(a, b)$

1.1 Probability Density Function $f(x)$

$$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{Otherwise} \end{cases}$$

1.2 Cumulative Distribution Function $F(x)$

$$F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & x > b \end{cases}$$

Proof.

$$\begin{aligned} F(X) &= \int_a^b f(u)du \\ &= \left[\frac{u}{(b-a)}\right]_a^x \\ &= \frac{x}{b-a} - \frac{a}{b-a} \\ &= \frac{x-a}{b-a} \end{aligned}$$

QED

1.3 Moment Generating Function $M(t) = E(e^{tX})$

$$M(t) = \frac{e^{tb} - e^{ta}}{t(b-a)}$$

Proof.

$$\begin{aligned}
M(t) &= E(e^{tX}) \\
&= \int_a^b e^{tx} f(x) dx \\
&= \int_a^b \frac{e^{tx}}{(b-a)} \\
&= \left[\frac{e^{tx}}{t(b-a}) \right]_a^b \\
&= \frac{e^{tb} - e^{ta}}{t(b-a)}
\end{aligned}$$

QED

1.4 $E(X)$

$$E(X) = \frac{a+b}{2}$$

Proof. Symmetry Argument makes this proof *trivial* ***QED***

Proof. Integration

$$\begin{aligned}
E(X) &= \int_a^b x f(x) dx \\
&= \int_a^b \frac{x}{b-a} dx \\
&= \left[\frac{x^2}{2(b-a)} \right]_a^b \\
&= \frac{1}{2(b-a)} [b^2 - a^2] \\
&= \frac{1}{2(b-a)} [(b-a)(b+a)] \\
&= \frac{a+b}{2}
\end{aligned}$$

QED

1.5 $Var(X)$

$$Var(X) = \frac{(b-a)^2}{12}$$

Proof. Integration

$$
\begin{aligned}
E(X^2) &= \int_a^b x^2 f(x)dx \\
&= \int_a^b \frac{x^2}{b-a}dx \\
&= \left[\frac{x^3}{3(b-a)}\right]_a^b \\
&= \frac{1}{3(b-a)}[b^3 - a^3] \\
&= \frac{1}{3(b-a)}[(b-a)(b^2+ab+a^2)] \\
&= \frac{a^2+ab+b^2}{3}
\end{aligned}
$$

Then we minus the two to obtain

$$
\begin{aligned}
Var(X) &= \frac{a^2+ab+b^2}{3} - \frac{(a+b)^2}{4} \\
&= \frac{a^2+ab+b^2}{3} - \frac{a^2+2ab+b^2}{4} \\
&= \frac{4a^2+4ab+4b^2}{12} - \frac{3a^2+6ab+3b^2}{12} \\
&= \frac{a^2-2ab+b^2}{12} \\
&= \frac{(b-a)^2}{12}
\end{aligned}
$$

QED

2 Exponential $X \sim Exp(\lambda)$

2.1 Probability Density Function $f(x)$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{Otherwise} \end{cases}$$

2.2 Cumulative Distribution Function $F(x)$

$$F(x) = 1 - e^{-\lambda x} \text{ where } x > 0$$

Proof.

$$\begin{aligned} F(x) &= \int_0^x \lambda e^{-\lambda u} du \\ &= \left[-e^{-\lambda u}\right]_0^x \\ &= (-e^{-\lambda x}) - (-1) \\ &= 1 - e^{-\lambda x} \end{aligned}$$

QED

2.3 Moment Generating Function $M(t) = E(e^{tX})$

$$M(t) = \frac{\lambda}{\lambda - t}$$

Proof.

$$\begin{aligned} M(t) &= E(e^t) \\ &= \int_{\mathbb{R}} e^{tx} f(x) dx \\ &= \int_0^\infty e^{tx} \cdot \lambda e^{-\lambda x} dxs \\ &= \int_0^\infty \lambda e^{(t-\lambda)x} dx \\ &= \left[\frac{\lambda}{(t-\lambda)} e^{(t-\lambda)x}\right]_0^\infty \\ &= \frac{\lambda}{\lambda - t} \end{aligned}$$

QED

2.4 $E(X)$

$$E(X) = \frac{1}{\lambda}$$

Proof.

$$\begin{aligned}
E(X) &= \int_0^\infty xf(x) \\
&= \int_0^\infty x\lambda e^{-\lambda x}dx \\
&= [-xe^{-\lambda x}]_0^\infty - \int_0^\infty -e^{-\lambda x} \\
&= \left[-\frac{1}{\lambda}e^{-\lambda x}\right]_0^\infty \\
&= 0 - \left(-\frac{1}{\lambda}\right) \\
&= \frac{1}{\lambda}
\end{aligned}$$

QED

2.5 $Var(X)$

$$Var(X) = \frac{1}{\lambda^2}$$

Proof.

$$
\begin{aligned}
E(X^2) &= \int_0^\infty x^2 f(x)dx \\
&= \int_0^\infty x^2 \lambda e^{-\lambda x} dx \\
&= \left[-x^2 e^{-\lambda x}\right]_0^\infty - \int_0^\infty -2xe^{-\lambda x} dx \\
&= 2\int_0^\infty xe^{-\lambda x} \\
&= 2\left[-\frac{1}{\lambda} xe^{-\lambda x}\right]_0^\infty - 2\int_0^\infty -\frac{1}{\lambda} e^{-x} \\
&= 2\int_0^\infty \frac{1}{\lambda} e^{-\lambda x} dx \\
&= 2\left[-\frac{1}{\lambda^2} e^{-\lambda x}\right]_0^\infty \\
&= \frac{2}{\lambda^2}
\end{aligned}
$$

$$
\begin{aligned}
Var(X) &= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 \\
&= \frac{1}{\lambda^2}
\end{aligned}
$$

QED

3 Gamma $X \sim \Gamma(\alpha, \lambda)$

3.1 Probability Density Function $f(x)$

$$f(x) = \begin{cases} \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} & x \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$

Where we have

$$\begin{aligned} \Gamma(\alpha) &= \int_0^\infty x^{\alpha-1} e^(- x) dx \\ &= (\alpha - 1)! \end{aligned}$$

3.2 Moment Generating Function $M(t) = E(e^{tX})$

$$M(t) = \frac{\lambda^\alpha}{(\lambda - t)^\alpha}$$

3.3 $E(X)$

$$E(X) = \frac{\alpha}{\lambda}$$

3.4 $Var(X)$

$$Var(X) = \frac{\alpha}{\lambda^2}$$

4 Beta $X \sim B(\alpha, \beta)$

4.1 Probability Density Function $f(x)$

$$f(x) = \begin{cases} \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} & x \in (0,1) \\ 0 & \text{Otherwise} \end{cases}$$

Where we have

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$$

4.2 $E(X)$

$$E(X) = \frac{\alpha}{\alpha + \beta}$$

4.3 $Var(X)$

$$Var(X) = \frac{\alpha\beta}{(\alpha\beta)^2(\alpha + \beta + 1)}$$

5 Normal $X \sim N(\mu, \sigma^2)$

5.1 Probability Density Function $f(x)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

Where we have $x \in \mathbb{R}$

5.2 Moment Generating Function $M(t)$

$$M(t) = exp\left[\mu t - \frac{\sigma^2 t^2}{2}\right]$$

5.3 $E(X)$

$$E(X) = \mu$$

5.4 $Var(X)$

$$Var(X) = \sigma^2$$

Also for normal distributions we can have that X can become the standard normal distribution $Z \sim N(0, 1)$ by the calculation

$$\frac{X-\mu}{\sigma^2}$$