This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/JLT.2016.2525781, Journal of Lightwave Technology

Simple Generation of Gamma, Gamma-Gamma and K Distributions with Exponential Autocorrelation Function

Dima Bykhovsky

***Abstract*—The simulation of a free-space optical (FSO) communication channel in the presence of strong turbulence typically requires the generation of channel states with a K or gamma-gamma distribution and a predefined autocorrelation function. In this paper we propose a simple and effective simulator of the strong-turbulence FSO channel that addresses the influence of the temporal covariance effect. Specifically, the proposed simulator provides K and gamma-gamma distributed values with the exponential autocorrelation function and a prescribed correlation time. This simulator is based on the numerical solution of the first-order stochastic differential equation (SDE). The simulated channel states are generated by a simple discrete-time differential equation and the simulator performance is analyzed in the paper.**

***Index Terms*—free-space optical communication, turbulence, stochastic differential equation, channel simulator, gamma distribution, gamma-gamma distribution, K distribution, negative exponential distribution**

I. Introduction

THE issue of the generation of time-correlated non-Gaussian processes has long been a subject of interest in diverse fields of application, such as radars [1], [2], RF communication channel modeling [3]–[5], optical channel modeling [6]–[9] and signal processing [10].

In communication, the main application of such methods is the provision of effective simulation modeling for stationary channel conditions. The use of such a model may significantly ease the communication system design, since it enables the simulation of channel states according to predefined first and second order statistics, namely channel distribution and autocovariance and/or power spectral density.

Free-space optical communication (FSO) is a subject of persistent interest, due to its high communication speed, low latency and unlicensed spectrum [11]–[13]. In FSO, the use of channel modeling may significantly reduce the number of preliminary experiments that are required for the design and prototyping of the transceiver system. The main benefit of such modeling is the possibility to verify and/or compare different communication and modulation schemes for different channel conditions.

In this paper, we propose a simple numerical method for the modeling of strongly turbulent channel conditions by the generation of gamma, gamma-gamma and K distributed processes with an exponential autocorrelation function. This method requires a minimal number of essential required parameters that are inserted into a discrete-time differential equation. Regarding the autocorrelation function, it has previously been shown that a simple exponential function may be used to effectively approximate the rigorous turbulent-channel expressions [7], [8], [14]–[17].

We next summarize some of the currently existing approaches. Liu and Munson [18] proposed a general method that uses an iteratively-calculated memoryless non-linear transformation (MNLT) of a white Gaussian noise (WGN) source. In [2, Sec. 5], a correlated Gamma process with exponential autocorrelation was produced by a simple approximated MNLT of a correlated Gaussian process. However, for improved accuracy of both the probability density function (PDF) and the autocorrelation functions, an intensive theoretical analysis with sufficiently long equations is required. Another method, by Filho and Yacoub [19], is based on multi-parameter MNLT and rank sample rearranging. The limitation of this method lies in the fact that it requires generation and processing of a predefined realization sequence. It is important to note that the mentioned methods give appropriately *accurate* generation results, with accompanying limitations that are outlined in the text. Additional methods are related to the generation of correlated sequences, rather than to autocorrelation [20], [21]. In the recent scheme proposed by Kay [10], only symmetric distributions may be generated. Since the gamma distribution is defined only on the positive side of the real axis, this method is inappropriate.

SDE-based process generation was addressed in [4], [5]; however, a significant part of the presented results was more theoretical than numerical. Davidson *et. al.* [7] provided some preliminary results from modeling strong turbulence using an SDE-based approach. Modulation of two log-normal distributed processes (for small-scale and large-scale turbulence, respectively) with exponential autocorrelation functions was used to approximate the gamma-gamma process and was verified experimentally.

The rest of the paper is organized as follows. Section II provides the essential theory concerned with the applied SDE technique, followed by details of the commonly used turbulence channel stochastic models. Section III provides a description of the applied numerical solution. Section IV presents the simulation results, followed by conclusions in Section V.

