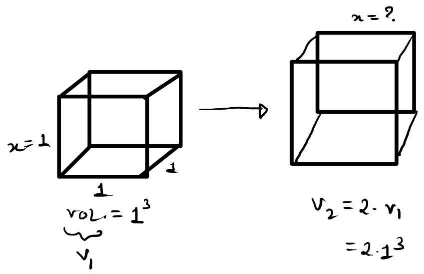
Number theory Vol-2 (Complex Analysis)

Lecture-1 (9/Sept) 2

1. Progression (It's all about cubic Eq^n)

1.a. $\sqrt[3]{\ }$ $\longrightarrow$ $\sqrt{-3}$: a timeline

* Antiquities { Egyptian, Indian, Greek } "Problem of doubling the cube"



* Diophantus (3 AD)
* Archimedes

* Wang Xiaotong (500) Jigu Suanjing math. treatise
  chinese
  created 25 cubic eq^n & solved { Numerical Sol^n }

  $x^3 = N$ type

  $V = x^3 = 2$ $\downarrow$ $x = \sqrt[3]{2} = ?$

Italians

* Fibonacci (1100) $x^3 + 2x^2 + 10x = 20$ Numerical Sol^n

* Scipione del Ferro (1460) Sol^n to "depressed cubic eq^n" Algebraic Sol^n

* Niccolo Tartaglia (1500)
  - $\to$ Projectile motion (Ballistic) $\longrightarrow$ subsummed into Galileo's work
  - $\to$ "secret to solving cubic eq^n"

* Gerolamo Cardano (1545) paper
  - $\longrightarrow$ Algebraic Sol^n to cubic polynomial (Ars magna)
  - $\longrightarrow$ saw emergence of wierd creatures (roots) $\sim \sqrt{-3}$ (didn't understand it)
  - $\longrightarrow$ wierd creatures are "as subtle as they are $\underbrace{\text{useless}}_{\text{Ignore}}$" / mental torture
  - $\longrightarrow$ 1st to use $-ve$ Number

  useless Roots

* Lodovico ferrari (1520)
  - servant to cardano
  - Sol^n to Quartic Eq^n

* Rafael Bombelli (1526) i
  - $\longrightarrow$ discoverer of Complex Numbers (C.N)
  - $\longrightarrow$ Arithmetic of CN (+ - x)

    i = plus of minus $\sqrt{-3} = \overset{i}{\sqrt{-1}}\sqrt{3} = i\sqrt{3}$

    $-i$ = minus of minus

French

* François viete (1540) $\longrightarrow$ used trig. to solve cubic Eq^n

* René Descartes (1596) $\longrightarrow$ what's up ?! (extended viete's work) i = imaginary number / useless at that time

* Euler (1700) $\longrightarrow$ $e^{i\theta}$ people started taking CN seriously
  created a boom of $\sqrt{-1}$

* Gauss (1780)

* Abel - Ruffini theorem (1800) $\longrightarrow$ $\boxed{\nexists \text{ analytical sol}^n \text{ to eq}^n \text{ degree} \geq 5}$

* William Hamilton (1800) $\longrightarrow$ " a lot is going on b/w Euler - Hamilton "

  Extension to C.N $\longrightarrow$ Quaternions. (After C.N)

1.b. Ferro-Tartaglia-Cardano's (FTC's) Solution to Depressed cubic eq^n

(Ref: Algebra vol.1)

* $f(t) = t^3 + pt + q = 0$ 'Depressed Cubic Eq^n' (cubic in t) ; Roots of a cubic polynomial = ?

The Big Question

* Introduce u, v : $u + v = t$ (def 1)

* $(u+v)^3 + p(u+v) + q = 0 \Rightarrow u^3 + v^3 + \underbrace{3uv(u+v) + p(u+v)}_{(3uv+p)(u+v)} + q = 0 \Rightarrow u^3 + v^3 + (3uv+p)(u+v) + q = 0$ — ①

cubic in u, v
(Bigger trouble)

* $(3uv + p) = 0$ (Cardano's Condition) $\Rightarrow u^3 + v^3 + q = 0$ }
def.2 Master stroke

$u^3 + v^3 = -q$ = sum of roots
$\underbrace{uv}_{u^3v^3 = -p^3/27} = -\frac{p}{3}$ = Product of Roots.
{ old quadratic Roots trick

* $x_1 = u^3$, $x_2 = v^3$ (def) $\Rightarrow$ $\underbrace{(x-u^3)(x-v^3)}_{\shortparallel} = 0$
Roots

$x^2 - \underbrace{(u^3+v^3)}_{-q}x + \underbrace{(uv)^3}_{-\frac{p}{3}} = 0 \Rightarrow \boxed{x^2 + qx - \frac{p^3}{27} = 0}$ — ②

Quad. in x

Sol^n / Roots are $(x_1, x_2) = (u^3, v^3)$

Quad. Form.

* $x_{1,2} = \dfrac{-q \pm \sqrt{q^2 + 4\frac{p^3}{27}}}{2} = -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$

$x_1 = u^3 = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \Rightarrow u = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$

$x_2 = v^3 = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \Rightarrow v = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$

* $$\boxed{t = u + v = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{p^2}{4} + \frac{p^3}{27}}}}$$

a Root of the depressed cubic $t^3 + pt + q = 0$

Cardano's formula.

Meaning / Nature of Root. ?

Lecture 2 (10/Sept) 2.15

* $t^3 + pt + q = 0$; $u + v \equiv t$; Roots of cubic Eq^n

$u^3 + v^3 + (3uv+p)(v+u) + q = 0$

* $3uv + p = 0$ 'Cardano's Cond^n' $\Rightarrow uv = -p/3 \Rightarrow u^3v^3 = -\frac{p^3}{27} = \prod$ roots

$\underbrace{u^3}_{\alpha} + \underbrace{v^3}_{\beta} = -p = \sum$ roots

* $(x-\alpha)(x-\beta) = 0 \xrightarrow[\text{C-C}]{} x^2 + qx - \frac{p^3}{27} = 0$

Naive guess

$\alpha \equiv u^3 = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \Rightarrow u = \sqrt[3]{u^3} \longrightarrow$ 3 roots $\{u_1\ u_2\ u_3\}$

$\beta \equiv v^3 = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \Rightarrow v = \sqrt[3]{v^3} \longrightarrow$ 3 roots $\{v_1\ v_2\ v_3\}$

$x^2 = 1$

$x = +1$

-1

* $t = u + v = \{t_1\ t_2\ t_3\ t_4\ t_5\ t_6\ t_7\ t_8\ t_9\}$ 9 indep. Roots (not possible)

* C.C $\Rightarrow$ $uv = \frac{-p}{3} \Rightarrow v = \frac{-p}{3u}$ (dependent) $\longrightarrow$ Reduces To 3 indep. SoV / Roots. (How ?)

* $t = u + v = \sqrt[3]{\quad + \sqrt[2]{\quad}} + \sqrt[3]{\quad - \sqrt[2]{\quad}}$ 1 real Root via Cardano's Initial Analysis.

$\nexists$ Img # yet!

1.c. Cube Root of 1 (unity)

* $x^2 = 1 \Rightarrow x = \pm 1$

$x^2 = N \Rightarrow x = \pm\sqrt{N}$ $\underbrace{N > 0}_{\text{Serious stance}}$, $N \neq 0$ (Later : Grassmann numbers)

* $x^3 = 1 \Rightarrow x_1 = +1 \in \mathbb{R}$

$x = x_2, x_3 = ? \in$ (Something other Than $\mathbb{R}$) non real

Plane Algebra

* $x^3 - 1^3 = 0 \Rightarrow (x-1)(x^2+x+1) = 0$ $\to x - 1 = 0 \Rightarrow x_1 = 1 \in \mathbb{R}$

$\hookrightarrow x^2 + x + 1 = 0 \Rightarrow x_{1,2} = \frac{-1 \pm \sqrt{1-4}}{2}$

$x_{1,2} = \frac{-1 \pm \sqrt{-3}}{2} \notin \mathbb{R}$

$(a \pm b)^3 = a^3 \pm b^3 \pm 3ab(a \pm b)$

$a^3 \mp b^3 = (a \mp b)(a^2 + b^2 \pm ab)$ □

Note: (neglect) ignore what is happening inside $\sqrt[2]{-}$
not put it = 0

$x^3 = 1 \Rightarrow \{\overset{1}{\omega^0}, \omega^1, \omega^2\}$; $\boxed{\omega \equiv \frac{-1 + \sqrt{-3}}{2}}$

* $\omega^2 = \left(\frac{-1+\sqrt{-3}}{2}\right)^2 = \frac{1 + (\sqrt[2]{-3})^2 - \sqrt{-3}}{4} = \frac{1 - 3 - \sqrt{-3}}{4} = \frac{-1-\sqrt{-3}}{2} \equiv \omega^* = \bar{\omega}$ Notation Alert! $(\sqrt[2]{x})^2 = x$

* $\omega^* \notin \mathbb{R}$; $\{\omega, \omega^2\}$ Conjugate pairs. (ω, ω^*) ' not really indep of each other '

* $\frac{1}{\omega} = \frac{1}{\frac{-1+\sqrt{-3}}{2}} = \frac{2}{-1+\sqrt{-3}} \times \frac{-1-\sqrt{-3}}{-1-\sqrt{-3}} = \frac{2(-1-\sqrt{-3})}{(-1)^2 - (\sqrt{-3})^2} = \frac{2(-1-\sqrt{-3})}{1-(-3)} = \frac{-1-\sqrt{-3}}{2} = \omega^2 = \omega^*$

* $\boxed{\omega^2 = \omega^* , \quad \frac{1}{\omega} = \omega^*}$ $\omega^3 = 1$

1.d. cube root of N

* $x^3 = N \longrightarrow \{\psi_1, \psi_2, \psi_3\}$ 3 roots / SoV

$x^3 - (\sqrt[3]{N})^3 = 0 \Rightarrow (x - \sqrt[3]{N})(x^2 + (\sqrt[3]{N})^2 + x\sqrt[3]{N}) = 0$ $\to x_1 = \psi_1 = \sqrt[3]{N} \in \mathbb{R}$

$\hookrightarrow x = \frac{-\sqrt[3]{N} \pm \sqrt{(\sqrt[3]{N})^2 - 4(\sqrt[3]{N})^2}}{2}$

$= \frac{-\sqrt[3]{N} \pm \sqrt[2]{-3(\sqrt[3]{N})^2}}{2}$

$= \frac{-\sqrt[3]{N} \pm \sqrt[3]{N}\sqrt{-3}}{2} = \left(\frac{-1 \pm \sqrt{-3}}{2}\right)\sqrt[3]{N}$

$\sqrt[2]{yx^2} = x\sqrt{y}$

ex:

$x^3 = 27 \Rightarrow x = \underbrace{3}_{\mathbb{R}}, \underbrace{\underbrace{3\omega}_{x_2}, 3\omega^*}_{\text{non } \mathbb{R}}$

$x_2^3 = (3\omega)^3 = 27\omega^3 = 27$

$\psi_2 = \omega\sqrt[3]{N} = \omega\psi_1$

$\psi_3 = \omega^*\sqrt[3]{N} = \omega^*\psi_1$

* ψ_1 Real , $\{\psi_2, \psi_2^*\}$ Conjugate pairs (non real) : $\psi_2^* = (\omega\psi_1)^* = \omega^*\underbrace{\psi_1^*}_{\psi_1 \ (\because \text{ it is Real})} = \omega^*\psi_1 = \psi_3 \Rightarrow \psi_2^* = \psi_3$

$\boxed{\begin{matrix} C_1^* = C_2 \\ R_1^* = R \end{matrix}}$

1.e. Viète Substitution (1550s) / 1 $\mathbb{R}$ & 2 non $\mathbb{R}$ roots

* $t^3+pt+q=0$; $t \equiv x - \frac{p}{3x}$

$$\left(x-\frac{p}{3x}\right)^3 + p\left(x-\frac{p}{3x}\right)+q=0 \Rightarrow x^3 - \frac{p^3}{27x^3} - p\,\cancel{\left(x-\frac{p}{3x}\right)} + p\left(\cancel{x}-\frac{p}{3x}\right)+q=0 \Rightarrow (x^3)^2 - \frac{p^3}{27}+q x^3=0$$

* $(x^3)^2+q x^3-\frac{p^3}{27}=0 \Rightarrow x^3 = \dfrac{-q\pm\sqrt{q^2+\frac{4p^3}{27}}}{2} = \underbrace{-\frac{q}{2}\pm\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}_{\equiv A} \longrightarrow x=\begin{Bmatrix}x_1\\x_2\\x_3\end{Bmatrix}$

* t
  - $\to t_1 = x_1-\frac{p}{3x_1} = ?$
  - $\to t_2 = x_2-\frac{p}{3x_2} = ?$
  - $\to t_3 = x_3-\frac{p}{3x_3} = ?$

Solve:

* $\boxed{x^3=A}$; $A \equiv -\frac{q}{2}\pm\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}$
  - $+ \Rightarrow -\frac{q}{2}+\sqrt{\quad} \equiv u^3$
  - $- \Rightarrow -\frac{q}{2}-\sqrt{\quad} \equiv v^3$

* $x^3=A \Rightarrow x_1=\sqrt[3]{A}$ (2 soln: $+$, $-$), $x_2=\left(\frac{-1+\sqrt{-3}}{2}\right)\sqrt[3]{A}$ ($+$, $-$), $x_3=\left(\frac{-1-\sqrt{-3}}{2}\right)\sqrt[3]{A}$ ($+$, $-$)

Lecture-3 (14/sept) 1.5

* $t^3+pt+q=0 \xrightarrow[t=x-\frac{p}{3x}]{\text{Viète subs.}} (x^3)^2+q(x^3)-\frac{p^3}{27}=0 \Rightarrow x^3 = \underbrace{-\frac{q}{2}\pm\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}_{\equiv A} \Rightarrow x^3=A$ $\qquad t=u+v$, u^3+v^3

$u^3 \equiv -\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}$, $v^3 \equiv -\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}$, $u^3+v^3=-q$; $u^3v^3=-\frac{p^3}{27} \Rightarrow \boxed{uv=-\frac{p}{3}} \Rightarrow 3uv+p=0$ Cardano's Cond^n

* $x^3=A \Rightarrow$ Roots are $x_1=\sqrt[3]{A}$ ($+$, $-$: 2 soln), $x_2=\underbrace{\left(\frac{-1+\sqrt{-3}}{2}\right)}_{\omega\text{ (cube root of 1)}}\sqrt[3]{A}$, $x_3=\underbrace{\left(\frac{-1-\sqrt{-3}}{2}\right)}_{\omega^*}\sqrt[3]{A}$

