Multiphase flow using level set method
Final Project of ME697

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April 30, 2021

1 Introduction

The level set method was proposed by S.Osher and J. A. Sethian [2], which is a simple and useful method for capturing and analyzing the motion of an interface. And the level set method has been shown to be able to simulate a wide variety of problems, such as compressible and incompressible (possibly reacting) flow, Stefan problems, kinetic crystal growth, epitaxial growth of thin films, vortex-dominated flows, and extensions to multiphase motion [1].

The purpose of this project is to compute the motion of the interface and simulate multiphase flow using the level set method. The curvature flow is used to validate the codes, and the bubble rising is simulated using the developed codes.

2 Methodology

I would like to work on developing a 2-D program to compute the motion of the interface Γ using level set method, and then combine it with the Navier-stokes equations by adding the surface tension force to simulate the two phase flow. The interface is captured as the zero level set of a smooth function $\phi(\mathbf{x}, t)$, i.e. $\Gamma(t) = \{\mathbf{x} \mid \varphi(\mathbf{x}, t) = 0\}$. ϕ is negative inside the interface, and positive outside the interface, and is zero on the interface $\Gamma(t)$ [1]. The governing equation for the motion of the interface is,

$$\frac{\partial \varphi}{\partial t} + \mathbf{v} \cdot \nabla \varphi = 0, \tag{1}$$

Here, I use the upwind scheme to discretize the convection term. For example, in x direction, if $u_x \geq 0$, then $\Delta_x \phi = \phi_i - \phi_{i-1}$, otherwise $\Delta_x \phi = \phi_{i+1} - \phi_i$.

Then the discretized equation can be written as,

$$\frac{\phi^{n+1} - \phi^n}{\Delta t} + u_x \frac{\Delta_x^n \phi}{\Delta x} + u_y \frac{\Delta_y^n \phi}{\Delta y} = 0 \tag{2}$$

The characteristic function χ is defined from ϕ by a 1-D Heaviside function.

$$\chi = H(\varphi), \tag{3}$$

Where, $H(\varphi) \equiv 0$ if $\varphi < 0$, $H(\varphi) \equiv 1$ if $\varphi > 0$. If $|\nabla \phi| = 1$ for all t, we can use a smeared out characteristic function, $\chi_s(\phi)$. $\chi_s(\phi) = 0$, if $\phi < -\epsilon$, $\chi_s(\phi) = 1/2 + \phi/(2\epsilon) + 1/2\pi \sin(\pi\phi/\epsilon)$, if $-\epsilon \leq \phi \leq \epsilon$; $\chi_s(\phi) = 1$, if $\phi > \epsilon$. Then $\delta_\Gamma(n_\Gamma) \approx \delta_s(\phi) = d\chi_s(\phi)/d\phi = 1/(2\epsilon)(1 + \cos(\pi\phi/\epsilon))$, $\mathbf{n_\Gamma} = \nabla\phi/|\nabla\phi| = \nabla\phi$, curvature $\kappa = \nabla \cdot n_\Gamma = \Delta\phi$. Here we take $\epsilon = 3/2\Delta x$

The distance re-initialization procedure replaces ϕ by the signed distance function $d(\mathbf{x}, t)$, which is the signed distance of $\mathbf{x}$ to the closest point on Γ, and it satisfies $|\nabla d| = 1$. And it is the steady state solution ($\tau \to \infty$) of the equation,

$$\begin{gathered} \frac{\partial \psi}{\partial \tau} + \operatorname{sign}(\varphi)(|\nabla\psi| - 1) = 0 \\ \psi(\mathbf{x}, 0) = \varphi(\mathbf{x}, t) \end{gathered} \tag{4}$$

where,

$$\operatorname{sign}(\varphi) = \operatorname{sign}_\epsilon(\varphi) = \frac{\phi}{\sqrt{\phi^2 + \varepsilon^2}},$$

where, I set $\varepsilon = 0.00001$. Combining the level set equation to the N-S equations by density ρ and surface tension force σ. Density is obtained by the characteristic function, $\rho = \chi_s \rho_2 + (1 - \chi_s)\rho_1$, where ρ_2 is the density of the drop, rho_1 is the density of the surrounding fluid. The surface tension force, $\sigma = \delta_s(\phi)\gamma\kappa n$, where γ is the surface tension, κ is the curvature, n is the unit normal, $n = \nabla\phi$ if $|\nabla\phi| = 1$. At, last, the conservation of linear momentum equation for two-phase flow can be written as,

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho\nabla\cdot(\mathbf{u}\otimes\mathbf{u}) = -\nabla p + \mu\nabla\mathbf{u} + \rho\mathbf{g} + \gamma\Delta\phi\delta_s(\phi)\nabla\phi. \tag{5}$$

Solving the N-S equations using projection method, same as in the lectures.

