EE352 Automatic Control

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Contents

- Complex variable concept
- Differential Equations
- Laplace Transforms
- Impulse Response and Transfer Function of Linear Systems
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Complex variable

- The complex variable *s* has two components as Real $\Re$ and Imaginary $\Im$ which are quadrature to each other. It is presented as

$$s = \sigma + j\omega$$

where *j* is the complex operator and $\sigma, \omega \in \Re$.

- A function *G* is said to be a function of *s* if for every value of *s*, there exist one or more corresponding values of *G(s)* such that

$$G(s) = \Re G(s) + j\Im G(s)$$

- If for every value of *s* there is only one corresponding value of *G(s)* in the *G(s)* plane, then the function *G(s)* is said to be a **Single Valued Function**.
- For example, $G(s) = \frac{1}{s(s+1)}$ has only one value of *G(s)* for every *s*.

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Analytic function

- A function *G(s)* of the complex variable *s* is called an analytic function in the region of *s - plane* if the function and all its derivatives exist in the region.
- For example, $G(s) = \frac{1}{s(s+1)}$ is analytic at every point in the *s-plane* except at *s = 0* and *s = -1*.
- On the other hand, *G(s) = s + 2* function is analytic at every point in the finite *s-plane*.
- **Singularities** of a function are the points in the *s - plane* at which the function or its derivatives do not exist.
- For example, $G(s)$=1/s(s+1) has singularities at *s = 0* and *s = -1*

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Poles and Zeros of a function

- If a function *G(s)* is analytic and single valued in the neighborhood of s_i, it is said to have a **Pole** of order *r* if the limit

  $$\lim_{s \to s_i} (s - s_i)^r G(s) \text{ has finite non-zero value.}$$

- In otherwords the denominator of $G(s) = \frac{Q(s)}{P(s)}$ i.e., *P(s)* must have $(s - s_i)^r$ so that $G(s) \to \infty$
- If $r = 1$ then s_i is called ***Simple Pole***.
- If a function *G(s)* is analytic and single valued in the neighborhood of s_i, it is said to have a **Zero** of order *r* if the limit

  $$\lim_{s \to s_i} (s - s_i)^{-r} G(s) \text{ has finite non-zero value.}$$

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Worked example

- Consider the system described by the s – function $G(s) = \dfrac{10(s+2)}{s(s+1)(s+3)^2}$

$\lim_{s\to 0}(s-0)^1 G(s) = 20/9$, Hence *s = 0* is a **Simple Pole**.

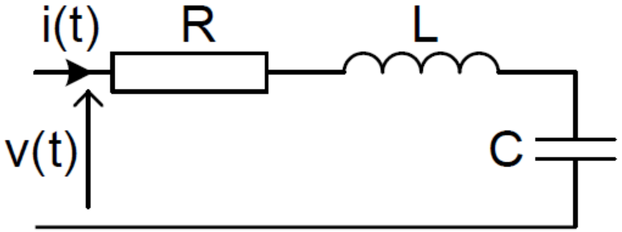
$\lim_{s\to -1}(s+1)^1 G(s) = -10/4$, Hence *s = -1* is a **Simple Pole**.

$\lim_{s\to -3}(s+3)^2 G(s) = -5/3$, Hence *s = -3* is a **Double Pole**.

$\lim_{s\to -2}(s+2)^{-1} G(s) = 5$, Hence *s = -2* is a **Zero of order 1**.

Linear Ordinary Differential Equations

- A wide range of engineering systems involve continuously varying signals which are modeled by differential equations.
- They generally involve derivatives and integrals of dependent variables with respect to the independent variables.
- For example, in the series RLC network case: The integro-differential equation describing the relationship between the applied voltage and the corresponding current can be given as



$$v(t) = Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_0^t i(t)dt$$

where *t,* i.e. the *time*, is the independent variable, *i(t) current* is the dependent variable and *v(t) i.e., input* is the forcing function.

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ODE contd.,

- The integro-differential equation can be converted to a second order differential equation by differentiating both sides, which gives

$$\frac{dv(t)}{dt} = L\frac{d^2 i(t)}{dt} + R\frac{di(t)}{dt} + \frac{1}{C}i(t)$$

- Generally, an n^th order linear system with constant coefficients can be represented as

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + a_{n-2}\frac{d^{n-2}y(t)}{dt^{n-2}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) = \mathrm{f}(t)$$

f(t) can be *u(t)* or it can also be a differential polynomial.