1.e.1 Reduction of 6 cases $\longrightarrow$ 3 Roots

Case I : $x_1=\sqrt[3]{A}$

a. $A=-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}} \equiv u^3$, $x_1=\sqrt[3]{u^3}=u$; $t_1 \equiv x_1-\frac{p}{3x_1} = u-\underbrace{\frac{p\,v}{3u\,v}}_{\frac{\cancel{p}v}{\cancel{3}(-\frac{\cancel{p}}{\cancel{3}})}} = u+v \in \mathbb{R}$ $\left(\because uv=-\frac{p}{3}\right)$

b. $A=-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}} \equiv v^3$, $x_1=\sqrt[3]{v^3}=v$; $t_2 = x_1-\underbrace{\frac{p}{3x_1}}_{\frac{p\,u}{3v\,u}=\frac{pu}{3(-\frac{p}{3})}} = v+u = t_1 \in \mathbb{R}$

Case II : $x_2=\left(\frac{-1+\sqrt{-3}}{2}\right)\sqrt[3]{A} \equiv \omega\sqrt[3]{A}$

a. $A=-\frac{q}{2}+\sqrt{\quad} \equiv u^3$; $x_2=\omega\sqrt[3]{u^3}=\omega u$; $t_3=x_2-\frac{p}{3x_2}=\omega u-\frac{p}{3\omega u}=\omega u+\omega^* v \notin \mathbb{R}$ $\qquad \frac{1}{\omega}=\omega^*$

b. $A = -\frac{q}{2} - \sqrt{} \equiv v^3$; $x_2 = \omega_2 \sqrt[3]{v^3} = \omega v$; $t_4 = x_2 - \underbrace{\frac{p}{3x_2}}_{\frac{p}{3\omega v}\,u \,=\, \frac{\cancel{p}\,\omega^* u}{3(-p/3)}} = \omega v + \omega^* u \notin \mathbb{R}$

$\frac{p}{3\omega u} v = \frac{pv}{3\omega(-\frac{p}{3})} = -\frac{v}{\omega} = -v\omega^*$

Case II : $x_3 = \left(\frac{-1-\sqrt{-3}}{2}\right)\sqrt[3]{A} = \omega^* \sqrt[3]{A}$

a. $A = -\frac{q}{2} + \sqrt{} \equiv u^3$; $x_3 = \omega^* \sqrt[3]{u^3} = \omega^* u$, $t_5 = x_3 - \underbrace{\frac{p}{3x_3}}_{\frac{1}{\omega^*} v} = \omega^* u + \omega v = t_4 \notin \mathbb{R}$

$\frac{1}{\omega^*} = \omega$

b. $A = -\frac{q}{2} - \sqrt{} \equiv v^3$; $x_3 = \omega^* v$; $t_6 = x_3 - \frac{p}{3x_3} = \omega^* v + \omega u = t_3 \notin \mathbb{R}$

$*$ $$\boxed{t^3 + pt + q = 0 \longrightarrow t = \begin{cases} u+v \\ \omega u + \omega^* v \\ \omega^* u + \omega v \end{cases}}$$

$u^3 = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$ $\qquad \omega = \frac{-1+\sqrt{-3}}{2}$

$v^3 = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$ $\qquad \omega^* = \frac{-1-\sqrt{-3}}{2}$

3 Roots of depressed cubic eqn

1.c.2 Nature of Roots

$*$ $\Delta \equiv \frac{q^2}{4} + \frac{p^3}{27}$

$*$ $\Delta > 0 \Rightarrow u^3, v^3$ both Real $\Rightarrow t = \{u+v, \underbrace{\omega u + \omega^* v, \omega^* v + \omega u}_{\notin \mathbb{R}}\}$

Case I

Case II

$*$ $\Delta = 0 \Rightarrow u^3 = v^3 \Rightarrow u = v \Rightarrow t = \{\underset{\text{simple root}}{2u}, \underset{\text{double root}}{-u, -u}\}$ 'double root' (degeneracy) $p \neq 0$

$\omega + \omega^* = \frac{-1+\sqrt{-3}}{2} + \frac{-1-\sqrt{-3}}{2} = -1$

if $p = q = 0 \Rightarrow t^3 = 0$ "triple root of cubic" (trivial but not really!!)

$*$ $\Delta < 0 \Rightarrow u^3, v^3$ both "$\sqrt{-1}$" { Casus Irreducibilis of Cardano / Avoided in Ars Magna

"Irreducible case"

$*$ $u^3 = -\frac{q}{2} + \sqrt{\Delta} \notin \mathbb{R}$

$v^3 \equiv -\frac{q}{2} - \sqrt{\Delta} \notin \mathbb{R}$

Schematically : (Bombelli's Idea)

clever trick $\qquad \sqrt{-n} \equiv in$ → Eulerian notation (later)

$*$ $u^3 \equiv a + \sqrt{-b} \Rightarrow u = \sqrt[3]{a + \sqrt{-b}} \equiv m + in$ $\qquad$ Trick ! { Note : cube root of C.N is a C.N

$v^3 \equiv a - \sqrt{-b} \Rightarrow v = \sqrt[3]{a - \sqrt{-b}} \equiv m - in$

$*$ $t_1 = u + v = 2m$ ✓

$t_2 = \omega u + \omega^* v = \underline{\omega(m+in) + \omega^*(m-in)} = ?$

HW

$t_3 = \omega^* u + \omega v = \underline{\omega^*(m+in) + \omega(m-in)} = ?$

Lecture-4 (15/Sept) 1.30

$\ast$ $t_1 = u+v = 2m$

$t_2 = \omega u + \omega^* v = \omega(m+in) + \omega^*(m-in) = \omega m + i\omega n + \omega^* m - i\omega^* n = -m + in(\omega - \omega^*) = -m + in\sqrt{-3}$
$= -m - n\sqrt{3}$

$t_3 = \omega^* u + \omega v = \omega^*(m+in) + \omega(m-in) = \underbrace{(\omega^* + \omega)}_{-1} m + in\underbrace{(\omega^* - \omega)}_{-i\sqrt{3}} = -m + n\sqrt{3}$

$\{t_1\ t_2\ t_3\} \in \mathbb{R}$

$\omega = \frac{-1+\sqrt{-3}}{2}$

$\omega^* = \frac{-1-\sqrt{-3}}{2}$

$\omega + \omega^* = -1$

$\omega - \omega^* = \sqrt{-3} = i\sqrt{3}$

1.e.3. Irreducible Case of Cardano
→ Bombelli's method (a) (L'Algebra in 1572),
→ Abraham de Moivre (1670) (b.) - Later De Moivre's theorem
(Calc cube root of CN $\sqrt[3]{a+ib}$)

Bombelli's method

$\ast$ $t^3 = 15t + 4 \Rightarrow t^3 - 15t - 4 = 0$ $p = -15$, $q = -4$

$t = u+v = \sqrt[3]{-\frac{q}{2} + \sqrt{\underbrace{\frac{q^2}{4} + \frac{p^3}{27}}_{4 - \frac{15^3}{27} = -121}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} = \underbrace{\sqrt[3]{2 + \sqrt{-121}}}_{\sqrt[3]{2+11i}} + \sqrt[3]{2 - \sqrt{-121}}$

$\Delta \equiv \frac{q^2}{4} + \frac{p^3}{27} = -121 < 0$

I.C

$\sqrt[3]{2+11i} \sim$ "Complex" $\# \sim m + in$
Painful #

$\ast$ $\sqrt[3]{2+11i} \equiv m + in \Rightarrow 2 + 11i = (m+in)^3 = m^3 - in^3 + 3imn(m+in) = (m^3 - 3mn^2) + i(-n^3 + 3m^2n) \Rightarrow \begin{cases} m(m^2 - 3n^2) = 2 \\ n(3m^2 - n^2) = 11 \end{cases}$

$\sqrt[3]{2-11i} \equiv m - in \Rightarrow 2 - 11i = (m-in)^3 \Rightarrow ??$ $\Rightarrow$ $m(m^2 - 3n^2) = 2$
$n(3m^2 - n^2) = 11$

$\Downarrow$ GROSS !!
$m, n = ?$

$\ast$ old skool : Hit & trial $t^3 = 15t + 4$ $\exists$ 1 real Sol^n which you can find via Algebra/geometry.

(a cubic polynomial always cuts at atleast 1 point on the x axis)

Quadr. $y(x) = 0$ → Root

$t = 0$: $0 \neq 4$
$t = 1$: not
:
$t = 4$: $4^3 = 15 \cdot 4 + 4 = 64$ ✓ $\boxed{t = 4 \text{ Sol}^n}$

cubic $= 0$

$\ast$ $t = 4 = u + v = \sqrt[3]{2+11i} + \sqrt[3]{2-11i} = m + \cancel{in} + m - \cancel{in} = 2m \Rightarrow \boxed{m = 2}$, $m(m^2 - 3n^2) = 2 \longrightarrow n = \pm 1 \to n = 1$

Cardano inspired!

this cancellation of $\sqrt{-1}$
is very crucial to our process

$\ast$ $\sqrt[3]{2 \pm 11i} = 2 \pm i$ → 4, → v

$t_1 = 2 + \cancel{i} + 2 - \cancel{i} = 4 \in \mathbb{R}$

$t_2 = \omega u + \omega^* v = \omega(2+i) + \omega^*(2-i) = -2 - \sqrt{3} \in \mathbb{R}$

$t_3 = \omega^* u + \omega v = \omega^*(2+i) + \omega(2-i) = -2 + \sqrt{3} \in \mathbb{R}$

Real numbers

$\omega = \frac{-1 + \sqrt{-3}}{2} = \frac{-1 + i\sqrt{3}}{2}$

$\omega^* = \frac{-1 - i\sqrt{3}}{2}$

Note : $\Delta \equiv \frac{q^2}{4} + \frac{p^3}{27}$
→ $> 0 \Rightarrow \{t_i\} = \{u+v, \omega u + \omega^* v, \omega v + \omega^* u\}$ 1 real / 2 "Imaginary" (Painful)
→ $= 0 \Rightarrow \{t_i\} = \{2u, -u, -u\}$
→ $< 0 \Rightarrow \{t_i\} \in \mathbb{R}$ 3 distinct / real

Lecture-5 (16/sept) 2

1.f Trigonometric solution to depressed cubic (by Viète)

* $t^3+pt+q=0$

choice of u: Linear terms in $\cos\theta$ cancel

* $\boxed{t \equiv u\cos\theta}$; $u \equiv 2\sqrt{\frac{-p}{3}}$; $u^2 = 4\left(\frac{-p}{3}\right)$; $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$u^2 = 4\left(\frac{-p}{3}\right)$

$u^3 = 8\left(\frac{-p}{3}\right)\sqrt{\frac{-p}{3}}$

* $u^3\cos^3\theta + pu\cos\theta + q = 0 \Rightarrow \cos^3\theta + \underbrace{\frac{p}{u^2}}_{-\frac{3}{4}}\cos\theta + \frac{q}{u^3} = 0 \Rightarrow \underbrace{4\cos^3\theta - 3\cancel{\cos\theta}}_{\cos 3\theta + 3\cancel{\cos\theta}} + \underbrace{\frac{4q}{u^3}}_{\frac{4q}{8_2}\left(\frac{3}{-p}\right)^{1/2}\left(\frac{3}{-p}\right)} = 0$

$\boxed{\cos 3\theta = \frac{3q}{2p}\sqrt{\frac{-3}{p}}} \Rightarrow \theta = \frac{1}{3}\cos^{-1}\left(\frac{3q}{2p}\sqrt{\frac{-3}{p}}\right)$

* $t = 2\sqrt{\frac{-p}{3}}\cos\left\{\overbrace{\frac{1}{3}\cos^{-1}\left(\frac{3q}{2p}\sqrt{\frac{-3}{p}}\right)}^{\text{Angle}}\right\} \in \mathbb{R}$

* $\boxed{t_k = 2\sqrt{\frac{-p}{3}}\cos\left\{\frac{1}{3}\cos^{-1}\left(\frac{3q}{2p}\sqrt{\frac{-3}{p}}\right) - \frac{2\pi k}{3}\right\}}$

Later (I.T.F) not very difficult (Can also do using de Moivre's thm)

$k = 0,1,2$ T sol^n for 3 real Roots

$\Delta \equiv \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 < 0$

1.f.1. Hyperbolic Trig. sol^n (2 cases) $p \neq 0$

$p>0$

* $\boxed{t = u\sinh\theta}$; $u \equiv 2\sqrt{\frac{p}{3}}$; $u^2 = 4\frac{p}{3}$, $u^3 = 8\sqrt{\frac{p}{3}}\left(\frac{p}{3}\right)$

* $t^3+pt+q=0 \Rightarrow u^3\sinh^3\theta + pu\sinh\theta + q = 0 \Rightarrow \sinh^3\theta + \underbrace{\frac{p}{u^2}}_{\frac{3}{4}}\sinh\theta + \underbrace{\frac{q}{u^3}}_{\frac{q}{8}\sqrt{\frac{3}{p}}\left(\frac{3}{p}\right)} = 0$

$\underbrace{4\sinh^3\theta + 3\sinh\theta} + \frac{3q}{2p}\sqrt{\frac{3}{p}} = 0$

* $4\sinh^3\theta + 3\sinh\theta = 4\left(\frac{e^\theta - e^{-\theta}}{2}\right)^3 + 3\left(\frac{e^\theta - e^{-\theta}}{2}\right) = \frac{e^{3\theta} - e^{-3\theta}}{2} = \sinh 3\theta$

* $\sinh 3\theta = -\frac{3q}{2p}\sqrt{\frac{3}{p}} \Rightarrow \theta = \frac{1}{3}\sinh^{-1}\left(-\frac{3q}{2p}\sqrt{\frac{3}{p}}\right)$

$t = 2\sqrt{\frac{p}{3}}\sinh\left\{\frac{1}{3}\sinh^{-1}\left(-\frac{3q}{2p}\sqrt{\frac{3}{p}}\right)\right\}$

$\boxed{t = -2\sqrt{\frac{p}{3}}\sinh\left\{\frac{1}{3}\sinh^{-1}\left(\frac{3q}{2p}\sqrt{\frac{3}{p}}\right)\right\}}$ $p>0$

1 real Root $\Delta \equiv \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 > 0$

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

$\theta \to i\theta$

$\cosh\theta \equiv \cos i\theta = \frac{e^{\theta} + e^{-\theta}}{2}$

$\sin i\theta = \frac{e^{-\theta} - e^{\theta}}{2i} \cdot \frac{i}{i}$

$\sin i\theta = i\underbrace{\left(\frac{e^\theta - e^{-\theta}}{2}\right)}_{\sinh\theta}$

$\sin i\theta = i\sinh\theta$

$\cosh\theta \equiv \frac{e^\theta + e^{-\theta}}{2}$

$\sinh\theta \equiv \frac{e^\theta - e^{-\theta}}{2}$

$\cosh(-\theta) = \cosh\theta$ Even

$\sinh(-\theta) = -\sinh\theta$ odd

Case II

* $\boxed{t \equiv u\cosh\theta}$, $u \equiv 2\sqrt{\frac{-p}{3}}$ $p<0$ $u^2 = 4\left(\frac{-p}{3}\right)$ $u^3 = 8\left(\frac{-p}{3}\right)\sqrt{\frac{-p}{3}}$