3 Codes description

- First, I initialize the shape as a circle located at the center of the domain. I implement numerically the level set equation to capture the interface, which involves implement the discretized form of Eq. (1). This is denoted as the interface propagation process in the codes. For the curvature flow, the governing equation is modified, and the discretization of this equation is shown in validation section.
- After propagation, I have the reinitialization process to replace ϕ with a signed distance, so that $|\nabla\phi| = 1$. This process involves implementation of discretized equation of Eq. (4).
- Then I calculate the density using the characteristic function. And then calculate the surface tension, and add it to the N-S equation, as shown in Eq. (5), and the N-S equation is solved using the projection method, same as the codes in the lectures.

  More details of comments are in the codes.

4 Validation and Results

4.1 Initialization

The initial condition of the level set of the signed distance function ϕ and the characteristic function χ are shown in the following figure, Fig 1.

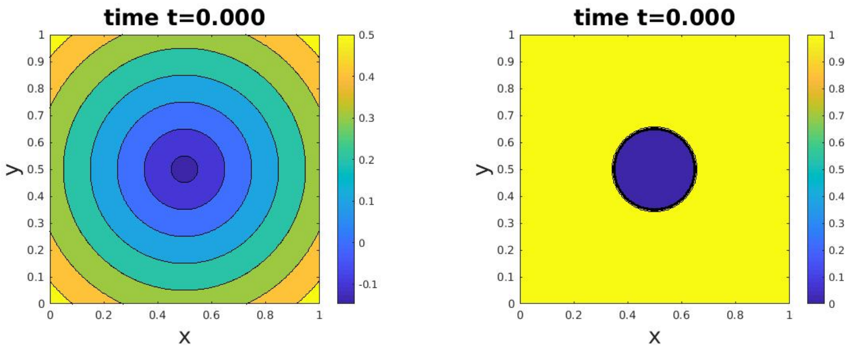

Figure 1: Initial condition of the drop or bubble, left is the distanced function ϕ, right is the sign of ϕ to represent the shape of the drop.

4.2 Curvature flow

First I test the codes by simulating the curvature flow. The governing equation of curvature flow using level set method can be written as,

$$\frac{\partial\phi}{\partial t} = D\Delta\phi, \tag{6}$$

where, D is the a positive constant. In the form of discretization, it can be written as,

$$\frac{\phi^{n+1}_{i,j} - \phi^{n}_{i,j}}{dt} = D\left(\frac{\phi^{n+1}_{i+1,j} - 2\phi^{n+1}_{i,j} + \phi^{n+1}_{i-1,j}}{dx^2} + \frac{\phi^{n+1}_{i,j+1} - 2\phi^{n+1}_{i,j} + \phi^{n+1}_{i,j-1}}{dy^2}\right) \tag{7}$$

Using the SOR (successive over relaxation) method to solve this equation, with the relaxation factor $\omega = 1.5$. The algorithm is written as,

$$\phi_{i,j}^{n+1} = (1-\omega)\phi_{i,j}^{n+1} + (\omega)\{\frac{1}{(1+\frac{2Ddt}{dx^2}+\frac{2Ddt}{dy^2}))}\{\phi_{i,j}^{n} + dtD(\frac{\phi_{i+1,j}^{n+1}+\phi_{i-1,j}^{n+1}}{dx^2} + \frac{\phi_{i,j+1}^{n+1}+\phi_{i,j-1}^{n+1}}{dy^2})\}\}. \quad (8)$$

