LINMA2370 - Homework 2

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1 Analyze the tree cover dynamics

The equation for the tree cover dynamics developed in Part I reads:

$$\frac{dT}{dt} = r(R)T\left(1 - \frac{T}{k}\right) - m_n T \frac{h_n}{T + h_n} - m_f T \frac{h_f^p}{T^p + h_f^p}, \tag{1}$$

where

$$r(R) = r_m \frac{R}{h_R + R}$$

We are searching for the equilibrium of our system. The notion of invariant set is a generalization of the concept of equilibrium. It's given by the following definition:

A set $\mathcal{X} \times U \subseteq \mathbb{R}^n \times \mathbb{R}^m$ is said to be (positively) invariant for the dynamical system $\dot{x} = f(x, u)$ if, for all $x_0 \in \mathcal{X}$ and for all input signal $t \mapsto u(t) \in U$, the trajectory $t \mapsto x(t, x_0, u(t))$ remains in $\mathcal{X}$ for all $t \geq t_0$ whenever it is defined.

In our problem, the set of Tree cover T has a domain in [0, 100]. So, we would like to know if for all input the trajectory remains in this interval.
For that we can use the Bony's Theorem that help us check whether or not the trajectories are leaving the domain when we are on it's boundary.

For starter, we define the normal vector called *the Bony's outward normal vector*. A Bony's outward normal vector to $\mathcal{X} \subseteq \mathbb{R}^n$ at $x \in \partial\mathcal{X}$ is a vector $n \in \mathbb{R}^n$ such that $n = \lambda(y - x)$ where $\lambda > 0$ and y is the center of an open ball $B \subseteq \mathbb{R}^n$ such that $x \in \partial B$ and $B \cap \mathcal{X} = \emptyset$; if no such open ball exists, $\mathcal{X}$ has no outward normal vector at x.
And the Bony's theorem reads as follow.
Theorem 7.12 (Bony's theorem on invariant sets). Let f be locally Lipschitz continuous vector field defined on an open set $\Omega \subseteq \mathbb{R}^n$, and $\mathcal{X}$ a closed set of Ω (i.e. the intersection of a closed set of $\mathbb{R}^n$ with Ω). If $\langle f(x), n(x)\rangle \leq 0$ for every $x \in \partial\mathcal{X}$ and every vector $n(x)$ outward normal to $\mathcal{X}$ at x, then $\mathcal{X}$ is (positively) invariant for f.
We then have to prove that the function T is locally Lipschitz continuous on the closed set $\mathcal{X}$ = [0, 100]. We can use Proposition 0.24 to prove that it's Lipschitz continuous and then use the Definition 0.23 of the course notes to prove that it is also locally Lipschitz continuous.
Proposition 0.24 If U is an open convex subset of X and $f : U \to Y$ is differentiable, then f is Lipschitz continuous if and only if f' is bounded.
A function f is bounded if there exist a real number M such that $|f(x)| \leq M$. We have seen in the part 1 the value of the derivative of T over different values of R, here is the absolute value of this function.
We can see in the figure 1 that $|\frac{dT}{dt}| \leq M$ with M = 0.044*T = 4,4. So we have proven that the function T is Lipschitz continuous.
Next the locally Lipschitz continuity,
Definition 0.23 (Lipschitz continuity). A function $f : U \to Y$, where U is a subset of X, is said to be Lipschitz continuous if its Lipschitz constant is finite:

$$\mathrm{Lip}(f) := \sup_{\substack{x,y \in U \\ x \neq y}} \frac{\|f(x) - f(y)\|_Y}{\|x - y\|_X} < \infty.$$

