LINMA2450 - Homeworks

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1 Analysis of a basic unit commitment problem

Question 1.1 Write the Lagrangian dual and characterize $L(\pi)$ in term of convexity and differentiability

For our unit commitment problem, the formulation of the Lagrangian dual is, for an arbitrary π_t :

$$L(\pi) = \min_{p,u,v,w} \sum_{g\in\mathcal{G}} \sum_{t\in\mathcal{T}} \left[C_g^P p_{g,t} + C_g^{NL} u_{g,t} + C_g^{SU} v_{g,t}\right] + \sum_{t\in\mathcal{T}} \pi_t (D_t - \sum_{g\in\mathcal{G}} p_{g,t}) \tag{1}$$

The lagrangian dual problem becomes :

$$\max_{\pi} L(\pi) = \max_{\pi} \min_{p,u,v,w} \sum_{g\in\mathcal{G}} \sum_{t\in\mathcal{T}} \left[C_g^P p_{g,t} + C_g^{NL} u_{g,t} + C_g^{SU} v_{g,t}\right] + \sum_{t\in\mathcal{T}} \pi_t (D_t - \sum_{g\in\mathcal{G}} p_{g,t}) \tag{2}$$

In term of convexity, the lagrangian dual $L(\pi)$ is concave. Indeed, this function is linear with respect to our parameter π and since the Lagrangian dual can be seen as the pointwise minimum of a set of linear functions, $L(\pi)$ is concave. It is also non differentiable since $L(\pi)$ contains binary variables.

Question 1.2 Provide a reason why it is computationally interesting to relax this constraint in particular ? Make it explicit in the expression of the Lagrangian dual problem.

The idea of Lagrangian relaxation, for the unit commitment problem, consists in including the demand and capacity constraints together with corresponding Lagrange multipliers into the objective function, so that the original problem decomposes into I independent single unit subproblems of lower dimension.
In this way the coupling constraints are removed from the equation.
One should of course prefer dualizing constraints that make the problem hard to solve, so that the auxiliary problem becomes simpler. This will typically be the case of coupling constraints etc.
Optimization of a continuous but not necessarily differentiable convex function can be achieved using a subgradient algorithm.

Question 1.3 Provide the analytical expression of the subgradient of the Lagrangian function $L(\pi)$

From the course note, proposition 6 of chapter 7, we know that $(d - Dx(\pi))$ is a subgradient of

$$z(\pi) = \{\max c^T x \qquad \text{st.} Ax \le b, Dx \le d, x \in \mathbb{Z}_+^n\}$$

where $x(\pi) \in X$ is the value of the optimum x for the relaxation with vectors π. In our case, the subgradient can be written as the following vector :

$$\partial f(p,u,v,w) = \left(D_0, -\sum_{g\in\mathcal{G}} p_{g,0}, ..., D_{\mathcal{T}} - \sum_{g\in\mathcal{G}} p_{g,\mathcal{T}} \right) \tag{3}$$

Question 1.4 Write the pseudo-code of a subgradient algorithm that seek to optimize the Lagrangian function and would allow you to compute the optimum Lagrangian multiplier π_t^*

Algorithm 1 Subgradient method

Choose initial π^0
for k **do** = 0,1,2...
  $\pi^{k+1} = \max\left[\pi^k - h_k \partial f\big(p(\pi^k), u(\pi^k), v(\pi^k), w(\pi^k)\big), 0\right]$
end for

We begin by setting our lagrangian multipliers vector π_t. Once it is initialised, we can perform the subgradient algorithm given above.

2 Dantzig-Wolfe column generation algorithm

Question 2.1 Write the full Master Problem1 of D-W applied to the problem (1). Explain why it amounts to solving the Lagrangian relaxation we are interested in.

Instead of optimizing on the variable π and x on the Lagrangian problem, we can instead reformulate the problem by letting our previous variables $p_{g,t}$ expressed as a combination of extreme point $z^i_{p_{g,t}}$:

$$p_{g,t} = \sum_i^{I_p} z^i_{p_{g,t}} \lambda^i_{p_{g,t}}$$

we can do the same for $u_{g,t}$,$v_{g,t}$,$w_{g,t}$ by doing so, the reformulation of our IP problem with Dantzig-Wolfe becomes :

$$\sum_{t\in\mathcal{T}} \min_\lambda \quad \sum_{g\in\mathcal{G}} \left[C_g^P (\sum_{i=1}^I z^i_{p_{g,t}} \lambda_i) + C_g^{NL} u_{g,t} (\sum_{i=1}^I z^i_{u_{g,t}} \lambda_i) + C_g^{SU} v_{g,t} (\sum_{i=1}^I z^i_{v_{g,t}} \lambda_i) \right]$$

