Chapter 5: Eigenvalues, Eigenvectors, and Invariant Subspaces

Exercises 5A - Invariant Subspaces

Problem 1

Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V.

(a) Prove that if $U \subseteq \operatorname{null} T$, then U is invariant under T.

(b) Prove that if $\operatorname{range} T \subseteq U$, then U is invariant under T.

Proof. (a) Consider $u \in U$. $Tu = 0$. $0 \in U$. So $T(u) \in U$. Hence, U is invariant under T.

(b) Consider $u \in U$. $T(u) \in range\ T \subseteq U$. Hence $T(u) \in U$. Hence U is invariant under T. □

Problem 2

Suppose $T \in \mathcal{L}(V)$ and $V_1, \ldots V_m$ are invariant subspaces of T. Prove $V_1 + \ldots + V_m$ is invariant under T.

Proof. Consider an arbitrary element $v \in V_1 + \ldots + V_m$. It can be expressed as $v = v_1 + \ldots + v_m$, where $v_i \in V_i$. $T(v) = T(v_1 + \ldots + v_m) = T(v_1) + \ldots + T(v_m)$. Since V_i is invariant under T, $T(v_i) \in V_i$. Hence $T(v) \in V_1 + \ldots + V_m$ if $v \in V_1 + \ldots + V_m$. □

Problem 3

Suppose $T \in \mathcal{L}(V)$. Prove that the intersection of every collection of subspaces of V invariant under T is invariant under T.

Proof. Consider an arbitrary element v in the intersection of the invariant subspaces $((V_1, V_2, \ldots))$. Since each of the subspaces are invariant, $T(v) \in V_i$. Hence $T(v) \in V_1 \cap V_2 \cap \ldots$. Hence intersection of invariant subspaces under some linear transformation is invariant. □

Problem 4

Prove or give a counterexample: If V is finite-dimensional and U is a subspace of V that is invariant under every operator on V, then $U = \{0\}$ or $U = V$.

Proof. Suppose to the contrary that U is neither the trivial subspace, nor the full space. Since $U \neq \{0\}$, there is at least one basis vector in U. Since $U \neq V$, there exists at least one vector $v \notin U$. Consider any operator that maps u to v. Obviously this map does not keep U invariant. □

Problem 5

Suppose $T \in \mathcal{L}(\mathbb{R}^2)$ defined by $T(x, y) = (-3y, x)$. Find the eigenvalues of T.

Proof. Suppose to the contrary that U is neither the trivial subspace, nor the full space. Since $U \neq \{0\}$, there is at least one basis vector in U. Since $U \neq V$, there exists at least one vector $v \notin U$. Consider any operator that maps u to v. Obviously this map does not keep U invariant. $\square$