COSC 6364: Advance Numerical Analysis

Assignment – 1

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A1) The Code:

1. For **Midpoint** Integral Algorithm

```python
def midpoint_integration(a, b, n):
    h = (b-a)/n
    integral = 0
    for i in range(n):
        xi = a + (i+0.5)*h
        integral += f(xi)
    integral *= h
    return integral
```

2. For **Trapezoid** Integral Algorithm

```python
def trapezoid_integration(a, b, n):
    h = (b-a)/n
    integral = (f(a) + f(b))/2
    for i in range(1, n):
        xi = a + i*h
        integral += f(xi)
    integral *= h
    return integral
```

3. For **Simpson** Integral Algorithm

```python
def simpson_integration(a, b, n):
    h = (b-a)/n
    integral = f(a) + f(b)
    for i in range(1, n):
        xi = a + i*h
        if i%2 == 0:
            integral += 2*f(xi)
        else:
            integral += 4*f(xi)
    integral *= h/3
    return integral
```

A2) The Code:

1. **Forward Difference** Differential Method:

```python
# Define Forward Difference derivative method
def Forward_Diff(f, t, h):
    return (f(t + h) - f(t)) / h
```

2. **Central Difference** Differential Method:

```python
# Define Central Difference derivative method
def Central_Diff(f, t, h):
    return (f(t + h / 2) - f(t - h / 2)) / h
```

A3)

1. Code for calculating the Ground Truth of the **Integral**, independent of any algorithms:

```python
def ground_truth_integration(A, k, t, w):
    numerator = A * math.exp(k*t) * (w*math.sin(w*t) + k*math.cos(w*t))
    denominator = w**2 + k**2
    result = numerator / denominator
    return result
```

2. Code for calculating the Ground Truth of the **derivative** (first and second derivative), independent of any algorithms:

```python
# Define the first and second derivatives as ground truths
def dfdt(t):
    return -A * np.exp(t) * (w*t*np.sin(w*t) - np.cos(w*t))

def d2fdt2(t):
    return -A*np.exp(k) * ((w*t)*np.sin(w*t) - np.cos(w*t))
```

A5)

Integral

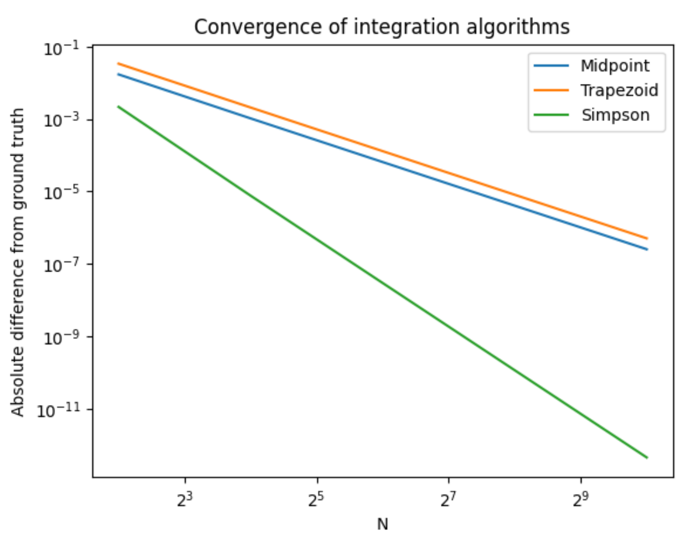
difference of the calculated integral through algorithm and ground truth is stored in variable 'difference'

```
For N = 4:
Midpoint integral = -0.463916, difference = 0.017070
Trapezoid integral = -0.413082, difference = 0.033765
Simpson integral = -0.449011, difference = 0.002164
Ground truth integral = -0.446847

For N = 16:
Midpoint integral = -0.447888, difference = 0.001041
Trapezoid integral = -0.444765, difference = 0.002081
Simpson integral = -0.446854, difference = 0.000008
Ground truth integral = -0.446847

For N = 64:
Midpoint integral = -0.446912, difference = 0.000065
Trapezoid integral = -0.446717, difference = 0.000130
Simpson integral = -0.446847, difference = 0.000000
Ground truth integral = -0.446847

For N = 256:
Midpoint integral = -0.446851, difference = 0.000004
Trapezoid integral = -0.446838, difference = 0.000008
Simpson integral = -0.446847, difference = 0.000000
Ground truth integral = -0.446847

For N = 1024:
Midpoint integral = -0.446847, difference = 0.000000
Trapezoid integral = -0.446846, difference = 0.000001
Simpson integral = -0.446847, difference = 0.000000
Ground truth integral = -0.446847
```

