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Numerical Strategies for Solving the Nonlinear Rational Expectations Commodity Market Model

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Abstract. In this paper, I compare the accuracy, efficiency and stability of different numerical strategies for computing approximate solutions to the nonlinear rational expectations commodity market model. I find that polynomial and spline function collocation methods are superior to the space discretization, linearization and least squares curve-fitting methods that have been preferred by economists in the past.

Key words: nonlinear rational expectations models, numerical methods

Economists have long been interested in commodity storage, its role in the intertemporal allocation of commodity supplies and its impact on commodity price dynamics. The earliest work on markets for storable commodities sought to explain the strong, regular relationship exhibited by commodity spot-futures price spreads and commodity stock levels. Later work sought to derive normative rules prescribing the amounts of a commodity that should be stored in order to maximize societal welfare. More recent work has focused on the stabilizaton and welfare distribution effects of private speculative storage and government buffer stock interventions.

Many recent commodity markets studies have employed dynamic stochastic partial equilibrium models in which agents form price expectations rationally. The rational expectations commodity market model possesses certain features that are essential for a useful theory of price formation in markets for storable commodities. One important feature of the model is that it endogenizes futures prices by structurally linking their formation to the optimizing behavior of commodity stockholders, producers and consumers. Another important feature of the model is that it allows producers and storers to revise their price expectations given exogeneous changes in their market environment. This feature is essential if the model is to be used to predict the impacts of changes in market structure or shifts in government policy.

Applications of the rational expectations commodity market model have been impeded by the analytical difficulties associated with solving it. The commodity rational expectations model presents a functional equation problem whose solution takes the form of an entire function rather than a finite-dimensional vector of prices and quantities. The functional equation that underlies the nonlinear rational expectations commodity market model is known to have a unique solution (Scheinkman and Schechtman). The solution, however, is not expressible in closed form and cannot be derived exactly using standard algebraic techniques. The nonlinear rational expectactions commodity market model can only be solved numerically on a computer.

In this paper, I examine various numerical strategies for computing approximate solutions to the nonlinear rational expectations commodity market model. One of my goals is to show economists having limited experience with numerical methods how numerical approximation, integration and rootfinding techniques may be combined to solve the model accurately and efficiently[1] Another goal is to criticize constructively the numerical strategies that have been used by economists in the past to solve the model. Specifically, I demonstrate that space discretizaton, linear approximation and least squares curve-fitting techniques that have been popular among economists are either inefficient or incapable of providing precise approximations to the true solution. As an alternative, I show that spline function and polynomial collocation methods are highly efficient and can provide extremely precise approximations to the true solution.[2]

The paper is organized as follows. I first introduce a simple rational expectations model of a market for a storable commodity. I then characterize the model's rational expectations equilibrium and explain why it cannot be derived using standard algebraic techniques. Then, I present alternative numerical methods for computing an approximate equilibrium for the model. I conclude by comparing the strengths and weaknesses of the various methods with reference to a specific numerical example.

1. The Commodity Market Model

The centerpiece of the classical theory of storage is the competitive intertemporal arbitrage equation

$$\delta E_t p_{t+1} - p_t = c(x_t). \tag{1}$$

The intertemporal arbitrage equation asserts that, in equilibrium, the discounted expected appreciation in the commodity price p_t must equal the unit cost of storage $c(x_t)$. Dynamic equilibrium in the commodity market is enforced by competitive expected-profit-maximizing storers. When expected appreciation exceeds the storage cost, the attendant profits induce storers to increase their stockholdings until the equilibrium is restored and inversely.

According to classical theory, the unit storage cost $c(x_t)$ is a nondecreasing function of the amount stored x_t (see Figure 1). The unit storage cost represents the marginal physical cost of storage less the marginal 'convenience yield', which is the amount processors are willing to pay to have sufficient stocks available to avoid costly production adjustments. If stock levels are high, the marginal convenience

