SOLVING RATIONAL EXPECTATIONS MODELS

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ABSTRACT

In this chapter, we present theoretical foundations of main methods solving rational expectations models with a special focus on perturbation approaches. We restrict our attention to models with a finite number of state variables. We first give some insights on the solution methods for linear models. Second, we show how to use the perturbation approach for solving non-linear models. We then document the limits of this approach. The perturbation approach, while it is the most common solution method in the macroeconomic literature, is inappropriate in a context of large fluctuations (large shocks or regime switching) and of strong non-linearities (e.g. occasionally binding constraints). The former case is then illustrated extensively by studying regime switching models. We also illustrate the latter case by studying existing methods for solving rational expectations models under the Zero Lower Bound constraint, i.e. the condition of non negativity of the nominal interest rate. Finally, we end up with a brief presentation of global methods which are alternatives when the perturbation approach fails in solving models.

1 INTRODUCTION

This chapter presents main methods for solving rational expectations models. We especially focus on their theoretical foundations rather than on their algorithmic implementations which are well described in Judd (1996). The lack of theoretical justifications in the literature motivates this choice. Amongst other methods, we particularly expound the perturbation approach in the spirit of the seminal papers by Woodford (1986) and Jin and Judd (2002). While most researchers make an intensive use of this method, its mathematical foundations are rarely evoked and sometimes misused. We thus propose a detailed discussion of advantages and limits of the perturbation approach for solving rational expectations models.

Micro-founded models are based on the optimizing behavior of economic agents. Agents adjust their decisions in order to maximize their inter-temporal objectives (utility, profits and so on). Hence, the current decisions of economic agents depend on their expectations of the future path of the economy. In addition, models often include a stochastic part implying that economic variables cannot be perfectly forecasted.

The rational expectation hypothesis consists of assuming that agents' expectations are the best expectations conditional on the structure of the economy and the information available to agents. Precisely, they are modeled by the expectation operator $\mathbb{E}_t$ defined by

$$\mathbb{E}_t(z_{t+1}) = \mathbb{E}(z_t|\Omega_t)$$

where Ω_t is the information set at t and z_t is a vector of economic variables.

Here, we restrict the scope of our analysis to models with a finite number of state variables. In particular, this chapter does not study models with heterogenous agents in which the number of state variables is infinite. We refer to Den Haan et al. (2010b) and Guvenen (2011) for a survey on this topic. When the number of state variables is finite, first-order conditions of maximization combined with market clearing lead to inter-temporal relations between future, past and current economic variables which can often be written as

$$\mathbb{E}_t g(z_{t+1}, z_t, z_{t-1}, \varepsilon_t) = 0 \tag{1}$$

where g is a function, z_t represents the set of state variables, and ε_t is a stochastic process.

The question is then to characterize the solutions of (1) and to find *determinacy conditions* i.e. conditions ensuring the existence and the uniqueness of a bounded solution.

Section **3** presents methods for solving model (1) when g is linear. The seminal paper by Blanchard and Kahn (1980), generalized by Klein (2000) gives a condition ensuring the existence and uniqueness of a solution in simple algebraic terms (see Theorem 1). Other approaches consider undetermined coefficients (Uhlig, 1999) or rational expectations errors (Sims, 2002) based methods.

The linear case appears as the cornerstone of the perturbation approach when g is smooth enough. In section **4**, we recall the theoretical foundations of the perturbation approach (Theorem 4). The determinacy conditions for model (1) result locally from a linear approximation of model (1) and the first-order expansion of the solution is derived from the solution of the linear one. This strategy is called *linearization*. We show how to use the perturbation approach to compute higher order Taylor expansions of the solution (Lemma 2) and highlight the limits of such local results (sections **4.4.1** and **4.4.2**).

When g presents discontinuities triggered by structural breaks or binding constraints for instance, the problem cannot be solved by the classical perturbation approach. We describe two main classes of models for which the perturbation approach is challenged: regime switching and the ZLB. Section **5** depicts the different methods to solve models with markovian transition probabilities. Although there is an increasing number of articles dealing with these models, there are only limited findings on determinacy conditions for such models. We present the main attempts and results: an extension of Blanchard and Kahn (1980), the method of undetermined coefficients or direct resolution. The topical Zero Lower Bound case illustrates the problem of occasionally binding constraints. We describe the different approaches to solve models including the condition on the positivity of the interest rate, either locally (section **6.2**) or globally (section **6.3**).

We end this chapter by a brief presentation of the global methods used to solve model (1). The aim here is not to give an exhaustive description of these methods (see Judd (1996); Heer and Maussner (2005) for a very detailed exposition on this subject) but to show why and when they can be useful and fundamental. Most of these methods rely on projection methods, i.e. consist of finding an approximate solution in a specific class of functions.

2 THEORETICAL FRAMEWORK

We consider general models of the form

$$\mathbb{E}_t g(z_{t+1}, z_t, z_{t-1}, \varepsilon_t) = 0$$

where $\mathbb{E}_t$ denotes the rational-expectations operator conditionally on the past values $(z_{t-k}, \varepsilon_{t-k})_{k\geq 1}$ of the endogenous variables and the current and past values of the exogenous shocks. The variable z denotes the endogenous variables and is assumed to evolve in a bounded set F of $\mathbb{R}^n$. We assume that the stochastic process ε takes its values in a bounded set V (containing at least two points) and that

$$\mathbb{E}_t(\varepsilon_{t+1}) = 0$$

It should be noticed that ε is not necessarily normally distributed even if it is often assumed in practice. Strictly speaking, Gaussian distributions are ruled out by the boundedness assumption. Nevertheless, it is often possible to replace Gaussian distributions by truncated ones.

Such an expression can be derived from a problem of maximization of an inter-temporal objective function. This formulation covers a wide range of models. In particular, for models with lagged endogenous variables $x_{t-1}, \cdots, x_{t-p}$, it suffices to introduce the vector $z_{t-1} = [x'_{t-1}, x'_{t-2}, \cdots, x'_{t-p}]'$ to rewrite them as required.

First, we present an example to illustrate how we can put a model in the required form. Then we present the formalism behind such models. Finally, we introduce the main general concepts pertaining to the resolution.

2.1 An example

We recall in this part how to cast practically a model under the form (1). In the stochastic neoclassical growth model, households choose consumption and capital to maximize lifetime utility

$$\mathbb{E}_t \sum_{k=0} \beta^k U(c_{t+k})$$

where c_t is the consumption, U is the utility function, and β is the discount factor. Output is produced using only capital:

$$c_t + k_t = a_t k_{t-1}^{\alpha} + (1-\delta) k_{t-1}$$

where k_t is the capital, a_t is the total factor productivity, α is the capital share and $\delta \in (0,1)$ is the depreciation rate of capital. We assume that a_t evolves recursively, depending on an exogenous process ε_t as follows:

$$a_t = a_{t-1}^{\rho_a} \exp(\varepsilon_t), \qquad \varepsilon_t \sim \mathcal{N}(0, \sigma^2)$$

Using techniques of dynamic programming (Stokey et al., 1989), we form the lagrangian $\mathcal{L}$, where λ_t is the Lagrange multiplier associated to the constraint on output:

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[U(c_t) - \lambda_t (c_t + k_t - a_t k_{t-1}^{\alpha} - (1-\delta) k_{t-1}) \right].$$

The necessary conditions of optimality lead to:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_t} &: \quad \lambda_t = U'(c_t) \\
\frac{\partial \mathcal{L}}{\partial k_t} &: \quad \lambda_t = \beta \mathbb{E}_t \left[(\alpha a_{t+1} k_t^{\alpha-1} - (1-\delta)) \lambda_{t+1} \right] \\
\frac{\partial \mathcal{L}}{\partial \lambda_t} &: \quad c_t + k_t - a_t k_{t-1}^{\alpha} - (1-\delta) k_{t-1} = 0.
\end{aligned}$$

Then defining $z_t = [a_t, c_t, \lambda_t, k_t]'$, the model can be rewritten as:

$$\mathbb{E}_t g(z_{t+1}, z_t, z_{t-1}, \varepsilon_t) = \mathbb{E}_t \begin{bmatrix} a_t - a_{t-1}^{\rho_a} \exp(\varepsilon_t) \\ \lambda_t - U'(c_t) \\ \beta \lambda_{t+1} [\alpha a_{t+1} k_t^{\alpha-1} - (1-\delta)] - \lambda_t \\ c_t + k_t - a_t k_{t-1}^{\alpha} - (1-\delta) k_{t-1} \end{bmatrix} = 0. \tag{2}$$

2.2 Formalism

We present two main theoretical ways of depicting solutions of model (1): either as functions of all past shocks (Woodford, 1986) or as a policy function, h, such that $z_t = h(z_{t-1}, \varepsilon_t)$ (Jin and Judd, 2002; Juillard, 2003). The second way is more intuitive and corresponds to the practical approach of resolution. The first angle appears more appropriate to handle general theoretical problems. In each case, we have to deal with infinite-dimension spaces: sequences spaces in the first case, functional spaces in the second. We will mainly adopt sequential views.

In model (1), we say that z is an endogenous variable and ε is an exogenous variable, since

$\pi(\varepsilon_t|\varepsilon^{t-1}, z^{t-1}) = \pi(\varepsilon_t)$. Solving the model consists in finding $\pi(z_t|\varepsilon^t, z^{t-1})$.

The sequential approach: Following Woodford (1986), we can look for solutions of model (1) as functions of the history of all the past shocks. We denote by Σ, the sigma field of V. Let V^∞ denote the product of an infinite sequence of copies of V and Σ^∞ is the product sigma field (Loève, 1977, p. 137). Elements $\varepsilon^t = (\varepsilon_t, \varepsilon_{t-1}, \cdots)$ of V^∞ represent infinite histories of realizations of the shocks. We can represent the stochastic process ε_t by a probability measure $\pi : \Sigma^\infty \to [0, 1]$. For any sets $A \in \Sigma$ and $S \in \Sigma^\infty$, we define $AS = \{as \in V^\infty,\ a \in A,\ s \in S\}$ where as is a sequence with first element a, second element the first element of s and so on. By the Radon-Nikodym theorem (Loève, 1977, p. 133), for any $A \in \Sigma$, there exists a measurable function $\pi(A|\cdot) : U^\infty \to [0, 1]$ such that:

$$\forall S \in \Sigma^\infty, \qquad \pi(AS) = \int_S \pi(A|\varepsilon^{t-1}) d\pi(\varepsilon^{t-1})$$

Here $\pi(A|\varepsilon^{t-1})$ corresponds to the probability that $\varepsilon_t \in A$, given a history ε^{t-1}. For each $\varepsilon^{t-1} \in V^\infty$, $\pi(\cdot|\varepsilon^{t-1})$ is a probability measure on (V, Σ), thus π defines a Markov process on V^∞ with a time-invariant transition function.