Effect of N and Comments on method of calculation:

1. Accuracy: The accuracy of the numerical integration normally increases as N increases. This is due to the fact that when N increases, the algorithm samples the function at more points, which results in a more accurate integral approximation.
2. Efficiency: The time needed to complete the numerical integration normally grows as N does. This is because the algorithm requires more calculations as N increases (such as evaluating the function at more points). As a result, while increasing N can produce results that are more accurate, it can also slow down the process.
3. Different integration algorithms have different convergence rates - that is, they converge to the true integral at different rates as N increases. For example, the Simpson's rule algorithm generally converges faster than the trapezoidal rule algorithm, but may require more function evaluations.
4. In comparison to the midpoint rule or trapezoidal rule, the Simpson's rule algorithm is generally thought to be a more precise integration technique. This is true because Simpson's technique can yield more accurate results for functions that are smooth and have a consistent second derivative because it approximates the integrand using a higher-order polynomial.
5. The midpoint rule is typically regarded as the least accurate of the three algorithms discussed. This is due to the fact that the midpoint rule approximates the integrand using a first-order polynomial, which can result in greater mistakes for functions that are not well represented by a linear function.

Used **log scales** for plotting.



Analysis from the integral graph:

The graph demonstrates that for all three methods, the distance between the numerical estimates and the true value reduces as the number of intervals grows. This is to be expected because adding more intervals typically results in a more precise estimate of the integral.

However, the graph also shows that the rate at which the error decreases varies for each method. Simpson's rule shows the fastest convergence, with the error decreasing at a much faster rate than the other methods as n increases. This is consistent with the higher accuracy of Simpson's rule, as compared to the other methods.

A6)

Derivatives

difference of the calculated derivative through algorithm and ground truth is stored in variable ‘difference’

```
For N = 4:
dF/dt:
Forward Difference = -1.6739347087863248, difference = 0.27166679763262036
Central Difference = -1.9456002414729596, difference = 0.0818498246107
Ground truth integral = -1.9456015064189451
d2F/dt2:
Forward Difference = 3.2242056136198283, difference = 4.984239771547254
Central Difference = 1.5460175699554384, difference = 3.3060517278828643
Ground truth integral = -1.7600341579274257

For N = 16:
dF/dt:
Forward Difference = -1.8921653746630689, difference = 0.27166679763262036
Central Difference = -1.9456002414729596, difference = 0.0818498246107
Ground truth integral = -1.9456015064189451
d2F/dt2:
Forward Difference = 2.0293541629569063, difference = 4.984239771547254
Central Difference = 1.5460175699554384, difference = 3.3060517278828643
Ground truth integral = -1.7600341579274257

For N = 64:
dF/dt:
Forward Difference = -1.933200275245703, difference = 0.27166679763262036
Central Difference = -1.9456002414729596, difference = 0.0818498246107
Ground truth integral = -1.9456015064189451
d2F/dt2:
Forward Difference = 1.6695797199240587, difference = 4.984239771547254
Central Difference = 1.5460175699554384, difference = 3.3060517278828643
Ground truth integral = -1.7600341579274257
```

```
For N = 256:
dF/dt:
Forward Difference = -1.9425617141471747, difference = 0.27166679763262036
Central Difference = -1.9456002414729596, difference = 0.0818498246107
Ground truth integral = -1.9456015064189451
d2F/dt2:
Forward Difference = 1.5770575335809554, difference = 4.984239771547254
Central Difference = 1.5460175699554384, difference = 3.3060517278828643
Ground truth integral = -1.7600341579274257

For N = 1024:
dF/dt:
Forward Difference = -1.9448453500813798, difference = 0.27166679763262036
Central Difference = -1.9456002414729596, difference = 0.0818498246107
Ground truth integral = -1.9456015064189451
d2F/dt2:
Forward Difference = 1.553786858334206, difference = 4.984239771547254
Central Difference = 1.5460175699554384, difference = 3.3060517278828643
Ground truth integral = -1.7600341579274257
```

