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ELEC 677 - HW3

3.1) Partial Correlations and Gaussian graphical models

$S_m = \{K1, \dots, K_m\}$

$X_{S_m} = [X_{K1}, \dots, X_{Km}]^T$

$$\boxed{\rho_{ij|S_m} = \frac{\sigma_{ij|S_m}}{\sqrt{\sigma_{ii|S_m} \cdot \sigma_{jj|S_m}}}} \quad (3.1)$$

$W_1 = [X_1, X_2]^T, \quad W_2 = X_{S_m}$

$$\mathrm{cov}\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

$$\Sigma_{11|2} = \Sigma_{11} - \Sigma_{12} \cdot \Sigma_{22}^{-1} \cdot \Sigma_{21} \quad (3.2)$$

(a) Use (3.2) to derive a closed-form expression for the partial covariance matrix $\Sigma_{XY|Z}$.

$m = 1 \rightarrow X_{S_m} = Z, \quad X_1 = X, \quad X_2 = Y.$

$$\mathrm{cov}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \mathbb{E}[(X-\mu_X)(X-\mu_X)] & \mathbb{E}[(X-\mu_X)(Y-\mu_Y)] & \mathbb{E}[(X-\mu_X)\cdot(Z-\mu_Z)] \\ \mathbb{E}[(Y-\mu_Y)(X-\mu_X)] & \mathbb{E}[(Y-\mu_Y)(Y-\mu_Y)] & \mathbb{E}[(Y-\mu_Y)(Z-\mu_Z)] \\ \mathbb{E}[(Z-\mu_Z)(X-\mu_X)] & \mathbb{E}[(Z-\mu_Z)\cdot(Y-\mu_Y)] & \mathbb{E}[(Z-\mu_Z)(Z-\mu_Z)] \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_X^2 & \Sigma_{XY} & \Sigma_{ZX} \\ \Sigma_{XY} & \sigma_Y^2 & \Sigma_{YZ} \\ \Sigma_{ZX} & \Sigma_{YZ} & \sigma_Z^2 \end{bmatrix} = \begin{bmatrix} 1 & \Sigma_{XY} & \Sigma_{ZX} \\ \Sigma_{XY} & 1 & \Sigma_{YZ} \\ \Sigma_{ZX} & \Sigma_{YZ} & 1 \end{bmatrix}.$$

$$\Sigma_{11} = \begin{bmatrix} 1 & \Sigma_{XY} \\ \Sigma_{XY} & 1 \end{bmatrix}, \quad \Sigma_{12} = \Sigma_{21}^{T} = \begin{bmatrix} \Sigma_{ZX} \\ \Sigma_{YZ} \end{bmatrix}, \quad \Sigma_{22} = [1]$$

$$\Sigma_{XY|Z} = \begin{bmatrix} 1 & \Sigma_{XY} \\ \Sigma_{XY} & 1 \end{bmatrix} - \begin{bmatrix} \Sigma_{ZX} \\ \Sigma_{YZ} \end{bmatrix} \cdot [1] \cdot \begin{bmatrix} \Sigma_{ZX} & \Sigma_{YZ} \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & \Sigma_{XY} \\ \Sigma_{XY} & 1 \end{bmatrix} - \begin{bmatrix} \Sigma_{ZX}^2 & \Sigma_{ZX} \cdot \Sigma_{YZ} \\ \Sigma_{YZ} \cdot \Sigma_{ZX} & \Sigma_{YZ}^2 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 - \Sigma_{ZX}^2 & \Sigma_{XY} - \Sigma_{YZ} \cdot \Sigma_{ZX} \\ \Sigma_{XY} - \Sigma_{YZ} \cdot \Sigma_{ZX} & 1 - \Sigma_{YZ}^2 \end{bmatrix} \quad \square$$

But $\sigma_X^2 = \sigma_Y^2 = \sigma_Z^2 = 1$, thus $\rho_{XY} = \Sigma_{XY}$, (...)

$$\Sigma_{XY|Z} = \begin{bmatrix} 1 - \rho_{ZX}^2 & \rho_{XY} - \rho_{YZ} \cdot \rho_{ZX} \\ \rho_{XY} - \rho_{YZ} \cdot \rho_{ZX} & 1 - \rho_{YZ}^2 \end{bmatrix}$$

