MATH7501 Exercise sheet 0 — solutions and marking notes

Use the marking scheme below to check your solutions and mark your own work. The purpose of this is to give you an idea of whether you are adequately prepared for this course, so it is in your interest to do it honestly: make a serious attempt at the questions before looking at these solutions. If you find it difficult to mark your own work, you could instead find a friend and mark each other's work using the guidelines given below.

As a rough guide, if you score at least 18 out of 25 on this exercise sheet, you should be adequately prepared. If you score less than 13 out of 25, you should spend quite a lot of time revising the material in Chapter 1 of the MATH7501 lecture notes.

1. The elements of the sample space are individual outcomes of an experiment; those of the event space are *events* i.e. collections of outcomes (or subsets of the sample space). *1 mark*

   Marking notes: for this mark, you need to express clearly *the difference between the elements of the sample and event spaces.*

   If we toss a coin once, the sample space is $\Omega = \{H, T\}$. The event space is $\mathcal{F} = \{\emptyset, \{H\}, \{T\}, \Omega\}$. *2 marks*

   Marking notes: 1 mark each for Ω and $\mathcal{F}$. No half marks — they're either right or wrong!

2. The first axiom is that for any E, $P(E) \geq 0$. In this case we have $P(E) = \sum_{\{i:\omega_i \in E\}} p_i$, and the question tells us that the $\{p_i\}$ are non-negative. $P(\cdot)$ satisfies the first axiom, therefore. *1 mark*

   For the second axiom, we have $P(\Omega) = \sum_{\{i:\omega_i \in \Omega\}} p_i$, which is 1 according to the question. Hence $P(\cdot)$ also satisfies the second axiom. *1 mark*

   For the final axiom, note that if E and F are disjoint, by definition they contain no outcomes in common. Therefore $P(E \cup F) = \sum_{\{i:\omega_i \in E \cup F\}} p_i = \sum_{\{i:\omega_i \in E\}} p_i + \sum_{\{i:\omega_i \in F\}} p_i = P(E) + P(F)$, and $P(\cdot)$ satisfies the third axiom. *2 marks*

   Marking notes: 1 mark for noting that the outcomes in $E \cup F$ can be split into those in E and those in F; and 1 mark for using this to show that $P(E \cup F) = P(E) + P(F)$.

3. From the hint given in the question, we can write $\Omega = E \cup E^c$. Now $P(\Omega) = 1$ (axiom 2), so $P(E \cup E^c) = 1$. But E and E^c are disjoint events; hence $P(E \cup E^c) = P(E) + P(E^c)$ by axiom 3. Thus we have $P(E) + P(E^c) = 1$ and $P(E^c) = 1 - P(E)$, as required. *3 marks*

   Marking notes: 1 mark for noting that $P(E \cup E^c) = 1$, 1 mark for noting that $P(E \cup E^c) = P(E) + P(E^c)$, and 1 mark for rearranging to obtain the required result.

4. First, notice that $E \cup F$ can be written as $E \cup [F \cap (E \cap F)^c]$ (i.e. to be in $E \cup F$, outcomes must be in either E or in the bit of F that is not in its intersection with E). Thus $P(E \cup F) = P(E) + P\left(F \cap (E \cap F)^c\right)$ by axiom 3. Notice next that $F = (E \cap F) \cup [F \cap (E \cap F)^c]$ (i.e. outcomes in F are either in $E \cap F$ are in the remainder of F. Again therefore, by axiom 3 we must have $P(F) = P(E \cap F) + P\left(F \cap (E \cap F)^c\right)$ so that $P\left(F \cap (E \cap F)^c\right) = P(F) - P(E \cap F)$. Substituting this into the previous result yields $P(E \cup F) = P(E) + P(F) - P(E \cap F)$ as required. *4 marks*

Comment on solution: as here, an appropriate representation in terms of mutually exclusive events is often the key to deriving results by making use of axiom 3. The logic in the solution above is most easily seen by drawing a Venn diagram — if you used a Venn diagram to help with your own solution, this is fine but your solution must *make explicit use of axiom 3 to be complete.*

Marking notes: 1 mark for the decomposition into 'E' and 'F but not E'; 1 mark for using this to write $P(E \cup F)$ as the sum of two parts. 1 mark for getting $P(F$ but not $E)$ $= P(F) - P(E \cap F)$. 1 final mark for putting everything together.

5. If E and F are mutually exclusive, then $P(E \cap F) = 0$. If they are independent then $P(E \cap F) = P(E)P(F)$. Therefore, to be both mutually exclusive and independent we must have $P(E)P(F) = 0$: this *is* possible, if at least one of $P(E)$ and $P(F)$ is zero. *3 marks*

   Marking notes: 1 mark for using definition of mutual exclusivity, 1 mark for definition of independence, 1 mark for the conclusion that at least one of $P(E)$ and $P(F)$ must be zero.

6. If $P(E|F) > P(E)$ then $P(E \cap F)/P(F) > P(E)$ (definition of conditional probability). But $P(E \cap F) = P(F|E)P(E)$ (conditional probability again — or generalized multiplication law if you like). Thus, if $P(E|F) > P(E)$ then $P(F|E)P(E)/P(F) > P(E)$ i.e. $P(F|E)/P(F) > 1$ and $P(F|E) > P(F)$ as required. *3 marks*

   Marking notes: 1 mark for using definition of conditional probability to get $P(E|F)$, 1 mark for using definition of conditional probability to get $P(E \cap F)$, 1 mark for rearranging to obtain desired result.

   ***NB** you might think that we need the condition $P(E) > 0$ in order to cancel $P(E)$ from both sides in the penultimate step — in fact however, this is implied by the inequality $P(E|F) > P(E)$ since we must have $P(E|F) > 0$ so that $P(E \cap F) > 0$, and $P(E) \geq P(E \cap F)$.*

   This result says that if the occurrence of F makes E more likely then the occurrence of E makes F more likely — in other words, E and F tend to occur together more often than if they were independent. *1 mark*

7. Let D be the event 'individual has the disease' and E the event 'positive test result obtained'. We are given $P(D) = 0.01$, $P(E|D) = 0.95$, $P(E|D^c) = 0.06$.

   Using the Law of Total Probability, we have $P(E) = P(E|D)P(D) + P(E|D^c)P(D^c) = (0.95 \times 0.01) + (0.06 \times 0.99) = \mathbf{0.0689}$. *1 mark*

   Now using Bayes' Theorem, we have $P(D|E) = P(E|D)P(D)/P(E) = (0.95 \times 0.01)/0.0689 = \mathbf{0.138}$ (3 d.p.). *1 mark*

   Comment on solution: this is an example of an 'assignment of probabilities to different potential causes' problem, whence the use of Bayes theorem should have been obvious — see pp13–14 of lecture notes.

8. Let W_i denote the event 'white ball drawn on ith draw', R_i denote the event 'red ball drawn on the ith draw'.

   (a) $P(W_2) = P(W_2|R_1)P(R_1) + P(W_2|W_1)P(W_1)$ (Law of Total Probability) $= (1/7 \times 5/6) + (2/7 \times 1/6) = 1/6$. *1 mark*

   (b) $P(R_1|W_2) = \frac{P(W_2|R_1)P(R_1)}{P(W_2)} = \frac{1/7 \times 5/6}{1/6} = 5/7$. *1 mark*

Comment on solution: part (a) is an example of a problem that would be easy if you knew the colour of the first ball drawn — hence the use of the Law of Total Probability (pp12–13 of lecture notes). Part (b) is another 'potential causes' problem.