* $t^3+pt+q=0 \Rightarrow u^3\cosh^3\theta + pu\cosh\theta + q = 0 \Rightarrow \cosh^3\theta + \underbrace{\frac{p}{u^2}}_{-\frac{3}{4}}\cosh\theta + \underbrace{\frac{q}{u^3}}_{\frac{q}{8}\left(\frac{-3}{p}\right)\sqrt{\frac{-3}{p}}} = 0$

* $4\cosh^3\theta - 3\cosh\theta - \frac{3q}{2p}\sqrt{\frac{-3}{p}} = 0$

* $4\cosh^3\theta - 3\cosh\theta = 4\left(\frac{e^{\theta}+e^{-\theta}}{2}\right)^3 - 3\left(\frac{e^{\theta}+e^{-\theta}}{2}\right) = \frac{1}{2}\left(e^{3\theta}+e^{-3\theta}+3\cancel{(e^{\theta}+e^{-\theta})}\right) - \frac{3}{\cancel{2}}\cancel{(e^{\theta}+e^{-\theta})} = \cosh 3\theta$

* $\cosh 3\theta = \frac{3q}{2p}\sqrt{\frac{-3}{p}} \Rightarrow \theta = \frac{1}{3}\cosh^{-1}\left\{\frac{3q}{2p}\sqrt{\frac{-3}{p}}\right\}$

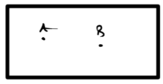
$$\boxed{t = 2\sqrt{\frac{-p}{3}}\cosh\left\{\frac{1}{3}\cosh^{-1}\left(\frac{3q}{2p}\sqrt{\frac{-3}{p}}\right)\right\}} \qquad p<0$$

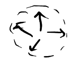
1 real Root $\qquad \Delta \equiv \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 > 0$

Lecture – 6 (17/sept)

1.f.2. Application of depressed cubic in Cosmology (Trig. sol^n of Viete) $\qquad$ $1\,\text{Mpc} \sim 3\times10^{24}\,\text{cm}$

* Cosmological principle $\Rightarrow$ Homogeneous / Isotropic at Large scale
  - Homogeneous: Every pt identical
  - Isotropic: Every dir^n is identical





* $\underbrace{R_{ab} - \frac{1}{2}g_{ab}R}_{\text{"Geometry"}} = \underbrace{\kappa T_{ab}}_{\text{"Matter"}}$ $\qquad a,b = 0,1,2,3 \qquad \#\ \text{Eq}^n = 16$

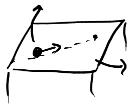
$4 \times 4 = 16$

00	11	22	33
01	02	03	
12	13		
23			

* g_{ab} = FRW metric ; $x^a = \{t\ r\ \theta\ \varphi\}$

$a(t) \in$ FRW metric 'Scale factor / expansion factor' $\}$ Geometry

Matter $\quad \frac{d^2y}{ds^2} = 0$ (accel^n)



flat geometry

Not flat $\quad \frac{d^2r}{ds^2} \neq 0$ "accel^n"

* Geodesic Eq^n $\longrightarrow$ Γ christoffel symbol $\longrightarrow$ R_{ab}, $R \Rightarrow$ LHS known

* T_{ab} = "Ideal fluid" $\{\rho, p\}$ (density / pressure) $\Rightarrow$ RHS known (Cosmological fluid)
  $= f(\text{velocity})$

* $R_{ab} - \frac{1}{2}g_{ab}R = \kappa T_{ab} \xrightarrow{\text{"Black Magic"}}$ $\boxed{\begin{aligned}\dot{a} &= a(\rho, p)\\ \ddot{a} &= a(\rho, p)\end{aligned}}$ Cosmological fluid Eq^n

$x' = \dot{x} = \frac{dx}{dt}$

* $\ddot{a} = -\frac{4\pi G}{3}(\rho + 3p)a$; $\dot{a}^2 = \frac{8\pi G\rho}{3}a^2 - \text{const}$ (Actual course on Cosmology)

* $H(t) \equiv \frac{\dot{a}(t)}{a(t)}$ Hubble's parameter (not a const)

$\Omega \equiv \frac{\rho(t)}{\rho_0}$ density parameter, $\rho_0 \to$ constant

* $\underset{\text{total substance}}{\rho_T} = \underset{\text{matter}}{\rho_m} + \underset{\text{radiation}}{\rho_r} + \underset{\text{C.C}}{\rho_\Lambda} \longrightarrow H^2 = (\ )^3 + (\ )^4 + (\ )^2 + (\ )^0$

* Lemaitre Model : choice of specific values of parameters to fit data (observations)

$\underset{\text{Expansion}}{\Omega_\Lambda} - \underset{\text{Matter}}{\Omega_m} \begin{cases} = 0 & \text{No accel}^n/\text{decel}^n \\ < 0 & \text{deceleration} \\ > 0 & \text{accel}^n \text{ univ.}\end{cases}$

* $\boxed{f(x) = \Omega_\Lambda x^3 + (1 - \Omega_m - \Omega_\Lambda)x + \Omega_m = 0}$ turning pt Eq^n $\qquad x = \frac{a(t)}{a(t_0)}$ (Then / Now)

cubic eq^n in x

$\downarrow$

* $y^3 - \frac{3}{4}y + \frac{\Omega_m - 1}{4\Omega_m} = 0$; $y \equiv \left(\frac{\Omega_\Lambda}{4\Omega_m}\right)^{1/3}$ $\qquad \Omega_\Lambda = \frac{\rho_\Lambda}{\rho_0}$

Cubic Eq^n Problems $\downarrow$

* $y^3 + py + q = 0$; $p = \frac{-3}{4}$, $q = \frac{\Omega_m - 1}{4\Omega_m}$; $\underset{\text{descriminant}}{\Delta \equiv \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} \longrightarrow$ undeterminable if info about Ω_m is not provided

$\boxed{p<0}$

Solution of depressed cubic / Nature of Roots

* $0 < \Omega_m \leq \frac{1}{2} \Rightarrow q_{max} = \frac{\frac{1}{2}-1}{2\cdot4(\frac{1}{2})} = -\frac{1}{4} < 0 \Rightarrow \Delta_{max} = \left(\frac{\frac{1}{4}}{2}\right)^2 + \left(-\frac{3}{4}\cdot\frac{1}{3}\right)^3 = \left(\frac{1}{8}\right)^2 - \frac{1}{4^3} = 0$

$\boxed{\Delta \geq 0}$, $p<0$

$q = \frac{\frac{1}{4}-1}{4(\frac{1}{4})} = -\frac{3}{4} < 0$

$\Delta = \left(\frac{-\frac{3}{4}}{2}\right)^2 + \left(-\frac{1}{4}\right)^3$

$\left(\frac{9}{8}\right)^2 - \left(\frac{1}{4}\right)^3 = \frac{5}{4} > 0$

$\boxed{\dot{a}_0 > 0 \quad \text{Expansion}}$

Now

$$y = 2\sqrt{\frac{-p}{3}}\cosh\left\{\frac{1}{3}\cosh^{-1}\left(\frac{3q}{2p}\sqrt{\frac{-3}{p}}\right)\right\}$$

$$\left(\frac{\Omega_\Lambda}{4\Omega_m}\right)^{1/3} = 2\sqrt{\frac{1}{4}}\cosh\left\{\frac{1}{3}\cosh^{-1}\left(\frac{3(\Omega_m-1)}{4\Omega_m\cdot 2\cdot(-\frac{3}{4})}\sqrt{4}\right)\right\}$$

$$\boxed{\Omega_\Lambda = 4\Omega_m \cosh^3\left\{\frac{1}{3}\cosh^{-1}\left(\frac{1-\Omega_m}{\Omega_m}\right)\right\}}$$

$p<0$

* $\frac{1}{2} < \Omega_m < 1 \Rightarrow q_{max} \rightarrow \Delta_{max} \overset{?}{\lessgtr} 0 \qquad \Delta \overset{?}{\gtrless} 0 \qquad p<0 \qquad y = ?$ HW

Lecture-7 (22/Sept)

* g_{ab} = FRW metric ; $a(t) \in g_{ab}$ scale factor 'Geometry'

* T_{ab} = "Cosmological fluid" = Ideal fluid : $\{\rho, p\}$ ⟶ Components of Cosmo.
  all filling — density, pressure

- → Matter (m)
  - Baryonic (standard Model) ✓
  - Non Baryonic (Beyond SM)
- → Radiation (r)
  - Photon
  - Neutrinos
- → Vacuum (Λ) = Cosmological Const = C.C

* Einstein's field Eq^n ⟶ $\dot{a} = a(p,\rho)$, $\ddot{a} = a(\rho, p)$ Coupled/non linear D.E

* $\rho_T = \rho_m + \rho_r + \rho_\Lambda$ total density

$H(t) \equiv \frac{\dot{a}(t)}{a(t)} \longrightarrow H^2 = H_0^2\left(\Omega_m(\ )^3 + (\ )^4 + (\ )^2 + (\ )^0\right)$

$$\boxed{y^3 \underbrace{-\frac{3}{4}}_{p}y + \underbrace{\frac{\Omega_m-1}{4\Omega_m}}_{q} = 0}$$

; $y \equiv \left(\frac{\Omega_\Lambda}{4\Omega_m}\right)^{1/3}$; $\Delta \equiv \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3$, $q \equiv \frac{\Omega_m - 1}{4\Omega_m}$

$P \equiv -3/4 < 0$

* $y^3 + py + q = 0$

sol^n to depressed cubic eq^n

* Case I : $0 < \Omega_m \leq \frac{1}{2} \Rightarrow q_{max} = -\frac{1}{4} < 0 \qquad \Delta_{max} = 0$; Say $\Omega_m = 1/4 \Rightarrow q = -\frac{3}{4} < 0$

$\Delta > 0$, $p < 0$

$\Delta = \left(\frac{-3/4}{2}\right)^2 + \left(\frac{-1}{4}\right)^3 = \left(\frac{3}{8}\right)^2 - \left(\frac{1}{4}\right)^3 = \frac{1}{8} > 0$

* $$y = 2\sqrt{\frac{-p}{3}}\cosh\left\{\frac{1}{3}\cosh^{-1}\left(\frac{3q}{2p}\sqrt{\frac{-3}{p}}\right)\right\}$$

* Case II : $\frac{1}{2} < \Omega_m \leq 1 \Rightarrow q_{max} = 0 \Rightarrow \Delta_{max} = \left(\frac{-\frac{1}{4}}{2}\right)^2 = \frac{1}{64} > 0$; Say $\Omega_m = \frac{3}{4} \Rightarrow q = \frac{\frac{3}{4}-1}{3} = \frac{-1}{12} < 0$

$\Delta < 0 \qquad p < 0 \qquad$ 3 real/distinct roots

$\Delta = \left(\frac{-\frac{1}{12}}{2}\right)^2 + \left(\frac{-1}{4}\right)^3 = \frac{-1}{72} < 0$

1 real sol^n

* $$y = 2\sqrt{\frac{-p}{3}}\cos\left\{\frac{1}{3}\cos^{-1}\left(\frac{3q}{2p}\sqrt{\frac{-3}{p}}\right)\right\} , \quad y \equiv \left(\frac{\Omega_\Lambda}{4\Omega_m}\right)^{1/3}$$

$$\left(\frac{\Omega_\Lambda}{4\Omega_m}\right)^{1/3} = 2\sqrt{\frac{1}{4}}\cos\left\{\frac{1}{3}\cos^{-1}\left(\frac{3(\Omega_m-1)}{4\Omega_m\, 2(-\frac{3}{4})}\sqrt{4}\right)\right.$$

* $\boxed{\Omega_\Lambda = 4\Omega_m \cos^3\left\{\frac{1}{3}\cos^{-1}\left(\frac{1-\Omega_m}{\Omega_m}\right)\right\}}$ .... More Math/Modelling .... $t_{univ} \sim 13.7$ bn yr.

$\exists$ Big bang origin (pt)

* recent obs : $(\Omega_m, \Omega_\Lambda) = (0.3, 0.7)$ interesting fact

1.g. General cubic Eq^n $\longrightarrow$ depressed cubic eq^n { Tschirnhaus' Transformation, 1683 'Contemporary of Spinoza / Leibniz / Newton / Huygens

* $f(x) = ax^3 + \underbrace{bx^2}_{} + cx + d = 0$ cubic machine

* $\boxed{x = t - \frac{b}{3a}}$

$x = t - \frac{b}{3a}$ → coeff of the term you want to get rid of; → coeff of the max. power; # of degree of poly. = 3 (cubic)

* $a\left(t - \frac{b}{3a}\right)^3 + b\left(t - \frac{b}{3a}\right)^2 + c\left(t - \frac{b}{3a}\right) + d = 0$

$a\left\{t^3 - \frac{b^3}{27a^3} - \frac{tb}{a}\left(t - \frac{b}{3a}\right)\right\} + b\left\{t^2 + \frac{b^2}{9a^2} - \frac{2}{3}\frac{tb}{a}\right\} + c\left(t - \frac{b}{3a}\right) + d = 0$

$at^3 - \frac{b^3}{27a^2} - \cancel{t^2 b} + \frac{tb^2}{3a} + \cancel{t^2 b} + \frac{b^3}{9a^2} - \frac{2tb^2}{3a} + ct - \frac{bc}{3a} + d = 0$ cubic in t with no t^2 term

$at^3 + \underbrace{\left(\frac{b^2}{3a} - \frac{2b^2}{3a} + c\right)}_{}t + \left(\frac{b^3}{9a^2} - \frac{b^3}{27a^2} - \frac{bc}{3a} + d\right) = 0$

$at^3 + \left(c - \frac{b^2}{3a}\right)t + \left(\frac{2}{27}\frac{b^3}{a^2} - \frac{bc}{3a} + d\right) = 0 \Rightarrow t^3 + \underbrace{\left(\frac{c}{a} - \frac{b^2}{3a^2}\right)}_{p}t + \underbrace{\left(\frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a}\right)}_{q} = 0$

* $\boxed{t^3 + pt + q = 0}$ depressed / reduced / Tschirnhaus transformed cubic eq^n ..... Cardano takes up...

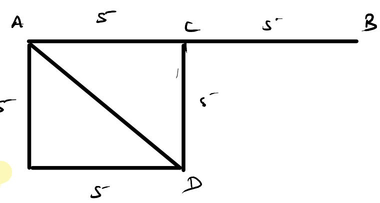
* Generalized version of Tschirnhaus transf^n $f(x) = x^n + a_1 x^{n-1} \dots + a_n = 0$ (Later: Algebra 3 transf^n of Eq^n) after Binomial thm.

