Supplemental Material

Appendix A

Proof pf Proposition 1

Proof. Assume that all agents truthfully submitted their bids. Suppose player $i \in N$ submits his true valuation v_{ij} for player j's meta-item, or s_i for his own meta-item. When he is exchanged with the other agent, then he is supposed to make payment $p_{ij} = v_{ij} - [\pi(N) - \pi(N\backslash\{i\})]$; when he chooses to escrow with the platform, then his payment will be $p_i = -s_i - [\pi(N) - \pi(N\backslash\{i\})]$. If agent i bids $v_{ij}^{'} \neq v_{ij}$ or $s_i^{'} \neq s_i$, the results will be changed, which can be discussed in four scenarios.

Scenario 1: Agent i who originally exchanges meta-item with agent j, may be allocated agent k's meta-item if he submits valuation $v_{ij}^{'} \neq v_{ij}$. The related payment in this case is $p_{ik}^{'} = v_{ik}^{'} - [\pi^{'}(N) - \pi(N\backslash\{i\})]$, where $\pi^{'}(N)$ denotes the welfare resulting from the allocation, calculated as follows:

$$\pi^{'}(N) = v_{ik}^{'} + \sum_{l \in N^{'}\backslash\{i\}} \sum_{j \in N^{'}\backslash\{i\}} x_{lj}^{'} v_{lj} + \sum_{m \in N^{'}\backslash\{i,k\}} y_m^{'}(b_m - s_m) \tag{28}$$

Here, $x_{lj}^{'}$ and $y_m^{'}$ represent the optimal solution to $\pi^{'}(N)$, $N^{'}$ is the set of resulting winning players, l is any winning player other than player i, and m is any player who consigns their meta-items to the platform other than agents i, k. For player i who benefits from submitting the false bid, the following condition must be met:

$$\pi(N) - \pi(N\backslash\{i\}) < v_{ik} - p_{ik}^{'} \tag{29}$$

Substituting for $p_{ik}^{'}$ yields

$$\pi(N) < v_{ik} + \sum_{l \in N^{'}\backslash\{i\}} \sum_{j \in N^{'}\backslash\{i\}} x_{lj}^{'} v_{lj} + \sum_{m \in N^{'}\backslash\{i,k\}} y_m^{'}(b_m - s_m) \tag{30}$$

The inequality clearly challenges the assumption that $\pi(N)$ signifies an optimal allocation of meta-items.

Scenario 2: Agent i who originally exchanges meta-item with agent j, may consign his meta-item with the platform if he submits valuation $s_i^{'} \neq s_i$. The related payment in this case is $p_i^{'} = -s_i^{'} - [\pi^{'}(N) - \pi(N\backslash\{i\})]$, where $\pi^{'}(N)$ denotes the welfare resulting from the allocation, calculated as follows:

$$\pi^{'}(N) = b_i - s_i^{'} + \sum_{l \in N^{'}\backslash\{i\}} \sum_{j \in N^{'}\backslash\{i\}} x_{lj}^{'} v_{lj} + \sum_{l \in N^{'}\backslash\{i\}} y_l^{'}(b_l - s_l) \tag{31}$$

Here, $x_{lj}^{'}$ and $y_l^{'}$ represent the optimal solution to $\pi^{'}(N)$, $N^{'}$ is the set of resulting winning agents, l is any winning agent other than agent i. The following condition must be met for agent i who benefits from submitting the false bid:

$$\pi(N) < \pi^{'}(N) + s_i^{'} - s_i \tag{32}$$

The inequality also contradicts the optimal allocation premise of $\pi(N)$.

Scenario 3: Agent i who originally consigns meta-item with the platform may submit an unreal cost $s_i^{'} \neq s_i$. Then the payment will be $p_i^{'} = -s_i^{'} - [\pi^{'}(N) - \pi(N\backslash\{i\})]$, and the corresponding welfare is

$$\pi^{'}(N) = b_i - s_i^{'} + \sum_{l \in N^{'} \backslash \{i\}} \sum_{j \in N^{'} \backslash \{i\}} x_{lj}^{'} v_{lj} + \sum_{l \in N^{'} \backslash \{i\}} y_l^{'}(b_l - s_l) \tag{33}$$

Again, $x_{lj}^{'}$, $y_l^{'}$, $\pi^{'}(N)$ and $N^{'}$ retain the analogous meaning as those in Scenario 2. For agent i who benefits from submitting the false bid, the following condition must be met:

$$p_i^{'} < -s_i - [\pi(N) - \pi(N \backslash \{i\})] \tag{34}$$

Substituting for $p_i^{'}$ yields

$$\pi(N) < b_i - s_i + \sum_{l \in N^{'} \backslash \{i\}} \sum_{j \in N^{'} \backslash \{i\}} x_{lj}^{'} v_{lj} + \sum_{l \in N^{'} \backslash \{i\}} y_l^{'}(b_l - s_l) \tag{35}$$

The inequality also contradicts the premise that $\pi(N)$ implies an optimal assignment of meta-items.