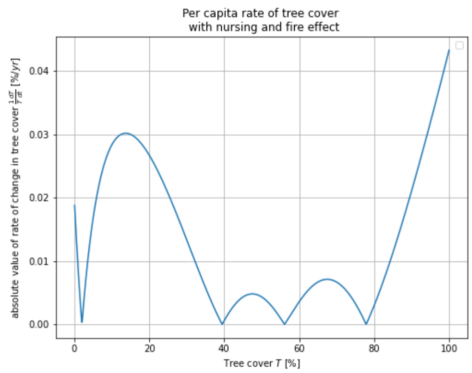


Figure 1

The function f is said to be locally Lipschitz continuous if, for every $x \in U$, there exists $r \in (0, \infty)$ such that f is Lipschitz continuous on $\{y \in U \mid \|y - x\|_X \leq r\}$.
Here the norm on the set $X = [0,\, 100]$ so we use the norm as the euclidean distance and by taking $U =]0,\, 100[$ we can easily see that any closed ball on the derivative of T satisfies the proposition 0.24. Thus we have proved that the function T is locally Lipschitz continuous and we can use the Bony's theorem (Theorem 7.12).
We look at the closure of the domain of X which are 0 and 100 and find the open balls B_1 and B_2 such that their closure are 0 and 100 respectively. We also look at the trajectory of T and compute its scalar product with the Bony's outward normal vectors $n_1(x)$ and $n_2(x)$.
By definition the outward normal vector, we have $n(x) = \lambda(y - x)$ hence,

$$n_1(0) = \lambda(y - 0)$$

and

$$n_2(x) = \lambda(y - 100)$$

such that y is the center of the open balls and λ is a strictly positive number.
The trajectory function on the figure 2 is a function of one variable so its scalar product is just a multiplication.

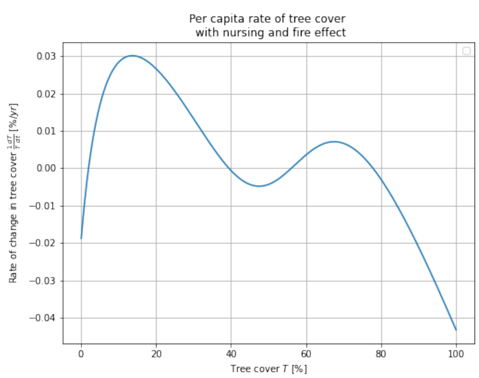


Figure 2

We must not forget to multiply this function by T to get the real trajectory. Then for T = 0 we have $< 0, n_1(0) >$ which equal to zero for every vector $n_1(0)$ outward normal. Similarly for the other x = 100

$$\in \partial \mathcal{X}$$

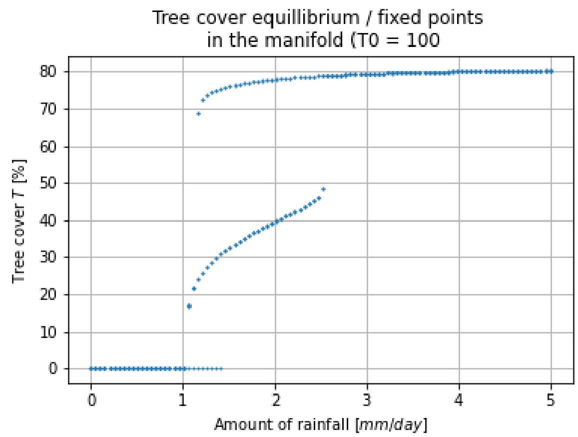
we have the scalar product of $< f(100), n_2(100) >$. As we see in figure 2 the value at this point is negative and is equal to -0.044*100 and by definition $n_2(100) = \lambda(y - 100)$. We know that y is the center of the open ball B_2 such that $B_2 \cap \mathcal{X} = \emptyset$ then y $>$ 100. Hence we have $n_2(100) > 0$ and $< f(100), n_2(100) > < 0$ for every vector $n_2(100)$ outward normal.

We have thus proven that [0, 100] is invariant on T under this dynamics.

