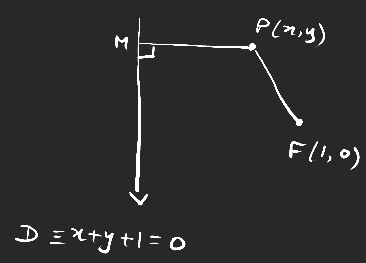
△ Cartesian geometry (vol-4) Part 2 ┌ Ellipse └ Hyperbola

Lecture-1 (8/Jul) 2.

1. Ellipse

Ex: $|PF| = e|MP|$ $\boxed{e<1}$ $e = \frac{1}{\sqrt{2}}$ locus Constraint eqⁿ



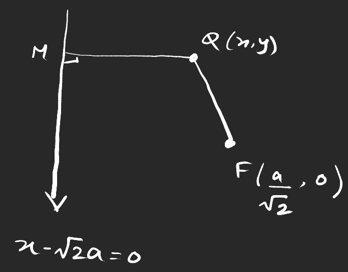
$$\sqrt{(x-1)^2+y^2} = \frac{1}{\sqrt{2}}\frac{|x+y+1|}{\sqrt{1+1}} \Rightarrow \left((x-1)^2+y^2\right)4 = (x+y+1)^2$$

$$4(x^2+y^2-2x) = x^2+y^2+1+2xy+2y+2x$$

$$\boxed{3x^2+3y^2-2xy-10x-2y+3=0}$$ Eqⁿ of locus (Eqⁿ of ellipse)

Journey to std. ellipse $e = \frac{1}{\sqrt{2}}$

Sample !



* $|QF|^2 = e^2|QM|^2 \Rightarrow \left(x-\frac{a}{\sqrt{2}}\right)^2+y^2 = \frac{1}{2}\frac{|x-\sqrt{2}a|^2}{\sqrt{1}}$

$$x^2+\frac{a^2}{2}+y^2-\cancel{\frac{2xa}{\sqrt{2}}} = \frac{1}{2}\left(x^2+2a^2-\cancel{2\sqrt{2}ax}\right)$$

$$\frac{x^2}{2}+y^2 = \frac{a^2}{2} \Rightarrow \frac{x^2}{a^2}+\frac{y^2}{a^2/2} = 1$$

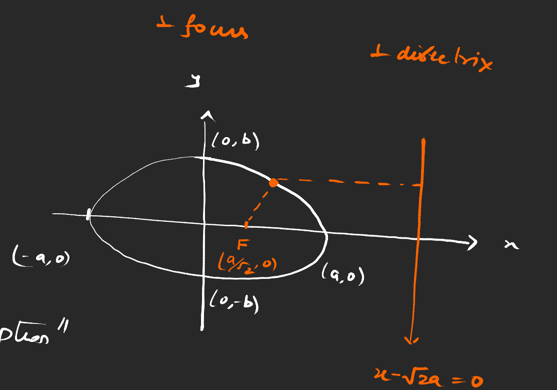
* $\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$ Eqⁿ of Ellipse $e<1$

$b^2 \equiv \frac{a^2}{2} = a^2\left(1-\underbrace{\left(\frac{1}{\sqrt{2}}\right)^2}_{e}\right) = a^2(1-e^2)$ $\downarrow b = \frac{a}{\sqrt{2}}\uparrow$ $a>b$

$b^2 \equiv a^2(1-e^2)$

$\boxed{a>b}$

Curve tracing $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $a>b$



- $x=0 \Rightarrow y=\pm b$
  $y=0 \Rightarrow x=\pm a$
- $\left\{\begin{matrix} x\to -x \\ y\to -y\end{matrix}\right\} \xrightarrow{\text{curve}}$ symmetric "Incomplete description"
- origin not satisfied ⇒ doesn't pass thr. O
- $x^2 = a^2\left(1-\frac{y^2}{b^2}\right) \Rightarrow x = \pm\frac{a}{b}\sqrt{b^2-y^2} \Rightarrow$ if $\underbrace{y^2>b^2}_{y>b \text{ or } y<-b} \Rightarrow$ img. values $x \in \mathbb{C}$ $-b$ b

  curve won't go here

  $y^2 = b^2\left(1-\frac{x^2}{a^2}\right) \Rightarrow y = \pm\frac{b}{a}\sqrt{a^2-x^2} \Rightarrow$ if $\underbrace{x^2>a^2}_{x>a \text{ or } x<-a} \Rightarrow$ img. values $y \in \mathbb{C}$ $-a$ a
- $y=\pm b \Rightarrow x=0$, $x=\pm a \Rightarrow y=0$

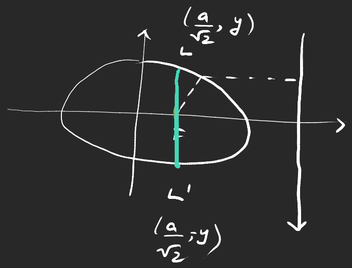
  $y: \pm b \to 0$ as $x: 0\to \pm a$ ⇒ $y\downarrow$ $x\uparrow$ closed curve

Naming

- Vertex V = pt of intersection of C with axes $= (a, 0)$ $\quad$ # vertices = 2
- Focus F = fixed pt. $= \left(\frac{a}{\sqrt{2}}, 0\right)$ $\quad$ # focus = 1
- Dirx D = fixed straight line $\equiv x - \sqrt{2}a = 0$
- LR = 2(LF)

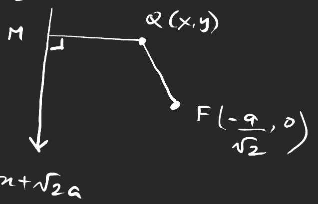
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \xrightarrow{x = a/\sqrt{2}} y = \frac{b}{\sqrt{2}} = \frac{a}{2}$$

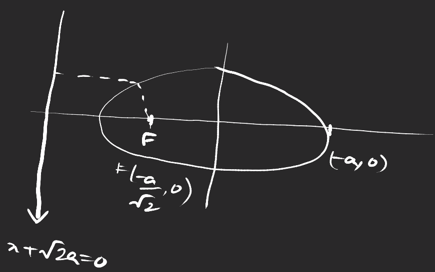
$\boxed{LR = a}$



sample 2

$e = 1/\sqrt{2}$





$$|QF|^2 = \frac{1}{2}|QM|^2 \Rightarrow \left(x + \frac{a}{\sqrt{2}}\right)^2 + y^2 = \frac{1}{2}|x + \sqrt{2}a|^2$$

$$\Downarrow$$

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

Curve tracing : same as 1

$V(-a, 0)$

$F\left(-\frac{a}{\sqrt{2}}, 0\right)$

dirx : $x + \sqrt{2}a = 0$

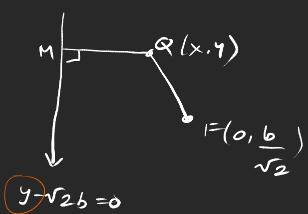
There are diff. situations (diff dirx / focus) corresponding to same eqn of the curve.

$$\Downarrow$$

$\exists$ 2 diff. FOCII

2 diff. DIRECTRIXES

Sample 3



$e = \frac{1}{\sqrt{2}}$, $|QF|^2 = \frac{1}{2}|QM|^2$

$$\downarrow$$

$$x^2 + \left(y - \frac{b}{\sqrt{2}}\right)^2 = \frac{1}{2}(y - \sqrt{2}b)^2$$

$$2\left(x^2 + y^2 + \frac{b^2}{2} - \cancel{\sqrt{2}yb}\right) = y^2 + 2b^2 - \cancel{2\sqrt{2}yb}$$

$$2x^2 + y^2 = b^2 \Rightarrow x^2 + \frac{y^2}{2} = \frac{b^2}{2}$$

$$\frac{x^2}{b^2/2} + \frac{y^2}{b^2} = 1$$

$$\Downarrow$$

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1} \qquad a^2 = \frac{b^2}{2} = b^2(1 - e^2)$$

$\downarrow \because e < 1$

$\boxed{b > a}$

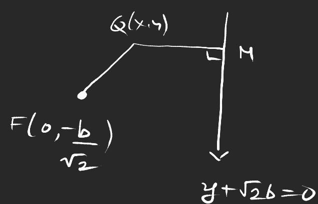
- $V(0, \pm b)$

$F(0, b/\sqrt{2})$

$D \equiv y - \sqrt{2}b = 0$

$\longleftrightarrow$ algebraically identically

Sample 4



$e = \frac{1}{\sqrt{2}}$, $|QF|^2 = \frac{1}{2}|QM|^2$

$$\downarrow$$

$$x^2 + \left(y + \frac{b}{\sqrt{2}}\right)^2 = \frac{1}{2}|y + \sqrt{2}b|^2$$

$$2\left(x^2 + y^2 + \frac{b^2}{2} + \cancel{\sqrt{2}yb}\right) = y^2 + 2b^2 + \cancel{2\sqrt{2}by}$$

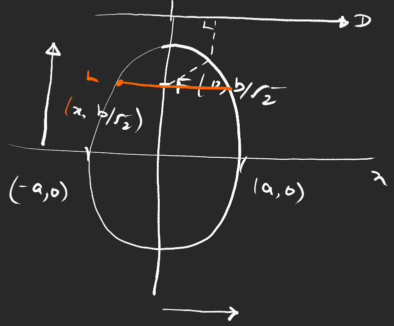
$$2x^2 + y^2 = b^2 \Rightarrow x^2 + \frac{y^2}{2} = \frac{b^2}{2}$$

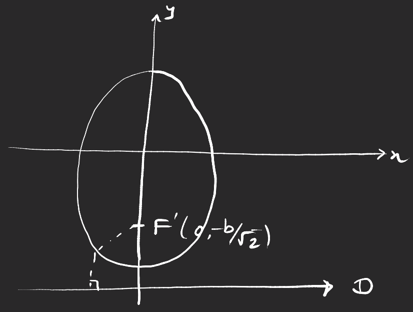
$$\frac{x^2}{b^2/2} + \frac{y^2}{b^2} = 1$$

curve tracing is same as ③

y

$\boxed{LR = b} = 2(LF)$





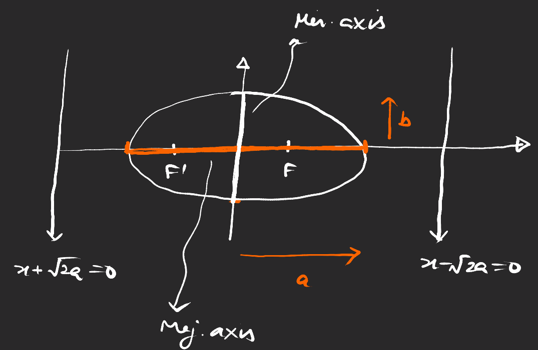
* NOTE : to remove / lift off the degeneracy in the $(1^{st}, 2^{nd})$, $(3^{rd}, 4^{th})$ case, we introduce 2 focii & 2 directrixes

Lecture-2 (9/JUL) 2

Summary

*

$\equiv$



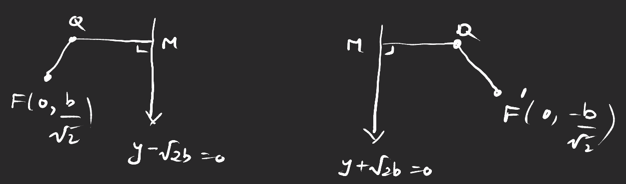
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$a > b$

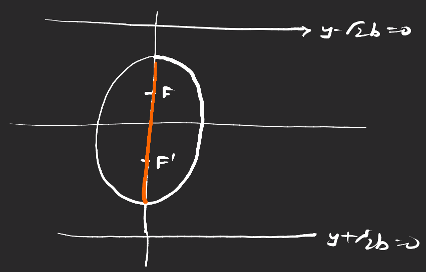
Maj. axis $= 2a$

Min. axis $= 2b$

*



$\equiv$



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$b > a$

Maj. axis $= 2b$

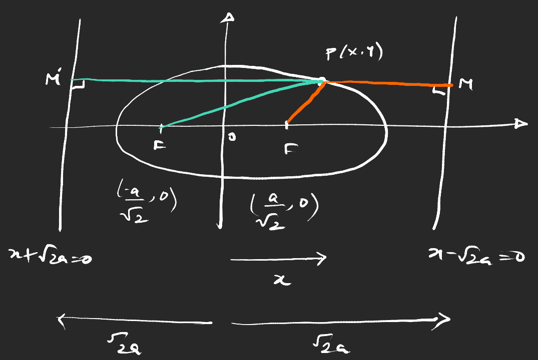
Min. " $= 2a$

* FP = focal dist. $\rightarrow FP = \frac{1}{\sqrt{2}} PM = \frac{1}{\sqrt{2}}(\sqrt{2}a - x)$

$F'P$ = focal dist $\rightarrow F'P = \frac{1}{\sqrt{2}} PM' = \frac{1}{\sqrt{2}}(\sqrt{2}a + x)$

$\boxed{\text{Thm: } FP + F'P = 2a = \text{const}}$

sum of focal distance is const for an ellipse



$e = \frac{1}{\sqrt{2}}$

1. Formal method

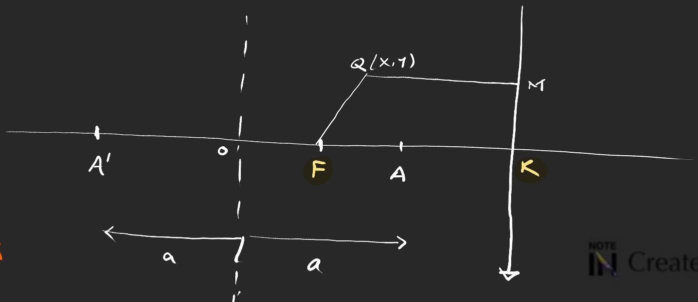
Set up

* $\because$ ellipse does cut at 2 pts on x

$OA = OA' = a$; O = mid pt of AA'

* $|QF| = e|QM|$ most fundamental eq^n $e < 1$

* Divide FK internally & externally in $e : 1$ } TRICK
(by A) (by A')



Coord. geo.

