No :

Groups & Fields

$* : G \times G \to G$ is a function.

Check if the followings are groups

(01) $(\mathbb{R}, +)$ ✓

$e = 0$

~~Property 1~~ $\bar{a} = -a$

Commutativity also exists. ∴ an Abelian group

(02) $(\mathbb{R}, \cdot)$ ✗

$e = 1$

$\bar{a} = a^{-1} = \frac{1}{a}$ ($\frac{1}{0}$ is not possible)

(03) $(\mathbb{R} - \{0\}, \cdot)$ ✓

$e = 1$

$\bar{a} = a^{-1} = \frac{1}{a}$

Commutativity exists. ∴ an Abelian group

(04) $(\mathbb{B}, +)$ ✗ , $\mathbb{B} = \{0, 1\}$, $+ = \text{or}$

Boolean algebra

$e = 0$

$\bar{a}$ does not exist for 1

+	0	1
0	0	1
1	1	1

(05) G = 2x2 real invertible matrices $= GL_2(\mathbb{R})$ under matrix mult

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; a, b, c, d \in \mathbb{R}, \; \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \neq 0 \right.$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in G \; . \; \therefore \; G \neq \phi$$

$$e = I$$

$$\bar{a} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = (ad - bc) \begin{pmatrix} d & -b \\ -b & a \end{pmatrix} \in G$$

Commutativity does not exists.

$\therefore$ G is a group (Not an Abelian group).

Note: Associativity works even for non-square matrices.

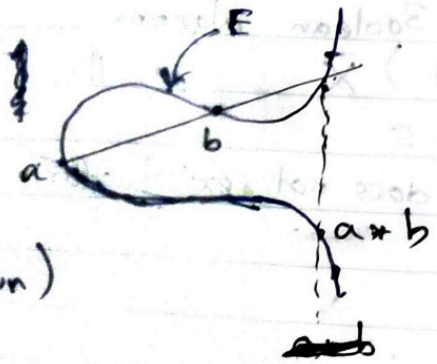
(06) Elliptic curve

$$E = \{ (x, y) \mid y^2 = x^3 + bx + c \; ; \; a, b \in \mathbb{R} \}$$

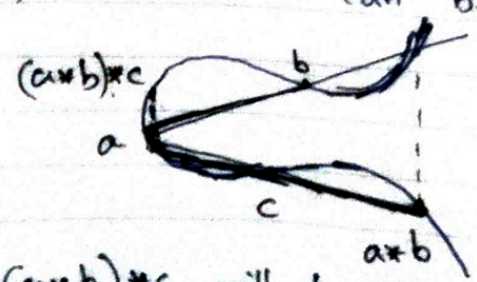
$\mathcal{O}$ = point at infinity.

$$G = E \cup \mathcal{O}$$

$*$ = Join a & b & get $a * b$ as in the figure (point addition)



$a * (b * c) = (a * b) * c$ can be represented as below



$(a * b) * c$ will be the same point as $a * (b * c$

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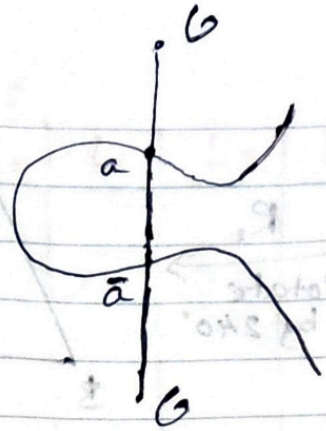
$e = G$

$\bar{a}$ exists



Join 'a' & G until it meets the curve again & draw a vertical again to get 'a' again

Commutativity also exists.

Join $\bar{a}$ & $\bar{\bar{a}}$ & go upwards to meet G at the curve. Draw the vertical to meet G at the other side

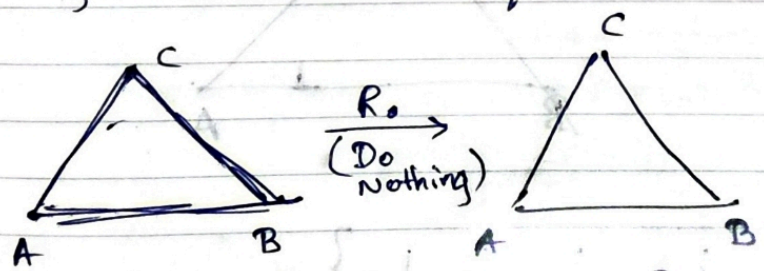
$\therefore (G, *)$ is an Abelian group.

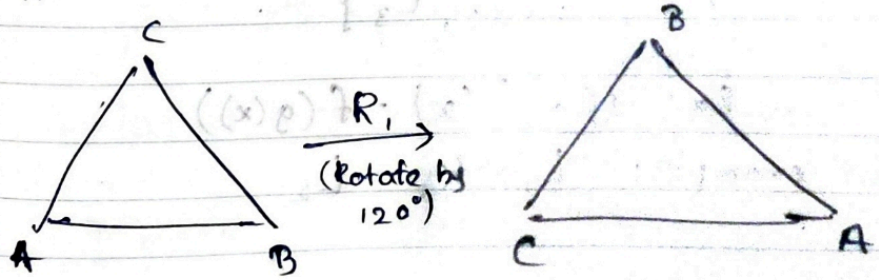
Theorem : $(G, *)$ be a group & $a \in G$, then $\bar{\bar{a}} = a$

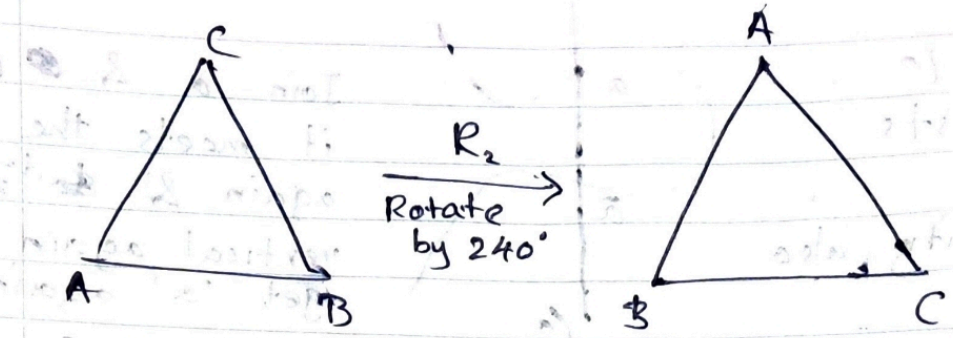
Proof:

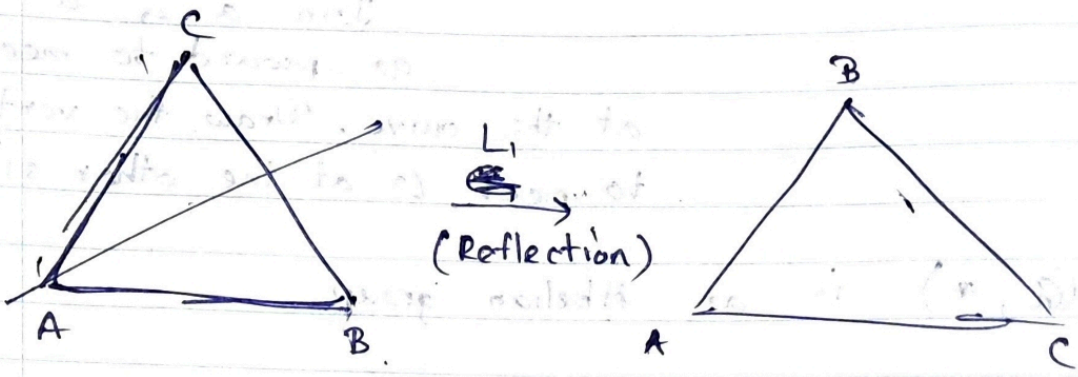
$$\bar{a} * \bar{\bar{a}} = e$$
$$a * (\bar{a} * \bar{\bar{a}}) = a * e$$
$$(a * \bar{a}) * \bar{\bar{a}} = a$$
$$e * \bar{\bar{a}} = a$$
$$\bar{\bar{a}} = a$$

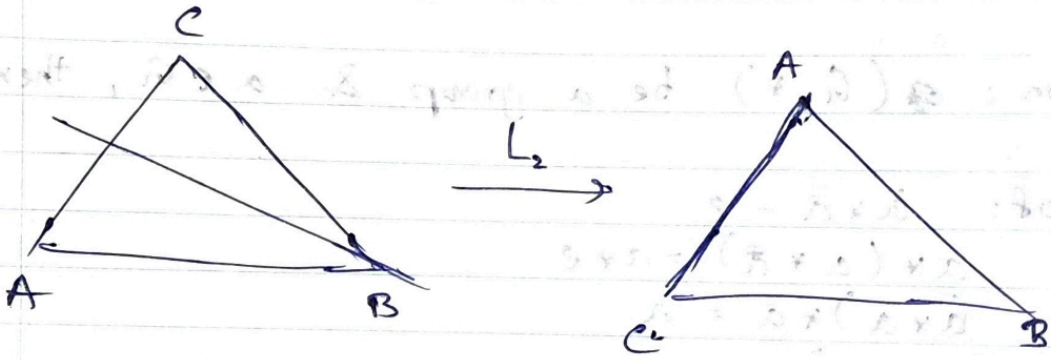
(07) Symmetries of an equilateral triangle.

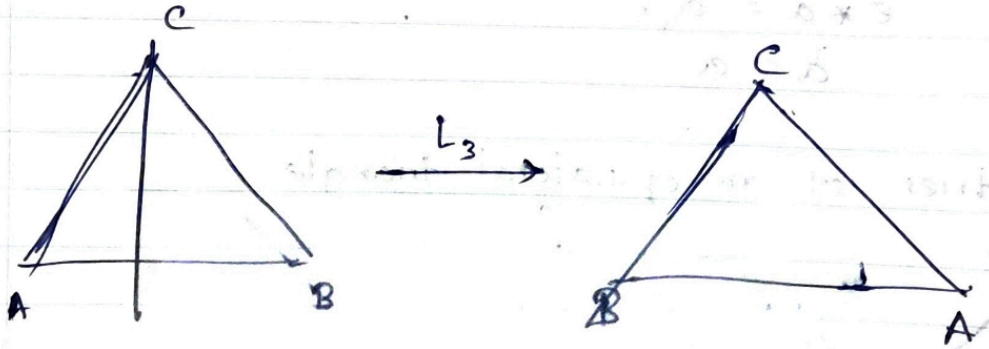












$$G = \{R_0, R_1, R_2, L_1, L_2, L_3\}$$

composition

$* = \circ$ (Eg: $(f \circ g)(x) = f(g(x))$

In this example; $L_1 \circ R_1 = L_2$

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2nd op$^{\underline{rn}}$

$\circ$	R_0	R_1	R_2	L_1	L_2	L_3
R_0	R_0	R_1				
R_1						
R_2						
L_1						
L_2						
L_3						

1st op$^{\underline{rn}}$

Fields

For a field $(\mathbb{F}, +, \cdot)$;

>> $(\mathbb{F}, +)$ is an Abelian group

>> $(\mathbb{F} - \{e_+\}, \cdot)$ is an Abelian group

>> $\cdot ; \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ is a function

>> $\forall a, b, c \in \mathbb{F}$; $a \cdot (b+c) = a \cdot b + a \cdot c$

(01) $(\mathbb{R}, +, \cdot)$ ✓

(02) $(\mathbb{R}, \cdot, +)$ ✗ $(\mathbb{R}, \cdot)$ is not an Abelian group

$(\mathbb{R} - \{1\}, +)$ is not an Abelian group.

(There is no 1).

Here e (which is 1) should be considered

Note

In a field $(F, +, \cdot)$ $(F - \{e_+\}, \cdot)$ should be an abelian group.

~~$\mathbb{F} \times \mathbb{F} \to \mathbb{F}$ is also not correct~~

~~Distributive law also do not exist because 1 does not~~

$a + (b.c) = (a+b) \cdot (a+c)$ also do not exist.

(03) $(\mathbb{B}, +, \cdot)$ ✗

$(\mathbb{B}, +)$ is not an Abelian group

$(\mathbb{B} - \{0\}, \cdot)$ is an Abelian group

$\mathbb{B} \times \mathbb{B} \to \mathbb{B}$ is a function

Distributive law exists

(04) $(\mathbb{B}, \cdot, +)$ ✗

$(\mathbb{B}, \cdot)$ is not an Abelian group

- $(\mathbb{B} - \{1\}, +)$ is an Abelian group

$\mathbb{B} \times \mathbb{B} \to \mathbb{B}$ is a function

$a + (b \cdot c) = (a+b) \cdot (a \cdot c)$ exists.

← Rational numbers

(05) $(\mathbb{Q}, +, \cdot)$ ✓ $\left(\frac{p}{q}\right)$

- $(\mathbb{Q}, +)$ is an Abelian group

$\left(e_+ = \frac{0}{1} \qquad \left(\frac{p}{q}\right)^{-} = -\frac{p}{q}\right)$

- ~~$(\mathbb{Q} - \{0\}, \cdot)$~~ $\left(\mathbb{Q} - \left\{\frac{0}{1}\right\}, \cdot\right)$ is an Abelian group

$\left(e_\cdot = 1 = \frac{1}{1} \qquad \left(\frac{p}{q}\right)^{-1} = \frac{q}{p}\right)$

~~p is not~~

$\mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ is a function

Distributive law exists

← Complex numbers

(06) $(\mathbb{C}, +, \cdot)$

note: Vectors & complex numbers behave in the same manner in 2D

$e_+ = 0 + 0i$

- $(\mathbb{C}, +)$ is an abelian group

- $(\mathbb{C} - \{0 + i0\}, \cdot)$ is an abelian group

$(x+iy) \cdot (a+ib) = (xa - by + ayi + xbi)$ ~~in 2D~~

$\overline{(a+ib)} = \frac{1}{a+ib} = \frac{a-ib}{a^2+b^2}$ $\because a^2+b^2 \neq 0$

$\Rightarrow a, b \neq 0$ at the same time. i.e ~~a, b~~ $0 + i0$ do not exist.

- $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a func^n
- Distributive law exists

Theorem

Let $(\mathbb{F}, +, \cdot)$ be a field. Then

$0 \cdot a = 0$ | or | $e_+ \cdot a = e_+$

$0 \neq 1$ | | $e_+ \neq e$.

$(-a) \cdot b = -(a \cdot b)$ | | $(-a) \cdot b = -(a \cdot b)$ $\because$ $\bar{a}^+ = -a$

Proof:

$$a \cdot 1 = a$$
$$a \cdot (1 + 0) = a$$
$$a \cdot 1 + a \cdot 0 = a$$
$$a + a \cdot 0 = a$$
$$-a + (a + a \cdot 0) = -a + a$$
$$-a + a + a \cdot 0 = 0$$
$$0 + a \cdot 0 = 0$$
$$a \cdot 0 = 0$$
$$0 \cdot a = 0$$

$(\mathbb{F} - \{0\}, \cdot)$ is an Abelian group.

$1 \in \mathbb{F} - \{0\}$

$\Rightarrow 1 \neq 0$

Note: For groups: Let $(G, *)$ is a group. Then,

e is unique

$\bar{a}$ is unique

Note

$\#: V \times V \to V$ (vector add^{n}) $\quad \cdot: F \times F \to F$ (number multi^{pln})

$+: F \times F \to F$ (number add^{n}) $\quad *: F \times V \to V$ (scalar multi^{pln})

Def: $(V, \#, *)$ over $(F, +, \cdot)$ is a vector space iff;

1) $(V, \#)$ is an Abelian group
2) $(F, +, \cdot)$ is a field
3) $*: F \times V \to V$ is a func^{n}
4) $\forall a, b \in F, \forall u \in V;\ (a+b) * u = (a * u) \# (b * u)$
5) $\forall a \in F, \forall u, w \in V;\ a * (u \# w) = (a * u) \# (a * w)$
6) $\forall a, b \in F, \forall u \in V;\ (a \cdot b) * u = a * (b * u)$
7) $\forall u \in V;\ 1 * u = u$

$\uparrow$ e.

(01) $(\mathbb{R}^2, \#, *)$ over $(\mathbb{R}, +, \cdot)$ a vector space?

1) $u = (x, y), w = (p, q) \in \mathbb{R}^2$

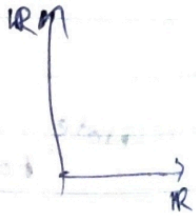
$u \# w = (x, y) \# (p, q) = (x+p, y+q) \in \mathbb{R}^2$
$= (x, y) + (p, q)$

$e_{\#} = (0, 0) \qquad \bar{a}^{\#} = (-x, -y) \in \mathbb{R}^2$
$= -(x, y)$

Note

$A \times B = \{(a, b) \mid a \in A, b \in$

$A^2 = A \times A$
$= \{(a, b) \mid a, b \in A$



$(\mathbb{R}^2, \#)$ is an Abelian group

2) $(\mathbb{R}, +, \cdot)$ is a field

3) $u = (x, y) \in \mathbb{R}^2 = V$, $a \in \mathbb{R} = F$

$a * u = a * (x, y) = (a \cdot x, a \cdot y) \in \mathbb{R}^2$

$* : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ is a func^{n}

4) $a, b \in \mathbb{R} = F$, $u = (x, y) \in \mathbb{R}^2 = V$

$(a+b) * u = (a+b) * (x, y)$
$= ((a+b) \cdot x, (a+b) \cdot y)$
$= (a \cdot x + b \cdot x, a \cdot y + b \cdot y)$

$$= (a\cdot x, a\cdot y) \# (b\cdot x, b\cdot y)$$

$$= (a * (x,y)) \# (b * (x,y))$$

$$= (a * u) \# (b * u)$$

5) ✓

6) ✓

7) $1 * u = 1 * (x,y)$

$= (1\cdot x, 1\cdot y)$

$= (x,y) = u$

(02) $(\mathbb{R}^2, \#, *)$ over $(\mathbb{C}, +, \cdot)$

1) ✓

2) ✓

3) $a = p+qi \in \mathbb{C}$, $u = (x,y) \in \mathbb{R}^2$

$a * u = (p+qi) * (x,y)$

$= ((p+qi)\cdot x, (p+qi)\cdot y) \notin \mathbb{R}^2$ $\boxed{\text{Note. } x = x + i0}$

$\therefore$ the above is not a vector space.

(03) $(\mathbb{C}^2, \#, *)$ over $(\mathbb{C}, +, \cdot)$

1) ✓

2) ✓

3) $a = p+qi \in \mathbb{C}$, $u = (x+iy, s+it) \in \mathbb{C}^2$

$a * u = ((p+qi)\cdot(x+iy), (p+qi)\cdot(s+it)) \in \mathbb{C}^2$

$* : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ is a func^n

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4) ✓

5) ✓

6) ✓

7) ✓ $1 * u = (1+0i) * (x,y) = (x,y) = u$

$\therefore$ Above is a vector space

Real 2x2 matrices

(04) $(\mathbb{R}^{2\times2}, \#, *)$ over $(\mathbb{R}, +, \cdot)$.

$u = \begin{pmatrix} x & y \\ s & t \end{pmatrix} \in \mathbb{R}^{2\times2}$ $w = \begin{pmatrix} l & m \\ p & q \end{pmatrix} \in \mathbb{R}^{2\times2}$

1) $u \# w = \begin{pmatrix} x & y \\ s & t \end{pmatrix} \# \begin{pmatrix} l & m \\ p & q \end{pmatrix} = \begin{pmatrix} x+l & m+y \\ s+p & t+q \end{pmatrix} \in \mathbb{R}^{2\times2}$

$e_\# = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{2\times2}$

$\bar{u}^\# = \begin{pmatrix} -x & -y \\ -s & -t \end{pmatrix} \in \mathbb{R}^{2\times2}$

2) $(\mathbb{R}, +, \cdot)$ is a field.