We define the functional $\mathcal{N}$ by:

$$\mathcal{N}(\phi) = \int_V g(\phi(\varepsilon\varepsilon^t), \phi(\varepsilon^t), \phi(\varepsilon^{t-1}), \varepsilon_t)\pi(\varepsilon|\varepsilon^t)d\varepsilon \tag{3}$$

Looking for a solution of model (1) is equivalent to find a function Φ solution of $\mathcal{N}(\phi) = 0$.

The recursive approach: Following the approach presented in Jin and Judd (2002), we consider $\mathcal{S}$ the set of functions acting on $F \times V$ with values in $F \times V$. We assume that the shock ε_t follows a distribution law $\mu(\cdot, \varepsilon^{t-1})$. Then, we define the functional $\tilde{\mathcal{N}}$ on $\mathcal{S}$ by:

$$\tilde{\mathcal{N}}(h)(z, \varepsilon) = \int_V g(h(h(z, \varepsilon), \tilde{\varepsilon}), h(z, \varepsilon), \tilde{\varepsilon})\mu(\tilde{\varepsilon}, \varepsilon)d\tilde{\varepsilon} \tag{4}$$

In this framework, looking for a solution of model (1) corresponds to finding a function h in $\mathcal{S}$ such that $\tilde{\mathcal{N}}(h) = 0$. In practice, this approach is the most used, since it leads to solutions' spaces with lower dimension. This approach underlies the numerical methods implemented in Dynare (Juillard, 1996), a well known software for solving rational expectations models.

2.3 Definitions

Adopting the sequential approach described in Woodford (1986), we introduce the type of solutions that we are interested in.

Definition 1 *A stationary rational expectations equilibrium (s.r.e.e.) of model* (1) *is an essentially bounded, measurable function* $\phi : V^\infty \to F$ *such that:*

1. $||\phi||_\infty = ess \sup_{V^\infty} \|\phi(\varepsilon^t)\| < \infty$
2. *If u is a U valued stochastic process associated with the probability measure π, then $z_t = \phi(u^t)$ is solution of* (1) *i.e.* $\mathcal{N}(\phi) = 0$.

Furthermore, this solution is a steady state if ϕ is constant.

A crucial question is the existence and uniqueness of a bounded solution called *determinacy*.

Definition 2 *We say that model* (1) *is determinate if there exists a unique s.r.e.e.*

In terms of the recursive approach *à la* Jin and Judd (2002), it is equivalent to look for a stable measurable function h on $F \times V$ with values in $F \times V$ which is solution of the model. It is worth noticing that solutions of model (1) may respond to the realizations of a *sunspot variable*, i.e. a random variable that conveys no information on technology, preferences, and endowments and thus does not directly enter the equilibrium conditions for the state variables (Cass and Shell, 1983).

Definition 3 *The deterministic model associated to model* (1) *is:*

$$g(z_{t+1}, z_t, z_{t-1}, 0) = 0 \tag{5}$$

A constant *s.r.e.e.* of the deterministic model, $\bar{z} \in F$, is called a (deterministic) steady state. This point satisfies:

$$g(\bar{z}, \bar{z}, \bar{z}, 0) = 0 \tag{6}$$

An equation as equation (6) is a non-linear multivariate equation and solving it can be challenging. Such an equation can be solved by iterative methods of Newton type, simple, by blocks, or with an improvement of the Jacobian conditioning.

For the example presented in section **2.1**, the computation of the steady state is simple.

$$\bar{a} = 1, \quad \bar{k} = \left[\frac{1}{\alpha}\left(\frac{1}{\beta} + (1-\delta)\right)\right]^{\frac{1}{\alpha-1}}, \quad \bar{c} = \bar{k}^{\alpha} - \delta\bar{k}, \quad \bar{\Lambda} = U'(\bar{c}) \tag{7}$$

In the reminder of this chapter, we do not tackle issues raised by multiple steady-states and mainly focus on the dynamic around an isolated steady-state.

3 LINEAR RATIONAL EXPECTATION MODELS

In this part, we review some aspects of solving linear rational expectations models, since they are the cornerstone of the perturbation approach.

We consider the following model:

$$g^1 \mathbb{E}_t z_{t+1} + g^2 z_t + g^3 z_{t-1} + g^4 \varepsilon_t = 0, \qquad \mathbb{E}_t \varepsilon_{t+1} = 0 \tag{8}$$

where g^i is a matrix, for $i \in \{1, \cdots, 4\}$. We present three important methods for solving these models: the benchmark method of Blanchard and Kahn (1980), the method of undetermined coefficients of Uhlig (1999) and a method developed by Sims (2002) exploiting the rational expectations errors. The aim of this section is to describe the theory behind these three methods and to show why they are theoretically equivalent. We focus on the algebraic determinacy conditions rather than on the computational algorithms, which have been extensively depicted. Then, we illustrate these three methods in a simple example.

3.1 The approach of Blanchard and Kahn (1980)

The first method was introduced by Blanchard and Kahn (1980). In this seminal paper, the authors lay the theoretical foundations.

Existence and uniqueness are then characterized by comparing the number of explosive roots to the number of forward-looking variables.

Following Blanchard and Kahn (1980) and their extension by Klein (2000), we rewrite (8) under the form:

$$A\mathbb{E}_t \begin{bmatrix} z_{t+1} \\ z_t \end{bmatrix} = B \begin{bmatrix} z_t \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} g^4 \\ 0 \end{bmatrix} \varepsilon_t \tag{9}$$

where $A = \begin{pmatrix} g^1 & g^2 \\ 0 & I_n \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -g^3 \\ I_n & 0 \end{pmatrix}$

We remind the main result on existence and uniqueness of a stable solution for this kind of linear model due to the seminal paper by Blanchard and Kahn (1980) and extended to a model with non-invertible matrix g^1 by Klein (2000).

Theorem 1 *If the number of explosive generalized eigenvalues of the pencil $< A, B >$ is exactly equal to the number of forward variables, n, and if the rank condition* (11) *is satisfied, then there exists a unique stable solution of model* (9)*.*

Let us provide an insight into the main ideas of the proof. We consider the pencil (A, B) defined in equation (9) and introduce its real generalized Schur decomposition. When A is invertible, generalized eigenvalues coincide with the standard eigenvalues of matrix $A^{-1}B$. Following Klein (2000), there exist unitary matrices Q and Z, quasi triangular matrices T and S such that:

$$A = QTZ \quad \text{and} \quad B = QSZ$$

For a matrix $M \in \mathcal{M}_{2n}(\mathbb{R})$, we write M by blocks of $\mathcal{M}_n(\mathbb{R})$:

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

We rank the generalized eigenvalues such that $|T_{ii}| > |S_{ii}|$ for $i \in [1, n]$ and $|S_{ii}| > |T_{ii}|$ for $i \in [n+1, 2n]$ which is possible if and only if the number of explosive generalized eigenvalues is n.

Equation (9) leads to:

$$TZ\mathbb{E}_t \begin{bmatrix} z_t \\ z_{t+1} \end{bmatrix} = SZ \begin{bmatrix} z_{t-1} \\ z_t \end{bmatrix} + Q' \begin{bmatrix} g^4 \\ 0 \end{bmatrix} \varepsilon_t$$

Taking the last n lines, we get that:

$$(Z_{21}z_{t-1} + Z_{22}z_t) = S_{22}^{-1}T_{22}\mathbb{E}_t(Z_{21}z_t + Z_{22}z_{t+1}) - T_{22}^{-1}Q'_{12}g^4\varepsilon_t \tag{10}$$

We assume that:

$$Z_{22} \text{ is full rank and thus invertible} \tag{11}$$

Looking for bounded solutions, we iterate equation (10) to obtain:

$$z_t = -Z_{22}^{-1}Z_{21}z_{t-1} - Z_{22}^{-1}S_{22}^{-1}Q'_{12}g^4\varepsilon_t \tag{12}$$

By straightforward computations, we see that:

$$Q'_{22} = S_{22}Z_{22} \qquad Z_{22}^{-1}Z_{21} = Z'_{11}T_{11}^{-1}S_{11}(Z'_{11})^{-1}$$

This shows, that when the Blanchard Kahn conditions are satisfied, there exists a unique bounded solution.

Reciprocally, if the number of explosive eigenvalues is strictly smaller than n, there exists several solutions of model (8). On the contrary, if the number of explosive eigenvalues is strictly higher than n, there is no solution. This strategy links explicitly the determinacy condition and the solution to a Schur decomposition. We notice in particular that the solution is linear and recursive.

The algorithm of solving used by Dynare relies on this Schur decomposition Juillard (1996).

3.2 Undetermined coefficients

This approach is presented in Uhlig (1999) and Christiano (2002) and consists in looking for solutions of (8) under the form

$$z_t = Pz_{t-1} + Q\varepsilon_t \tag{13}$$

with $\rho(P) < 1$ where we denote by ρ the spectral radius of a matrix. The previous approach in section **3.1** has shown that the stable solutions of a linear model can be written under the form (13). Introducing (13) into model (8) leads to:

$$(g^1P^2 + g^2P + g^3)z + (g^1PQ + g^2Q + g^4)\varepsilon = 0, \quad \forall\varepsilon \in V, \quad \forall z \in F$$

Thus, (13) is satisfied if and only if:

$$g^1P^2 + g^2P + g^3 = 0 \tag{14}$$

$$g^1PQ + g^2Q + g^4 = 0 \tag{15}$$

Uhlig (1999) obtains the following characterization of the solution:

Theorem 2 *If there is a stable recursive solution of model* (8)*, then the solution* (13) *satisfies:*

(i) The matrix P is solution of the quadratic matrix equation

$$g^1 P^2 + g^2 P + g^3 = 0$$

and the spectral radius of P is smaller than 1*.*

(ii) The matrix Q satisfies the matrix equation:

$$(g^1 P + g^2) Q = -g^4$$

The method described in Uhlig (1999) is based on the computation of roots of the quadratic matrix equation (14), which is in practice done by computing generalized eigenvalues of the pencil $< A, B >$, defined in section **3.1**.

Higham and Kim (2000) make the explicit link between this approach and the one of Blanchard and Kahn (1980) in the following result. Introducing matrices $< A, B >$ and the Schur decomposition as in section **3.1**, they show that:

Theorem 3 *With the notations of section* **3.1***, all the solutions of* (14) *are given by* $P = Z_{22}^{-1} Z_{21} = Z'_{11} T_{11}^{-1} S_{11} (Z'_{11})^{-1}$.

The proof is based on standard manipulations of linear algebra. We refer to Higham and Kim (2000) for the details. Moreover, by simple matrix manipulations, we can show that $Q'_{12}(g^1 P + g^2) = S_{22} Z_{22}$, the rank condition (11) implies that $(g^1 P + g^2)$ is invertible, and Q is defined uniquely by (15). The method of undetermined coefficients leads to manipulate matrix equations rather than iterative sequences, but the computational algorithm is similar, and is depicted by Uhlig (1999).