Lecture-8 (23/Sept) 2

1.h Troubled Cardano (Ch-37 "On the Rule for Postulating a -ve #") Ref Ars Magna

* Puzzle : 10 divided into 2 parts, pdt is 40

10 → 5, 5 : square = 25 = Pdt. □



10, A 5 C 5 B, 5, 5, 5, D

get! $\frac{5(AB)}{2}$ = 25 = $\underbrace{10}_{AB} + 15$ = Area = (AC)(CD) = 40 = 4(AB)

want ! { 40 = 4(AB) | $\underbrace{\frac{5}{2}AB - 4AB}_{\text{Remainder}} = -\frac{3}{2}AB$ Side is -ve

I imagine it ! ↑ to extract what is not even there ?!

$\boxed{\text{How much is extra? = subtract 40 from 25} \longrightarrow \text{Remainder} = 25 - 40 = -15 < 0}$

↓ Sol^n

$\sqrt{-}$ = sophistry (deception) ; = i ; = 'Bombelli' ; = Heck

Pdt $= 40 = \underbrace{(5 - \sqrt{-15})}_{x}\underbrace{(5 + \sqrt{-15})}_{y} = 25 + 5\cancel{\sqrt{-15}} - 5\cancel{\sqrt{-15}} - (\sqrt{-15})^2$

$= 25 - (-15) = 25 + 15 = 40$

$\left[(-x)^{1/2}\right]^2 = -x$

* $x + y = 10$, $xy = 40$ $\Rightarrow$ $x(10 - x) = 40$ $\Rightarrow$ $-x^2 + 10x - 40 = 0$ $\Rightarrow$ $x^2 - 10x + 40 = 0$

Practice 1

Q1. $x^3 - 15x = 126$ (using Cardano's 1st principles)

* Cardano's principle

Step 1 Introduce 2 more variable: $x = u + v$

Step 2 Cardano's Condition — put one term (after analysis 'CLEVERLY' to 0)

Step 3: Get the Condition for sum & Pdt of Roots & of a quadratic eq^n ans

Step 4: Roots are easy cubic eq^n of form $u^3 = N$, $v^3 = M$

Step 5: Sol^n to depressed cubic $x = u + v$: $uv = \alpha$ (C.C)

Calculations:

* $(u+v)^3 - 15(u+v) = 126 \Rightarrow u^3 + v^3 + 3uv(u+v) - 15(u+v) = 126$

$u^3 + v^3 + \underbrace{(3uv - 15)}(u+v) = 126$

$\because u+v \neq 0,\ u^3+v^3 \neq 0$

$3uv - 15 = 0$ 'C.C' $\Rightarrow \boxed{uv = 5}$

* $\underbrace{u^3+v^3}_{\alpha} = 126$, $\underbrace{u^3v^3}_{\beta} = 125$

* Roots are u^3, v^3 of a quadratic eq^n $\begin{cases}(x-\alpha)(x-\beta) = 0 \\ x^2 - \alpha x - \beta x + \alpha\beta = 0\end{cases}$

$y^2 - (\alpha+\beta)y + \alpha\beta = 0$

$y^2 - 126y + 125 = 0 \Rightarrow y^2 - 125y - y + 125 = 0 \Rightarrow y(y-125) - 1(y-125) = 0$

$(y-1)(y-125) = 0$

* $y_1 = \alpha = 1 = u^3$

$y_2 = \beta = 125 = v^3$

* $u = \sqrt[3]{125}$ → $u_1 = 5$, $u_2 = 5\omega$, $u_3 = 5\omega^*$

$v = \sqrt[3]{1}$ → $v_1 = 1$, $v_2 = \omega$, ω^*

Ref: 1-c, 1-d

Solution to depressed cubic:

* $x = u + v$: $uv = 5$

$\omega = \dfrac{-1+\sqrt{-3}}{2}$, $\omega^* = \dfrac{-1-\sqrt{-3}}{2}$

$\omega^3 = 1 \Rightarrow \omega = \dfrac{1}{\omega^2} = \dfrac{1}{\omega^*} \Rightarrow \omega\omega^* = 1$ ✓

* $x_1 = 5 + 1 = 6$ ✓ $\longrightarrow 5(1) = 5$ C.C. ✓

$x_2 = 5 + \omega \longrightarrow 5\omega \neq 5$

$x_3 = 5 + \omega^*$

$x_4 = 5\omega + 1 \longrightarrow 5\omega(1) = 5\omega \neq 5$

$x_5 = 5\omega + \omega = 6\omega$

$x_6 = 5\omega + \omega^* \longrightarrow (5\omega)(\omega^*) = 5(\omega\omega^*) = 5$ CC ✓

$x_7 = 5\omega^* + 1$

$x_8 = 5\omega^* + \omega \longrightarrow 5\omega^*(\omega) = 5(\omega^*\omega) = 5$ CC ✓

$x_9 = 5\omega^* + \omega^* = 6\omega^*$

$\{5,\ 5\omega + \omega^*,\ 5\omega^* + \omega\}$ Solution

---

wherefore x equals $1\frac{1}{2}$ and this, in the negative, is what Francis has, and I have $10\frac{1}{2}$, and such are the *aurei* sought for.

Problem III

Likewise, if I say I have 12 *aurei* more than Francis and the square of mine is 128 more than the cube of Francis' *aurei*, we let Francis have

[2] 1570 and 1663 have 422x.

$-x$ and I have 12 *aurei* minus x. The square of mine is $144 + x^2 - 24x$, and this is equal to $-x^3 + 128$. Therefore

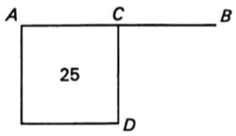
$$16 + x^2 + x^3 = 24x,$$

wherefore x is 4, and so much in the negative is what Francis lacks. I have 8 *aurei* of my own.

Rule II

The second species of negative assumption involves the square root of a negative. I will give an example: If it should be said, Divide 10 into two parts the product of which is 30 or 40, it is clear that this case is impossible. Nevertheless, we will work thus: We divide 10 into two equal parts, making each 5. These we square, making 25. Subtract 40, if you will, from the 25 thus produced, as I showed you in the chapter on operations in the sixth[3] book, leaving a remainder of -15, the square root of which added to or subtracted from 5 gives parts the product of which is 40. These will be $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$.

Demonstration



In order that a true understanding of this rule may appear, let AB be a line which we will say is 10 and which is divided in two parts, the rectangle based on which must be 40. Forty, however, is four times 10; wherefore we wish to quadruple the whole of AB. Now let AD be the square of AC, one-half of AB, and from AD subtract $4AB$, ignoring the number.[4] The square root of the remainder, then — if anything remains — added to or subtracted from AC shows the parts. But since such a remainder is negative, you will have to imagine $\sqrt{-15}$ — that is, the difference between AD and $4AB$ — which you add to or subtract from AC, and you will have that which you seek, namely $5 + \sqrt{25 - 40}$ and $5 - \sqrt{25 - 40}$, or $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$. Putting aside the mental tortures involved,[5] multiply

geometric meaning

[3] 1570 and 1663 have "fourth."

[4] *absque numero.*

[5] *dimissis incruciationibus.* We may perhaps suspect Cardano of indulging in a play on words here, for this can also be translated "the cross-multiples having canceled out," with the whole sentence then reading, "Multiply $5 + \sqrt{-15}$ by $5 - \sqrt{-15}$ and, the cross-multiples having canceled out, the result is $25 - (-15)$, which is +40. *Cf.* the translation of this passage by Professor Vera Sanford in David Eugene Smith, *A Source Book in Mathematics* (New York, 1959 reprint), I, 202.

Q2. $f(x) = x^3 - 6x^2 + 11x - 6 = 0$ (using factorization / long division method) Math 7-8

not really helpful at all times

* $\boxed{f(x) = f_{linear} \cdot f_{quad.}}$

Calc. for f_{linear}:

* Guess a solution (this is the catch with this method) $\longrightarrow$ if numerical methods applied $\longrightarrow$ computability speed

* $f_{linear} = (x-1)$

* $\underbrace{x^3-6x^2+11x-6}_{\overset{!}{0}} = (x-1)\, f_{Quad} \Rightarrow \quad f_{Quad} = \dfrac{x^3-6x^2+11x-6}{x-1} = x^2-5x+6$

$$\begin{array}{r}
x^2-5x+6 \\
x-1 \overline{)\, x^3 - 6x^2 + 11x - 6} \\
\vdots \\
\overline{\quad 0 \quad}
\end{array}$$

* $x_1 = 1 \;,\; x_2 = 2 \;,\; x_3 = 3 \quad \in \mathbb{R}$

Lecture-9 (24/Sept)

Q3. $f(x) \equiv x^3 + 3x^2 - 10x + 8 = 0$ Roots of cubic eqn = ?

* Step 1 : Reduce the full cubic → depressed cubic via Tschirnhaus' transformation $x = t-1$

* $t^3 - 13t + 20 = 0$ ✓

* Step 2 : Follow Cardano's method ⇒ Roots = $\{ \; . \quad . \quad . \; \}$

* $t = u+v \;;\quad 3uv - 13 = 0$ 'CC' $\Rightarrow u^3+v^3 = -20$ $\left.\begin{array}{l}\to \alpha+\beta = -20 \\ \to \alpha\beta = \left(\frac{13}{3}\right)^3\end{array}\right\}$ $x^2 + 20x + \frac{2197}{27} = 0$

$\downarrow$

$x = \dfrac{-20 \pm \sqrt{400 - \frac{4 \cdot 2197}{27}}}{2}$

$(\alpha, \beta) = -10 \pm \sqrt{100 - \frac{2197}{27}} = -10 \pm \sqrt{\frac{503}{27}}$

(u^3, v^3)

* $a \equiv \sqrt[3]{-10 + \sqrt{\frac{503}{27}}} \;,\; b \equiv \sqrt[3]{-10 - \sqrt{\frac{503}{27}}}$

* $x_1 = a+b-1 \;,\; x_2 = a\omega + b\omega^* - 1 \;,\; x_3 = a\omega^* + b\omega - 1$

* $u = \sqrt[3]{-10 + \sqrt{\frac{503}{27}}} \;,\; v = \sqrt[3]{-10 - \sqrt{\frac{503}{27}}}$

Q4. $x^3 + 6x = 20$

* $x_1 = \underbrace{\sqrt[3]{10 + \sqrt{108}}}_{(a)} + \underbrace{\sqrt[3]{10 - \sqrt{108}}}_{(b)}$

$x_2 = a\omega + b\omega^* \;,\; x_3 = a\omega^* + b\omega$

HW

Q5. $x^3 - 7x + 6 = 0$

* $x = u+v \longrightarrow u^3+v^3+(3uv-7)(u+v) = -6$

$x_1 = \underbrace{\sqrt[3]{-3 + \frac{10}{9}\sqrt{-3}}}_{=a} + \underbrace{\sqrt[3]{-3 - \frac{10}{9}\sqrt{-3}}}_{b}$

$\left| \; \sqrt{\frac{-100}{27}} = \frac{10}{9}\sqrt{-3} \right.$

$x_2 = \omega a + \omega^* b \;,\; x_3 = \omega^* a + \omega b$

Lecture-10 (25/sept) 2

2. Prerequisites

2.1 Machines (Ref: algebra vol Lect 29..)

Machine = function = Transformation
$\parallel$
Mapping
$\parallel$
change = Becoming

Rational $\equiv$ Ratio + analysis

Functions

Algebra

1. Polynomial machine: x, x^2, $-x^3$, x^{2n}/x^{2n-1}
2. Rational Machine ($\frac{p}{q}$): $\frac{1}{x}$, $-\frac{1}{x^2}$, $\frac{1}{x^{2n}}$
3. Irrational Machine: $\sqrt{x}$, $\sqrt[3]{x}$, $x^{1/2n}$, $x^{\frac{1}{2n-1}}$
4. Piecewise Machine: $\to |x|$, $\to$ signum, $\lfloor x \rfloor$ $\lceil x \rceil$ ... & more

Transcendental Machines

1. Trigonometric / Inverse Trig.
2. Exp. a^x, e^x, 10^x
3. Log. $\log_{10}$, $\ln_e$

2.1.a Polynomial Machine

* $y = f(x) = \sum_{i=0}^{n} a_i x^i$; $a_i, x \in \mathbb{R}$, $\boxed{n \in \mathbb{N}}$ n can't be ∞

2.1.a.1 Constant Machine

* $i = 0 \Rightarrow y = f(x) = a_0$

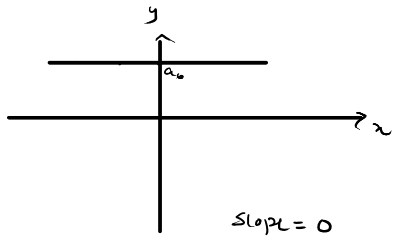
intuition $\to$ $f = \{ (x, \underbrace{f(x)}_{a_0}) : x \in \mathbb{R} \}$

$= \{ (x, a_0) : x \in \mathbb{R} \} = \{ (1, a_0), (2, a_0), (9, a_0), (\sqrt{\pi}, a_0) \dots \}$

$\to y = a_0 (x)^0$

x	1	2	3	
y	a_0	a_0	a_0	...

'Naive'

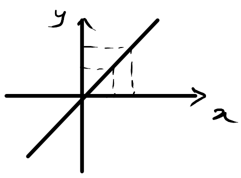


Slope = 0

2.1.a.2 Identity Machine

* $y = f(x) = x = \{ (x, x) : x \in \mathbb{R} \}$

$\underset{i/p}{x} \longrightarrow \underset{o/p}{x} \Rightarrow$ Equilibrium



Slope = 1

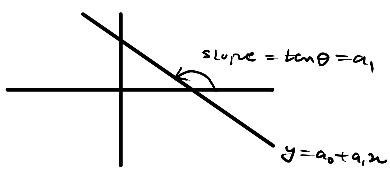
$\tan\theta = \frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta y} = 1$

2.1.a.3 Linear Machine

* $y = f(x) = a_0 + a_1 x$

* if $a_0 = 0 \Rightarrow f(x) = a_1 x \xrightarrow[\text{Scaling}]{x \to \lambda x \text{ (Scale factor)}} f(\lambda x) = a_1 \lambda x = \lambda f(x)$

$\boxed{f_{scaled} \propto f_{orig.}}$



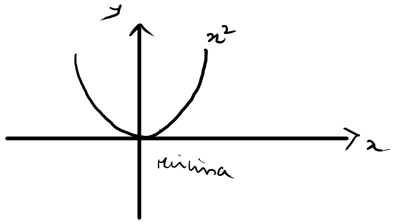
Slope = a_1

$\tan\theta = \frac{\Delta y}{\Delta x} = \frac{\Delta}{\Delta x}(a_1 x + a_0)$

$= \frac{\Delta}{\Delta x}(a_1 x) + \cancel{\frac{\Delta a_0}{\Delta x}}^{0} = a_1 \frac{\Delta x}{\Delta x} = a_1$

2.1.a.4 Quadratic Machine

* $y = f(x) = a_0 + a_1 x + a_2 x^2 \xrightarrow[\substack{(a_0, a_1) = 0 \\ a_2 = 1}]{} \boxed{f(x) = x^2}$ Parabola $a_2 \geq 0$



* $f(-x) = f(x) \Rightarrow$ Even function / Parity invariance / $\underbrace{\text{Reflection}}_{\text{y axis as mirror}}$ transf^n invariant

$\Downarrow$

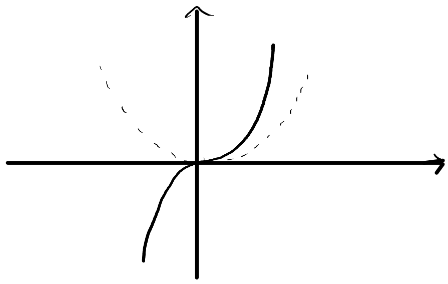
symmetric about y axis

* Dom f = I/P = $\mathbb{R}$

Range $f = \mathbb{R}^+ \cup \{0\} = [0, \infty)$

2.1.a.5 Cubic Machine

* $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \xrightarrow[\substack{a_0, a_1, a_2 = 0 \\ a_3 = 1}]{\text{set}} \boxed{f(x) = x^3}$



* $f(-x) = -f(x) \Rightarrow$ Odd function / Parity = -1

* $f = \{(x, x^3) : x \in \mathbb{R}\} = \{(2, 8), (3, 27), \dots, (-2, -8), \dots (-3, -27) \dots\}$

crawls up "faster" relative to x^2 intuitive

$\shortparallel$

1 order faster increment!