$x - |OK| = 0$ Directrix

$\longleftrightarrow$ $2a$

- $\frac{FA}{AK} = \frac{e}{1} \Rightarrow FA = e|AK|$
- $\frac{FA'}{A'K} = \frac{e}{1} \Rightarrow FA' = e|A'K|$

Calculation

for dirx

$*$ $FA + FA' = 2a$

$\Downarrow$

$e(\underbrace{AK}_{OK-\cancel{OA}} + \underbrace{A'K}_{OK+\cancel{OA'}}) = 2a \Rightarrow 2eOK = 2a$

$\Downarrow$

$\boxed{|OK| = \frac{a}{e}}$

Directrix

for focus

$\underbrace{FA'}_{\cancel{A'O}+OF} - \underbrace{FA}_{\cancel{OA}-OF} = e(A'K - AK) = e(A'A) = e(2a)$

$\Downarrow$

$2|OF| = 2ea \Rightarrow \boxed{|OF| = ea}$

"focus"

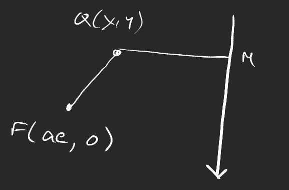
Eqn

$*$ $|QF|^2 = e^2|QM|^2 \Rightarrow (x-ae)^2 + y^2 = e^2\left|x - \frac{a}{e}\right|^2$

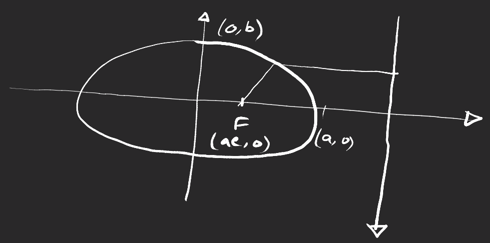
$x^2 + a^2e^2 - \cancel{2aex} + y^2 = e^2\left(x^2 + \frac{a^2}{e^2} - \frac{2a}{e}x\right) = e^2x^2 + a^2 - \cancel{2aex}$

$(1-e^2)x^2 + y^2 = a^2(1-e^2) \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1 \Rightarrow \boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$ Eqn of Ellipse

$b^2 \equiv a^2(1-e^2)$



Curve Tracing : same as before



2nd focus / 2nd Directrix

$*$ $\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1 \Rightarrow \frac{x^2}{\cancel{a^2}} = 1 - \frac{y^2}{a^2(1-e^2)} = \frac{a^2(1-e^2) - y^2}{\cancel{a^2}(1-e^2)}$

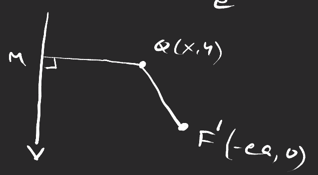
$(1-e^2)x^2 = a^2 - a^2e^2 - y^2 \Rightarrow x^2 + a^2e^2 + y^2 = a^2 + e^2x^2$

$\underbrace{x^2 + 2aex + a^2e^2}{} + y^2 = a^2 + 2aex + e^2x^2 \Rightarrow (x+ae)^2 + y^2 = \underbrace{(ex+a)^2}_{e^2\left(x+\frac{a}{e}\right)^2}$

$(x+ea)^2 + y^2 = e^2\left|x + \frac{a}{e}\right|^2$

$|QF'|^2 = e^2|QM|^2 \Rightarrow \boxed{|QF'| = e|QM|}$

$e < 1$



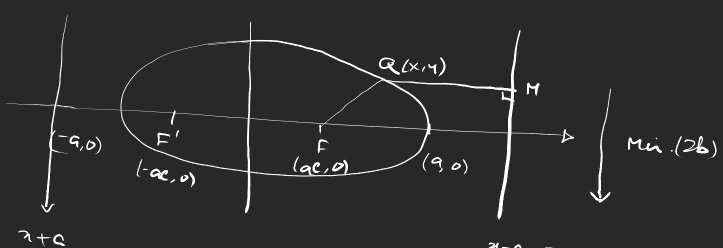
Analysis

1. Eccentricity

$*$ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$

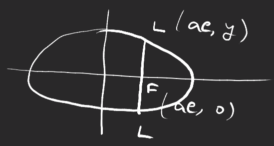
$b^2 = a^2(1-e^2) \Rightarrow b^2 = a^2 - a^2e^2 \Rightarrow a^2e^2 = a^2 - b^2$

$*$ $\boxed{e = \sqrt{1 - \left(\frac{b}{a}\right)^2}} \Rightarrow e = \sqrt{1 - \left(\frac{Mn}{Mj}\right)^2}$



2. LR :

* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \xrightarrow{x=ae} \frac{(ae)^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow y^2 = \underbrace{b^2(1-e^2)}_{\frac{b^2}{a^2}} = \frac{b^4}{a^2} \Rightarrow \boxed{y=\frac{b^2}{a}}$

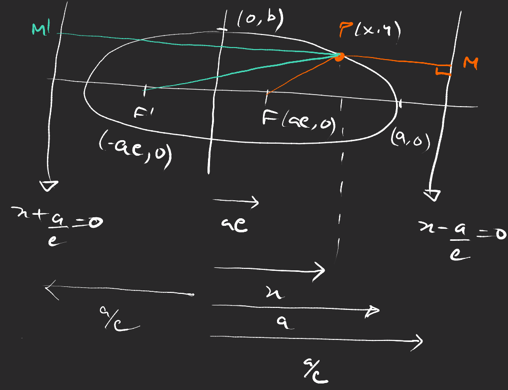


$\boxed{LR = \frac{2b^2}{a} = 2a(1-e^2)}$ ✓ ✓

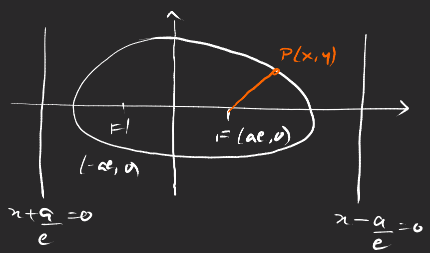
3. Thm. of focal dist.

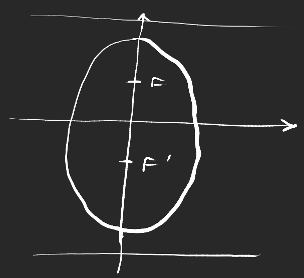
* $FP = ePM = e\left(\frac{a}{e}-x\right) = a-ex$
$F'P = ePM' = e\left(\frac{a}{e}+x\right) = a+ex$ } Focal distance

* $\boxed{FP + F'P = 2a = \text{const}}$ Sum of focal dist.



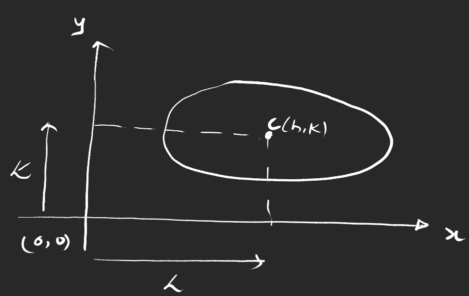
Comparison (Dictionary)





	$a>b$	$a<b$
Eqⁿ :	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
	$b^2=a^2(1-e^2)$ → Trick $\boxed{a \leftrightarrow b}$	$a^2=b^2(1-e^2)$
	$LR = \frac{2b^2}{a}$	$LR = \frac{2a^2}{b}$
	f dist $= (a+ex)$	f dist $= b \pm ey$
	Maj $= 2a$	Maj $= 2b$
	Min $= 2b$	Min $= 2a$
	$e=\sqrt{1-\frac{b^2}{a^2}}$	$e=\sqrt{1-\frac{a^2}{b^2}}$
	$C(0,0)$	$C(0,0)$

* C : $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ centre (h,k)

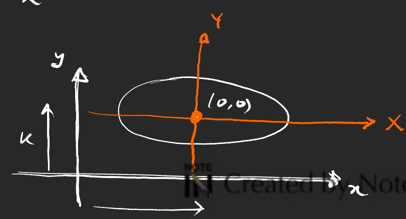


check

Coord transf : $x \longrightarrow X = x-h$ $\quad x = X+h$
$y \longrightarrow Y = y-k$ $\quad y = Y+k$

$\boxed{C \longrightarrow C' \equiv \frac{X^2}{a^2}+\frac{Y^2}{b^2}=1}$

Ellipse with Center $(0,0)$ in (X,Y) coord sys.



Lecture-3 (10 Jul) 2

Practice

Q1. $C \equiv x^2+4y^2+2x+16y+13=0$ what geometry does C correspond to? "Perfecting the sq. method"

$*$ $\underbrace{x^2+2x+1}-1+4(\underbrace{y^2+4y+4}-4)+13=0$

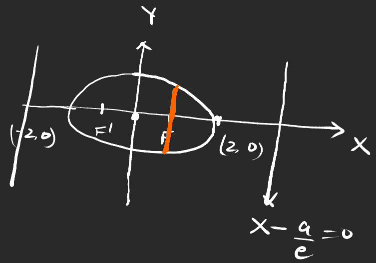
$(x+1)^2+4(y+2)^2=4 \Rightarrow \dfrac{(x+1)^2}{2^2}+\dfrac{(y+2)^2}{1^2}=1$ 'Ellipse'

$$\boxed{\frac{(x+1)^2}{2^2}+\frac{(y+2)^2}{1^2}=1}$$

Coord. transf^n: $y \to Y \equiv y+2$, $x \to X \equiv x+1$ $\to$ $y=Y-2$, $x=X-1$

(Trivial) New Coord sys

Ellipse $\dfrac{X^2}{2^2}+\dfrac{Y^2}{1^2}=1$



$a=2$, $b=1$

$C(0,0)$

$V(\pm a,0) = (\pm 2,0)$

$LR = \dfrac{2b^2}{a} = 1$

$e = \sqrt{1-\dfrac{b^2}{a^2}} = \dfrac{\sqrt3}{2} < 1$

$F(\pm ae,0) = \boxed{F(\pm\sqrt3,0)}$

Directrix: $X \pm \dfrac{a}{e}=0 \Rightarrow \boxed{X\pm\dfrac{4}{\sqrt3}=0}$

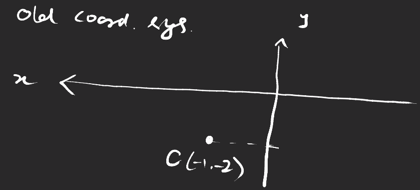
$\boxed{X=\pm\dfrac{4}{\sqrt3}}$

Eq^n of LR: $X\pm ae=0 \Rightarrow \boxed{X\pm\sqrt3=0}$

$\boxed{X=\pm\sqrt3}$

New

Old coord. sys.



$C\overset{XY}{(0,0)} \longrightarrow C'\overset{x\ \ y}{(-1,-2)}$

$V(\pm2,0) \longrightarrow V_1(1,-2)$, $V_2(-3,-2)$

$LR=1$ Invariant

$e=\dfrac{\sqrt3}{2}$ "

$F\underset{X\ \ Y}{(\pm\sqrt3,0)} \longrightarrow F'(\pm\sqrt3-1,-2)$

Dir: $X=\pm\dfrac{4}{\sqrt3} \longrightarrow x=\pm\dfrac{4}{\sqrt3}-1$ $\to x=\dfrac{4}{\sqrt3}-1$, $\to x=-\dfrac{4}{\sqrt3}-1$

Eq^n of LR: $X=\pm\sqrt3 \longrightarrow x=\pm\sqrt3-1$ $\to x=\sqrt3-1$, $\to x=-\sqrt3-1$

Q2. Ellipse $\mathcal{E}$: $C(0,0)$, $F(\pm1,0)$, $e=\frac12$; Eq^n of $\mathcal{E}$ = ? $x^2+\dfrac{4y^2}{3}=1$

$F(\pm ae,0)=F(\pm1,0)$, $e=\dfrac12$ $\longrightarrow$ $\boxed{a=2}$

$b^2=a^2(1-e^2)=4\left(1-\dfrac14\right)=3$ $\Rightarrow$ $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1 \longrightarrow \dfrac{x^2}{4}+\dfrac{y^2}{3}=1$

HW

Q3. $C(1,2)$, $F_1(6,2)$, passes thru pt $(4,6)$ Eq^n of ellipse?

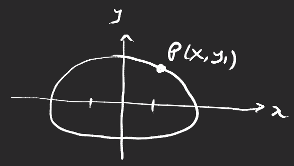
Q4. $F(\pm2,3)$, semi maj. axis $=\sqrt5$ Eq^n of ellipse?

Q5. Ellip $\mathcal{E}$: axes are along coord axes, $F(0,\pm4)$, $e=\dfrac45$ Eq^n = ?

Q6. Arc. is in form of semi-ellipse. It is 8m wide, 2m high at the center. Find hgt of the arch at pt 1.5m from one end.

2. Ellipse (the real deal)

A. Rel. pos of a pt wrt ellipse



* $\mathcal{E} \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, $P(x_1, y_1)$

* $\mathcal{E}(x_1, y_1) = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 = 0$ ON

> 0 OUT

< 0 IN

B. Parametric eqⁿ of ellipse

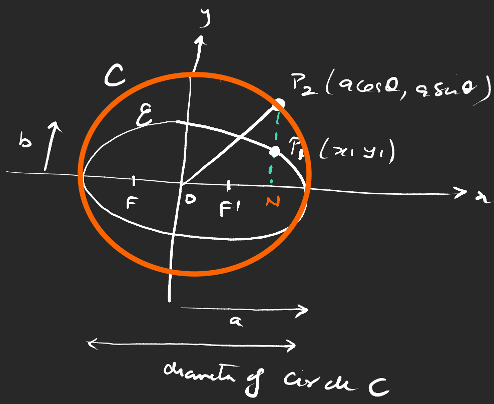
* $\left.\begin{aligned} x &= a\cos\theta \\ y &= b\sin\theta \end{aligned}\right\} \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ Ellipse

θ = Eccentric Angle

$x^2 + y^2 = a^2 \rightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$

$\left\{\begin{aligned} x &= a\cos\theta \\ y &= a\sin\theta \end{aligned}\right\}$

C. Auxiliary circle



* Auxiliary Circle ≡ circle w/ diameter = Maj axis

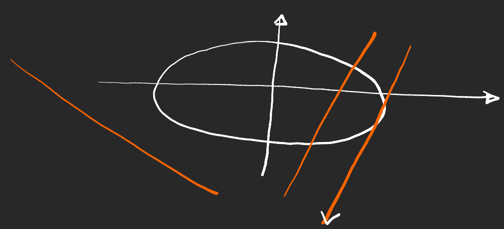
* $C \equiv x^2 + y^2 = a^2$, $\mathcal{E} \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$C(P_2): x_2^2 + y_2^2 = a^2$ $\quad \mathcal{E}(P_1): \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$

$\frac{x_2^2}{a^2} + \frac{y_2^2}{a^2} = 1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$

for line P_2N $\boxed{x_1 = x_2} \rightarrow \boxed{\frac{y_2}{y_1} = \frac{a}{b}}$ $\Rightarrow \boxed{\frac{\ell(P_1N)}{\ell(P_2N)} = \frac{b}{a} = \frac{\text{semi min. axis}}{\text{semi maj axis}}}$

D. Straight line intersecting with ellipse



* $\mathcal{E} \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $b^2 = a^2(1-e^2)$, $y = mx + c$

* $\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1 \Rightarrow \frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$

$b^2x^2 + a^2(m^2x^2 + c^2 + 2mcx) = a^2b^2 \Rightarrow \boxed{(b^2 + a^2m^2)x^2 + 2mca^2x + a^2c^2 - a^2b^2 = 0}$ Quad in x

* $x_{1,2} = \dfrac{-2mca^2 \pm \sqrt{4m^2c^2a^4 - 4(b^2 + a^2m^2)(c^2 - b^2)a^2}}{2(b^2 + a^2m^2)}$ Common x

$= \dfrac{-2mca^2 \pm \sqrt{\cancel{4m^2c^2a^4} - 4a^2b^2c^2 + 4a^2b^4 - \cancel{4a^4m^2c^2} + 4a^4m^2b^2}}{2(b^2 + a^2m^2)}$

$= \dfrac{-2mca^2 \pm \sqrt{4a^2b^2(-c^2 + b^2 + a^2m^2)}}{2(b^2 + a^2m^2)} = \dfrac{-2mca^2 \pm 2ab\sqrt{-c^2 + b^2 + a^2m^2}}{2(b^2 + a^2m^2)}$

→ D = 0 same roots → tangency

→ $\boxed{D > 0 \quad \text{2 distinct roots}}$ → secancy

→ D < 0 2 img. roots

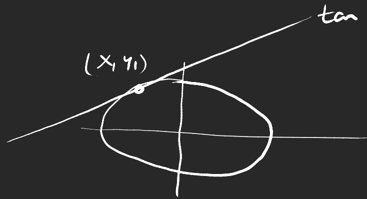
Ref "Eqⁿ of chord"

* $\boxed{D = 0 \Rightarrow c^2 = a^2m^2 + b^2}$ tangency condition

* $c^2 > a^2m^2+b^2 \rightarrow$ line cuts 2 pts
$c^2 < a^2m^2+b^2 \rightarrow$ line won't cut

