LINMA2370 - Homework 1

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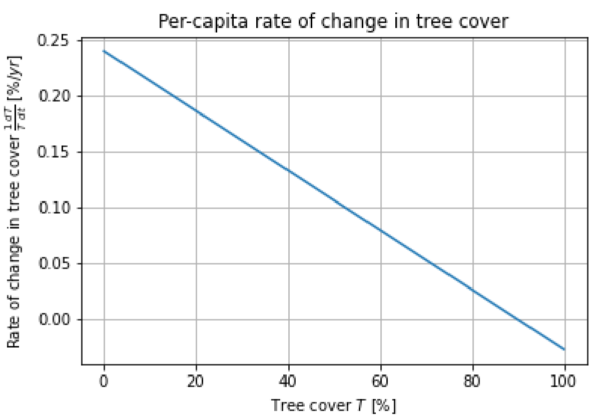
1 Question 1

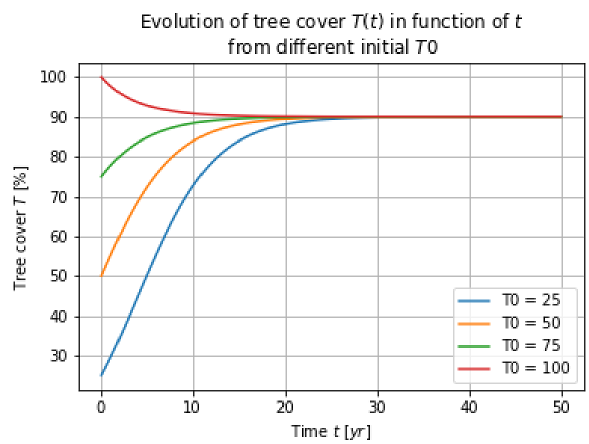
From the logistic growth equation for tree cover given in the statement (Eq 1) we can write the code of this differential equation in Python. By using the default values for different parameters $(R0, r_m, h_R, k)$ from the Table 1 in the statement, we can plot the per-capita rate of change in tree cover against the tree cover as shown in the Figure 1a.

In this figure, we observe that there is 1 equilibria $T^* = 90\%$ which correspond to the maximum carrying capacity of the land in term of the fraction of tree cover k. When $T < 90\%$, the rate of change in tree cover is positive while when $T > 90\%$. Moreover, the variation of the rate $\frac{d}{dT}\left(\frac{1}{T}\frac{dT}{dt}\right)$ is constant. As we approach the equilibria $T = 90\%$, the variation's amplitude become smaller and smaller.
Thus, we have 1 equilibria and it is stable and attractive. An another illustration of this equilibria is given in Figure 1b.

$$\frac{1}{T}\frac{dT}{dt} = r_m \frac{R}{h_R + R}\left(1 - \frac{T}{k}\right) \tag{1}$$



(a)



(b)