3.3 Rational expectations errors based methods

Here we depict the approach of Sims (2002) for the model (8) and explain how it is consistent with the previous methods.

Introducing $\eta_t = z_t - \mathbb{E}_{t-1} z_t$ and $y_t = \mathbb{E}_t z_{t+1}$, we rewrite equation (8) as

$$g^1 y_t + g^2 z_t + g^3 z_{t-1} + g^4 \varepsilon_t = 0$$

$$z_t = y_{t-1} + \eta_t, \quad \mathbb{E}_t \eta_{t+1} = 0, \quad \mathbb{E}_t \varepsilon_{t+1} = 0$$

The model is rewritten in Sims framework as:

$$AY_t + BY_{t-1} + \begin{bmatrix} 0 \\ \mathbb{I}_n \end{bmatrix} \eta_t + \begin{bmatrix} \mathbb{I}_n \\ 0 \end{bmatrix} g^4 \varepsilon_t = 0 \tag{16}$$

where A and B are defined in section **3.1**, and $Y_t = \begin{bmatrix} y_t \\ z_t \end{bmatrix}$.

Here the shocks η_t are not exogenous but depend on the endogenous variables Y_t. By iterating expectations of relation (16), we can express Y_t as a function of ε_t and η_t. Since η_t depends on Y_t, we obtain an equation on η_t. The model (16) is then determinate if this equation admits a unique solution.

Let us show that this approach is equivalent to the method of Blanchard and Kahn (1980). Considering the Schur decomposition of the pencil $< A, B >$, there exists $\tilde{n} \in \{1, 2n\}$ such that $A = QTZ$ and $B = QSZ$, with $|T_{ii}| > |S_{ii}|$ for $i \in \{1, \cdots, \tilde{n}\}$ and $|S_{ii}| > |T_{ii}|$ for $i \in \{\tilde{n}+1, \cdots, 2n\}$. We will show that there exists a unique solution of (16) if and only if $\tilde{n} = n$ and Q'_{12} is invertible.

Introducing stable and unstable subspaces of the pencil $< A, B >$, we define $Y_t = Z' \begin{bmatrix} s_t \\ u_t \end{bmatrix}$, then the stable solutions of equation (16) are given by the following system:

$$s_t = 0, \qquad u_t = T_{11}^{-1} S_{11} u_{t-1} + Q'_{21}\eta_t + Q'_{11} g^4 \varepsilon_t, \qquad Q'_{22}\eta_t = -Q'_{12} g^4 \varepsilon_t \tag{17}$$

The linear system (17) admits a unique solution if and only if Q'_{22} is a square matrix i.e. $\tilde{n} = n$, and invertible (rank condition). We find again conditions of Theorem 1. The approach of Sims (2002) avoids the distinction between predetermined and forward-looking variables.

In this part, we have presented different approaches of solving linear models, relying on a simple determinacy condition. This algebraic condition is easy to check, even in the case of large scale models.

3.4 An example

In this part, we depict these three methods in a simple example. We consider the following univariate model, variant of an example studied in Iskrev (2008) :

$$\theta^2 \kappa z_t = \theta^2 \mathbb{E}_t z_{t+1} + (\theta\kappa - 1) z_{t-1} + \varepsilon_t$$

where $0 < \theta < 1$ and $\kappa > 0$. We can rewrite this model as follwos:

$$\begin{bmatrix} \mathbb{E}_t z_{t+1} \\ z_t \end{bmatrix} = \begin{bmatrix} \kappa & \frac{1-\theta\kappa}{\theta^2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z_t \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} \frac{1}{\theta^2\kappa} \\ 0 \end{bmatrix}$$

The matrix $\begin{bmatrix} \kappa & \frac{1-\rho\kappa}{\rho^2} \\ 1 & 0 \end{bmatrix}$ has two eigenvalues $1/\theta$ and $(\theta\kappa - 1)\theta$. There is one predetermined variable, thus according to Blanchard and Kahn's conditions, the model is determinate if and only if $\frac{\theta\kappa-1}{\theta} < 1$, i.e. if $\kappa < (\theta+1)/\theta$; in this case, the solution is given by:

$$z_t = \frac{\theta\kappa - 1}{\theta} z_{t-1} + \frac{1}{\theta}\varepsilon_t$$

The approach of Uhlig (1999) consists in looking for $(p, q) \in \mathbb{R}^2$ such that $y_t = py_{t-1} + q\varepsilon_t$, and $|p| < 1$. Then p and q are solutions of the equations

$$\theta^2 p^2 - \kappa\theta^2 p + (\theta\kappa - 1) = 0 \qquad \theta^2(p-\kappa)q = 1$$

which admit a unique solution $p = \frac{\theta\kappa-1}{\theta} \in]-1, 1[$ if $\kappa < (\theta+1)/\theta$.

For the method of Sims (2002), we define $y_t = \mathbb{E}_t z_{t+1}$, and $\eta_t = z_t - y_{t-1}$. The model is then rewritten as:

$$\begin{bmatrix} \frac{\theta^2\kappa}{\theta\kappa-1} & \frac{-\theta^2}{\theta\kappa-1} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z_t \\ y_t \end{bmatrix} = \begin{bmatrix} z_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} \frac{\varepsilon_t}{\theta\kappa-1} \\ \eta_t \end{bmatrix}.$$

When $(\theta\kappa - 1)\theta < 1$, the matrix on the left-hand-side of the former equality has a unique eigenvalue smaller than 1 (θ). Projecting on the associated eigenspace, we get that:

$$\frac{\theta\kappa - 1}{\theta} z_{t-1} - y_{t-1} + \frac{\theta\kappa - 1}{\theta(\kappa\theta - 1)}\varepsilon_t - \eta_t = 0$$

Thus, replacing η_t by $z_t - y_{t-1}$, we get that:

$$z_t = \frac{\theta\kappa - 1}{\theta} z_{t-1} + \frac{1}{\theta}\varepsilon_t.$$

3.5 Comparison of the three methods

From a numerical point of view, the algorithms induced by these three methods lead to globally equivalent solutions, we refer the reader to Anderson (2006) for a detailed comparison of the different algorithms.

Blanchard and Kahn (1980) and Sims (2002) approaches are particularly useful to build sunspots solutions when the determinacy conditions are not satisfied, as it is done for instance in Woodford (1986) for the first method, or in Lubik and Schorfheide (2001) for the second one.

Uhlig (1999) clearly makes the link between linear rational expectations and matricial Ricatti equations, which are widely used in control theory. He also allows for a more direct insight on the transition matrix. Besides, this approach lays the foundations of indeterminate coefficient methods.

4 PERTURBATION APPROACH

This section is devoted to the *linearization*, method that we can use to solve non linear, smooth enough rational expectations models (1) in the neighborhood of a steady state (see Definition 1).

We assume that the function g is smooth enough (C^1) in all its arguments, and we assume that there exists a locally unique steady-state $\bar{z}$ such that:

$$g(\bar{z}, \bar{z}, \bar{z}, 0) = 0$$

We will solve model (1) by a perturbation approach. To this aim, we introduce a scale parameter $\gamma \in \mathbb{R}$ and consider the model:

$$\mathbb{E}_t g(z_{t+1}, z_t, z_{t-1}, \gamma\varepsilon_t) = 0 \tag{18}$$

When $\gamma = 0$, model (18) is the deterministic model (5) and when $\gamma = 1$, model (18) is the original model (1). We first explain the underlying theory of linearization, mainly developed by Woodford (1986); Jin and Judd (2002) and show an example. Then, we study higher-order expansions. Finally, we discuss the limits of such a local resolution.

4.1 From linear to non linear models : theory

In this section, we explain in details how to solve non linear rational expectations models using a perturbation approach. Although, linearization is well-known and widely used to solve non-linear rational expectations models, the theory underlying this strategy and the validity domain of this approach are not necessarily well-understood in practice. We rely on the works of Woodford (1986); Jin and Judd (2002).
We define the functional $\mathcal{N}$ by:

$$\mathcal{N}(\phi,\gamma)=\int_V g(\phi(\varepsilon\varepsilon^t),\phi(\varepsilon^t),\phi(\varepsilon^{t-1}),\gamma\varepsilon_t)\pi(\varepsilon|\varepsilon^t)d\varepsilon \tag{19}$$

By definition of the steady state, we see that the constant sequence $\phi_0(u)=\bar{z}$ for any $u\in U^\infty$ satisfies:

$$\mathcal{N}(\phi_0,0)=0$$

Perturbation approaches often rely on the implicit function theorem. Let us remind of a version of this result in Banach spaces.

Theorem 4 ***[Abraham et al. (1988)]*** *Let E,F,G be 3 Banach spaces, let $U\subset E,V\subset F$ be open and $f:U\times V\to G$ be C^r, $r\geq 1$. For some $x_0\in U$, $y_0\in V$ assume $D_yf(x_0,y_0):F\to G$ is an isomorphism. Then there are neighborhoods U_0 of x_0 and W_0 of $f(x_0,y_0)$ and a unique C^r map $g:U_0\times W_0\to V$ such that, for all $(x,w)\in U_0\times W_0$*

$$f(x,g(x,w))=w$$

This Theorem is an extension of a familiar result in finite dimension spaces, to infinite complete normed vector spaces (Banach spaces). Some statements of this Theorem impose to check that $D_y f(x_0, y_0)$ is a homeomorphism, i.e. a continuous isomorphism with continuous inverse. We claim that, due to the Banach isomorphism Theorem, it suffices to assume that $D_y f(x_0, y_0)$ is a linear continuous isomorphism.

We apply now this Theorem to the functional $\mathcal{N} : \mathcal{B} \times \mathbb{R} \to \mathbb{R}^n$ in appropriate Banach spaces. As we are looking for bounded solutions, we introduce $\mathcal{B}$, the set of essentially bounded, measurable functions $\Phi : V^\infty \to \mathbb{R}^n$. $\mathcal{B}$, with the infinite norm

$$\|\Phi\|_\infty = \operatorname*{ess\,sup}_{u \in U^\infty} \|\Phi(u)\|$$

$\mathcal{B}$ is a Banach space (see Dunford and Schwartz, 1958, section III.6.14). $\mathbb{R}$ with $|\cdot|$ is also a Banach space.

The regularity of g ensures that the functional $\mathcal{N}$ is C^1.

We introduce the operators lag $\mathcal{L}$ and lead $\mathcal{F}$ defined in $\mathcal{B}$ by:

$$\mathcal{F} : \quad \Phi \mapsto ((\varepsilon^t) \mapsto \int_V H(\varepsilon\varepsilon^t)\pi(\varepsilon|\varepsilon^t)d\varepsilon) \tag{20}$$

$$\mathcal{L} : \quad \Phi \mapsto ((\varepsilon^t) \mapsto \Phi(\varepsilon^{t-1})) \tag{21}$$

We notice that $\mathcal{F}$ and $\mathcal{L}$ have the following straightforward properties.