$c = am$
tangency for parabola

E. Tangent to Ellipse



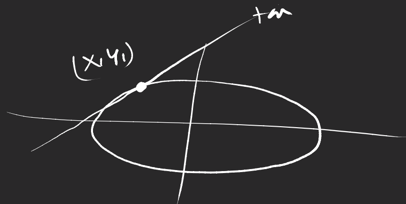
* tangent $\equiv$ $c^2 = a^2m^2+b^2 \Rightarrow x_{1,2} = -\dfrac{\cancel{2}mca^2}{\cancel{2}(b^2+a^2m^2)} = -\dfrac{a^2mc}{b^2+a^2m^2} = -\dfrac{a^2m}{\sqrt{b^2+a^2m^2}}$

$y_1 = mx_1 + c = -\dfrac{a^2m^2}{\sqrt{b^2+a^2m^2}} + \sqrt{a^2m^2+b^2} = \dfrac{b^2}{\sqrt{b^2+a^2m^2}}$

{ note $c = \pm\sqrt{a^2m^2+b^2}$
then taken +ve }

* $\boxed{\text{Point of contact } (x_1, y_1) = \left(\dfrac{-a^2m}{\sqrt{b^2+a^2m^2}}, \dfrac{b^2}{\sqrt{b^2+a^2m^2}}\right) = \left(\dfrac{-a^2m}{c}, \dfrac{b^2}{c}\right)}$

Equation



* $y - y_1 = m(x - x_1)$; $x_1 = -\dfrac{a^2m}{c}$, $y_1 = \dfrac{b^2}{c}$

$\Downarrow$ $\Downarrow$

$y - y_1 = -\dfrac{b^2x_1}{a^2y_1}(x - x_1)$; $\dfrac{x_1}{y_1} = -\dfrac{a^2m}{b^2} \Rightarrow \boxed{m = -\dfrac{b^2x_1}{a^2y_1}}$

$a^2y_1y - a^2y_1^2 = -b^2x_1x + b^2x_1^2$

$a^2y_1y + b^2x_1x = \underbrace{b^2x_1^2 + a^2y_1^2}_{a^2b^2}$

$\boxed{\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1}$ Eq^n of tan (I)

$(x_1 y_1)$ needed
POC

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$\Downarrow$ $(x_1 y_1)$ satisfied

$\dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} = 1$

$b^2x_1^2 + a^2y_1^2 = a^2b^2$

Lecture-4 (11 | Jul) 2.5

* $y = mx + c$, $c^2 = a^2m^2 + b^2$

$\Downarrow$

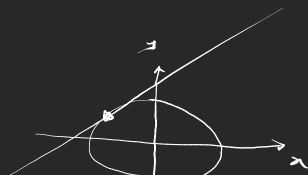
$\boxed{y = mx \pm \sqrt{a^2m^2+b^2}}$ Eq^n of tan (II)

m needed

* tangent $\equiv$ $\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1 \longrightarrow \boxed{\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1}$ Eq^n of tan (III)

$P(x_1 y_1) = (a\cos\theta, b\sin\theta)$

Parametric form



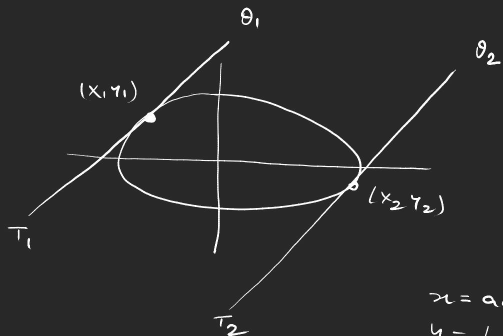
$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Comment on || tangents

* || tan. $\Rightarrow m_{T_1} = m_{T_2} \Rightarrow -\dfrac{b^2x_1}{a^2y_1} = -\dfrac{b^2x_2}{a^2y_2}$

$m = -\dfrac{b^2x_1}{a^2y_1}$

$\boxed{x_1y_2 = x_2y_1}$



$a\cos\theta_1\, b\sin\theta_2 = a\cos\theta_2\, b\sin\theta_1 \Rightarrow \sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1 = 0$

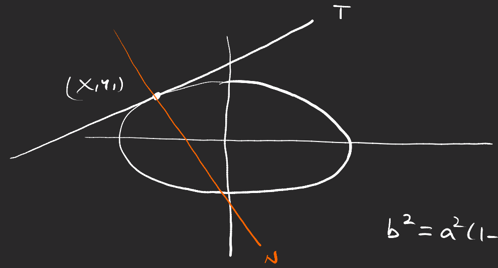
$\Downarrow$

$\sin(\theta_1 - \theta_2) = \sin\pi \Rightarrow \boxed{\theta_1 - \theta_2 = \pi}$

Cond^n for || tangents

$x = a\cos\theta$
$y = b\sin\theta$

F. Normal to Ellipse



* $m_T = -\frac{b^2 x_1}{a^2 y_1} \Rightarrow m_N = \frac{a^2 y_1}{b^2 x_1}$

* $y - y_1 = m_N(x-x_1) = \frac{a^2 y_1}{b^2 x_1}(x-x_1)$

$b^2 x_1 y - b^2 x_1 y_1 = a^2 y_1 x - a^2 y_1 x_1 \Rightarrow b^2 x_1 y - a^2 y_1 x = \underbrace{b^2 x_1 y_1 - a^2 y_1 x_1}_{\underbrace{(b^2-a^2)}_{-a^2e^2} x_1 y_1}$

$b^2 = a^2(1-e^2)$

$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$

$b^2x_1^2 + a^2y_1^2 = a^2b^2$

$\frac{b^2 y}{y_1} - \frac{a^2 x}{x_1} = -a^2e^2 \Rightarrow \boxed{\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2e^2}$

1 pt form of Normal

(x_1,y_1) needed

To the slope form...

* $y = m_N x + c_N$

Calc. POC in terms of m_N

$$(x_1,y_1) = \left(\frac{\mp a^2 m_T}{\sqrt{b^2+a^2m_T^2}}, \frac{\pm b^2}{\sqrt{b^2+a^2m_T^2}}\right) = \left(\frac{\pm \frac{a^2}{m_N}}{\sqrt{b^2+\frac{a^2}{m_N^2}}}, \frac{\pm b^2}{\sqrt{b^2+\frac{a^2}{m_N^2}}}\right) = \left(\overbrace{\frac{\pm a^2}{\sqrt{a^2+b^2m_N^2}}}^{x_1}, \overbrace{\frac{\pm b^2 m_N}{\sqrt{a^2+b^2m_N^2}}}^{y_1}\right)$$

* Normal $\equiv \frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2e^2 = (a^2-b^2)$

$m_N\cdot m_T = -1$

$m_T = \frac{-1}{m_N}$

$b^2 = a^2(1-e^2)$

$$\frac{\cancel{a^2}x\sqrt{a^2+b^2m_N^2}}{\pm \cancel{a^2}} - \frac{\cancel{b^2}y\sqrt{a^2+b^2m_N^2}}{\pm\cancel{b^2}m_N} = a^2-b^2$$

$$\pm x - \frac{y}{\pm m_N} = \frac{a^2-b^2}{\sqrt{a^2+b^2m_N^2}} \Rightarrow \mp\frac{y}{m_N} \pm x = \frac{(a^2-b^2)}{\sqrt{a^2+b^2m_N^2}} \Rightarrow \mp\frac{y}{m_N} = \mp x + \frac{(a^2-b^2)}{\sqrt{a^2+b^2m_N^2}}$$

$$\mp y = \mp x m_N + \frac{(a^2-b^2)m_N}{\sqrt{a^2+b^2m_N^2}} \Rightarrow \boxed{y = m_N x \mp \underbrace{\frac{(a^2-b^2)m_N}{\sqrt{a^2+b^2m_N^2}}}_{\equiv C_N} \Rightarrow y = m_N x + C_N}$$

$(\mp)\cdot(\mp) = +$

$(\mp)\cdot + = \mp$

$(\mp)\ - = \pm$

Slope form

m_N needed

* Thumb rule: $C_T = \pm\sqrt{a^2m_T^2+b^2}$; $m_N\cdot m_T = -1$

$C_N = \mp\frac{(a^2-b^2)m_N}{\sqrt{a^2+b^2m_N^2}}$ Normal Cond^n

To the parametric form...

* Normal $\equiv \frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2-b^2$, $x = a\cos\theta$, $y = b\sin\theta$ $P(x_1,y_1) = (a\cos\theta, b\sin\theta)$

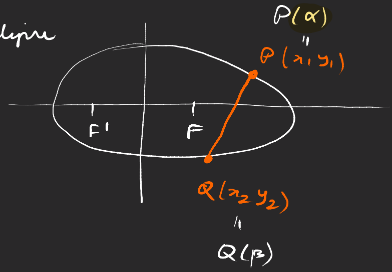
$\boxed{a\sec\theta\, x - b\,\mathrm{cosec}\,\theta\, y = a^2-b^2}$

Parametric form

θ : needed

G. Eq^n of chord

"$P(\alpha)$" a point α on an ellipse
$(a\cos\alpha, b\sin\alpha)$



* PQ chord : $P(\underset{x_1}{a\cos\alpha}, \underset{y_1}{b\sin\alpha})$

$Q(\underset{x_2}{a\cos\beta}, \underset{y_2}{b\sin\beta})$

* $$y - b\sin\alpha = \frac{b(\sin\beta - \sin\alpha)}{a(\cos\beta - \cos\alpha)}(x - a\cos\alpha)$$

$$= \frac{b}{a}\frac{\cancel{2}\cos\frac{\beta+\alpha}{2}\cancel{\sin\left(\frac{\beta-\alpha}{2}\right)}^{(-1)}}{\cancel{2}\sin\frac{\beta+\alpha}{2}\cancel{\sin\left(\frac{\alpha-\beta}{2}\right)}}(x - a\cos\alpha)$$

C D $\frac{C+D}{2}$ $\frac{C-D}{2}$

$S + S = 2SC$

$S - S = 2CS$

$C + C = 2CC$

$C - C = -2SS$*

$$\frac{a}{b}(y - b\sin\alpha) = \frac{-\cos\frac{\alpha+\beta}{2}}{\sin\frac{\alpha+\beta}{2}}(x - a\cos\alpha) \Rightarrow \frac{a}{b}\sin\frac{\alpha+\beta}{2}\,y - a\sin\alpha\sin\frac{\alpha+\beta}{2} = -x\cos\frac{\alpha+\beta}{2} + a\cos\alpha\cos\frac{\alpha+\beta}{2}$$

* $$\boxed{\frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\left(\frac{\alpha-\beta}{2}\right)}$$

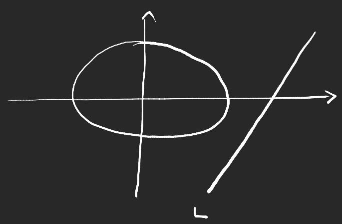
Eq^n of chord.
in terms of eccentric angle (α, β)

(or) eq^n of st. line joining 2 pts
on ellipse with diff. eccentric angle

Lecture-5 (12 Jul) 2

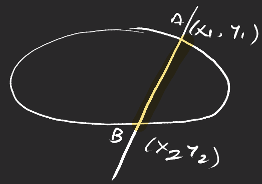
H. Miscellany

4.1 length of chord



* $\mathcal{E} \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $y = mx + c$

$$(b^2 + a^2m^2)x^2 + 2mca^2x + a^2c^2 - a^2b^2 = 0 \Rightarrow x_{1,2} = \frac{-2mca^2 \pm 2ab\sqrt{-c^2 + b^2 + a^2m^2}}{2(b^2 + a^2m^2)} \longrightarrow \boxed{D > 0}$$



* $$x_1 + x_2 = \frac{-2mca^2}{(b^2 + a^2m^2)} \checkmark , \quad x_1x_2 = \frac{a^2(c^2 - b^2)}{(b^2 + a^2m^2)} \checkmark$$

length of chord $|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(x_2 - x_1)^2 + m^2(x_2 - x_1)^2}$

$$= (x_2 - x_1)\sqrt{1 + m^2}$$

$$= \sqrt{(x_2 + x_1)^2 - 4x_2x_1}\,\sqrt{1 + m^2}$$

$y_2 = mx_2 + c$

$y_1 = mx_1 + c$

$(y_2 - y_1)^2 = m^2(x_2 - x_1)^2$

$$= \frac{\sqrt{1 + m^2}}{(b^2 + a^2m^2)}\sqrt{4m^2c^2a^4 - 4a^2c^2b^2 - 4a^4c^2m^2 + 4a^2b^4 + 4a^4b^2m^2}$$

$(x_2 + x_1)^2 = x_2^2 + x_1^2 + 2x_2x_1 - 2x_2x_1 + 2x_2x_1$

$(x_2 + x_1)^2 = (x_2 - x_1)^2 + 4x_2x_1$

$$= \frac{\sqrt{1 + m^2}}{(b^2 + a^2m^2)}\,2ab\sqrt{a^2m^2 + b^2 - c^2}$$

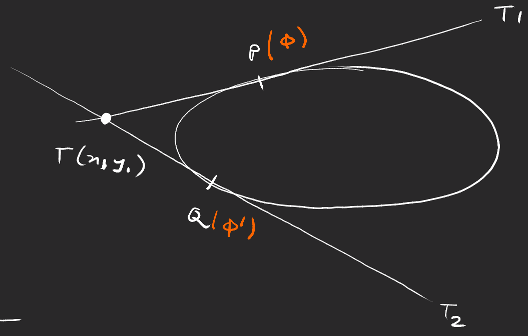
$$x_2 - x_1 = \sqrt{(x_2 + x_1)^2 - 4x_2x_1}$$

$$= \sqrt{\frac{4m^2c^2a^4}{(b^2 + a^2m^2)^2} - \frac{4a^2(c^2 - b^2)}{(b^2 + a^2m^2)}}$$

$$|AB| = \frac{2ab\sqrt{1 + m^2}\sqrt{a^2m^2 + b^2 - c^2}}{(b^2 + a^2m^2)}$$ length of chord

$$= \sqrt{\frac{4m^2c^2a^4 - 4a^2(c^2 - b^2)(b^2 + a^2m^2)}{(b^2 + a^2m^2)^2}}$$

H.2. Point of intersection of 2 tangents



* $T_1 \equiv \frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi - 1 = 0$

$T_2 \equiv \frac{x}{a}\cos\phi' + \frac{y}{b}\sin\phi' - 1 = 0$

solve:

* $$\frac{x}{\begin{vmatrix} \frac{\sin\phi}{b} & -1 \\ \frac{\sin\phi'}{b} & -1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} \frac{\cos\phi}{a} & -1 \\ \frac{\cos\phi'}{a} & -1 \end{vmatrix}} = \frac{1}{\begin{vmatrix} \frac{\cos\phi}{a} & \frac{\sin\phi}{b} \\ \frac{\cos\phi'}{a} & \frac{\sin\phi'}{b} \end{vmatrix}}$$

$$\frac{x}{-\frac{\sin\phi}{b} + \frac{\sin\phi'}{b}} = \frac{-y}{-\frac{\cos\phi}{a} + \frac{\cos\phi'}{a}} = \frac{1}{\frac{\cos\phi\sin\phi'}{ab} - \frac{\sin\phi\cos\phi'}{ab}}$$

$$\frac{xb}{\sin\phi' - \sin\phi} = \frac{ya}{\cos\phi - \cos\phi'} = \frac{ab}{\sin(\phi'-\phi)}$$

for x:

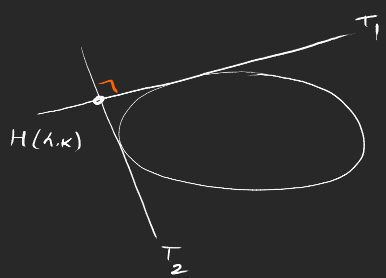
$S+S = 2SC$
$S-C = 2CS$
$C+C = 2CC$
$C-C = 2SS$*

* $$x = a \cdot \frac{\cancel{2}\sin\frac{\phi'+\phi}{2}\cancel{\cos\frac{\phi'-\phi}{2}}}{\cancel{2}\sin\frac{\phi'-\phi}{2}\cancel{\cos\frac{\phi'-\phi}{2}}} \Rightarrow x = a\frac{\sin\frac{\phi'+\phi}{2}}{\sin\frac{\phi'-\phi}{2}}$$

* $$y = b \cdot \frac{2\sin\frac{\phi+\phi'}{2}\cancel{\sin\frac{\phi'-\phi}{2}}}{2\cancel{\sin\frac{\phi'-\phi}{2}}\cos\frac{\phi'-\phi}{2}} \Rightarrow y = b\frac{\sin\frac{\phi'+\phi}{2}}{\cos\frac{\phi'-\phi}{2}}$$

(x, y) Point of Intersection

H3 Director Circle



* $T_1 \perp T_2$, locus of point of intersection = ?