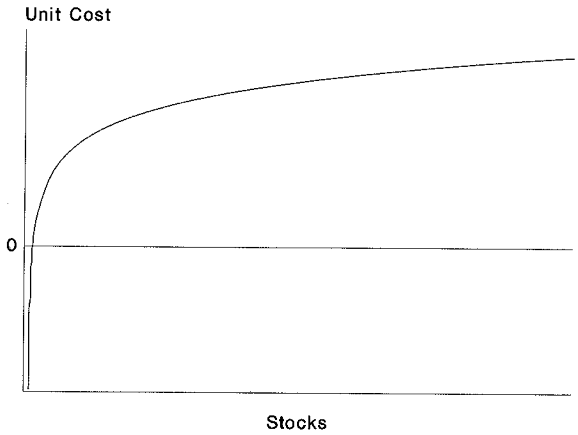

Figure 1. Cost of storage.

yield is zero and the unit storage cost equals the physical storage cost. As stock levels approach zero, however, the marginal convenience yield rises, eventually resulting in a negative unit storage cost (Kaldor; Working 1948, 1949; Williams). The classical model has received strong empirical support over the years and captures the key stylized fact of markets for storable commodities: the coincidence of negative intertemporal price spreads and low, but positive, stock levels (e.g., Brennan; Telser).

The modern theory of storage extends the classical model to a partial equilibrium model of price-quantity determination by appending supply, demand and market clearing conditions to the intertemporal arbitrage equation (e.g., Gardner; Williams and Wright). For the sake of discussion, let us consider a simple agricultural commodity market model with lagged production. Denote quantity consumed by q_t, acreage planted by a_t, per-acre yield by y_t, available supply by s_t, and the per-period discount factor by δ. Assume that the market clearing price is a decreasing function of the quantity consumed

$$p_t = p(q_t), \tag{2}$$

that acreage planted is an increasing function of the expected discounted per-acre revenue the following period

$$a_t = a(\delta E_t p_{t+1} y_{t+1}), \tag{3}$$

that available supply is either consumed in the current period or stored

$$s_t = q_t + x_t \tag{4}$$

and that the supply available next period will be the sum of current carryout and next period's harvest

$$s_{t+1} = x_t + a_t \cdot y_{t+1}. \tag{5}$$

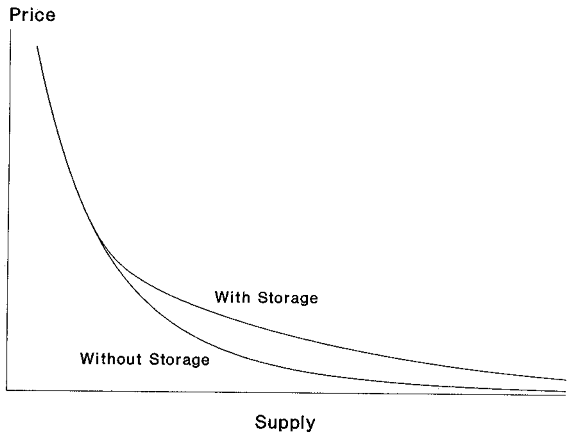

Figure 2. Equilibrium price.

The modern storage model is closed by assuming that price expectations are consistent with the other structural assumptions of the model. The so-called rational expectations assumption endogenizes the expected future price while preserving internal consistency of the model (Muth, Lucas).

The solution of the nonlinear rational expectations commodity market model is illustrated in Figures 2–4. These figures show, respectively, equilibrium price, carryout and acreage in terms of available supply. For comparison, Figure 1 also shows the inverse consumpton demand function $p(\cdot)$, which gives the market price that would prevail in the absence of storage. At low supply levels, there is effectively no storage and the equilibrium price coincides with the inverse consumption demand function. Over this range, acreage supply is not significantly affected by variations in available supply. At sufficiently high supply levels, incentives for speculative storage begin to appear. Over this range, the equilibrium price, which reflects both consumption and storage demand, exceeds the inverse consumption demand function. Higher stock levels reduce price expectations and diminish the incentives to plant.

2. Rational Expectation Equilibrium

The nonlinear rational expectatons commodity market model cannot be solved using standard algebraic techniques. To see this, let $\lambda(s)$ denote the equilibrium price implied by the model for a given available supply s (see Figure 1). Having the equilibrium price function $\lambda(\cdot)$, the rational ex-ante expected price and per-acre revenue could be computed by integrating over the yield distribution