1. $\mathcal{F}\mathcal{L} = \mathbf{1}$

2. $|||\mathcal{F}||| = 1$ and $|||\mathcal{L}||| = 1$

where $|||\cdot|||$ is the operator norm associated to $\|\cdot\|_\infty$.

To apply Implicit Function Theorem, we compute $D_{\Phi_0}\mathcal{N}(\Phi_0, 0)$.

$$D_{\Phi_0}\mathcal{N}(\Phi_0, 0)H = g^1\mathcal{F}H + g^2H + g^3\mathcal{L}H$$

To check if $D_{\Phi_0}\mathcal{N}(\Phi_0, 0)$ is invertible, we consider $\Psi \in \mathcal{B}$, and study whether there exists a unique solution of the equation:

$$D_{\Phi_0}\mathcal{N}(\Phi_0, 0)H = \Psi \tag{22}$$

Equation (22) can be rewritten as:

$$g^1 \mathcal{F}H + g^2 H + g^3 \mathcal{L}H = \Psi$$

where g^1 (respectively g^2, g^3) is the first-order derivative with respect to the first variable (second, third). We refer to the method and the notations described in section **3.1**.

Introducing the pencil $< A, B >$ and its Schur decomposition, we rewrite (22) as:

$$\underbrace{\begin{pmatrix} g_2 & g_1 \\ I_n & 0 \end{pmatrix}}_{A} \begin{pmatrix} H \\ \mathcal{F}H \end{pmatrix} = \underbrace{\begin{pmatrix} -g_3 & 0 \\ 0 & I_n \end{pmatrix}}_{B} \mathcal{L} \begin{pmatrix} H \\ \mathcal{F}H \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Psi$$

For any $(\varepsilon^t) \in V^\infty$, defining $z_t = H(\varepsilon^t)$ and $z_{t+1} = \mathcal{F}H(\varepsilon^t)$, and $\Psi_t = \Psi(\varepsilon^t)$, we have to find bounded processes z_t such that:

$$A \begin{pmatrix} z_t \\ z_{t+1} \end{pmatrix} = B \begin{pmatrix} z_{t-1} \\ z_t \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Psi_t$$

Then, $D_{\Phi_0}\mathcal{N}(\Phi_0, 0)$ is invertible if and only if the number of explosive generalized eigenvalues of the pencil $< A, B >$ is exactly equal to n. Moreover, the solution z_t is given by:

$$z_t + Z_{22}^{-1} Z_{21} z_{t-1} = Z_{22}^{-1} \sum_{k=0}^{\infty} (S_{22}^{-1} T_{22})^k S_{22}^{-1} Q'_{12} \mathbb{E}_t \Psi_{t+k}$$

which finally gives:

$$D_{\Phi_0}\mathcal{N}(\Phi_0, 0)^{-1} = (1 + Z_{22}^{-1} Z_{21} \mathcal{L})^{-1} Z_{22}^{-1} (1 - S_{22}^{-1} T_{22} \mathcal{F})^{-1} S_{22}^{-1} Q'_{12}$$

Application of Implicit Function Theorem leads to the following result, which is an extension of Woodford (1986) when g^1 is non invertible:

Theorem 5 *If the linearized model in $\bar{z}$ is determinate, then for γ small enough, there exists a unique s.r.e.e for model* (18)*. Moreover, if not, we have:*

(1) If the number of explosive generalized eigenvalues of the pencil $< A, B >$ is smaller than n, the linearized model is indeterminate, and there is both

(i) a continuum of s.r.e.e near $\bar{z}$, in all of which the endogenous variables depend only upon the history of the exogenous shocks.

(ii) a continuum of s.r.e.e near $\bar{z}$, in which the endogenous variables respond to the realizations of a stationary sunspot process as well as to the exogenous shocks.

(2) If the number of explosive generalized eigenvalues of the pencil $< A, B >$ is greater than n, for any γ small enough, no s.r.e.e exists near $\bar{z}$.

The previous result is an equivalence result, we have only detailed that the determinacy of the linearized model implies the local determinacy of the non linear model. The reciprocal is a bit tricky and uses an approach similar to the Constant Rank Theorem. We refer to Woodford (1986) for more details.

Theorem 5 shows that the determinacy condition for model (18) around the steady state $\bar{z}$ is locally equivalent to the determinacy condition for the linearized model in $\bar{z}$.

To expound this result in functional terms (Jin and Judd, 2002), we have the following result.

Proposition 1 *If the linearized model is determinate, the solution of Theorem* 5 *is recursive.*

For a fixed γ, let (z_t) a solution of model (18) for a sequence ε^t, there exists a unique function Φ such that:

$$z_t = \Phi(\varepsilon^t, \gamma)$$

We define

$$\mathrm{Im}(\Phi(\cdot,\gamma)) = \{z \in F \quad | \quad \exists \varepsilon^0 \in V^\infty \text{ such that } z = \Phi(\varepsilon^0)\}$$

For any $z \in \mathrm{Im}(\Phi(\cdot,\gamma))$, we define the function $\mathcal{Z}$ by

$$\mathcal{Z}(z,\varepsilon,\gamma) = \Phi(\varepsilon\varepsilon^0,\gamma), \text{ where } z = \Phi(\varepsilon^0,\gamma)$$

We consider the sequence $\tilde{z}_t = \mathcal{Z}(z_{t-1},\varepsilon_t)$ for $t > 0$ and $\tilde{z}_t = z_t$ for $t \leq 0$. The sequence $\tilde{z}_t$ is solution of model (18). By uniqueness of the solution, $\tilde{z}_t = z_t$ for any $t > 0$. Thus $z_t = \mathcal{Z}(z_{t-1},\varepsilon_t)$ for any $t > 0$. In addition, computing $D_{\Phi_0}\mathcal{N}(\Phi_0,0)^{-1}D_\gamma\mathcal{N}(\Phi_0,0)$ leads to the first order expansion of the solution:

$$z_t = Pz_{t-1} + \gamma Q\varepsilon_t + o(\gamma)$$

where P and Q are given in equation (13).

4.2 Applications: a Fisherian model of inflation determination

Consider an economy with a representative agent, living for an infinite number of periods, facing a trade-off between consuming today and saving to consume tomorrow. The agent maximizes its utility function $\mathbb{E}_t \sum_{k=0}^{\infty} \beta^k \frac{c_{t+k}^{1-\sigma}}{1-\sigma}$, where, β is the discount factor, C_t the level of consumption and σ the inter-temporal elasticity of substitution. Then maximizing utility function under the budget constraints leads to:

$$\mathbb{E}_t \frac{c_t}{c_{t+1}}^{\sigma} \frac{r_t}{\pi_{t+1}} = \frac{1}{\beta} \tag{23}$$

where r_t is the gross risk-free nominal interest rate and π_{t+1} is the inflation. In addition, we assume that $\frac{c_{t+1}}{c_t} = \exp(a_{t+1})$ where a_t is an exogenous process.

$$a_t = \rho a_{t-1} + \varepsilon_t$$

Defining $r_t = \bar{r}\exp(\hat{r}_t)$ and $\pi_t = \bar{\pi}\exp(\hat{\pi}_t)$, with $\bar{\pi} = \beta\bar{r}$, we rewrite equation (23) as:

$$\mathbb{E}_t[\exp(\hat{r}_t - \hat{\pi}_{t+1} - \sigma a_{t+1})] = 1$$

If we assume that $\hat{r}_t$ follows a Taylor rule:

$$\hat{r}_t = \alpha\hat{\pi}_t$$

The vector of variables $z = [\hat{r}, \hat{\pi}, a]'$ satisfies the model:

$$\mathbb{E}_t g(z_{t+1}, z_t, z_{t-1}, \varepsilon_t) = \mathbb{E}_t \begin{bmatrix} \exp(\hat{r}_t - \hat{\pi}_{t+1} - \sigma a_{t+1}) - 1 \\ \hat{r}_t - \alpha\hat{\pi}_t \\ a_t - \rho a_{t-1} - \varepsilon_t \end{bmatrix} = 0 \tag{24}$$

$g(0,0,0,0) = 0$ and the first order derivatives of g in $(0,0,0,0)$ are given by:

$$g^1 = \begin{bmatrix} 0 & -1 & -\sigma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad g^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad g^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho \end{bmatrix}$$

Then we obtain the following Taylor principle(Woodford, 2003):

Lemma 1 *If $\alpha > 1$ and for a small variance of ε, the model* (24) *is determinate.*

The proof is immediate, it suffices to compute associated matrices A and B, and the generalized eigenvalues of the pencil $< A, B >$ and to apply Theorem 5.

4.3 Solving higher order solutions

In this part, we do not focus on the practical computations of high order solutions. These aspects are developed in Jin and Judd (2002) and Schmitt-Grohe and Uribe (2004). First, we show the theoretical interest of computing expansions at higher order. Second, we show that if the linearized model is determinate, and if the model is smooth, then the solution admits an asymptotic expansion at any order (Lemma 2), similarly to results of Jin and Judd (2002) or Kowal (2007).

4.3.1 Theoretical necessity of a quadratic approximation: the example of the optimal monetary policy

A very important application of model (18) is the evaluation of alternative monetary policy rules and the concept of optimal monetary policy. There is a consensus in the literature that a desirable monetary policy rule is one that achieves a low expected value of a discounted loss function, where the losses at each period are a weighted average of quadratic terms depending on the deviation of inflation from a target rate and in some measure of output relative to its potential. This loss function is often derived as a quadratic approximation of the level of expected utility of the representative household in the rational-expectations equilibrium associated with a given policy. This utility function is then rewritten as:

$$U(z,\gamma\varepsilon) = U(\bar{z},0) + U_z(\bar{z},0)(z-\bar{z}) + \gamma U_\varepsilon(\bar{z},0)\varepsilon + \frac{1}{2}(z-\bar{z})' U_{zz}(\bar{z},0)(z-\bar{z}) \tag{25}$$

$$+\gamma(z-\bar{z})' U_{z\varepsilon}(\bar{z},0)\varepsilon + \gamma^2 \frac{1}{2}\varepsilon' U_{\varepsilon\varepsilon}(\bar{z},0)\varepsilon + O(\gamma^3)$$

Then, if we consider a first-order solution of the model $z(\gamma) = z_0 + \gamma\Phi + O(\gamma)$, then due to the properties of composition of asymptotic expansions, we obtain in general only a first-order expansion of the utility function (25), which is not sufficient to compute optimal monetary policy. On the contrary, a second order expansion of z allows for computing a second-order expansion in equation (25). For a complete description, we refer the reader to chapter 6 of Woodford (2003) or Kim et al. (2003).