∵ there is no info of P.O.C → slope form is the way

$\perp \Downarrow m_1 \cdot m_2 = -1$

* $T_1 \equiv y = mx + c = mx + \sqrt{a^2m^2+b^2}$

$\downarrow$

$T_2 \equiv y = \left(-\frac{1}{m}\right)x + c = -\frac{1}{m}x + \sqrt{a^2\left(-\frac{1}{m}\right)^2+b^2}$

H satisfied (h, k)

* $k = mh + \sqrt{a^2m^2+b^2} \longrightarrow k - mh = \sqrt{a^2m^2+b^2} \xRightarrow{sq.} k^2 + m^2h^2 - 2mhk = a^2m^2+b^2$

$k = -\frac{1}{m}h + \sqrt{\frac{a^2}{m^2}+b^2} \longrightarrow mk + h = \sqrt{a^2+b^2m^2} \Rightarrow m^2k^2 + h^2 + 2mhk = a^2+b^2m^2$

+

$k^2+h^2+m^2(h^2+k^2) = a^2+b^2+m^2(a^2+b^2)$

$(k^2+h^2)\cancel{(1+m^2)} = (a^2+b^2)\cancel{(1+m^2)}$

$h^2+k^2 = a^2+b^2$

* $\boxed{H(x, y) : \quad x^2+y^2 = a^2+b^2}$ locus = circle

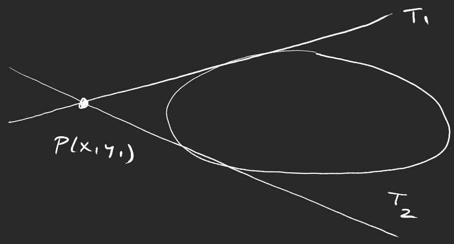
└ Director circle

Center = $(0,0)$ = center of ellipse

Rad = $\sqrt{a^2+b^2}$

(Existence of 2 tangents)

H.4 Pair of 2 tangents from a pt. (Criterion)



* Tangent $\equiv y = mx + \sqrt{a^2m^2+b^2}$ satisfied P

* $y_1 = mx_1 + \sqrt{a^2m^2+b^2} \Rightarrow (y_1 - mx_1)^2 = a^2m^2+b^2$

$y_1^2 + m^2x_1^2 - 2mx_1y_1 = a^2m^2+b^2 \Rightarrow \boxed{(x_1^2-a^2)m^2 - 2x_1y_1m + (y_1^2-b^2) = 0}$

- $D>0 \rightarrow$ 2 tangents
- $D=0$
- $D<0$

$\{b^2 - 4ac\}$

* for existence of 2 tang. $\Rightarrow 4x_1^2y_1^2 - 4(x_1^2-a^2)(y_1^2-b^2) > 0$

$\cancel{x_1^2y_1^2} - \cancel{x_1^2y_1^2} + x_1^2b^2 + y_1^2a^2 - a^2b^2 > 0$

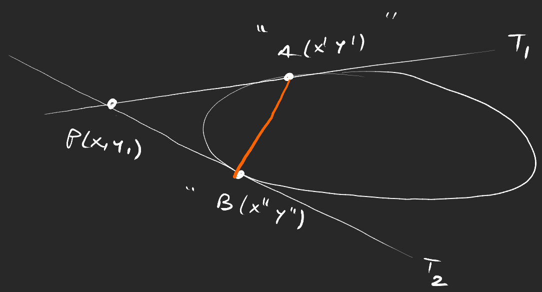
$x_1^2b^2 + y_1^2a^2 - a^2b^2 > 0 \Rightarrow \underbrace{\boxed{\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 > 0}}_{\mathcal{E}(P)} \Rightarrow$ pt is outside

* for 2 tangents w/ same slope $\Rightarrow \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 = 0 \Rightarrow \mathcal{E}(P) = 0 \Rightarrow$ pt is on the ellipse

* " No real tangents $\Rightarrow \underbrace{\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 < 0}_{\mathcal{E}(P)} \Rightarrow$ pt is inside the ellipse

2 tangents from P not possible

H.5 Eq^n of chord of Contact



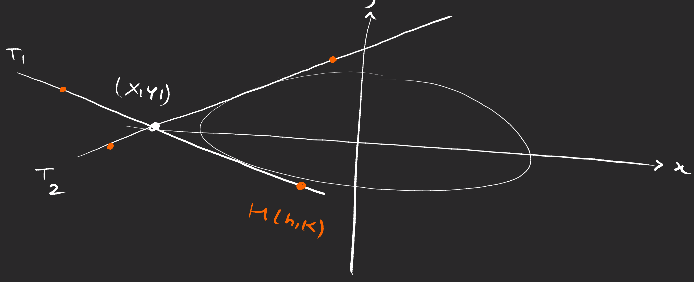
$$\left.\begin{aligned} * \quad T_1 &\equiv \frac{xx'}{a^2} + \frac{yy'}{b^2} = 1 \\ * \quad T_2 &\equiv \frac{xx''}{a^2} + \frac{yy''}{b^2} = 1 \end{aligned}\right\} P(x_1,y_1) \text{ satisfied}$$

$\Downarrow$

$$\left.\begin{aligned} \frac{x_1x'}{a^2} + \frac{y_1y'}{b^2} &= 1 \\ \frac{x_1x''}{a^2} + \frac{y_1y''}{b^2} &= 1 \end{aligned}\right\} \boxed{\text{Eq}^n \ AB \equiv \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1}$$

(x_1,y_1) pt of intersection

H.6 Eq^n of Pair of tangents.



* $H(h,k)$ pt on tangent

* $y - y_1 = \frac{k-y_1}{h-x_1}(x-x_1) \Rightarrow y = \left(\frac{k-y_1}{h-x_1}\right)x - \frac{x_1(k-y_1)}{h-x_1} + y_1$

$y = \left(\frac{k-y_1}{h-x_1}\right)x \ \frac{-x_1k + \cancel{x_1y_1} + y_1h - \cancel{x_1y_1}}{h-x_1} \Rightarrow y = \underbrace{\left(\frac{k-y_1}{h-x_1}\right)}_{\leftarrow m \rightarrow}x + \underbrace{\left(\frac{hy_1 - kx_1}{h-x_1}\right)}_{\leftarrow c \rightarrow}$

* if T_1H line is a tangent $\Rightarrow y = mx + \sqrt{a^2m^2+b^2}$

Comparing

$$m = \frac{k-y_1}{h-x_1} \quad , \quad \left(\frac{hy_1-kx_1}{h-x_1}\right)^2 = \underbrace{a^2m^2+b^2}_{c}$$

$$* \quad \left(\frac{hy_1-kx_1}{h-x_1}\right)^2 = a^2\left(\frac{k-y_1}{h-x_1}\right)^2 + b^2 \xrightarrow{(h,k)\to(x,y)} \boxed{(xy_1-yx_1)^2 = a^2(y-y_1)^2+b^2(x-x_1)^2}$$

quad. eqⁿ in x, y.

Combined eqⁿ of Pair of tangents.

HW

$$\boxed{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right) = \left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2}$$

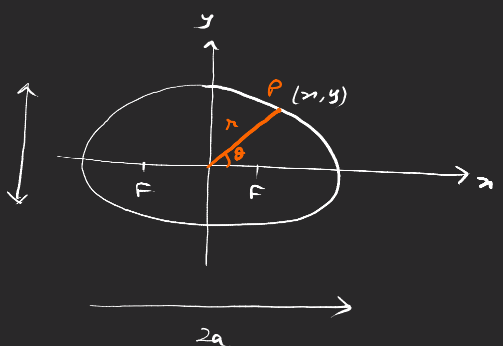
$$|||$$

$$\boxed{\mathcal{E}\,\mathcal{E}_1 = T^2}$$

Lecture-6 (13/Jul) 2

H.7 Polar form of ellipse

a) from center



$* \quad P: \; x = r\cos\theta \; , \; y = r\sin\theta$

$$\mathcal{E} \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \Rightarrow r^2(b^2\cos^2\theta + a^2\sin^2\theta) = a^2b^2$$

$$r = \frac{ab}{\sqrt{(a\sin\theta)^2+(b\cos\theta)^2}} = \frac{\cancel{a}b}{\cancel{a}\sqrt{\sin^2\theta+\underbrace{\frac{b^2}{a^2}}\cos^2\theta}} = \frac{b}{\sqrt{\sin^2\theta+\cos^2\theta-e^2\cos^2\theta}}$$

$$b^2 = a^2(1-e^2) = a^2-a^2e^2$$

$$\frac{b^2}{a^2} = 1-e^2$$

$$\frac{a^2}{b^2} = \frac{1}{1-e^2}$$

$$\boxed{\mathcal{E}: \; r = \frac{b}{\sqrt{1-e^2\cos^2\theta}}}$$

Polar eqⁿ from centre of ellipse

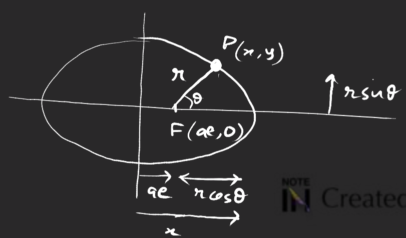
$r = r(\theta)$

Practice

$$* \quad r = \frac{ab}{b\sqrt{\frac{a^2}{b^2}\sin^2\theta+\cos^2\theta}} = \frac{a\sqrt{1-e^2}}{\sqrt{\sin^2\theta+\cos^2\theta-e^2\cos^2\theta}} = \frac{a\sqrt{1-e^2}}{\sqrt{1-e^2\cos^2\theta}}$$

nasty expression.

b) from Focus



$* \quad P: \; x = ae + r\cos\theta \; , \; y = r\sin\theta$

$$* \quad \mathcal{E}: \frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$

$(ae + r\cos\theta)^2 b^2 + a^2 r^2 \sin^2\theta = a^2 b^2$

$b^2 = a^2(1-e^2)$

$b^2 = a^2 - a^2e^2$

$a^2 - b^2 = a^2e^2$

$a^2b^2e^2 + r^2b^2\cos^2\theta + 2aerb^2\cos\theta + a^2r^2\underbrace{\sin^2\theta}_{1-\cos^2\theta} = a^2b^2$

$a^2r^2 + 2aerb^2\cos\theta - \underbrace{(a^2-b^2)}_{a^2e^2}r^2\cos^2\theta = \underbrace{a^2b^2(1-e^2)}_{b^2 \cdot b^2} \Rightarrow (ar)^2 = \underbrace{b^4 + (aer\cos\theta)^2 - 2aerb^2\cos\theta}_{(b^2 - aer\cos\theta)^2}$

$ar = b^2 - aer\cos\theta \Rightarrow \boxed{r = \dfrac{b^2/a}{1+e\cos\theta} = \dfrac{\ell}{1+e\cos\theta}}$

$r = r(\theta)$

Polar form of ellipse

$LR = \dfrac{2b^2}{a}$

$\ell \equiv \dfrac{LR}{2} = \dfrac{b^2}{a}$ semi latus Rectum

$r = r(\ell, e)$

rectilinear form

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

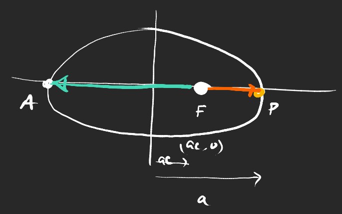
- x, y
- $a, b, b = b(e)$
- Quad. in x & y
- Can't use Trig.

polar form

$\dfrac{1}{r} = \dfrac{1+e\cos\theta}{\ell}$

- r, θ
- ℓ, e
- Linear in r
- Involve trig fn

Terminologies



$r_{per} = FP = a(1-e)$ near Perigee

$r_{ap} = FA = a(1+e)$ Apogee away from / far

F = sun $\begin{cases}\text{perihelion} \\ \text{Aphelion}\end{cases}$

Practice

Q1. $\mathcal{E} \equiv 9x^2 + 16y^2 = 144$

$L \equiv y = x + \lambda$: L touches Ellipse $\mathcal{E}$ $\lambda = ?$

* $\dfrac{9x^2}{144} + \dfrac{16y^2}{144} = 1 \Rightarrow \dfrac{x^2}{4^2} + \dfrac{y^2}{3^2} = 1$ $a = 4, b = 3$

* $\boxed{L = T \equiv y = mx + c \ : \ c^2 = a^2m^2 + b^2}$

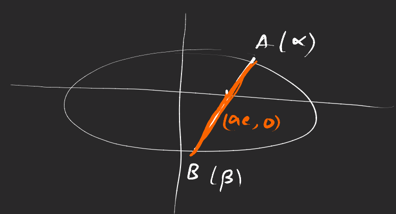
for L to be a T, $y = x + \lambda \longrightarrow m = 1$

$\boxed{c = \lambda} \longrightarrow \lambda = c = \sqrt{a^2m^2 + b^2} = \sqrt{a^2 + b^2} = 5$

Q2. $\mathcal{E} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

α, β are eccentric angles of end pts of focal chord

$\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2} = ?$



* eq^n of chord : $\dfrac{x}{a}\cos\dfrac{\alpha+\beta}{2} + \dfrac{y}{b}\sin\dfrac{\alpha+\beta}{2} = \cos\dfrac{\alpha-\beta}{2}$

* Chord passes $F(ae, 0) \Rightarrow e\cos\dfrac{\alpha+\beta}{2} = \cos\dfrac{\alpha-\beta}{2} \Rightarrow \boxed{\dfrac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}} = \dfrac{e}{1}}$

??