$$E_t p_{t+1} = E_y \lambda(x_t + a_t \cdot y), \tag{6}$$

$$E_t p_{t+1} y_{t+1} = E_y \lambda(x_t + a_t \cdot y) \cdot y. \tag{7}$$

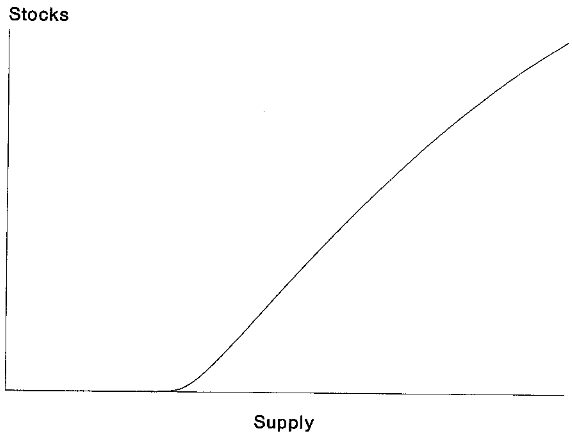

Figure 3. Equilibrium storage.

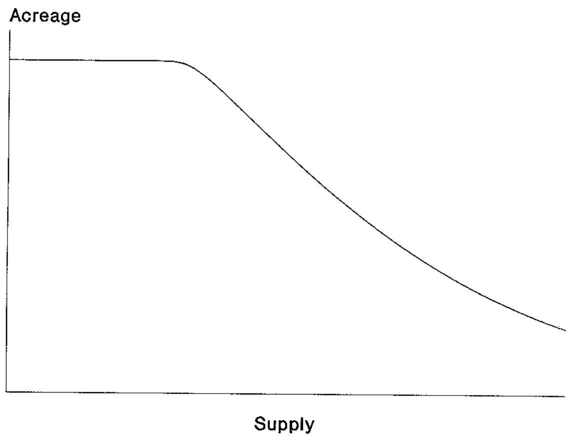

Figure 4. Equilibrium acreage.

Appending these two equations to (1)–(4) results in a system of six nonlinear algebraic equations that, in principle, could be solved for the six unknowns.

Unfortunately, the equilibrium price function $\lambda(\cdot)$ is not known *a priori* and deriving it, the key to solving the commodity market model, is a nontrivial functional equation problem. Combining (1)–(7), we see that $\lambda(\cdot)$ must simultaneously satisfy an infinite number of conditions. Specifically, for every realizable supply s,

$$\lambda(s) = p(s - x), \tag{8}$$

where stock x and acreage a solve

$$\delta E_y \lambda(x + a \cdot y) - p(s - x) = c(x), \tag{9}$$

$$a = a(\delta E_y \lambda(x + a \cdot y) \cdot y). \tag{10}$$

Functional equations can be solved in a variety of ways.[3] An approach that sometimes works is to posit that $\lambda(\cdot)$ is a polynomial of given order, fix its coefficients by imposing an equal number of conditions, and hope that all remaining conditions implied by the theory are satisfied. This approach, called the method of undetermined coefficients, is known to work if the consumption demand, acreage supply, and cost of storage functions are assumed linear and if the nonnegatively of stocks is ignored. The nonlinearity of the storage cost function and the nonnegativity of stocks, however, are two essential stylized facts that must be captured by any useful theory of commodity market dynamics. Thus, the undetermined coefficients approach to solving the rational expectations commodity market model is not generally fruitful.[4]

3. Numerical Solution Strategies

To solve the commodity market model numerically one must approximate the infinite-dimensional functional equation problem posed by (8)–(10) with a finite-dimensional problem that can be solved on a computer. The process of forming such an approximation is called discretization. There are many discretization strategies one can employ. I begin by discussing a strategy called the collocation method. The collocation method replaces the infinite-dimensional functional equation problem with a finite-dimensional nonlinear equation problem (Judd 1991, 1992).

The collocation method involves three steps: First, $\lambda(\cdot)$ is approximated using a finite linear combination of known functions $\phi_0, \phi_1, \phi_2, \ldots, \phi_n$, called the basis functions

$$\lambda(s) \approx \sum_{j=0}^{n} c_j \phi_j(s). \tag{11}$$

Second, (8)–(10) are required to hold exactly at only finitely-many points $s_0, s_1, s_3, \ldots, s_n$, called the collocation nodes. Third, using Gaussian quadrature principles, the continuous yield distribution is replaced with an approximating m-point discrete distribution. The values $y_1, y_2, \ldots, y_m$ assumed by the discrete yield variable and the associated probabilities $w_1, w_2, \ldots, w_m$ are fixed by requiring the discrete yield distribution to possess the same first $2m - 1$ moments as the original yield distribution.[5]

