AI1103-Assignment 4

Name: Avula Mohana Durga Dinesh Reddy , Roll Number: CS20BTECH11005

Download all Latex codes from

https://github.com/DineshAvulaMohanaDurga/AI1103/blob/main/assignment_4/main.tex

1 Question

(CSIR-UGC-NET June 2015 Q 104)Let X and Y be random variables with joint cumulative distribution function $F_{XY}(x, y)$. Then which of the following cnditions are sufficient for $(x_0, y_0) \in R^2$ to be a point of continuity of F_{XY}?

1) $p_{XY}(x = x_0, y = y_0) = 0$
2) Either $p_{XY}(x = x_0) = 0$ or $p_{XY}(y = y_0) = 0$.
3) $p_{XY}(x = x_0) = 0$ and $p_{XY}(y = y_0) = 0$.
4) $p_{XY}(x = x_0, y \leq y_0)$=0 and $p_{XY}(x \leq x_0, y = y_0)$=0.

2 Answer

Let $F_{XY}(x, y)$ be joint cumulative distribution function and $P_{XY}(x, y)$ be joint probability distribution function.

Lemma 2.1. *$F_{XY}(x, y)$ is continuous at (x_0, y_0) iff all*

1) $p_{X,Y}(x = x_0, y \leq y_0)$
2) $p_{X,Y}(x \leq x_0, y = y_0)$
3) $p_{X,Y}(x = x_0, y = y_0)$

exists and is finite.

Proof. One of the unique properties of cdf is that if cdf is continuous at a point then its differentiable at that point.i.e

$$\text{continuity of cdf} \Rightarrow \text{ differentiability of cdf.} \tag{2.0.1}$$

one of the properties of a function f(x) is that if its differentiable then its continuous.i.e

$$\text{differentiability of f}(x) \Rightarrow \text{ continuity of f}(x). \tag{2.0.2}$$

$\therefore$from equations 2.0.1 and 2.0.2 we can say that

$$\text{continuity of cdf} \Leftrightarrow \text{ differentiability of cdf.} \tag{2.0.3}$$

So the conditions of differentiabilitty of cdf and the conditions of continuity of cdf are the same.
Lets check the conditions for differentiability of cdf.
For a function f(x, y) to be differentiable at (x_0, y_0)

1) $f_x(x, y)$ and $f_y(x, y)$ should exist and be finite.
2) $$\lim_{t \to 0} \frac{f(x_0 + u_1 t, y_0 + u_2 t) - f(x_0, y_0)}{t}$$
   should exist and be finite for all unit vector u=$u_1 \hat{i} + u_2 \hat{j}$

So for cdf to be differentiable at (x_0, y_0)

1) $$\lim_{h \to 0} \frac{F_{XY}(x_0 + h, y_0) - F_{XY}(x_0, y_0)}{h} = p_{XY}(x = x_0, y \leq y_0) \text{ should exist and be finite.Similarly,} \tag{2.0.4}$$
   $$\lim_{k \to 0} \frac{F_{XY}(x_0, y_0 + k) - F_{XY}(x_0, y_0)}{k} = p_{XY}(x \leq x_), y = y_0) \text{ should exist and be finite} \tag{2.0.5}$$
2) $$\lim_{t \to 0} \frac{F_{X,Y}(x_0 + u_1 t, y_0 + u_2 t) - F_{X,Y}(x_0, y_0)}{t} = p_{X,Y}(x = x_0, y = y_0) \text{ should exist and be finite} \tag{2.0.6}$$

From equations 2.0.4 , 2.0.5 , 2.0.6 we can prove that lemma 2.1 is true. ◻

Now lets verify the options

1) Option 1 fails to satisfy the lemma 2.1 as it defines only 3 of lemma 2.1 but not 1,2.
   Counter-example:-

Let's consider an example similar to dirac-delta function.

$$P_{XY}(x,y) = \infty \quad (x,y) = (x_0, y_1) \ \text{where } y_1 < y_0$$
$$= 0 \quad \text{otherwise} \tag{2.0.7}$$

Here $P_{XY}(x_0, y_0)$ is 0 still F_{XY} is not continuous at (x_0, y_0) parellel to X-axis

$$F_{XY}(x,y) = 1 \qquad x > x_0, y > y_1$$
$$= 0 \qquad otherwise \tag{2.0.8}$$

So option 1 is false

2) Option 2 also fails to satisfy lemma 2.1 as it defines only 3 and either 1 or 2 of lemma 2.1.
Counter-example:-
Lets consider another function similar to dirac delta function

$$P_{XY}(x,y) = \infty \quad (x,y) = (x_1, y_0) \ \text{where } x_1 < x_0$$
$$= 0 \quad \text{otherwise} \tag{2.0.9}$$

Here $p_{XY}(Y = y_0) = 0$ but F_{XY} is not continuous at (x_0, y_0).

$$F_{XY}(x,y) = 1 \qquad x > x_1, y > y_0$$
$$= 0 \qquad otherwise \tag{2.0.10}$$

So option 2 is also false

3) Option 3 satisfies lemma 2.1.
So option 3 is true
4) Option 4 also satisfies lemma 2.1.
So option 4 is also true

Hence correct options are 3,4.