4.3.2 Some insights on the expansion of the solution

Most of the papers dealing with high-order expansions introduce tensorial notations, which are very useful but would be slightly heavy in this chapter. Thus, we illustrate the main ideas with a naive approach which stems directly from the Implicit Function Theorem.

Lemma 2 *We assume that the function g in model* (18) *is C^r, if the linearized model in $\bar{z}$ is determinate, then the solution Φ admits an asymptotic expansion in γ until order r.*

$$\Phi(\gamma) = \Phi(0) + \sum_{n=1}^{r} \gamma^n a_n + o(\gamma^n) \tag{26}$$

where the functions $a_k \in C^0(V^\infty)$ are computed recursively.

This result shows that local determinacy and smoothness for the model ensure an expansion at each order. Let us give some details on the proof of Lemma 2. The real function $\eta(\alpha) = \alpha \mapsto \mathcal{N}(\Phi(\alpha), \alpha)$ is C^r; its derivative of order $n \leq r$ is zero and we can show, by an immediate recursion, that there exists a function η_n on $(C^0(V^\infty))^n$ such that:

$$\eta^{(n)}(\alpha) = D_\Phi \mathcal{N}(\Phi(\alpha), \alpha)\Phi^{(n)}(\alpha) + \eta_n(\Phi^{(n-1)}(\alpha), \cdots, \Phi(\alpha)) = 0$$

Applying this identity for $\alpha = 0$ leads to:

$$\Phi^{(n)}(0) = -D_\Phi \mathcal{N}(\Phi(0), 0)^{-1}\eta_n(\Phi^{(n-1)}(0), \cdots, \Phi(0))$$

since $D_\Phi \mathcal{N}(\Phi(0), 0)$ is invertible. Thus, the functions (a_n) in formula (26) are given by:

$$a_n = \frac{\Phi^{(n)}(0)}{n!}$$

Lemma 2 shows that, under determinacy condition, and for a smooth function, it is possible to obtain a Taylor asymptotic expansion of the solution in the scale parameter at any order. We refer to Jin and Judd (2002) and Kowal (2007) for a more detailed analysis.

4.3.3 What are the advantages of higher order computations ?

Of course, an expansion at higher order provides a more accurate approximation of the solution in the neighborhood around the deterministic steady state, as soon as the perturbation approach is valid in this neighborhood. However, it should be noticed that an higher order approximation does not change the size of the domain on which the perturbation approach is valid, and thus can be uninformative if this domain is very small.

4.4 Limits of the perturbation approach

It is worth noticing that previous results are local i.e. only valid on a small neighborhood of the steady-state. We illustrate these limits by two examples. The first one shows that if the size of the shocks is not so small, the conditions obtained by linearization can be evident; this is notably the case when the shocks are discrete. The second example illustrates that if the model is locally determinate, it does not exclude complex behaviors in a larger neighborhood.

4.4.1 Small perturbation

The existence and uniqueness of the solution is an asymptotic result, and remains valid for a "small" γ. Refinements of the implicit function Theorem can give a quantitative condition on γ (Holtzman, 1970) but, the conditions of validity of the linearization in terms of size of the shocks are never checked.

As an illustration, we consider the following model

$$\mathbb{E}_t(\pi_{t+1}) = \alpha_{s_t}\pi_t \tag{27}$$

This model corresponds to a simplified Fisherian model of inflation determination as described in section **4.2**, π_t is the inflation and α_{s_t} is a parameter taking two values α_1 and α_2. We assume that the reaction to inflation evolves stochastically between these two values. A monetary policy regime is a distinct realization of the random variable s_t and we recall that we say that a monetary policy regime is active if $\alpha_i > 1$ and passive if $\alpha_i < 1$, following the terminology of Leeper (1991). We assume that the process s_t is such that:

$$p(s_t = 1) = p, \qquad p(s_t = 2) = 1 - p$$

We define

$$\bar{\alpha} = p\alpha_1 + (1-p)\alpha_2, \qquad \Delta\alpha = \alpha_2 - \alpha_1$$

We illustrate some limits of Theorem 5 by considering the model (27) as a perturbation of

$$\mathbb{E}_t(\pi_{t+1}) = \bar{\alpha}\pi_t \tag{28}$$

Theorem 5 gives determinacy conditions for the following perturbed model:

$$\mathbb{E}_t(\pi_{t+1}) = \bar{\alpha}\pi_t + \gamma\Delta\alpha\frac{(\alpha_{s_t} - \bar{\alpha})}{\Delta\alpha}\pi_t \tag{29}$$

when the scale parameter γ is small enough.

Lemma 3 *Determinacy conditions for models 27 and 29 :*

1. *A sufficient condition for applying Theorem* 5 *to model* (29) *for a small* γ *is that:*

$$p\alpha_1 + (1-p)\alpha_2 > 1$$

2. *A sufficient condition for determinacy of model* (27) *is that*

$$\frac{p}{\alpha_1} + \frac{1-p}{\alpha_2} < 1$$

These two conditions are represented in figure 1*.*

Let us give some details on the proof of Lemma 3. The first point is immediate (see section

Figure 1: Determinacy conditions depending on γ

4.2). It remains to show the second point.

To do that, following the method presented in section **4**, we introduce the functional $\mathcal{N}$ acting on the set of continuous functions on $\{1,2\}^\infty$ and defined by:

$$\mathcal{N}(\Phi,\gamma)(s^t) = p\Phi(1s^t) + (1-p)\Phi(2s^t) - \bar{\alpha}\Phi(s^t) + \gamma\Delta\alpha(p-2+s_t)\Phi(s^t)$$

First, we check easily that, for $\bar{\alpha} > 1$, $\mathcal{N}(\Phi,0) = 0$ admits a unique solution o, and that $D_\Phi\mathcal{N}(o,0)$ is invertible. To apply Theorem 5 for any $\gamma \in [0,1]$, we have to find conditions ensuring that $D_\Phi\mathcal{N}(\Phi_0,\gamma)$ remains invertible. We compute $D_\Phi\mathcal{N}(\Phi,\gamma)$.

$$D_\Phi\mathcal{N}(\Phi,\gamma)H(s^t) = pH(1s^t) + (1-p)H(2s^t) - \bar{\alpha}H(s^t) + \gamma\Delta\alpha(p-2+s_t)H(s^t) \tag{30}$$

and look for a condition ensuring that for any $|\gamma| \in [0,1]$, $D_\Phi\mathcal{N}(\Phi,\gamma)$ is invertible. We compute iteratively the solution of $D_\Phi\mathcal{N}(\Phi,\gamma)H(s^t) = \Psi(s^t)$:

$$H(s^t) = \frac{1}{s_t}\left[h(s^t) + \sum_{j=1}^{k}\sum_{s^j\in\{1,2\}^j}\left(\frac{p}{\alpha_1}\right)^{\nu(s^j)}\left(\frac{1-p}{\alpha_2}\right)^{j-\nu(s^j)}\Psi(s^js^t)\right]$$

where $\nu(s^j) = \#(\{s \in s^j = 1\})$. A sufficient condition for invertibility of $D_\Phi\mathcal{N}(\Phi,\gamma)$ is then that:

$$\frac{p}{\alpha_1} + \frac{1-p}{\alpha_2} < 1$$

In figure 1, we display the determinacy conditions with respect to policy parameters α_1 and α_2 for model (27) in blue line and in dashed black line for model (29). Determinacy conditions of the linearized model appear to be tangent to the determinacy conditions of the original regime switching model at $\alpha_1 = \alpha_2 = 1$. The indeterminacy region (in the south-west) for the linearized model is included in that of the original one. However, for some policy parameters, determinacy conditions are satisfied for the linearized model while the regime switching model is indeterminate. Nonetheless, there is no contradiction as determinacy conditions for the linearized model only ensure existence and uniqueness of a stable solution for small - and hence perhaps smaller than one - γ. Thus we can not rely on such perturbation approach for solving regime switching models.

This example shows that, in a context of switching parameters, applying a perturbation approach around the constant parameter case is generally inadequate. Nevertheless, it does not mean that the perturbation approach cannot be employed in this context for other purposes. For example, Foerster et al. (2011) describe an algorithm to solve non-linear Markov-switching models by a perturbation approach. In this paper, the perturbation approach aims at simplifying the non-linearity of the model and not the Markov-switching process of the parameters[1]. Finally, Barthélemy and Marx (2011) use a perturbation approach to make the link between a standard linear Markov-Switching model, and a non-linear regime switching model in which transition probabilities may depend on the state variables.

4.4.2 Local versus global solutions

In addition, Theorem 5 gives results of existence and uniqueness only locally. Precisely, it means that there exists a neighborhood V of the steady state, in which there is a unique bounded solution ; it does not exclude that in a bigger neighborhood, there are more complex dynamics, as for instance the chaotic behaviors described in Benhabib et al. (2004). To give some insights on the limits risen by Benhabib et al. (2004), we present some results on the following sequence

$$u_{t+1} = \chi u_t^2 \times [1 - u_t^2], \qquad \chi = 3.4$$

There exists $0 < u_- < u_+ < 1$ such that

Proposition 2 *The sequence (u_t) has the following properties:*

- *For $u_0 \in [0, u_-]$, the sequence is convergent.*

- *For* $u_0 \in [u_-, u_+]$*, the sequence* (u_t) *is a 2-cycle, i.e. there exists* $0 < c_0 < c_1 < 1$ *such that:*

$$\lim_{t \to +\infty} u_{2t} = c_0, \qquad \lim_{t \to +\infty} u_{2t+1} = c_1$$

The proof of this proposition relies on the non-linear theory of recurrent real sequences defined by

$$u_{t+1} = f(u_t)$$

We display the graphs of f and the iterated function $f \circ f$ in figure 2. In figure 3, we represent the adherence values of (u_t) depending on the initial value u_0. This figure shows that if u_0 is large enough, the sequence does not converge any more.
This example highlights that, depending on the size of the neighborhood, there can be a locally unique stable solution, but more complex behaviors if we enlarge the neighborhood. Benhabib et al. (2004) exhibits a similar example where the model is locally determinate but presents chaotic features. More generally, we refer the reader to the study of logistic map in Ausloos and Dirickx (2006), or to Benhabib et al. (2004) for more formal definitions.

Figure 2: The functions f and $f \circ f$

Figure 3: Adherence values of (u_t) depending on the initial value u_0

5 DEALING WITH STRUCTURAL BREAKS: THE CASE OF REGIME SWITCHING MODELS

A necessity to model behavioral changes lead to consider rational expectations models in which parameters can switch between different values depending on the regime of economy. A way to formalize this assumption is to introduce some regimes labeled by s_t, s_t taking discrete values in $\{1, \cdots, N\}$. The model, then can be written as:

$$\mathbb{E}_t[f_{s_t}(z_{t+1}, z_t, z_{t-1}, \varepsilon_t)] = 0. \tag{31}$$

Let us assume that the random variables $s_t \in \{1, \cdots, N\}$ follow a Markov process with transition probabilities $p_{ij} = p(s_t = j | s_{t-1} = i)$.