The discretization scheme employed in the collocation method replaces the original functional equation problem with a nonlinear equation problem. Specifically, the approximation to $\lambda(\cdot)$ in (11) is derived by solving the $3n + 3$ equations

$$\sum_{j=0}^{n} c_j \phi_j(s_i) = p(s_i - x_i), \tag{12}$$

$$\delta \sum_{k=1}^{m} \sum_{j=0}^{n} w_k c_j \phi_j(x_i + a_i y_k) - p(s_i - x_i) = c(x_i), \tag{13}$$

$$a_i = a\left(\delta \sum_{k=1}^{m} \sum_{j=0}^{n} w_k c_j \phi_j(x_i + a_i y_k) y_k\right), \tag{14}$$

for the $3n + 3$ unkowns $c_i, a_i, x_i, i = 0, 1, 2, \dots, n$.

The nonlinear equation system (12)–(14) can be solved using an iterative nonlinear rootfinding technique such as Newton's or Broyden's method (Judd 1991; Atkinson). One can also use a simple function iteration algorithm:

(0) Initial Step: Select the degree of approximation n; select the basis functions $\phi_0, \phi_1, \dots, \phi_n$; select the collocation nodes $s_0, s_1, \dots, s_n$; and guess the values of $c_0, c_1, \dots, c_n$.
(1) Solution Step: Holding the coefficients $c_0, c_1, \dots, c_n$ fixed, find, for each $i = 0, 1, \dots, n$, the x_i and a_i that solve the nonlinear equation system (13)–(14).
(2) Update Step: Holding the stock levels $x_0, x_1, \dots, x_n$ fixed, find the coefficients $c'_0, c'_1, \dots, c'_n$ that solve the linear equation system

$$\sum_{j=0}^{n} c'_j \phi_j(s_i) = p(s_i - x_i), \tag{15}$$

for $i = 0, 1, \dots, n$.
(3) Convergence Check: If $|c'_j - c_j| < \tau$ for all j and some convergence tolerance τ, set $c_j = c'_j$ for all j and stop; otherwise set $c_j = c'_j$ for all j and return to step 1.

I recommend solving (13)–(14) using Newton's method and (15) using L-U factorization, the methods most commonly used to solve general linear and nonlinear equation systems, respectively (see Atkinson or Press).

Collocation schemes differ in how the collocation nodes and basis functions are selected. I examine four such schemes, denoting the minimum realizable supply by $\underline{s}$ and the maximum realizable supply by $\bar{s}$.[6]

3.1. Uniform-node Polynomial Approximation

In this scheme, the collocation nodes are equally spaced, $s_i = \underline{s} + i \cdot w$, where $w = (\bar{s} - \underline{s})/n$, and the basis functions are the standard monomials $\phi_i(s) = s^i$.

3.2. Chebychev Polynomial Approximation

In this scheme, the collocation nodes are the Chebychev nodes

$$s_i = 0.5(\underline{s} + \bar{s}) + 0.5(\bar{s} - \underline{s})\cos\left(\frac{i + 0.5}{n + 1}\right) \tag{15}$$

and the basis functions are the Chebychev polynomials, which are recursively defined by $\phi_0(s) = 1, \phi_1(s) = s$, and

$$\phi_{i+1}(s) + 2s\phi_i(s) - \phi_{i-1}(s). \tag{16}$$

3.3. Linear Spline Approximation

In this scheme, the collocation nodes are equally spaced, $s_i = \underline{s} + i \cdot w$, where $w = (\bar{s} - \underline{s})/n$, and the basis functions are the so-called tent functions

$$\phi_i(s) = \begin{cases} 1 - |s - s_i|/w & \text{for } s_i - w \leqslant s \leqslant s_i + w, \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

3.4. Cubic Spline Approximation

In this scheme, the collocation nodes are equally spaced, $s_i = \underline{s} + i \cdot w$, where $w = (\bar{s} - \underline{s})/n$, and the basis functions are the so-called cubic b-splines

$$\phi_i(s) = \begin{cases} \frac{2}{3}(1 - 6q^2(1-q)) & \text{if } q = \frac{|s-s_i|}{w} \leqslant 1, \\ \frac{4}{3}(1-q)^3 & \text{if } 1 \leqslant q = \frac{|s-s_i|}{w} \leqslant 2, \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

For comparison purposes, I also examine three other numerical solution strategies that are not collocation methods, but which appear extensively in the literature. These schemes also involve discretizing the problem, that is, replacing it with a soluble finite-dimensional problem.