3) $a * u = a * \begin{pmatrix} x & y \\ s & t \end{pmatrix} = \begin{pmatrix} a\cdot x & a\cdot y \\ a\cdot s & a\cdot t \end{pmatrix} \in \mathbb{R}^{2\times2}$

$* : \mathbb{R} \times \mathbb{R}^{2\times2} \longrightarrow \mathbb{R}^{2\times2}$ is a func^n

4) ✓

5) ✓

6) ✓

7) ✓

$\therefore$ Above is a vector space

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(05) $(\mathbb{R}^{2\times 2}, \#, *)$ over $(\mathbb{R}, \cdot, +)$

(06) $(\mathbb{R}^{2\times 2}, \#, *)$ over $(\mathbb{R}-\{0\}, \cdot, +)$

1) ✓

2) ✓

Th^{rm}: $(V, \#, *)$ over $(F, +, \cdot)$ is a vector space, then $0 * u = e \ \forall u \in V$

(Consider $e_\# = e$, $\bar{u}^\# = \bar{u}$, $e_+ = 0$, $\bar{a}^+ = -a$, $e_\cdot = 1$, $\bar{a}^\cdot = a^{-1}$)

Proof: $u = 1 * u$ (rule no. 7)

$= (1+0) * u$

$= (1*u) \# (0*u)$ (rule 4)

$= u \# (0*u)$

$\bar{u} \# u = \bar{u} \# (u \# (0*u))$

$e = (\bar{u} \# u) \# (0*u) = e \# (0*u)$

$= 0 * u$

Th^{rm}: $a \in \mathbb{F}$; $a * e = e$

Th^{rm}: $a \in \mathbb{F}$; $u \in V$

$a * u = e \Rightarrow a = 0$ or $u = e$

Th^{rm}: $(-1) * u = \bar{u}$

Proof; $e = 0 * u$

$= ((-1)+1) * u$

$= ((-1)*u) \# (1*u)$

$= ((-1)*u) \# u$

$e \# \bar{u} = (((-1)*u) \# u) \# \bar{u}$

$= ((-1)*u) \# (u \# \bar{u})$

$\bar{u} = ((-1)*u) \# e$

$\bar{e}^+ + u = \bar{u}^\# = \bar{u} = ((-1)*u)$

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Note

Elliptic Group $E = \{(x,y) \mid y^2 = x^3 + bx + c ;\ \left(\frac{b}{3}\right)^3 + \left(\frac{c}{2}\right)^2 \neq 0\}$

$G = E \cup \mathcal{O}$

Thrms applied to $(\mathbb{R}^+, \#, *)$ over $(\mathbb{R}, +, \cdot)$

$x, y \in \mathbb{R}^+ = V$, $a \in \mathbb{R} = F$

$x \# y = x \cdot y$

$a * x = x^a$

$e_\cdot = 1$ $e_\# = e$

(01) $0 * x = x^0 = 1 = e_\cdot = e_\# = e$

(02) $a * e = a * 1 = 1^a = 1$

(03) $a * x = e$ ($\Rightarrow a = 0$ or $x = e$)

$x^a = 1$

$\Rightarrow x = 1$ or $a = 0$ # $(a > 0)$

$\Rightarrow x = e$

(04) $(-1) * x$

$= x^{-1}$

$= \frac{1}{x}$

$= \overline{x^\cdot}$

$= \overline{x^\#}$

$= \overline{x}$

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(*) From Here on we will write
$x, y \in V$

1.) $x \# y \equiv x + y$ (vector add$^{\underline{n}}$)

2) $a * x \equiv a \cdot x = ax$ (scalar multip$^{\underline{ion}}$)

3) $e_{\#} = e \equiv \underline{0}$ (zero vector)

4) $\overline{x^{\#}} = \bar{x} = -x$ (neg. vector)

Also; $0 = e_+$, $1 = e_\cdot$, $-a = \bar{a}^+$, $a^{-1} = \bar{a}^\cdot$
$\because a \in F$

5) $(V, \#, *)$ over $(F, +, \cdot)$ as
$(V, +, \cdot)$ over $(F, +, \cdot)$ or as
V over F or as
V if F is implied

With new notations: $(V, +, \cdot)$ over $(F, +, \cdot)$ is a vector space iff

(01) $(V, +)$ is an Abelian group

1.1) $V \neq \phi$

1.2) $+ : V \times V \to V$ is a func$^{\underline{n}}$ (Vector add$^{\underline{n}}$)

1.3) $\forall x, y, z \in V$; $x + (y + z) = (x + y) + z$

1.4) $\exists\ \underline{0} \in V$, $\forall x \in V$; $\underline{0} + x = x + \underline{0} = x$

1.5) $\forall x \in V$, $\exists\ -x \in V$; $(-x) + x = x + (-x) = e$

1.6) $\forall x, y \in V$; $x + y = y + x$

(02) $(F, +, \cdot)$ is a field

(03) $\cdot : F \times V \to V$ is a func$^{\underline{n}}$ (scalar multip$^{\underline{n}}$

(04) $\forall a, b \in F$; $\forall x \in V$; $(a + b) \cdot x = (a \cdot x) + (b \cdot x$

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(05) $\forall a \in F;\ \forall x, y \in V;\ a\cdot(x+y) = a\cdot x + a\cdot y$

(06) $\forall a, b \in F,\ \forall x \in V;\ (a\cdot b)\cdot x = a\cdot(b\cdot x)$

(07) $\forall x \in V;\ 1\cdot x = x$

<u>Definition:</u> Let V be a vec. space over F. & S be a non-empty subset of V.
Then S is a subspace of V iff S is a vec. space over F.

$\subseteq V$

5

With new notations: $(S, +, \cdot)$ over $(F, +, \cdot)$ is a vector space iff

G1) S is an Abl. group

✓ 1.1) $S \neq \emptyset$

✓ 1.2) $+ : S \times S \to S$ is a func^n (+ should be close) (S is a subset of V)

✓ 1.3) S is a subset of V

✓ 1.4) $0 \cdot \underline{x} = \underline{0}$ (Any vector multiplied by zero is zero vector) and ③

$\therefore \exists\, \underline{0} \in S,\ \forall x \in S;\ \underline{0} + x = x + \underline{0} = x$

1.5) $\forall x \in S,\ \exists\, -x \in S';\ (-x) + x = x + (-x) = \underline{0}$

$(-1)\cdot x = x$ and ③

✓ 1.6) $\forall x, y \in S;\ x + y = y + x$

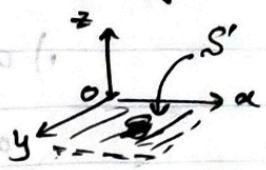
✓ (02) $(F, +, \cdot)$ is a field

✓ (03) $\cdot : F \times S \to S$ is a func^n (need $\cdot$ to be closed)

✓ (04) $\forall a, b \in F;\ \forall x \in S';\ (a+b)\cdot x = (a\cdot x) + (b\cdot x)$

✓ (05) ✓ (06) ✓ (07) $\forall x \in S;\ 1\cdot x = x$

Eg: $(\mathbb{R}^3, +, \cdot)$ over $(\mathbb{R}, +, \cdot)$ is a vector space. Is $S = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$ a subspace of $\mathbb{R}^3$?



(01) $S' \neq \phi$ ← not null

(02) $S \subseteq \mathbb{R}^3$ ← subset of $\mathbb{R}^3$

vector addition is closed → (03) $(x_1, y_1, 0) + (x_2, y_2, 0) = (x_1 + x_2, y_1 + y_2, 0) \in S'$

closed under scalar multiplic^{n} → (04) $a \cdot (x, y, 0) = (a \cdot x, a \cdot y, a \cdot 0) \in S'$

$a \in \mathbb{R}$, $(x, y, 0) \in S$

Eg: $(\mathbb{R}^{2\times 2}, +, \cdot)$ over $(\mathbb{R}, +, \cdot)$ is a vector space. Is $S = GL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, ad - bc \neq 0 ; a, b, c, d \in \mathbb{R} \right\}$ a subspace?

✓ (01) $S \neq \phi$

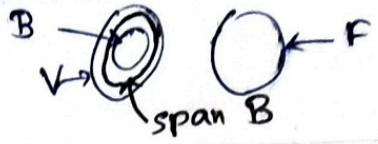
✓ (02) $S \subseteq \mathbb{R}^{2\times 2}$

✗ (03) $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \notin S'$

∴ not a subspace

✗ (04) $0 \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \notin S'$



<u>Definition</u>: Let $B = \{x_1, x_2, \dots, x_n\}$ be a non-empty finite subset of the vec. space V over F.

1) $a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n = \sum_{i=1}^{n} a_nx_n$

where $a_i \in F$ is a linear comb^{n} of B

2) Collec^{n} of all linear combinations of B is called the span of B.

$$\text{span } B = \{ \sum_{i=1}^{n} a_i x_i \mid x_i \in B,\ a_i \in F \}$$

$$B \subseteq (\text{span } B)$$

3) Iff (span $B = V$), we say that B spans V

4) B is linearly independant iff

$$\cancel{\sum_{i=1}^{n} a_i \cdot x_i = 0 \Rightarrow \forall i,\ a_i = 0}$$

$$\forall (a_1, a_2, \dots, a_n) \in F^n;\ \left(\sum_{i=1}^{n} a_i \cdot x_i = 0 \Rightarrow \forall i,\ a_i = 0 \right)$$

5) B is linearly dependant

$\Leftrightarrow$ B is not linearly independant

$\Leftrightarrow \exists (a_1, a_2, \dots, a_n) \in F^n$ s.t. $\left(\sum_{i=1}^{n} a_i \cdot x_i = 0 \text{ and } \exists i,\ a_i \neq 0 \right)$

Note

If $S = \emptyset$, then $\text{span}(\emptyset) = \{\bar{0}\}$

Eg: $(1,2) \in F^2$; $1(i) + 2(2i) = 5i \neq \underline{0}$

$(0,0) \in F^2$; $0i + 0(2i) = \underline{0} \Rightarrow a_1 = a_2 = 0$

$(-2,1) \in F^2$; $(-2)i + 1(2i) = \underline{0}$ and $a_1 = -2 \neq 0$

(and $a_2 = 1 \neq 0$)

Linearly Dependant

Eg: $B = \{(1,0), (0,1)\} \subseteq R^2$; $F = \mathbb{R}$

1) $a_1, a_2 \in \mathbb{R} = F$. Is B Linearly ind.?

$a_1 x_1 + a_2 x_2 = a_1(1,0) + a_2(0,1)$

$= (a_1, a_2) = \underline{0} = (0,0)$

$\Rightarrow a_1 = 0$ and $a_2 = 0$

$\Rightarrow a_1 = a_2 = 0$

$\therefore$ Lin. indep.

2) span $B = \mathbb{R}^2$?

span $B \ni a_1 x_1 + a_2 x_2 = (a_1, a_2) = \mathbb{R}^2$

$\Rightarrow$ span $B \subseteq \mathbb{R}^2$

$\mathbb{R}^2 \ni (a,b) = a(1,0) + b(0,1) \in$ span B

$\Rightarrow \mathbb{R}^2 \subseteq$ span B

$\Rightarrow \mathbb{R}^2 =$ span B

~~Definition : Let $B = \{x_1, x_2, \dots, x_n\}$ be a non empty finite sub set of a vector space V over F.~~

$$\cancel{\text{span } B = \left\{ \sum_{i=1}^{n} a_i x_i \mid x_i \in V,\ a_i \in F \right\}}$$

~~B spans V iff span B = V~~

	L.I.?	spans V?
$B = \{(1,0,0), (1,1,0)\}$	✓	×
$B = \{(1,0,0), (0,1,0), (0,0,1), (1,1,1)\}$	×	✓
$B = \{(1,0,0), (0,1,0), (0,0,1)\}$	✓	✓
$B = \{(1,0,0), (2,0,0)\}$	×	×

Definition : Let B is a non-empty subset of a vector space V over F, B is a basis of V iff it is L.I. and spans V (span $B = V$)

Note.

Basis is not unique.

But the no. of elements in a basis is same.

No. of elements of ~~basis~~ a basis is called the dimension of V (or $\dim V$)

Note: If three are given, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ also suggests L.I.

Eg: $V = \mathbb{R}^3$, $F = \mathbb{R}$

$B_1 = \{(1,0,0), (0,1,0), (0,0,1)\}$ This is a basis

$B_2 = \{(1,0,0), (1,1,0), (2,3,4)\}$

To check for L.I, do row op^{rns} for the matrix

$$B_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 3 & 4 \end{pmatrix} \xrightarrow[-2R_1+R_3 \to R_3]{-R_1+R_2 \to R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 4 \end{pmatrix}$$

$$\xrightarrow{-3R_2+R_3 \to R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \xrightarrow{R_3/4 \to R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

~~If L.I, a null matrix can~~

$\therefore$ this is linear independant & spans V

L.I can be also checked using the determinent.

$\det(B_2) = 4 \neq 0$ If determinant is zero $\Rightarrow$ linear dependant

Eg: $B_3 = \{(1,1,0), (2,1,1), (4,3,1)\}$

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 4 & 3 & 1 \end{pmatrix} \xrightarrow[-4R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \xrightarrow{-R_2+R_3 \to R_3}$$

This is linearly dependant $\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Finding the dependance;

$$\bar{R}_3 = \underline{0} = -\bar{R}_2 + \bar{R}_3$$
$$= -(-2R_1 + R_2) + (-4R_1 + R_3)$$
$$\underline{0} = -2R_1 - R_2 + R_3$$
$$R_3 = 2R_1 + R_2$$

$$(-2)R_1 + (-1)R_2 + (1)R_3 = \underline{0}$$

(Non of the coefficients are zero)

<u>Theorem</u>: Let V be a vector space over F with $\dim V = n$. Then;

1) ~~Any~~ Any subset of V with more than n elements is linearly ~~dependant~~ independant

2) Any subset of V with less than n elements will not span V.

<u>Proof</u>:

Let $C = \{y_1, y_2, \dots, y_m\} \subseteq V$

Let $B = \{x_1, x_2, \dots, x_n\}$ be a basis of V

~~Case 01: $m > n$~~

Case 01: $m > n$

$$\cancel{\underline{0} = b_1y_1 + b_2y_2 + \dots + b_my_m}$$

$$\underline{0} = b_1y_1 + b_2y_2 + \dots + b_my_m$$

$$= b_1(a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n)$$
$$+ b_2(a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n)$$
$$\vdots$$
$$+ b_m(a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n)$$

$$y_j = \sum_{i=1}^{m} a_{ji}x_i$$

$\rightarrow 2 \rightarrow$

$$\underline{0} = (b_1 a_{11} + b_2 a_{21} + \dots + b_m a_{m1}) x_1$$
$$+ (b_1 a_{12} + b_2 a_{22} + \dots + b_m a_{m2}) x_2$$
$$\vdots$$
$$+ (b_1 a_{1n} + b_2 a_{2n} + \dots + b_m a_{mn}) x_n$$

$$\Rightarrow b_1 a_{11} + b_2 a_{21} + \dots + b_m a_{m1} = 0 \quad \text{—— (1)}$$

$$b_1 a_{12} + b_2 a_{22} + \dots + b_m a_{m2} = 0 \quad \text{—— (2)}$$

$$\vdots$$

$$b_1 a_{1n} + b_2 a_{2n} + \dots + b_m a_{mn} = 0 \quad \text{—— (3)}$$

$$\begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix}_{n \times m} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\Rightarrow \forall j,\ b_j = 0$.

$\exists j,\ b_j \neq 0$, $\therefore$ C is linearly ~~in~~dependant

Case 02: $m < n$, ~~span C ≠ V~~

$V \ni x = b_1 y_1 + b_2 y_2 + \dots + b_m y_m$

$$A_1 x_1 + A_2 x_2 + \dots + A_n x_n = (\quad) x_1 + (\quad) x_2 + \dots + (\quad) x_n$$

the same as before

Then
$$\begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix}_{n \times m} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}_{m \times 1} = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix}_{n \times 1}$$

$m < n$

$\Rightarrow$ no sol^n. for b_j

$\therefore$ span C ≠ V

Note. row rank = no. of independant rows (non zero vectors)

rank =

column rank = no. of independant columns

~~Note. Sol^ns to AX = b~~

Note. Sol^ns to $A_{n\times m} X_{m\times 1} = b_{n\times 1}$

no. of unknowns

no. of eq^ns

(01) $\text{rank}(A \mid b) = \text{rank}\, A = r$

(i) $r = m \Rightarrow$ unique sol^n

(ii) $r < m \Rightarrow$ infinite sol^ns

(iii) $r > m$ is not possible

(02) $\text{rank}(A \mid b) > \text{rank}\, A \Rightarrow$ no sol^n

(03) $\text{rank}(A \mid b) < \text{rank}\, A$ is not possible.

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(01) $\frac{dy}{dx} - \frac{1}{2y} = 0$

$2y\,dy = dx$

$y^2 = x + c$

$y = \pm\sqrt{x+c}$; $c \in \mathbb{R}$

(a) $(V, \#, *)$, $(F, +, \cdot)$

$(V, \#)$ is Abelian?

$(\pm\sqrt{x_1+c}) + (\pm\sqrt{x_2+c}) \notin V$

$\therefore$ not abelian

$\therefore$ not a vector space.

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(b) V is not a v.s. $\therefore$ $\{\sqrt{x}, -\sqrt{x}\}$ is not a basis.

(c) $a_1\sqrt{x} + a_2(-\sqrt{x}) = 0$

$\sqrt{x}(a_1 - a_2) = 0$

$\Rightarrow$ $a_1 = a_2$ (not always zero)

$\therefore$ line. dep.

(d) V contains ~~ex~~ sol^{ns} of the form $\pm\sqrt{x+c}$.
It cannot be expressed as a linear comb^{n} of $\sqrt{x}$ & $-\sqrt{x}$. $\therefore$ false

(e) $c = 0$; $\pm\sqrt{x} \in V$

(02) See the lecture note

(b), (c) $\rightarrow$ Theorem 2 $(n = p^k)$

(d) $\rightarrow$ Group with only the identity element

(e) $\rightarrow$ V.S. with zero vector

(04) $W = \text{span}\{1, x\}$, $p(x) = 2 + 3x + 4x^2$

$\therefore$ any func^{n} in W is of the form

$q(x) = a + bx$ | $a, b \in \mathbb{R}$

best approximation $\rightarrow$ minimizes the distance

orthogonal proj^{n}

$e(x) = p(x) - q(x)$

$= (2-a) + (3-b)x + 4x^2$

$q(x)$ is the best approximation

$\therefore$ $(e(x), 1) = 0$ and $(e(x), x) = 0$

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$$(e(x), 1) = \int_{-1}^{1} ((2-a) + (3-b)x + 4x^2) \cdot 1 \, dx = 0$$

$$(2-a)[x]_{-1}^{1} + (3-b)\left[\frac{x^2}{2}\right]_{-1}^{1} + 4\left[\frac{x^3}{3}\right]_{-1}^{1} = 0$$

$$(2-a) \cdot 2 + (3-b) \cdot 0 + 4 \cdot \frac{2}{3} = 0$$

$$4 - 2a + \frac{8}{3} = 0$$

$$a = \frac{10}{3}$$

$$(e(x), x) = \int_{-1}^{1} ((2-a) + (3-b)x + 4x^2) x \, dx = 0$$

$$(2-a)\frac{x^2}{2}\Big|_{-1}^{1} + (3-b)\frac{x^3}{3}\Big|_{-1}^{1} + \cancel{4}\frac{x^4}{\cancel{4}}\Big|_{-1}^{1} = 0$$

$$0 + (3-b) \cdot \frac{2}{3} + 0 = 0$$

$$b = 3$$

$$\therefore q(x) = \frac{10}{3} + 3x$$

(05) (a) $\mathbb{Q}$ ← U v.s. over $\mathbb{Q}$ ← F field ?

$\mathbb{Q} \neq \phi$

$\mathbb{Q} \subseteq \mathbb{R}$ ← V

Let $\frac{p_1}{q_1}, \frac{p_2}{q_2} \in \mathbb{Q}$

$$\frac{p_1}{q_1} + \frac{p_2}{q_2} \in \mathbb{Q}$$

$$\frac{a}{b} \in \mathbb{Q}$$

$$\frac{a}{b} \cdot \frac{p}{q} \in \mathbb{Q}$$

$\therefore$ $\mathbb{Q}$ is a subspace.

(b) $\mathbb{Z}$ of $\mathbb{Q}$ over $\mathbb{Z}$

U V F

$\mathbb{Z} \neq \phi$

$\mathbb{Z} \subseteq \mathbb{Q}$

$z_1, z_2 \in \mathbb{Z}$

$z_1 + z_2 \in \mathbb{Z}$

~~$\frac{p}{q} \cdot z \notin \mathbb{Z}$ always also~~ $\mathbb{Z}$ is not a field.

$\therefore$ not a subspace

(c) $\mathbb{R}$ of $\mathbb{C}$ over $\mathbb{R}$

$\mathbb{R} \neq \phi$

$\mathbb{R} \subseteq \mathbb{C}$

$a, b \in \mathbb{R}$; $a+b \subseteq \mathbb{R}$

$a.b \in \mathbb{R}$

$\therefore$ a subspace

(d) $\frac{p}{q} . a \in \mathbb{R}$; $a \in \mathbb{R}$, $\frac{p}{q} \in \mathbb{Q}$

$\therefore$ subspace

(e) $\mathbb{C} \notin \mathbb{R}$

(f) $\frac{p}{q} . z \notin \mathbb{Z}$

(g) $a . z \notin \mathbb{Z}$

06 (a) $m^2 - m - 6 = 0$

$m = 3 \quad , \quad m = -2$

$$y = c_1 e^{-2x} + c_2 e^{3x}$$

$$a_1(e^{-2x} + e^{3x}) + a_2(-2e^{-2x} + 3e^{3x}) = 0$$

$$(a_1 - 2a_2)e^{-2x} + (a_1 + 3a_2)e^{3x} = 0$$

$\Rightarrow a_1 - 2a_2 = 0 \quad , \quad a_1 + 3a_2 = 0$

$a_1 = 2a_2 \qquad 5a_2 = 0$

$a_2 = 0$

$a_1 = 0$

When $c_1 = c_2 = 1 \quad , \quad y = e^{-2x} + e^{3x}$

$c_1 = -2, \; c_2 = 3 \quad , \quad y = -2e^{-2x} + 3e^{3x}$

(b) Obvious

~~(c) $a_1(x^2 + x + 1) + a_2(x - 1) = 0$~~

~~$a_1 x^2 + (a_1 + a_2)x + a_1 - a_2 = 0$~~

(c) $R_2[x] \quad , \quad \dim[V] = 3$

no. of elements $= 2$

$\therefore$ does not span $R_2[x]$

(d) $a \cdot \pi = 0 \Rightarrow a = 0$

$\{\pi\}$ spans $\mathbb{R}$?

~~$a\pi \in \mathbb{R} \Rightarrow \text{span } B \in \mathbb{R}$~~

Let $r \in \mathbb{R} \qquad \pi r \in \mathbb{R}$

~~$\mathbb{R} \ni a\pi$~~ $\qquad \therefore \{\pi\}$ spans $\mathbb{R}$

(e) $y'' - y' - 6y = 1 \quad , \quad y = y_c(x) + y_p(x)$

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(12) (a) $(\mathbb{C}^{n\times n}, \#, *)$ over $(\mathbb{C}, +, \cdot)$

matrix add^n — scalar multipl^n of a matrix

$(\mathbb{C}^{n\times n}, \#)$?

$$\begin{pmatrix} x & y \\ s & t \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} x+a & y+b \\ s+c & t+d \end{pmatrix}.$$

$\because x, y, s, t, a, b, c, d \in \mathbb{C}$

This is in $\mathbb{C}^{n\times n}$

Matrix addition is associative

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = e$$

$$\bar{a} = \begin{pmatrix} -x & -y \\ -s & -t \end{pmatrix}$$

$\therefore (\mathbb{C}^{n\times n}, \#$ is Abelian$)$ ✓

~~$\therefore$ V is a vector space over F~~

$(\mathbb{C}, +, \cdot)$ is a field (can be proved) ✓

~~(2) when # is matrix multiplication~~

$* : \mathbb{C} * \mathbb{C}^{n\times n} \longrightarrow \mathbb{C}^{n\times n}$ ✓

$$(xi+yj)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{C}^{n\times n}$$

$\because a, b, c, d \in \mathbb{C}$

$\forall a, b \in \mathbb{C}, \forall x \in \mathbb{C}^{n\times n}$;

$(a+b) * x$ ~~= (a+b)~~

$\underbrace{(a*x)} \# (b*x)$

matrix add^n

complex number × matrix = complex matrix

$\therefore (a+b) * x = (a*x) \# (b*x)$

$\forall a \in \mathbb{C}, \forall x, y \in \mathbb{C}^{n \times n}$

$$(a * x) \# (a * y)$$
$$= (a * x) + (a * y)$$
$$= a * (x + y)$$
$$= a * (x \# y)$$

$\therefore$ V is a vector space over F

(e) When # is the matrix multiplic$^{\underline{n}}$, inverse does not exist for all.