D. Bykhovsky is with the Electrical and Electronics Engineering Department, Shamoon College of Engineering, Beer-Sheva, Israel.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/JLT.2016.2525781, Journal of Lightwave Technology

II. Theory

A. Synthesis of SDE

We start with the essential theory required for the generation of an arbitrary stationary distribution with an exponential correlation function. The Itô form SDE may be given by [4], [5, Eq. (7.46)]

$$\dot{x} = f(x) + g(x)\xi(t) \tag{1}$$

with some initial condition $x(t_0) = x_0$, where $\xi(t)$ is a normal uncorrelated WGN process and $f(x)$ and $g(x)$ are time-independent deterministic functions. Whenever the exponential autocorrelation function is required, $f(x)$ and $g(x)$ are given by [4, Eq. (48),(50)], [5]

$$g^2(x) = -\frac{2}{\tau_c w_x(x)} \int_0^x (s - m_x)\, w_x(s) ds \tag{2a}$$

$$f(x) = \frac{g^2(x)}{2}\frac{d}{dx} \ln\left[g^2(x) w_x(x)\right], \tag{2b}$$

where $w_x(x)$ is the desired probability density function (PDF) and m_x is the average, given by

$$m_x = \int_0^\infty s w_x(s) ds. \tag{3}$$

The resulting normalized covariance function is given by [4], [5]

$$C_{xx}(\tau) = \exp\left(-\frac{|\tau|}{\tau_c}\right), \tag{4}$$

where the correlation time, τ_c, is defined as $C_{xx}(\tau_c) = \exp(-1)$ [1].

B. Gamma Distribution

The general expression for the gamma distribution is given by

$$w_x(x;\alpha,\theta) = \frac{x^{\alpha-1}\exp\left(-\frac{x}{\theta}\right)}{\Gamma(\alpha)\,\theta^\alpha}, \quad x,\alpha,\theta > 0, \tag{5}$$

where $\Gamma(z)$ is the gamma function. The application of the technique above yields the following functions:

$$f(x) = \frac{1}{\tau_c}\left[\alpha\theta - x(t)\right] \tag{6a}$$

$$g(x) = \sqrt{\frac{2x(t)\theta}{\tau_c}} \tag{6b}$$

that are substituted in (1). Finally, The SDE is given by

$$\dot{x} = \frac{1}{\tau_c}\left(\alpha\theta - x(t)\right) + \sqrt{\frac{2x(t)\theta}{\tau_c}}\xi(t). \tag{7}$$

[1] A lengthy proof of the properties of Eqs. (2)-(3) is provided in [4], [5], [22] and is omitted here for brevity.

C. Gamma-Gamma Distribution

The gamma-gamma distribution model is frequently used to describe the condition of moderate to strong scintillations. This distribution is related to the multiplication (or modulation) of small-scale and large-scale statistically independent factors, $I = XY$, each of which is described by a gamma distribution of the form [23, Ch. 9, Eq. (134-135)]

$$p_x(x;\alpha) = \frac{\alpha\,(\alpha x)^{\alpha-1}}{\Gamma(\alpha)}\exp(-\alpha x) = w_x(x;\alpha,1/\alpha) \tag{8a}$$

$$p_y(y;\beta) = \frac{\beta\,(\beta y)^{\beta-1}}{\Gamma(\beta)}\exp(-\beta y) = w_y(y;\beta,1/\beta). \tag{8b}$$

These distributions are related to the general expression of the gamma distribution (5) by substitution of $\theta = 1/\alpha$ and $\theta = 1/\beta$, correspondingly.

The resulting gamma-gamma distribution is given by [23, Ch. 9, Eq. (137)]

$$p_I(I;\alpha,\beta) = \frac{2\,(\alpha\beta)^{(\alpha+\beta)/2}}{\Gamma(\alpha)\Gamma(\beta)} I^{(\alpha+\beta)/2-1} K_{\alpha-\beta}\left(2\sqrt{\alpha\beta I}\right), \tag{9}$$

where $K_n(z)$ is the modified Bessel function of the second kind and the parameters α and β are related to the physical effects of large-scale and small-scale scintillations. The resulting scintillation index, σ_I^2, describes fluctuations (scintillations) of the received optical power, I, as measured by a point receiver and is defined as [23, Ch. 9, Eq. (9),(139)]

$$\sigma_I^2 = \frac{\langle I^2\rangle}{\langle I\rangle^2} - 1 = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\alpha\beta}, \tag{10}$$

where $\langle\rangle$ stands for averaging over time.