3.4.1. *Space Discretization*

In this scheme, the equilibrium conditions are ‘integrated’ to form the equivalent stochastic dynamic optimization problem

$$\max E \sum_t \delta^t \left\{ \int_0^{s_t - x_t} p(\cdot) - \int_0^{x_t} c(\cdot) - \int_0^{a_t} a^{-1}(\cdot) \right\}, \tag{19}$$

$$\text{s.t.} \quad s_{t+1} = x_t + a_t \cdot y_{t+1},$$
$$s_t \geqslant x_t \geqslant 0; a_t \geqslant 0.$$

The supply, stock and acreage levels are then restricted to finite sets and the discretized problem is solved using discrete dynamic programming. This technique is used in Gustafson and in Gardner, who pioneered numerical analysis of the commodity market model.

3.4.2. *Least-Squares Curve Fitting*

In this scheme, $\lambda(\cdot)$ is approximated with a low-order polynomial. Using function iteration, the polynomial coefficients are sequentially updated by solving the equilibrium conditions (8)–(10) at n equally space nodes and fitting the polynomial to the computed prices using ordinary least squares. This techniques is used in Williams and Wright.

3.4.3. *Global Linearization*

In this scheme, the acreage supply, consumption demand, and storage cost functions are replaced with their 1st-order Taylor approximations about the certainty-equivalent steady-state. Ignoring the nonnegativity of stocks, the solution to the linearized problem is computed and taken as an approximate solution to the original problem. This technique is equivalent to approximating (19) with a linear-quadratic control problem, a strategy that has been used extensively by macroeconomists to compute approximate solutions to stochastic optimal growth problems (Kendrick; Taylor and Uhlig; Christiano; Rausser).

4. Comparison of Discretization Methods

In order to compare the seven discretization methods discussed above, I refer to an illustrative numerical example. Below, I report computational experience with a market model in which $p(q) = q^{-5}, a(r) = r^{0.8}, c(x) = 0.6 + 0.1\ln(x)$, and $\delta = 0.9$, and in which yield is distributed as the log of a normal $(0, 0.04)$. The elasticities and yield variability are roughly representative of major world grain markets with the quantities and prices normalized so that the longrun mean price, acreage and harvest are each approximately 1.

Table I gives the execution time and approximation error associated with the seven discretization schemes for different numbers of nodes. Figure 5 graphs approximation error against execution time for selected methods. Approximation error is defined as the maximum absolute difference between the 'true' $\lambda(\cdot)$ function and its approximant at points spaced 0.001 units apart over the approximation interval [0.5, 2.0]. Execution times for the collocation schemes are based on function iteration implemented on a 90 megahertz Pentium microcomputer using the Lahey Professional FORTRAN compiler 5.1 under MS-DOS 6.0. As the 'true' function, I used a superapproximant computed on a Cray supercomputer using quadruple precision. Ex post bounds computed for the superapproximant assured that its error was several orders of magnitude lower than those presented in Table I and Figure 5, ensuring the adequacy of the estimated errors.

The linearization method is perhaps the easiest to implement because the discretized problem can be solved directly by hand. The $\lambda(\cdot)$ function of the linearized problem, in particular, is linear and its coefficients can be obtained via the method of undetermined coefficients. As seen in Table I, linearization is the fastest computational method, taking less than one-tenth seconds. Unfortunately, linearization also yields, by far, the poorest approximation. The principal problem with linearization is that the storage cost function, the key structural feature of the commodity market model, cannot be adequately approximated globally by a linear function.

Table I. Execution times and approximation error for different discretization nethods.