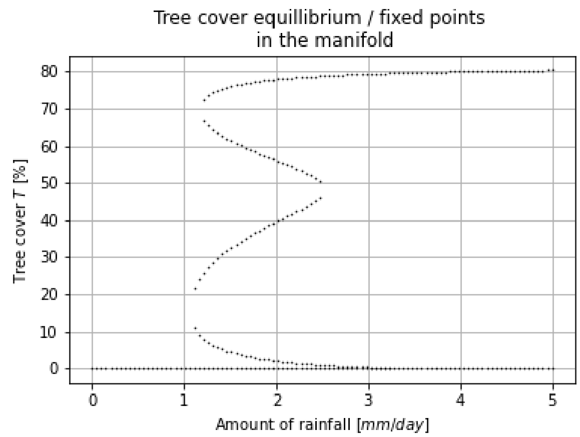
The (implicit) equation defining the set of equilibria $(\bar{T}, \bar{R})$ is given by the following equation,

$$0 = r_m \frac{R}{h_R + R} T \left(1 - \frac{T}{k}\right) - m_n T \frac{h_n}{T + h_n} - m_f T \frac{h_f^p}{T^p + h_f^p},$$

We have computed the graph of those equilibria, describing the tree cover at rainfall values in the range of [0 5mm/day] similarly to the first part of the project. Here's our resulting graphs,



(a) Attractive equilibria of the system



(b) Attractive and repulsive equilibria of the system

By superimposing the figures 3a and 3b we can see where the repulsive equilibria resides. All the blue parts are attractive ones and the resulting grey part are repulsive.

2 Extended model of tree cover dynamics

Up until now we have consider a dynamical model as a function of the current tree-cover T and the rainfall parameter as an input. However, in reality there is a phenomenon called 'vegetation-rainfall feedback' that increases the amount of rainfall locally where there are high number of trees. Using this new information we can better our model as such,
The tree cover evolves as:

$$\frac{dT}{dt} = r(R) T \left(1 - \frac{T}{k}\right) - m_n T \frac{h_n}{T + h_n} - m_f T \frac{h_f^p}{T^p + h_f^p},$$

while the amount of rainfall is determined by:

$$\frac{dR}{dt} = r_R \left(\left(R_{\text{constant}} + b \frac{T}{k} \right) - R \right)$$

with our new parameter $r_R, R_{constant}$ and b we can simulate the evolution over 600 years with different pairs of initial values $(T_0, R_0, R_{constant})$.
We obtain the graph 4 which show the vegetation-rainfall feedback. Indeed we see that by starting with a high percentage of tree cover the result after 600 years keep this percentage high even with a really low amount of starting rainfall.
This is because this high number of tree favors a great amount of rainfall as we can see in its definition : the derivative over time of the rainfall will keep growing. Its a great feedback loop when the percentage of tree cover is big.
Now, for a very low tree coverage, the results are similar to the equilibria found in the old formula (we see similarity with figure 3a). With a small variation for the amount of rainfall between 0.5 and 1.5 that yield better tree coverage, if the tree cover is very low it will stay that way because there isn't enough trees to activate the phenomenon of 'vegetation-rainfall feedback' thus the simulation behave as before.

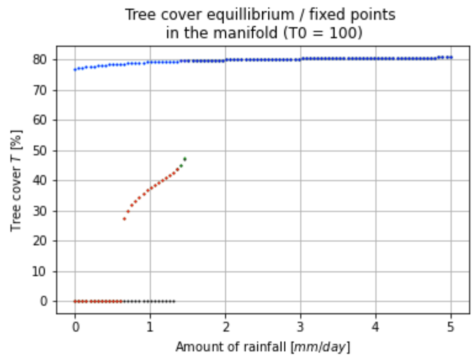


Figure 4: Equilibria after 600 years from simulation the new two-dimensional tree cover dynamics

3 Analyze and visualize GeoSpatial remote sensing data

We were asked using the geospatial data to predict the evolution of the tree cover on the region of South America, Africa, and Australasia. In particular how those tree cover changes from 2014 to 2044. The Figure 5 is result of our computation.