D. K Distribution

The K distribution is also useful for scintillation modeling. This distribution is derived from a modulation process of a negative exponential distribution and a gamma distribution, $I = XY$, where [23, Ch. 9.9.1]

$$p_x(x;b) = \frac{1}{b}\exp(-\frac{x}{b}) = w_x(x;1,b) \tag{11a}$$

$$p_y(y;\alpha) = \frac{\alpha\,(\alpha y)^{\alpha-1}}{\Gamma(\alpha)}\exp(-\alpha y) = w_y(y;\alpha,1/\alpha) \tag{11b}$$

and the resulting K distribution expression is given by

$$p_I(I;b,\alpha) = \frac{2\alpha}{\Gamma(\alpha)}(\alpha I)^{(\alpha-1)/2} K_{\alpha-1}(2\sqrt{\alpha I}). \tag{12}$$

III. Numerical Solution

The desired process, $x(t)$, is defined only for the positive part of the real axis, both because of the required PDF and because of the $\sqrt{x(t)}$ term in the SDE (7). However, preservation of such a positive domain restriction is not trivial and, in general, it is not maintained by default numerical methods, provided by commonly used packages such as Mathematica (`ItoProcess` command) or Matlab [24].

It transpired that the resulting SDE (7) is well known as the Cox-Ingersoll-Ross (CIR) process that is widely applied in economic studies [25, Eq. (17)]. The stable solution of such

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/JLT.2016.2525781, Journal of Lightwave Technology

a process is currently the subject of significant interest [26]–[28].

The proposed method for numerical solution involves the use of the implicit (backward) solution scheme. Following [26], we used the implicit Milstein scheme that preserves the SDE solution domain, while converging to the corresponding numerical solution. The implicit Milstein scheme may be described by [26, Eq. (40)]

$$\begin{aligned} x_{k+1} = x_k + f(x_{k+1})\Delta t + g(x_k)\sqrt{\Delta t}\xi_k \\ + \frac{1}{2} g(x_k) g'(x_k) \left(\Delta t \xi_k^2 - \Delta t\right) \end{aligned}, \tag{13}$$

where indexes k and $k+1$ are related to the values at times t_k and t_{k+1}, respectively, Δt is the sampling time and ξ_k are samples of the WGN process.

A. Gamma Distribution

The substitution of functions $f(x)$ and $g(x)$ results in a simple first order differential equation

$$x_{k+1} = x_k + \frac{\alpha\theta}{\tau_c}\Delta t - x_{k+1}\frac{\Delta t}{\tau_c} + \sqrt{\frac{2x_k\theta\Delta t}{\tau_c}}\xi_k + \frac{\theta\Delta t}{2\tau_c}\left(\xi_k^2 - 1\right) \tag{14}$$

with solution

$$x_{k+1} = \frac{x_k\tau_c + \alpha\theta\Delta t + \theta\Delta t\left(\xi_k^2 - 1\right)/2 + \left[2x_k\theta\tau_c\Delta t\right]^{1/2}\xi_k}{\Delta t + \tau_c} \tag{15}$$

that can be easily evaluated.

B. Gamma-Gamma & K Distribution

The generation of a gamma-gamma process requires the multiplication of two gamma distributions. These distributions may be derived from (15) by substitution of $\theta = 1/\alpha$ and $\theta = 1/\beta$ correspondingly. For example, for (8a)

$$x_{k+1} = \frac{x_k\tau_c + \Delta t + \Delta t\left(\xi_k^2 - 1\right)/2\alpha + \left[2x_k\tau_c\Delta t/\alpha\right]^{1/2}\xi_k}{1 + \Delta t}. \tag{16}$$

Given two gamma processes, x_k and y_k, with parameters α, β and τ_c, respectively, the gamma-gamma process, I_k, is simply the multiplication of the solutions of two gamma processes

$$I_k = x_k y_k,\ \forall k. \tag{17}$$

The K distribution may be generated in the same fashion as a gamma-gamma one, replacing the corresponding parameters in (15) in order to produce gamma and negative exponential distributions.