(13) ~~(c)~~

Any subset of V	L.I.	spans V
no. of elements = m,		
$m < n$	✓	✗
$m > n$	✗	✓
$m = n$	✓	✓

(13) See theorem 8, 9, 10, 11, 12, 13

(15) (b) $cx = 0$

$c = 0$ always iff $x \neq 0$

Property	Preserved in a subset	superset
Linear independance	✓	?
Linear dependance	?	✓
Spanning the V.S.	?	✓
Not spanning the V.S.	✓	?

<u>Theorem</u>

Subset of a L.I. set is L.I.

Proof: $B = \{x_1, x_2, \dots, x_{n-1}, x_n\}$ be L.I. in a v.s. V.

$$B' = \{x_1, x_2, \dots, x_{n-1}\} \subseteq B$$

Let span $B' \ni a_1x_1 + a_2x_2 + \dots + a_{n-1}x_{n-1} = \underline{0}$

span $B \ni a_1x_1 + a_2x_2 + \dots + a_{n-1}x_{n-1} + \underset{a_n}{0}\, x_n = \underline{0}$

$\Rightarrow a_1 = a_2 = \dots = a_{n-1} = a_n = 0$

$\Rightarrow B'$ is linearly independant.

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Theorem : A superset of a L.D. set is L.D.

Proof : $B = \{x_1, x_2, \dots, x_n\}$ be L.D. in v.s. V

$$B' = \{x_1, x_2, \dots, x_n, x_{n+1}\} \supseteq B$$

span $B' \ni a_1x_1 + a_2x_2 + \dots + a_nx_n + 0x_{n+1} = \underline{0}$

span $B \ni a_1x_1 + a_2x_2 + \dots + a_nx_n = \underline{0}$

$\Rightarrow \exists\, a_i \neq 0, \ 1 \leq i \leq n$

$\Rightarrow \exists\, a_i \neq 0, \ 1 \leq i \leq n+1$

$\Rightarrow B'$ is L.D.

Theorem : A superset of a spanning set is spanning.

Proof : $B = \{x_1, x_2, \dots, x_n\}$ be spanning V

$$B' = \{x_1, x_2, \dots, x_n, x_{n+1}\} \supseteq B$$

B is spanning V

$\Leftrightarrow \forall x \in V; \exists\, a_i \in F; x = a_1x_1 + a_2x_2 + \dots + a_nx_n \in$ span$B$

$x = a_1x_1 + a_2x_2 + \dots + a_nx_n + \underbrace{0}_{\in F}x_{n+1} \in$ spanB'

$\Rightarrow V = \text{span}\, B'$

$\Rightarrow B'$ spans V.

Theorem : Subset of a non-spanning set is non-spanning

Proof : $B = \{x_1, x_2, \dots, x_n\}$ be non-spanning

$B' = \{x_1, x_2, \dots, x_{n-1}\} \subsetneq B$

Assume B' is spanning V.

$\forall x \in V$; $\exists a_i \in F$ s.t. $x = a_1x_1 + a_2x_2 + \dots + a_{n-1}x_{n-1} \in \text{span}B'$

$x = a_1x_1 + a_2x_2 + \dots + a_{n-1}x_{n-1} + 0\, x_n \in \text{span}\, B$

$\Rightarrow$ span $B = V$; i.e. B is spanning

#

$\therefore$ B' is not spanning V.

Theorem : We can extend a L.I. subset to make a basis.

Proof : Let $B \subseteq V$ be L.I.

If span $B = V$, then B is a basis.

Otherwise span $B \neq V$ and $\exists x \in V$ s.t.

$x \notin$ span B.

claim: $B' = B \cup \{x\}$ is also L.I.

with $B = \{x_1, x_2, \dots, x_n\}$

Consider span $B' \ni a_1x_1 + a_2x_2 + \dots + a_nx_n + ax = \underline{0}$

$a_i : a \in F$

Assume $a \neq 0$

$x = -a^{-1}a_1x_1 - a^{-1}a_2x_2 \dots - a^{-1}a_nx_n$

$= b_1x_1 + b_2x_2 + \dots + b_nx_n \in$ span B , $b_i \in F$

#

$\therefore a = 0$

span $B' \ni a_1x_1 + a_2x_2 + \dots + a_nx_n + 0x = \underline{0}$

$\Rightarrow a_1x_1 + a_2x_2 + \dots + a_nx_n = \underline{0}$

$\Rightarrow a_1 = a_2 = \dots = a_n = a = 0$

span $B \ni$

$\rightarrow \rightarrow$

$B' = B \cup \{x\}$, $x \in \text{span}\, B$ *
is linearly independant.

Keep doing this to get a basis.

<u>Theorem</u> We can reduce a spanning superset to a basis

[Q]

Proof: Let $B = \{x_1, x_2 \dots, x_n\}$ be spanning V.

If B is L.I., then B is a basis.
Otherwise B is L.D i.e.
with $a_1x_1 + a_2x_2 + \dots + a_nx_n = \underline{0}$; $\exists\, a_i \neq 0$
(say $a_n \neq 0$)

$$x_n = -a_n^{-1}a_1x_1 - a_n^{-1}a_2x_2 \dots - a_n^{-1}a_{n-1}x_{n-1}$$
$$= b_1x_1 + b_2x_2 + \dots + b_{n-1}x_{n-1}$$
$$; \; b_j \in F$$

Since B is spanning V,
$\forall\, x \in V$, $\exists\, c_i \in F$ s.t.

[Q]

$$x = c_1x_1 + c_2x_2 + \dots + c_{n-1}x_{n-1} + c_nx_n \in \text{span}\, B$$
$$= c_1x_1 + c_2x_2 + \dots + c_{n-1}x_{n-1} + c_n(b_1x_1 + b_2x_2 + \dots + b_{n-1}x_{n-1})$$
$$= (c_1 + c_nb_1)x_1 + (c_2 + c_nx_2) + \dots + (c_{n-1} + c_nb_{n-1})x_{n-1}$$
$$= d_1x_1 + d_2x_2 + \dots + d_{n-1}x_{n-1} \in \text{span}\, B'$$

$$B' = \{x_1, x_2, \dots, x_{n-1}\} \not\subset B$$

$\Rightarrow$ span $B' = V$, B' is spanning V

Keep doing this to get a Basis.

* B is a basis of V iff
  1) $B \neq \phi$
  2) $B \subseteq V$
  3) B is L.I.
  4) B spans $V \equiv \text{span}\, B = V$

[Q] Is $\{\underline{0}\}$ a v.s? ← This is the only one element vector space.

~~Let~~ ~~S~~ ~~be~~ Let $S = \{\underline{0}\}$ be a subset of a v.s. V.

$S \neq \phi$, $S \subseteq V$

$\underline{0} + \underline{0} = \underline{0} \in S$

~~$a \cdot \underline{0}$~~ $a \cdot \underline{0} = \underline{0} \in S$

$\therefore$ S is a subspace of $V \Rightarrow S$ is a v.s.

[Q] Is $\{\underline{0}\}$ a basis?

$a \cdot \underline{0} = \underline{0} \not\Rightarrow a = 0$

$\therefore$ L.D. $\therefore$ Not a basis.

* Any set containing $\underline{0}$ is linearly dependant.

* $\text{span}\, B = \left\{ \sum_{i=1}^{n} a_i x_i \;\middle|\; a_i \in F ;\ x_i \in B \right\} = \underline{0}$

Any empty sum is zero;

$$\sum_{i=1}^{n} i = \frac{n}{2}(n+1) \qquad \sum_{i=1}^{1} i = 1$$

$$\sum_{i=0}^{0} i = 0$$

* $\dim\{\underline{0}\} = 0$.
Points are zero dimensional

* Empty product is one

$$\prod_{i=1}^{0} i = 1 \qquad (0! = 1)$$

* Eg. $\mathbb{R}(x)$ = set of polynomials in x with coefficients in $\mathbb{R}$. ~~Is above a v.s. over $\mathbb{R}$~~.
$V = \mathbb{R}(x)$ is a vector space over $\mathbb{R}$.

The above can be a basis with infinite no. of elements.

$$B = \{1, x, x^2, x^3, \dots\}$$

The potential dangers are the span can be L.D. (Since the above is $\sum_{x \in \mathbb{R}} x$ we cannot calculate it)

$\therefore$ do the span for finite subsets of B

Take the spans of $B_1 = \{1, x\}$, $B_2 = \{x, x^2\}$, ...

For linearly independance, same can be done

L.I. if every finite subset is L.I.

Although the set is infinite, every vector should be ~~describ~~ able to describe by a finite ~~num of vectors~~ sum of basis vectors

$\therefore$ B can be a basis. Such a basis is called a Hamel basis.

Def A Hamel basis of a vector space is similar to a finite basis, where we only take finite sums (finite linear combinations)

$$\text{span}B = \{ \sum_{\text{finite}}' a_i x_i \mid a_i \in F ; x_i \in \underbrace{B'}_{\text{finite}} \subseteq B \}$$

$$\sum_{\text{finite}}' a_i x_i = \underline{0} \Rightarrow \forall i, a_i = 0$$

* Finite bases are Hamel bases
* Every vector space has Hamel basis.

Theorem : Every v.s. has a Hamel basis.

Hamel basis:
A basis with possibly infinite no. of elements where linear comb^{ns} (for L.I. & spanning) are taken over finite subsets.

Eg: $\mathcal{C}$ = set of real valued continuous func^{ns} over $\mathbb{R}$

$f, g \in \mathcal{C}$, $a \in F = \mathbb{R}$

1) $f + g \in \mathcal{C}$; $(f+g)(x) = f(x) + g(x)$

2) $af \in \mathcal{C}$; $(af)(x) = af(x)$

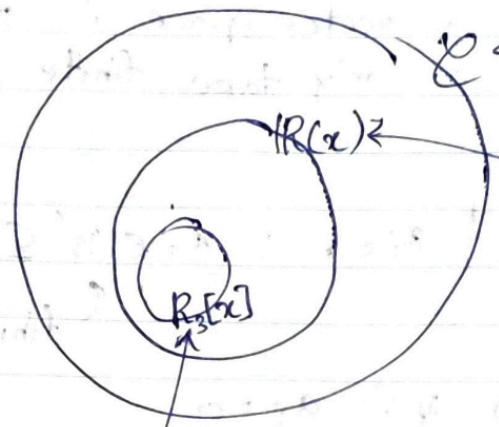
3) $-f \in \mathcal{C}$; $(-f)(x) = -f(x)$

4) $0 \in \mathcal{C}$; $\forall x$, $0(x) = 0$

$\rightarrow$

This is a vector space..

~~∴ it has a Hamel basis~~



uncountable Hamel basis

Countable infinite Hamel basis
Eg: $\{1, x, x^2, x^3, \ldots\}$

Finite basis
Eg: $\{1, x, x^2, x^3\}$

<u>Defin^{n}</u>: Set A is countable iff there is a one to one function. ~~$f: A \to \mathbb{R}^+$~~

$f: A \to \mathbb{Z}^+ = \{1, 2, 3, 4, \ldots\}$

Eg: $\mathbb{Z}^+$

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

$\mathbb{Q}^+ =$

$$\begin{array}{cccc} \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} \\ \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} \\ \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

<u>Note</u> If a subset of a set is uncountable, the larger set is also uncountable.

Hamel base B of V over F:
$V = \text{span}(\text{span}\, B)$

eg: $\mathbb{R}$

$(0, 1) \subseteq \mathbb{R}$

$0.a_{11}a_{12}a_{13}\ldots$

$0.a_{21}a_{22}a_{23}\ldots$

$0.a_{31}a_{32}a_{33}\ldots$

$\vdots$

$0.b_1 b_2 b_3 \ldots$

b_1 is ~~not equal~~ a_{11}, $b_2 \neq a_{22}$, $b_3 \neq a_{33}$

no place in the above list

$\therefore$ $\mathbb{R}$ is ~~not~~ also not countable.

	Size	Sum	Example
Finite	Finite	Finite	$\{i, j, k\}$ in $\mathbb{R}^3$
Hamel	• Finite • Countably infinite • Uncountable	Finite	$\{1, x, x^2, \ldots\}$ in $\mathbb{R}[x]$
Schauder	Countably infinite	Infinite	$\{e^{ikx} \mid k \in \mathbb{Z}\}$ for Fourier series

eg: V = set of sol^{ns} to $y'' - 5y' + 6y = 0$ over $\mathbb{R}$

$0'' - 5\,0' + 6\,0 = 0$; $0 \in V$

$y_1, y_2 \in V$; $(y_1 + y_2)'' - 5(y_1 + y_2)' + 6(y_1 + y_2)$

$= y_1'' + y_2'' - 5y_1' - 5y_2' + 6y_1 + 6y_2$

$= \underbrace{y_1'' - 5y_1' + 6y_1}_{=0} + \underbrace{y_2'' - 5y_2' + 6y_2}_{=0}$

sol^{ns} to the func^n

$= 0$

$k \in \mathbb{R}$; $(ky_1)'' - 5(ky_1)' + 6(ky_1)$

$= k(y_1'' - 5y_1' + 6y_1)$

$= k \times 0 = 0$

$\Rightarrow ky_1 \in V$

$$y = e^{\alpha x} \Rightarrow \alpha^2 e^{\alpha x} - 5\alpha e^{\alpha x} + 6 e^{\alpha x} = 0$$

$$\underbrace{e^{\alpha x}}_{\neq 0} (\alpha^2 - 5\alpha + 6) = 0$$

$$\therefore \alpha = 2 \text{ or } \alpha = 3$$

$$\therefore y = a e^{2x} + b e^{3x} \; ; \; a, b \in \mathbb{R}$$

$$B = \{ e^{2x}, e^{3x} \} \quad V = \text{span} B$$

$$f(x) = e^{2x} \; , \; g(x) = e^{3x}$$

$$\cancel{ay_1 + by_2} \; af + bg = 0$$

$$\Rightarrow a = b = 0$$

$\therefore$ linearly independant.

$$af(x) + bg(x) = 0$$

$$\Leftrightarrow \forall x \, (ae^{2x} + be^{3x}) = 0$$

$$\Rightarrow \forall x \, (2ae^{2x} + 3be^{3x} = 0)$$

derivative

$$\forall x, \begin{pmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Wronskien

$$\begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = e^{5x} \neq 0$$

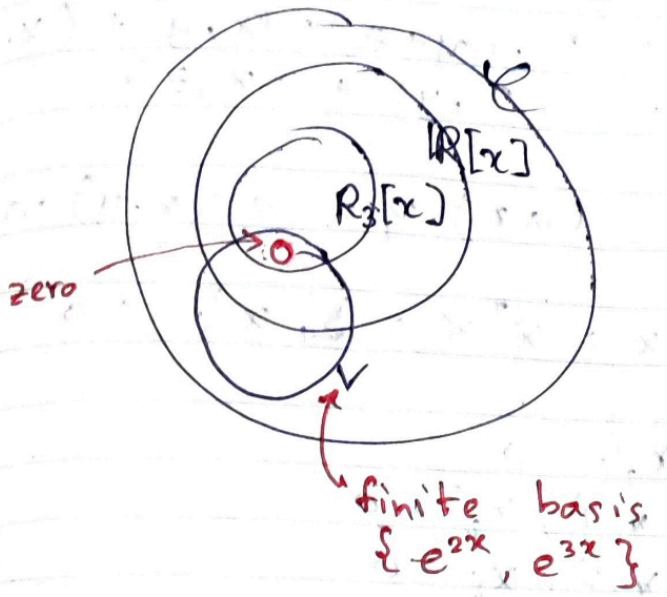
$$\Rightarrow a = b = 0$$

Hamel basis : finite / infinite , countable / uncountable

Schauder basis : infinite , countable

infinite linear combns

finite linear combns



Def Let V be a vector space over the field F which has an order $\geq$. An inner product $\langle \cdot , \cdot \rangle$ satisfies.

1) $\langle \cdot , \cdot \rangle : V \times V \longrightarrow F$ is a function
2) $\langle x, y \rangle = \langle y, x \rangle, \ \forall x, y \in V$
3) $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle ; \ \forall x, y, z \in V$
4) $\langle ax, y \rangle = a \langle x, y \rangle \ ; \ x, y \in V \ ; \ a \in F$
5) $\langle x, x \rangle \geq 0$ and
   $\langle x, x \rangle = 0 \iff x = \underline{0}$

Norm of x

$\|x\| = \sqrt{\langle x, x \rangle}$

Eg: $V = \mathbb{R}^n, \ F = \mathbb{R}$

$x = (x_1, x_2, \dots x_n), \ y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$

$\langle x, y \rangle = x \cdot y = x^T y = \sum_{i=1}^{n} x_i y_i$

1) $\langle \cdot , \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ a func^n

2) $\langle x, y \rangle = x \cdot y = \langle y, x \rangle$

3) $\langle x, y+z \rangle = x \cdot (y+z) = (x \cdot y) + (x \cdot z)$
$= \langle x, y \rangle + \langle x, z \rangle$

4) $\langle ax, y \rangle = (ax) \cdot y = a \cdot (x \cdot y) = a \langle x, y \rangle$

5) $\langle x, x \rangle = x \cdot x = x^T x = \sum_1^n x_i^2 \geq 0$

$\langle x, x \rangle = \sum_1^n x_i^2 = 0$
$\Rightarrow \forall i, \ x_i^2 = 0$
$\Rightarrow \forall i, \ x_i = 0$
$\Rightarrow x = \underline{0}$

<u>Def</u> Let V be a vector space over the field $F = \mathbb{C}$.
An inner product $\langle \cdot, \cdot \rangle$ satisfies,

1) $\langle \cdot, \cdot \rangle : V \times V \to F$ is a func^n
2) $\langle x, y \rangle = \overline{\langle y, x \rangle}$; $\forall x, y \in V$
3) $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$; $\forall x, y, z \in V$
4) $\langle ax, y \rangle = \bar{a} \langle x, y \rangle$ $\forall x, y \in V$, $a \in F$
5) $\langle x, x \rangle \in \mathbb{R}$ and $\langle x, x \rangle \geq 0$
$\langle x, x \rangle = 0 \Leftrightarrow x = \underline{0}$

$\mathbb{C}$
$\mathbb{R}$
$\mathbb{R} \subseteq \mathbb{C}$

Eg: $V = \mathbb{C}^n$, $F = \mathbb{C}$

for complex, conjugate is taken

Note
$\bar{x}^T$
$= \overline{x^T}$
$= x^H$

1) $\langle x, y \rangle = \bar{x}^T y = \sum_1^n \bar{x}_i y_i$

$\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ is a func^n

2) $\langle x, y \rangle = \sum_1^n \bar{x}_i y_i = \overline{\sum_1^n \bar{y}_i x_i} = \overline{\langle y, x \rangle}$

$\langle x, y \rangle = x^T y = \sum_i x_i y_i$

3) $\langle x, y+z \rangle = \sum_i \bar{x}_i \langle y_i + z_i \rangle$

$= \sum_i \bar{x}_i y_i + \sum_i \bar{x}_i z_i$

$= \langle x, y \rangle + \langle x, z \rangle$

4) $\langle ax, y \rangle = \sum_i \bar{a} \bar{x}_i y_i = \bar{a} \sum_i \bar{x}_i y_i = \bar{a} \langle x, y \rangle$

5) $\langle x, x \rangle = \sum_i \bar{x}_i x_i = \sum_i (x_i)^2 \geq 0$

and $\langle x, x \rangle = \sum_i |x_i|^2 = 0 \Leftrightarrow \forall i ; |x_i|^2 = 0$

$\Leftrightarrow \forall i ; |x_i| = 0$

$\Leftrightarrow \forall i ; x_i = 0$

$\Leftrightarrow x = \underline{0}$

<u>Theorem</u> : $\langle \underline{0}, x \rangle = \langle 0x, x \rangle = \bar{0} \langle x, x \rangle = 0 \langle x, x \rangle = 0$

$\overline{\langle x, \underline{0} \rangle} = 0 \Rightarrow \langle x, \underline{0} \rangle = \bar{0} = 0$

<u>Theorem</u> : $\langle x, ay \rangle = \overline{\langle ay, x \rangle} = \bar{\bar{a}} \overline{\langle y, x \rangle}$

$\cancel{= \bar{\bar{a}} \langle \bar{y}, x \rangle}$

$= a \langle x, y \rangle$

<u>Theorem</u> : $\langle x+y, z \rangle = \overline{\langle z, x+y \rangle}$

$= \overline{\langle z, x \rangle + \langle z, y \rangle}$

$= \overline{\langle z, x \rangle} + \overline{\langle z, y \rangle}$

$= \langle x, z \rangle + \langle y, z \rangle$

$$|\sum_i x_i y_i| \leq \sqrt{\sum_i x_i^2}\sqrt{\sum_i y_i^2}$$

<u>Th^{rm}</u> Cauchy - Schwarz inequality.