$$\text{st.} \quad \sum_{g\in\mathcal{G}} \sum_{i=1}^I z^i_{p_{g,t}} \lambda_i = 1$$

$$\sum_{i=1}^I \lambda^i_{p_{g,t}} = 1$$

$$p_{g,t} = \sum_{i=1}^I z^i_{p_{g,t}} \lambda_i \geq 0$$

$$u_{g,t} = \sum_{i=1}^I z^i_{u_{g,t}} \lambda_i \in \{0,1\}$$

$$v_{g,t} = \sum_{i=1}^I z^i_{v_{g,t}} \lambda_i \in \{0,1\}$$

$$w_{g,t} = \sum_{i=1}^I z^i_{w_{g,t}} \lambda_i \in \{0,1\}$$

$$\lambda_i \geq 0$$

where $(z^i_{p_{g,t}}, z^i_{u_{g,t}}, z^i_{v_{g,t}}, z^i_{w_{g,t}})$ are extreme point from the set described by the constrain from our initial problem (1c) up to (1f).

Question 2.2 Provide the expression of the reduced costs, the Restricted Mas- ter Problem and the subproblems2. Give the pseudo-code of D-W algorithm.

Restricted Master Problem :

The restricted master problem is obtained once we choose a subset $I' \subseteq I$ of extreme points which are available.

$$\sum_{t\in\mathcal{T}} \min_{\lambda} \quad \sum_{g\in\mathcal{G}} \left[C_g^P (\sum_{i=1}^{I'} z^i_{p_{g,t}} \lambda_i) + C_g^{NL} u_{g,t} (\sum_{i=1}^{I'} z^i_{u_{g,t}} \lambda_i) + C_g^{SU} v_{g,t} (\sum_{i=1}^{I'} z^i_{v_{g,t}} \lambda_i) \right]$$

$$\text{st.} \quad \sum_{g\in\mathcal{G}} \sum_{i=1}^{I} z^i_{p_{g,t}} \lambda_i = 1$$

$$\sum_{i=1}^{I} \lambda^i_{p_{g,t}} = 1$$

$$p_{g,t} = \sum_{i=1}^{I'} z^i_{p_{g,t}} \lambda_i \in \mathbb{R}^+$$

$$u_{g,t} = \sum_{i=1}^{I'} z^i_{u_{g,t}} \lambda_i \in \{0,1\}$$

$$v_{g,t} = \sum_{i=1}^{I'} z^i_{v_{g,t}} \lambda_i \in \{0,1\}$$

$$w_{g,t} = \sum_{i=1}^{I'} z^i_{w_{g,t}} \lambda_i \in \{0,1\}$$

$$\lambda_i \geq 0$$

Sub-problems

We can also derive the sub-problem from the master problem. Here, π_t^* and π_0^* are our dual variables :

$$\max_{p,u,v,w} = \sum_{t\in\mathcal{T}} \pi_t^* p_{g,t} - \sum_{t\in\mathcal{T}} \left[C_g^P p_{g,t} + C_g^{NL} u_{g,t} + C_g^{SU} v_{g,t} \right] - \pi_0^*$$

Reduced cost

From the sub-problem, we can obtain the reduced cost

$$\sum_{t\in\mathcal{T}} \pi_t^* p_{g,t} - \sum_{t\in\mathcal{T}} \left[C_g^P p_{g,t} + C_g^{NL} u_{g,t} + C_g^{SU} v_{g,t} \right] \tag{4}$$

Pseudocode of D-W algorithm

Algorithm 2 Pseudocode of D-W algorithm

```
while RC > 0 or first iteration do
    λ_i^* ← Solve the restricted linear programming master problem (RLM)
    π^*, π_0^* ← Get the dual variables.
    RC ← compute the reduce cost (4)
    if RC ≤ 0 then
        stop and return λ^*
    end if
    if RC >0 then
        if π_t^* Σ_{g∈G} p_{g,t} − D_t = 0 then
            stop and return λ^*
        end if
    else
        z^{k+1} ← (p^*, u^*, v^*, g^*)
        Add the column in the shape of (cz^{k+1}, Az^{k+1}, 1)^T. Where the parameter c is from the objective function (1a in the statement) and A comes from our constrain (1b in the statement) Then we can perform the restricted linear programming master problem.
    end if
end while
```