IV. Simulation Results

Simulations [2]. were carried out to validate the theoretical results. All simulations included 5×10^5 samples with sampling time $\Delta t = 10^{-4}$ sec and were repeated 100 times for the presented covariance function results. The initial values, x_0, were set to the mean value of the required process.

[2]The supplementary Matlab source code is published at: https://bykhov.github.io/gamma_sde/

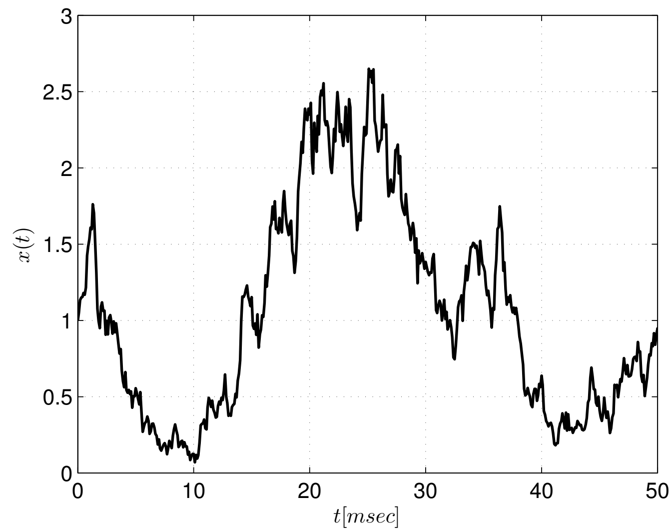


Fig. 1. Example of time-correlated gamma process ($\alpha = 1.05, \tau_c =$ 20msec).

A. Gamma Process

We started with the validation of gamma process properties. The example of a gamma process with PDF $p_x(x;\alpha)$ (8a) that was generated by (16) with $\alpha = 1.05$ and $\tau_c = 20$msec is presented in Fig. 1. The PDF of this process is presented in Fig. 2a with a close-up of the tail of the distribution. An additional PDF for the similar process with $\alpha = 5$ is presented in Fig. 2b. The presented histograms are based on the empirical cumulative distribution function (CDF) evaluated by the Kaplan-Meier estimate algorithm. The values of α were chosen in accordance with [23, Ch. 9.10.1] so as to be realistic. The resulting distributions exhibit a significant similarity to the analytical PDF (8).

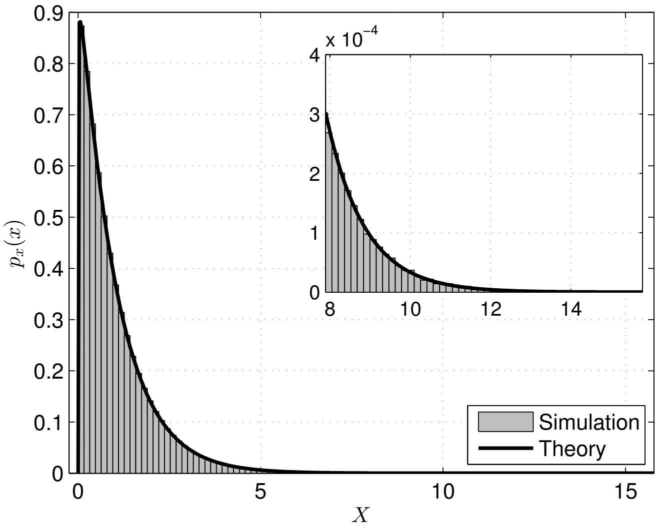
In order to quantitatively measure the discrepancy between the theoretical and simulated PDFs, a one-sample Kolmogorov-Smirnov test [29] was applied between numerically generated values and an analytical Gamma distribution. Since this test is designed for uncorrelated sequences and its adaptation for correlated sequences is non-trivial [30], 103 samples were randomly picked from the original sequence. The resulting null-hypothesis was not rejected at 1% confidence level.