→ Partial Covariance Matrix

Then follow (3.1) to combine the elements of the matrix to show that:

$$\rho_{ij|S_m} = \frac{\sigma_{ij|S_m}}{\sqrt{\sigma_{ii|S_m} \cdot \sigma_{jj|S_m}}}$$

→ In this exercise,

$$\sigma_{ij|S_m} = \rho_{XY} - \rho_{YZ} \cdot \rho_{ZX}$$

$$\sigma_{ii|S_m} = 1 - \rho_{ZX}^2$$

$$\sigma_{jj|S_m} = 1 - \rho_{YZ}^2$$

Finally,

$$\boxed{\rho_{XY|Z} = \frac{\rho_{XY} - \rho_{YZ}\cdot\rho_{ZX}}{\sqrt{(1-\rho_{ZX}^2)\cdot(1-\rho_{YZ})}}} \blacksquare$$

(b) $$\min_{\beta_{ZX},\,\beta_{ZY}} \mathbb{E}\left[\left(Z - \beta_{ZX}\cdot X - \beta_{ZY}\cdot Y\right)^2\right]$$

$$\mathbb{E}\left[(Z - \beta_{ZX}\cdot X - \beta_{ZY}\cdot Y)\cdot(Z - \beta_{ZX}\cdot X - \beta_{ZY}\cdot Y)\right] =$$

$$= \mathbb{E}\Big[Z^2 - \beta_{ZX}\cdot X\cdot Z - \beta_{ZY}\cdot Y\cdot Z - \beta_{ZX}\cdot X\cdot Z + \beta_{ZX}^2\cdot X^2 + \boxed{\beta_{ZX}\cdot\beta_{ZY}\cdot X\cdot Y} - \beta_{ZY}\cdot Y\cdot Z + \boxed{\beta_{ZY}\cdot\beta_{ZX}\cdot Y\cdot X} + \beta_{ZY}^2\cdot Y^2\Big] =$$

$$= \mathbb{E}[Z^2] + \beta_{ZX}^2\cdot\mathbb{E}[X^2] + \beta_{ZY}^2\cdot\mathbb{E}[Y^2] - 2\cdot\beta_{ZX}\cdot\mathbb{E}[X\cdot Z] - 2\beta_{ZY}\cdot\mathbb{E}[Y\cdot Z] + 2\beta_{ZX}\cdot\beta_{ZY}\cdot\mathbb{E}[X\cdot Y] =$$

$$MSE = \boxed{1 + \beta_{ZX}^2 + \beta_{ZY}^2 - 2\beta_{ZX}\cdot\rho_{ZX} - 2\beta_{ZY}\cdot\rho_{ZY} + 2\beta_{ZX}\cdot\beta_{ZY}\cdot\rho_{XY}} \blacksquare$$

(c) Optimal predictor $[\hat{\beta}_{ZX}, \hat{\beta}_{ZY}]^T$

$$\nabla\cdot\nabla MSE = \begin{bmatrix} \dfrac{\partial^2 MSE}{\partial^2\beta_{ZX}} & \dfrac{\partial^2 MSE}{\partial\beta_{ZX}\,\partial\beta_{ZY}} \\ \dfrac{\partial^2 MSE}{\partial\beta_{ZX}\,\partial\beta_{ZY}} & \dfrac{\partial^2 MSE}{\partial^2\beta_{ZY}} \end{bmatrix} = \begin{bmatrix} 1 & \rho_{XY} \\ \rho_{XY} & 1 \end{bmatrix} \blacksquare$$

(d) Solve the system:

$$\begin{pmatrix} 1 & \rho_{XY} \\ \rho_{XY} & 1 \end{pmatrix} \cdot \begin{bmatrix} \hat{\beta}_{ZX} \\ \hat{\beta}_{ZY} \end{bmatrix} = \begin{bmatrix} \rho_{ZX} \\ \rho_{ZY} \end{bmatrix}$$

$$\begin{cases} \hat{\beta}_{ZX} + \rho_{XY} \cdot \hat{\beta}_{ZY} = \rho_{ZX} & \text{I} \\ \rho_{XY} \cdot \hat{\beta}_{ZX} + \hat{\beta}_{ZY} = \rho_{ZY} & \text{II} \end{cases}$$