(Later curve sketching (+ Calculus) = More Rigorous way)

* "Symmetric about opp. quadrant or origin"

* Dom $f = \mathbb{R}$, Range $f = \mathbb{R}$

2.1.a.6 Generalized Even / odd powers of x

* $f(x) = x^{2n}$

* $f(-x) = f(x)$ Even function $\Rightarrow$ sym about y axis

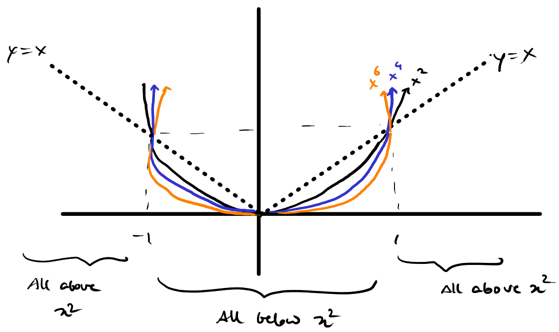
* Dom $f = \mathbb{R}$, Range $f = \mathbb{R}^+ \cup \{0\}$

* $x^2 < x^4 < x^6 < x^8 \dots \quad x \in (-\infty, -1) \cup (1, \infty)$ say $x = 2$

$(2^2 < 2^4 < 2^6 \dots)$ all curves above x^2

$x^2 > x^4 > x^6 > \dots \quad x \in (-1, 1) \quad x = \frac{1}{2}$ (say)

$\left(\frac{1}{2^2} > \frac{1}{2^4} > \frac{1}{2^6} \dots\right)$ all curves below x^2

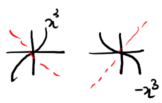


<u>Lecture 11</u> (27/Sept) 2

* $f(x) = x^{2n-1}$

$f(-x) = (-x)^{2n-1} = -f(x) \Rightarrow$ odd fu^n

* "sym. about opp. quadrant"



Power Analysis

* $x < x^3 < x^5 < x^7$ $x \in (1, \infty)$

$x > x^3 > x^5 > x^7$ $x \in (0,1)$

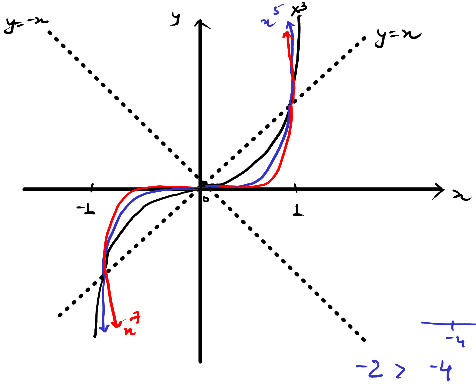
$\left(\frac{1}{2} > \frac{1}{2^3} > \frac{1}{2^5} > \frac{1}{2^7}\uparrow\right)$

$x < x^3 < x^5 < x^7$ $x \in (-1, 0)$

$\left(\frac{-1}{2} < \frac{-1}{2^3} < \frac{-1}{2^5} < \frac{-1}{2^7}\right)$

$x > x^3 > x^5 > x^7$ $x \in (-\infty, -1)$

$\left(-2 > -2^3 > -2^5 > -2^7\right)$



$-2 > -4$

$\frac{-1}{2} < \frac{-1}{8}$

$-8 < -2$

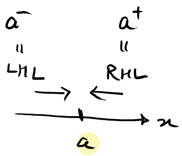
Dom f : $\mathbb{R}$

Range f : $\mathbb{R}$

2.1.6. Rational Machines $f(x) = \frac{P(x)}{Q(x)}$ dom $f = \mathbb{R} - \{x : Q(x) = 0\}$ 'general property'

2.1.6.1. Reciprocal Machine / Inverse proportionality / Rectangular hyperbola

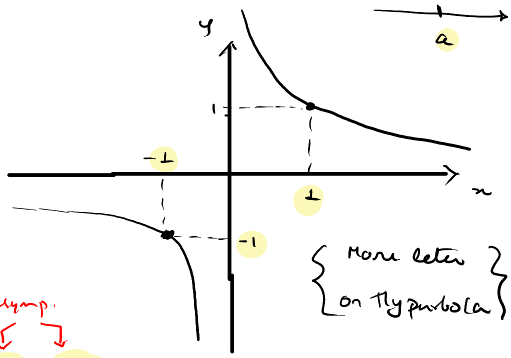
* $f(x) = \frac{1}{x}$, $Q(x) = x$ linear $\xrightarrow{x=0}$ $Q(x) = 0$



* $f(-x) = -f(x) \Rightarrow$ curves land in opp. quadrant

* $\lim_{x\to 0^+} \frac{1}{x} = +\infty$, $\lim_{x\to 0^-} \frac{1}{x} = -\infty$

$\lim_{x\to +\infty} \frac{1}{x} = 0$ $\lim_{x\to -\infty} \frac{1}{x} = 0$



* dom $f = \mathbb{R} - \{0\}$

Range f = o/p = $\mathbb{R} - \{0\}$

$\exists$ a special creature = Asymptote $\left\{ 2 \text{ asymp.} \right.$ $x=0$ $y=0$

= St. lines

that a curve always tries to approach but never hits ($\nexists$ Rect b/w the curved Asymp.)

{ More later on Hyperbola }

$s^2 = y^2 - x^2$ Minkowski

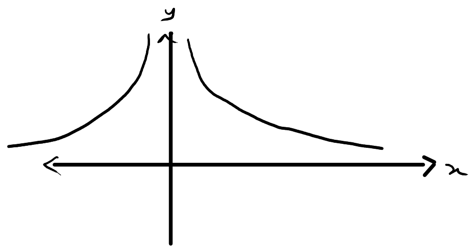
2.1.6.2 Reciprocal sq.

* $f(x) = \frac{1}{x^2}$

* $f(-x) = f(x)$ Even fun $\Rightarrow$ curves land in adjacent quad.

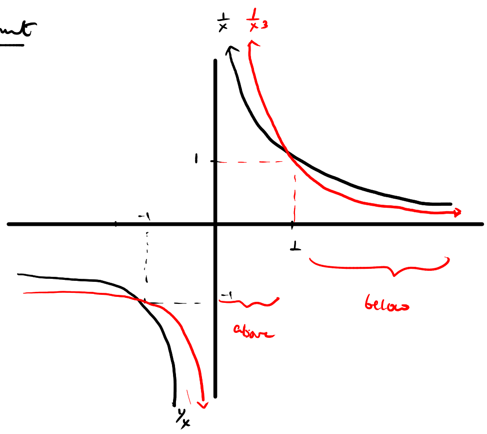
Dom. $f = \mathbb{R} - \{0\}$

Range $f = (0, \infty)$



Comment

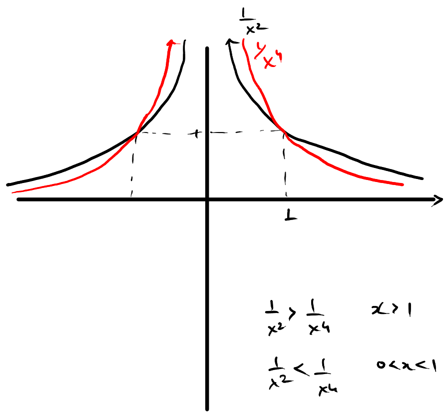
*



$f(x) = \frac{1}{x^{2n-1}}$

$\frac{1}{x} > \frac{1}{x^3}\uparrow$

Asymptotes = { $x=0$, $y=0$ }



$\frac{1}{x^2} > \frac{1}{x^4}$ $x > 1$

$\frac{1}{x^2} < \frac{1}{x^4}$ $0 < x < 1$

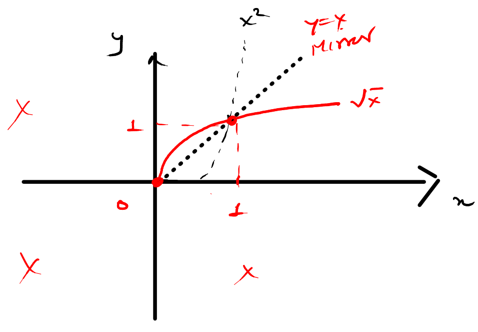
$f(x) = \frac{1}{x^{2n}}$

$\frac{1}{x^2} > \frac{1}{x^4}$

Asymp. = { $x=0$, $y=0$ }

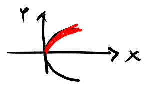
2.1.C Irrational Machine (Power of $x \notin \mathbb{Z}$) $\sqrt[3]{1}$ $(\sqrt[3]{-1})^3$

2.1.C.1 Sq. root Machine



* $y = f(x) = \sqrt{x}$; $x \notin \mathbb{C}$

$\left(y^2 = x \Rightarrow x = f(y) = y^2 \text{ Parabola}\right)$ flash back $\rightarrow$ square $f(-y) = f(y)$ Even



* Dom f : $\mathbb{R}^+ \cup \{0\} = [0, \infty)$ ✓ Problem $\xrightarrow{\text{sol}^n}$ go into mental torture. $\mathbb{C}$

Rang f : $\mathbb{R}^+ \cup \{0\} = [0, \infty)$

* $y_1 = x^2$, $y_2 = x^{1/2}$ Inverses of each other

↳ Mirror img. about $y=x$

$y = \sqrt{x}$ $\quad$ $y = x$

$x=1 \downarrow$ $\quad$ $x=1 \downarrow$ $\quad$ (inspection)

$y=1$ $\quad$ $y=1$ $\quad$ Algebraic Root

2: Just like solving simult. eq^n

$\sqrt{x} = x \Rightarrow x^2 = x$

(later) Graphical method

$x^2 - x = 0$

$x(x-1) = 0$

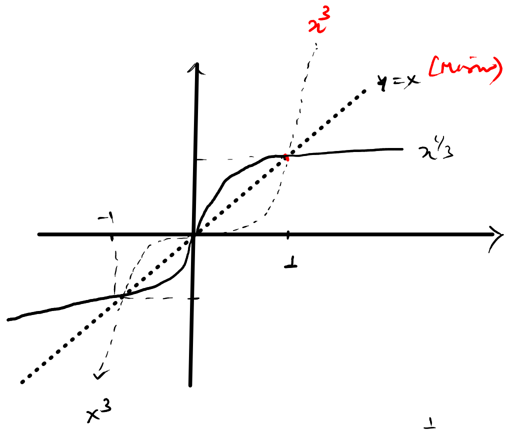
$x = 0, \; x = 1$

$\downarrow \quad \downarrow$

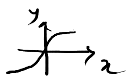
$y = 0 \quad y = 1$

Lecture-12 (29 | Sept) 2

2.1.C.2 Cube Root Machine



* $y = f(x) = \sqrt[3]{x} = x^{1/3}$: $x \notin \mathbb{C}$; $f : \underset{\text{Dom.}}{\mathbb{R}} \rightarrow \underset{\text{Codomain}}{\mathbb{R}}$

($y^3 = x \Rightarrow x = f(y) = y^3$ cube ) $\rightarrow$ cube $f(-y) = -f(y)$ odd fun^n

* $f_1(x) = x^3$. $f_2(x) = x^{1/3}$ $\{$ $\underbrace{\text{Inverses}}_{3 \times \frac{1}{3} = 1}$ of each other / Mirror img.

$x \in (0,1)$	$x^{1/3} > x^3$
$x \in (1, \infty)$	$x^3 > x^{1/3}$
$x \in (-1, 0)$	$x^{1/3} > x^3$
$x \in (-\infty, -1)$	$x^{1/3} > x^3$

$\left(\frac{1}{27}\right)^{\frac{1}{3}} > \left(\frac{1}{27}\right)^3$

usual case

$\left(\frac{-1}{27}\right)^{1/3} > \left(\frac{-1}{27}\right)^3$

$(-27)^{1/3} > (-27)^3$

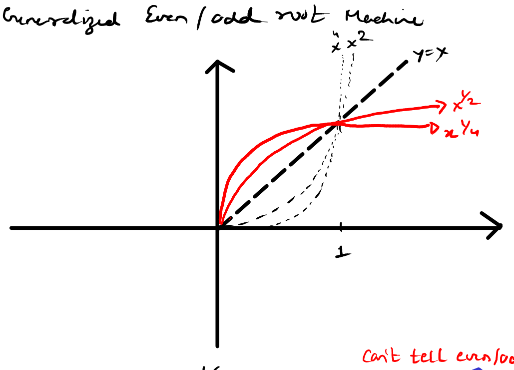
* Dom $f = \mathbb{R}$

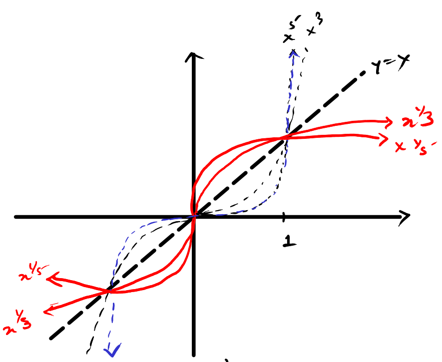
Range $f = \mathbb{R}$ $\xrightarrow{\text{Catch}}$ $x \notin \mathbb{C} \Rightarrow$ Range $f = \mathbb{C}$ $\quad$ $\sqrt[3]{1} = \overbrace{1, \omega, \omega^*}$

* $f(-x) = \sqrt[3]{-x} = \sqrt[3]{-\alpha^3} = -\underset{\sim}{\alpha} = f(x)$

$\sqrt[3]{-8} = -2$

2.1.C.3 Generalized Even / odd root Machine





Mirror Imgs. $\begin{cases} f(x) = x^{1/2n} \\ g(x) = x^{2n} \end{cases}$ Even Root $\rightarrow$ $f(-x)$ Not allowed by Domain (can't tell even/odd) ; Even $\rightarrow g(-x) = g(x)$

$f(x) = x^{\frac{1}{2n-1}}$ odd. Root $\rightarrow$ $f(-x) = -f(x)$ odd fun^n

$g(x) = x^{2n-1}$ odd $\rightarrow g(-x) = -g(x)$

2.1.d Piecewise linear (PL) / Segmented / Hybrid Machines

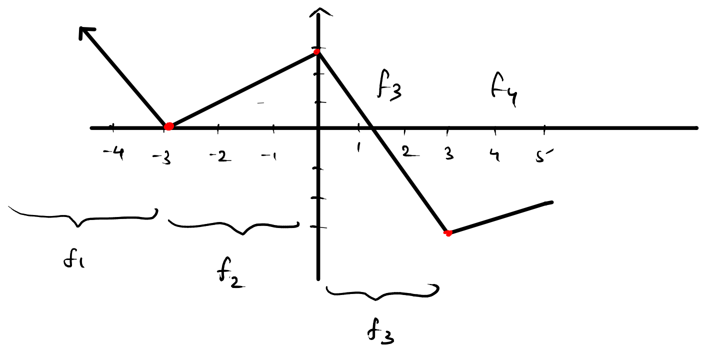
Smooth / Compact

* Defined by multiple sub-functions / Def. by cases

Example

$$f(x) = \begin{cases} -x-3 & x \le -3 \\ x+3 & -3 < x < 0 \\ -2x+3 & 0 \le x < 3 \\ \underbrace{0.5x-4.5}_{\text{Linear functions (4 piece)}} & x \ge 3 \end{cases}$$

parts



Analysis

* $f_1 = -x-3$ — dom. $f_1 = (-\infty, -3]$ — It is not a 'Line'. It is a segment / Ray / "Vectors" ↳ Not much used jargon

Range $f_1 = \mathbb{R}^+ \cup \{0\} = [0, \infty)$

Slope $= -1$

$Y = mx + c$

Lecture 13 (30/sept) 2

* $\underbrace{f_2}_{y} = x+3$ — dom $f_2 = (-3, 0)$ — Not a Line but a segment.