The autocorrelation functions for $\tau_c = 5$ msec and 20 msec for $\alpha = 1.05$ are presented in Fig. 3. The results show a small discrepancy in covariance functions for $\tau > 2\tau_c$.

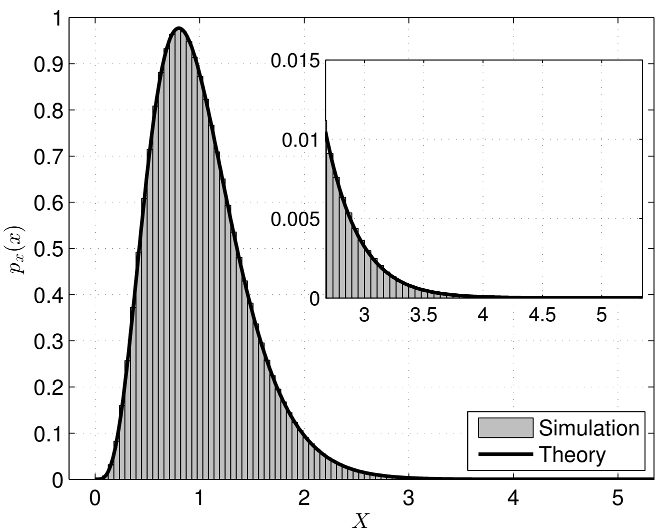
B. Gamma-Gamma & K Processes

The gamma-gamma process was generated by multiplication (17) of two gamma processes (8) with $\alpha = 5$, $\beta = 1.05$ and $\tau_c = 20$ msec. The PDF of the resulting signal is presented in Fig. 4, with a close-up of the tail of the distribution, and its covariance is presented in Fig. 5. Both graphs show similar results to those for the gamma distribution with a greater difference between covariance values for different realizations.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/JLT.2016.2525781, Journal of Lightwave Technology



(a) $\alpha = 1.05$.



(b) $\alpha = 5$.

Fig. 2. The empirical PDF of the generated gamma process and its theoretical plots for different α values ($\tau_c = 20$msec).

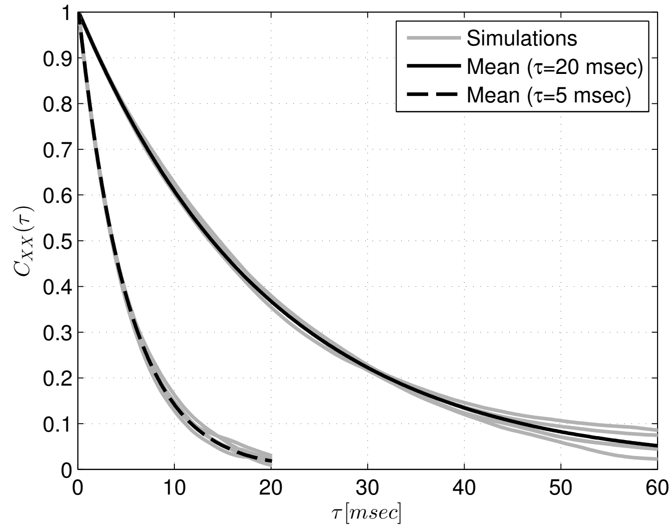


Fig. 3. Auto-covariance of the generated gamma process ($\alpha = 1.05$) that includes five simulation plots and the mean covariance for each of $\tau_c = 5$ msec and 20 msec values.