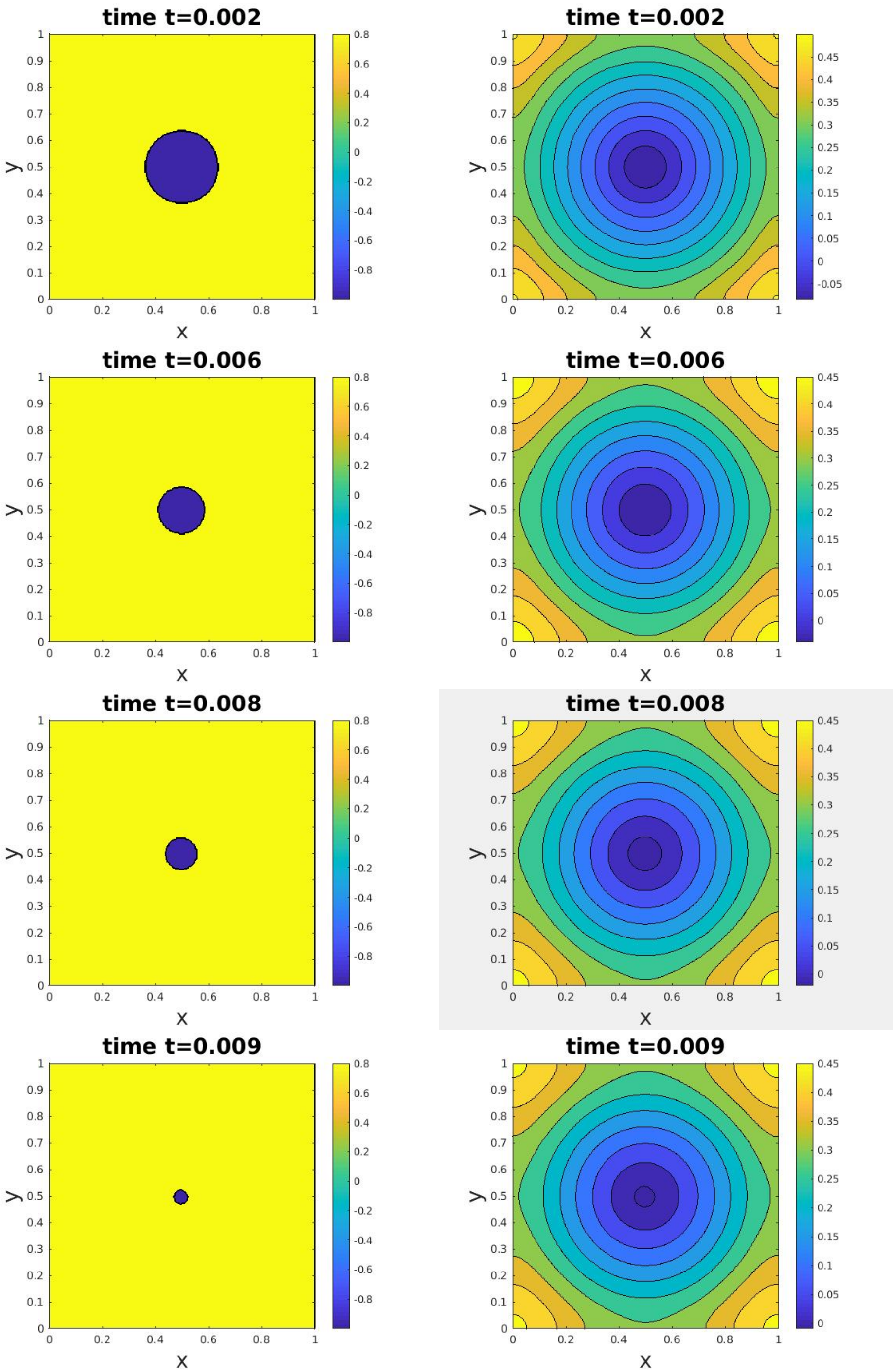

Figure 2: Curvature flow with the initial shape as a circle with $r_0 = 0.15$.

Setting D=1, dt=0.00001,and the initial radius of the circle $r_0 = 0.15$ and the cell number is 256×256, the shape of the circle is represented by the sign of ϕ. The contraction of the circle at different times are shown in Fig. 2, left column represents the $sign(\phi)$, right column represents the level set of ϕ.

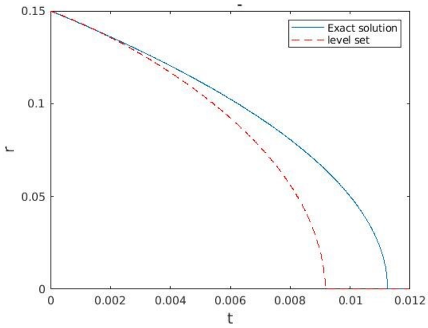

Figure 3: Evolution of radius of the circle with initial radius $R_0 = 0.15$ compared to exact analytical solution.

