

# Summary of Continuous Distributions for a random Variable $X$

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## 1 Uniform $X \sim U(a, b)$

### 1.1 Probability Density Function $f(x)$

$$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{Otherwise} \end{cases}$$

### 1.2 Cumulative Distribution Function $F(x)$

$$F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & x > b \end{cases}$$

*Proof.*

$$\begin{aligned} F(X) &= \int_a^b f(u) du \\ &= \left[ \frac{u}{(b-a)} \right]_a^x \\ &= \frac{x}{b-a} - \frac{a}{b-a} \\ &= \frac{x-a}{b-a} \end{aligned}$$

***QED***

### 1.3 Moment Generating Function $M(t) = E(e^{tX})$

$$M(t) = \frac{e^{tb} - e^{ta}}{t(b-a)}$$

*Proof.*

$$\begin{aligned}
 M(t) &= E(e^{tX}) \\
 &= \int_a^b e^{tx} f(x) dx \\
 &= \int_a^b \frac{e^{tx}}{(b-a)} \\
 &= \left[ \frac{e^{tx}}{t(b-a)} \right]_a^b \\
 &= \frac{e^{tb} - e^{ta}}{t(b-a)}
 \end{aligned}$$

***QED***

## 1.4 $E(X)$

$$E(X) = \frac{a+b}{2}$$

*Proof.* Symmetry Argument makes this proof *trivial*

***QED***

*Proof.* Integration

$$\begin{aligned}
 E(X) &= \int_a^b x f(x) dx \\
 &= \int_a^b \frac{x}{b-a} dx \\
 &= \left[ \frac{x^2}{2(b-a)} \right]_a^b \\
 &= \frac{1}{2(b-a)} [b^2 - a^2] \\
 &= \frac{1}{2(b-a)} [(b-a)(b+a)] \\
 &= \frac{a+b}{2}
 \end{aligned}$$

***QED***

## 1.5 $Var(X)$

$$Var(X) = \frac{(b-a)^2}{12}$$

*Proof.* Integration

$$\begin{aligned}
 E(X^2) &= \int_a^b x^2 f(x) dx \\
 &= \int_a^b \frac{x^2}{b-a} dx \\
 &= \left[ \frac{x^3}{3(b-a)} \right]_a^b \\
 &= \frac{1}{3(b-a)} [b^3 - a^3] \\
 &= \frac{1}{3(b-a)} [(b-a)(b^2 + ab + a^2)] \\
 &= \frac{a^2 + ab + b^2}{3}
 \end{aligned}$$

Then we minus the two to obtain

$$\begin{aligned}
 Var(X) &= \frac{a^2 + ab + b^2}{3} - \frac{(a+b)^2}{4} \\
 &= \frac{a^2 + ab + b^2}{3} - \frac{a^2 + 2ab + b^2}{4} \\
 &= \frac{4a^2 + 4ab + 4b^2}{12} - \frac{3a^2 + 6ab + 3b^2}{12} \\
 &= \frac{a^2 - 2ab + b^2}{12} \\
 &= \frac{(b-a)^2}{12}
 \end{aligned}$$

***QED***

## 2 Exponential $X \sim \text{Exp}(\lambda)$

### 2.1 Probability Density Function $f(x)$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{Otherwise} \end{cases}$$

### 2.2 Cumulative Distribution Function $F(x)$

$$F(x) = 1 - e^{-\lambda x} \text{ where } x > 0$$

*Proof.*

$$\begin{aligned} F(x) &= \int_0^x \lambda e^{-\lambda u} du \\ &= [-e^{-\lambda u}]_0^x \\ &= (-e^{-\lambda x}) - (-1) \\ &= 1 - e^{-\lambda x} \end{aligned}$$

***QED***

### 2.3 Moment Generating Function $M(t) = E(e^{tX})$

$$M(t) = \frac{\lambda}{\lambda - t}$$

*Proof.*

$$\begin{aligned} M(t) &= E(e^t) \\ &= \int_{\mathbb{R}} e^{tx} f(x) dx \\ &= \int_0^{\infty} e^{tx} \cdot \lambda e^{-\lambda x} dx \\ &= \int_0^{\infty} \lambda e^{(t-\lambda)x} dx \\ &= \left[ \frac{\lambda}{(t-\lambda)} e^{(t-\lambda)x} \right]_0^{\infty} \\ &= \frac{\lambda}{\lambda - t} \end{aligned}$$

***QED***

## 2.4 $E(X)$

$$E(X) = \frac{1}{\lambda}$$

*Proof.*

$$\begin{aligned} E(X) &= \int_0^{\infty} x f(x) dx \\ &= \int_0^{\infty} x \lambda e^{-\lambda x} dx \\ &= [-x e^{-\lambda x}]_0^{\infty} - \int_0^{\infty} -e^{-\lambda x} dx \\ &= \left[ -\frac{1}{\lambda} e^{-\lambda x} \right]_0^{\infty} \\ &= 0 - \left( -\frac{1}{\lambda} \right) \\ &= \frac{1}{\lambda} \end{aligned}$$

