Gaussian Quadrature Weights and Abscissae

This page is a tabulation of weights and abscissae for use in performing Legendre-Gauss quadrature integral approximation, which tries to solve the following function

$$\int_a^b f(x)dx = \sum_{i=1}^{\infty} w_i f(x_i) \simeq \sum_{i=1}^{n} w_i f(x_i)$$

by picking approximate values for n, w_i and x_i. While only defined for the interval [-1,1], this is actually a universal function, because we can convert the limits of integration for any interval [a,b] to the Legendre-Gauss interval [-1,1]:

$$\int_a^b f(x)dx = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}x_i + \frac{b+a}{2}\right)dx \simeq \frac{b-a}{2}\sum_{i=1}^{n} w_i \cdot f\left(\frac{b-a}{2}x_i + \frac{b+a}{2}\right)$$

The summation function is called the Legendre-Gauss quadrature rule because the abscissae x_i in the Gauss quadrature function for [-1,1] are defined as the roots of the Legendre polynomial for n:

$$P_n(x) = \frac{1}{2\pi i}\oint (1-2tx+r^2)^{-1/2}t^{-n-1}dt$$

with the weights w_i coming from the following function:

$$w_i = -\frac{2}{(1-x_i^2)[P_n'(x_i))]^2}$$

(The procedure is explained in more detail here, with a very accessible video lecture on the theory behind the Gauss quadrature rule here)

The tables provided here give the values for x_i and w_i for n=2 through n=64, with an internal decimal precision of 256, limited to 16 decimals due to floating point number limits in PHP. These tables were made using the numbers that are generated by the following series of Mathematica instructions:

```
symboliclegendre[n_, x_] := Solve[LegendreP[n, x] == 0];
legendreprime[n_, a_] := D[LegendreP[n, x], x] /. x -> a;
weights[n_, x_] := 2/((1 - x^2) legendreprime[n, x]^2);

(*how many terms should be generated*)
h = 64;

(* what numerical precision is desired? *)
precision = 256;

str = OpenWrite["lgvalues.txt"];
Write[str, "abcissae"];
Do[Write[str]; Write[str, n]; Write[str];
  nlist = symboliclegendre[n, x];
  xnlist = x /. nlist;
  Do[Write[str, N[Part[xnlist, i], precision]], {i, Length[xnlist]}];, {n, 2, h}];
Write[str, "weights"];
Do[Write[str]; Write[str, n]; Write[str];
  slist := symboliclegendre[n, x];
  xslist = x /. slist;
  Do[Write[str, N[weights[n, Part[xslist, i]], precision]], {i, Length[xslist]}];, {n, 2, h}];
Close[str];
```

If you need higher precision than is presented on this web page you can either download the abscissae and weights files below, or you can run this program in Mathematica yourself, with higher precision and/or higher *h* values.

- abscissae (607KB, easily converted to not-PHP)
- weights (615KB, easily converted to not-PHP)

Weights and Abscissae Tables for n=2 to n=64

n = 2

jump to n = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64

i	weight - w_i	abscissa - x_i
1	1.0000000000000000	-0.5773502691896257
2	1.0000000000000000	0.5773502691896257

n = 3

jump to n = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64

i	weight - w_i	abscissa - x_i
1	0.8888888888888888	0.0000000000000000
2	0.5555555555555556	-0.7745966692414834
3	0.5555555555555556	0.7745966692414834

n = 4

jump to n = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64

i	weight - w_i	abscissa - x_i
1	0.6521451548625461	-0.3399810435848563
2	0.6521451548625461	0.3399810435848563
3	0.3478548451374538	-0.8611363115940526
4	0.3478548451374538	0.8611363115940526

n = 5

jump to n = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64

i	weight - w_i	abscissa - x_i
1	0.5688888888888889	0.0000000000000000
2	0.4786286704993665	-0.5384693101056831
3	0.4786286704993665	0.5384693101056831
4	0.2369268850561891	-0.9061798459386640
5	0.2369268850561891	0.9061798459386640

n = 6

jump to n = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64

i	weight - w_i	abscissa - x_i

i	weight - w_i	abscissa - x_i
1	0.3607615730481386	0.6612093864662645
2	0.3607615730481386	-0.6612093864662645
3	0.4679139345726910	-0.2386191860831969
4	0.4679139345726910	0.2386191860831969
5	0.1713244923791704	-0.9324695142031521
6	0.1713244923791704	0.9324695142031521

n = 7

jump to n = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64

i	weight - w_i	abscissa - x_i
1	0.4179591836734694	0.0000000000000000
2	0.3818300505051189	0.4058451513773972
3	0.3818300505051189	-0.4058451513773972
4	0.2797053914892766	-0.7415311855993945
5	0.2797053914892766	0.7415311855993945
6	0.1294849661688697	-0.9491079123427585
7	0.1294849661688697	0.9491079123427585