- eq^n of ellipse
- focus of ellipse
- Chord eq^n
- Focal chord ($\neq$ LR) eq^n
- Eccentric angle $\{x = a\cos\theta, \ y = b\sin\theta\}$
- Trigono. formula

• $\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2} = \dfrac{2\sin\frac{\alpha}{2}\sin\beta/2}{2\cos\frac{\alpha}{2}\cos\beta/2} = \dfrac{\cos\frac{\alpha-\beta}{2} - \cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha+\beta}{2} + \cos\frac{\alpha-\beta}{2}} = \dfrac{\cos\frac{\alpha-\beta}{2} - \cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2} + \cos\frac{\alpha+\beta}{2}}$

C D (+) (−)

$= ?$

$S+S = 2SC$
$S-S = 2CS$
$C+C = 2CC$
$C-C = 2SS$ *

A+B A-B A B

Q3. $E \equiv 3x^2+4y^2=12$ find the eq^n's of tangents to E which are $\perp$ to line $L \equiv y+2x=4$

eq^n of ellip

$$\frac{x^2}{2^2}+\frac{y^2}{(\sqrt{3})^2}=1 \quad \begin{cases} a^2=4 \\ b^2=3 \end{cases}$$

st line $\perp$ cond^n

$L \equiv y=-2x+4$

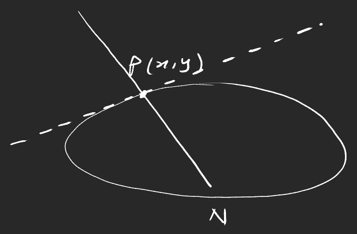
$m_L=-2 \Rightarrow \boxed{m_T=\frac{1}{2}}$

eq^n of tangent : $y=m_Tx+c=m_Tx\pm\sqrt{a^2m_T^2+b^2}$

$\boxed{y=\frac{1}{2}x\pm 2} \Rightarrow \boxed{x+2y\pm 4=0}$ 2 eq^n's of tans.

Lecture 7 (14 Jul) 2:15

Q4. $E \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $L \equiv lx+my=n$ if L is normal to E under what condition ?



* $P(x,y) \to P(a\cos\theta, b\sin\theta)$ $\boxed{N \equiv ax\sec\theta - by\,\mathrm{cosec}\,\theta = a^2-b^2}$

$N \equiv \frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2 \longrightarrow$ if $L=N$ equ^n must be identical

$$\left.\begin{aligned} N &\equiv \frac{a}{\cos\theta}x-\frac{b}{\sin\theta}y=a^2-b^2 \\ L &\equiv lx+my=n \end{aligned}\right\} \Rightarrow \frac{l}{\frac{a}{\cos\theta}}=\frac{m}{-\frac{b}{\sin\theta}}=\frac{n}{a^2-b^2}$$

$L_1: a_1x+b_1y+c_1=0$
$L_2: a_2x+b_2y+c_2=0$
$L_1=L_2 \Rightarrow \frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

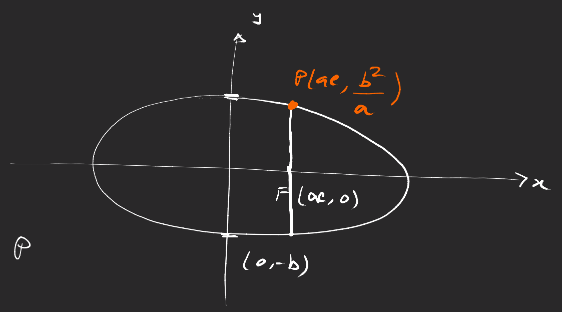
* $$\left.\begin{aligned} \cos\theta &= \frac{a}{l}\frac{n}{a^2-b^2} \\ \sin\theta &= -\frac{b}{m}\frac{n}{a^2-b^2} \end{aligned}\right\} \Rightarrow \boxed{\frac{n^2}{(a^2-b^2)^2}\left(\frac{a^2}{l^2}+\frac{b^2}{m^2}\right)=1}$$

sq and add & apply $\cos^2\theta+\sin^2\theta=1$

Q5. $E \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

if the normal at an end of a LR passes thru. + extremity of minor axis

$e = ?$



let

* $N \equiv \frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$ (x_1,y_1) P.O.C

* N passes thru the end coord. of LR $\left(P\left(ae, \frac{b^2}{a}\right)\right) \Rightarrow$ N satisfies P

x_1 y_1

$\frac{a^2x}{ae}-\frac{b^2y}{b^2/a}=a^2-b^2 \Rightarrow \boxed{\frac{ax}{e}-ay=a^2-b^2}$

* N satisfies minor axis coord $(0,-b)$ $\{$ not $(0,b)$ $\because$ geometry

$\boxed{ab = a^2 - b^2} \Rightarrow a^2b^2 = (a^2-b^2)^2 \Rightarrow \cancel{a^4}(1-e^2) = (\cancel{a^2}e^2)^2$

$b^2 = a^2(1-e^2)$
$b^2 = a^2 - a^2e^2$
$a^2 - b^2 = a^2e^2$

$e^4 + e^2 - 1 = 0 \Rightarrow (e^2)^2 + e^2 - 1 = 0 \Rightarrow e^2 = \dfrac{-1 \pm \sqrt{1+4}}{2} \Rightarrow e = \pm\sqrt{\dfrac{-1\pm\sqrt{5}}{2}}$

$\boxed{e = \sqrt{\dfrac{-1+\sqrt{5}}{2}}}$ $\because$ $0 < e < 1$

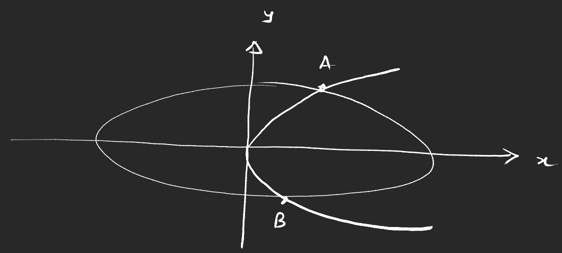
HW

Q6. $\mathcal{E} \equiv \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

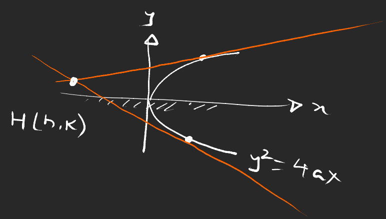
$\mathcal{P} \equiv y^2 = 4ax$

if tangents to parabola $\mathcal{P}$ intersect $\mathcal{E}$ at A, B.

find the locus of point of intersection of tangents at A, B



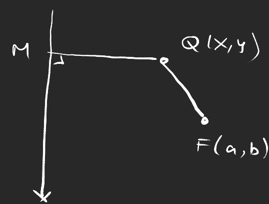
* Locus of $H(h,k) = ?$



2. Hyperbola

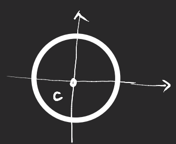
2.1 Remark on Previous Conics

* $|QF| = e|QM|$

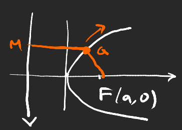


$D \equiv lx + my + n = 0$

$e = 0$



$e = 1$

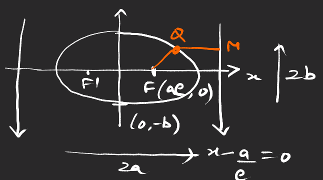


$x + a = 0$

focus = 1
dirx = 1

$y^2 = 4ax$

$e < 1$



2a $\quad x - \dfrac{a}{e} = 0$

focus = 2
dirx = 2

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

limits

FYI: $0 \times 0 = 0$

$a \times b = 0 \Rightarrow a = 0$ or $b = 0$

$b = \dfrac{0}{a} \sim b = \infty$

$b^2 = a^2(1-e^2)$ "illegal parlance"

$e = 0$

- $\boxed{b = a} \Rightarrow$ maj/min. distinction disappear.
- $F(ae,0) = F(0,0) = F' = C(0,0)$

$\exists$ only 1 "focus" ie center

- Direx: $x - \dfrac{a}{e} = 0 \longrightarrow$ "$D \equiv \infty$" line at ∞ "DNE"

Circle = special ellipse

$e = 1$

$b^2 = a^2(1-e^2)$
$a^2 = \dfrac{b^2}{1-e^2}$

I. • $b = 0 \Rightarrow \boxed{a = \infty} \Rightarrow$ one focus is at ∞

II. • $F(ae,0) = F(a,0) \Rightarrow$ there is a focus at finite point

III. • Direx: $x - \dfrac{a}{e} = 0 \Rightarrow x - a = 0$

there is one finite dirx

IV. • $a = \infty$
$F'(a,0)$ $\Rightarrow$ dir^x D^*: $x - \dfrac{a}{e} = 0$

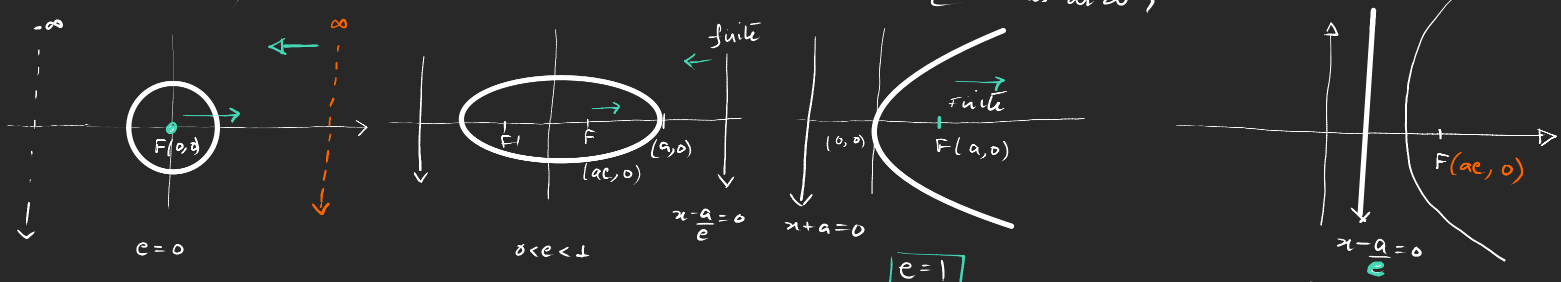
$\downarrow \lim_{a\to\infty}$

"$D^* = \infty$ line at ∞" "DNE"

$\boxed{\begin{array}{l}\exists \ \# \text{focus} = 1 \\ \# \text{dir}^x = 1\end{array}}$

Lecture 8 (15/Jul) 2.10

{1 pair of Dir^x & Focus at ∞}



e = 0 0 < e < 1 $\boxed{e=1}$ e > 1

knob

* $e: 0 \longrightarrow \infty \Rightarrow$ Dir$x\big|_{e=0} \longrightarrow +\infty$, "degeneracy" $F(0,0)=F'(0,0)=C(0,0)$

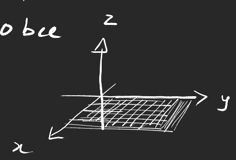
As $e\uparrow \Rightarrow$ Dir$x\big|_{0<e<1} \longrightarrow$ 2 finite lines , F splits into 2 "degeneracy is lifted off" F(ae,0) F'(-ae, 0) } Dirx is on the right of focus

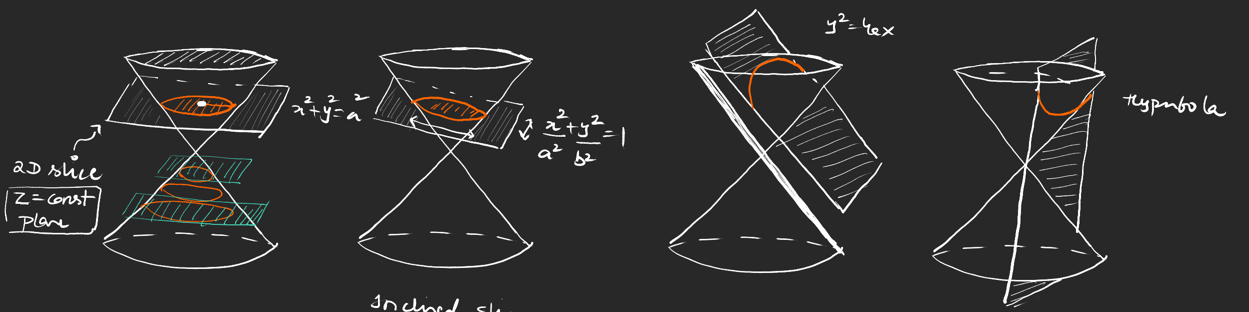
e = 1 $\Rightarrow$ where dir^x crosses over & changes sides with Focus (limiting case)

e > 1 $\Rightarrow$ No turning back. } Dirx is on the left of Focus

2.2 What's "CONIC" about it ?

* $\mathbb{R}^3$: ambient space $\longrightarrow \exists$ 4 quadratic curves { • Menaechmus, student of Plato (350 bce) • APOLLONIUS, "the great geometer" 260 bce





Inclined slice made axis stretch & squeeze

unified

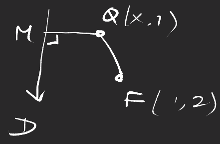
Ellipse - Hyperbola equivalence in $\mathbb{C}P^2$ (complex projective plane)

$\mathcal{E}: x^2+y^2=1 \xrightarrow[y=i\omega]{\mathbb{C}} x^2-\omega^2=1$ } Hyperbola

*

point Line Crossed Lines

Q1. $F(1,2)$ $D \equiv x+y+1=0$, $e = \frac{3}{2}$ locus of $Q(x,y)$



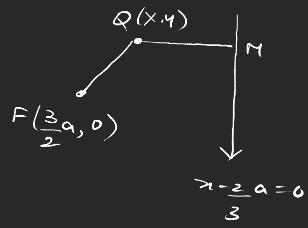
* $|QF|^2 = e^2 |QM|^2 \Rightarrow (x-1)^2+(y-2)^2 = \frac{9}{4} \frac{|x+y+1|^2}{(\sqrt{2})^2}$

$8\{x^2+y^2-2x+1+4-4y\} = 9\{x^2+y^2+1+2xy+2y+2x\}$

$\boxed{x^2+y^2+18xy+34x+50y-31=0}$ Eq^n of Hyperbola $(e>1)$

Journey to std. hyperbola

Type I : $e = 3/2$



* $|QF|^2 = e^2 |QM|^2 \Rightarrow (x-\frac{3}{2}a)^2 + y^2 = \frac{9}{4} |x - \frac{2}{3}a|^2$

$4(x^2+\frac{9}{4}a^2 - \cancel{3ax} + y^2) = 9(x^2 + \frac{4}{9}a^2 - \cancel{\frac{4ax}{3}}) \Rightarrow -5x^2+4y^2 = -5a^2$

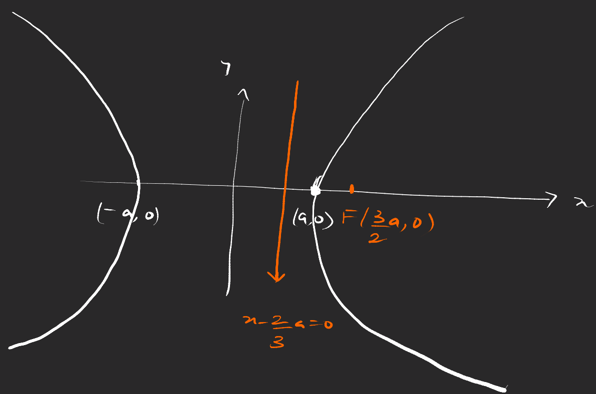
$\Downarrow$

$$\frac{x^2}{a^2} - \frac{y^2}{\left(\frac{5}{4}a^2\right)} = 1$$

* $\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$ $b^2 \equiv \frac{5}{4}a^2 = a^2\left(\frac{9-4}{4}\right) = a^2\left(\left(\frac{3}{2}\right)^2-1\right) = a^2(e^2-1)$

$\boxed{b^2 \equiv a^2(e^2-1)}$ $e>1$, $b>a$

Curve tracing : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



* $x=0 \Rightarrow y^2 = -b^2 \Rightarrow y \in \mathbb{C}$ won't cut y axis

$y=0 \Rightarrow \boxed{x = \pm a}$

* symmetric in x & y ($\because$ even in x, y)) (
* $O(0,0)$ not satisfied $\Rightarrow$ won't pass thru O
* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow x = \pm a\sqrt{1+\frac{y^2}{b^2}} = \pm\frac{a}{b}\sqrt{y^2+b^2}$