V is a v.s. over $F = \mathbb{C}$; $u, v \in V$

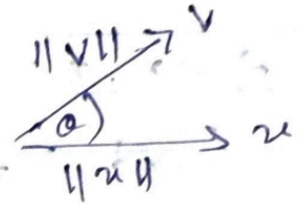
$$|\langle u, v\rangle| \leq \|u\| \, \|v\| \; ; \because \|u\| = \sqrt{\langle u, u\rangle}$$

$\cancel{|\sum_i x_i y_i| \leq \sqrt{\sum_i x_i^2}\sqrt{\ }}$ Induced norm

Eg: $V = \mathbb{R}^2$, $\cancel{x, y \in}$ $u, v \in \mathbb{R}^2$

$$|u \cdot v| = |\|u\| \, \|v\| \cos\theta|$$

$$|\langle u, v\rangle| \leq \|u\| \, \|v\|$$



<u>Proof</u>: $\|au + bv\|^2$; $a, b \in \mathbb{C}$

$$= \langle au + bv, au + bv\rangle$$

$$= \langle au + bv, au\rangle + \langle au + bv, bv\rangle \quad \text{axiom}$$

$$= \langle au, au\rangle + \langle bv, au\rangle + \langle au, bv\rangle + \langle bv, bv\rangle \quad (\text{a th}^{rm})$$

$$= \bar{a}a\langle u, u\rangle + \bar{b}a\langle v, u\rangle + \bar{a}b\langle u, v\rangle + \bar{b}b\langle v, v\rangle$$

$$= |a^2| \, \|u\|^2 + \underbrace{\bar{b}a\langle v, u\rangle}_{= \overline{\bar{a}b\langle u, v\rangle}} + \bar{a}b\langle u, v\rangle + |b^2| \, \|v\|^2$$

Let $a = \|v\|^2$, $b = -\langle v, u\rangle \in \mathbb{C} = F$

$$0 \leq \|au + bv\|^2$$

$$= \|v\|^4 \|u\|^2 - \overline{\langle v, u\rangle} \, \|v\|^2 \langle v, u\rangle - \|v\|^2 \langle v, u\rangle\langle u, v\rangle + |-\langle v, u\rangle|^2 \|v\|^2$$

$$= \|v\|^4 \|u\|^2 - \|v\|^2 |\langle v, u\rangle|^2 - \cancel{\|v\|^2 |\langle v, u\rangle|^2} + \cancel{|\langle v, u\rangle|^2 \|v\|^2}$$

$$= \|v\|^2 (\|v\|^2 \|u\|^2 - |\langle v, u\rangle|^2) \geq 0$$

$v \neq \underline{0} \Rightarrow \|v\| > 0 \Rightarrow \|v\|^2 \|u\|^2 - |\langle v, u\rangle|^2 \geq 0$

$$\Rightarrow |\langle v, u\rangle|^2 \leq \|v\|^2 \|u\|^2$$

$$\Rightarrow |\langle v, u\rangle| \leq \|v\| \|u\|$$

All these are ≥ 0

$$v = \underline{0} \Rightarrow |\langle v, u\rangle| = |\langle \underline{0}, u\rangle|$$
$$= |0|$$
$$= 0$$
$$\leq 0 = 0\|u\| = \|v\| \|u\|$$

Eg: $V = \mathcal{C}[a, b]$, $F = \mathbb{R}$

$$\langle f, g\rangle = \int_a^b f(x) g(x) dx$$

Cauchy-Schwarz inequality;

$$\left|\int_a^b f(x) g(x) dx\right| \leq \sqrt{\int_a^b (f(x))^2 dx} \sqrt{\int_a^b (g(x))^2 dx}$$

Generalzn:

$$\left|\int_a^b f(x) g(x) dx\right| \leq \left(\int_a^b (f(x))^p dx\right)^{1/p} \left(\int_a^b (f(x))^q dx\right)^{1/q}$$

$$\because \frac{1}{p} + \frac{1}{q} = 1$$

Eg: $V = \mathbb{R}^{m \times n}$, $V = \mathbb{R}$

$$\langle A, B\rangle = Tr(A^H B)$$

Note

Trace of $A_{n \times n}$ is the sum of diagonal elements of A $\left(\sum_{i=1}^{n} a_{ii}\right)$

For example,
$m = 3$, $n = 2$

$$\langle A, B\rangle = Tr(A^H B)$$

$$= Tr\begin{pmatrix} \bar{a}_{11} & \bar{a}_{21} & \bar{a}_{31} \\ \bar{a}_{12} & \bar{a}_{22} & \bar{a}_{32} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$$

$\rightarrow \rightarrow$

Note $Tr(AB) = Tr(BA)$

$$\langle A, B \rangle = \bar{a}_{11} b_{11} + \bar{a}_{21} b_{21} + \bar{a}_{31} b_{31} + \bar{a}_{12} b_{12} + \bar{a}_{22} b_{22} + \bar{a}_{32} b_{32}$$

$$= \sum_{i,j}' \bar{a}_{ij} b_{ij}$$

$$= \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{12} \\ b_{23} \\ \vdots \end{pmatrix}$$

Cauchy - Schmarz ;

$$\left| \sum_{i,j}' \bar{a}_{ij} b_{ij} \right| \leq \sqrt{\sum_{i,j}' |a_{ij}|^2} \sqrt{\sum_{i,j}' |b_{ij}|^2}$$

$$\langle A, B \rangle = Tr(B A^H)$$

$$= Tr \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} \begin{pmatrix} \bar{a}_{11} & \bar{a}_{21} & \bar{a}_{31} \\ \bar{a}_{12} & \bar{a}_{22} & \bar{a}_{32} \end{pmatrix}$$

$$= \bar{a}_{11} b_{11} + \bar{a}_{12} b_{12} + \bar{a}_{21} b_{21} + \bar{a}_{22} b_{22} + \bar{a}_{31} b_{31} + \bar{a}_{32} b_{32}$$

(The same answer)

(Prove the above is an inner product)

Def^n: 1) x, y are orthogonal iff $\langle x, y \rangle = 0$

2) x, y are orthonormal iff
$\langle x, y \rangle = 0$ for $x \neq y$ and $\|x\| = \|y\| = 1$

3) a set B is orthogonal iff $\langle x, y \rangle = 0 \ \forall x \neq y$

4) set B is orthonormal iff
$\langle x, y \rangle = 0 \ \forall \ x \neq y$ & $\|x\| \ \forall \ x, y \in B$

Th^{em}: $\underline{0} \notin B$ and B is orthogonal then B is L.I.

Note $\left\| \frac{x}{\|x\|} \right\| = \frac{1}{\|x\|} \|x\| = 1, \ x \neq \underline{0}$

($x \neq \underline{0}$ & orthogonal means orthonormal ?)

Proof:

$$B = \{ x_1, x_2, x_3, \dots, x_n \}$$

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = \underline{0}$$

$$\langle x_1, \underline{0} \rangle = \langle x_1, a_1 x_1 + a_2 x_2 + \dots + a_n x_n \rangle$$

$$0 = a_1 \langle x_1, x_1 \rangle + a_2 \cancelto{0}{\langle x_1, x_2 \rangle} + \dots + a_n \cancelto{0}{\langle x_1, x_n \rangle}$$

$$0 = a_1 \|x_1\|^2$$

$$\Rightarrow a_1 = 0 \quad ; \quad \because x_1 \neq \underline{0}$$

Similarly $a_2 = a_3 = \dots = a_n = 0$

~~$\therefore a_i = 0 \ \forall i$~~

$\Rightarrow$ B is linearly independant.

Note B is orthonormal $\Rightarrow$ B is L.I.

Reverse is not true.

Eg: $B = \{ \underline{i}, \underline{i} + \underline{j} \}$

orthogonal vectors $\xrightarrow[\text{by the length}]{\text{dividing}}$ orthonormal vectors.

Th$^{\underline{rm}}$: If $B = \{x_1, x_2, \dots, x_n\}$ is an orthonormal basis of V and if $u \in V$, then

$$u = a_1 x_1 + a_2 x_2 + \dots + a_n x_n$$

$$\langle x_1, u \rangle = a_1 \langle x_1, x_1 \rangle + a_2 \cancel{\langle x_1, x_2 \rangle}^{\,0} + \dots + a_n \cancel{\langle x_1, x_n \rangle}^{\,0}$$

$$= a_1 \underbrace{\|x_1\|^2}_{=1 \text{ (orthonormal)}}$$

$$= a_1$$

$$u = \sum_{i=1}^{n} \langle x_i, u \rangle x_i$$

If vectors are not already orthogonal

if we start with a L.D. set, we will get a $\underline{0}$ from the Gram-Schmidt orth$^{\underline{glzn}}$

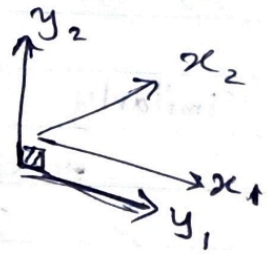
Gram-Schmidt Orthogonalization

Let $B = \{x_1, x_2, \dots, x_n\}$ be L.I. basis

To do: convert $B \Rightarrow C = \{\hat{y}_1, \hat{y}_2, \dots, \hat{y}_n\}$ (orthonormal)

let $y_1 = x_1$, $\hat{y}_1 = \dfrac{y_1}{\|y_1\|}$

parallel vector to x_1



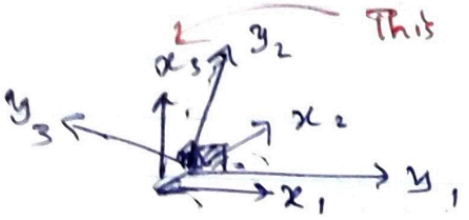
$$x_2 = a_1 y_1 + y_2$$

$$\langle y_1, x_2 \rangle = \langle y_1, a_1 y_1 + y_2 \rangle$$
$$= a_1 \langle y_1, y_1 \rangle + \cancel{\langle y_1, y_2 \rangle}^{\,0}$$
$$= a_1 \|y_1\|$$

$$y_2 = x_2 - a_1 y_1$$
$$= x_2 - \frac{\langle y_1, x_2 \rangle}{\|y_1\|^2} y_1$$

$$y_2 = x_2 - \left\langle \frac{y_1}{\|y_1\|}, x_2 \right\rangle \frac{y_1}{\|y_1\|}$$

$$= x_2 - \langle \hat{y}_1, x_2 \rangle \hat{y}_1$$



This can't be on the same plane of (If not it would be L.I.)

$$x_3 = b_1 y_1 + b_2 y_2 + y_3$$

$$\langle y_1, x_3 \rangle = b_1 \langle y_1, y_1 \rangle + b_2 \cancelto{0}{\langle y_1, y_2 \rangle} + \cancelto{0}{\langle y_1, y_3 \rangle}$$

$$= b_1 \|y_1\|^2$$

$$b_1 = \frac{\langle y_1, x_3 \rangle}{\|y_1\|^2}, \quad b_2 = \frac{\langle y_2, x_3 \rangle}{\|y_2\|^2}$$

$$y_3 = x_3 - \langle \hat{y}_1, x_3 \rangle \hat{y}_1 - \langle \hat{y}_2, x_3 \rangle \hat{y}_2$$

Similarly $\boxed{y_k = x_k - \sum_{j=1}^{k-1} \langle \hat{y}_j, x_k \rangle \hat{y}_j}$

Eg: $B = \{ \underline{i}, \underline{i} + \underline{j} \}$.

$y_1 = x_1 = \underline{i}$ $\qquad x_2 = \underline{i} + \underline{j}$

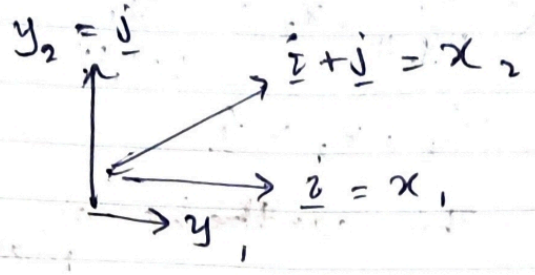
$$\hat{y}_1 = \frac{y_1}{\|y_1\|} = \frac{\underline{i}}{\|\underline{i}\|} = \frac{\underline{i}}{1} = \underline{i}$$

$$y_2 = x_2 - \langle \hat{y}_1, x_2 \rangle \hat{y}_1$$

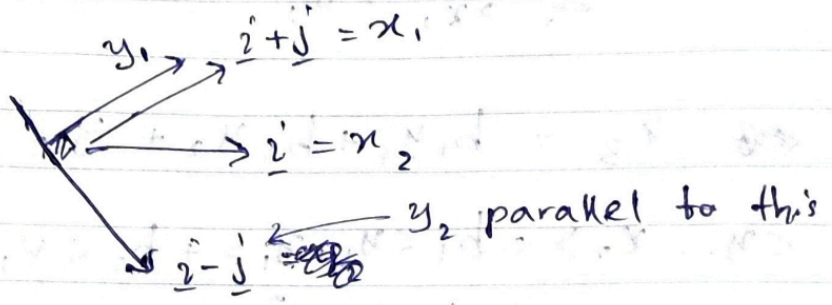
$$\underline{i} + \underline{j} - \langle \underline{i}, \underline{i} + \underline{j} \rangle \underline{i}$$

$$\underline{i} + \underline{j} - [\underline{i} (\underline{i} + \underline{j})] \underline{i}$$

$$\underline{i} + \underline{j} - (1 + 0) \underline{i} = \underline{j}$$



Eg: $B = \{ \underline{i}+\underline{j} , \underline{i} \}$



$$y_1 = x_1 = \underline{i}+\underline{j} \quad , \quad \hat{y}_1 = \frac{y_1}{\|y_1\|} = \frac{\underline{i}+\underline{j}}{\|\underline{i}+\underline{j}\|} = \frac{\underline{i}+\underline{j}}{\sqrt{1^2+1^2}} = \frac{\underline{i}+\underline{j}}{\sqrt{2}}$$

$$y_2 = x_2 - \langle \hat{y}_1 , x_2 \rangle \hat{y}_1$$

$$= \underline{i} - \left\langle \frac{\underline{i}+\underline{j}}{\sqrt{2}} , \underline{i} \right\rangle \left(\frac{\underline{i}+\underline{j}}{\sqrt{2}} \right)$$

$$= \underline{i} - \frac{1}{\sqrt{2}} \cdot \left(\frac{\underline{i}+\underline{j}}{\sqrt{2}} \right)$$

$$= \frac{\underline{i}-\underline{j}}{2}$$

$$\|y_2\| = \left\| \frac{\underline{i}-\underline{j}}{2} \right\| = \frac{1}{2}\sqrt{1^2+(-1)^2} = \frac{1}{\sqrt{2}}$$

$$\hat{y}_2 = \frac{y_2}{\|y_2\|} = \frac{(\underline{i}-\underline{j})/2}{1/\sqrt{2}} = \frac{\underline{i}-\underline{j}}{\sqrt{2}}$$

Note

$W = \{\underline{i}\} \subseteq \mathbb{R}^3$

Here $(W^\perp)^\perp = \text{span}\{\underline{i}\}$
$\supseteq \underline{i}$

$W^\perp = \text{span}\{\underline{j}, \underline{k}\}$

All the linear combns of $\underline{j}$ & $\underline{k}$ are normal to $\underline{i}$

Def^n Let W be a non-empty subset of V over F.

$W^\perp = W$ perpendicular
$= \{u \in V \mid \langle u, w \rangle = 0 ; \forall w \in W\}$

Th^{rm}: $W \subseteq (W^\perp)^\perp$

Proof:

$w \in W \Rightarrow \langle w, x \rangle = 0 ; \forall x \in W^\perp$

$\Leftrightarrow \langle x, w \rangle = 0 ; \forall x \in W^\perp$

$\Rightarrow w \in (W^\perp)^\perp$

$\Rightarrow W \subseteq (W^\perp)^\perp$

(Reverse is not true)

$(W^\perp)^\perp \not\subseteq W$

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Th^{rm}: $W \cap W^{\perp} \neq \phi \Rightarrow W \cap W^{\perp} = \{\underline{0}\}$

Proof: $x \in W \cap W^{\perp}$

$\Rightarrow x \in W$ and $x \in W^{\perp}$

$\Rightarrow \langle x, x \rangle = 0$

$\Leftrightarrow \|x\| = 0$

$\Leftrightarrow x = \underline{0}$

Th^{rm}: $W^{\perp}$ is a subspace of V

1) $W^{\perp} = \{x \in V \mid \langle x, w \rangle = 0 \ \forall w \in W\} \subseteq V$

2) $\langle \underline{0}, w \rangle = \underline{0} \ \forall w \in W \Rightarrow \underline{0} \in W^{\perp}$

$\Rightarrow W^{\perp} \neq \phi$

3) $x, y \in W^{\perp}$; $w \in W$

$\langle x+y, w \rangle = \langle x, w \rangle + \langle y, w \rangle = 0 + 0 = 0$

$\Rightarrow x+y \in W^{\perp}$

4) $a \in \mathbb{F}$, $x \in W^{\perp}$, $w \in W$

$\langle ax, w \rangle = \bar{a} \langle x, w \rangle = \bar{a} \times 0 = 0$

$\Rightarrow ax \in W^{\perp}$

$\therefore W^{\perp}$ is a subspace of V.

If W is a subspace, it should have $\underline{0}$, meaning $W \cap W^{\perp} = \{\underline{0}\}$.

When the second order polynomials goes to zero, they form a v.s. with two L.I. sol^{ns}.

~~Q 2 08~~

~~$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) dx$~~

$B = \{ x_1, x_2, x_3 \dots \}$, $C = \{ \hat{y}_1, \hat{y}_2, \hat{y}_3 \dots \}$

6.1) $\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) dx$

$C = \{ 1, x, \frac{1}{2}(3x^2-1), \dots \} = \{ P_n(x) \}_{n=0}^{\infty}$

$P_n(x)$ = Legendre polynomials.

They also satisfy
$(1-x^2) y'' - 2xy' + n(n+1) y = 0$

6.2) $\langle f, g \rangle = \int_{0}^{\infty} e^{-x} f(x) g(x) dx$

$C = \{ 1, -x+1, \frac{1}{2}(x^2-4x+2), \dots \} = \{ L_n(x) \}_{n=0}^{\infty}$

$L_n(x)$ = Legurre polynomials.

They satisfy $xy'' + (1-x)y' + ny = 0$

6.3) $\langle f, g \rangle = \int_{-\infty}^{\infty} e^{-x^2} f(x) g(x) dx$

$C = \{ 1, 2x, 4x^2-2, \dots \} = \{ H_n(x) \}_{n=0}^{n=\infty}$

$H_n(x)$ = Hermit polynomials.

$H_n(x)$ satisfies $y'' - 2xy' + 2ny = 0$

6.4) $\langle f, g \rangle = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x) g(x) dx$

$C = \{ 1, x, 2x^2-1, 4x^3-3x, \dots \} = \{ T_n(x) \}_{n=0}^{\infty}$

$T_n(x)$ = chyhyshev polynomial.

They satisfy $(1-x) y'' - xy' + n^2 y = 0$

Note

$$T_n(\cos\theta) = \cos n\theta$$
$$T_2(\cos\theta) = \cos 2\theta = 2\cos^2\theta - 1$$
$$T_2(x) = 2x^2 - 1$$

$$0 = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_n(x) T_m(x)\, dx \quad ; \quad m \neq n$$

When $x = \cos\theta$

$$0 = \int_{\pi}^{0} \frac{1}{\sqrt{1-\cos^2\theta}} \cos n\theta \cos m\theta (-\sin\theta)\, d\theta$$

$$= \int_{\pi}^{0} \frac{1}{\sin\theta} \cos n\theta \cos m\theta (-\sin\theta)\, d\theta$$

$$0 = \int_{0}^{\pi} \cos n\theta \cos m\theta\, d\theta \quad ; \quad m \neq n$$

When $m = n$

$$\int_0^{\pi} \cos^2 n\theta\, d\theta = \int_0^{\pi} \frac{1+\cos 2n\theta}{2}\, d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{\sin 2n\theta}{2n}\right]_0^{\pi} \quad ; n \neq 0$$

$$= \frac{\pi}{2}$$

When $n = 0$: $\int_0^{\pi} dx = \pi$

$$f(\theta) = \sum_n{}' a_n \cos n\theta$$

$$\int_0^{\pi} f(\theta)\cos m\theta \, d\theta = \int_0^{\pi} \left(\sum_n{}' a_n \cos n\theta\right)\cos m\theta \, d\theta$$

$$= \sum_n{}' a_n \int_0^{\pi} \cos n\theta \cos m\theta \, d\theta$$

when $n \neq m$ this will be zero

$$= a_m \int_0^{\pi} \cos^2 m\theta \, d\theta + 0$$

$$m = 0: \quad a_0 = \int_0^{\pi} f(\theta)\cos 0 \, d\theta = \frac{1}{\pi}\int_0^{\pi} f(\theta)\, d\theta$$

$$m \neq 0 \qquad a_m = \frac{2}{\pi}\int_0^{\pi} f(\theta)\cos m\theta \, d\theta$$

Norms:

There are norms which are not derived from inner products.

Eg: Supremum norm

$$\| f \|_{\infty} = \sup\{ f(x) \mid x \in [a,b] \}$$

in $\mathcal{C}[a,b]$ = continuous func^{ns} on $[a,b]$.

Def: Let V be a v.s. over $F = \mathbb{C}$. $x, y \in V$, $a \in F$
A norm $\|\cdot\|$ on V is s.t.

1) $\|\cdot\| : V \to F$
2) $\|x\| \geq 0$ and $\|x\| = 0 \iff x = \underline{0}$
3) $\|ax\| = |a| \, \|x\|$
4) $\|x+y\| \leq \|x\| + \|y\|$
5) $\|x \cdot y\| \leq \|x\| \|y\|$ if V has a multip$^{\underline{n}}$ (like in a Ring)

Triangular inequality: $\|x+y\| \leq \|x\| + \|y\|$

Proof of trang$^{\underline{r}}$ inequality in an inner product space.

$$\|x+y\|^2 = \langle x+y, x+y \rangle$$
$$= \langle x, x \rangle + \langle x, y \rangle + \underbrace{\langle y, x \rangle}_{\overline{\langle x, y \rangle}} + \langle y, y \rangle$$

$$\|x+y\|^2 = \|x\|^2 + 2\,\mathrm{Re}\langle x, y \rangle + \|y\|^2$$
$$\leq \|x\|^2 + 2\,|\langle x, y \rangle| + \|y\|^2$$
$$\leq \|x\|^2 + 2\,\|x\|\,\|y\| + \|y\|^2$$
$$= (\|x\| + \|y\|)^2$$

Note.
$z + \bar{z}$
$= 2\mathrm{Re}\, z$
$\leq 2|z|$

Note.

