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Homework 3

COMP 5212
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Problem 1

(a) Proof as the following:

$$\begin{aligned}\frac{\partial g}{\partial z} &= -\frac{1}{(1+e^{-z})^2} \cdot \left(e^{-z}\right) \cdot -1 \\ &= \frac{(e^{-z})}{(1+e^{-z})^2} \\ &= \frac{1}{1+e^{-z}} \cdot \frac{e^{-z}}{1+e^{-z}} \\ &= g(z)(1-g(z))\end{aligned}$$

(b) Proof as follows:

$$\begin{aligned}1-g(z) &= \frac{e^{-z}}{1+e^{-z}} \\ &= \frac{1}{e^{z}+1} \\ &= g(-z)\end{aligned}$$

Problem 2

(a) As g is convex, we have the following relation (equation 1),

$$g\left(t\left[\langle w_1, x\rangle + y\right] + (1-t)\left[\langle w_2, x\rangle + y\right]\right) \leq tg(\langle w_1, x\rangle + y) + (1-t)g(\langle w_2, x\rangle + y) \quad (1)$$

$\forall t \in [0,1]$ and $\forall w_1, w_2 \in \mathbb{R}^d$. Therefore, we can do the following substidution:

$$\begin{aligned}f(tw_1 + (1-t)w_2) &= g(\langle tw_1 + (1-t)w_2, x\rangle + y) \\ &= g\left(t\left[\langle w_1, x\rangle + y\right] + (1-t)\left[\langle w_2, x\rangle + y\right]\right) \\ &\leq tg(\langle w_1, x\rangle + y) + (1-t)g(\langle w_2, x\rangle + y) \\ &= tf(w_1) + (1-t)f(w_2)\end{aligned}$$

the equality holds when equality of equation 1 holds. Conclude that f is also convex if g is convex.

(b) $\forall t \in [0,1]$ and $x_1, x_2 \in \mathbb{R}^d$, we have

$$\begin{aligned}g(tx_1 + (1-t)x_2) &= max_{i\in[r]} f_i(tx_1 + (1-t)x_2) \\ &\leq t \cdot max_{i\in[r]} f_i(x_1) + (1-t) \cdot max_{i\in[r]} f_i(x_2) \\ &= tg(x_1) + (1-t)g(x_2)\end{aligned}$$

The equality holds if and only if $argmax_{i\in[r]} f_i(x_1) = argmax_{i\in[r]} f_i(x_2)$. Therefore g is also a convex function.