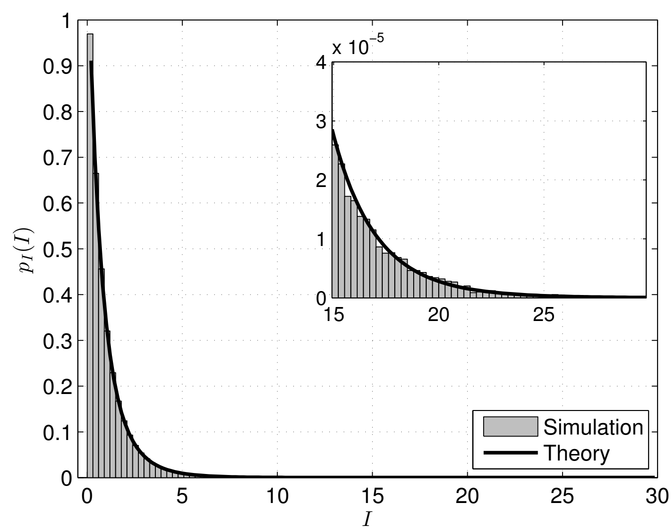


Fig. 4. The empirical PDF of the generated gamma-gamma process and its theoretical plot for $\alpha = 5$, $\beta = 1.05$ and $\tau_c = 20$ msec.

The K process is based on a multiplication of the generated negative exponential distribution process and the gamma process. An example of the PDF of the generated negative distribution process is presented in Fig. 6 and exhibits a significant similarity to the analytical PDF (11a). The results for the K process are similar to those for the gamma-gamma process and are not presented, for brevity.

C. Convergence & Numerical Stability Issues

The solution of the gamma process generation SDE (7) has to be bounded by $x(t) > 0$ for any values of constants α, θ and τ_c. However, the theoretical (and practical) limit for the positive solution of (7) is given by $\alpha > 0.5$ and does not depend on the θ and τ_c values [26].

When the negative exponential distribution (11a) and the related K distribution are of interest, the numerical accuracy of the resulting PDF decreases with the value of parameter b. Empirically, reasonably accurate results for the applied numerical scheme (13) may be reached for $b > 0.15$.

With regard to the sampling time, Δt, the SDE solution is expected to be more accurate for smaller values of Δt. Moreover, it is reasonable to assume the relation $\tau_c \gg \Delta t$.

V. Summary & Conclusions

A mathematically simple solution is proposed for the modeling of moderate to strong turbulence channel conditions that are described by gamma-gamma and K distributions and an exponential autocorrelation function. The channel states are generated by repeated calculation of a simple discrete differential equation that is based only on a WGN sequence, correlation time and distribution parameters. SDE-provided simulation results show the resemblance of the generated channel states to the theoretical predictions.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/JLT.2016.2525781, Journal of Lightwave Technology

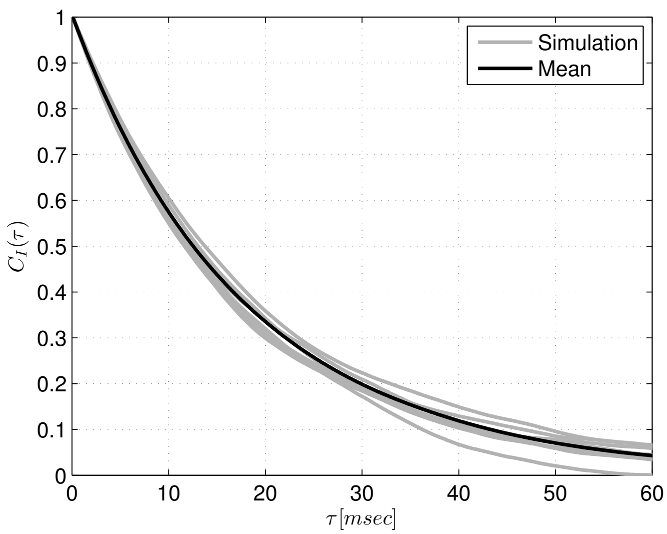


Fig. 5. Auto-covariance of the generated gamma-gamma process that includes five examples and the mean covariance. The required correlation time is $\tau_c = 20$ msec.