$y=0 \Rightarrow x = \pm a$; $y\uparrow \Rightarrow x\uparrow \Rightarrow$ curve won't close

* $y = \pm b\sqrt{\frac{x^2}{a^2}-1} = \pm\frac{b}{a}\sqrt{x^2-a^2}$

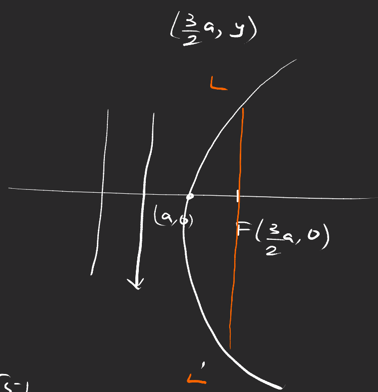
if $x \in [-a,a] \Rightarrow y \in \mathbb{C} \Rightarrow$ curve DNE b/w $[-a,a]$

Naming

* Vertex $(\pm a, 0)$ $b = \frac{\sqrt{5}}{2}a$ Later $\rightarrow$ $b^2 = \frac{5}{4}a^2 \Rightarrow \frac{2b^2}{a} = \underbrace{\frac{5}{2}a}_{LR}$

Focus $(\frac{3}{2}a, 0)$

Dirx : $x - \frac{2}{3}a = 0$



"Hack"

* $LR = 2|LF| = b\sqrt{5} = \frac{5}{2}a = \frac{2b^2}{a}$ (Later)

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\downarrow x = \frac{3}{2}a$

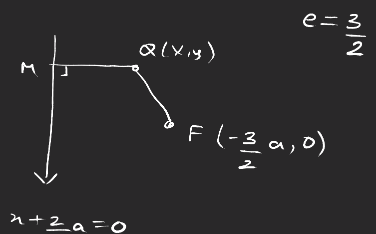
$\frac{9}{4} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{y^2}{b^2} = \frac{5}{4} \Rightarrow y = \pm\frac{\sqrt{5}}{2}b$

$\| $

$|LF|$

Lecture-9 (16 Jul) 1.5

Type 2

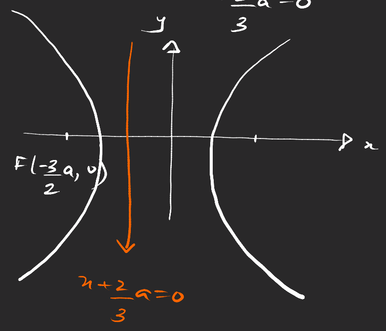


$*$ $|QF|^2 = e^2|QM|^2 \Rightarrow (x + \frac{3}{2}a)^2 + y^2 = \frac{9}{4}\,|x + \frac{2}{3}a|^2 \Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$b^2 = \frac{5}{4}a^2 = a^2(e^2 - 1)$

$*$ curve tracing - same as type 1

$V(-a, 0)$, $F(-\frac{3}{2}a, 0)$, $D \equiv x + \frac{2a}{3} = 0$, $LR = \frac{2b^2}{a}$



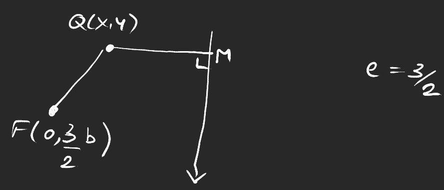
Type 3

$*$ $|QF|^2 = e^2|QM|^2 \Rightarrow \boxed{\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1}$

$e > 1$

$a^2 = \frac{5}{4}b^2 = b^2(e^2 - 1)$

$\boxed{a > b}$



$*$ curve tracing

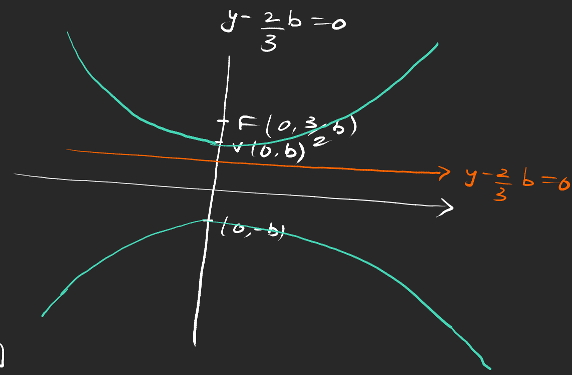
- $x = 0 \Rightarrow \boxed{y = \pm b}$

  $y = 0 \Rightarrow x^2 = -a^2 \Rightarrow$ won't cut x axis

- sym. in x, y
- Doesn't pass thru O
- $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1 \Rightarrow x = \pm a\sqrt{\frac{y^2}{b^2} - 1} = \pm\frac{a}{b}\sqrt{y^2 - b^2}$

  if $y^2 - b^2 < 0 \Rightarrow y \in [-b, b] \Rightarrow$ curve DNE b/w $y \in [-b, b]$

- $y = \pm b\sqrt{1 + \frac{x^2}{a^2}} = \pm\frac{b}{a}\sqrt{x^2 + a^2} \longrightarrow x\uparrow\ y\uparrow$ Not closed curve



$*$ $V(0, b)$, $a = \frac{\sqrt{5}}{2}b$

$F(0, \frac{3}{2}b)$

$D \equiv y - \frac{2}{3}b = 0$

'Hack'

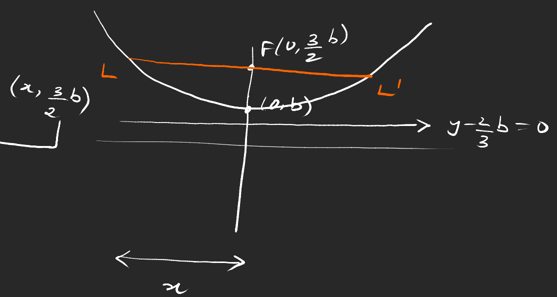
$*$ $\boxed{LR = 2(LF) = \sqrt{5}a = \frac{5}{2}b = \frac{2b^2}{a}}$

$(x, \frac{3}{2}b)$

$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$

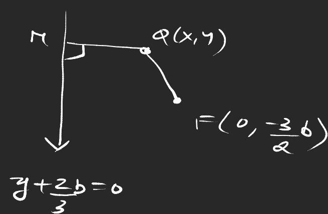
$\Downarrow$

$x = \frac{\sqrt{5}}{2}a$



Type 4

$*$ $|QF|^2 = e^2|QM|^2 \Rightarrow \cdots\cdots \Rightarrow \boxed{\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1}$



curve tracing: same as type 3

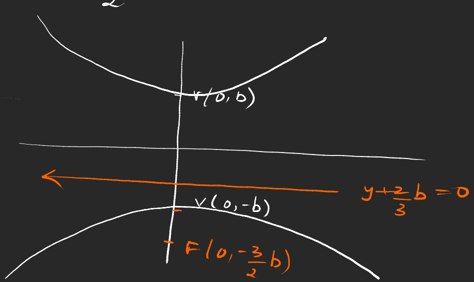
$a^2 = \frac{5}{2}b^2 = b^2(e^2 - 1)$ $a > b$

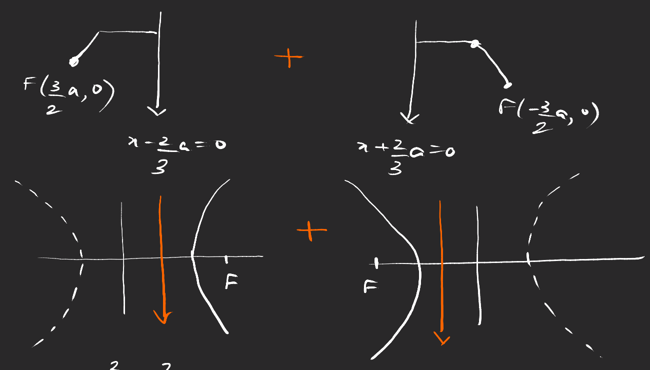
$V(0, -b)$

$F(0, -\frac{3}{2}b)$

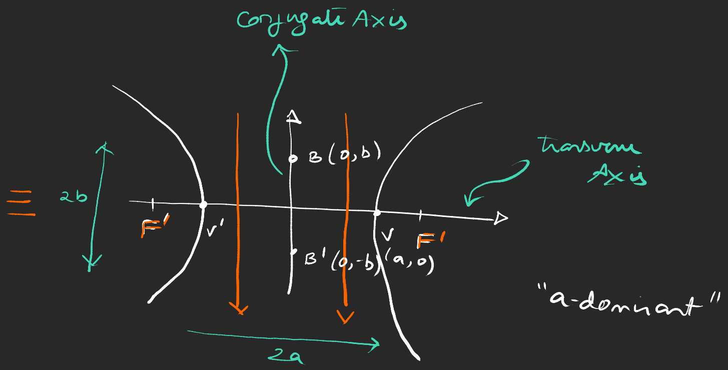
$D \equiv y + \frac{2}{3}b = 0$

$LR = \frac{2b^2}{a}$ ✓



* 

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $\qquad$ $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

≡ 

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $\qquad$ $b^2 \equiv a^2(e^2-1)$

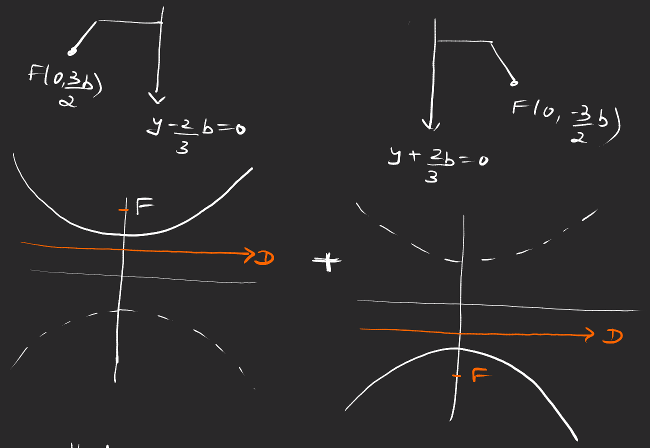
$LR = \frac{2b^2}{a}$

Principal Axes of Hyperbola

- Transverse Axis ≡ Axis in which both foci lie

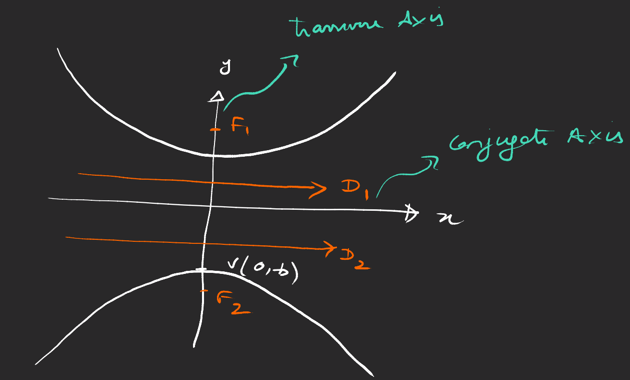
$\boxed{|VV'| = 2a}$

- Conjugate Axis ≡ ⊥ to transverse

* 

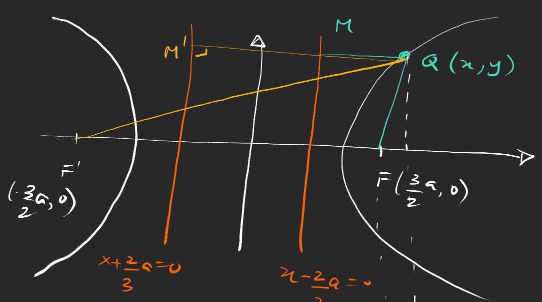
dirx = 1 $\qquad$ # D = 1

F = 1 $\qquad$ # F = 1

≡ 

$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$, $a^2 \equiv b^2(e^2-1)$, $LR = \frac{2b^2}{a}$

Focal distance theorem

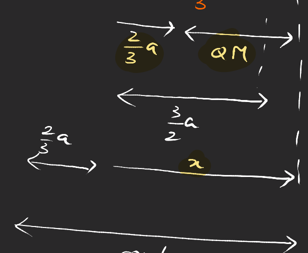


* $|QF| = e|QM| = \frac{3}{2}QM$ $\qquad$ $QM = x - \frac{2}{3}a$

$|QF'| = e|QM'| = \frac{3}{2}QM'$ $\qquad$ $QM' = x + \frac{2}{3}a$

$FQ = \frac{3}{2}\left(x - \frac{2}{3}a\right) = \frac{3}{2}x - a$

$F'Q = \frac{3}{2}\left(x + \frac{2}{3}a\right) = \frac{3}{2}x + a$



* $\boxed{F'Q - FQ = 2a}$ diff of focal dist is const.

Comment on Special type of Hyperbola

* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $b^2 = a^2(e^2-1)$

$\mathbb{R} \to \mathbb{C}$

$y \to iy$

'Rectangular Hyperbola'

∃ special value of $\boxed{e = \sqrt{2}} \Rightarrow b = a \Rightarrow \boxed{x^2 - y^2 = a^2}$ "Pythagoras" in $\mathbb{C}$

Super-special & useful $\qquad$ "Circle" in $\mathbb{C}$

$\boxed{\text{Pythagoras}}$

circle

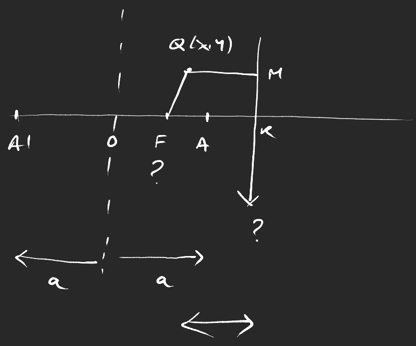
$x^2 + y^2 = a^2$ $\xleftarrow{a=b}$ ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

" ?? " weird Analog $\xleftarrow{a=b}$ $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ Hyperbola

Lecture-10 (17/Jul) 2

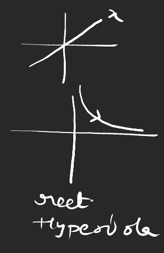
A Formal Approach to Hyperbola

Set up



$f(x) = x$

$\boxed{y = f(x) = \frac{1}{x}}$



$|OA = OA' = a$

* divide FK internally / externally $(e:1)$

* $|QF| = e|QM|$, $e > 1$

* $\frac{FA}{AK} = \frac{e}{1} \Rightarrow FA = e\,AK$

  $\frac{FA'}{A'K} = \frac{e}{1} \Rightarrow FA' = e\,A'K$

* Add

$$\underbrace{FA + FA'}_{2a} = e(\underbrace{AK}_{OK - \cancel{OA}} + \underbrace{A'K}_{\cancel{A'O} + OK})$$

$\Downarrow$

$\boxed{|OK| = \frac{a}{e}}$ dir x coord

subtract

$$\underbrace{\underbrace{FA'}_{\cancel{A'O} + OF} - \underbrace{FA}_{\cancel{AO} - OF}}_{2(OF)} = e\,(\underbrace{A'K - AK}_{\underbrace{(A'O + \cancel{OK}) - (\cancel{OK} - OA)}_{2a}})$$

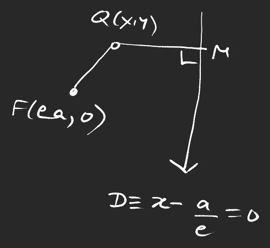
$\boxed{|OF| = ea}$ focus coord

Calc

* $|QF|^2 = e^2|QM|^2 \Rightarrow (x - ea)^2 + y^2 = e^2\left|x - \frac{a}{e}\right|^2$