Normed space is a v.s. with a norm.
If you are in a normed space, "distance" between x & y can be found as
$d(x, y) = \|x - y\|$ where d is called a matric

$\rightarrow \rightarrow$

If we don't have a norm, we define matric ~~as~~ on V as

1) $d : V \times V \longrightarrow \mathbb{R}$
2) $d(x,y) \geq 0$ & $d(x,y) = 0 \iff x = y$.
3) $d(x,y) = d(y,x)$
4) $d(x,y) \leq d(x,z) + d(z,y)$

Eg: $V = \mathbb{R}$

$$d(x,y) = |x-y|$$

Eg: In ~~general~~ special relativity

$$d^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 + (ic^2 \Delta t^2)$$

speed of light

Matric special relativity

$$d^2 = -\Delta x^2 - \Delta y^2 - \Delta z^2 + c^2 \Delta t^2 \geq 0$$

$$c^2 \geq \frac{\Delta x^2 + \Delta y^2 + \Delta z^2}{\Delta t^2}$$

$$c^2 \geq v^2$$

In a matric space, we can do limits.

$$\lim_{x \to a} f(x) = L$$

$$\iff \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x, \ |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

If there is a norm $\longrightarrow \|x-a\| < \delta \longrightarrow \|f(x) - L\| < \varepsilon$

If there is a matric $\longrightarrow$ or $d(x,a) < \delta \longrightarrow$ or $d(f(x), L) < \varepsilon$

Cauchy Seq. x_n is s.t.

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{Z}^+ \ \forall m,n \in \mathbb{Z}^+ ; m,n > N \Rightarrow \|x_n - x_m\| < \varepsilon$$

or $d(x_n, x_m) < \varepsilon$

~~Cauchy seq. are measuring~~

Cauchy seq. are important in spaces where there is no limit.

$u : \mathbb{Z}^+ \longrightarrow V$

input is an integer

If every cauchy seq. converge to a point in V, we say that V is complete.

Eg: $\mathbb{Q}$ is not complete.

For example $u_n = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} \to e$

(Converging sequences are cauchy)

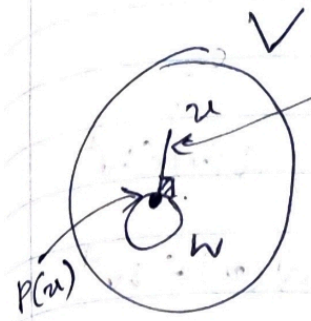
This is not an element of $\mathbb{Q}$.

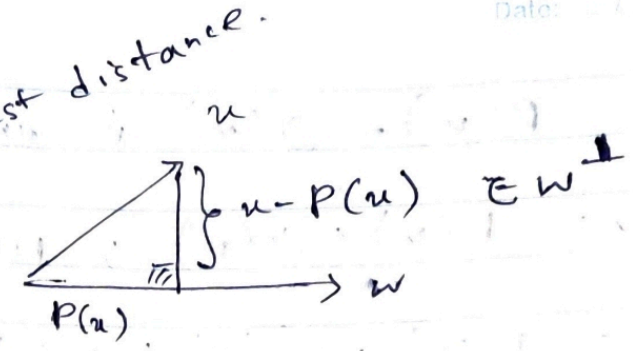
$\mathbb{R}$ is complete.

(Limits of cauchy seq of $\mathbb{Q}$ are added to make it complete).

<u>Def^n</u>: A Hilbert space is a complete inner product space.

<u>Def^n</u>: A Banach space is a complete ~~normal~~ normed space.





Thm: Let W be a ~~sup~~ subspace of V with an orthonormal basis $\{w_1, w_2, \dots, w_n\}$. Then any $u \in V$ can be uniquely written as $u = P(u) + Q(u)$ where $P(u) = \sum_1^n \langle w_j, u \rangle w_j$, $w_j \in W$ and $Q(u) = u - P(u) \in W^\perp$. $P(u)$ is the best approximation to $u \in W$ i.e., $\|u - w\| \geq \|u - P(u)\|$ for all $w \in W$

Proof: $\langle w_k, u - P(u) \rangle = \langle w_k, u - \sum_1^n \langle w_j, u \rangle w_j \rangle$

$$= \langle w_k, u \rangle - \sum_1^n \langle w_j, u \rangle \langle w_k, w_j \rangle$$

$$= \langle w_k, u \rangle - \langle w_k, u \rangle$$

$$= 0$$

$w = \sum_1^n a_j w_j \in W$ (W is a basis)

$$\langle u - P(u), w \rangle = \langle u - P(u), \sum_1^n a_j w_j \rangle$$

$$\cancel{= \langle u - P(u), \sum_1^n}$$

$$= \sum_1^n a_j \langle u - P(u), w_j \rangle$$

$$= \sum_1^n a_j (0)$$

$$= 0$$

$\Rightarrow u - P_u = Q(u) \in W^\perp$

Thm $x = P(x) + Q(x)$ is unique.

Proof: $x = P(x) + Q(x)$; $P(x) \in W$, $Q(x) \in W^{\perp}$

$x = P'(x) + Q'(x)$; $P'(x) \in W$, $Q'(x) \in W^{\perp}$

$x - x = \underline{0}$

$P(x) - P' + Q(x) - Q'(x) = \underline{0}$

$P(x) - P'(x) = Q'(x) - Q(x)$

$\in W \quad \in W \quad \in W^{\perp} \quad \in W$

~~$\Rightarrow -P(x) = -Q(x)$~~

$P(x) - P'(x) \in W$, $Q'(x) - Q(x) \in W^{\perp}$

$\therefore P(x) - P'(x) = 0$ and $Q'(x) - Q(x) = 0$

$\Rightarrow P(x) = P'(x)$ $\quad Q'(x) = Q(x)$

In this context we write $V = W \oplus W^{\perp}$ ← direct sum

Proving $x - P(x)$ is the shortest:

$\|x - w\|^2$; $w \in W$

$= \|\underbrace{x - P(x)}_{Q(x) \in W^{\perp}} + \underbrace{P(x) - w}_{R(x) \in W}\|^2$

$= \langle Q(x) + R(x), Q(x) + R(x) \rangle$

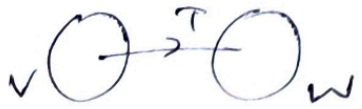
$= \|Q(x)\|^2 + 2\,\mathrm{Re}\,\langle \underbrace{Q(x), R(x)}_{=0} \rangle + \|R(x)\|^2$

$= \|Q(x)\|^2 + \|R(x)\|^2$

$\geq \|Q(x)\|^2$

$= \|x - P(x)\|^2$ $\quad \Rightarrow \|x - w\| \geq \|x - P(x)\|$

or $d(x, w) \geq d(x, P(x))$



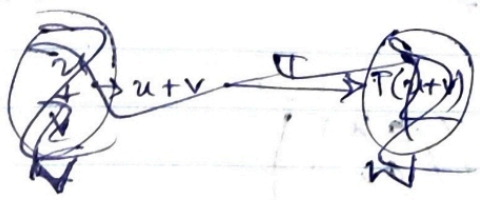
Def^n: Let V and W be v.s. over F. A mapping T is a Linear Transformation iff

(01) $T : V \to W$ is a func^n

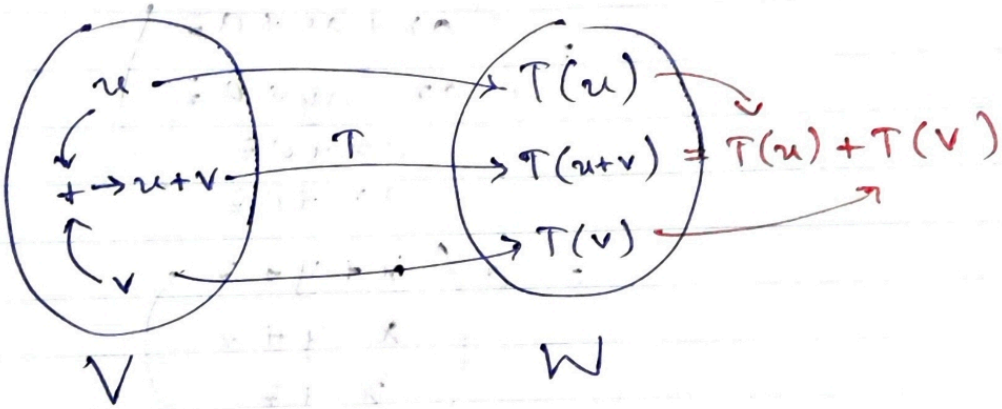
(02) $\forall\, u, v \in V$; $T(u+v) = T(u) + T(v)$

(03) $\forall\, u \in V$, $\forall\, a \in F$; $T(au) = a\,T(u)$

(04) $T(au + bv) = a\,T(u) + b\,T(v)$



Note (02)



Eg: $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$, $F = \mathbb{R}$

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x + y + z \\ x - y + z \\ x + z \\ x + z \end{pmatrix}$$

(01) $T : \mathbb{R}^3 \to \mathbb{R}^4$ is a function since for every $u \in \mathbb{R}^3$, $T(u)$ is a unique element in $\mathbb{R}^4$

(02) $$T\left(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\right) = T\left(\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}\right)$$

$$= T\begin{pmatrix} (x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) \\ (x_1 + x_2) - (y_1 + y_2) + (z_1 + z_2) \\ (x_1 + x_2) + (z_1 + z_2) \\ (x_1 + x_2) + (z_1 + z_2) \end{pmatrix} \longrightarrow$$

$$= \begin{pmatrix} x_1+y_1+z_1 \\ x_1-y_1+z_1 \\ x_1+z_1 \\ x_1+z_1 \end{pmatrix} + \begin{pmatrix} x_2+y_2+z_2 \\ x_2-y_2+z_2 \\ x_2+z_2 \\ x_2+z_2 \end{pmatrix}$$

$$= T\left(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}\right) + T\left(\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\right)$$

(03) $$T\left(a\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = T\left(\begin{pmatrix} ax \\ ay \\ az \end{pmatrix}\right)$$

$$= \begin{pmatrix} ax+ay+az \\ ax-ay+az \\ ax+az \\ ax+az \end{pmatrix}$$

$$= a\begin{pmatrix} x+y+z \\ x-y+z \\ x+z \\ x+z \end{pmatrix}$$

$$= aT\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right)$$

~~Let $T: V \to W$ be a L.T. over F~~

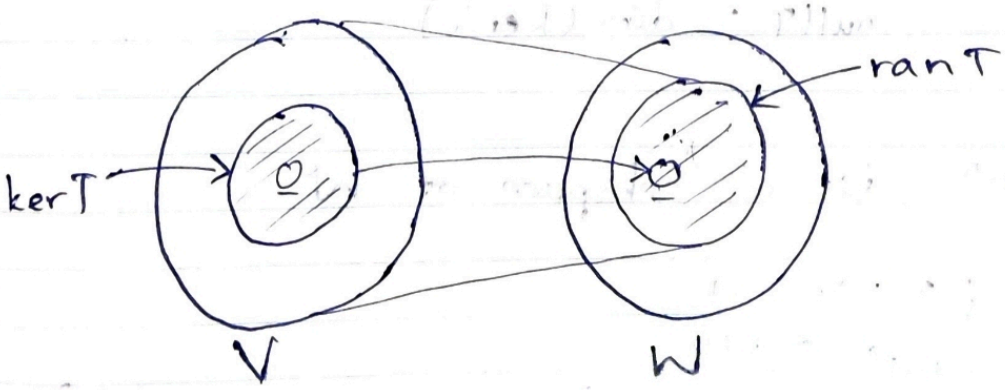
Thm $T(\underline{0}) = \underline{0}$

Proof: $T(\underline{0}) = T(0u)$
$= 0\,T(u)$
$= \underline{0}$

Thm $T(-u) = -T(u)$, $u \in V$

Proof: $T(-u) = T((-1)u)$
$= (-1)\,T(u)$
$= -T(u)$

Def^n Let $T: V \to W$ be a L.T. over F.
Kernel of $T = \ker T = \{u \in V \mid T(u) = \underline{0}\}$
range of $T = \operatorname{ran} T = \{T(u) \mid u \in V\} \subseteq W$



Th^{rm} ker$T$ is a subspace of V

Proof:

1) $\ker T \subseteq V$
2) $T(\underline{0}) = \underline{0} \Rightarrow \underline{0} \in \ker T \Rightarrow \ker T \neq \emptyset$
3) $u, v \in \ker T$; $u, v \in V \Rightarrow u+v \in V$
   $T(u+v) = T(u) + T(v) = \underline{0} + \underline{0} = \underline{0}$
   $\Rightarrow u+v \in \ker T$
4) $u \in \ker T$; $a \in F$; $u \in V$
   $T(au) = aT(u) = a\,\underline{0} = \underline{0}$
   $\Rightarrow au \in \ker T$

* $\therefore$ ker$T$ is a subspace of V
ker$T$ is a vector space
$\therefore$ it has a basis
No. of elements in it (dim (ker$T$)) is called nullT or nullity of T.
nullT = dim (ker$T$)

Th^{rm} ran$T$ is a subspace of W

Proof 1) $T : V \to W$
ran$T \subseteq W$
2) ran$T \neq \emptyset$ (Since T is a func^{n})
3) $T(u), T(v) \in$ ran$T$; $u, v \in V$
   $T(u) + T(v) = T(u+v) \in$ ran$T$; $u+v \in V$
4) $T(u) \in$ ran$T$; $u \in V$, $a \in F$
   $aT(u) = T(au) \in$ ran$T$; $au \in V$

$\therefore$ ran$T$ is a subspace of W.

ran T is a v.s.

So it has a basis

No. of elements in it (dim (ran T)) is called the rank T (rank of T)

$$\text{rank } T = \dim(\text{ran } T)$$

Eg: $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^4, \quad F = \mathbb{R}$

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+y+z \\ x-y+z \\ x+z \\ x+z \end{pmatrix}$$

$$\ker T : T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+y+z \\ x-y+z \\ x+z \\ x+z \end{pmatrix} = \underline{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x+y+z=0 \quad \text{— (1)}$$
$$x-y+z=0 \quad \text{— (2)}$$
$$x+z=0 \quad \text{— (3)}$$
$$x+z=0 \quad \text{— (4)}$$

(1) + (2) $\quad 2x+2z=0 \iff x+z=0 \iff z=-x$

$$2y=0 \iff y=0$$

$$\ker T \ni \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ -x \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \in \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right\}$$

$$\ker T = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right\}$$

$\left\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right\}$ is a basis. $\dim(\ker T) = 1$

Method II

$$\begin{pmatrix} x+y+z \\ x-y+z \\ x+z \\ x+z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow[i=1,2,3]{-R_4+R_i \to R_i} \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{R_1+R_2 \to R_2} \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow[R_2 \leftrightarrow R_4]{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} x+z=0 \\ y=0 \end{array}$$

<u>ranT</u>

$$\begin{pmatrix} x+y+z \\ x-y+z \\ x+z \\ x+z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= x\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + y\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + z\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\text{ranT} \ni T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) \in \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\right\}$$

$$= \text{span}\left\{\underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}_{T\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right)}, \underbrace{\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}}_{T\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right)}\right\}$$

$$= S$$

$\text{ranT} \subseteq S$

~~$S \subseteq \text{ranT}$~~

$$S \ni a\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = aT\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) + bT\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right)$$

$$= \underbrace{T\left(\underbrace{a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{\in \mathbb{R}^3}\right)}_{\in \text{ranT}}$$

$\Rightarrow S \subseteq \text{ranT}$

$S = \text{ranT}$

$$\therefore \quad \text{ranT} = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}\right\} \leftarrow \text{basis}$$

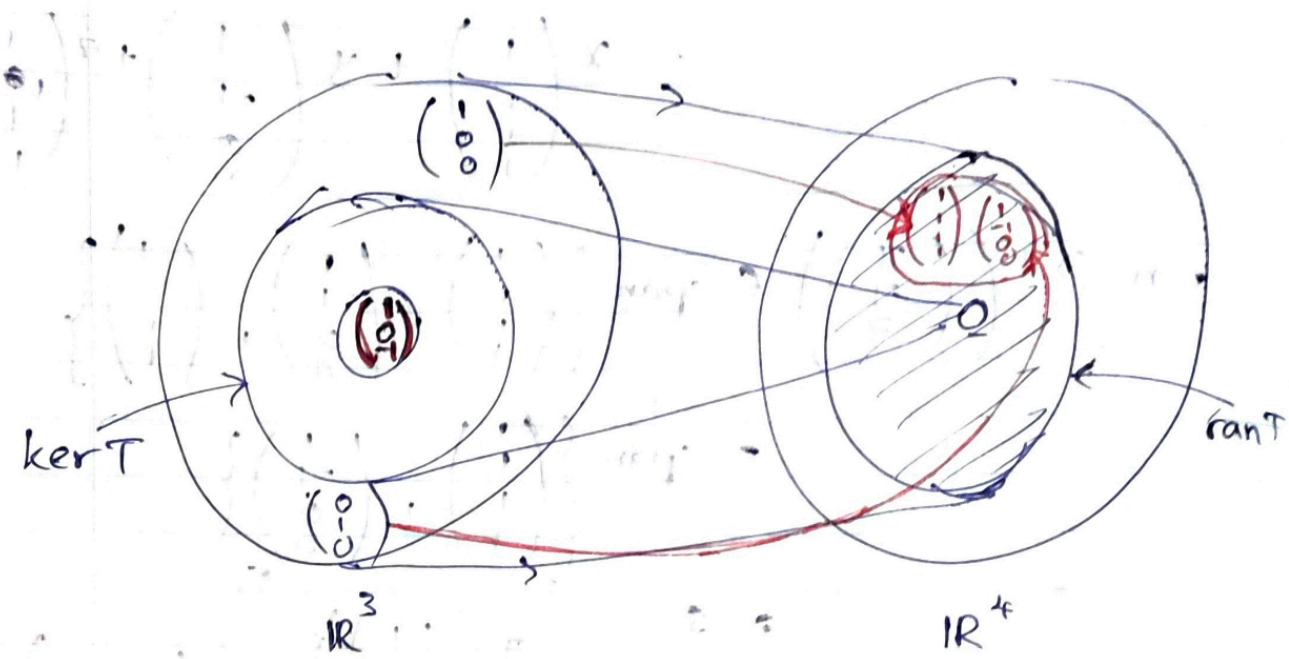
$$\text{rankT} = \dim(\text{ranT}) = 2$$

* null T + rank T $\quad T: \mathbb{R}^3 \to \mathbb{R}^4$

$= 1 + 2$

$= 3$

$= \dim(\mathbb{R}^3)$



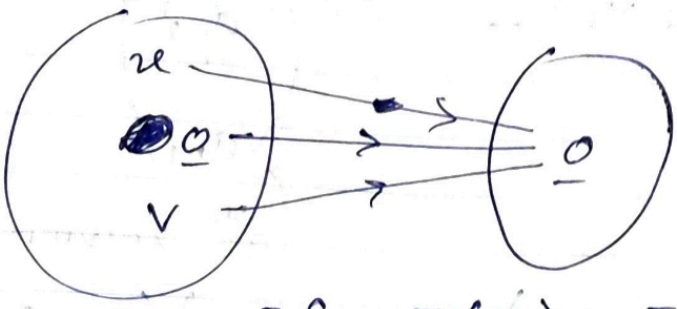
<u>Th^rm</u> : Dimension Th^rm :

Let $T: V \to W$ be a L.P.

Then $\text{null}\, T + \text{rank}\, T = \dim(V)$

Note

If $\ker T$ has only one element T should be one to one. If not it is many to one



If $T(u) = T(v) = \underline{0}$

~~If u T(u) and T(v) T(u) T(v)~~

$T(u) = T(v)$

$T(u-v) = \underline{0}$

$\therefore$ $u-v$ should be inside $\ker T$.

$\therefore$ It should be many to one.

Def Matrix of a L.T.

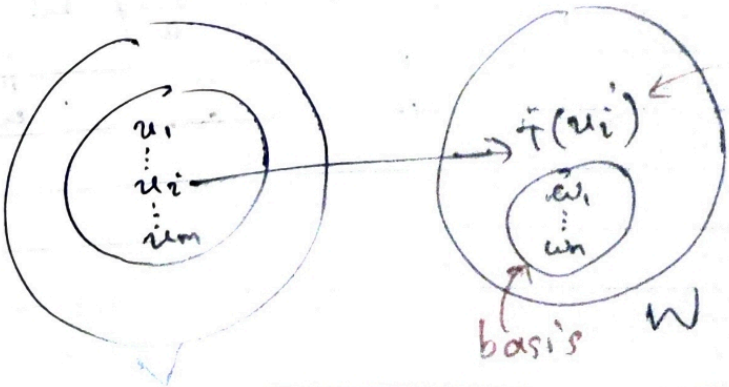
Let $T: V \longrightarrow W$ be a L.T. over F and $(u_i)_{i=1}^{m} = (u_1, u_2, u_3, \ldots, u_m)$ be an ordered basis of V and $(w_j)_{j=1}^{n} = (w_1, w_2, \ldots, w_n)$ be an ordered basis of W.

A matric $A \in F^{n \times m}$ s.t.

$(T(u_1)\, T(u_2) \ldots T(u_m)) = (w_1 w_2 \ldots w_n) A$

is called the matric of T w.r.t. (u_i) & (w_j)

We write $A = T_{(u_i),(w_j)}$



This can be written using w_j (since it is a basis)

$$W \ni T(u_i) = a_{1i}w_1 + a_{2i}w_2 + a_{3i}w_3 + \cdots + a_{ni}w_n$$

$$= (w_1\ w_2 \cdots w_n)\underbrace{\begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{pmatrix}}_{i^{th}\text{ column of } A}$$

$$\big(T(u_1)\ T(u_2) \ldots T(u_m)\big) = (w_1\ w_2 \ldots w_n)\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}_{n\times m}}_{\ni F^{n\times m}}$$

Eg: $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+y+z \\ x-y+z \\ x+z \\ x+z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$

Let $(u_i)_{i=1}^{3} = \left(\underset{u_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}, \underset{u_2}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}, \underset{u_3}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}\right)$

$(w_j)_{j=1}^{4} = \left(\underset{w_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}, \underset{w_2}{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}, \underset{w_3}{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}, \underset{w_4}{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}\right)$

$$T(v_3) = T(u_1) = T\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right) = \begin{pmatrix}1\\1\\1\\1\end{pmatrix} = 1\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + 1\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + 1\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + 1\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

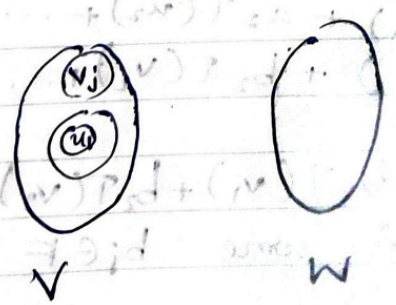
$$= \begin{pmatrix}\omega_1 & \omega_2 & \omega_3 & \omega_4\end{pmatrix}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

$$T(v_2) = T(u_2) = T\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\right) = \begin{pmatrix}1\\-1\\0\\0\end{pmatrix} = \begin{pmatrix}\omega_1 & \omega_2 & \omega_3 & \omega_4\end{pmatrix}\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}$$

$$A = \begin{pmatrix}1 & 1 & 1\\1 & -1 & 1\\1 & 0 & 1\\1 & 0 & 1\end{pmatrix}$$

Dimension Thrm (Proof)

Let null $T = \dim(\ker T) = m$ and $(u_i)_{i=1}^{m}$ be a basis of $\ker T$. We can extend that basis to a basis of V by adding $\{v_j\}_{j=1}^{k}$ to it by selecting v_j outside $\ker T$



Now $\{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_k\}$ is a basis of V and $\dim(V) = m+k$.

