Proof of CART Pruning

In the CART algorithm, define

$$g_k(t) = \frac{C(t) - C(T_t)}{|T_t| - 1} \tag{0.1}$$

for any interior node t in the kth round. In the meanwhile, define $\mathcal{M}_k = \{t_1, \cdots, t_n\}$ as the set of interior nodes in the kth round. Suppose $a \stackrel{\text{def}}{=} t_1$ is pruned in the kth round. Define $\mathcal{F}_a$ as the collection of nodes whose root is a and $\mathcal{M}_{k+1} = \mathcal{M}_k \backslash \mathcal{F}_a$. Then in the kth round, we have $\alpha_k = g_k(a)$. By Lemma 1, we have $\alpha_{k+1} \stackrel{\text{def}}{=} \min_{b \in \mathcal{M}_{k+1}} g_{k+1}(b) > \alpha_k$. For any $\alpha \in [\alpha_k, \alpha_{k+1})$, for any interior node $b \in \mathcal{M}_{k+1}$ it holds $g_{k+1}(b) > \alpha$. As a result, if we prune any $b \in \mathcal{M}_{k+1}$, the total loss will increase.

Lemma 1. *In the kth round of the CART algorithm, it holds,*

$$\min_{b \in \mathcal{M}_{k+1}} g_{k+1}(b) > g_k(a) \tag{0.2}$$

Proof. It suffices to prove for any $b \in \mathcal{M}_{k+1}$, we have $g_{k+1}(b) > g_k(a)$. Define $\mathcal{N}_a$ as the collection of nodes from the node a up to the root, i.e., a is the child of any node in $\mathcal{N}_a$. We prove the conclusion respectively for $b \notin \mathcal{N}_a$ (left panel of Figure 1) and $b \in \mathcal{N}_a$ (right panel of Figure 1).

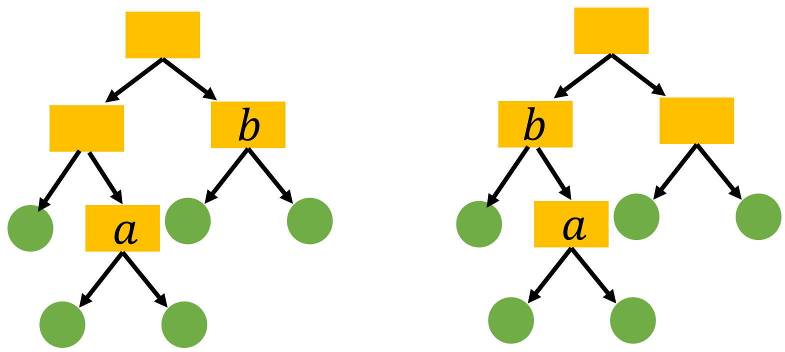

Figure 1: Left panel: $b \notin \mathcal{N}_a$; right panel: $b \in \mathcal{N}_a$.

Case 1. ($b \notin \mathcal{N}_a$)

In this case, pruning T_a does not influence $C(T_b)$ and $C(b)$. As a result, $g_{k+1}(b) = g_k(b) > g_k(a)$ by definition.

Case 2. ($b \in \mathcal{N}_a$)

Define $T_b' = T_b \backslash T_a$ as the tree whose root is b after T_a is pruned. The following equation holds,

$$C(T_b') = \big\{C(T_b) - C(T_a)\big\} + C(a),$$
$$|T_b'| = |T_b| - |T_a| + 1$$

Then we have

$$g_{k+1}(b) = \frac{C(b) - C(T_b')}{|T_b'| - 1} = \frac{C(b) - C(T_b) + C(T_a) - C(a)}{|T_b| - |T_a|}.$$

It can be derived that

$$
\begin{aligned}
g_{k+1}(b) - g_k(a) &= \frac{C(b) - C(T_b) + C(T_a) - C(a)}{|T_b| - |T_a|} - \frac{C(a) - C(T_a)}{|T_a| - 1} \\
&= Z^{-1}\Big\{[C(b) - C(T_b) + C(T_a) - C(a)][|T_a| - 1] - [C(a) - C(T_a)][|T_b| - |T_a|]\Big\}
\end{aligned}
$$

$\stackrel{\text{def}}{=} Z^{-1}\Delta$, where $Z = (|T_b| - |T_a|)(|T_a| - 1) > 0$. It suffices to show $\Delta > 0$, which is verified as follows:

$$
\begin{aligned}
\Delta &= |T_a|C(b) - C(b) - |T_a|C(T_b) + C(T_b) + |T_a|C(T_a) - C(T_a) - |T_a|C(a) + C(a) - \\
&\quad |T_b|C(a) + |T_a|C(a) + |T_b|C(T_a) - |T_a|C(T_a) \\
&= |T_a|C(b) - C(b) - |T_a|C(T_b) + C(T_b) - C(T_a) + C(a) - |T_b|C(a) + |T_b|C(T_a) \\
&= (|T_a| - 1)C(b) - (|T_a| - 1)C(T_b) + (|T_b| - 1)C(T_a) - (|T_b| - 1)C(a) \\
&= (|T_a| - 1)(C(b) - C(T_b)) - (|T_b| - 1)(C(a) - C(T_a)) \\
&= \Big(\frac{C(b) - C(T_b)}{|T_b| - 1} - \frac{C(a) - C(T_a)}{|T_a| - 1}\Big)Z' \\
&= (g_k(b) - g_k(a))Z' > 0
\end{aligned}
$$

where $Z' = (|T_a| - 1)(|T_b| - 1) > 0$.

$\square$