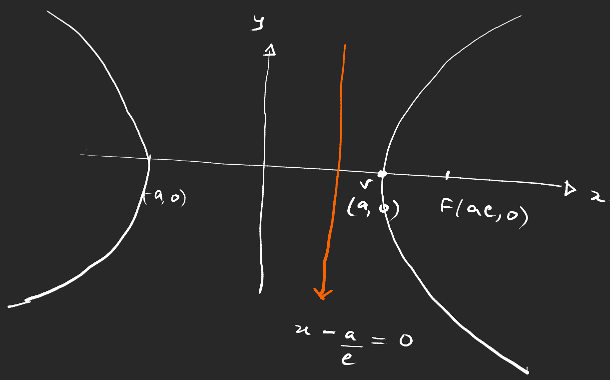
$x^2 + e^2a^2 - \cancel{2}eax + y^2 = e^2\left(x^2 + \frac{a^2}{e^2} - 2\frac{a}{e}x\right)$

$(1 - e^2)x^2 + y^2 = a^2(1 - e^2) \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$ $e > 1$



$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1 \Rightarrow \boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$, $b^2 \equiv a^2(e^2 - 1)$

Curve tracing : same



2nd focus / 2nd dirx

* $\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1 \Rightarrow x^2 = \frac{a^2(e^2-1) + y^2}{e^2 - 1}$

$(e^2 - 1)x^2 = a^2(e^2 - 1) + y^2 \Rightarrow e^2x^2 - x^2 = a^2e^2 - a^2 + y^2$

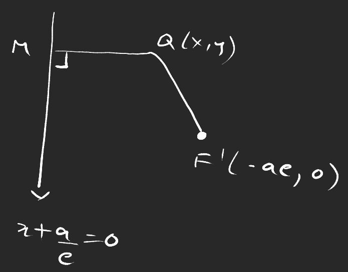
$x^2 - e^2x^2 = -a^2e^2 + a^2 - y^2 \Rightarrow x^2 + a^2e^2 + 2aex = x^2e^2 + a^2 - y^2 + 2aex$

$(x + ae)^2 + y^2 = \underbrace{(xe + a)^2}_{e^2\left(x + \frac{a}{e}\right)^2} \Rightarrow |QF|^2 = e^2|QM|^2$

$\Downarrow$

$\boxed{|QF| = e|QM|}$ $e > 1$

Assume new,



B. Important properties / Result

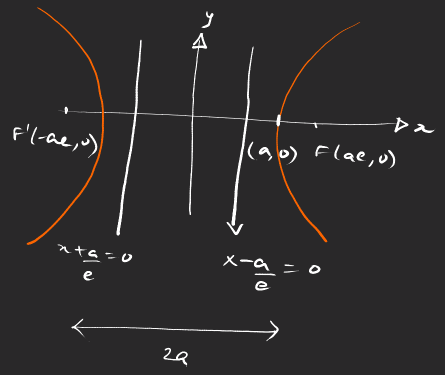
* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $b^2 \equiv a^2(e^2-1) \Rightarrow b^2 = a^2e^2 - a^2 \Rightarrow a^2+b^2 = a^2e^2 \Rightarrow \boxed{e = \sqrt{1+\left(\frac{b}{a}\right)^2}}$

* Transverse axis $= 2a$

Conjugate Axis $= 2b$

$$\boxed{e = \sqrt{1+\left(\frac{\text{Conjugate}}{\text{transverse}}\right)^2}}$$

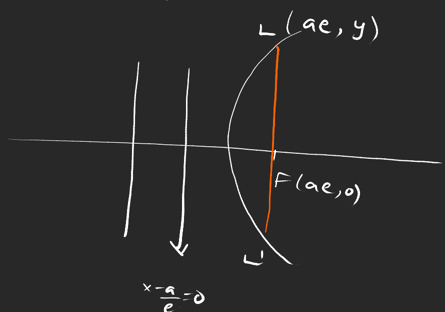
$$b^2 = a^2(e^2-1)$$



* $LR = 2|LF| = 2b\sqrt{e^2-1} = 2b \cdot \frac{b}{a}$

$\boxed{LR = \frac{2b^2}{a}}$ Latus Rectum

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (x = ae)$$
$$\Downarrow$$
$$\frac{y^2}{b^2} = e^2 - 1$$
$$y = b\sqrt{e^2-1}$$



* $|FQ| = e|QM| = e\left(x - \frac{a}{e}\right) = \boxed{ex - a}$

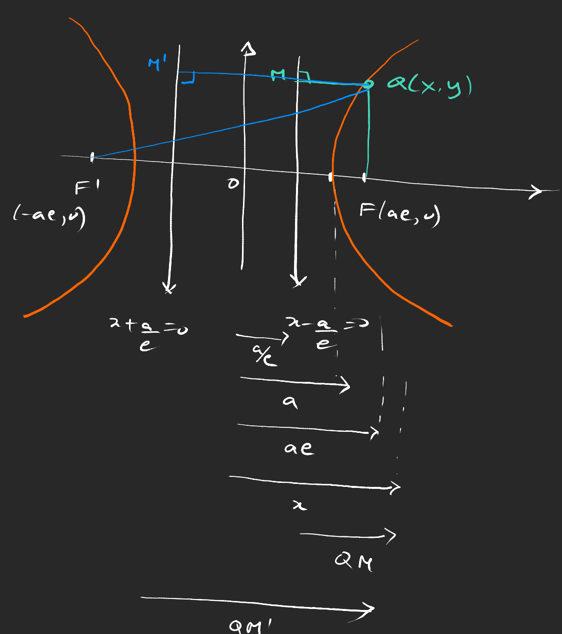
$|F'Q| = e|QM'| = e\left(x + \frac{a}{e}\right) = \boxed{ex + a}$

$QM = x - \frac{a}{e}$

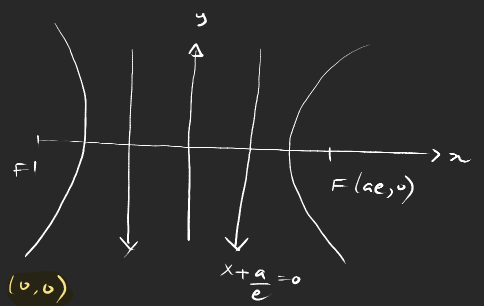
$QM' = x + \frac{a}{e}$

* $\boxed{|F'Q| - |FQ| = 2a = \text{const}}$ focal dist. thm.

diff. of focal dist is const



C. Comparison



centre $(0,0)$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $\quad b^2 \equiv a^2(e^2-1)$ $\quad \boxed{b > a}$

$LR = \frac{2b^2}{a}$

Focal dist $= ex \pm a$

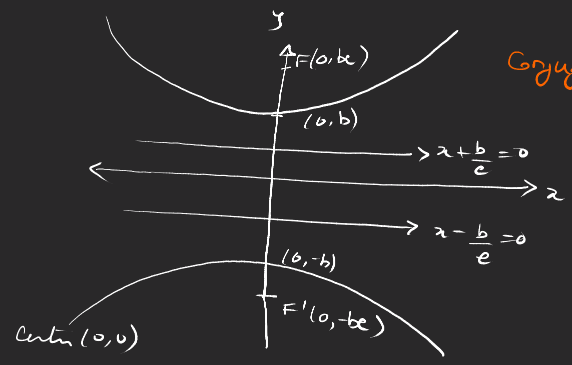
transv. Axis $= 2a$

Conjugate Axis $= 2b$

$e = \sqrt{1 + \frac{b^2}{a^2}}$

$\boxed{a \leftrightarrow b}$

$\boxed{x \leftrightarrow y}$



Conjugate Hyperbola

Centre $(0,0)$

$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$, $a^2 \equiv b^2(e^2-1)$ $\quad \boxed{a > b}$

$LR = \frac{2a^2}{b}$

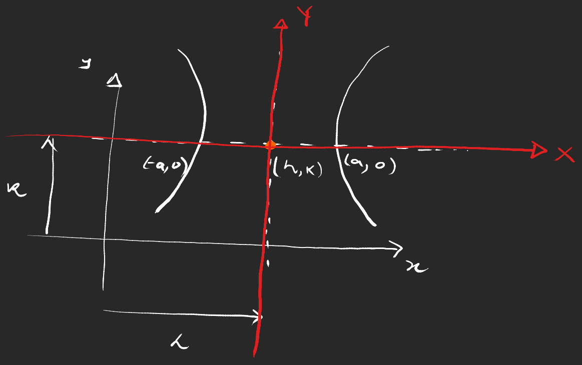
Focal dist $= ey \pm b$

T.A $= 2b$

C.A $= 2a$

$e = \sqrt{1 + \frac{a^2}{b^2}}$

Comment on off centre :



$*$ $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$ centred at (h,k)

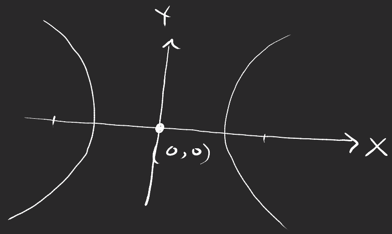
$\downarrow$ make a coord. transf^n.

$(x,y) \to (X,Y)$: $X \equiv x-h$, $Y \equiv y-k$ $\Rightarrow$ $x = X+h$, $y = Y+k$

$X=0$, $Y=0$ $\Rightarrow$ $x=h$, $y=k$

$$\boxed{\begin{pmatrix} h \\ k \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ 0 \end{pmatrix}}$$

$$\boxed{\mathcal{E}' \equiv \frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1}$$



Lecture-11 (20|Jul) 1.20

23 Hyperbola (the real deal)

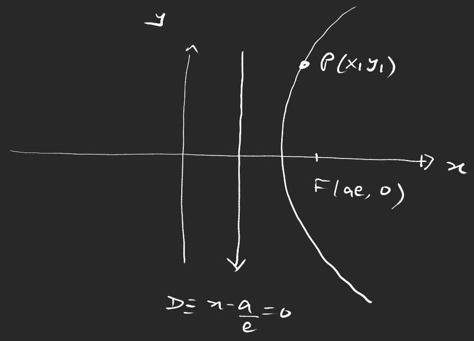
A. Position of a pt. relative to hyperbola (Intuitive)

$*$ $\mathcal{H} \equiv \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, $P(x_1, y_1)$

$\dfrac{x_1^2}{a^2} - \dfrac{y_1^2}{b^2} - 1$ $= 0$ on the $\mathcal{H}$

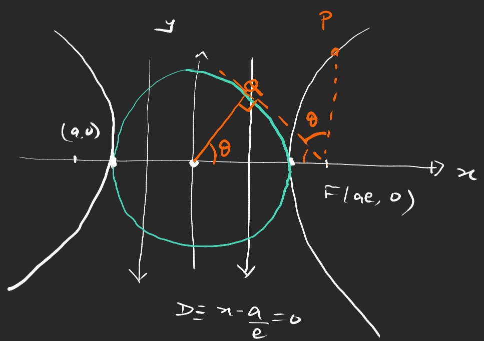
> 0 outside $\mathcal{H}$

< 0 inside $\mathcal{H}$



B. Auxiliary Circle

$*$ $C \equiv x^2 + y^2 = a^2$ centre $= (0,0)$, Rad $= a$ } C auxiliary circle



C. Parametric eq^n of Hyperbola

$*$ $P(x,y) \longrightarrow P(\theta)$

$$\boxed{\begin{aligned} x &= a\sec\theta \\ y &= b\tan\theta \end{aligned}} \iff \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Parametric eq^n (1)

θ parameter

Another way to parametrize

$*$
$$\boxed{\begin{aligned} x &= a\cosh\theta \\ y &= b\sinh\theta \end{aligned}} \Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Param eq^n in Terms of Hyperbolic T-fun^n (2) — THE BEST !

$\cos^2\theta + \sin^2\theta = 1$

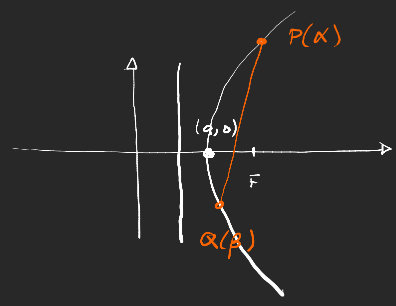
$\downarrow$

$\cosh^2\theta - \sinh^2\theta = 1$

D. Eqⁿ of chord

* PQ chord : $P(a\sec\alpha, b\tan\alpha)$ (x_1, y_1)

$Q(a\sec\beta, b\tan\beta)$ (x_2, y_2)



$$y - b\tan\alpha = \frac{\frac{b}{a}(\tan\beta - \tan\alpha)}{(\sec\beta - \sec\alpha)}(x - a\sec\alpha)$$

$$\frac{y - b\sin\alpha}{\cos\alpha} = \frac{b}{a}\left(\frac{\frac{\sin\beta}{\cos\beta} - \frac{\sin\alpha}{\cos\alpha}}{\frac{1}{\cos\beta} - \frac{1}{\cos\alpha}}\right)\left(\frac{x - a}{\cos\alpha}\right)$$

$$\left(\frac{x}{a}\cos\alpha - 1\right)\cancel{a}$$

$$\frac{\cancel{a}}{b}(y\cos\alpha - b\sin\alpha) = \underbrace{\frac{\sin\beta\cos\alpha - \cos\beta\sin\alpha}{\cos\alpha - \cos\beta}}\overbrace{(x\cos\alpha - a)}$$

$$\frac{\sin(\beta-\alpha)}{2\cdot\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}} = \frac{2\cancel{\sin\frac{\beta-\alpha}{2}}\cos\frac{\beta-\alpha}{2}}{2\sin\frac{\alpha+\beta}{2}\cancel{\sin\frac{\beta-\alpha}{2}}} = \frac{\cos\frac{\beta-\alpha}{2}}{\sin\frac{\beta+\alpha}{2}}$$

$\longrightarrow$

$C \quad S \quad \frac{(+)}{2} \quad \frac{(-)}{2}$

$S+S = 2SC$

$S-S = 2CS$

$C+C = 2CC$

$C-C = 2SS^*$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$ ✓

* $\frac{y}{b}\cos\alpha\sin\frac{\alpha+\beta}{2} - \sin\alpha\sin\frac{\alpha+\beta}{2} = \frac{x}{a}\cos\alpha\cos\frac{\beta-\alpha}{2} - \cos\frac{\beta-\alpha}{2}$ $\rightsquigarrow$ $ax+by+c=0$

$$\frac{x}{a}\cos\frac{\beta-\alpha}{2} - \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \frac{1}{\cos\alpha}\underbrace{\left\{\cos\frac{\beta-\alpha}{2} - \sin\alpha\sin\frac{\alpha+\beta}{2}\right\}}_{\text{|| P.T} \; \cos\frac{\alpha+\beta}{2}}$$

HW

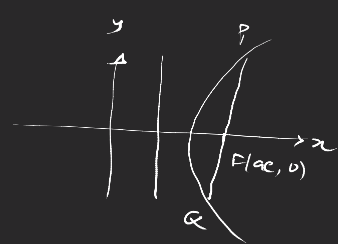
$\cos 2\theta = 2\cos^2\theta - 1$

$= 1 - 2\sin^2\theta$

* $$\boxed{\frac{x}{a}\cos\frac{\alpha-\beta}{2} - \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha+\beta}{2}}$$ ✓

Comment on Focal chord

* Focal ch. ⟹ $F(ae, 0)$ satisfied



$$e\cos\frac{\alpha-\beta}{2} = \cos\frac{\alpha+\beta}{2} \Rightarrow e = \frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}} = \frac{\cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{\cos\frac{\alpha}{2}\cos\frac{\beta}{2} + \sin\frac{\alpha}{2}\sin\frac{\beta}{2}}$$