Method	Number of nodes	Execution time (seconds)	Maximum absolute error
Chebychev Polynomial	10	0.028	0.5E − 03
	20	0.082	0.5E − 03
	30	0.143	0.1E − 03
	50	0.363	0.2E − 04
	100	1.291	0.1E − 06
	150	2.774	0.2E − 08
Uniform Polynomial	10	0.022	0.1E − 02
	20	0.055	0.3E − 03
	30	2.543	0.4E − 02
Linear Spline	100	0.176	0.2E − 03
	200	0.351	0.5E − 04
	300	0.522	0.2E − 04
	500	0.879	0.7E − 05
	1000	1.752	0.2E − 05
	1500	2.620	0.8E − 06
Cubic Spline	50	0.149	0.7E − 05
	100	0.297	0.2E − 06
	200	0.599	0.1E − 07
	400	1.187	0.8E − 09
	600	1.780	0.2E − 09
	1000	2.966	0.2E − 10
Least Squares	100	0.396	0.4E − 01
	200	0.791	0.4E − 01
	300	1.191	0.4E − 01
	400	1.582	0.4E − 01
	500	2.076	0.4E − 01
	600	2.351	0.4E − 01
Space Discretization	10	0.610	0.5E + 01
	20	0.461	0.2E + 01
	50	7.800	0.4E − 00
	100	30.380	0.1E − 00
	150	71.120	0.4E − 01
	200	129.510	0.3E − 01
Linearization		0.0	2.8E + 03

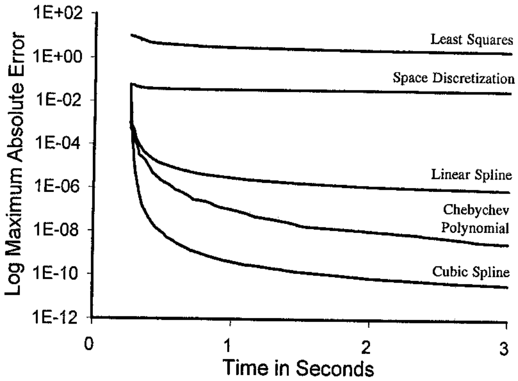

Figure 5. Approximation error as a function of execution time.

A good global approximation, however, is essential because stochastic production ensures that the supply varies over a broad range of values. The more nonlinear the model or the more variable the production, the poorer the approximation provided by linearization. A troubling result implied by the linearized model is that storage is negative at sufficiently low supply levels.

Space discretization is perhaps the next easiest method to implement numerically because the enumerative search procedure used to find the equilibrium carryout and acreage at each supply node is easy to code. In principle, space discretization can provide arbitrarily accurate approximations by increasing the number of supply, stock and acreage nodes. The effort required to achieve a given degree of accuracy with space discretization, however, typically is orders of magnitude greater than with other methods. For example, as seen in Table I, the accuracy afforded by space discretization in over two minutes of computation (200 nodes) can be achieved with Chebychev, linear spline and cubic spline interpolation in less than one-tenth seconds.

There are two basic problems with space discretization. First, the equilibrium acreage and stock at each supply node need not coincide with a knot on the acreage-stock grid. As such, the accuracy of the computed equilibrium is limited by the coarseness of the grid. Second, the enumerative search procedure used to find the approximate equilibrium acreage and stock ignores available information about the location of the solution. Collocation methods, in contrast, can use rootfinding techniques that exploit derivative information to pinpoint the equilibrium stock and acreage at each supply node more quickly and more accurately.

The least squares curve-fitting method is perhaps the most intuitively appealing to economists because it employs a notion of approximation commonly encountered in econometric practice. This approach, however, has severe shortcomings. Most notably, the true equilibrium price function need not be well approximated by a low-order polynomial. In this case, the least squares approximant does not yield a good approximation, regardless of how many data nodes are used for fitting. As the number of nodes is increased, the least squares approximant of a given order converges to a limit. The limiting approximant, however, need not coincide with the true function. As seen in Figure 5, the best least squares approximant, in this case a fourth order polynomial, provides a fit that is five to six orders of magnitude worse than that afforded by Chebychev and cubic spline interpolation in a fraction of a second.

In moving to a collocation scheme, one might be tempted first to try a simple polynomial approximation at equally spaced points. This is the first of the four collocation methods discussed above. One important result from numerical analysis, however, is that uniform nodes do not necessarily yield good polynomial approximations. In particular, there are examples of well-behaved functions for which the approximation error explodes as the number of uniform nodes is increased. Another important result from numerical analysis is that computing the monomial coefficients of an interpolating polynomial is a notoriously ill-conditioned problem, that is, it is severely affected by rounding errors. Problems of ill-conditioning arose quickly when solving the example market model. As seen in Table I, the approximation error actually increased as the number of nodes went from 20 to 30; the algorithm, moreover, would not converge for more than 35 nodes.