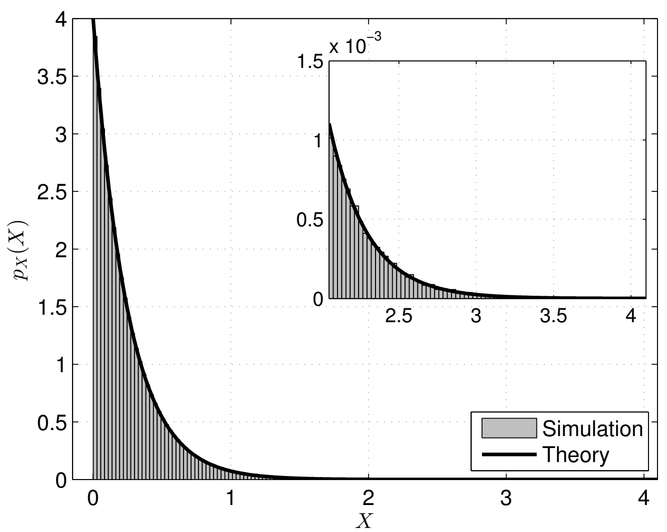


Fig. 6. The PDF of the generated negative exponential distribution with $b = 0.25$.

Acknowledgment

We want to thank Prof. Vladimir Lyandres for inspiring discussions.

References

[1] K. Ward, R. Tough, and P. Shepherd, "Modelling sea clutter: correlation, resolution and non-Gaussian statistics," in *Radar 97 (Conf. Publ. No. 449)*, pp. 95–99, Oct 1997.

[2] R. J. A. Tough and K. D. Ward, "The correlation properties of gamma and other non-Gaussian processes generated by memoryless nonlinear transformation," *Journal of Physics D: Applied Physics*, vol. 32, no. 23, pp. 3075–3084, 1999.

[3] V. Kontorovich and V. Lyandres, "Stochastic differential equations: an approach to the generation of continuous non-Gaussian processes," *IEEE Transactions on Signal Processing*, vol. 43, pp. 2372–2385, Oct 1995.

[4] S. Primak, V. Lyandres, and V. Kontorovich, "Markov models of non-Gaussian exponentially correlated processes and their applications," *Phys. Rev. E*, vol. 63, May 2001.

[5] S. Primak, V. Kontorovitch, and V. Lyandres, *Stochastic methods and their applications to communications: stochastic differential equations approach*. John Wiley & Sons, 2005.

[6] B. Epple, "Simplified channel model for simulation of free-space optical communications," *IEEE/OSA Journal of Optical Communications and Networking*, vol. 2, no. 5, pp. 293–304, 2010.

[7] F. Davidson, C. Juan, and A. Hammons, "Channel fade modeling for free-space optical links," in *Millitary Communication Conference (MILCOM)*, pp. 802–807, Oct 2010.

[8] A. Jurado-Navas, J. Maria, M. Castillo-Vazquez, and A. Puerta-Notario, *Numerical Simulations of Physical and Engineering Processes*, ch. 7, A Computationally Efficient Numerical Simulation for Generating Atmospheric Optical Scintillations, pp. 157–180. InTech, Sep 2011. dx.doi.org/10.5772/25139.

[9] D. Bykhovsky, "Free-space optical channel simulator for weak-turbulence conditions," *Applied Optics*, vol. 54, pp. 9055–9059, Oct. 2015.

[10] S. Kay, "Representation and generation of non-Gaussian wide-sense stationary random processes with arbitrary PSDs and a class of PDFs," *IEEE Trans. Signal Process.*, vol. 58, pp. 3448–3458, Jul 2010.

[11] D. Bykhovsky, D. Elmakayes, and S. Arnon, "Experimental evaluation of free space links in the presence of turbulence for server backplane," *Journal of Lightwave Technology*, vol. 33, pp. 2923–2929, Jul. 2015.

[12] M. Kashani, M. Uysal, and M. Kavehrad, "A novel statistical channel model for turbulence-induced fading in free-space optical systems," *Journal of Lightwave Technology*, vol. 33, pp. 2303–2312, June 2015.