There is a hot debate concerning the good techniques to solve Markov Switching rational expectations models. The main contributions are due to Farmer et al. (2009b,a, 2007, 2010a,b) and to Davig and Leeper (2007). The former papers focus on mean-square stable solutions, relying on some works in optimal control while the latter is trying to solve the model by mimicking Blanchard and Kahn (1980). Nevertheless, Farmer et al. (2010a) casts doubts on this second approach. First, we present the existing results, explain their limits and present some complements due to Svennson and Williams (2009); Barthélemy and Marx (2011).

In this part, we focus on linear Markov-Switching models, Barthélemy and Marx (2011) show how to deduce determinacy conditions for a non-linear Markov Switching model from a linear one.

5.1 A simple example

To begin with, we present a Fisherian model of inflation determination with regime switching in monetary policy, following Davig and Leeper (2007). This model is studied in Davig and

Leeper (2007); Farmer et al. (2009a).

As in sections **4.2** and **4.4.1**, the log-linearized asset-pricing equation can be written as

$$i_t = \mathbb{E}_t \pi_{t+1} + r_t \tag{32}$$

where r_t is the equilibrium ex-ante real interest rate, and we assume that it follows an exogenous process:

$$r_t = \rho r_{t-1} + v_t \tag{33}$$

with $|\rho| < 1$ and v is a zero-mean i.i.d random variable with bounded support.

Monetary policy rule follows a simplified Taylor rule, adjusting the nominal interest rate in response to inflation. The reaction to inflation evolves stochastically between regimes,

$$i_t = \alpha_{s_t} \pi_t \tag{34}$$

For the sake of simplicity, we assume that $s_t \in \{1, 2\}$. Regime s_t follows a Markov chain with transition probabilities $p_{ij} = P(s_t = j | s_{t-1} = i)$. We assume that the random variables s and v are independent. In addition, we assume that the non-linearity induced by regime switching is more important than the non-linearity of the rest of the model. Thus, we neglect in this section the effects of log-linearization.

In the case of a unique regime $\alpha_1 = \alpha_2$, the model is determinate if $\alpha > 1$ (see sections **4.2** and **4.4.1**), and in this case, the solution is:

$$\pi_t = \frac{1}{\alpha - \rho} r_t$$

Farmer et al. (2009a) show the following result:

Theorem 6 ***[Farmer et al. (2009a)]*** *The model* (32),(33) *and* (34) *admits a unique* *bounded solution if and only if all the eigenvalues of* $\begin{bmatrix} |\alpha_1|^{-1} & 0 \\ 0 & |\alpha_2|^{-1} \end{bmatrix} \times \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$ *are inside the unit circle.*

This result is explicitly derived by Davig and Leeper (2007) when $|\alpha_i| > p_{ii}$ for $i = 1, 2$. They call the determinacy conditions the Long Run Taylor Principle. The model (32),(33) and (34) is determinate if and only if

$$(1 - |\alpha_2|)p_{11} + (1 - |\alpha_1|)p_{22} + |\alpha_1||\alpha_2| > 1 \tag{35}$$

This condition is represented in figure 4 and shows that if the active regime is very active, there is a room of manoeuver for the passive regime to be passive, and this room of manoeuver is all the larger as the active regime is absorbant (i.e. that p_{11} is high).

Equation (35) illustrates how the existence of another regime affects the expectations of the agents and thus helps stabilizing the economy. Intuitively, it means that assuming that there exists a stabilizing regime, we can deviate from it either briefly with high intensity, or modestly for a long period (Davig and Leeper, 2007).

Figure 4: Long Run Taylor principle for $p_{11} = 0.6$ and $p_{22} = 0.5$

5.2 Formalism

Linear Markov-switching rational expectations models can generally be written as follows:

$$A_{s_t}\mathbb{E}_t(z_{t+1}) + B_{s_t}z_t + C_{s_t}z_{t-1} + D_{s_t}\varepsilon_t = 0 \tag{36}$$

The regime s_t follows a discrete space Markov chain, with transition matrix p_{ij}. The element p_{ij} represents the probability that $s_t = j$ given $s_{t-1} = i$ for $i, j \in \{1, \cdots, N\}$. We assume that ε_t is mean-zero, i.i.d and independent from s_t.

5.3 The approach of Davig and Leeper (2007)

Davig and Leeper (2007) find determinacy conditions for forward-looking models with Markov switching parameters ($C_i = 0$ for any i). They introduce a state-contingent variable $z_{it} = z_t \mathbb{1}_{s_t=i}$ and consider the stacked vector $Z_t = [z_{1t}, z_{2t}, \cdots, z_{Nt}]'$, to rewrite model (36) as a linear model in variable Z_t.

In the absence of shocks ε_t, the method of the authors (equations (80) and (81) p.631 Davig and Leeper, 2007) consists in assuming:

$$\mathbb{E}_t(z_{t+1}\mathbb{1}_{s_t=i}) = \sum_{j=1} p_{ij} z_{j(t+1)} \tag{37}$$

They introduce this relation in model (36) to transform the Markov model into a linear one. Finally, they solve the transformed model by using usual linear solving methods (see section **4**).

However, Equation (37) is not true in general as its right-hand-side is not zero when $i \neq s_t$ contrary to the left-hand-side. In the familiar new-Keynesian model with switching monetary policy rule, Farmer et al. (2010a) exhibit two bounded solutions for a parameter combination satisfying Davig and Leeper's determinacy conditions. Branch et al. (2007) and Barthélemy and Marx (2012) show that these conditions are actually valid for solutions, which only depend on a finite number of past regimes, called Markovian. Consequently, if these conditions are satisfied but multiple bounded solutions co-exist, it necessarily means that one solution is Markovian the others are not.

The question risen by this simplified approach is whether restricting to Markovian solutions makes economic sense or not. The strategy of Davig and Leeper (2007) relies on this question, see Davig and Leeper (2010),Branch et al. (2007) and Barthélemy and Marx (2012).

The question raised by Farmer et al. (2010a) in their comment to Davig and Leeper (2007) is whether restricting to Markovian solutions makes economic sense or not. For further details on this debate we refer to Davig and Leeper (2010), Branch et al. (2007) and Barthélemy and Marx (2012).

5.4 The strategy of Farmer et al. (2010b)

The contribution of Farmer et al. (2010b) consists in describing the complete set of solutions. For the sake of the presentation, we describe their results for invertible purely forward-looking models, i.e. when for any i, $C_i = 0$, $A_i = \mathbb{I}$ and B_i is invertible. Under these assumptions, the model (36) turns to be as follows:

$$\mathbb{E}_t z_{t+1} + B_{s_t} z_t + D_{s_t} \varepsilon_t = 0 \tag{38}$$

For this model, Farmer et al. prove the following result.

Theorem 7 (Farmer et al. (2009b)) *Any solution of model* (38) *can be written as follows:*

$$z_t = -B_{s_t}^{-1} D_{s_t} \varepsilon_t + \eta_t$$

$$\eta_t = \Lambda_{s_{t-1},s_t} \eta_{t-1} + V_{s_t} V'_{s_t} \gamma_t$$

Where for $s \in \{1,2\}$, V_s is a $n \times k_s$ matrix with orthonormal columns, $k_s \in \{1, \cdots, n\}$. γ_t is an arbitrary sunspot process such that:

$$\mathbb{E}_{t-1}(V_{s_t} V'_{s_t} \gamma_t) = 0$$

And for $(i,j) \in \{1,2\}^2$, there exist matrices $\Phi_{i,j} \in \mathcal{M}_{k_i \times k_j}(\mathbb{R})$ such that:

$$\tilde{B}_i V_i = \sum_{j=1}^{2} p_{ij} V_j \Phi_{i,j} \tag{39}$$

The strength of this result is that it exhaustively describes all the solutions. When the model embodies backward-looking components, solutions are recursive. The strategy of Farmer et al. (2010b) extends that of Sims (2002) (see section **3.3**) and Lubik and Schorfheide

(2004). Apart from purely forward-looking models presented above, finding all the solutions mentioned by Farmer et al. (2010b) requires employing numerical methods.

Following the influential book by Costa et al. (2005), Cho (2011) and Farmer et al. (2010b) argue that the convenient concept of stability in the Markov switching context is mean square stability:

Definition 4 (Farmer et al. (2009b)) *A process z_t is mean-square stable (MSS) if and only if there exists a vector m and a matrix Σ such that*

1. $\lim_{t \to +\infty} \mathbb{E}_0(z_t) = m$,
2. $\lim_{t \to +\infty} \mathbb{E}_0(z_t z_t') = \Sigma$,

This stability concept is less stringent than the boundedness (see Definition 1). On the one hand, checking that a solution is mean-square stable is easy (see Cho, 2011; Farmer et al., 2010b). On the other hand, this concept does not rely on a norm and hence, does not allow for applying perturbation approach.

5.5 Method of Undetermined coefficients (Svennson and Williams)

Svennson and Williams adopt an approach similar to the method of Uhlig (1999) and, consistently with the results of Farmer et al., look for solutions of model (36) under the form:

$$Z_t = P_{s_t} Z_{t-1} + Q_{s_t} \varepsilon_t$$

Introducing the matrices P_i and Q_i into the model (36) leads to a quadratic matrix system

$$A_1(p_{11}P_1 + p_{12}P_2)P_1 + B_1P_1 + C_1 = 0$$

$$A_2(p_{21}P_1 + p_{22}P_2)P_2 + B_2P_2 + C_2 = 0$$

However this system is more complex to solve than the Ricatti-type matrix equation presented in Uhlig (1999) and section **3**. Solving such equations involves computation-based methods.

5.6 The approach of Barthélemy and Marx

Barthélemy and Marx (2012) give general conditions of determinacy for purely forward looking models with the usual definition of stability (see definition 1):

$$\mathbb{E}_t z_{t+1} + B_{s_t} z_t + D_{s_t} \varepsilon_t = 0 \tag{40}$$

Contrary to Davig and Leeper (2007), the authors do not restrict the solutions space to Markovian solutions.

For a fixed operator norm on $\mathcal{M}_n(\mathbb{R})$, $|||.|||$, they introduce the matrix S_p defined for $p \geq 2$ by:

$$S_p = \left(\sum_{(k_1,\cdots,k_{p-1})\in\{1,\cdots,N\}^{p-1}} p_{ik_1} \cdots p_{k_{p-1}j} |||B_i^{-1} B_{k_1}^{-1} \cdots B_{k_{p-1}}^{-1}||| \right)_{ij} \tag{41}$$

They give the following determinacy condition for the existence of a unique s.r.e.e. of model (40):

Proposition 3 ***[Barthélemy and Marx (2012)]*** *There exists a unique bounded solution for model* (40) *if and only if*

$$\lim_{p\to+\infty} \rho(S_p)^{1/p} < 1$$

In this case, the solution is the solution found by Davig and Leeper (2007).