Scenario 4: Agent i who originally consigns meta-item with the platform may exchange meta-item with agent k if he submits valuation $v_{ij}^{'} \neq v_{ij}$. Then the payment will be $p_{ik}^{'} = v_{ik}^{'} - [\pi^{'}(N) - \pi(N \backslash \{i\})]$, and the corresponding welfare is

$$\pi^{'}(N) = v_{ik}^{'} + \sum_{l \in N^{'} \backslash \{i\}} \sum_{j \in N^{'} \backslash \{i\}} x_{lj}^{'} v_{lj} + \sum_{m \in N^{'} \backslash \{i,k\}} y_m^{'}(b_m - s_m) \tag{36}$$

As mentioned before, the following condition must be met for agent i:

$$\pi(N) < \pi^{'}(N) + v_{ik} - v_{ik}^{'} \tag{37}$$

This inequality also contradicts the notion that $\pi(N)$ signifies the best possible allocation.

In each of the four scenarios, the argument holds true for any participant in N. Thus, these scenarios confirms the incentive compatibility. ■

Proof pf Proposition 2

Proof. Clearly, the profits of each agent could be expressed as $\pi(N) - \pi(N \backslash \{i\})$. Specifically, When $\pi(N) - \pi(N \backslash \{i\}) = 0$, agent i remains indifferent towards participation. Conversely, in cases where $\pi(N) - \pi(N \backslash \{i\}) > 0$, agent i experiences a strictly superior outcome, compelling he chooses to participate. Regardless of the two scenarios, agent i opts to participate, thus confirming the individual rationality of the SCE-VCG auction. ■

Proof of Lemma 1

Proof. We argue by induction on $|N|$. If $|N| = 2$, let $N^{'} = N$, then the left-side of Eq.23 equals to π, but the right-side of Eq.23 equals to 2π. Thus, the result holds when $|N| = 2$.

We next consider the case with $|N| = 3$ and $N^{'} = N$. The left-side of Eq.23 equals to π. Then, we try to make the right-side of Eq.23 as small as possible; that is, $\pi - \pi(N \setminus i) < \pi$ for any i. For simplicity, we say $N = \{A, B, C\}$, and the value vector $v = \{v_{AB} = a, v_{BA} = a^{'}, v_{BC} = b, v_{CB} = b^{'}, v_{CA} = c, v_{AC} = c^{'}\}$. So, the right-side of Eq.23 equals to $\pi - \pi(N \setminus \{A\}) + \pi - \pi(N \setminus \{B\}) + \pi - \pi(N \setminus \{C\})$. Suppose the substitute condition holds. We obtain that $\pi \geqslant \pi - \pi(N \setminus \{A\}) + \pi - \pi(N \setminus \{B\}) + \pi - \pi(N \setminus \{C\})$; that is, $\pi = a + b + c \leq a^{'} + b^{'} + c^{'}$, which causes a contradiction. Note that, ties are broken arbitrarily, so π is the unique optimal solution. Thus, the result holds when $|N| = 3$.

Similarly, in the case with $|N| = 4$ and $N^{'} = N = \{A, B, C, D\}$, if the substitute condition holds, then we have that

$$\pi(\{A,B,C\}) + \pi(\{A,B,D\}) + \pi(\{A,C,D\}) + \pi(\{B,C,D\}) \geqslant 3\pi \tag{38}$$

Observe that each winner is covered by at most three exchange cycles. Then, by a simple construction, the left-side of Eq.38 contains at most three feasible solutions in the same time. Clearly, the constructed feasible solutions cannot be the same and must be less than π. So, the sum of the constructed feasible solutions is strictly less than 3π, which causes a contradiction. Thus, the result holds when $|N| = 4$.

Finally, in the case with $|N| = k$ and $N^{'} = N$, if the substitute condition holds, then we obtain that

$$\sum_{i \in N} \pi(N \setminus i) \geqslant (h-1)\pi \tag{39}$$

Likewise, the left-side of Eq.39 contains at most $h-1$ feasible solutions, implying a contradiction. Thus, the result holds when $|N| = h$.

