AI1103-Assignment 4

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Download all Latex codes from

https://github.com/DineshAvulaMohanaDurga/AI1103/blob/main/assignment_4/main.tex

1 Question

(CSIR-UGC-NET June 2015 Q 104)Let X and Y be random variables with joint cumulative distribution function $F_{XY}(x, y)$. Then which of the following cnditions are sufficient for $(x_0, y_0) \in \mathbb{R}^2$ to be a point of continuity of F_{XY}?

1) $p_{XY}(x = x_0, y = y_0) = 0$
2) Either $p_{XY}(x = x_0) = 0$ or $p_{XY}(y = y_0) = 0$.
3) $p_{XY}(x = x_0) = 0$ and $p_{XY}(y = y_0) = 0$.
4) $p_{XY}(x = x_0, y \leq y_0)$=0 and $p_{XY}(x \leq x_0, y = y_0)$=0.

2 Answer

Let $F_{XY}(x, y)$ be joint cumulative distribution function and $P_{XY}(x, y)$ be joint probability distribution function.

Lemma 2.1. *$F_{XY}(x, y)$ is continuous at (x_0, y_0) iff all*

$$p_{X,Y}(x = x_0, y \leq y_0) \tag{1}$$
$$p_{X,Y}(x \leq x_0, y = y_0) \tag{2}$$
$$p_{X,Y}(x = x_0, y = y_0) \tag{3}$$

exists and is finite.

Proof. One of the unique properties of cdf is

$$\text{continuity of cdf} \Leftrightarrow \text{ differentiability of cdf.} \tag{2.0.1}$$

Lets check the conditions for differentiability of cdf. For cdf to be differentiable at (x_0, y_0)

1)
$$\lim_{h\to 0} \frac{F_{XY}(x_0 + h, y_0) - F_{XY}(x_0, y_0)}{h} \in \mathbb{R}$$
$$\Rightarrow p_{XY}(x = x_0, y \leq y_0) \in \mathbb{R} \tag{2.0.2}$$

2)
$$\lim_{k\to 0} \frac{F_{XY}(x_0, y_0 + k) - F_{XY}(x_0, y_0)}{k} \in \mathbb{R}$$
$$\Rightarrow p_{XY}(x \leq x_), y = y_0) \in \mathbb{R} \tag{2.0.3}$$

3)
$$\lim_{t\to 0} \frac{F_{X,Y}(x_0 + u_1 t, y_0 + u_2 t) - F_{X,Y}(x_0, y_0)}{t} \in \mathbb{R}$$
$$\Rightarrow p_{X,Y}(x = x_0, y = y_0) \in \mathbb{R} \tag{2.0.4}$$

From equation 2.0.1 ,2.0.2 ,2.0.3 ,2.0.4 we can say that lemma 2.1 is true. □

Now lets verify the options

1) Option 1 $\Rightarrow$ (3) ,but Option 1 $\not\Rightarrow$ (1) & (2) .Hence it fails lemma 2.1.
   Counter-example:-
   Let's consider an example similar to dirac-delta function.
   $$P_{XY}(x, y) = \infty \quad (x, y) = (x_0, y_1), y_1 < y_0$$
   $$= 0 \quad \text{otherwise} \tag{2.0.5}$$
   Here $P_{XY}(x_0, y_0)$ is 0 still F_{XY} is not continuous at (x_0, y_0) parellel to X-axis
   $$F_{XY}(x, y) = 1 \qquad x > x_0, y > y_1$$
   $$= 0 \qquad otherwise \tag{2.0.6}$$
   So option 1 is false
2) Option 2 $\Rightarrow$ (3) ,either (1) or (2) .Hence it fails lemma 2.1.
   Counter-example:-
   Lets consider another function similar to dirac delta function
   $$P_{XY}(x, y) = \infty \quad (x, y) = (x_1, y_0), x_1 < x_0$$
   $$= 0 \quad \text{otherwise} \tag{2.0.7}$$
   Here $p_{XY}(Y = y_0) = 0$ but F_{XY} is not continuous at (x_0, y_0).
   $$F_{XY}(x, y) = 1 \qquad x > x_1, y > y_0$$
   $$= 0 \qquad otherwise \tag{2.0.8}$$

So option 2 is also false

3) Option 3 satisfies lemma 2.1.
   So option 3 is true
4) Option 4 also satisfies lemma 2.1.
   So option 4 is also true

Hence correct options are 3,4.