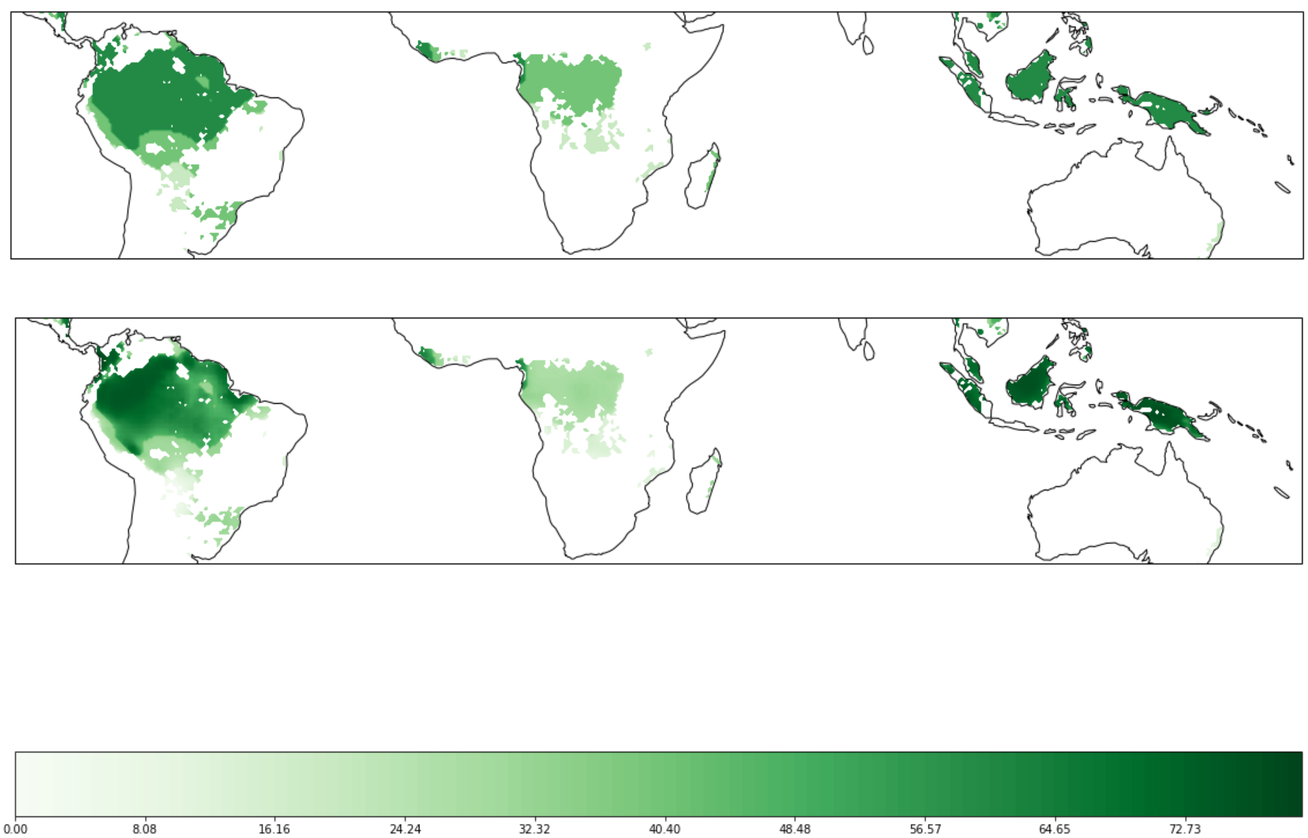


Figure 5: the tree cover after 30 years. Above we have the current tree cover (2014), and bellow, the prediction for the year 2044.

We observe that based on our simulation, Amazonian tropics are either in a forest state, or savannah state.