$\rightarrow \rightarrow$

Claim: $\{T(v_j)\}_{j=1}^{k}$ is a basis of ran T

Proof:

$$a_1 T(v_1) + a_2 T(v_2) + \dots + a_k T(v_k) = \underline{0}; \; a_j \in F$$

$$\Rightarrow T(a_1 v_1 + a_2 v_2 + \dots + a_k v_k) = \underline{0}$$

$$\Rightarrow a_1 v_1 + a_2 v_2 + \dots + a_k v_k \in \ker T$$

$$\Rightarrow a_1 v_1 + a_2 v_2 + \dots + a_k v_k = b_1 u_1 + b_2 u_2 + \dots + b_m u_m$$

for some $b_i \in F$

$$\Rightarrow -b_1 u_1 - b_2 u_2 - \dots - b_m u_m + a_1 v_1 + a_2 v_2 + \dots + a_k v_k = \underline{0}$$

$$\Rightarrow -b_1 = -b_2 = \dots = -b_m = a_1 = a_2 = \dots = a_k = 0$$

$$\Rightarrow a_1 = a_2 \dots = a_k = 0$$

$$\Rightarrow \{T(v_j)\}_{j=1}^{k} \text{ is L.I}$$

$T(u) \in$ ran T; $u \in V$

$$T(u) = T(a_1 u_1 + a_2 u_2 + \dots + a_m u_m + b_1 v_1 + b_2 v_2 + \dots + b_k v_k)$$
$$= a_1 \cancel{T(u_1)} + a_2 \cancel{T(u_2)} + \dots + a_m \cancel{T(u_m)}$$
$$+ b_1 T(v_1) + b_2 T(v_2) + \dots + b_k T(v_k)$$

$$\text{ran}T \ni T(u) = b_1 T(v_1) + b_2 T(v_2) + \dots + b_k T(v_k)$$

for some $b_j \in F$

$$\Rightarrow \text{ran}T = \text{span}\{T(v_j)\}_{j=1}^{k}$$

$$\Rightarrow \{T(v_j)\}_{j=1}^{k} \text{ is a basis of ran}T$$

$$\Rightarrow \text{rank}\,T = \dim(\text{ran}T) = k$$

$$\dim(V) = m + k$$
$$= \dim(\ker T) + \dim(\operatorname{ran} T)$$
$$= \operatorname{null} T + \operatorname{rank} T$$

Matrix of a Linear Transformation

$T : V \to W$ over F

$$T : V \longrightarrow W$$

ordered basis $(u_i)_{i=1}^{m}$ $\quad (w_j)_{j=1}^{n}$

matrix $A \in F^{m \times n}$ s.t. $(T(u_i))_{i=1}^{m} = (w_j)_{j=1}^{n}$. A is called the matrix of T w.r.t. (u_i), (w_j). We write ~~A = T(u~~ $A = T_{(u_i),(w_j)}$

Eg: $T : \mathbb{R}_2[x] \to \mathbb{R}_2[x]$, $F = \mathbb{R}$, $V = W = \mathbb{R}_2[x]$

$$T[p(x)] = xp'(x) + p(x)$$

$$(u_i)_{i=1}^{3} = (w_i)_{i=1}^{3} = \{1, x, x^2\}$$

$$T(1) = x(0) + 1 = 1$$
$$= 1(1) + 0(x) + 0(x^2)$$
$$= (1 \;\; x \;\; x^2)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$T(x) = x(1) + x = 2x = (1 \;\; x \;\; x^2)\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

$$T(x^2) = x(2x) + x^2 = 3x^2 = (1 \;\; x \;\; x^2)\begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

$$A \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Eg: $T : \mathbb{R}_2[x] \longrightarrow \mathbb{R}_2[x]$

$T[p(x)] = x p'(x) + p(x)$

$(u_i)_{i=1}^{3} = (w_j)_{j=1}^{3} = \{1, 1+x, 1+x^2\}$

$$T(1) = \begin{pmatrix} 1 & 1+x & 1+x^2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} T(1+x) &= x(1) + (1+x) \\ &= 2x+1 \\ &= 2(x+1) - 1 \\ &= \begin{pmatrix} 1 & 1+x & x+x^2 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} T(x+x^2) &= x(1+2x) + (x+x^2) \\ &= 3x^2 + 2x \\ &= 3(x^2+x) - (x+1) + 1 \\ &= \begin{pmatrix} 1 & 1+x & x+x^2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \end{aligned}$$

$(T(1) \quad T(1+x) \quad T(x+x^2))$

$$A' = \begin{pmatrix} 1 & 1+x & x+x^2 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix}$$

No:

Date:

Change of bases

$T: V \longrightarrow W$ over F

ordered basis $(u_i)_{i=1}^{m}$, $(w_j)_{j=1}^{n} \longrightarrow A$

P

new ordered basis $(u_i')_{i=1}^{m}$ $(w_j')_{j=1}^{n} \longrightarrow A'$ Q

$$V \ni u_i' = a_{1i}u_1 + a_{2i}u_2 + \dots + a_{mi}u_m$$

$$= (u_1 \; u_2 \dots u_m) \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{pmatrix} \leftarrow i^{th} \text{ column of } P$$

$$(u_1' \; u_2' \dots u_m') = (u_1 \; u_2 \dots u_m) P$$

$$(u_i')_{i=1}^{m} = (u_i)_{i=1}^{m} P \quad \text{—} (1)$$

$$T(u_i') = T(a_{1i}u_1 + a_{2i}u_2 + \dots + a_{mi}u_m)'$$

$$= a_{1i}T(u_1) + a_{2i}T(u_2) + \dots + a_{mi}T(u_m)$$

$$= (T(u_1) \; T(u_2) \dots T(u_m)) \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{pmatrix} \leftarrow i^{th} \text{ column of } P$$

$$(T(u_1') \; T(u_2') \dots T(u_m')) = (T(u_1) \; T(u_2) \dots T(u_m)) P$$

$$(T(u_i')_{i=1}^{m}) = (T(u_i))_{i=1}^{m} P \quad \text{—} (2)$$

$$P \in F^{m \times m}$$

Also $w_j' = b_{1j}w_1 + b_{2j}w_2 + \dots + b_{nj}w_n$

$$= (w_1 \; w_2 \dots w_n) \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix} \leftarrow i^{th} \text{ column of } Q$$

$$(w_1' \; w_2' \cdots w_n') = (w_1 \; w_2 \cdots w_n) Q$$

$$(w_j')_{j=1}^{n} = (w_j)_{j=1}^{n} Q \quad \text{--- (3)}$$

$$Q \in F^{n \times n}$$

Now $(T(x_i'))_{i=1}^{m} = (w_j')_{i=1}^{n} A'$

$$T(x_j))_{i=1}^{m} P = (w_j)_{j=1}^{n} Q A'$$

$$(w_j)_{j=1}^{n} AP = (w_j)_{j=1}^{n} Q A'$$

$$\underbrace{(w_j)_{j=1}^{n}}_{1 \times n} \underbrace{(AP}_{n \times m} - \underbrace{QA')}_{m \times m} = \underbrace{0}_{1 \times m} = (\underline{0} \; \underline{0} \cdots \underline{0})_{1 \times m}$$

$$\Rightarrow (AP - QA') = 0_{n \times m}$$

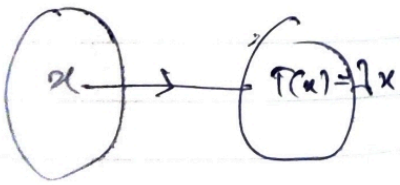
$$\Rightarrow AP = QA'$$

$$\boxed{\Rightarrow A' = Q^{-1} AP}$$

<u>Def^n</u> <u>Eigen values</u> / Eigen vectors.

Let $T: V \longrightarrow V$

$\lambda \in F$ is an eigen value of T & $x \in V - \{\underline{0}\}$ is a corresponding eigen vector of T iff $T(x) = \lambda x$



* $T(\underline{0}) = \underline{0} = \lambda(\underline{0})$

($\underline{0}$ is an eigenvector for any λ) ← This is wrong

Eg: $T : \mathbb{R}_2[x] \to \mathbb{R}_2[x]$, $F = \mathbb{R}$

$$T(p(x)) = x p'(x) + p(x)$$

$$T(p(x)) = x p'(x) + p(x) = \lambda p(x)$$

$$x p'(x) = (\lambda - 1) p(x)$$

$$\int \frac{p'(x)}{p(x)} dx = \int \frac{(\lambda - 1)}{x} dx$$

$$\ln p(x) = (\lambda - 1) \ln x + \ln a$$

$$p(x) = a x^{\lambda - 1} \in \mathbb{R}_2[x] = V$$

λ	1	2	3
$p(x)$	a	ax	ax^2

$\lambda = 1$, $p(x) = a$

$$T(p(x)) = T(a) = x(0) + a = a = 1 \cdot a = 1 \cdot p(x)$$

$$p(x) = a = a \cdot 1 \in \text{span}\{1\} - \{0\}$$

$\lambda = 2$, $p(x) = ax$

$$T(p(x)) = T(ax) = x(a) + ax = 2ax = 2(ax) = 2p(x)$$

$$p(x) = ax = a(x) \in \text{span}\{x\} - \{0\}$$

$\lambda = 3$, $p(x) = ax^2$

$$T(p(x)) = T(ax^2) = x(2ax) + (ax^2) = 3ax^2 = 3(ax^2) = 3p(x)$$

$$p(x) = ax^2 = a(x^2) \in \text{span}\{x^2\} - \{0\}$$

Def^n Let $A \in F^{n \times n}$

Let $T: F^n \longrightarrow F^n$; $F = F$ a linear transformation defined by $T(x) = Ax$.

So the eigen values/vectors of T are given by $T(x) = \lambda x$

$$Ax = \lambda x, \; x \neq 0$$

$$(A - \lambda I)x = \underline{0}, \; x \neq \underline{0}$$

$\exists x \neq 0;\; B_{n \times n}\, x = \underline{0}$ $\Longleftrightarrow$ $\det B = 0$

$\det(A - \lambda I) = \underline{0}$, find λ. Substitute in $Ax = \lambda x$ or in $(A - \lambda I)x = \underline{0}$ & find $\underline{x}$

Eg: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$$\det(A - \lambda I) = \det\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$$

$$= \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)(3-\lambda) = 0$$

$\Leftrightarrow \lambda = 1, 2, 3$ are the eigen values.

$\lambda = 1$ $\quad Ax = 1\lambda$

$$(A - 1I)x = \underline{0}$$

$$0 = \begin{pmatrix} 1-1 & 0 & 0 \\ 0 & 2-1 & 0 \\ 0 & 0 & 3-1 \end{pmatrix} x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$b = 0$$
$$2c = 0$$

$$x = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda = 1, \quad x = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$$

$$Ax = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = 1 \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = \lambda x$$

$\lambda = 2 \qquad Ax = 2\lambda$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \quad -a = 0, \quad c = 0$$

$$x = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}$$

$$\lambda = 2, \quad x = \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}$$

$\lambda = 3, \quad Ax = 3\lambda$

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \quad \begin{matrix} -2a = 0 \\ -b = 0 \end{matrix}$$

$$x = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}$$

$$\lambda = 3, \quad x = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} = c \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in \operatorname{span}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\} - \left\{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right\}$$

No: Date:

Def

Eigen space of λ. $T: V \to V$ over F

$$V_\lambda = \{ x \mid T(x) = \lambda x \}, \quad \underline{0} \in V_\lambda$$

Th^rm: ~~If eigen values / vectors exists then~~ V_λ is a subspace of V (If λ is existing)

Proof : 1) $V_\lambda \subseteq V$

~~2) V_λ is non-empty (assumption)~~

2) $\underline{0} \in V_\lambda \Rightarrow V_\lambda \neq \phi$

3) $u, v \in V_\lambda \Rightarrow T(u) = \lambda u$
$T(v) = \lambda v$

$\Rightarrow T(u) + T(v) = \cancel{\lambda(u+v)}\ \lambda u + \lambda v$
$\Rightarrow T(u+v) = \lambda(u+v)$
$\Rightarrow u+v \in V_\lambda$

4) $u \in V_\lambda, \ a \in F$

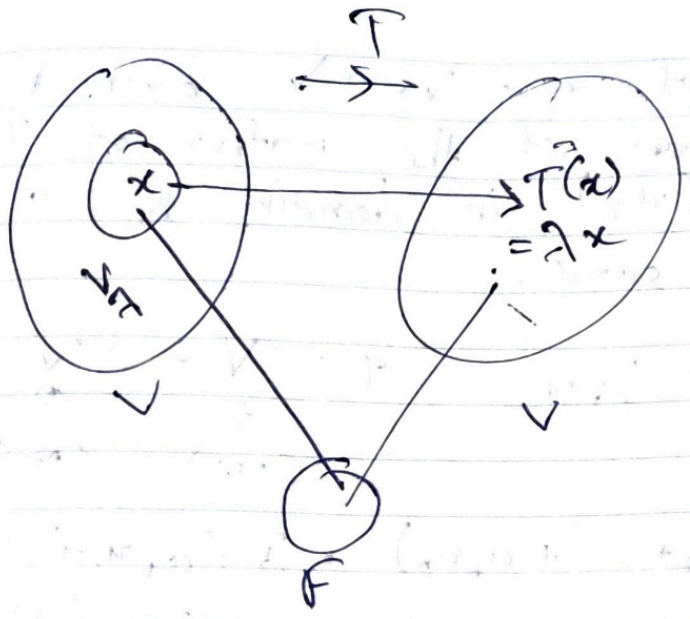
$\Rightarrow T(u) = \lambda u$
$\Rightarrow a\,T(u) = a\lambda u$
$\Rightarrow T(au) = \lambda(au)$
$\Rightarrow au \in V_\lambda$

$\therefore V_\lambda$ is a subspace of V.

~~Note: If there are no~~

$\therefore V_\lambda$ is a v.s. over F.

So it will have a basis & we can call the ~~elements~~ basis elements ~~eigen vectors~~ ~~assoc~~ ($x_\lambda \neq \underline{0}$) the eigen vectors associated with λ.



For matrices $A \in F^{n \times n}$ the eigen values / vectors are the same for the L.T

$T : F^n \to F^n$ over F

defined by $T(x) = Ax = \lambda x$

Note

Previously

$T : \mathbb{R}_3[x] \to \mathbb{R}_3[x]$ over $\mathbb{R}$

$T(p(x)) = x\,p'(x) + p(x)$

$(u_i) = (1, x, x^2) = (w_j)$

matrix of T, $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

eigen vectors:

$\lambda = 1$: $\{1, 0, 0\}$ $\qquad$ $\lambda = 3$: $\{0, 0, 1\}$

$\lambda = 2$: $\{0, 1, 0\}$

For A:

λ	1	2	3
v	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Thm: Eigen value of $T: V \to V$ over F & the eigen values of the matrix of T with the same basis (u_i) in domain & co-domain V are the same.

Proof: $T(x) = \lambda x \qquad T: \underset{(u_i)}{V} \longrightarrow \underset{(v_i)}{V}$

$$\Leftrightarrow T(a_1u_1 + a_2u_2 + \dots + a_nu_n) = \lambda(a_1u_1 + a_2u_2 + \dots + a_nu_n)$$

$$\Leftrightarrow a_1T(u_1) + a_2T(u_2) + \dots + a_nT(u_n) = \lambda(a_1u_1 + a_2u_2 + \dots + a_nu_n)$$

$$\Leftrightarrow (T(u_1) + T(u_2) + \dots + T(u_n)) \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \lambda(u_1 u_2 \dots u_n) \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

$$\Leftrightarrow (T(u_i))\,\underline{a} = \lambda(u_i)\,\underline{a}$$

$$\Leftrightarrow (u_i)\,A\,\underline{a} = \lambda(u_i)\,\underline{a}$$

$$\Leftrightarrow A\underline{a} - \lambda\underline{a} = \underline{0}$$

$$\Leftrightarrow A\underline{a} = \lambda\underline{a}$$

$$x = a_1u_1 + a_2u_2 + \dots + a_nu_n$$

$$= (u_1 u_2 \dots u_n) \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

$*$ $x = (u_i)\,\underline{a}$, x is a eigen ~~value~~ vector for T & $\underline{a}$ is an eigen vector for A

~~$\lambda = 1 : x = 2t_i$~~

$\lambda = 1 : y = (1, x, x^2)\begin{pmatrix}1\\0\\0\end{pmatrix} = 1$

$T(y) = T(1) = x(1)' + 1 = x(0) + 1 = 1 = 1 \cdot 1$
$= \lambda \cdot 1$
$= \lambda \cdot y$

$\lambda = 2 : y = (1 \;\; x \;\; x^2)\begin{pmatrix}0\\1\\0\end{pmatrix} = x$

$T(y) = T(x) = x(x)' + x = x + x$
$= 2x$
$= \lambda \cdot y$

$\lambda = 3 : y = (1 \;\; x \;\; x^2)\begin{pmatrix}0\\0\\1\end{pmatrix} = x^2$

$T(y) = T(x^2) = x(x^2)' + x^2 = 3x^2$
$= \lambda \cdot y$

$\therefore$ For T:

λ	1	2	3
y	1	x	x^2

Previously for A:

λ	1	2	3
v	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\end{pmatrix}$

related

Ex: Change of basis

$$T : V \longrightarrow V, \quad T(p(x)) = x\,p'(x) + p(x)$$

$$(u_i') = (w_j') = (1,\ 1+x,\ 1+x,\ x^2)$$

$$\begin{tikzcd} (u_i) \arrow[r, "A"] \arrow[d, "P"] & (w_j) \arrow[d, "Q"] \\ (u_i') & (w_j') \end{tikzcd}$$

$$= (1 \quad x \quad x^2)\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P = Q$$

$$A' = Q^{-1} A P$$

$T(1) = 1$ ~~$= (1,\ 1+x,\ 1+x,\ x^2)$~~

$T(1+x) = x(1) + 1 + x = 1 + 2x = 2(1+x) - 1$

$T(1+x+x^2) = x(1+2x) + 1 + x + x^2 = 3x^2 + 2x + 1$

$= 3(x^2+x+1) - (x+1) - 1$

$$(T(u_i')) = (w_j') \cdot \underbrace{\begin{pmatrix} 1 & -1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix}}_{A'}$$

A':

λ	1	2	3
v	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

For T: $y = (u_i)\,\underline{a}$

$\lambda = 1$: $y = (1,\ 1+x,\ 1+x+x^2)\begin{pmatrix}1\\0\\0\end{pmatrix} = 1$

$\lambda = 2$: $y = (1,\ 1+x,\ 1+x+x^2)\begin{pmatrix}-1\\1\\0\end{pmatrix} = x$

$\lambda = 3$: $y = (1,\ 1+x,\ 1+x+x^2)\begin{pmatrix}0\\-1\\1\end{pmatrix} = x^2$

Def^n: A & A' square matrices are called similar iff $\exists$ an invertible matrix P s.t. $A' = P^{-1}AP$ $\quad (A = PA'P^{-1} = (P^{-1})^{-1}AP^{-1})$

Th^{rm}: Similar matrices have same eigen values

Proof: $\det(A' - \lambda I)$

$= \det(P^{-1}AP - \lambda P^{-1}P)$

$= \det(P^{-1}(A - \lambda I)P)$

$= \det P^{-1} \cdot \det(A - \lambda I) \cdot \det P$

$= \dfrac{1}{\cancel{\det(P)}} \cdot \det(A - \lambda I) \cdot \cancel{\det P}$

$(\det P \neq 0,\ P$ is invertible$)$

$PP^{-1} = I$

$|PP^{-1}| = |I|$

$|P|\,|P^{-1}| = 1$

$|P^{-1}| = \dfrac{1}{|P|}$

$= \det(A - \lambda I)$

i.e. A & A' have the same characteristic polynomial. $\therefore$ same eigen values.

Ex: $A = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix}$, $\lambda = 1, 2, 3$

$$\sum \lambda = 1 + 2 + 3 = \text{trace } A$$
$$\prod \lambda = 1 \cdot 2 \cdot 3 = \det A$$

Proof: (In general) $A \in F^{n \times n}$

Characteristic polynomial
$$f(x) = \det(xI - A) \leftarrow n^{th} \text{ deg polynomial}$$

$$= \begin{vmatrix} x - a_{11} & -a_{12} & \cdots \\ -a_{21} & x - a_{22} & \cdots \\ \vdots & \vdots & \ddots \\ & & x - a_{nn} \end{vmatrix}_{n \times n}$$

n roots

$$= (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n)$$

$$= x^n - (\lambda_1 + \lambda_2 + \cdots + \lambda_n) x^{n-1} + \cdots + (-1)^n (\lambda_1 \lambda_2 \cdots \lambda_n).$$

$$= (x - a_{11})(x - a_{22}) \cdots (x - a_{nn}) + O(x^{n-2})$$

order

$$\Rightarrow \lambda_1 + \lambda_2 + \cdots + \lambda_n = a_{11} + a_{22} + \cdots + a_{nn}$$

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} = \text{trace } A$$

$$f(0) = \det(0 - A) = \det(-A) = (-1)^n \det A$$
$$= (-1)^n \lambda_1 \lambda_2 \cdots \lambda_n$$

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$$\det A = \lambda_1 \lambda_2 \cdots \lambda_n = \prod_{i=1}^{n} \lambda_i$$

Thrm: If $\lambda_1 \neq \lambda_2$ (If the eigen values are different, then the associated eigen vectors are L.I.