Range $f_2 = (0, 3)$

slope $= +1$

* $f_3 = -2x+3$ — dom $f_3 = [0, 3)$

Range $f_3 = [3, -3)$

slope $= -2$

* $f_4 = 0.5x - 4.5$ — dom $f = [3, \infty)$

Range $f = [-3, \infty)$

Slope $= \frac{1}{2}$

Remark : points where slope change occurs → Breakpoints / Changepoint / Threshold values / Knots

{ App. : digital signal processing / waveform Analysis / Electronics / Info. theory / statistics

* P.L functions are defined on collection of Intervals on each of which function is an "Affine" function.
* if domain of P.L function is "Compact" => ∃ finite Collection of Intervals.

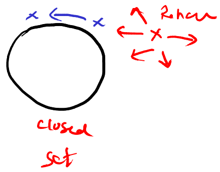
Detour to Real Analysis , Geometry , Topology

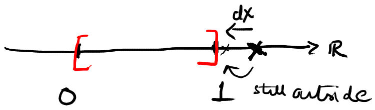
α. Closed / Bounded / Compact Space

* Closed Set ≡ an interval which contains all Limit points — 'Set theoretic Intuition'

ex : $[a,b]$, $\underbrace{[0,1] \equiv I}_{\text{unit Interval}}$, $[1, \infty) \sim$ Ray/vector

* If you are "outside" a closed set, you may move a small amt in any dir^n & still stay outside the set — 'geometrical'





Math. Analog of a closed system (Isolated)

* Bounded Set : in which $\exists$ upper bound & lower bound

$\exists$ a finite $\underbrace{\text{"measure"}}_{\sim \text{length}}$

$[a,b] \longrightarrow$ measure $= b-a =$ finite
$[1,\infty) \longrightarrow$ measure $= \infty$

* Compact Space = generalization of closed / Bounded subset of Euclidean space

= $\nexists$ punctures / missing end pts / limiting values.

"are exactly same as"

$$\boxed{\begin{array}{c} S \text{ is closed \& Bounded} \equiv S \text{ is Compact} \\ S \subset \mathbb{R}^n \end{array}}$$

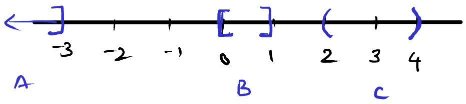
Heine-Borel theorem (Real Analysis, 1895)
student of Gauss / Dirichlet

Ex : 1. $[0,1]$ Compact

2. $(0,1)$ no boundary $\Rightarrow$ not Compact

3. $\mathbb{Q}$ ∞ many punctures corresponding to $\mathbb{I}$ $\mathbb{Q}$ $\Rightarrow$ not Compact

4. $\boxed{\mathbb{R} = (-\infty, \infty) \quad \text{missing end pts}}$ $\Rightarrow$ not Compact Real No. line

5. 

$A = (-\infty, -3]$ Not bounded $\Rightarrow$ Not Compact

$B = [0,1]$ Bounded / closed $\Rightarrow$ compact

$C = (2,4)$ missing End pts $\Rightarrow$ Not Compact
Not closed

$\Downarrow$

$2.01 \in C$
$2.001 \in C$
$\vdots$
∞ regression $\Rightarrow \nexists$ a finite Boundary

$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{(x_1, \dots, x_n)}$

6. Affinely Extended $\mathbb{R}$ # system $= \mathbb{R} + \underbrace{(\pm\infty)}_{\text{As Actual Number}} = \overline{\mathbb{R}} = [-\infty, \infty] = \mathbb{R} \cup \{-\infty, \infty\}$ 'Compact' (desirable property)

"Dedekind-MacNeille Completion of $\mathbb{R}$ (1937)"

<u>Arithmetic operation on $\overline{\mathbb{R}}$</u>

1. $a \pm \infty = \pm\infty + a = \pm\infty$ $\qquad a \mp \mp\infty$

2. $a \cdot (\pm\infty) = \pm\infty \cdot a = \pm\infty$ $\qquad a \in (0, +\infty]$

3. $\pm\infty = \pm\infty \cdot a = \mp\infty$ $\qquad a \in [-\infty, 0)$

4. $\dfrac{a}{\pm\infty} = 0$ $\qquad a \in \mathbb{R}$

5. $\dfrac{\pm\infty}{a} = \pm\infty$ $\qquad a \in (0, \infty)$

6. $\dfrac{\pm\infty}{a} = \mp\infty$ $\qquad a \in (-\infty, 0)$

7. $e^{-\infty} = 0$

8. $\ln(0) = -\infty$

9. $\tan^{-1}(\pm\infty) = \pm\dfrac{\pi}{2}$

10. $\tanh(\pm\infty) = ?$ <u>HW</u>

$\vdots$

$\exists$ Indeterminant forms

$\log_a(0) = x$
$a^x = 0$
$x = -\infty$

Lecture-14 (1/oct)

(Surgery)
+ connected sums
"
cut & paste

β. Manifold / It's building blocks / Operations ($\times$ / gluing)

* Manifold (mfd.) ≡ nice / smooth geometric figure that generalizes curves & surfaces to arbitrary dimension.

* Mfd. can be build from :

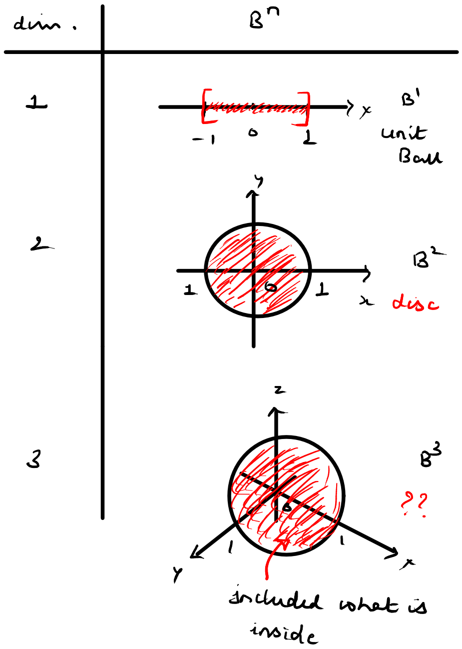
Basic building blocks

1. n-Ball $B^n = \{ (x_1, \dots x_n) \in \mathbb{R}^n : \sum_i x_i^2 \leq 1 \}$ filled

$B^1 = \{ x \in \mathbb{R} : \underbrace{x^2 \leq 1}_{-1 \leq x \leq 1} \}$ 1-ball

$B^2 = \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \}$ 2 ball

$B^3 = \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1 \}$ 3 ball



* $[0, 1]$ ≡ unit Interval = half of B^1
"
I (filled)

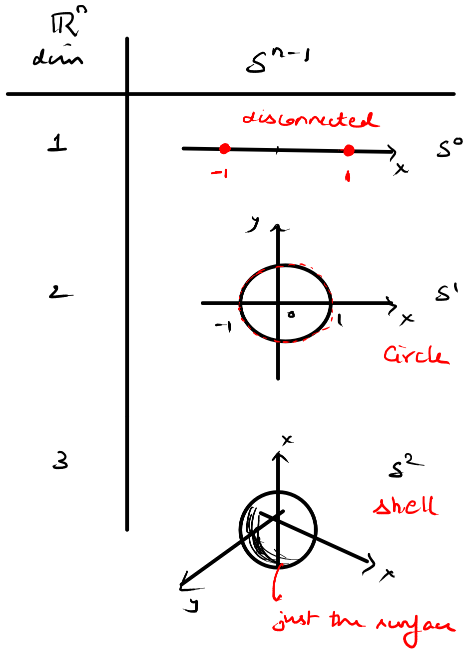
2. (n-1)-sphere $S^{n-1} = \{ (x_1, \dots x_n) \in \mathbb{R}^n : \sum_i x_i^2 = 1 \}$ unfilled

$S^0 = \{ x \in \mathbb{R} : \underbrace{x^2 = 1}_{x = \pm 1} \}$ 0-sphere (0 dim. sphere)

$S^1 = \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \}$ 1-sphere (circle)
Eq^n of circle

$S^2 = \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \}$ 2-sphere
Eq^n of sphere

$S^8 \to$ 8-sphere in $\mathbb{R}^9$
(9-dim euclidean space)

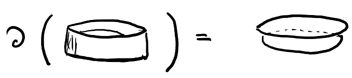


Useful symbols :

* ∂ ≡ "Boundary of" { useful in Calculus — Stokes' thm. $\int_V \xrightarrow{\text{red.}} \int_{\partial V}$
$\underbrace{\partial V}_{\text{boundary of volume}}$
"
surface

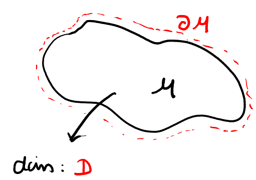
$\partial (B^n) = S^{n-1}$

$\partial (B^3) = S^2$


2 disconnected circles

* $\boxed{\partial = \text{operator that drops the dimension}}$
(1)

if $\dim (M) = D \Rightarrow \dim (\partial M) = D - 1$



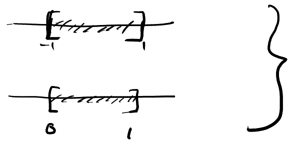
$\dim (\partial M) = D - 1$

* $\approx$ "Homeomorphic to" = "Topologically the same as"

ex 1

B'

I



0 1

$B' \approx I$

"proof" : I = half of B' = $\underbrace{\frac{1}{2}}_{k}(B')$ → $I = k\,B'$ ↳ scale factor

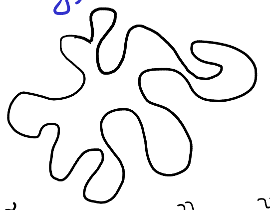
↓

$I \approx B'$

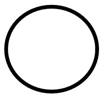
* $\boxed{\text{In topology} \rightarrow \text{visualize all spaces made of a "material" that is Elastic (stretch) \& Magical (you can go thru it)}}$

stretching (scaling) doesn't do anything to topology

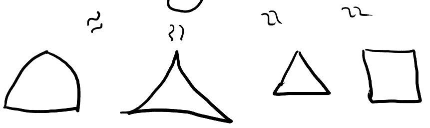
ex 2 :



$\approx$



S^1



* these building blocks generate interesting mfd via

Operations of . .

1. Product (x)

* $m\text{-cube} = I^m = \underbrace{I \times I \times \ldots \times I}_{m} \approx B^m$

* $I \times I = \{ (x,y) \in \mathbb{R}^2 : 0 \le x, y \le 1 \} = I^2$

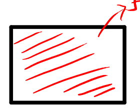
$= \{ (0,0), (0,1), (1,0), (1,1) \ldots$

$(\frac{\pi}{4}, 0) \; (\frac{\pi}{4}, 1) \ldots \}$

$I \times I \times I = \{ (x,y,z) \in \mathbb{R}^3 : 0 \le x,y,z \le 1 \} = I^3$

$= \{ (\frac{\pi}{4}, 0, 1), (\frac{\pi}{4}, 1, 2) \ldots \}$

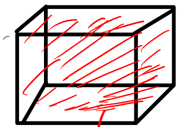
$I' = [0,1]$



filled

I^2

unit square



I^3

cube

filled

(kind)

$f \circ g$ = some kind of creatures

K' K' K'

General feature of rule of composition in Math

$\underset{\sim}{A} \cdot \underset{\sim}{B}$ = Number

weirder than

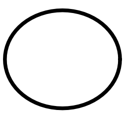
$\underset{\sim}{A} \times \underset{\sim}{B}$ = Vector

* $I^m \approx B^m \Rightarrow \underset{\sim}{\partial} I^m \approx \partial B^m = S^{m-1}$

surface / boundary of m cube

boundary of m-cube is (m-1) sphere

*



$S^1 = \{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \}$

Circle

* $S^1 \times S^1$ = shape / draw it ? HW

Lecture-15 (2/oct) 1

* $B^n = \{(x_1 \ldots x_n) \in \mathbb{R}^n : \sum_i x_i^2 \leq 1\}$ Balls are filled
* $S^{n-1} = \{(x_1 \ldots x_n) \in \mathbb{R}^n : \sum x_i^2 = 1\} \rightarrow \partial(B^n) = S^{n-1}$ spheres are unfilled
* $I = [0,1]$ unit interval is filled

} Building blocks of mfd.

