MATH7501: Exercise 4 Solutions

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1 Question 1 (5 MARKS)

Let X denote the bulb lifetime. $\boldsymbol{X \sim Exp(\lambda) \implies E(X) = \frac{1}{\lambda}}$

In this case $\boldsymbol{E(X) = 500hrs \iff \lambda = \frac{1}{500} \implies X \sim Exp(\frac{1}{500})}$

We also note that in the exponential distribution $F_x(x) = P(X \leq x) = 1 - e^{-\lambda x}$

1.1 Proportion of bulbs have lifetimes in excess of 50 hours?

So the required probability

$$= P(X > 50)$$

$$= 1 - P(X \leq 50)$$

$$= 1 - (1 - e^{\frac{-50}{500}})$$

$$= e^{-0.1}$$

$$= \boxed{\mathbf{0.905}} \text{ (3 s.f)}$$

1.2 Percentage of boxes that meet guarantee

Let Y be the number of bulbs in boxes of 10 where lifetime exceeds 50 hours

Then $\boldsymbol{Y \sim Bin(10, 0.905)}$ or $\boldsymbol{Y \sim Bin(10, e^{-0.1})}$ (either acceptable)

$$P(Y \geq 9) = P(Y = 9) + P(Y = 10)$$

$$= \binom{10}{9}(0.905)^9(1 - 0.095)^1 + \binom{10}{10}(0.905)^{10}(1 - 0.095)^0$$

$$= 0.755 \text{ (3 s.f)}$$

But because we want the percentage of boxes we get the final answer to be $\boxed{\mathbf{75.5\%}}$ of boxes meet the guarantee

2 Question 2 (7 MARKS)

Let T be the journey time of Mrs Smith's Journey in minutes

Let X be the cloud coverage (takes values from 0 to 1) Then $\boldsymbol{X \sim B(\frac{1}{2}, 1)}$

Note if $\boldsymbol{X \sim B(\alpha, \beta)}$ then

$$E(X) = \frac{\alpha}{\alpha + \beta}$$

$$P(X \leq x) = \frac{1}{B(\alpha, \beta)} \int_0^x x^{\alpha-1}(1-x)^{1-\beta} dx$$

where we have

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{1-\beta} dx$$

And this makes sense as we want to get a probability between 0 and 1 and so we are normalising it by dividing it by the whole range of the intergal (from 0 to 1)

2.1 Expected journey time

As for T it takes values 10 and 15

$$P(T = 10) = P(X \leq 0.4)$$
$$P(T = 15) = P(X > 0.4)$$

So in our case

$$
\begin{aligned}
B\left(\frac{1}{2}, 1\right) &= \int_0^1 x^{-\frac{1}{2}}(1-x)^{1-1}dx \\
&= \int_0^1 x^{-\frac{1}{2}}dx \\
&= \left[2x^{\frac{1}{2}}\right]_0^1 \\
&= 2
\end{aligned}
$$

So to find $P(X \leq 0.4)$

$$
\begin{aligned}
B\left(\frac{1}{2}, 1\right) &= \int_0^{0.4} x^{-\frac{1}{2}}(1-x)^{1-1}dx \\
&= \int_0^{0.4} x^{-\frac{1}{2}}dx \\
&= \left[2x^{\frac{1}{2}}\right]_0^{0.4} \\
&= 2\sqrt{0.4} \\
&= 0.6325 \text{ (4 s.f)}
\end{aligned}
$$

Then

$$
\begin{aligned}
E(T) &= 10P(T=10) + 15P(T=15) \\
&= 10(0.6325) + 15(1-0.6325) \\
&= \boxed{\mathbf{11.8}} \text{ (3 s.f)}
\end{aligned}
$$

2.2 Comparing to time when cloud coverage is the expected cloud coverage

The expected cloud coverage is given by

$$E(X) = \frac{\alpha}{\alpha + \beta}$$

$$= \frac{\frac{1}{2}}{\frac{1}{2} + 1}$$

$$= \frac{1}{3}$$

Then $E(X) < 0.4$ which means she takes the direct route which is 10 minutes. Here the expected journey time is longer than the journey time for expected cloud cover

3 Question 3 (3 MARKS)

Surivor function $S(x) = 1 - F(x) = P(X \geq x)$

3.1 Show that $\int_0^\infty S(x)dx = E(X)$

First recall the by parts formula

$$\int_0^\infty u \cdot' vdx = [u \cdot v]_0^\infty - \int_0^\infty u \cdot v' dx$$

$$\int_0^\infty S(x)dx = \int_0^\infty 1 \cdot S(x)dx$$

$$= [x \cdot S(x)]_0^\infty - \int_0^\infty -xf(x)dx$$

Now as $x \to 0$ we have $xS(x) = xP(X \geq x) \to 0$ and by the assumption we can say the same for when $x \to \infty$ So we now have

$$\int_0^\infty S(x)dx = -\int_0^\infty -xf(x)dx$$

$$= \int_0^\infty xf(x)dx$$

and as we are only taking *non-negative values* we have that the above is $E(X)$ so $\int_0^\infty S(x)dx = E(X)$ as required

3.2 Show that this is the case with the exponential distibution

For exponential distribtion where $X \sim Exp(\lambda)$ we have $E(X) = \frac{1}{\lambda}$

We want to achieve the same using the integral $\int_0^\infty S(x)dx$

$$= \int_0^\infty 1 - F(x)dx \text{ where } F(x) = 1 - e^{-\lambda x} \text{ in exponential distibution}$$

$$= \int_0^\infty 1 - (1 - e^{-\lambda x})dx$$

$$= \int_0^\infty e^{-\lambda x}dx$$

$$= \left[-\frac{1}{\lambda}e^{-\lambda x}\right]_0^\infty$$

$$= \left[0 - (-\frac{1}{\lambda})\right]$$

$$= \boxed{\boldsymbol{\frac{1}{\lambda}}}$$

4 Question 4 (5 MARKS)

4.1 Show that $M_Y(t) = e^{tb}M_X(at)$ for continous random variable X

X is a continous random variable therefore the mgf = $M_X(t) = E(e^{tX})$

Suppose $Y = aX + b$ with mgf of $M_Y(t)$ then we have

$$\begin{aligned} M_Y(t) &= E(e^{tX}) \\ &= E(e^{t(aX+b)}) \\ &= E(e^{(at)X}e^{bt}) \\ &= e^{bt}E(e^{(at)X}) \\ &= \boxed{\boldsymbol{e^{tb}M_X(at)}} \end{aligned}$$

as required

4.2 Say for $X \sim \Gamma(\alpha, \lambda)$ and $Y = 3X$ show that Y too has a Gamma distibution and state its parameters

From (a):

$$M_Y(t) = e^{tb}M_X(at) \text{ (where here } a = 3 \text{ and } b = 0)$$

$$= \frac{\lambda^\alpha}{(\lambda - 3t)^\alpha}$$

$$= \frac{\lambda^\alpha}{3^\alpha(\frac{\lambda}{3} - t)^\alpha}$$

$$= \frac{(\frac{\lambda}{3})^\alpha}{(\frac{\lambda}{3} - t)^\alpha}$$

By comparsion with mgf of a Gamma distibution conclude that Y is also a Gamma distribution where $\boxed{\boldsymbol{Y \sim \Gamma(\alpha, \frac{\lambda}{3})}}$