***QED***

## 2.5 $Var(X)$

$$Var(X) = \frac{1}{\lambda^2}$$

*Proof.*

$$\begin{aligned} E(X^2) &= \int_0^\infty x^2 f(x) dx \\ &= \int_0^\infty x^2 \lambda e^{-\lambda x} dx \\ &= \left[ -x^2 e^{-\lambda x} \right]_0^\infty - \int_0^\infty -2x e^{-\lambda x} dx \\ &= 2 \int_0^\infty x e^{-\lambda x} dx \\ &= 2 \left[ -\frac{1}{\lambda} x e^{-\lambda x} \right]_0^\infty - 2 \int_0^\infty -\frac{1}{\lambda} e^{-\lambda x} dx \\ &= 2 \int_0^\infty \frac{1}{\lambda} e^{-\lambda x} dx \\ &= 2 \left[ -\frac{1}{\lambda^2} e^{-\lambda x} \right]_0^\infty \\ &= \frac{2}{\lambda^2} \end{aligned}$$

$$\begin{aligned} Var(X) &= \frac{2}{\lambda^2} - \left( \frac{1}{\lambda} \right)^2 \\ &= \frac{1}{\lambda^2} \end{aligned}$$

***QED***

### 3 Gamma $X \sim \Gamma(\alpha, \lambda)$

#### 3.1 Probability Density Function $f(x)$

$$f(x) = \begin{cases} \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} & x \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$

Where we have

$$\begin{aligned} \Gamma(\alpha) &= \int_0^\infty x^{\alpha-1} e^{-x} dx \\ &= (\alpha - 1)! \end{aligned}$$

#### 3.2 Moment Generating Function $M(t) = E(e^{tX})$

$$M(t) = \frac{\lambda^\alpha}{(\lambda - t)^\alpha}$$

*Proof.*

$$\begin{aligned} M(t) &= E(e^{tX}) \\ &= \int_0^\infty e^{tx} f(x) dx \\ &= \int_0^\infty \frac{1}{\Gamma(\alpha)} x^{\alpha-1} \lambda^\alpha e^{-(\lambda-t)x} dx \\ &= \frac{\lambda^\alpha}{(\lambda - t)^\alpha} \int_0^\infty \frac{1}{\Gamma(\alpha)} x^{\alpha-1} (\lambda - t)^\alpha e^{-(\lambda-t)x} dx \end{aligned}$$

However, provided that  $t < \lambda$  so  $\lambda - t > 0$  the integrand here is the density of  $\Gamma(\alpha, \lambda - t)$  hence would equal 1. Hence can conclude that

$$M(t) = \frac{\lambda^\alpha}{(\lambda - t)^\alpha}$$

***QED***

#### 3.3 $E(X)$

$$E(X) = \frac{\alpha}{\lambda}$$

*Proof.* From MGF

$$\begin{aligned} M(t) &= \lambda^\alpha (\lambda - t)^\alpha \\ M'(t) &= \lambda^\alpha \alpha (\lambda - t)^{-(\alpha+1)} \\ M'(0) &= \frac{\lambda^\alpha \alpha}{\lambda^{\alpha+1}} \\ &= \frac{\alpha}{\lambda} \end{aligned}$$

***QED***

### 3.4 $Var(X)$

$$Var(X) = \frac{\alpha}{\lambda^2}$$

*Proof.*

$$\begin{aligned} M'(t) &= \lambda^\alpha \alpha (\lambda - t)^{-(\alpha+1)} \\ M''(t) &= \lambda^\alpha \alpha (\alpha + 1) (\lambda - t)^{-(\alpha+2)} \\ M''(0) &= \frac{\lambda^\alpha \alpha (\alpha + 1)}{\lambda^{\alpha+2}} \\ &= \frac{\alpha(\alpha + 1)}{\lambda^2} \end{aligned}$$

Then we use  $Var(X) = E(X^2) - [E(X)]^2$  to get the result

$$\begin{aligned} Var(x) &= \frac{\alpha(\alpha + 1)}{\lambda^2} - \frac{\alpha^2}{\lambda^2} \\ &= \frac{\alpha}{\lambda^2} \end{aligned}$$

***QED***



## 4 Beta $X \sim B(\alpha, \beta)$

### 4.1 Probability Density Function $f(x)$

$$f(x) = \begin{cases} \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)} & x \in (0, 1) \\ 0 & \text{Otherwise} \end{cases}$$

Where we have

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx$$

### 4.2 $E(X)$

$$E(X) = \frac{\alpha}{\alpha + \beta}$$

### 4.3 $Var(X)$

$$Var(X) = \frac{\alpha\beta}{(\alpha\beta)^2(\alpha + \beta + 1)}$$

## 5 Normal $X \sim N(\mu, \sigma^2)$

### 5.1 Probability Density Function $f(x)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[ -\frac{(x - \mu)^2}{2\sigma^2} \right]$$

Where we have  $x \in \mathbb{R}$

### 5.2 Moment Generating Function $M(t)$

$$M(t) = \exp \left[ \mu t - \frac{\sigma^2 t^2}{2} \right]$$

### 5.3 $E(X)$

$$E(X) = \mu$$

### 5.4 $Var(X)$

$$Var(X) = \sigma^2$$

Also for normal distributions we can have that  $X$  can become the standard normal distribution  $Z \sim N(0, 1)$  by the calculation

$$\frac{X - \mu}{\sigma}$$