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With *f(t)* as a differential polynomial

- $\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + a_{n-2}\frac{d^{n-2}y(t)}{dt^{n-2}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) = b_m\frac{d^m u(t)}{dt^m} + b_{m-1}\frac{d^{m-1}u(t)}{dt^{m-1}} + b_{n-2}\frac{d^{m-2}u(t)}{dt^{m-2}} + \cdots + b_1\frac{du(t)}{dt} + b_0 u(t)$
- With all zero initial conditions, by taking Laplace transforms
- $G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0}$
- Singularities of $G(s)$ can be found by solving

$$s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0 = 0$$

- Hence, it is called the Characteristic Equation (CE)

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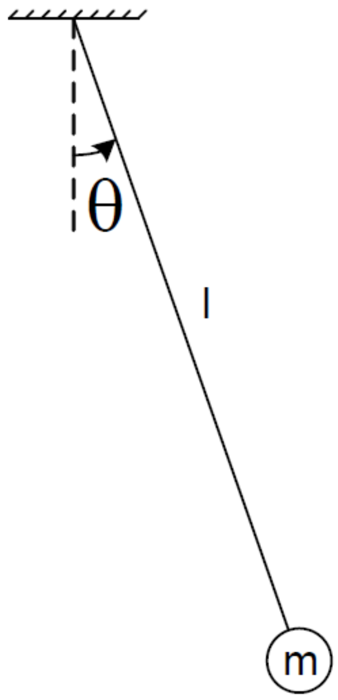
Nonlinear Differential Equations

- Many physical systems are nonlinear and they must be described by nonlinear differential equations.
- Consider a simple pendulum: The rotational motion can be described by $ml^2\ddot{\theta} + mgl\sin\theta = 0$
- Which results in the differential equation

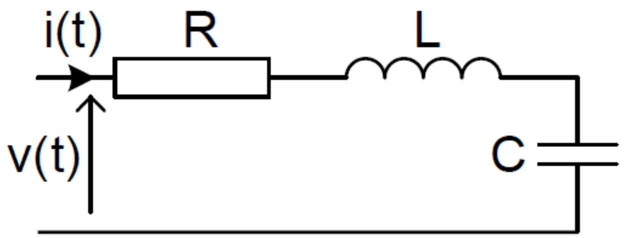
$$\frac{d^2\theta(t)}{dt^2} + \frac{g}{l}\sin\theta(t) = 0$$

- Since θ appears as a sinusoidal function, this system is non-linear.



State Equations

- In general, an n^{th} order differential equation can be decomposed in to n first order differential equations.
- They are simpler to solve and used very much in analytical studies in Control Systems.
- Consider the RLC circuit



$$v(t) = Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_0^t i(t)dt$$

- By taking $x_1(t) = \int_0^t i(t)dt$ and $x_2(t) = i(t)$

$$v(t) = Rx_2(t) + L\frac{dx_2(t)}{dt} + \frac{1}{C}x_1(t)$$

State Equations contd.,

- Augmenting both equations in to one matrix gives

$$\begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-1}{LC} & \frac{-R}{L} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} v(t),$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

which is in the standard **STATE SPACE** format

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t),$$

$$y(t) = Cx(t)$$

Where State matrix $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, System matrix A $= \begin{bmatrix} 0 & 1 \\ \frac{-1}{LC} & \frac{-R}{L} \end{bmatrix}$, Input matrix B $= \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}$,

Output matrix C $= \begin{bmatrix} 0 & 1 \end{bmatrix}$

n^{th} order system

- Consider $\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + a_{n-2}\frac{d^{n-2}y(t)}{dt^{n-2}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) = \mathrm{f}(t)$
- By selecting

$$x_1(t) = y(t)$$

$$x_2(t) = \frac{dy(t)}{dt}$$

$$x_3(t) = \frac{d^2y(t)}{dt^2}$$

$$\vdots$$

$$x_n(t) = \frac{d^{n-1}y(t)}{dt^{n-1}}$$

- The above n^{th} order differential equation can be reduced to an n number of first order differential equations

n^{th} order system contd.,

$$\frac{dx_1(t)}{dt} = x_2(t)$$

$$\frac{dx_2(t)}{dt} = x_3(t)$$

$$\frac{dx_3(t)}{dt} = x_4(t)$$

$$\vdots$$

$$\frac{dx_n(t)}{dt} = -a_0x_1(t) - a_1x_2(t) - a_2x_3(t) \ldots - a_{n-1}x_n(t) + f(t)$$

n^{th} order system contd.,

- By augmenting the n number of first order differential equations in to one matrix equation

$$\begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \\ \frac{dx_3(t)}{dt} \\ \vdots \\ \frac{dx_n(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} f(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

Where it is in the State Space form $\frac{dx(t)}{dt} = Ax(t) + Bu(t)$,

$$y(t) = Cx(t)$$

n^th order system contd.,

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$C = [1 \quad 0 \quad 0 \quad 0 \quad 0]$$

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State definitions

- In Control Theory, a set of first order differential equations derived as above are called **State Equations**.
- $x_1(t), x_2(t), x_3(t) \cdots x_n(t)$ are called **State Variables**.
- **State Variables** is the minimum set of variables needed to define the past present and future conditions of the system.

For a variable to become a state:

- At any initial time $t = t_0$ the state variables $x_1(t_0), x_2(t_0),$ $x_3(t_0) \cdots x_n(t_0)$ should define the initial state of the system.
- Once the input of the system for $t \geq t_0$ and the initial states defined above are specified, the state variables should completely define the future behavior of the system.
- The output of a system is a variable that can be measured, but a state variable does not always satisfy this requirement. For example, in an electric motor such state variables as winding current, rotor velocity, displacement, etc., can be measured physically and all these variables qualify as outputs.
- But on the other hand, the magnetic flux can also be a state while it can not be measured directly. Hence, we generally do not consider flux as an output.