Ex:

λ	1	2	3
v	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}-1\\1\\0\end{pmatrix}$	$\begin{pmatrix}-1\\-1\\1\end{pmatrix}$

L.I?

Proof: $a_1 v_1 + a_2 v_2 = \underline{0}$ — (1)

$$T(a_1 v_1 + a_2 v_2) = T(\underline{0})$$

$$a_1 T(v_1) + a_2 T(v_2) = \underline{0}$$

$$a_1 \lambda_1 v_1 + a_2 \lambda_2 v_2 = \underline{0} \quad \text{— (2)}$$

(1) $\times \lambda_1$ − (2) $\quad \underline{0} + a_2 \lambda_1 v_2 - a_2 \lambda_2 v_2 = \underline{0}$

$$a_2 \underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} \underbrace{v_2}_{\neq 0} = 0$$

$$\Rightarrow a_2 = 0$$

(1) $\times \lambda_2$ − (2) $\Rightarrow a_1 = 0$

$\therefore v_1, v_2$ are L.I.

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Is the reverse true?

Ex: $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda)(3-\lambda)(1-\lambda) = (3-\lambda)(1-\lambda)^2$$

Char. poly.

$$f(x) = (x-1)^2 (x-3) = 0 \Rightarrow \lambda = x = 1, 1, 3$$

$\lambda = 3$: $(A - \lambda I) = \begin{pmatrix} -2 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$

$\downarrow$ $R_1 + R_2 + R_3 \to R_3$, $R_1/2 \to R_1$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xleftarrow{R_1 + R_2 \to R_2} \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-x = 0 \qquad -y + z = 0$

$\Rightarrow x = 0 \qquad z = y$

$$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ y \end{pmatrix} = y \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$V_1 = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$ eigen vectors

$$\lambda = 1 : A - \lambda I = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{-R_2 + R_3 \to R_3} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow x + y + z = 0$$

$$z = -x - y$$

$$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ -x-y \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ -x \end{pmatrix} + \begin{pmatrix} 0 \\ y \\ -y \end{pmatrix}$$

$$= x \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$V_2 = \text{span}\left\{ \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{u_1}, \underbrace{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}_{u_2} \right\}$$

$$A(au_1 + bu_2) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \left(a \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ -a-b \end{pmatrix} =$$

$$= \begin{pmatrix} a \\ a + 2b - a - b \\ a + b - 2a - 2b \end{pmatrix}$$

$$= \begin{pmatrix} a \\ b \\ -a-b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$= 1(au_1 + bu_2)$$

$\rightarrow \rightarrow$

~~Ex: $A = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$~~

Matrix diagonalization ; $A \in \mathbb{R}^{3\times 3} \Rightarrow 3$ λ's

$$A v_1 = \lambda_1 v_2$$
$$A v_2 = \lambda_2 v_2$$
$$A v_3 = \lambda_3 v_3$$

$$(Av_1 \;\; Av_2 \;\; Av_3) = (\lambda v_1 \;\; \lambda v_2 \;\; \lambda v_3)$$

$$A(v_1 \;\; v_2 \;\; v_3) = (v_1 \;\; v_2 \;\; v_3)\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$$AP = P\Lambda$$
$$A = P\Lambda P^{-1} \quad \text{(If } P \text{ is invertible)}$$

This is called matrix diagonal$^{\underline{zn}}$ or eigen value decomp$^{\underline{sn}}$

Ex: $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

$$\Lambda = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}$$

$\lambda = 3 \quad \lambda = 1 \quad \lambda = 1$ $\qquad$ $\lambda = 3 \quad \lambda = 1 \quad \lambda = 1$

$$AP = P\Lambda$$
$$A = P\Lambda P^{-1}$$

Quiz 12:

Let $A = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

1) Find Eigen values / vectors
2) Is A diagonalizable?

$$|A - \lambda I| = 0$$

$$\left| \begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} 3-\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\big((2-\lambda)(2-\lambda) - 1\big) = 0$$

$$-(\lambda-1)(\lambda-3)^2 = 0$$

$$\lambda_1 = 1 \quad , \quad \lambda_2 = 3$$

$\lambda_1 = 1$: $A v_1 = \lambda v_1$

$(A - \lambda I) v_1 = 0$

$$\left(\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \xrightarrow{R_1/2 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{R_2 - R_1 \to R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \xrightarrow{R_3 - R_1 \to R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \xrightarrow{R_3 - R_2 \to R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

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$x = 0$

$\cancel{x+}\ y + z = 0$

$$v_1 = \begin{pmatrix} 0 \\ y \\ -y \end{pmatrix} = y \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$v_1 = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$\lambda_2 = 3$:

$\cancel{A = \lambda}$ $A v_2 = \lambda v_2$

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right) \xrightarrow[R_3 - R_1 \to R_3]{R_2 \leftrightarrow R_1} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right)$$

$$\xrightarrow[R_2/2 \to R_2]{R_3 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2 - R_2 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x = 0$

$y - z = 0$

$$v_2 = \begin{pmatrix} 0 \\ y \\ y \end{pmatrix} = y \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$v_2 = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \qquad P = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$\det P = 0$. P is not invertible

$\therefore$ A is not diagonalizable.

- We will focus on $A \in \mathbb{C}^{n \times n}$ from here on. But the concepts can be generalized to general linear transformations.

Def^n Let $A \in \mathbb{C}^{n \times n}$. Eigen values $\lambda_i \in \mathbb{C}$ and associated eigen vectors v_i of A are defined by

$$A v_i = \lambda_i v_i \quad ; \quad v_i \neq \underline{0}$$

Eigen space of $\lambda_i = V_{\lambda_i} = \{ v_i \mid A v_i = \lambda_i v_i \} \ni \underline{0}$

Th^{rm} V_{λ_i} is a subspace of $\mathbb{C}^n$

Eg: $A = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 \\ 2 & 4-\lambda \end{vmatrix} = (\lambda - 2)(\lambda - 5) = 0$$

$$\Leftrightarrow \lambda = 2 \ , \ \lambda = 5$$

$\lambda = 2$: $A - \lambda I = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow x + y = 0$$

$$v = \begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$V_2 = \text{span} \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

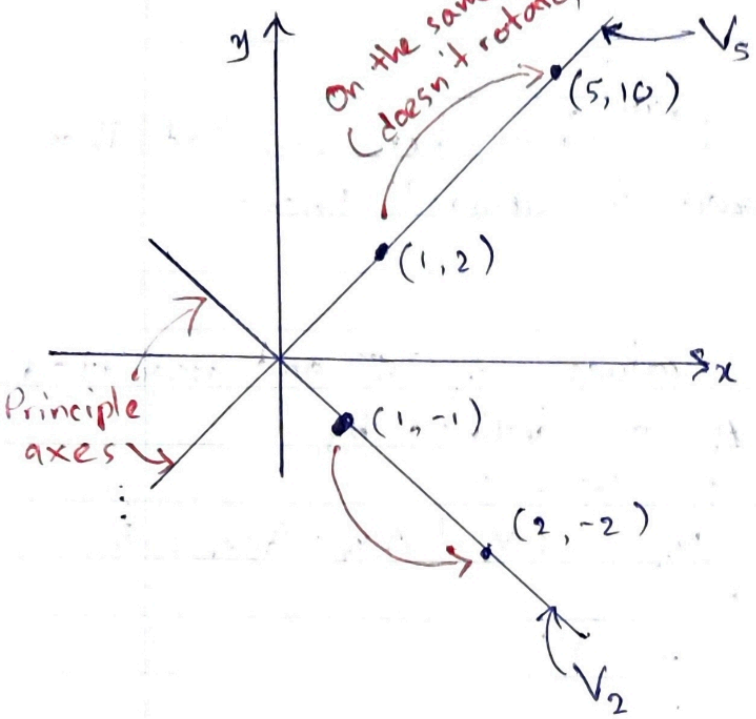
Eigen space

Eigen vector

Similarly for $\lambda = 5$: $\quad v = x\begin{pmatrix}1\\2\end{pmatrix}$

$$V_5 = \text{span}\left(\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}\right)$$

Think about a L.T $\quad T: \mathbb{R}^2 \to \mathbb{R}^2$



$$T(u) = Au$$

Not on the line (It rotates)

$$= \begin{pmatrix}3 & 1\\2 & 4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

$$T\left(\begin{pmatrix}2\\3\end{pmatrix}\right) = \begin{pmatrix}9\\16\end{pmatrix}$$

$$T\left(\begin{pmatrix}1\\2\end{pmatrix}\right) = \begin{pmatrix}5\\10\end{pmatrix} = 5\begin{pmatrix}1\\2\end{pmatrix}$$

$$T\left(\begin{pmatrix}1\\-1\end{pmatrix}\right) = \begin{pmatrix}2\\-2\end{pmatrix} = 2\begin{pmatrix}1\\-1\end{pmatrix}$$

<u>Def^n</u>: Characteristic polynomial of $A \in \mathbb{C}^{n\times n}$

$$f(x) = \det(xI - A) = (-1)^n \det(A - xI)$$

Eigen values / vectors always exist.

<u>Th^{rm}</u>: $\lambda_i \neq \lambda_j \Rightarrow v_i$ & v_j are L.I.

Ex: $\quad A = \begin{pmatrix}3 & 1\\2 & 4\end{pmatrix}$

$$f(x) = (x-2)(x-5) = x^2 - 7x + 10$$

$$f(A) = A^2 - 7A + 10I = 0$$

$$f(A) = (A - 2I)(A - 5I) = 0$$

^{n} ^{rm}: Caylay - Hamilton Th^{rm}

• A matrix satisfies its own characteristic eq^{n}, i.e. $f(A) = 0$

Proof:

$$|xI - A| I = (xI - A)\,\mathrm{adj}(xI - A)$$

$$f(x)\, I = (xI - A)\left(x^{n-1} B_0 + x^{n-2} B_1 + \dots + x^0 B_{n-1}\right)$$

Note

$$A^{-1} = \frac{\mathrm{adj}\,A}{|A|}$$

$$A\,\mathrm{adj}\,A = |A|\, I$$

$$= x^n B_0 + (B_1 - AB_0)x^{n-1} + (B_2 - AB_1)x^{n-2} + \dots + (B_{n-1} - AB_{n-2})x - AB_{n-1}x^0$$

$$= c_0 x^n + c_1 x^{n-1} + \dots + c_n x^0$$

$\therefore$ $c_0 = B_0$ — (0)

$c_1 = B_1 - AB_0$ — (1)

$c_2 = B_2 - AB_1$ — (2)

$\vdots$

$c_{n-1} = B_{n-1} - AB_{n-2}$ — (n-1)

$c_n = -AB_{n-1}$ — (n)

$A^n \times (0) + A^{n-1} \times (1) + \dots + A^0 \times (n)$ | $A^0 = I$

$c_0 A^n + c_1 A^{n-1} + c_2 A^{n-2} + \dots + c_n A^0$

$$= A^n B_0 + A^{n-1}(B_1 - AB_0) + A^{n-2}(B_2 - AB_1) + \dots + A^1(B_{n-1} - AB_{n-2}) - A^0(AB_{n-1})$$

$= 0$

$f(A) = 0$

Ex: $A = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$, $xI - A = \begin{pmatrix} x-3 & 1 \\ 2 & x-4 \end{pmatrix}$

$$\operatorname{adj}(xI - A) = \begin{pmatrix} +(x-4) & -2 \\ -1 & +(x-3) \end{pmatrix}^T$$

$$= \begin{pmatrix} x-4 & -2 \\ -1 & x-3 \end{pmatrix}$$

$$= x\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -4 & -2 \\ -1 & -3 \end{pmatrix}$$

$$= xB_0 + B_1$$

$$f(x) = x^2 - 7x + 10 = (x-2)(x-5) = 0$$

$$f(A) = A^2 - 7A + 10I = (A - 2I)(A - 5I) = 0$$

$$A^2 - 7A + 10I = 0$$

$$A^{-1}(A^2 - 7A + 10I) = A^{-1}0 \qquad (\because A^{-1} \text{ exists})$$

$$A - 7I + 10A^{-1} = 0$$

$$A^{-1} = -\frac{1}{10}(A - 7I) = \begin{pmatrix} 2/5 & -1/10 \\ -1/5 & 3/10 \end{pmatrix}$$

$$A^{-1}(A - 7I + A^{-1}) = 0$$

$$I - 7A^{-1} + A^{-2} = 0$$

$$A^{-2} = +7A^{-1} - I.$$

$\vdots$

Matrix Diagⁿ

Def: Minimal polynomial is the least degree monic (leading coefficient is 1) satisfied by A, i.e. $g(A) = 0$ and

if $h(A) = 0$, then ~~degree~~ $\deg g \le \deg h$.

so $\deg g \le \deg f$. By caylay - Hamilton $f(A) = 0$.

$\rightarrow \rightarrow$

Def^n: Matrix Diag^{lzn}.

$A \in \mathbb{C}^{n\times n}$ is diagonalizable iff there is a ~~eigen value decomposition~~ diagonal matrix Λ & an invertible matrix P s.t.

$$A = P \Lambda P^{-1}$$

Def^n: Eigen value decomp^{sn}

Suppose $Av_i = \lambda_i v_i$; v_i are L.I.

$$A(v_1 v_2 \cdots v_n) = (v_1 v_2 \cdots v_n)\begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

$$AP = P\Lambda$$

$$A = P\Lambda P^{-1}$$

Th^{rm}: If λ_i are dif^{nt} $\Rightarrow$ v_i are L.I.

So diag^{lzn} is always possible.

Ex: $A = \begin{pmatrix} 5 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = (5-\lambda)\left[(2-\lambda)^2 - 1\right]$$

$$= (5-\lambda)(3-\lambda)(1-\lambda)$$

$$f(x) = (x-1)(x-3)(x-5) = 0$$

$\lambda = 1$, $\lambda = 3$, $\lambda = 5$

$\therefore$ Diagonalizable.

m_i	1	1	1
λ	1	3	5
v	$\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$
a_i	1	1	1
g_i	1	1	1

Eigen values are dif^{nt} $\therefore$ L.I.

a_i = algebraic multiplicity of λ_i
= power of $(x-\lambda_i)$ in $f(x)$ (How many times λ_i occurs)

g_i = geometric multiplicity of λ_i
= no. of L.I. eigen vectors associated with λ_i
= $\dim(V_{\lambda_i})$ = dimension of eigenspace of λ_i

m_i = minimal multiplicity of A
= power of $(x-\lambda_i)$ in the minimal polynomial of A

How to find m_i?

Thrm: minimal polynomial $g(x)$ divides characteristic poly. $f(x)$

$f(x) = (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)$

$g(x) = (x-\lambda_1)$

Theorem says;
$g(x)$ & $f(x)$ have the same roots

Ex: $f(x) = (x-1)(x-3)(x-5)$

$g(x) = (x-1)^1(x-3)(x-5)$

only possibility

Eg: $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ $\quad |A - \lambda I| = (1-\lambda)^2(3-\lambda) = 0$

$\lambda = 1$ & $\lambda = 3$

$f(x) = (x-1)^2(x-3)$

$g(x) = (x-1)(x-3)$ or $g(x) = (x-1)^2(x-3)$

minimal poly

$g(A) = (A - I)(A - 3I)$

$= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -2 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\therefore g(x) = (x-1)^1(x-3)^1$

$(A - \lambda I)x = 0$

$\left[\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} - 1\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$

$= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$\downarrow$

$\cancel{x_1 + x_2 + x_3 = 0}$

$= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\downarrow$

$= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$x_1 + x_2 + x_3 = 0$

$x_1 = -x_2 - x_3$

$x = \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix}$

$= x_2\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$x = \text{span}\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}$

λ	1	3
v	$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$
a_i	2	1
g_i	2	1
m_i	1	1

degree in char. poly.

How many eigen values are associated with each λ

Ex: $A = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$, $|A - \lambda I| = (3-\lambda)^2(1-\lambda)$

$\lambda = 3$, $\lambda = 1$

$f(x) = (x-1)(x-3)^2 = 0$

$g(x) = (x-3)(x-1)$ or $g(x) = (x-3)^2(x-1)$

- $g(A) = (A - 3I)(A - I)$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \neq 0$$

- $g(A) = (A - 3I)^2 (A - I) = 0$

~~$g(x) = (x-3)^2(x-1)$~~

λ_i	1	3
v_i	$\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$
a_i	1	2
g_i	1	1
m_i	1	2

* Cond^{ns} for diagonalizability;

$a_i = g_i$ $(m_i = 1)$

(All Eigen vectors should be L.I)

Quiz

Applic^n of matrix diag^{lzn}

Solve the coupled first order sys^m of ODEs.

$$\dot{x} = -11x - 10y + 5z + 1$$
$$\dot{y} = 5x + 4y - 5z + 2$$
$$\dot{z} = -20x - 20y + 4z + 3$$
$$y(0) = x(0) = z(0) = 0$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -11 & -10 & 5 \\ 5 & 4 & -5 \\ -20 & -20 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\dot{X} = A\cancel{X} + b$$
$$\dot{X} = P\Lambda P^{-1} X + b$$
$$P^{-1}\dot{X} = \Lambda P^{-1} X + P^{-1} b$$
$$(P^{-1}X)^{\cdot} = \Lambda P^{-1} X + P^{-1} b$$
$$\dot{Y} = \Lambda Y + C$$

$$\Lambda = \begin{pmatrix} -6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad P = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

$$Y = \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \quad \cancel{\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} -6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}}$$

$$\cancel{+ \begin{pmatrix} 1 & -2 & -3 \\ 1 & -2 & -2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}$$

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} -6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} 1 & 1 & 0 \\ -2 & -2 & 1 \\ -3 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} 3 \\ -3 \\ -4 \end{pmatrix}$$

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$\dot{u} = -6u+3$ — (1) $\quad \dot{v} = 4v-3$ — (2)

$\dot{w} = -w-4$ — (3)

(1) : $\dot{u}+6u = 3$, $I = e^{\int 6\,dt} = e^{6t}$

$$\frac{d}{dt}\left(ue^{6t}\right) = 3e^{6t}$$

$$ue^{6t} = \int 3e^{6t}\,dt = 3\frac{e^{6t}}{6} + a$$

$$u = \frac{1}{2} + ae^{-6t}$$

$$Y = P^{-1}X$$

$$Y(0) = P^{-1}(X(0))$$

$$\begin{pmatrix} \frac{1}{2}+a \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ -2 & -2 & 1 \\ -3 & -2 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$u(0) = 0 = \frac{1}{2} + a \Rightarrow a = -\frac{1}{2}$$

$$u(t) = \frac{1}{2} - \frac{1}{2}e^{-6t}$$

(2) : $\dot{v} - 4v = -3$, $I = e^{\int -4\,dt} = e^{-4t}$

$$\frac{d}{dt}\left(ve^{-4t}\right) = -3e^{-4t}$$

$$ve^{-4t} = -\int 3e^{-4t} = \frac{-3e^{-4t}}{-4} + b$$

$$v = \frac{3}{4} + be^{+4t}$$

$$\begin{pmatrix} \cancel{\frac{3}{4}+b}\,u \\ \frac{3}{4}+b \\ w \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ -2 & -2 & 1 \\ -3 & -2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\frac{3}{4} + b = 0 \Rightarrow b = -\frac{3}{4}$$

$$v(t) = \frac{3}{4} - \frac{3}{4} e^{4t}$$

(3) : $\dot{w} + w = -4$, $I = e^{\int 1\,dt} = e^{t}$

$$w = -4 + c e^{-t}$$

$$\begin{pmatrix} u \\ v \\ -4+c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$c = 4$$

$$w(t) = -4 + 4e^{-t}$$

$$x = PY$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} - \frac{1}{2}e^{-6t} \\ 3/4 - 3/4\, e^{4t} \\ -4 + 4e^{-t} \end{pmatrix}$$

$$x = \frac{3}{4} - \frac{3}{4}e^{4t} + 4 - 4e^{-t}$$

$$= \frac{19}{4} - \frac{3}{4}e^{4t} - 4e^{-t}$$

$$y = -\frac{11}{4} - \frac{1}{2}e^{-6t} + \frac{3}{4}e^{4t} + 4e^{-t}$$

$$z = \frac{7}{4} - e^{-6t} - \frac{3}{4}e^{4t}$$

Eigenvalues / vectors for L.T. $T: V \longrightarrow V$ over F

$$T(v_i) = \lambda_i v_i \; ; \; \lambda_i \in F, \; v_i \in V - \{0\}$$

Eigen space of λ_i

$$V_{\lambda_i} = \{\text{set of all eigen vectors for } \lambda_i\} \cup \{\underline{0}\}$$
$$= \{v_i \in V \mid T(v_i) = \lambda_i v_i\}, \; T(\underline{0}) = \underline{0} = \lambda(\underline{0})$$
$$= \{v_i \in V \mid (T - \lambda_i I)(v_i) = \underline{0}\}$$
$$= \ker(T - \lambda_i I) = \text{span } B_i$$

V_{λ_i} is a subspace of V (if λ_i is existing). So it has a basis & the basis's elements ($\neq \underline{0}$) are the eigenvectors. No. of elements in the basis, $\dim V_{\lambda_i} = \dim(\ker(T - \lambda_i I)) = \text{null}(T - \lambda_i I)$ is called the geometric multiplicity of λ_i (or g_i). This is the no. of L.I. eigenvectors for λ_i.