* $I^n = \underbrace{I \times I \times \ldots \times I}_{n} \approx B^n$

$I^1 = [0,1]$

I^2 square

I^3 cube

* $T^m = \underbrace{S^1 \times S^1 \times S^1 \times \ldots \times S^1}_{m}$

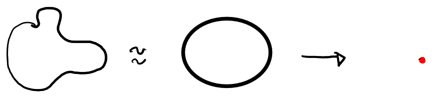
m-torus

Material: Elastic / Reversible

Comment

$T^1 = S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

also means homeomorphic to



1D closed curve (boundary)

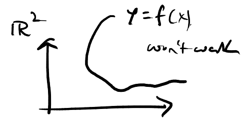
S^1 circle

Ability to be contracted to a point

$A = x$ — is Equal to

$A = \{ \ldots \}$ — is a set of

$A \sim B$ — is related to / is equivalent to

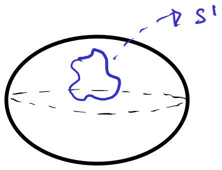


$y = y(s)$

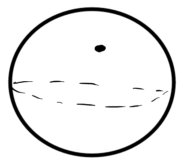
$x = x(s)$

1D open curve embedded in $\mathbb{R}^2$

*



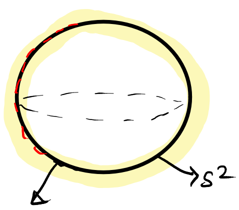
S^2 = shell / boundary of sphere



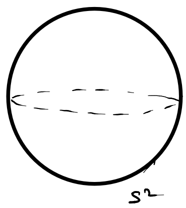
Contracted to a pt

$\nexists$ 1D curve to wrap S^2

S^1



closed surface S^2



S^2

S^2 on the surface can't be contracted to a point

$\exists$ 1 surface to wrap S^2

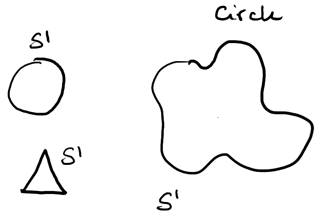
S^2

* $B^n = \{ (x_1, \dots x_n) \in \mathbb{R}^n : \sum_i x_i^2 \le 1 \}$, $I = [0, 1] \simeq B^1$

$S^{n-1} = \{ (x_1, \dots x_n) \in \mathbb{R}^n : \sum x_i^2 = 1 \} \longrightarrow \partial B^n = S^{n-1}$; $\times$ (pdt) Operator

* $I^n = I \times I \times \dots I$
  - $n = 1$ Line seg.
  - $n = 2$ square
  - $n = 3$ cube
  - $n = 4$ "Hyper" cube

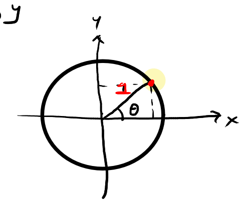
* $T^m = \underbrace{S^1 \times S^1 \times \dots S^1}_{m}$



* $S^1_a = \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \}$ Unit circle parametrized by x, y

$= \{ (\cos\theta, \sin\theta) : 0 \le \theta < 2\pi \}$ " " " by θ

$\dim (S^1_a) = 2$ 2D space



$x = \cos\theta$
$y = \sin\theta$

* $S^1_b = \{ (\cos\phi, \sin\phi) : 0 \le \phi < 2\pi \}$ unit circle parametrized by ϕ

$\dim (S^1_b) = 2$

* $S^1_a \times S^1_b = \{ (\cos\theta, \sin\theta, \cos\phi, \sin\phi) : 0 \le \theta < 2\pi, 0 \le \phi < 2\pi \} = T^2$

$\dim (S \times S) = 4$.

$\boxed{\text{Clifford Torus}}$

Resulting product space is 4D $(\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2)$

1850

Can't visualized.

* Clifford Torus (symm.) $\in \mathbb{R}^4$

↓ Reduction of dimension

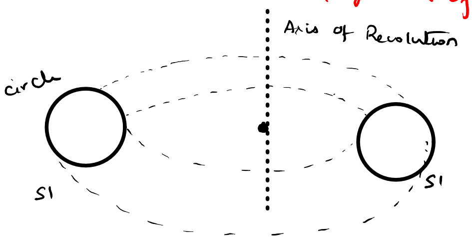
Famous version of

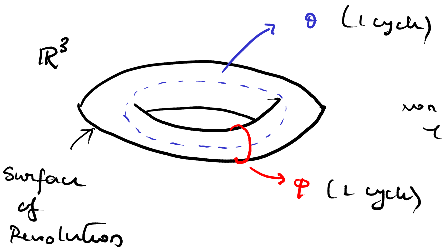
Torus $\in \mathbb{R}^3$

Torus is embedded in $\mathbb{R}^3$ → It is not filled

||

Reduced-dimension projection of Clifford Torus

* 

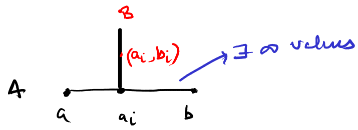


$A \simeq \mathbb{R}^1$

$A \times B = \{ (a, b) : a \in A, b \in B \}$ (set)

interval

$A = [a, b]$, $B = [c, d]$

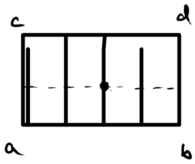


$A \times B \to (a_i, B)$ → $a_i \in A$ is crossed with 1 copy of B

$A = \{ x : a \le x \le b \}$ $\dim A = 1$

$B = \{ y : c \le y \le d \}$ $\dim B = 1$

$A \times B = \{ (x, y) : x \in A, y \in B \}$ $\dim (A \times B) = 2$ $= 1 + 1$



$\simeq \mathbb{R}^2$

$\dim A = 2$; $A = \{ (x, y) : a \le x, y \le b \}$

$\dim B = 2$; $B = \{ (z, w) : c \le z, w \le d \}$

$A \times B = \{ (x_1, x_2, x_3, x_4) : (x_1, x_2) \in A, (x_3, x_4) \in B \}$

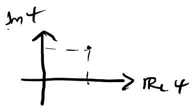
$\dim (A \times B) = 4$ → 4D Euclidean space

|| # tuples

↓ visual aid

3D Cuboid is a way to visualized.

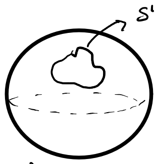
$\mathbb{R}^4$
- → $\mathbb{R}^2 \times \mathbb{R}^2$ → Real Represenⁿ
- → $\mathbb{C} \times \mathbb{C}$ → Complex Repreⁿ



$\underbrace{\mathbb{C}}_{2D} \sim \mathbb{R}^2$

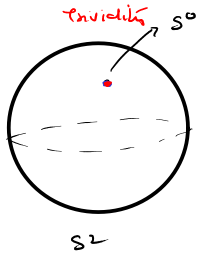
Lecture-17 (7/oct) 2.20

Topology : elastic material



S^2 surr. of sphere

Contract it to a pt

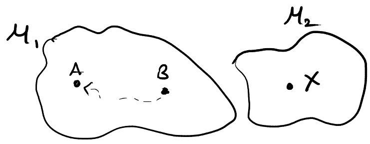


$\boxed{b_0 = 1}$

$\boxed{b_1 = 0}$ → 1D curve

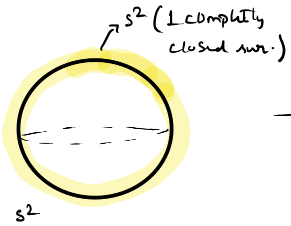
1st betti Number of S^2 (sphere)

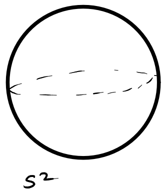
∄ 1D curves to wrap it



A & B are not topologically distinct

A & X are topologically distinct

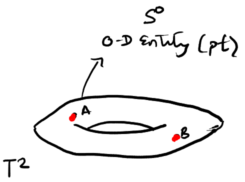




∃ 1 surr. to wrap it

(can't contract it)

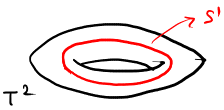
$\boxed{b_2 = 1}$ → 2D surf.

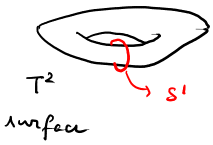


Can be contracted to a pt.

∃ 1

$\boxed{b_0 = 1}$ → 0D point to "wind" it



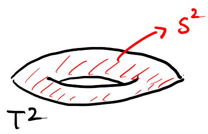


∃ 2 S^1 (1D curves) that can't be contracted to a pt

∃ 2

$\boxed{b_1 = 2}$

1D curves to wrap it (wind)



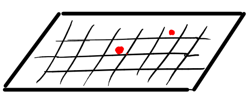
∃ 1 S^2 (2D surf.) to wrap it completely

$\boxed{b_2 = 1}$

∃ 1

2D surface to wrap it

Some simple topological spaces & their Betti # (by Henri Poincaré)

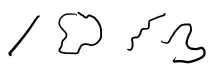


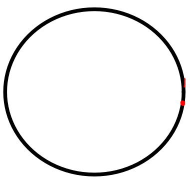
$\mathbb{R}^2$ (Given topology)

$b_0 = 1$ (∃ 1 0-D creature)

$b_1 = 0$ (∄ any 1-D creature on $\mathbb{R}^2$)

"circular" hole





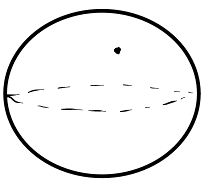
S^1

Circle

$b_0 = 1$ (∃ 1 0-D creature)

$b_1 = 1$ (∃ 1 1-D creature on S^1)

$b_{d>1}$ = Don't worry them.



S^2

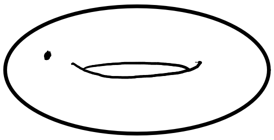
shell

$b_0 = 1$ (∃ 1 0-D crea-)

$b_1 = 0$ (∄ 1 1-D creature)

$b_2 = 1$ (∃ 2-D surf.)

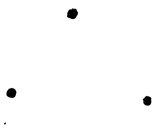
$b_{d>2}$ = Not concerned r.n. → "voids" ("cavities")

* 

T^2
2-torus
$=$
$S^1 \times S^1$

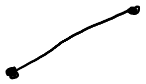
$b_0 = 1$ (∃ 1 0-D creature)
$b_1 = 2$ (∃ 2 1-D creature)
$b_2 = 1$ (∃ 2D surf.) → It is possible to wrap it
$b_{d \geq 2} =$ Not imp. r.n.

Recapitulatif

* 

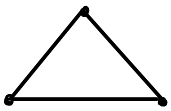
3 points

$b_0 = 3$ (∃ 3 0-D creature)
$b_1 = 0$ (∄ any 1-D creature)

* 

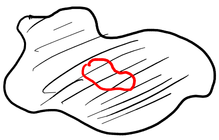
segment

$b_0 = 1$ (∃ 1 0-D creature)
$b_1 = 0$ (∄ 1-D creature)

* 

triangle

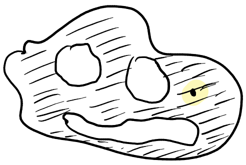
$b_0 = 1$ (∃ 1 0-D creature)
$b_1 = 1$ (∃ 1 1-D creature)

* 

solid 2-D blob

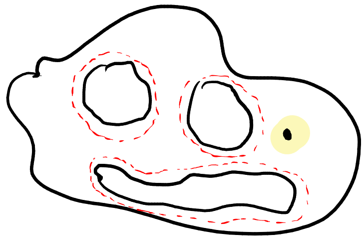
$b_0 = 1$
$b_1 = 0$

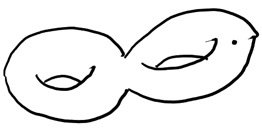
HW !!!

* 

2D solid blob with 3 holes

$b_0 = 1$
? $\boxed{b_1 = 3}$?



* 

Double torus

$b_0 = 1$
? $\boxed{b_1 = 4}$?
$b_2 = 1$

Imp. remarks

* Any surf. contractible to a pt ⇒ $\boxed{\text{Cohomology} = 0 \Rightarrow \text{Betti \#} = 0}$
(Can't wrap it)

!!! cutting edge !!!

* $\boxed{\text{Hole} \equiv \text{topo. structure : it prevents the obj. from being continuously shrunk to a point}}$

$=$

It is a disconnectivity in space

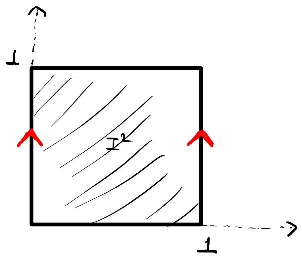
Lecture-18 (Slot) 1.5

Operations to build more mfd. ($\times$)

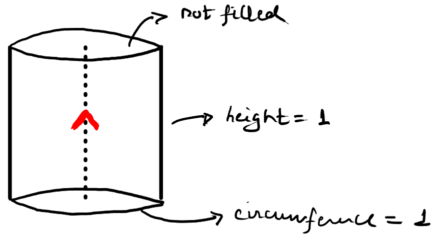
Building Blocks
B^n filled
S^{n-1} unfilled
$I = [0,1]$
$\downarrow$
I^m m-cube
T^m m-torus

* Quotient

* $I^2 = I \times I$



Identify
vertical Edges

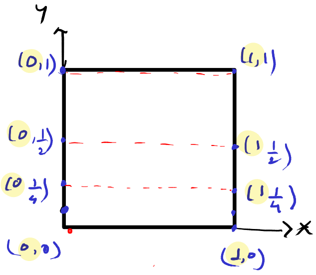


cylinder = $S^1 \times I$

I^2 is divided by its equivalence rel^n ... ~ ... (later)

$$\boxed{\underbrace{S^1 \times I}_{\text{Quotient Space}} = \underbrace{I^2 /}_{\text{Quotient Operator}} (0,y) \underbrace{\sim}_{\text{Identification}} (1,y) \quad \forall y}$$
(cylinder)

Rule : arrows (on Edges) are glued based on equivalence rel^n $\leftrightarrow$

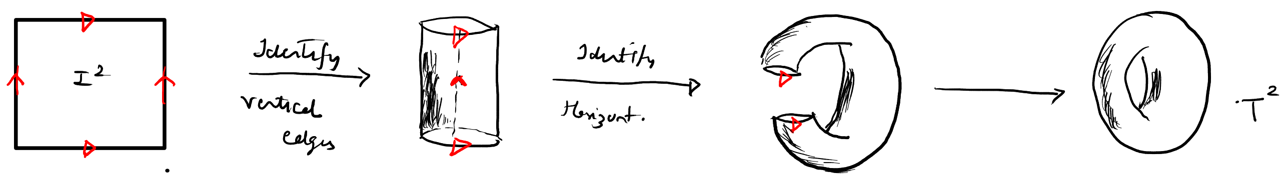


$\boxed{(0,y) \text{ ------- } (1,y) \text{ identified}}$ $\forall y$

$(1,y) \sim (0,y)$ $\boxed{x \sim y}$
$(0,y) \sim (1,y)$ $\boxed{y \sim x}$

" cylinder is the quotient space of the square by the given equivalence rel^n / Identification "
$\leftrightarrow$

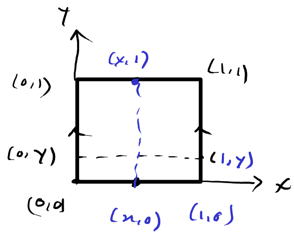
*



$$\boxed{I^2 \Big/ \begin{matrix} (0,y) \sim (1,y) & \forall y \\ (x,0) \sim (x,1) & \forall x \end{matrix} \approx T^2}$$

3D rep.