Example

Consider the transfer function

$$\frac{Y(s)}{U(s)} = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} = \frac{1}{s^3 + a_2 s^2 + a_1 s + a_0} . b_2 s^2 + b_1 s + b_0$$

$$= \frac{W(s)}{U(s)} \frac{Y(s)}{W(s)}$$

$$\frac{W(s)}{U(s)} = \frac{1}{s^3 + a_2 s^2 + a_1 s + a_0} \Rightarrow \dddot{w} = -a_2 \ddot{w} - a_1 \dot{w} - a_0 w + u$$

$$\frac{Y(s)}{W(s)} = b_2 s^2 + b_1 s + b_0 \Rightarrow y = b_2 \ddot{w} + b_1 \dot{w} + b_0 w$$

Selecting the states as

$$x_1 = w; x_2 = \dot{w}; x_3 = \ddot{w}$$

State equations are

$$\dot{x_1} = x_2;$$

$$\dot{x_2} = x_3;$$

$$\dot{x_3} = \dddot{w} = -a_2 \ddot{w} - a_1 \dot{w} - a_0 w + u = -a_2 x_3 - a_1 x_2 - a_0 x_1 + u$$

$$y = b_2 \ddot{w} + b_1 \dot{w} + b_0 w = b_2 x_3 + b_1 x_2 + b_0 x_1$$

Example contd.,

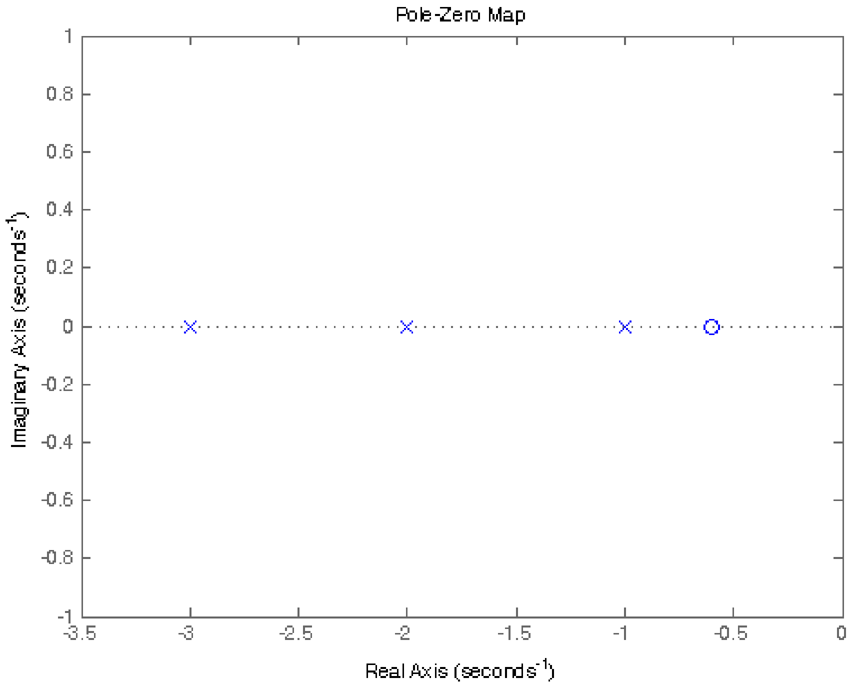
State space system

$$\dot{x_1} = x_2;$$
$$\dot{x_2} = x_3;$$
$$\dot{x_3} = \dddot{w} = -a_2\ddot{w} - a_1\dot{w} - a_0 w + u = -a_2 x_3 - a_1 x_2 - a_0 x_1 + u$$
$$y = b_2\ddot{w} + b_1\dot{w} + b_0 w = b_2 x_3 + b_1 x_2 + b_0 x_1$$
$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Impulse response

- A system can be characterized by its impulse response g(t), which is defined as the output when the input is a unit impulse function.
- Once the impulse response of a linear system is known, the output of the system y(t) with any input u(t) can be found by the transfer function.

Impulse response example



- Let the transfer function of a system be
- $G(s) = \frac{5s+3}{(s+1)(s+2)(s+3)}$
- By taking partial fractions
- $G(s) = \frac{-1}{(s+1)} + \frac{7}{(s+2)} + \frac{-6}{(s+3)}$
- Impulse input $\delta(t) = \begin{cases} 1 \; if \; t = 0 \\ 0 \; if \; t \neq 0 \end{cases}$ and $\Delta(s) = 1$
- By taking inverse Laplace transforms of $G(s)\Delta(s)$ the impulse response is $y(t) = -1e^{-t} + 7e^{-2t} - 6e^{-3t}$

$$y(t) = -1e^{-t} + 7e^{-2t} - 6e^{-3t}$$