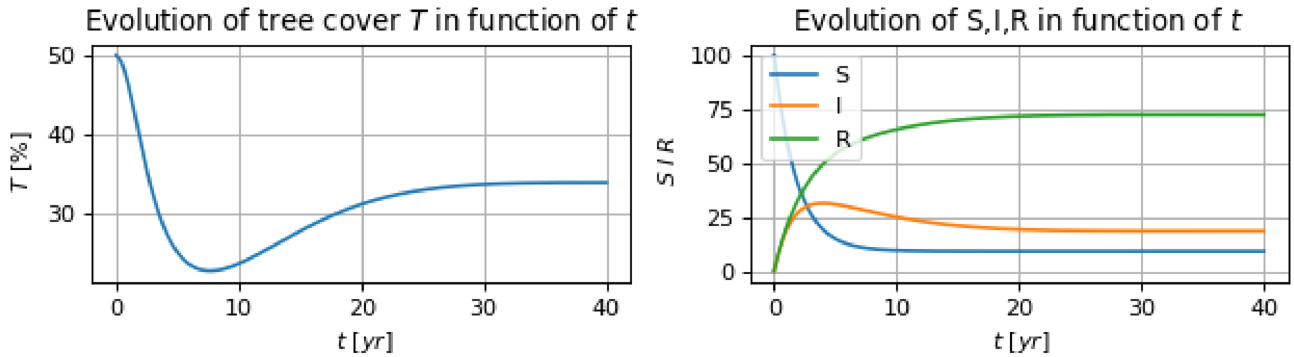


Figure 6: Evolution of tree cover T and S, I, R from the initial state

Indeed, from the hysteresis obtained in question 1 (Figure 3b), most of this regions' initial tree cover and rainfall is sufficiently big enough to make it converge to the attractive equilibrium which is a forest. This is especially visible in the north of this tropic.

However, in the Congo forest, due to its initial tree cover, closer to a savannah, and a lesser amount of rainfall, the forest remain a savannah.

4 Model human action

For this last question, we tried to model human action through a SIR model:

- $I \in [0, 100]$,,'infected' is the proportion of people who are cutting trees.
- $R \in [0, 100]$,'recovered' is the proportion of people who are planting trees.
- $S \in [0, 100]$, 'susceptible' is the proportion of people who currently do nothing. they can move either to the group I or R.

The model derived was the following:

$$\dot{T} = r(R)T\left(1 - \frac{T}{k}\right) - m_n T\frac{h_n}{T + h_n} - m_f T\frac{h_f^p}{T^p + h_f^p} + r_d IT + r_f R(100 - T)$$

$$\begin{cases} \dot{S} = & \mu(S + I + R) & -\mu S & -\alpha S(100 - T) & -\beta ST & \\ \dot{I} = & & -\mu I & & +\beta ST & -\gamma I(100 - T) \\ \dot{R} = & & -\mu R & +\alpha S(100 - T) & & +\gamma I(100 - T) \end{cases}$$

- $\alpha = 5\,10^{-3}$ is the transition rate from Susceptible (S) to Recovered (R) group
- $\beta = 5\,10^{-3}$ is the transition rate from Susceptible (S) to Infected (I) group
- $\gamma = 5\,10^{-4}$ is transition rate from Infected (I) to Recovered (R) group
- $r_d = 1\,10^{-2}$ is the rate of deforestation
- $r_f = 1\,10^{-3}$ is the rate of reforestation
- $\mu = 5\,10^{-2}$ is the death /movement rate of the population

With theses parameters, and with the initial tree cover $T0 = 50\%$ and $S0 = 100\%\, I0 = 0\%, R0 = 0\%$, we obtain the evolution in Figure 6. We see that Tree cover drop from 50% to around 25% then it stabilise around 34%. It is due to the initial increase in I, the part of the population who cut trees. I then decrease as T decrease while the number of S, proportion who is planting trees.