AI1103-Assignment 4

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Download all Latex codes from

https://github.com/DineshAvulaMohanaDurga/AI1103/blob/main/assignment_4/main.tex

1 Question

(CSIR-UGC-NET June 2015 Q 104)Let X and Y be random variables with joint cumulative distribution function $F_{XY}(x, y)$. Then which of the following cnditions are sufficient for $(x_0, y_0) \in R^2$ to be a point of continuity of F_{XY}?

1) $p_{XY}(x = x_0, y = y_0) = 0$
2) Either $p_{XY}(x = x_0) = 0$ or $p_{XY}(y = y_0) = 0$.
3) $p_{XY}(x = x_0) = 0$ and $p_{XY}(y = y_0) = 0$.
4) $p_{XY}(x = x_0, y \leq y_0)$=0
and $p_{XY}(x \leq x_0, y = y_0)$=0.

2 Answer

Let $F_{XY}(x, y)$ be joint cumulative distribution function and $P_{XY}(x, y)$ be joint probability distribution function.
From the definitoion of cummulative distribution function

$$pdf = \frac{d}{dx}cdf$$
$$P(x) = \frac{d}{dx}F(x)$$
$$or$$
$$F(x) = \int_{x=-\infty}^{x} P(x)\, dx \tag{2.0.1}$$
$$\Rightarrow F_{XY}(x, y) = \int_{y=-\infty}^{y}\int_{x=-\infty}^{x} p_{XY}(x, y)\, dxdy \tag{2.0.2}$$

Lemma 2.1. *$F_{XY}(x, y)$ is continuous at (x_0, y_0) iff $p_{XY}(x, y)$ is defined in the regions*

1) $-\infty < x < x_0, y = y_0$
2) x=x_0,$-\infty < y < y_0$
3) point (x_0, y_0)

Proof. For F to be continuous at (x, y)

$$\lim_{k\to 0}\lim_{h\to 0}\left(F_{XY}(x_0 + h, y_0 + k) - F_{XY}(x_0, y_0)\right) = 0 \tag{2.0.3}$$

$$\lim_{k\to 0}\lim_{h\to 0}\int_{y=-\infty}^{y_0+k}\int_{x=-\infty}^{x_0+h} p_{XY}(x, y)\, dxdy - \int_{y=-\infty}^{y_0}\int_{x=-\infty}^{x_0} p_{XY}(x, y)\, dxdy = 0 \tag{2.0.4}$$

On expanding the limits of integrals we get

$$\lim_{k\to 0}\lim_{h\to 0}\int_{y=y_0}^{y_0+k}\int_{x=-\infty}^{x_0} p_{XY}(x, y)\, dxdy + \int_{y=-\infty}^{y_0}\int_{x=x_0}^{x_0+h} p_{XY}(x, y)\, dxdy + \int_{y=y_0}^{y_0+k}\int_{x=x_0}^{x_0+h} p_{XY}(x, y)\, dxdy = 0 \tag{2.0.5}$$

Which is zero if $p_{XY}(x, y)$ is defined over the integrating region i.e $p_{XY}(x, y)$ should be defined over the intergrating region of

1) $$\lim_{k\to 0}\lim_{h\to 0}\int_{y=y_0}^{y_0+k}\int_{x=-\infty}^{x_0} p_{XY}(x, y)\, dxdy$$
2) $$\lim_{k\to 0}\lim_{h\to 0}\int_{y=-\infty}^{y_0}\int_{x=x_0}^{x_0+h} p_{XY}(x, y)\, dxdy$$
3) $$\lim_{k\to 0}\lim_{h\to 0}\int_{y=y_0}^{y_0+k}\int_{x=x_0}^{x_0+h} p_{XY}(x, y)\, dxdy$$

The integrating regions tend to be equal to

1) $-\infty < x < x_0, y = y_0$
2) x=x_0,$-\infty < y < y_0$
3) point (x_0, y_0)

Let these be regions 1,2,3 respectively
$\Rightarrow$ P(X, Y) should be defined over regions 1,2,3. i.e

1) p$_{XY}(-\infty < x < x_0, y = y_0)$
2) p$_{XY}(x = x_0, -\infty < y < y_0)$
3) p$_{XY}(x, y)$ should be defined

□

Now lets verify the options

1) Option 1 fails to satisfy the lemma 2.1 as it defines $p_{XY}(x, y)$ only in region 3 but not 1,2.
Counter-example:-
Let's consider an example similar to dirac-delta function.

$$P_{XY}(x, y) = \infty \quad (x, y) = (x_0, y_1) \text{ where } y_1 < y_0$$
$$= 0 \quad \text{otherwise} \tag{2.0.6}$$

Here $P_{XY}(x, y)$ is 0 still F_{XY} is not continuous at (x_0, y_0) parellel to X-axis

$$F_{XY}(x, y) = 1 \qquad x > x_0, y > y_1$$
$$= 0 \qquad otherwise \tag{2.0.7}$$

So option 1 is false

2) Option 2 also fails to satisfy lemma 2.1 as it defines $p_{XY}(x, y)$ only in region 3 and either region 1 or region 2.
If we take the same example in equation 2.0.6 it satisfies $p_{XY}(Y = y)$=0 i.e option 2 but F_{XY} is not continuous at (x, y).
So option 2 is also false
Another counter example is:-

$$P_{XY}(x, y) = \infty \quad (x, y) = (x_1, y_0) \text{ where } x_1 < x_0$$
$$= 0 \quad \text{otherwise} \tag{2.0.8}$$

This is also a function similar to dirac-delta function

3) Option 3 satisfies lemma 2.1.
So option 3 is true

4) Option 4 also satisfies lemma 2.1.
So option 4 is also true

Hence correct options are 3,4.