As shown in Fig. 2, the circle gradually contracts to a point. But it does not fit very well with the exact solution, as shown in Fig. 3. As the radius decreases, there is a larger difference between the numerical solution and the exact solution.

4.3 Bubble rising

Setting the density of the bubble $\rho_1 = 1$, and the density of the surrounding fluid $\rho_2 = 2$, the initial radius of the bubble is 0.15. As with the surface tension included, the codes can easily break down, the surface tension is set to be zero.

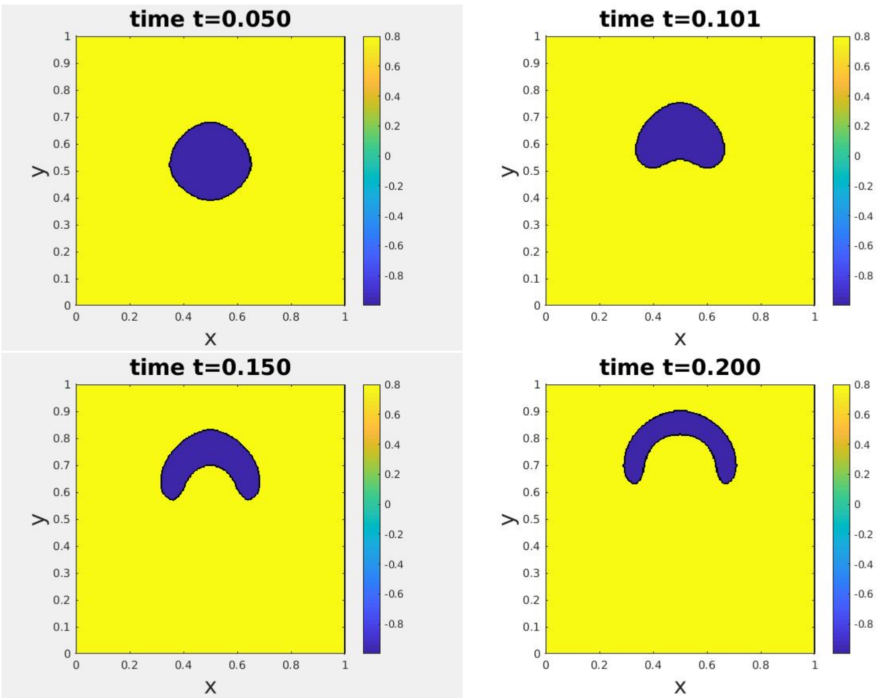

Figure 4: Bubble rising

The results is shown in Fig. 4.we can observe that the bubble rises slowly. As time goes by, the bubble gradually rises up as its density is smaller than the surrounding fluid and gets flatter, and it forms a shape of an arch at t=0.2.

Increasing the initial bubble radius to $R_0 = 0.25$, the results are in Fig. 5. Compared to the small bubble, large bubble is not as deformable, the bubble does not gets that flatter. But at t=2.0, the larger bubble also forms an arch shape.