Basic results from numerical analysis suggest that the Chebychev nodes, which are not equally spaced, are nearly optimal choices for polynomial interpolation nodes (Judd, 1991; Atkinson). Unlike uniform node interpolation, the approximation error associated with Chebychev node interpolation necessarily goes to zero as the number of nodes rises. The best way to express the polynomial that interpolates a function at the Chebychev nodes is to write it as the linear combination of the Chebychev polynomials. Because the Chebychev polynomials are orthogonal at the Chebychev nodes, the linear equation problem (15) can be solved quickly, accurately, and without problems of numerical stability that accompany a uniform polynomial approximation. The choice of basis functions is, in the vernacular of econometrics, designed to completely eliminate 'multicolinearity' among the 'regressors'. As seen in Figure 5, Chebychev polynomial interpolation is about ten orders of magnitude more accurate than the more popular least squares and space discretization techniques.

The simplest of the collocation shemes is based on linear spline approximation. In linear spline approximation, the values of the function are linearly interpolated at the collocation nodes to obtain the value of the function at no node points. The resulting approximant is continuous, but not continuously differentiable. Linear splines are extremely flexible and use only local information in fitting the approximant. In principle, linear splines can provide as accurate an approximation as desired by increasing the number of nodes (de Boor). Linear splines, moreover, preserve the monotonicity properties of the approximated function, a property not shared by the other collocation strategies discussed here. As can be seen in Table I and Figure 5, linear spline collocation methods provide, in only a short amount of execution time, an approximate solution that is six to seven orders of magnitude more accurate than that afforded by least squares and space discretization techniques.

Cubic spline function approximation offers the best choice over other collocation and non-collocation schemes. In cubic spline approximation, the basis functions are twice continuously differentiable, making them ideal for approximating smooth functions. Like linear splines, cubic splines are extremely flexible and use only local information in fitting the approximant. In principle, cubic splines can provide as accurate an approximation as desired by increasing the number of nodes (de Boor). As can be seen in Table I and Figure 5, cubic spline collocation methods provide, in only a short amount of execution time, an approximate solution that is two orders of magnitude more accurate than the best alternative, Chebychev polynomial interpolation and almost fifteen orders of magnitude more accurate than the more popular techniques of least squares and space discretization.

5. Comparison of Iteration Methods

If a collocation method is chosen to solve the commodity market model, the choice of iterative scheme for solving the equation system (12)–(14) remains. Alternatives to the function iteration algorithm used in the preceding section include Newton and quasi-Newton methods.

The function iteration method is effectively a contraction mapping algorithm and thus converges under very mild conditions on the initial guess for $\lambda(\cdot)$. Function iteration, however, converges only at a linear rate. The Newton method converges at a faster quadratic rate, but under more stringent conditions on the initial guess for $\lambda(\cdot)$. Although the Newton method typically requires fewer iterations than function iteration, each Newton iteration is typically more costly because derivatives must be computed.

Whether the Newton method is faster than function iteration overall is ultimately an empirical question. To illustrate the possible trade-offs between the two methods, I solve the market model using both methods, adopting Chebychev interpolation as the underlying discretization scheme. Table II reports the execution times for both methods for different numbers of nodes. As can be seen in Table II, if the model is solved once using five nodes, Newton's method solves the model in about half the time taken by function iteration. As the number of nodes increases, however, the number of derivatives that must be computed grows quadratically. Eventually, function iteration surpasses Newton's method in speed. With thirty nodes, for example, function iteration is thirteen percent faster than Newton's method.

Newton's method typically requires more development effort because the derivatives must be correctly coded. If the model is to be solved only once, the gains from Newton's method might be too meager to justify incurring the additional development cost. Substantial benefit can be realized from using Newton's method, however, if the model is to be solved many times. Multiple solutions are required if the model must be simulated for many different parameter values. Multiple solu-

Table II. Average execution time under alternative iterative schemes.