Based on eigenvalue computations, this Proposition extends Blanchard and Kahn's determinacy conditions to Markov switching models following the attempt by Davig and Leeper (2007). The advantage of Proposition 3 compared to previous methods is that it provides explicit ex-ante conditions ensuring existence and uniqueness of a bounded solution. However, this result suffers from two weaknesses. First, for some parameters combinations, this condition is numerically difficult to check. Second, this result only applies for purely forward-looking models.(we refer to Barthélemy and Marx, 2012, for more details).

To conclude this section, even if major advances have been made, this literature has not converged toward a unified approach yet. This lack of consensus reflects the extreme sensitivity of the results to the definition of the solutions' space and of the stability concept. On the one hand, Farmer et al. (2010b) show that mean square stability concept leads to very applicable and flexible techniques. But mean square stability does not rely on a well-defined norm and thus, does not allow for a perturbation approach. On the other hand, the concept of boundedness is consistent with the perturbation approach (see Barthélemy and Marx, 2011) and new results on determinacy are encouraging. However, this concept remains limited by the fact that the determinacy conditions are hard to compute and, at this stage, not generalized to models with backward-looking components.

6 DEALING WITH DISCONTINUITIES: THE CASE OF THE ZERO LOWER BOUND

Solving model (1) with a perturbation approach such as described above requires regularity conditions of the model. More specifically, applying Implicit Function Theorem requires that g is at least C^1 (see section **4**). However, certain economic models including for instance piece-wise linear functions do not fulfill this condition. One famous example of such models is a model taking into account the positivity of the nominal interest rate, the so-called Zero Lower Bound (ZLB). This part reviews existing methods to address technical issues raised by the ZLB.

6.1 An illustrative example with an explicit ZLB

Let us first present a monetary model including an explicit ZLB. Following most of recent literature studying the ZLB, we focus on standard New-Keynesian models as those described in Woodford (2003). To limit the complexity of the model, most papers log-linearize the structural equations (as it is the usual procedure to solve C^1 models by a perturbation approach) assuming that the non-linearity of these equations are secondary compared to the non-linearity introduced by the ZLB. This assumption is, however, a simplification which is theoretically unfounded. In such a model, the log-linear approximate equilibrium relations may be summarized by two equations, a forward-looking IS relation:

$$x_t = \mathbb{E}_t x_{t+1} - \sigma(i_t - \mathbb{E}_t \pi_{t+1} - r_t^n) \tag{42}$$

and a New Keynesian Phillips curve:

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t + u_t \tag{43}$$

Here π_t is the inflation rate, x_t is a welfare-relevant output-gap, and i_t is the absolute deviation of the nominal risk-free interest rate from the steady-state, r^*, the real interest rate consistent with the golden rule. Inflation and the output-gap are supposed to be zero at the steady-state. The term u_t is commonly referred to as a cost-push disturbance while r_t^n is the Wicksellian natural rate of interest. The coefficient κ measures the degree of price stickiness while σ is the inter-temporal elasticity of substitution; both are positive. $0 < \beta < 1$ is the discount factor of the representative household.

To allow bonds to coexist with money, one can ensure that the return of holding bonds is positive (in nominal term). This condition translates into:

$$i_t \geq -r^* \tag{44}$$

This condition triggers a huge non-linearity that violates the C^1 assumption required to use perturbation approach and the Implicit Function Theorem (see section **4**). To circumvent these difficulties, one can either solve analytically by assuming extra hypothesis on the nature of shocks (section **6.2**) or use global methods (section **6.3**).

6.2 Ad hoc linear methods

Following Jung et al. (2005), a large literature (Eggertsson and Woodford, 2003, 2006; Eggertsson, 2011; Christiano et al., 2011; Bodenstein et al., 2010) solves rational expectations model with the ZLB by postulating additional assumptions on the nature of the stochastic processes. In this seminal paper, the authors find the solution of equations (42), (43) and (44) with the assumption that the number of periods for which the natural rate of interest will be negative is known with certainty when the disturbance occurs. In this paper, monetary policy is supposed to minimize a Welfare Loss function.

$$\min \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^k (\pi_{t+k}^2 + \lambda x_{t+k}^2) \tag{45}$$

We refer to Woodford (2003) for more details on potential micro-foundation of such a loss function.

Eggertsson and Woodford (2003) show how the system can be solved when the natural interest rate is negative during a stochastic duration unknown at date t. This resolution strategy has been recently used by Christiano et al. (2011) to assess the size of the Government Spending Multiplier at the ZLB.

For the sake of clarity, we present a procedure to solve equations (42), (43) and (44) when monetary policy authority follows a simple Taylor rule as long as it is possible instead of minimizing the welfare loss (45):

$$i_t = \max(-r^*, \alpha\pi_t) \tag{46}$$

Figure 5: Scenario of negative real natural interest rate

Studying a contemporaneous Taylor rule rather than an optimized monetary policy like (45) prevents us to introduce backward-looking components in the model (through Lagrange multipliers), and hence, it reduces the size of the state-space Eggertsson and Woodford (2003) for more details.

We solve the model when the path of the future shocks is known (*perfect foresight equilibrium*) and make the assumption of a (potentially very) negative shock to the natural rate of interest during a finite number of periods τ. When the shock is small enough, the model can be solved using a standard backward-forward procedure as presented in section **4** since the ZLB constraint is never binding. In this case, the equilibrium is given by:

$$x_t = x_{t+1} - \sigma(i_t - r_t^n - \mathbb{E}_t \pi_{t+1})$$

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t$$

$$i_t = \alpha \pi_t$$

where $r_t^n = -r^l$ from $t = 0$ to $t = \tau$ and $r_t^n = 0$ afterwards (see figure 5).

Using notations of section **3** and denoting by Z_t the vector of variables $[x_t, \pi_t, i_t]'$, these equations can be written as follows:

$$A\mathbb{E}_t \begin{bmatrix} Z_t \\ Z_{t+1} \end{bmatrix} = B \begin{bmatrix} Z_{t-1} \\ Z_t \end{bmatrix} + C r_t^n$$

When r^l is small enough, we find the solution by applying the methods described in section **3**:

For $t \leq \tau$,

$$\forall t \leq \tau, \quad Z_t = Z_{22}^{-1} \sum_{k=0}^{\tau-t-1} (S_{22}^{-1} T_{22})^k S_{22}^{-1} Q_{12}' \, [\sigma, 0, 0]' \, r^l \qquad (47)$$

where Z_{22}, S_{22}, T_{22} and Q_{12} are matrices given by the Schur decomposition (see Equations (10) in section **3**).

For $t > \tau$, all variables are at the steady state as the model is purely forward-looking.

$$\forall t > \tau, \quad Z_t = 0 \tag{48}$$

As long as i_0 given by equation (47) is larger than $-r^*$, the ZLB constraint is never binding and the solution does not violate the constraint (44). When the shock r_l becomes large enough, the solution 47 is no more valid as the ZLB constraint binds.

In this case, let us define k the largest positive integer such that $i_{\tau-k}$, defined in equations (47) and (48) is larger than $-r^*$. Obviously, if $i_\tau < -r^*$ then $k = 0$, meaning that up to the reversal of the natural real interest rate shock, ZLB constraint is binding.

Because of the forward-looking nature of the model, for $t > \tau - k$ the solution of the problem is given by equations (47) and (48), and the ZLB does not affect the equilibrium dynamic.

For $t < \tau - k$, we need to solve the model by backward induction. The solution found for $t > \tau - k$ is the terminal condition and is sufficient to anchor the expectations. Thus, for $t < \tau - k$, the policy rate is stuck to its lower bound and the variables should satisfy the following dynamic (explosive) system:

$$x_t = x_{t+1} - \sigma(-r^* + r^l - \pi_{t+1})$$

and

$$\pi_t = \beta \pi_{t+1} + \kappa x_t$$

This can be easily rewritten as follows:

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \begin{pmatrix} 1 & \sigma \\ \kappa & \sigma\kappa + \beta \end{pmatrix} \begin{bmatrix} x_{t+1} \\ \pi_{t+1} \end{bmatrix} + \begin{pmatrix} \sigma \\ \sigma\kappa \end{pmatrix} (r^* - r^l)$$

Figure 6: Responses of endogenous variables to a negative real natural interest rate

Thus,

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \sum_{k=0}^{\tau-k-t-1} \begin{pmatrix} 1 & \sigma \\ \kappa & \sigma\kappa+\beta \end{pmatrix}^k \begin{pmatrix} \sigma \\ \sigma\kappa \end{pmatrix} (r^* - r^l) + \begin{pmatrix} 1 & \sigma \\ \kappa & \sigma\kappa+\beta \end{pmatrix}^{\tau-k-t} \begin{pmatrix} x_{\tau-k} \\ \pi_{\tau-k} \end{pmatrix}$$

Where $[x_{\tau-k}, \pi_{\tau-k}]'$ is given by equation (47) or is 0 if the shock is large enough ($k = 0$).

To illustrate this method, we compute the equilibrium dynamics after an unexpected shock on the negative real interest rate lasting 15 periods (see Figure 5). After the initial fall in the real natural interest rate, we assume that there is no uncertainty and the economic agents perfectly know the dynamic of this shock. We calibrate the model with common numbers: $\sigma = 1$, $\beta = 0.99$, $\kappa = 0.02$, $\alpha = 1.5$ and there is no inflation at the steady-state. Figure 6 displays the responses of output-gap, annualized inflation and annualized nominal interest rate with and without the ZLB. The size of the natural real interest rate shock is calibrated such as the economy hits the ZLB immediately and during 9 periods (thus, $\tau = 15$ and $k = 9$).

As we could expect, in the absence of the ZLB, the economy would suffer from a less severe crisis with lower deflation and higher output-gap. In our simulations, the gap between the dynamic equilibrium with and without the ZLB is quite huge suggesting a potential large effect of this constraint. However, this should be taken with care as this example is only illustrative and the model is very stylized.

The great interest of this approach is that we can understand all the mechanisms at work and we have a proper proof of existence and uniqueness of a stable equilibrium (proof by construction). However, this resolution strategy is only valid for a well known path of shocks and in a perfect foresight equilibrium. Even if Eggertsson and Woodford (2003) extend this method for negative natural interest rate shock with unknown but finite duration, this kind of methods only addresses issues raised by the ZLB partially. Indeed, this method is mute on the consequences of the ZLB in normal situation, when there are risks of liquidity trap and ZLB but when this risk had not still materialized. In such a situation, one can expect that economic agents' decisions are altered by the risk of reaching the ZLB in the future.