From the above argument, Lemma 1 has been proved. ■

Proof of Lemma 2

Proof. Suppose $|N| = h$ and let I denote the set of winners. If the substitute condition holds, then we have

$$\sum_{i \in I} \pi(N \setminus i) \geqslant (h-2)\pi \tag{40}$$

Let j be the (unique) loser. Observe that we can arbitrarily add new arcs ($\widetilde{v}_{ij}$ or $\widetilde{v}_{ji}$) to agent j and/or improve some values of agent j so as to guarantee that all agents are winners and $\pi(I) = \pi(I \cup \{j\})$. For example, we can add new arcs $\widetilde{v}_{ij}$ and $\widetilde{v}_{jk}$, satisfying $\widetilde{v}_{ij} + \widetilde{v}_{jk} = v_{ik}$. If the arcs v_{ij} and v_{jk} have already existed, then we can improve the weights of such arcs so that $\widetilde{v}_{ij} + \widetilde{v}_{jk} = v_{ik}$. Note that, when only v_{ij} has existed, we can add the arc $\widetilde{v}_{jk}$ such that $v_{ij} + \widetilde{v}_{jk} = v_{ik}$, where $\widetilde{v}_{jk}$ is negative if $v_{ij} > v_{ik}$. By such construction, the left-side of Eq.40 becomes weakly larger since there are more feasible solutions. Interestingly, the right-side of Eq.(40) is unaltered.

So far, we have found two optimal solutions, $\pi(I)$ and $\pi(I \cup \{j\})$. Then, we broke ties and choose $N = I \cup \{j\}$ as the unique set of winners. That is, $\pi(I \cup \{j\})$ is the unique optimal solution though the gap between $\pi(I)$ and $\pi(I \cup \{j\})$ can be completely ignored.

By simultaneously adding π to the left-side and the right-ride of Eq.40, we obtain

$$\sum_{i \in I} \pi(N \setminus i) + \pi(N \setminus \{j\}) \geqslant (h-1)\pi \tag{41}$$

The Eq.41 can be rewritten as

$$\sum_{i \in N} \pi(N \setminus i) \geqslant (h-1)\pi \tag{42}$$

According to Lemma 1, we argue that Eq.42 does not hold.

Thus, we have proved Lemma 2. ■

Proof of Theorem 1

Proof. By a similar construction stated in the proof of Lemma 2, we can find two optimal solutions, $\pi(I)$ and $\pi(I \cup (N \setminus I))$, where $N \setminus I$ is the set of losers in the solution $\pi(I)$. That is, we can find an optimal solution by adding arcs to the set of losers and/or improving the values of the losers so that the set of winners is N.

Likewise, if the substitute condition holds, then we have

$$\sum_{i \in I} \pi(N \setminus i) + \sum_{j \in N \setminus I} \pi(N \setminus \{j\}) \geqslant (h-1)\pi \tag{43}$$

Further, we get

$$\sum_{i \in N} \pi(N \setminus i) \geqslant (h-1)\pi \tag{44}$$

Clearly, the Eq.44 contradicts Lemma 1. Thus, we have proved Theorem 1. ■

Proof of Theorem 2

Proof. Observe that if the number of exchanges under maximal social welfare is strictly less than $|N|$, then the platform's surplus is identical whether $t = |N| - 1$ or $t = |N|$.

Suppose the number of exchanges under maximal social welfare is $|N|$ and the optimal solution is unique (ties are broken arbitrarily). Then, the platform's surplus under $t = |N|$ is given by

$$s(t = |N|) = \sum_{i \in N} \pi(N \setminus i) - (|N| - 1)\pi(N, t = |N|) \tag{45}$$

Now we consider the case with $t = |N| - 1$. In this case, the platform's surplus is given by

$$s(t = |N| - 1) = \sum_{i \in I'} \pi(N \setminus i) - (|I'| - 1)\pi(N, t = |N| - 1) \tag{46}$$

where I' is the set of winners.

Clearly, the Eq.45 can be rewritten as

$$s(t = |N|) = \sum_{i \in I'} \pi(N \setminus i) + \sum_{j \in N \setminus I'} \pi(N \setminus j) - (|N| - 1)\pi(N, t = |N|) \tag{47}$$

Also, notice that for any $j \in N \setminus I'$, we have

$$\pi(N \setminus j) = \pi(N, t = |N| - 1) \tag{48}$$

From Eq.45, Eq.47 and Eq.48, it follows that

$$s(t = |N| - 1) - s(t = |N|) = (|N| - 1)\left[\pi(N, t = |N|) - \pi(N, t = |N| - 1)\right] > 0 \tag{49}$$

The inequality above implies that the platform's surplus is strictly larger under $t = |N| - 1$ than that under $t = |N|$ if the number of exchanges under maximal social welfare is $|N|$.

Thus, we have proved Theorem 2. ■