Number of runs	Number of nodes	Newton's method	Successive approximation	Ratio
		Time in seconds		
1	5	0.033	0.060	0.55
	10	0.088	0.132	0.67
	15	0.203	0.231	0.88
	20	0.336	0.357	0.94
	25	0.511	0.478	1.07
	30	0.736	0.653	1.13
100	5	0.011	0.050	0.22
	10	0.025	0.098	0.25
	15	0.048	0.170	0.28
	20	0.079	0.268	0.30
	25	0.116	0.384	0.30
	30	0.165	0.514	0.32

tions are also required if the model is structurally estimated, since the model has to be resolved every time the parameters are perturbed by the hill-climbing routine seeking the maximum of the likelihood function (Miranda, 1994).

Table II illustrates the potential benefits from using Newton's method when the model must be solved many times. The table gives the average execution times for both Newton's method and function iteration in solving one hundred instances of the model. The instances are generated by varying the inverse demand parameter from -4 to -6 in equal increments. In solving the model using Newton's method, I updated the Jacobian every four iterations. In this way, Newton's method solved all the models in 22 to 32 percet of the time used by function iteration, depending on the number of nodes.

A potential compromise between the Newton method and function iteration is to use a quasi-Newton method. Broyden's method, the best known quasi–Newton method, converges at a superlinear rate, which is faster than the linear rate of function iteration but slower than quadratic of the Newton method. Broyden's method has the distinct advantage over Newton's method that it does not require the derivatives to be evaluated (or coded). Broyden's method, however, requires a good starting guess for the Jacobian of the nonlinear equation system.

6. Conclusion

In this paper I have examined alternative numerical approaches to solving the nonlinear rational expectations commodity market model. Using a simple numerical example, I have demonstrated the shortcomings of the techniques traditionally used by economists. Linearizaton and least squares curve-fitting methods are incapable of generating arbitrarily accurate approximations. Space discretizaton is extremely inefficient.

As an alternative, I have shown that collocation methods can solve the market model quickly and accurately. Collocation methods replace the underlying functional equation with a system of nonlinear equations that can be solved using standard rootfinding techniques, Collocation based on Chebychev polynomial and cubic spline interpolation, in particular, significantly outperformed all the other schemes examined.

It is arguable that some or all of the traditional techniques solve the simple market model to an acceptable degree of accuracy, in an acceptable period of time. However, it is also undeniable that the basic market model has been and will continue to be, extended to incorporate additional structural features such as government intervention, international trade, and multiple commodities. Also on there have been recent econometric efforts to estimate the nonlinear rational expectations market model subject to all the underlying structural restrictions.[7] These extensions demand far greater numerical efficiency in solving the commodity market model than has been realized in the past. The collocation method discussed in this paper and its multidimensional generalizations can provide the needed gains in efficiency.

Notes

[1] I will make the documented, self-contained FORTRAN code developed for the paper available to academic researchers. I will try to honor all requests placed via electronic mail to miranda.4@osu.edu

[2] Applications of numerical methods to solving dynamic programming problems include Taylor and Uhlig; Tauchen and Hussey; and Johnson et al. Taylor and Uhlig report the computational experience gained by various researchers working on the same stochastic optimal growth model. The researchers, however, used different computers and compilers and adopted different degrees of discretization and convergence tolerances. I hold all of these factors constant in the current paper.

[3] Under mild regularity assumptions, the Contraction Mapping Theorem guarantees one and only one function $\lambda(\cdot)$ simultaneously satisfies conditions (8)–(10) for all supply levels s (Miranda 1986; Scheinkman and Schechtman). Other researchers (e.g., Williams and Wright) focus on deriving the functonal relation between the expected price and the stock level. Their approach is theoretically equivalent to the one presented here since the expected price function can be derived from $\lambda(\cdot)$ and vice versa.

[4] Despite its inherent shortcomings, the linear rational expectations model has been used extensively in empirical commodity market studies (e.g., Hansen and Sargent; Goodwin and Sheffrin; Eckstein).

[5] Including the zeroth moment, the $2m$ moment conditions completely determine the $2n$ unknowns. Gaussian quadrature is discussed in most numerical analysis texts (e.g., Judd 1991; Atkinson).

[6] If the demand, supply, and cost functions possess standard curvature properties, the discretization of the yield stock, which is necessary for numerical work, naturally 'compactifies' the problem, implying a minimum and maximum realizable supply. Supply bounds can be derived analytically in theory, but in practice are more easily computed by trial and error using the program developed to solve the model.

[7] Recent efforts to structurally estimate nonlinear rational expectation commodity models include Miranda and Glauber.

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