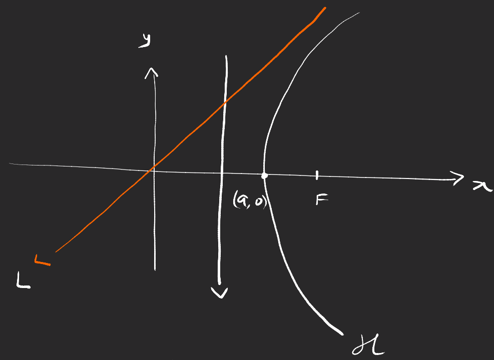
$$e = \frac{1 - \tan\frac{\alpha}{2}\tan\frac{\beta}{2}}{1 + \underbrace{\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}_{r}} \Rightarrow e = \frac{1-r}{1+r} \Rightarrow e + er = 1 - r \Rightarrow (1+e)r = 1-e$$

$\Downarrow$

$$\boxed{\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{1-e}{1+e}}$$

E. Hyperbola intersecting w/ a st. line ┌ tan. └ secancy

* $\mathcal{H} \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $L \equiv y = mx + c$, $b^2 = a^2(e^2-1)$



$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1 \Rightarrow b^2x^2 - a^2(m^2x^2 + c^2 + 2mcx) = a^2b^2$

$(b^2 - a^2m^2)x^2 - 2mca^2x - a^2(b^2+c^2) = 0$ Quad. eqn in x

$$x_{1,2} = \frac{2mca^2 \pm \sqrt{4a^4m^2c^2 + 4(b^2-a^2m^2)(b^2+c^2)a^2}}{2(b^2-a^2m^2)}$$

Lecture-12 (21|Jul) 1.5

* $$x_{1,2} = \frac{2mca^2 \pm \sqrt{\cancel{4a^4m^2c^2} + 4a^2b^4 + 4a^2b^2c^2 - 4a^4m^2b^2 - \cancel{4a^4m^2c^2}}}{2(b^2-a^2m^2)}$$

$$= \frac{2mca^2 \pm \sqrt{4a^2b^2(b^2+c^2-a^2m^2)}}{2(b^2-a^2m^2)} = \frac{2mca^2 \pm 2ab\sqrt{c^2+b^2-a^2m^2}}{2(b^2-a^2m^2)}$$

→ D = 0 same roots
→ D > 0 2 distinct roots
→ D < 0 2 img. roots

* D = 0 ⇒ $\boxed{c^2 = a^2m^2 - b^2}$ tangency cond^n

* D > 0 ⇒ $c^2 > a^2m^2 - b^2$ line will cut at 2 pts (secant)

D < 0 ⇒ $c^2 < a^2m^2 - b^2$ [∵ won't cut at all]

Parabola e = 1
$y^2 = 4ax$
Tang. c = am

Ellipse e < 1
Tang. $c^2 = a^2m^2 + b^2$ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $b^2 = a^2(1-e^2)$

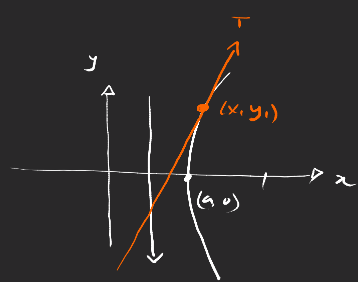
$\boxed{b^2 \to -b^2}$ trick

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $b^2 = a^2(e^2-1)$

F. Tangents to Hyperbola

* T ⇒ $c^2 = a^2m^2 - b^2$, $x_1 = \frac{mca^2}{b^2 - a^2m^2} = -\frac{ma^2}{\sqrt{a^2m^2-b^2}}$

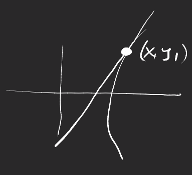
$y_1 = mx_1 + c = -\frac{m^2a^2}{\sqrt{a^2m^2-b^2}} + \sqrt{a^2m^2-b^2} = \frac{-b^2}{\sqrt{a^2m^2-b^2}}$



$$\boxed{\text{P.O.C} \equiv (x_1, y_1) = \left(\frac{-a^2m}{\sqrt{a^2m^2-b^2}}, \frac{-b^2}{\sqrt{a^2m^2-b^2}}\right) = \left(\frac{-a^2m}{c}, \frac{-b^2}{c}\right)}$$ Point of Contact

* EoT : $y - y_1 = m(x - x_1) \Rightarrow y - y_1 = \frac{b^2x_1}{a^2y_1}(x - x_1)$

$ya^2y_1 - a^2y_1^2 = b^2x_1x - b^2x_1^2 \Rightarrow a^2yy_1 - b^2xx_1 = \underbrace{a^2y_1^2 - b^2x_1^2}_{-a^2b^2}$



$\left.\begin{array}{l} x_1 = -\frac{a^2m}{c} \\ y_1 = -\frac{b^2}{c} \end{array}\right\} \Rightarrow \frac{x_1}{y_1} = \frac{a^2m}{b^2}$

⇓

$\boxed{m = \frac{b^2x_1}{a^2y_1}}$

* $\boxed{T \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1}$ 1 pt form (x₁,y₁) needed

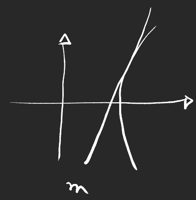
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ satisfies (x_1, y_1)

$\boxed{x_1^2 b^2 - a^2 y_1^2 = a^2 b^2}$

* EOT 2 : $y = mx + c$, $c^2 = a^2 m^2 - b^2$

$\boxed{T \equiv y = mx \pm \sqrt{a^2 m^2 - b^2}}$ slope form

m needed

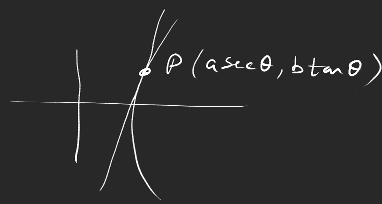


* EOT 3 : $x_1 = a \sec\theta$, $y_1 = b \tan\theta$

$\because T \equiv \frac{x x_1}{a^2} - \frac{y y_1}{b^2} = 1 \Rightarrow \boxed{T \equiv \frac{x \sec\theta}{a} - \frac{y \tan\theta}{b} = 1}$

Parametric form

θ needed



Comment on || tangents

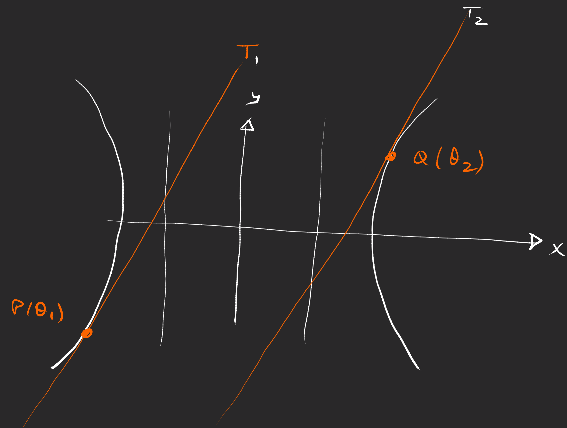
* $T_1 \parallel T_2 \Rightarrow m_{T_1} = m_{T_2}$ $m_{T_1} = \frac{b^2}{a^2}\frac{x_1}{y_1}$ $m_{T_2} = \frac{b^2}{a^2}\frac{x_2}{y_2}$

$\downarrow$

$x_1 y_2 = x_2 y_1$

$\Downarrow$

$\cancel{a} \sec\theta_1 \, b \tan\theta_2 = \cancel{b} \sec\theta_2 \, \cancel{a} \tan\theta_1$



$\frac{\sin\theta_2}{\cos\theta_1 \cancel{\cos\theta_2}} = \frac{\sin\theta_1}{\cos\theta_1 \cancel{\cos\theta_2}} \Rightarrow \sin\theta_1 - \sin\theta_2 = 0$ Trig eq^n

$\Downarrow$

$2 \cos\frac{\theta_1+\theta_2}{2} \sin\left(\frac{\theta_1-\theta_2}{2}\right) = 0$

$m_T = \frac{b^2}{a^2}\frac{x_1}{y_1}$

$x = a \sec\theta$
$y = b \tan\theta$

$S + S = 2SC$
$S - S = 2CS$

$\cos\frac{\theta_1+\theta_2}{2} = 0 = \cos\frac{\pi}{2}$

$\Downarrow$

$\boxed{\theta_1 + \theta_2 = \pi}$

non trivial

$\sin\left(\frac{\theta_1-\theta_2}{2}\right) = 0 = \sin 0$

$\Downarrow$

$\boxed{\theta_1 = \theta_2}$ ignore ($\because \theta_1 \neq \theta_2$)

note : θ_1, θ_2 are different points
not slopes

$AB = 0$

$A = 0$ or $B = 0$

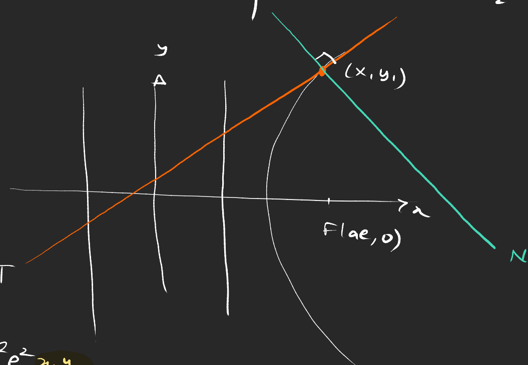
$0 \times 0 \neq 0$

Ellipse

$T_1 \parallel T_2$ $\theta_1 - \theta_2 = \pi$

G. Normal to Hyperbola

* $m_T = \frac{b^2}{a^2}\frac{x_1}{y_1}$; $m_N = \frac{-a^2}{b^2}\frac{y_1}{x_1}$



* $y - y_1 = m_N (x - x_1) \Rightarrow y - y_1 = \frac{-a^2 y_1}{b^2 x_1}(x - x_1)$

$b^2 x_1 y - b^2 x_1 y_1 = -a^2 y_1 x + a^2 x_1 y_1 \Rightarrow b^2 x_1 y + a^2 y_1 x = \underbrace{(a^2+b^2)}_{a^2 e^2} x_1 y_1 = a^2 e^2 x_1 y_1$

$\boxed{N \equiv \frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 e^2}$ |-pt form

(x_1, y_1)

$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$

$b^2 x_1^2 - a^2 y_1^2 = a^2 b^2$

$b^2 = a^2 (e^2 - 1)$
$= a^2 e^2 - a^2$

$\Downarrow$

$a^2 + b^2 = a^2 e^2$

is not the c for tangency

* $y = m_N x + c_N$, $m_T = -\dfrac{1}{m_N}$, $c_T = \pm\sqrt{a^2 m_T^2 - b^2}$

P.O.C ?

$$(x_1, y_1) = \left(\frac{-a^2 m_T}{c_T}, \frac{-b^2}{c_T}\right) = \left(\frac{-a^2 m_T}{\sqrt{a^2 m_T^2 - b^2}}, \frac{-b^2}{\sqrt{a^2 m_T^2 - b^2}}\right) = \left(\frac{\pm \frac{a^2}{m_N}}{\sqrt{\frac{a^2}{m_N^2} - b^2}}, \frac{\mp b^2}{\sqrt{\frac{a^2}{m_N^2} - b^2}}\right) = \left(\frac{\pm a^2}{\sqrt{a^2 - b^2 m_N^2}}, \frac{\mp b^2 m_N}{\sqrt{a^2 - b^2 m_N^2}}\right)$$

POC in terms of m_N

EON $N \equiv \dfrac{a^2 x}{x_1} + \dfrac{b^2 y}{y_1} = a^2 e^2 \Rightarrow \underbrace{\dfrac{\cancel{a^2} x}{\pm \cancel{a^2}}\sqrt{a^2 - b^2 m_N^2} + \dfrac{\cancel{b^2} y}{\mp \cancel{b^2} m_N}\sqrt{a^2 - b^2 m_N^2}}_{\left(\pm x \mp \frac{y}{m_N}\right)\sqrt{a^2 - b^2 m_N^2}} = \underbrace{a^2 e^2}_{(a^2 + b^2)}$

* $\mp \dfrac{y}{m_N} \pm x = \dfrac{a^2 + b^2}{\sqrt{a^2 - b^2 m_N^2}} \Rightarrow \mp y = \mp m_N x + \dfrac{m_N (a^2 + b^2)}{\sqrt{a^2 - b^2 m_N^2}}$

$\pm \pm = +$
$\pm \mp = -$
$\pm + = \pm$
$\pm - = \mp$

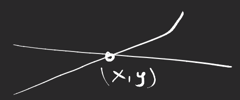
$$\boxed{y = m_N x \mp \underbrace{\frac{m_N (a^2 + b^2)}{\sqrt{a^2 - b^2 m_N^2}}}_{c_N}}$$

EON
slope form
m_N needed

* $x_1 = a\sec\theta$, $y_1 = b\tan\theta$

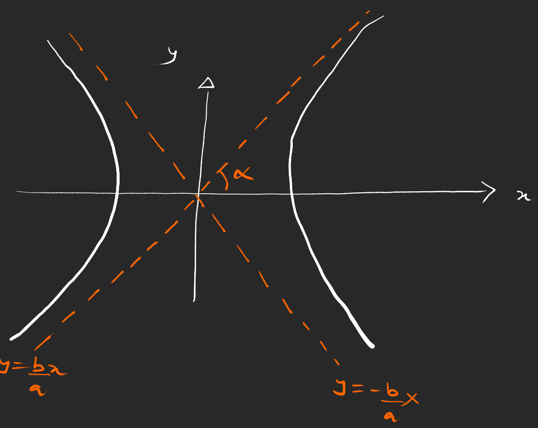
EON : $\dfrac{a^2 x}{x_1} + \dfrac{b^2 y}{y_1} = a^2 e^2 \longrightarrow \boxed{\dfrac{ax}{\sec\theta} + \dfrac{by}{\tan\theta} = a^2 e^2}$

Parametric form
θ: needed



"Beyond real deal"

H. Asymptote Eqⁿ



* $\mathcal{H} \equiv \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, $L \equiv y = mx + c$ $m \equiv m_T$

$$\boxed{x_{1,2} = \frac{2mca^2 \pm 2ab\sqrt{c^2 + b^2 - a^2 m^2}}{2(b^2 - a^2 m^2)}}$$

D > 0 secant
D = 0 tangent
D < 0 no exist

Quad eqⁿ

* $D = 0 \Rightarrow x_{1,2} = \dfrac{mca^2}{b^2 - a^2 m^2}$

* $\boxed{\text{if } b^2 - a^2 m^2 = 0 \Rightarrow x_{1,2} \to \frac{0}{0}, \; y \to \infty}$ due to Real Analysis   Indeterminate forms $\left(\frac{0}{0}\right)$

$\boxed{m = \tan\alpha = \pm\frac{b}{a}}$ slope of a special st. line

Asymptote ≡

if $2mca^2 = 0 \Rightarrow \boxed{c = 0}$ Intercept of sp. st. line $\Rightarrow$ passes thru Origin

* $\boxed{y = \pm\frac{b}{a}x}$ $\dfrac{y}{b} + \dfrac{x}{a} = 0$, $\dfrac{y}{b} - \dfrac{x}{a} = 0$ $\Rightarrow \boxed{\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 0}$

Eqⁿ of Asymptotes

∃ 2 asymptotes to a hyperbola.

not a hyperbola but st. line