[13] S. Rajbhandari, Z. Ghassemlooy, P. Haigh, T. Kanesan, and X. Tang, "Experimental error performance of modulation schemes under a controlled laboratory turbulence FSO channel," *Journal of Lightwave Technology*, vol. 33, pp. 244–250, Jan 2015.

[14] M. Uysal, S. M. Navidpour, and J. Li, "Error rate performance of coded free-space optical links over strong turbulence channels," *IEEE Communications Letters*, vol. 8, pp. 635–637, Oct. 2004.

[15] S. Navidpour, M. Uysal, and M. Kavehrad, "BER performance of free-space optical transmission with spatial diversity," *IEEE Transactions on Wireless Communications*, vol. 6, pp. 2813–2819, Aug. 2007.

[16] T. B. Reddy, R. Subadar, and P. Sahu, "Outage probability of SC receiver over exponentially correlated K fading channels," *IEEE Communications Letters*, vol. 14, pp. 118–120, Feb. 2010.

[17] K. P. Peppas, G. C. Alexandropoulos, C. K. Datsikas, and F. I. Lazarakis, "Multivariate gamma-gamma distribution with exponential correlation and its applications in radio frequency and optical wireless communications," *IET Microwaves, Antennas & Propagation*, vol. 5, pp. 364–371, Feb. 2011.

[18] B. Liu and J. Munson, D.C., "Generation of a random sequence having a jointly specified marginal distribution and autocovariance," *IEEE Transactions on Acoustics, Speech and Signal Processing*, vol. 30, pp. 973–983, Dec 1982.

[19] J. Filho and M. Yacoub, "Coloring non-Gaussian sequences," *IEEE Transactions on Signal Processing*, vol. 56, pp. 5817–5822, Dec 2008.

[20] E. Conte, M. Di Bisceglie, M. Lops, and G. Ricci, "Simulation of correlated random fields with gamma-distributed amplitude for SAR applications," in *International Remote Sensing: Global Monitoring for Earth Management (IGARSS)*, vol. 4, pp. 2397–2400, Jun 1991.

[21] Y. Dong, L. Rosenberg, and G. Weinberg, "A hybrid method for generating correlated gamma sequences for sea-clutter simulation," in *International Conference on Radar*, pp. 548–553, Sept 2013.

[22] S. Primak, "Generation of compound non-Gaussian random processes with a given correlation function," *Phys. Rev. E*, vol. 61, pp. 100–103, Jan. 2000.

[23] L. C. Andrews and R. L. Phillips, *Laser Beam Propagation through Random Media*. SPIE, 2nd ed., 2005.

[24] A. D. Horchler, "Matlab toolbox for the numerical solution of stochastic differential equations." https://github.com/horchler/SDETools, 2013. Version 1.2.

[25] J. C. Cox, J. Ingersoll, Jonathan E., and S. A. Ross, "A theory of the term structure of interest rates," *Econometrica*, vol. 53, pp. 385–407, Mar 1985.

[26] A. Neuenkirch and L. Szpruch, "First order strong approximations of scalar SDEs defined in a domain," *Numerische Mathematik*, vol. 128, no. 1, pp. 103–136, 2014.

[27] A. Alfonsi, "Simulation of the CIR process," in *Affine Diffusions and Related Processes: Simulation, Theory and Applications*, vol. 6 of *Bocconi & Springer Series*, pp. 67–92, Springer International Publishing, 2015.

[28] J.-F. Chassagneux, A. Jacquier, and I. Mihaylov, "An explicit Euler scheme with strong rate of convergence for financial SDEs with non-Lipschitz coefficients," *ArXiv e-prints*, May 2015.

[29] F. J. Massey, "The Kolmogorov-Smirnov test for goodness of fit," *Journal of the American Statistical Association*, vol. 46, pp. 68–78, Mar. 1951.

[30] M. S. Weiss, "Modification of the Kolmogorov-Smirnov statistic for use with correlated data," *Journal of the American Statistical Association*, vol. 73, pp. 872–875, Dec. 1978.