6.3 Global methods

To completely solve a model with ZLB, one can turn toward global methods or non-linear methods. Amongst the first to solve model with a ZLB using non-linear method, we count Wolman (2005) who uses the method of finite elements from McGrattan (1996) and Adam and Billi (2004, 2006) who solve a model very similar to the one presented above for the case with policy commitment (2004) and with discretionary policy (2006).

The clear advantage of global methods for studying an economy subject to the ZLB is that the full model can be solved and analyzed without assuming a particular form of shocks. Furthermore, with this approach one can study the influence of the probability to reach the ZLB in the future on current economic decisions and equilibria through the expectations channel.

The general strategy is to first determine the right concept of functions space in which the solution should be (here for example, the only state variable could be the shock r_t^n). Then, one can replace the variables by some unknown functions in equations (42), (43) and (46). The expectations are integral of these functions (the measure is the probability distribution of shock r_t^n), hence equations (42), (43) and (46) translate into:

$$x(r_t^n) = \int x(r_{t+1}^n) - \sigma(i(r_{t+1}^n) - r_t^n - \int \pi(r_{t+1}^n)) \tag{49}$$

$$\pi(r_t^n) = \beta \int \pi(r_{t+1}^n) + \kappa x(r_t^n) \tag{50}$$

$$i(r_t^n) = \max(\alpha\pi(r_t^n), -r^*) \tag{51}$$

Finally, solving for the equilibrium requires finding a fixed-point in the functions-space $[x(r_t^n), \pi(r_t^n), i(r_t^n)]$ of equations (49), (50) and (51).

To solve this kind of fixed-point problem one can either use projection methods as described in section **7.2** or in Judd (1996) or iterate the above system up to find a fixed-point as in Adam and Billi (2004). In this latter paper, the authors use finite elements. They define a grid and interpolate between nodes using a linear interpolation to compute the integrals. The algorithm they propose can be summed up as follows in our context:

1. guess initial function
2. compute the right hand side of equations (49), (50) and (51)
3. attribute a new value for the guess solution function equal to the left hand side
4. if the incremental gain is less than a targeted precision stop the algorithm, otherwise go to 2.

To fasten the algorithm, it seems natural to place more nodes around the ZLB (i.e. negative natural rate shocks) and less nodes for large positive natural rate shock as the model supposedly behaves linearly when probability of reaching the ZLB is very low.

Up to now, this computational oriented strategy is the only available method to solve rational expectations models with the ZLB. However, this method is limited since its outcome is not guaranteed by any theoretical background (no existence and uniqueness results). Besides, this method is numerically costly and suffers from the curse of dimensionality. It thus prevents from estimating a medium-scale DSGE model using such a strategy.

7 GLOBAL SOLUTIONS

When the model presents non smooth features or high variation of shocks, some alternatives based on purely computational approaches may reveal more appropriate (see section **6**. These approaches have been highly presented, compared and used in a more general framework than this chapter. In this section, we will present main methods used in the context of rational expectations models. The aim is not to give an exhaustive description of all the numerical available tools -they have been intensively presented in **?**, Heer and Maussner (2005) and Den Haan et al. (2010a)- or to compare the accuracy and performance of different methods (Aruoba et al., 2006), but rather to explain what type of method can be used to deal with models not regular enough to apply perturbation method. Typical examples are models with occasionally binding constraints (section **6**), or with large shocks. Finally, it should be noted that numerical methods may sometimes be mixed with perturbation method to improve the accuracy of solutions (Maliar et al., 2011).

It is worth noticing that most of these methods are very expensive in terms of computing time, and do not allow for checking the existence and uniqueness of the solution. We mainly distinguish three types of methods: value function iteration, projections and extended deterministic path methods.

7.1 Value Function Iteration

This method can be applied when the stochastic general equilibrium models are written under the form of an optimal control problem.

$$\max_{x\in\{(\mathbb{R}^n)^\infty\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(x_t, y_t) \quad \text{s.c.} \quad y_{t+1} = g(y_t, x_t, \varepsilon_{t+1}) \text{ with a fixed } y_0 \tag{52}$$

It is easy to see that first order conditions for model (52) and changes in notations lead to an equation like (1) but this formulation is more general. According to the Bellman principle (see Rust (1996)), such a program can be rewritten as

$$V(y_0) = \max_{x_0}[U(y_0, x_0) + \beta \mathbb{E}_0 V(y_1)] \tag{53}$$

When U and g satisfy some concavity conditions, it is possible to compute by iterations the value function V and the decision function h defined by:

$$h(y_t) = \arg\max_{x_t \in A}[U(y_t, x_t) + \beta \mathbb{E}_t V(g(y_t, x_t, \varepsilon_{t+1}))] \tag{54}$$

where A stands for the set of admissible solutions. This method consists in defining a bounded convex set of admissible values for y_t, containing the initial value and the steady state, and considering a grid G on this set. Then, we consider the sequence of functions V^n, defined recursively for $y \in G$ by:

$$V^{n+1}(y) = \max_{x}\{U(y, x) + \beta \mathbb{E}(V^n[g(y, x, \varepsilon)])\}$$

We refer the reader to **?** for the theoretical formalism and to Fernandez-Villaverde and Rubio-Ramirez (2006) for the algorithmic description. This method is computationally expensive since it is applied on grids, and hence may be applied only for relatively small models. Theoretical properties of convergence have been studied in Santos and Vigo (1998), and , the illustration of the method on the growth model is presented in Christiano (1990) and Barillas and Fernandez-Villaverde (2007).

7.2 Projection Method

Projection method consists in looking for an approximate solution of the form $z_t = h(z_{t-1}, \varepsilon_t)$. Assuming that the shocks ε_t follow a distribution law μ, problem (1) can be reformulated as solving the functional equation:

$$\mathcal{G}(h) = 0 \tag{55}$$

where $\mathcal{G}$ is defined as:

$$\mathcal{G}(h)(z,\varepsilon) = \int g(h(h(z,\varepsilon'),\varepsilon), h(z,\varepsilon), z)\mu(\varepsilon')d\varepsilon'$$

The core idea of projection method is to find an approximate solution $\hat{h}$ belonging to a finite-dimension functional space $\mathcal{S}$ (Judd, 1996).

Let us denote by $\{\Phi_i\}_{i\in\{1,\cdots,P\}}$ the basis of the vector space $\mathcal{S}$. In other words, we are looking for P parameters $(c_i)_{i\in\{1,\cdots,P\}}$ such that $\hat{h} = \sum_{i=1}^{P} c_i\Phi_i$ is close to the solution h.

There are four sets of issues that must be addressed by this approach (Judd, 1996; Fackler, 2005; Heer and Maussner, 2005). The first question is the choice of $\mathcal{S}$, for instance $\mathcal{S}$ can be the set of polynomials of degree smaller than d, the set of piecewise linear functions, or the set of spline functions. The second issue is the computation of the expectation operator; here, we have at our disposal all the numerical techniques dealing with integrals, mainly quadrature formula or Monte Carlo computations. The third question is to characterize the accuracy of the approximation, in other words the size of the residual function $\mathcal{G}(\hat{h})$. There are three main criteria, either we look for the function minimizing the L^2 norm of $\mathcal{G}(\hat{h})$ (least squares), or the zero of $\mathcal{G}(\hat{h})$ in a finite number of points (collocation), or the function such that $\mathcal{G}(\hat{h})$ is orthogonal to $\mathcal{S}$. Finally, the fourth issue concerns the way we can solve such a problem. These different steps lead to find the zero $(c_i)_{i\in\{1,\cdots,P\}}$ of a function; there are various root-finding algorithms, mainly based on iterative approaches (Newton, Broyden, fixed point ...).

This kind of method is used for solving models with occasionally binding constraints (see for instance Christiano and Fisher, 2000) or endogenous regime switching models (see Davig and Leeper, 2008). The accuracy and the computational cost of this method depend both on the dimension of $\mathcal{S}$ and on the choice of the basis $\{\Phi_i\}_{i\in\{1,\cdots,P\}}$. In practice, Christiano and Fisher (2000); Heer and Maussner (2005) refine this method with a Parametrized Expectations Algorithm.

7.3 Parametrized Expectations Algorithm

The Parametrized Expectations Algorithm consists in rewriting model (1) under the form (Juillard and Ocaktan, 2008):

$$\tilde{f}(\mathbb{E}_t[\phi(y_{t+1}, y_t)], y_t, y_{t-1}, \varepsilon_t) = 0 \tag{56}$$

We restrict the focus on solutions only depending on (y_{t-1}, ε_t). Defining h the expected solution such that:

$$\mathbb{E}_t[\phi(y_{t+1}, y_t)] = h(y_{t-1}, \varepsilon_t)$$

We can apply a projection method (described above) to h, and find an approximation $\hat{h}$ of h in an appropriate functional vector space. This method is described in Marcet and Marshall (1994) and applied in Christiano and Fisher (2000) for models with occasionally binding constraints.

7.4 Extended Deterministic Path Method

The Extended Path due to Fair and Taylor (1983) is a forward iteration method for solving models with a given path of shocks. It is close to the one presented in section **6**. As it does not include uncertainty, this method is unable to solve DSGE models.

Let us assume that we want to get the solution on a period $[0, p]$. Fix $T > 0$ large enough, and for any $t \in \{0, \cdots, p\}$, define $y_{T+s} = \bar{y}$ and $\varepsilon_{T+s} = 0$ for all $s > 0$. Then, for $t \in \{0, \cdots, p\}$, by fixing the terminal condition, we can numerically solve the model:

$$g(y_{t+s+1}, y_{t+s}, y_{t+s-1}, \varepsilon_{t+s}) = 0, \qquad \forall s > 0, y_{t+T} = \bar{y}$$

and get $y_t^T = h^T(y_{t-1}, \varepsilon_t)$. Love (2009) shows that the approximation error with this algorithm is reasonable for the stochastic growth model, this method has also been implemented for solving models with occasionally binding constraints, such as the ZLB (see Coenen and

Wieland, 2003; Adjemian and Juillard, 2010).

8 CONCLUSION

In this chapter, we have presented the main theories underlying the solving of rational expectations models with a finite number of state variables. We have insisted on the importance of the perturbation approach as this approach is based on solid theoretical framework. The interest of this approach is that it allows for checking existence and uniqueness of a stable solution. Moreover, we have tried to give some insights on the limits of this approach. This chapter ends up with a brief review of some important numerical approaches.

This chapter obviously raises a wide range of non solved essential questions. What is the size of the admissible domain to apply perturbation approach ? How to characterize the solutions of non linear Markov Switching rational expectation models and what are the determinacy conditions ? How to solve rational expectations models with ZLB without requiring global methods ?

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