From I: $\hat{\beta}_{ZX} = \rho_{ZX} - \rho_{XY} \cdot \hat{\beta}_{ZY}$

I in II:

$$\rho_{XY} \cdot (\rho_{ZX} - \rho_{XY} \cdot \hat{\beta}_{ZY}) + \hat{\beta}_{ZY} = \rho_{ZY}$$

$$\hat{\beta}_{ZY} - \rho_{XY}^2 \cdot \hat{\beta}_{ZY} = \rho_{ZY} - \rho_{XY} \cdot \rho_{ZX}$$

$$\boxed{\hat{\beta}_{ZY} = \frac{\rho_{ZY} - \rho_{XY} \cdot \rho_{ZX}}{(1 - \rho_{XY}^2)}}$$

And:

$$\boxed{\hat{\beta}_{ZX} = \frac{\rho_{ZX} - \rho_{XY} \cdot \rho_{ZY}}{(1 - \rho_{XY}^2)}}$$

$$\rho_{ZX|Y} = \frac{\rho_{ZX} - \rho_{ZY} \cdot \rho_{XY}}{\sqrt{(1 - \rho_{ZY}^2) \cdot (1 - \rho_{XY}^2)}}$$

The mumerators of $\hat{\beta}_{ZX}$ and $\rho_{ZX|Y}$ are equal, therefore, if $\rho_{ZX|Y} = 0$, them $\hat{\beta}_{ZX} = 0$, and vice-versa simce $(1 - \rho_{ZY}^2) \neq 0$ in this case.

The same goes for $\hat{\beta}_{ZY}$ and $\rho_{ZY|X}$.

3.2) Epidemics across the world

(a) Construct a network from flight data

See python file.

↳ Resultant network has 2905 nodes and 15645 edges in a single connected component.

(b) Plot the airport network

See python file.

Epidemic Model

- Graph $G(V, E)$, $V = \{v_1, \ldots, v_n\}$, $A = [a_{ij}]$.
- Random variable $X_i(t) \in \{0, 1\}$, $X(t) = [X_1(t), \ldots, X_n(t)]^T$

↳ Two possible stochastic transitions:

(1st) $\text{Prob}[X_i(t+\Delta) = 1 | X_i(t) = 0, X(t)] = 1 - \prod_{j \in N_i \wedge X_j(t)=1} (1 - \beta a_{ij}) =$

$$\approx \boxed{\sum_{j \in N_i} \beta a_{ij} X_j(t)}$$

(2nd) $\text{Prob}[X_i(t+\Delta) = 0 | X_i(t) = 1] = \gamma$

(c) Show that (3.9) is an upper bound for (3.8).

Start with the original transition probability:

$$1 - \prod_{j \in N_i \wedge X_j(t)=1} (1 - \beta a_{ij}) = 1 - \prod_{j \in N_i \wedge X_j(t)=1} (1 - \beta a_{ij} \cdot X_j(t))$$

Without loss of generality, let's represent this expression by the product of m terms of format $(1 - \beta \cdot a_{ij} \cdot X_j(t))$:

$$\doteq 1 - (\text{term } 1) \cdot (\text{term } 2) \cdot (\ldots) \cdot (\text{term } m) \quad \square$$

This long multiplication will have as result the sum of 2^m individual terms that can be arranged as follows:

$$= \cancel{1} - \left[\cancel{1} - \sum_j (\beta \cdot a_{ij} \cdot X_j(t)) + \sum_{j,k} (\beta^2 \cdot a_{ij} \cdot a_{ik} \cdot X_j(t) \cdot X_k(t)) - \ldots \right] =$$

$$= \underbrace{\sum_j (\beta \cdot a_{ij} \cdot X_j(t))}_{1^{st} \text{ order approximation}} - \sum_{j,k} (\beta^2 \cdot a_{ij} \cdot a_{ik} \cdot X_j(t) \cdot X_k(t)) + \sum_{j,k,l} (\beta^3 \cdot (\ldots)) - (\ldots) \quad \square$$

Since $\beta << 1$, the successive terms are monotonically decreasing, with alternating signals (starting with negative).

Thus,

$$\boxed{\sum_{j \in N_i} \beta a_{ij} X_j(t) \geq 1 - \prod_{j \in N_i \wedge X_j(t)=1} (1 - \beta a_{ij})} \quad \blacksquare$$