Figure 1: Per capita of change in tree cover and evolution of tree cover

2 Question 2

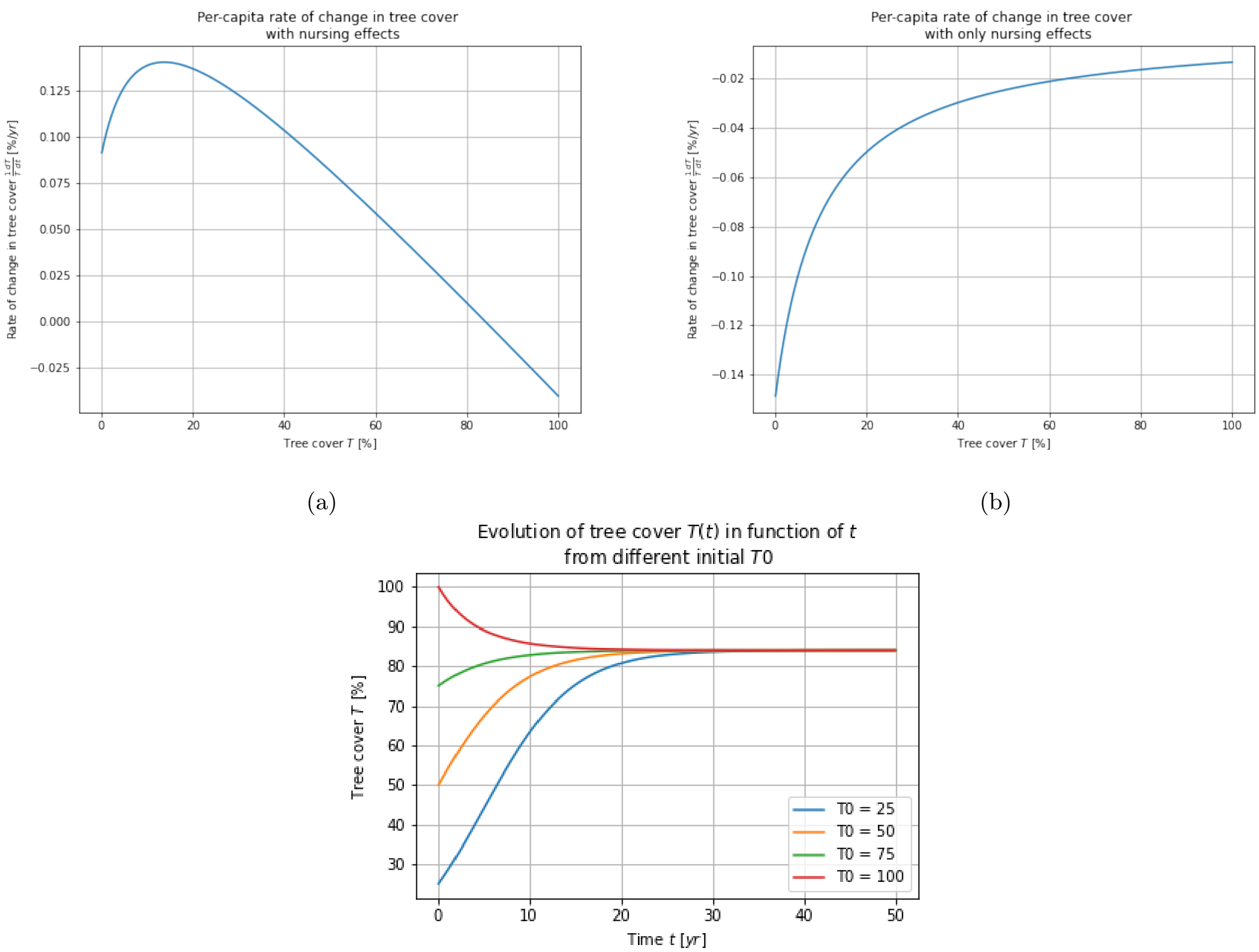
The nursing effect modeled with the Monod function (Eq 2). The plot of the per capita rate of change in T (without the logistic growth) against the tree cover T is given in Figure 2b.
We observe the rate of change remain negative for all $T \in (0, 100]$. Moreover, its variation is greater as T decrease which is consistent with the definition of the nursing effect.

$$\frac{1}{T}\frac{dT}{dt} = -m_n \frac{h_n}{T + h_n} \tag{2}$$

Combining both the Eq 1 and Eq 2, we can obtain the per-capita logistic growth for tree cover taking into account the nursing effect (Eq.3).

Afterward, with the same parameter as in the Question 1, we compute the plot of $\frac{1}{T}\frac{dT}{dt}$ against T is given in Figure 2a. We observe the change in tree cover for all value of $T \in (0, 100]$ is lower than in Figure 1a due to nursing effect. The variation of the change $\frac{d}{dt}\left(\frac{1}{T}\frac{dT}{dt}\right)$ is not constant as in 1a. Here, it increase then decrease constantly as shown in Figure 2b.
There is only 1 stable and attractive equilibria $T^* = 84.02\%$ in this model as illustrated in Figure 2c. Remark that this equilibria is lower than the previous one (90%) due to nursing effect.

$$\frac{1}{T}\frac{dT}{dt} = r_m \frac{R}{h_R + R}\left(1 - \frac{T}{k}\right) - m_n \frac{h_n}{T + h_n} \tag{3}$$



(a)

(b)

(c)