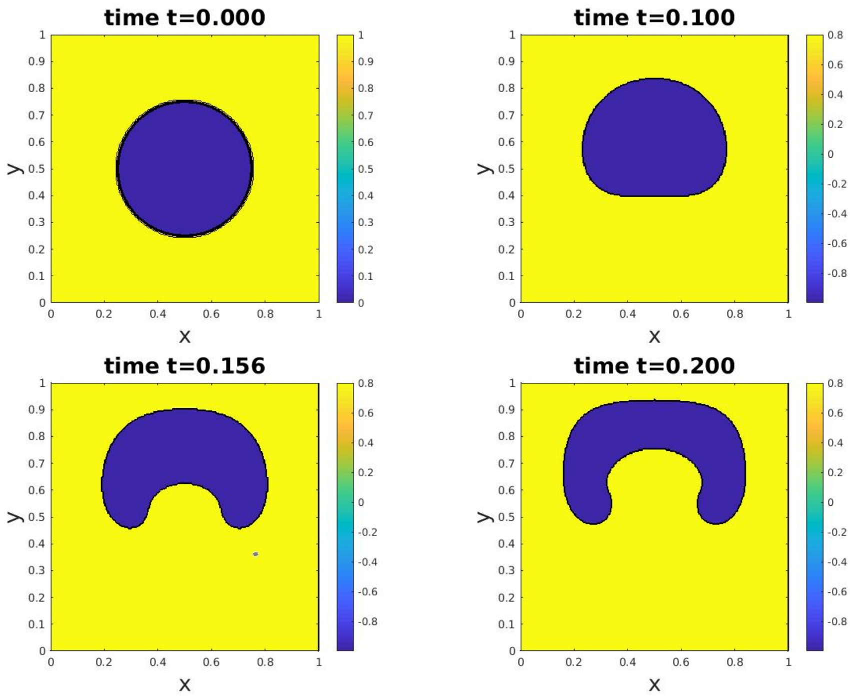

Figure 5: Bubble rising with larger radius, $R_0 = 0.25$

4.4 Droplet falling

Setting $\rho_1 = 2$, $\rho_2 = 1$, keep the initial radius $R_0 = 0.15$. The density inside the circle is larger that outside the circle, it becomes a droplet falling problem, the results are shown in Fig. 6. Compared to bubble with the radius, the droplet does not deform as much at t=0.101, but it deforms much more at t=0.2 as the drop get closer and closer to the wall.

Unlike the results of droplet falling using the front tracking method, the deformation of the droplet is not as large and dramatic. And there is no small droplets breaking out even without surface tension. Maybe the mesh is not small enough to capture the small features or this level set method can easily lose small features.

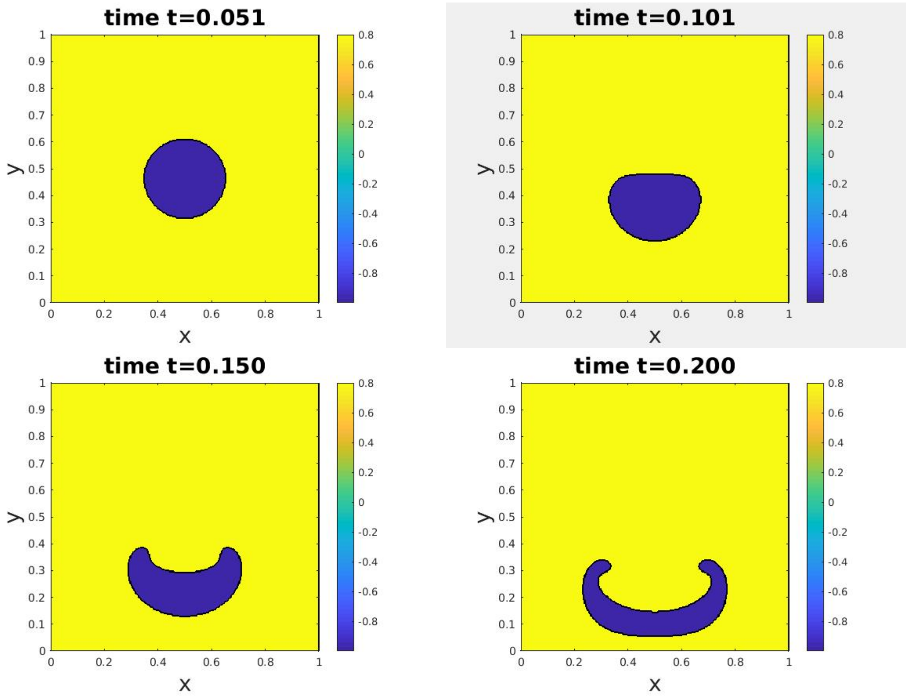

Figure 6: Droplet falling with initial radius, $R_0 = 0.15$.

5 Discussion/Comments

In my project, the contraction of the circle to a point can be simulated well, but contraction is a little faster than the exact solution as time increases when the radius gets smaller. The behaviors of bubble rising, such as the formation of an arch shape, can be simulated well using the level set method. Besides, the droplet falling can be simulated well qualitatively, but the small features are lost, such as the breaking out of small drops. Although a wide range application can be available using the level set method, this method of computing the interface is a very popular one currently. It is because that the signed distance property need to be restored through an extra process (reinitialization), which takes extra time. And also it can lose small features of the interface using level set method,. To make the simulation of multiphase flow using the level set method better, extra efforts are still needed.

References

[1] Stanley Osher and Ronald P Fedkiw. Level set methods: An overview and some recent results. *Journal of computational physics*, 169(2):463–502, 2001.

[2] Stanley Osher and James A Sethian. Fronts propagating with curvature-dependent speed: Algorithms based on hamilton-jacobi formulations. *Journal of Computational Physics*, 79(1):12–49, 1988.