Characteristic eq^n

$$f(x) = \det(xI - A) \qquad f(\lambda_i) = 0$$
$$= \cdots (x - \lambda_i)^{a_i} \cdots$$

a_i := algebraic multiplicity.

$$a_i \geq g_i \geq 1$$

Ex

$$Av = \lambda v$$
$$\Rightarrow A(Av) = A(\lambda v)$$
$$\Rightarrow A^2 v = \lambda(Av) = \lambda(\lambda v)$$
$$\Rightarrow A^2 v = \lambda^2 v$$

$$A^n v = \lambda^n v \quad ; n \in \mathbb{Z}^+ \cup \{0\}$$

If A is invertible;

$$A^n v = \lambda^n v \quad \forall n \in \mathbb{Z}$$

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If $p(x)$ is a polynomial

$$p(A)v = p(\lambda)v$$

$$Iv = 1 \cdot v$$
$$Av = \lambda v$$
$$A^2 v = \lambda^2 v$$
$$\vdots$$

$$(2x^2 + 3A + 4I)v$$
$$= (2\lambda^2 + 3\lambda + 4)v$$
$$\therefore p(A)v = p(\lambda)v$$

Def^n: Minimal polynomial $g(x)$ of $A \in \mathbb{C}^{n \times n}$ satisfies

1) $g(x)$ is monic (leading coefficient is 1)
$\Rightarrow g(x) \not\equiv 0$
2) $g(A) = \mathbb{O}$
3) If $h(x)$ is a non-zero ($h(x) \not\equiv 0$) polynomial with $h(A) = \mathbb{O}$, then $\deg h \geq \deg g$

Note: Cayley-Hamilton: $f(A) = \mathbb{O}$

$$\underset{\text{charact.}}{\deg f} \geq \underset{\text{minimal}}{\deg g}$$

Th^{rm}: $f(x)$ & $g(x)$ has the same roots (factors)

Proof: $f(\lambda) = 0$

$\Rightarrow Av = \lambda v \; ; \; v \neq 0$

$\Rightarrow \underbrace{g(A)}_{=0} v = g(\lambda) v$

$\underline{0} = g(\lambda) \cdot v \neq \underline{0}$

$\Rightarrow g(\lambda) = 0$

$g(\lambda) = 0$

$\Rightarrow g(x) = (x - \lambda) h(x) \quad ; \deg h < \deg g$

$g(A) = (A - \lambda I) h(A) \quad ; h(A) \not\equiv \mathbb{O}$

$\mathbb{O} = (A - \lambda I) \begin{pmatrix} \cdots \\ \vdots \end{pmatrix}$

$\mathbb{O} = (A - \lambda I) u$

$\Rightarrow \quad Au = \lambda u \; ; \; u \neq 0.$
$\Rightarrow \quad f(\lambda) = 0$

~~Ex: $f(x) = (x-1)^4 (x-2)$~~
~~$g(x) = (x-1)(x-2)$~~

<u>Th$^{\underline{rm}}$</u>: $g(x)$ divides $f(x)$

Proof: $\deg f \geq \deg g$

$f(x) = a(x)\, g(x) + b(x) \; , \quad \deg b \leq \deg g$

$f(A) = a(A)\, g(A) + b(A)$
$\mathbb{O} = a(A)\, \mathbb{O} + b(A)$
$b(A) = \mathbb{O}$

If $b(A) \neq 0$, that is a contradic$^{\underline{n}}$ since we have $b(A) = 0$ with $\deg b < \deg g$.

So only possibility is $b(x) \equiv 0$

$\therefore \; f(x) = a(x)\, g(x) + 0$
$\therefore \; g(x)$ divides $f(x)$.

<u>Note</u>.

$f(x) = \cdots (x - \lambda_i)^{a_i} \cdots$
$g(x) = \cdots (x - \lambda_i)^{m_i} \cdots \quad 1 \leq m_i \leq a_i$
m_i = minimal multiplicity of λ_i.
a_i = ~~alg~~ algebraic " " "
g_i = geometric " " " $= \dim V_{\lambda_i}$

$a_i - g_i \geq m_i - 1 \geq 0$

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Thm: The Following Are Equivalent (TFAE) For $A \in \mathbb{C}^{n\times n}$;

1) A is diagonalizable
2) $a_i = g_i \quad \forall i$
3) $m_i = 1 \quad \forall i$
4) $\mathbb{C}^n = V_{\lambda_1} \oplus V_{\lambda_2} \oplus V_{\lambda_3} \oplus \cdots$ direct sum

$\Rightarrow$ Union of all bases will be a basis for $\mathbb{C}^n$.

$\Rightarrow$ We have n L.I. eigenvectors.

Def^n: A is diagonalizable iff we have an invertible matrix $P \in \mathbb{C}^{n\times n}$ and a diagonal matrix $\Lambda \in \mathbb{C}^{n\times n}$ s.t. $A = P\Lambda P^{-1}$.

We find P, Λ by Eigenvalue decompos^n.

i.e. $A(v_1 v_2 \dots v_n) = (v_1 v_2 \dots v_n)\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ & & & \lambda_n \end{pmatrix}$

$$AP = P\Lambda$$

λ_1 can be equal to λ_2 as long as v_1 & v_2 are not equal

~~(i.e. P is invert.~~

P is invertible iff v_i are L.I.

Def^n: Let $A \in \mathbb{C}^{n\times n}$

1) Hermitian of A is $A^H = \bar{A}^T = \overline{A^T}$
2) A is Hermition iff $A = A^H$ ($\Leftrightarrow \bar{a}_{ij} = a_{ji}$)
3) A is symmetric iff $A = A^T$

If $A \in \mathbb{R}^{n\times n}$ & A is symmetric

$\Rightarrow A = \bar{A}^T = \overline{A^T} = A^H$

4) A is ~~unitary~~ unitary iff $A^{-1} = A^H$

i.e. $AA^H = A^HA = I$

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<u>Th^{rm}</u>: Let $A \in \mathbb{C}^{n\times n}$ can be Hermition (i.e $A = A^H$) then,

1) $\lambda_i \in \mathbb{R}$
2) $\lambda_i \neq \lambda_j \Rightarrow \langle v_i, v_j \rangle = 0$ i.e. v_i, v_j are orthogonal

$\Rightarrow$ v_i, v_j are L.I. $(v_i, v_j \neq \underline{0})$

<u>Note</u>.

$$\left\langle \underbrace{\frac{v_i}{\|v_i\|}}_{\omega_i}, \underbrace{\frac{v_j}{\|v_j\|}}_{\omega_j} \right\rangle = 1$$

we have $\langle \omega_i, \omega_j \rangle = 0$ & $\|\omega_i\| = \|\omega_j\| = 1$

$\Rightarrow$ ω_i, ω_j are orthonormal

3) A is diagonalizable (A will have n orthogonal eigenvecors) $a_i = g_i$

spectral th^{rm}

Proof of 1) & 2)

$Av = \lambda v, \ v \neq \underline{0}$

$Au = \mu u, \ u \neq \underline{0}$

Consider $\bar{\lambda} \langle v, u \rangle$
$= \bar{\lambda} v^H u$
$= v^H A u$
$= v^H \mu u = \mu v^H u$
$= \mu \langle v, u \rangle$

$Av = \lambda v$

$(Av)^H = (\lambda v)^H$

$v^H A^H = v^H \lambda^H$

$v^H A = \bar{\lambda} v^H$

$(\bar{\lambda} - \mu) \langle v, u \rangle = 0$

$\rightarrow \rightarrow$

If $\lambda = \mu$ $(v = u)$

$$(\bar{\lambda} - \lambda)\langle v, v\rangle = 0$$

$$(\bar{\lambda} - \lambda)\|v\|^2 = 0$$

$$\Rightarrow (\bar{\lambda} - \lambda) = 0 \Rightarrow \lambda = \bar{\lambda} \in \mathbb{R}$$

Now we have $(\lambda - \mu)\langle v, u\rangle = 0$.

If $\lambda \neq \mu \Rightarrow \langle v, u\rangle = 0$

Diagonalizing Hermition matrices.

$$AP = P\Lambda$$

$$(v_1\ v_2 \cdots v_n) = (v_1 v_2 \cdots v_n)\Lambda$$

$$\downarrow$$

$P \equiv (\omega_1\ \omega_2 \cdots \omega_n) \leftarrow \omega_i$ are orthonormal

$$\omega_i = \frac{v_i}{\|v_i\|}$$

$$A = P\Lambda P^{-1}$$

$$P^H P = \begin{pmatrix} \omega_1^H \\ \omega_2^H \\ \vdots \\ \omega_n^H \end{pmatrix} (\omega_1\ \omega_2 \cdots \omega_n) = \underbrace{\begin{pmatrix} \|\omega_1\|^2 & 0 & \cdots & 0 \\ 0 & \|\omega_2\|^2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & & \|\omega_n\|^2 \end{pmatrix}}_{= I}$$

$$P^H P = I = P P^H$$

$$\Rightarrow P^{-1} = P^H$$

P is unitary

$$A \operatorname{adj} A = I|A|$$

$$\operatorname{adj} A = (C_{ij})^T$$

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Def^n

Let V be a V.S. over F & $T: V \to F$ be a L.T (F is a v.s. over F). These are called functionals (covectors in physics). Collection of all functionals ~~over~~ on V is called the Dual space V^* of V. V^* is a v.s. over F. So it will have a basis too.

Eg: $V = \mathbb{R}^3$, $F = \mathbb{R}$.

A basis of V^* is $\{e^1, e^2, e^3\}$ with the property that for the basis $\{\underset{\underset{\underline{i}}{\shortparallel}}{e_1}, \underset{\underset{\underline{j}}{\shortparallel}}{e_2}, \underset{\underset{\underline{k}}{\shortparallel}}{e_3}\}$ of $V = \mathbb{R}^3$, we have

$$e^1(e_1) = 1, \quad e^1(e_2) = 0, \quad e^1(e_3) = 0.$$

In general, ~~$e^j(e_i) = \delta_i^j$~~

$$e^j(e_i) = \delta_i^j = \begin{cases} 1 & ; i = j \\ 0 & ; i \neq j \end{cases}$$

is the Kronker Delta Function.

Ex: Einstein Notation

$$\alpha^i \beta_i \text{ means } \sum_i \alpha^i \beta_i$$

Def^n Quadratic form

$$f = X^H A X, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$(= a_{11}x_1^2 + a_{12}x_1x_2 + \dots)$?

Def^n: $A \in \mathbb{C}^{n \times n}$ is

1) Positive definite ($A > 0$) iff $X^H A X > 0 \ \forall X \neq \underline{0}$

) Positive semidefinite ($A \geq 0$) iff $X^H A X \geq 0 \ \forall X \neq \underline{0}$

3) negative definite.
4) negative semidefinite

If A is Hermitian ($A = A^H$)

$$f = x^H A x$$
$$= x^H P \Lambda P^H x$$
$$= (P^H x)^H \Lambda (P^H x)$$
$$= Y^H \Lambda Y$$
$$= \lambda_1 |y_1|^2 + \lambda_2 |y_2|^2 + \dots + \lambda_3 |y_n|^2 > 0 \quad \text{if } \lambda_1 > 0$$

$$Y = P^H X = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \qquad X \neq \underline{0}$$

P contains eigen vectors

$$= \begin{pmatrix} v_1^H \\ v_2^H \\ \vdots \\ v_n^H \end{pmatrix} X$$

So $A > 0$ if $\lambda_i > 0$

In quiz 14:

$$f = 2x^2 + 12xy + y^2 - 4xz - 8yz - 3z^2$$
$$= 9y_1^2 - 6y_2^2 - 3y_3^2 = 0$$

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = P^H X = \begin{pmatrix} v_1^H \\ v_2^H \\ v_3^H \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

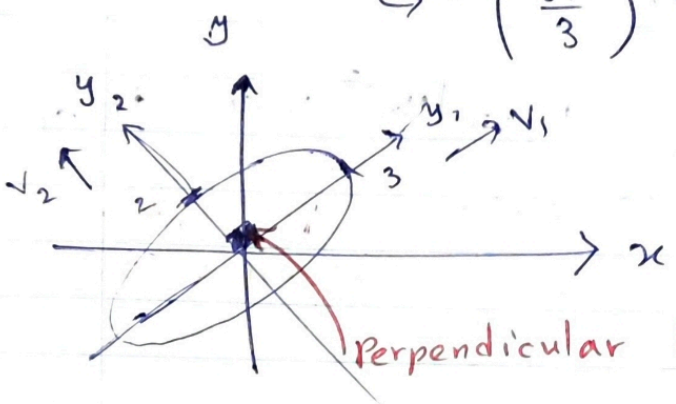
$y_2 - y_3$ plane $\Leftrightarrow y_1 = 0 \Leftrightarrow 0 = v_1^H x \Leftrightarrow v_1 \perp // y_1$

In 2D:

$$f(x,y) = ax^2 + bxy + cy^2$$
$$= \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
$$= X^H A X$$

Ex:

$$\hookrightarrow = \left(\frac{y_1}{3}\right)^2 + \left(\frac{y_2}{2}\right)^2 = 1$$



Perpendicular because eigen vectors are orthogonal

Ex: $A = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

λ	1	3	3
v	$\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$	$\cancel{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}$
a_i	1	2	
g_i	1	1	
m_i	1	2	

$Av_1 = 1V_1$

$AV_3 = 3V_3$

$$A - 3I = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{aligned} x &= 0 \\ -y + z &= 0 \\ y &= z \end{aligned}$$

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$$v_3 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ z \\ z \end{pmatrix} = z \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\text{null}(A - 3I) = 3 - \text{rank}(A - 3I) \quad \left(\begin{matrix}\text{rank} \\ \text{nullity} \\ \text{thm}\end{matrix}\right)$$
$$= 1$$

$$(A - 3I)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{0} = (A - 3I)^2 w = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Leftrightarrow y - z = 0$$
$$y = z$$

$$w = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ y \end{pmatrix} = \underbrace{x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{w_3} + \underbrace{y \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}_{v_3}$$

$$(A - 3I) v_3 = \underline{0} \Rightarrow (A - 3I)^2 v_3 = \underline{0}$$

$$(A - 3I) w_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & +1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = v_3$$

$$\therefore \quad A v_1 = 1 v_1$$
$$A v_3 = 3 v_3$$
$$A w_3 = 3 w_3 + v_3$$

$$A \underbrace{(v_1 \quad v_3}_{L.I} \quad w_3) = (v_1 \; v_2 \; w_3) \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}}_{J}$$

$$AP = PJ$$
$$A = PJP^{-1} \quad (\text{Jordan Decompos}^{n})$$

when A is not diagonalizable

$$J = \begin{pmatrix} \boxed{1}_{1\times 1} & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}_{2\times 2}$$

Eigen value decompos^n is a special case of Jordan decompos^n.

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Minimal polynomiale $g(x) = (x-3)^2(x-1)$

$$(A - 1I)v_1 = \underline{0}$$

$$\Rightarrow (A-3I)^2(A-1I)v_1 = \underline{0} \quad \text{—} (1)$$

$$\cancel{(A-1I)v_3 = Av_3 = v}$$

$$(A-1I)v_3 = Av_3 - v_3 = 3v_3 - v_3 = 2v_3$$

$$(A-3I)(A-1I)v_3 = (A-3I)(2v_3) = \underline{0}$$

$$\Rightarrow (A-3I)^2(A-1I)v_3 = \underline{0} \quad \text{—} (2)$$

$$(A-1I)w_3 = Aw_3 - w_3 = 3w_3 + v_3 - w_3 = v_3 + 2w_3$$

$$(A-3I)(A-1I)w_3 = (A-3I)(v_3 + 2w_3)$$
$$= 0 + 2(v_3)$$

$$(A-3I)^2(A-1I)w_3^2 = (A-3I)2v_3 = \underline{0} \quad \text{—} (3)$$

$$(A-3I)^2(A-1I)\underbrace{(v_1\ v_3\ w_3)}_{P} = (\underline{0}\ \underline{0}\ \underline{0})$$

$$\Rightarrow (A-3I)^2(A-1I) = 0$$

Example 22

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Ex: $A = \begin{pmatrix} 2 & 0 & 0 & 0 & 3 & 3 \\ 0 & 2 & 0 & 3 & 3 & 3 \\ 0 & 0 & 2 & 3 & 3 & 3 \\ 0 & 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$ $\quad |A - \lambda I| = \begin{pmatrix} 2-\lambda & & & & & \\ & 2-\lambda & & & & \\ & & 2-\lambda & & & \\ & & & 2-\lambda & & \\ & & & & 2-\lambda & \\ & & & & & 2-\lambda \end{pmatrix}$

$= (2-\lambda)^6$

Charac. poly: $f(x) = (x-2)^6$, $\quad a_2 = 6$

$$A - 2I = \begin{pmatrix} 0 & 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\underline{0} = (A - 2I)v = \begin{pmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} \iff v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 = 0 \\ x_5 = 0 \\ x_6 = 0 \end{pmatrix}$$

$$\text{span}\left\{ \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\bar{v}_1}, \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\bar{v}_2}, \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\bar{v}_3} \right\}$$

$$(A - 2I)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\underline{0} = (A - 2I)^2 w = \begin{pmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

$$w = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 = 0 \end{pmatrix} \in \text{span}\left\{ \bar{v}_1, \bar{v}_2, \bar{v}_3 \right\} + \left\{ \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{\bar{w}_1}, \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{\bar{w}_2} \right\}$$

$$(A - 3I)^3 = 0$$

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$$\cancel{(A-2I)^2} = \cancel{(A-2I)^3} = 0 = (A-2I)^3 u$$

$$= 0 \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

$$u = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} \in \text{span}\{\bar{v}_1, \bar{v}_2, \bar{v}_3\} \oplus \text{span}\{\bar{w}_1, \bar{w}_2\}$$

$$\oplus \underbrace{\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix}}_{u_1}$$

$$(A-2I)u_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \\ 0 \\ 0 \end{pmatrix} = w_1$$

$$\Rightarrow Au_1 = 2u_1 + w_1$$

$$(A-2I)w_1 = \begin{pmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 9 \\ 9 \\ 0 \\ 0 \\ 0 \end{pmatrix} = v_1$$

$$Aw_1 = 2w_1 + v_1$$

$$w_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$(A-2I)w_2 = \begin{pmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 0 \\ 0 \\ 0 \end{pmatrix} = v_2$$

$$Aw_2 = 2w_2 + v_2$$

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

Finally

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & \end{pmatrix} \leftarrow \begin{cases} Av_1 = 2v_1 \\ Aw_1 = 2w_1 + v_1 \\ Au_1 = 2u_1 + w_1 \end{cases} \quad \begin{matrix} Av_2 = 2v_2 \\ Aw_2 = 2w_2 + v_2 \\ Av_3 = 2v_3 \end{matrix} \rightarrow (2)$$

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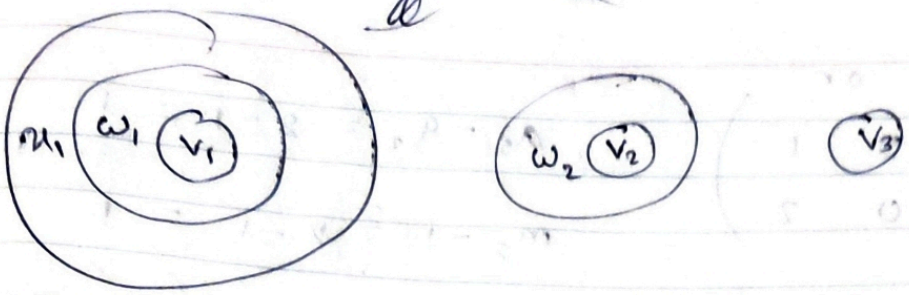

$$A\begin{pmatrix} v_3 & v_2 & w_2 & v_1 & w_1 & u_1 \end{pmatrix} = \begin{pmatrix} v_3 & v_2 & w_2 & v_1 & w_1 & u_1 \end{pmatrix}\begin{pmatrix} \boxed{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$$AP = PJ$$
$$A = PJP^{-1}$$

$\lambda_i = 3$ $\quad a_i = 6 = 1+2+3 = \sum_i$ sizes of Jordan blocks

$g_i = 3 =$ no. of Jordan blocks

$m_i = 3 =$ ~~size of~~ max size of a Jordan block

$g(x) = (x-2)^3$

Th^{rm}: For λ_i we have $a_i - g_i \geq m_i - 1$

Proof: Let the Jordan blocks have sizes $k_1, k_2, \dots, k_r \geq 1$

$$a_i = \sum_{j=1}^{r} k_j, \quad g_i = r, \quad m_i = \max\{k_j\} = k_\ell \text{ (say)}$$

$$a_i - g_i = \sum_{j=1}^{r} k_j - r = \sum_{j=1}^{r} k_j - \sum_{j=1}^{r} 1 = \sum_{j=1}^{r} (k_j - 1) \geq k_\ell - 1 = m_\ell - 1$$

* For 3×3 or lesser, we have $a_i - g_i = m_i - 1$

Eg: $J = \begin{pmatrix} \boxed{2} & 0 \\ 0 & \boxed{3} \end{pmatrix}$ $\quad a_2 - g_2 = 1 - 1 = 0$

$m_2 - 1 = 1 - 1 = 0$

or

$J = \begin{pmatrix} \boxed{2} & 0 \\ 0 & \boxed{2} \end{pmatrix}$ $\quad a_2 - g_2 = 2 - 2 = 0$

$m_2 - 1 = 1 - 1 = 0$

or

$$J = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \qquad a_2 - g_2 = 2 - 1 = 1$$

$$m_2 - 1 = 2 - 1 = 1$$

~~Quiz 15~~

* A square matrix 'A' is diagonalisable $\Leftrightarrow$ there is an invertible X with $X^{-1}AX = D$ (diagonal matrix
* $A = XDX^{-1}$ (Matrix decompos^{n})
Only works if A is diagonalisable

Also include $\mathbb{R}^{n\times n}$

* $A \in \mathbb{C}^{n\times n}$. There is alway a J for A. (J is not unique).
$A = XJX^{-1}$

J could be a diagonal matrix. But in general it is not.

* Usually, select Jordon blocks according to the algebraic multiplicities.

* If diagonalizable: $a_i = g_i$

* Note

$$\det(A) = \prod \lambda_i$$

Proof: $\Rightarrow f(x) = \det(xI - A) = (x-\lambda_1)(x-\lambda_2)\dots(x-\lambda_n)$

$$f(0) = \det(0I - A) = (-\lambda_1)(-\lambda_2)\cdots(-\lambda_n)$$

$$(-1)^n \det(A - 0) = (-1)^n(\lambda_1)(\lambda_2)\dots(\lambda_n)$$

$$\therefore \det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$$