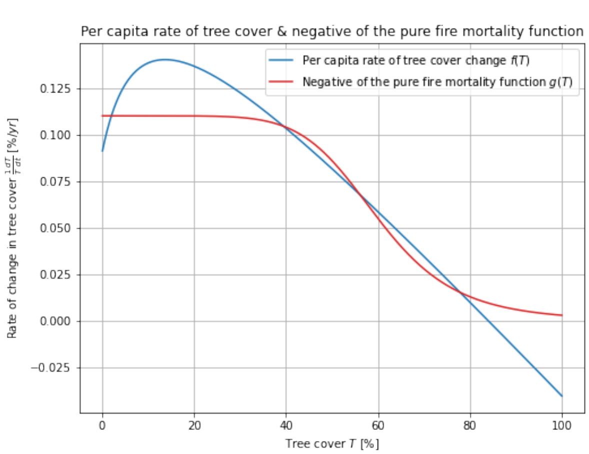
Figure 2

3 Question 3

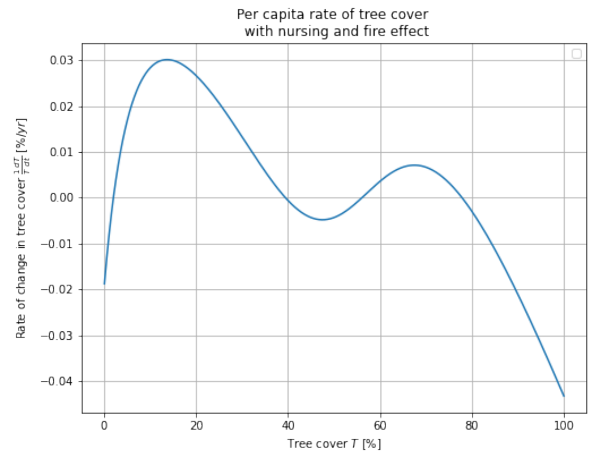
Adding Eq 3 with the fire effect (Eq 4) which model wildfire as a function of T, we obtain the final rate of change in tree cover (Eq 5). With the default parameters, we can plot $f(T)$ against $g(T)$ in Figure 3a where $f(T)$ which is the per-capita logistic growth for tree cover taking into account the nursing effect and $g(T)$ which is the pure fire mortality function.

$$\frac{1}{T}\frac{dT}{dt} = -m_f \frac{h_f^p}{T^p + h_f^p} \tag{4}$$

$$\frac{1}{T}\frac{dT}{dt} = r_m \frac{R}{h_R + R}\left(1 - \frac{T}{k}\right) - m_n \frac{h_n}{T + h_n} - m_f \frac{h_f^p}{T^p + h_f^p} \tag{5}$$



(a)



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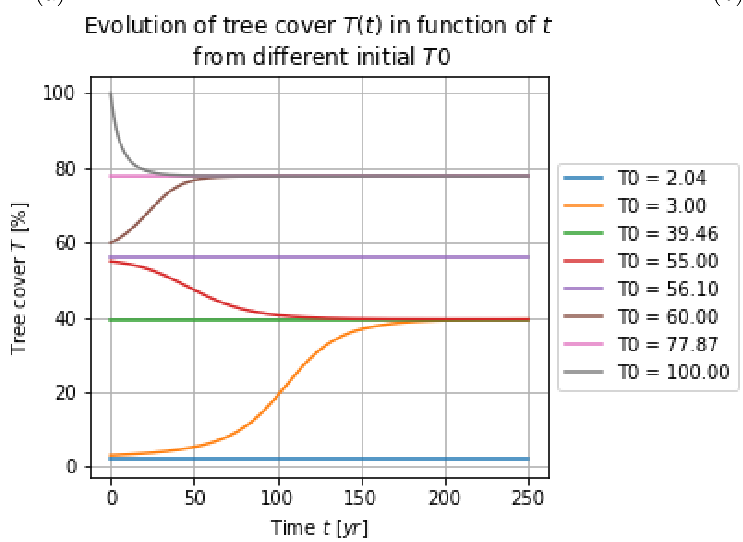


(c)