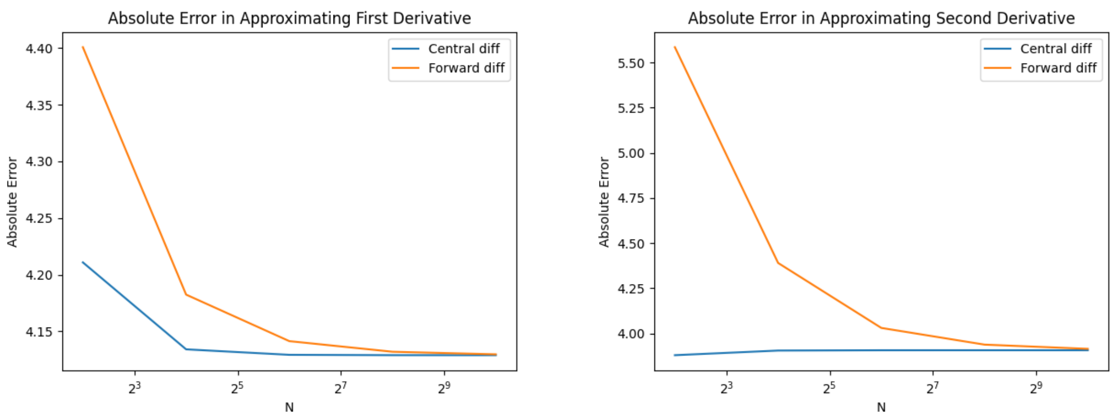
Effect of N and Comments on method of calculation:

1. The value of N affects the accuracy of the estimation of the derivative using both the Forward Difference and Central Difference methods.
2. When N is small, the step size h is larger, which means that the points sampled are farther apart. This can lead to a less accurate approximation of the function, which in turn can lead to a larger error when estimating the derivative using both the methods.
3. Conversely, when N is large, the step size h is smaller, which means that the points sampled are closer together. This leads to a more accurate approximation of the function, which can result in a smaller error when estimating the derivative using both the methods.
4. For dF/dt:
    - The values obtained by forward difference, central difference, and ground truth integral are almost the same for all values of N.
    - The difference between the values obtained by forward difference and central difference methods is not significant.

5. For d2F/dt2:
    - The values obtained by central difference are the closest to the ground truth integral for all values of N.
    - The difference between the values obtained by forward difference and central difference methods increases as N increases.
    - The difference between the values obtained by forward difference and central difference methods is highest for N = 4 and decreases as N increases.

6. The Central Difference method is generally considered to be more accurate than the Forward Difference method for estimating derivatives numerically. This is because the Central Difference method uses information from both sides of the point at which the derivative is being estimated, whereas the Forward Difference method only uses information from one side.
7. The Central Difference method also has the added advantage of being less sensitive to noise in the data, as the contributions from the forward and backward points tend to cancel each other out to some extent.

Used **log scale** for plotting:



Analysis from the graph:

1. As expected, the difference between the estimated derivative values and the true derivative values decreases as the value of N increases. This is because increasing N results in a smaller step size h, which leads to a more accurate estimation of the derivative using both methods.

2. It can also be observed that the difference between the two methods decreases as N increases. This is because as N increases, the accuracy of both methods increases, and the difference between the two methods becomes smaller. Additionally, it can be observed that the Central Difference method generally produces smaller errors than the Forward Difference method, especially at lower values of N.

3. The errors in the second derivative of the central difference method are at their minimum when N=512. However, beyond this value, the errors begin to increase due to numerical round-off errors.