n = 8

jump to n = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64

i	weight - w_i	abscissa - x_i
1	0.3626837833783620	-0.1834346424956498
2	0.3626837833783620	0.1834346424956498
3	0.3137066458778873	-0.5255324099163290
4	0.3137066458778873	0.5255324099163290
5	0.2223810344533745	-0.7966664774136267
6	0.2223810344533745	0.7966664774136267
7	0.1012285362903763	-0.9602898564975363
8	0.1012285362903763	0.9602898564975363

n = 9

jump to n = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64

i	weight - w_i	abscissa - x_i
1	0.3302393550012598	0.0000000000000000
2	0.1806481606948574	-0.8360311073266358
3	0.1806481606948574	0.8360311073266358
4	0.0812743883615744	-0.9681602395076261
5	0.0812743883615744	0.9681602395076261
6	0.3123470770400029	-0.3242534234038089
7	0.3123470770400029	0.3242534234038089

i	weight - w_i	abscissa - x_i
8	0.2606106964029354	-0.6133714327005904
9	0.2606106964029354	0.6133714327005904

n = 10

jump to n = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64

i	weight - w_i	abscissa - x_i
1	0.2955242247147529	-0.1488743389816312
2	0.2955242247147529	0.1488743389816312
3	0.2692667193099963	-0.4333953941292472
4	0.2692667193099963	0.4333953941292472
5	0.2190863625159820	-0.6794095682990244
6	0.2190863625159820	0.6794095682990244
7	0.1494513491505806	-0.8650633666889845
8	0.1494513491505806	0.8650633666889845
9	0.0666713443086881	-0.9739065285171717
10	0.0666713443086881	0.9739065285171717

n = 11

jump to n = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64

i	weight - w_i	abscissa - x_i
1	0.2729250867779006	0.0000000000000000
2	0.2628045445102467	-0.2695431559523450
3	0.2628045445102467	0.2695431559523450
4	0.2331937645919905	-0.5190961292068118
5	0.2331937645919905	0.5190961292068118
6	0.1862902109277343	-0.7301520055740494
7	0.1862902109277343	0.7301520055740494
8	0.1255803694649046	-0.8870625997680953
9	0.1255803694649046	0.8870625997680953
10	0.0556685671161737	-0.9782286581460570
11	0.0556685671161737	0.9782286581460570

n = 12

jump to n = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64

i	weight - w_i	abscissa - x_i
1	0.2491470458134028	-0.1252334085114689
2	0.2491470458134028	0.1252334085114689
3	0.2334925365383548	-0.3678314989981802
4	0.2334925365383548	0.3678314989981802
5	0.2031674267230659	-0.5873179542866175

i	weight - w_i	abscissa - x_i
6	0.2031674267230659	0.5873179542866175
7	0.1600783285433462	-0.7699026741943047
8	0.1600783285433462	0.7699026741943047
9	0.1069393259953184	-0.9041172563704749
10	0.1069393259953184	0.9041172563704749
11	0.0471753363865118	-0.9815606342467192
12	0.0471753363865118	0.9815606342467192

n = 13

jump to n = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64

i	weight - w_i	abscissa - x_i
1	0.2325515532308739	0.0000000000000000
2	0.2262831802628972	-0.2304583159551348
3	0.2262831802628972	0.2304583159551348
4	0.2078160475368885	-0.4484927510364469
5	0.2078160475368885	0.4484927510364469
6	0.1781459807619457	-0.6423493394403402
7	0.1781459807619457	0.6423493394403402
8	0.1388735102197872	-0.8015780907333099
9	0.1388735102197872	0.8015780907333099
10	0.0921214998377285	-0.9175983992229779
11	0.0921214998377285	0.9175983992229779
12	0.0404840047653159	-0.9841830547185881
13	0.0404840047653159	0.9841830547185881

n = 14

jump to n = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64

i	weight - w_i	abscissa - x_i
1	0.2152638534631578	-0.1080549487073437
2	0.2152638534631578	0.1080549487073437
3	0.2051984637212956	-0.3191123689278897
4	0.2051984637212956	0.3191123689278897
5	0.1855383974779378	-0.5152486363581541
6	0.1855383974779378	0.5152486363581541
7	0.1572031671581935	-0.6872929048116855
8	0.1572031671581935	0.6872929048116855
9	0.1215185706879032	-0.8272013150697650
10	0.1215185706879032	0.8272013150697650
11	0.0801580871597602	-0.9284348836635735
12	0.0801580871597602	0.9284348836635735