Figure 3

From the figure 3a, we can see 4 intersections between $f(T)$ and $g(T)$. Since the rate of change is equal to $f(T) - g(T)$, we deduce that there are 4 equilibria for T taking all effects into account (Eq 5 and Figure 3b). From the course note, an equilibrium $(\bar{x}, \bar{u})$ is said to be stable if :

$$\begin{aligned} &\forall \epsilon > 0 \; \exists \delta > 0 \text{ such that } \forall x_0 \in \Omega : \\ &\left\|x\left(t_0\right) - \bar{x}\right\| < \delta \Longrightarrow \left\|x\left(t, x\left(t_0\right), \bar{u}\right) - \bar{x}\right\| < \epsilon \; \forall t \geq t_0. \end{aligned} \tag{6}$$

And with this definition, using `scipy.optimize.fsolve()`, we obtain approximate values for equilibriums and deduce their stability:

- $T^* = 2.04$ This equilibria is an unstable one.
- $T^* = 39.46$ This equilibria is stable.
- $T^* = 56.10$ This equilibria is unstable
- $T^* = 77.87$ This equilibria is stable

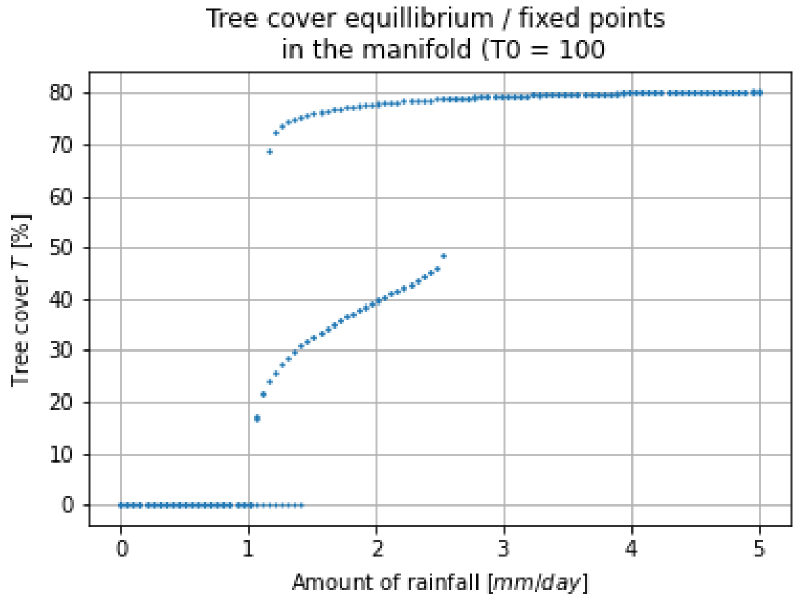


Figure 4

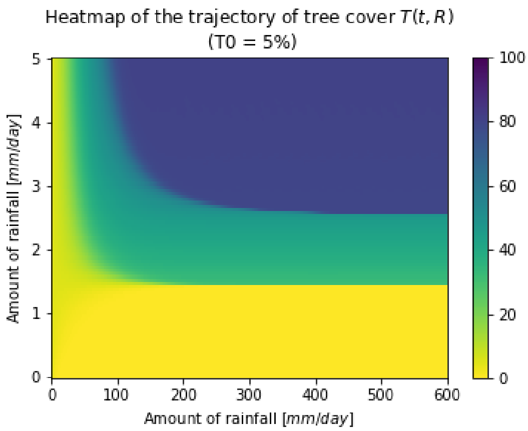
The figure 3c shows the evolution of $T(t)$ from different provided $T0$ initial points. Those points are arbitrarily chosen to illustrate stability:

- For our first point (T0 = 2.04) the equilibria is unstable because by taking another close starting point (T0 = 3.00) the curves diverges from each other as t growth. The same can be said for T0 = 56.10.
- On the other hand, we see that for T0 = 39.46 we can take another starting point relatively close to it such as T0 = 55.00 and we can see that both curve converge to a similar value. Same for the last solution to f(T) - g(T) which is T0 = 77.87. We can add that the stable equilibria are also attractive ones.

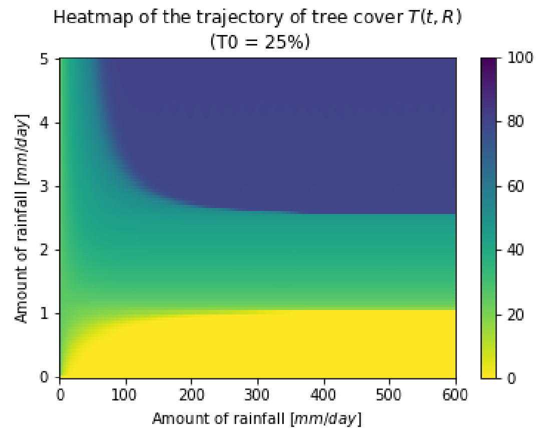
4 Question 4

In Figures 5a-5e, we plotted different heatmaps that correspond to the simulation of the differential equation for the rate of change in tree cover for 600 years which correspond to the solutions to Eq 5, with different initial tree cover $T0$.
We notice that the final tree cover T^* depend on its initial value $T0$. With a $R = 2$ mm/day, in Figures 5d and5e, we would have reached a Savannah (illustrated in green) while in Figures 5a,5b,5c we would have reached a dense forest (illustrated in navy blue).

Then we've been asked to take each starting pair (R, T0) and to collect the equilibrium points that correspond to t=600 year. We refer to those pairs as the fixed points. We plotted those fixed point in the manifold (T, R) as shown in Figure 4. We observe different biffurcations but this plot do not recovers all the fixed points of the dynamical system. Indeed, the fixed points in the plot represent only those who are attractive and stable. Those who are instable cannot be represented since it depend on directly on $T0$ and cannot be recovered at $t = 600$.



(a)



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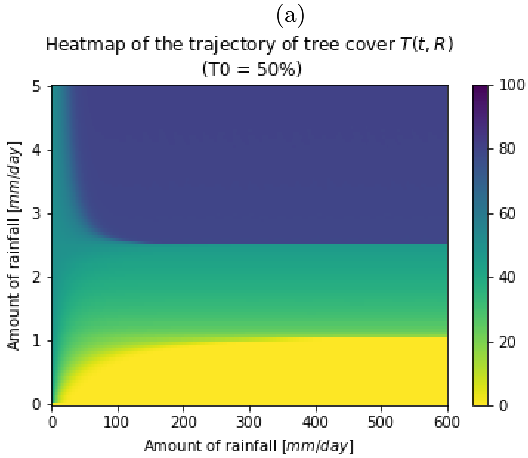

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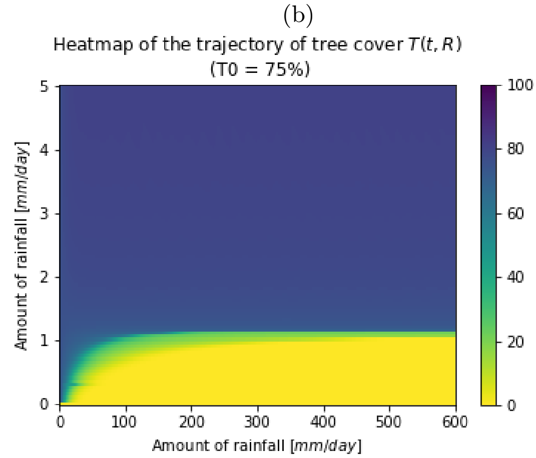
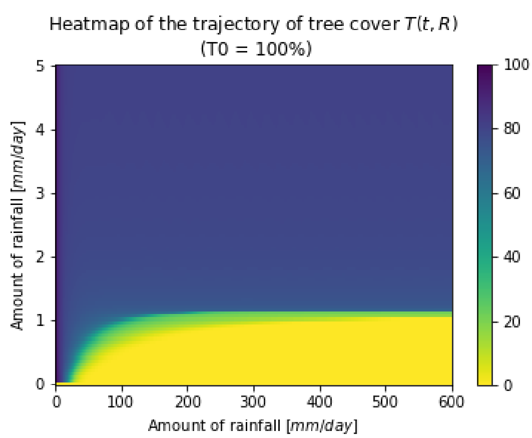


(d)



(e)

Figure 5