$T^2 = S^1 \times S^1 \rightarrow$ clifford torus



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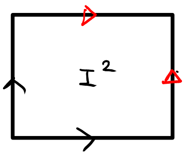
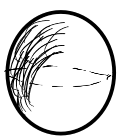
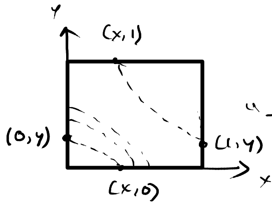


$\longrightarrow$ Hollow patty/ Pocket $\approx S^2$

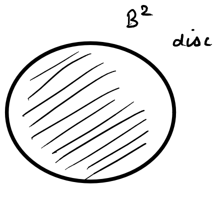


$$\boxed{I^2 \Big/ \begin{matrix} (0,y) \sim (x,0) \\ (1,y) \sim (x,1) \end{matrix} \quad \forall x, y \approx S^2}$$

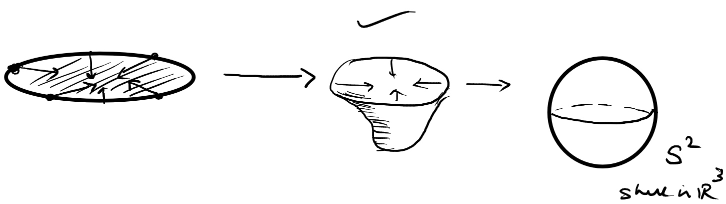


Identification
" Take your arrows Seriously "

*



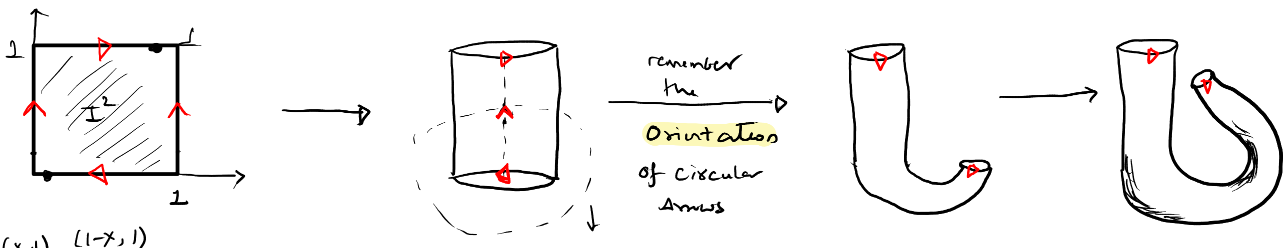
Identify
Antipodal points on $\underbrace{\text{Boundary}}_{\partial B^2}$
(Equivalence rel^n)

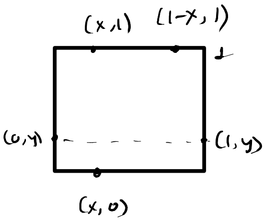


$B^2 /$ all pts on $\partial B^2 \approx S^2$

HW

*





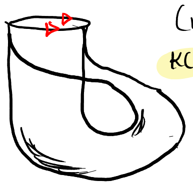
$$I^2 / \begin{matrix} (0,y) \sim (1,y) \quad \forall y \\ (x,0) \sim (1-x,1) \quad \forall x \end{matrix} = \text{Klein Bottle}$$

orientation preserving

orientation reversing

Quotient space of I^2 by $\Leftrightarrow$ (Equivalence) rel^n

Topological materials are Elastic / Penetrable



Self-intersecting obj.

Comment

$$\frac{R_{xyz}}{R_{xy}} = R_z$$

$$\left.\begin{matrix} * \ \underset{\sim}{v}_1 = v_x\hat{i} + v_y\hat{j} + v_z\hat{k} \\ \underset{\sim}{v}_2 = \underset{\sim}{v}_1 + u_x\hat{i} + u_y\hat{j} \end{matrix}\right\}$$

$$\frac{\mathbb{R}^3}{\mathbb{R}^2} = \mathbb{R}^1$$

$$\underset{\sim}{v}_2 - \underset{\sim}{v}_1 = u_x\hat{i} + u_y\hat{j}$$

Quotient Space (Part of the space that is unaltered)

In R^1

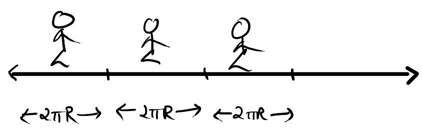
$\underset{\sim}{v}_1 \in \mathbb{R}^3$, $\underset{\sim}{v}_2 \in \mathbb{R}^3$, $\underset{\sim}{v}_1 - \underset{\sim}{v}_2 \in \mathbb{R}^2$

$\boxed{\underset{\sim}{v}_1 = \underset{\sim}{v}_2}$ $\quad v_1 \sim v_2$

γ. Compactification / Alexandroff extensions

1920 (introduced "Compactness")

Copies / Clones of Anubhav....



* Imagine a 1D world

∃ Copies/Clones of scenary after every 2πR dist (Premise)

on a line with special/strange Property

on a circle with circumference 2πR

{ ques: Are there really copies ?

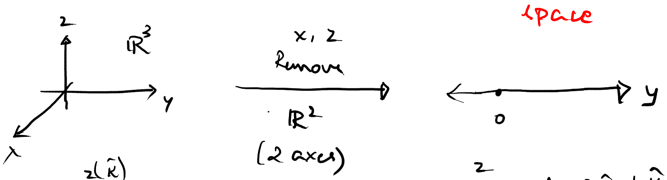
Lecture-19 (9/oct) 2

HW discussion

* $\frac{\mathbb{R}^3}{\mathbb{R}^2} = \mathbb{R}^3 \text{ mod } \mathbb{R}^2 = \mathbb{R}^3 \text{ by } \mathbb{R}^2 \equiv$ Collapse the $\mathbb{R}^2 \to 0$ out of $\mathbb{R}^3$

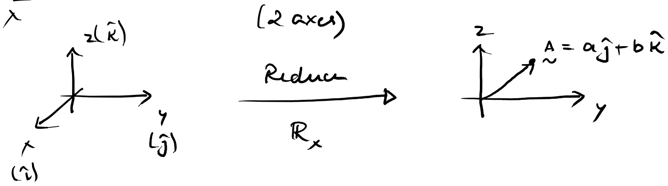
Resulting / Quotient space

ex: $\frac{\mathbb{R}^3}{\mathbb{R}^2} =$



$= \mathbb{R}_y$

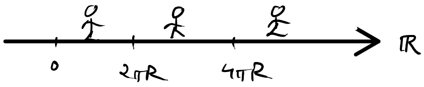
$\frac{\mathbb{R}^3}{\mathbb{R}} =$

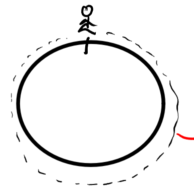


$= \mathbb{R}^2_{yz}$ $\quad \underset{\sim}{A} \in \mathbb{R}^2$

* V = vector space , N = vector subspace ; $\frac{V}{N} = V \text{ mod } N \equiv$ Collapse $N \to 0$ out of V

Contd. (Intuition)

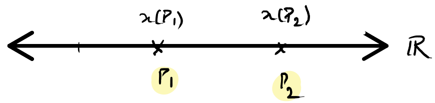
* 

∞ line with strange property $\xrightarrow{\text{Sol}^n}$ 

$\underbrace{\text{with strange property}}_{\exists\ \infty \text{ copies of scenery}}$

$\boxed{\infty \text{ str. line with "strange property"} \approx S^1}$

Math.

* 

$P_1 \sim P_2 \ : \ x(P_2) = x(P_1) + 2\pi R n \quad n \in \mathbb{Z}$

ex: $P_1 = 0$

$x(P_2) = 2\pi R$

$x(P_3) = x(P_2) + 2\pi R(2)$

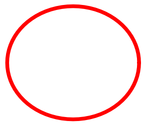
$= 4\pi R$

$\forall$ pt. P $\exists$ $x(P)$

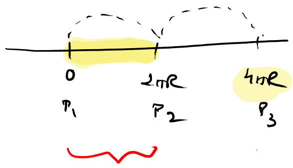
Coordinate (position value)

$\boxed{x \sim x + 2\pi R}$ Identification

$\Downarrow$

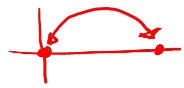


Circle



Fundamental domain (FD) = $\{ x : 0 \le x \le 2\pi R \}$

$\boxed{\text{Space} = \underbrace{\text{F.D} + \text{Boundary}}_{0 \le x \le 2\pi R} + \text{Quotient Identification}}$

"put the Boundary & glue" 

* Aim : To make a non compact space into a compact space

How : $\exists$ ways to control points from "going to ∞"
- by including pt at ∞
- Escape it

via : $\boxed{\text{Alexandroff extension}}$

Compact = $\nexists$ missing End pts / limiting val.

$\mathbb{R} = (-\infty, \infty)$

Not compact

1.1 Projectively extended $\mathbb{R}$ ($\hat{\mathbb{R}}$, $\mathbb{R}^*$)

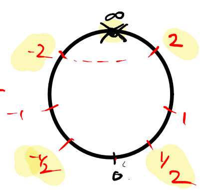
$\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$ ✓ (1 pt compactification of $\mathbb{R}$)

$\uparrow$

∞ is a pt no diff. than other

division by 0 possible $\quad \frac{a}{0} = \infty$
- $\frac{1}{0} = \infty$
- $\frac{1}{\infty} = 0$

0 1, 2 . . . . ∞ ⟶ ℝ

Topo. Circle 

$\boxed{\mathbb{R}^* \approx S^1 \text{ (circle)}}$

0, 1, 2, . . . ∞, 0, 1, 2, . . ∞,

* $[a, \infty] = \{ x : x \in \mathbb{R}, x \ge a \} \cup \{\infty\}$
* $[\infty, \infty] = \{\infty\}$
* $\lim_{x \to \infty^+} f(x) = L \equiv \lim_{x \to -\infty} f(x)$

1.2. Affinely extended $\mathbb{R}$ ($\overline{\mathbb{R}}$)

$\overline{\mathbb{R}} = [-\infty, \infty]$ distinguishable $-\infty, +\infty$ (2 pt. compactification of $\mathbb{R}$)

☆ Imp. Remark : Extension of line to solve problem of Perspective (= 2 ∥ lines do not intersect each other but seems to "at ∞")

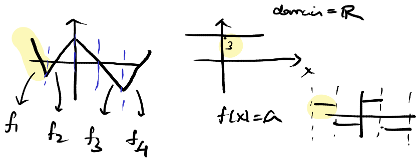
End of Detour to Real Analysis/topology/mfd.

Lecture-20 (13/oct) 2.5

2.1.a. Piecewise linear (PL) / segmented / hybrid machines (contd.)

'defined by cases'

χ = chi



A. Step function

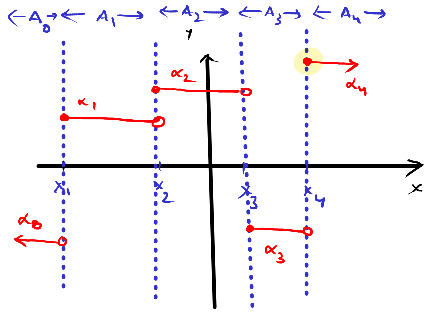
* Piecewise constant function with finitely many intervals.

* $f: \mathbb{R} \to \mathbb{R}$ is step $\Rightarrow$ $\boxed{f(x) = \sum_{i=0}^{n} \alpha_i \chi_{A_i}(x)}$, $n \geq 0$, $\alpha_i \in \mathbb{R}$, A_i interval

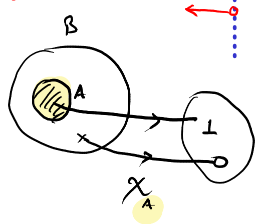
value of the const.

χ_A indicator function / characteristic fun^n

subfun^n (scaling of χ_A)



* $\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$ 'Map from subset to 1 & all other to 0.'



$A = [a,b)$

$\chi_A : x \longrightarrow \{0,1\}$

* A_i interval : 1. $A_i \cap A_j = \phi \quad \forall\, i \neq j$ piecewise disjoint

2. $\bigcup_{i=0}^{n} A_i \equiv A_1 \cup A_2 \cdots \cup A_n = \mathbb{R}$

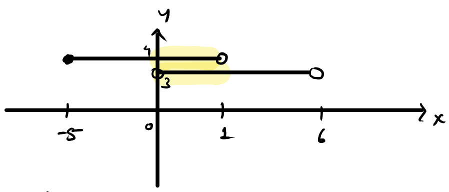
Capital Notation

$\sum_{i=1}, \prod_{i}, \bigcup_{i=1}, \bigcap_{i=1}$

Ex: $f(x) = 4\chi_{[-5,1)} + 3\chi_{(0,6)}$ $\xrightarrow[\text{steps}]{\text{avoid overlapping}}$ $f(x) = 0\chi_{(-\infty,-5)} + 4\chi_{[-5,0]} + 7\chi_{(0,1)} + 3\chi_{[1,6)} + 0\chi_{[6,\infty)}$

$\alpha_1 = 4$

$\alpha_2 = 3$



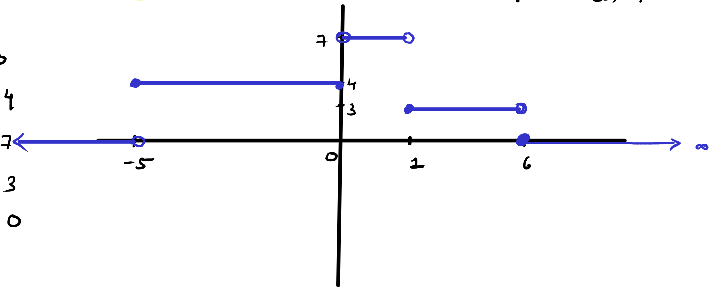
$\alpha_1 = 0$

$\alpha_2 = 4$

$\alpha_3 = 7$

$\alpha_4 = 3$

$\alpha_5 = 0$



$\underset{A_1}{[-5,1)} \cap \underset{A_2}{(0,6)} = (0,1) \neq \phi$

$[-5,1) \cup (0,6) \neq \mathbb{R}$

Criterion not satisfied

$A_i \cap A_j = \phi$, $\bigcup_{i=1} A_i = \mathbb{R}$

$f(x) = 4\begin{cases} 1 & x \in [-5,0] \cup (0,1) \\ 0 & x \notin \quad " \end{cases} + 3\begin{cases} 1 & x \in (0,1) \cup [1,6) \\ 0 & x \notin \quad " \end{cases}$

$f(x) \longrightarrow 4 + 3 = 7 \quad x \in (0,1)$

A.1 Constant step fun^n (Simplest / trivial)

* $f(x) = \sum_{i=1}^{n} \alpha_i \chi_{A_i}(x)$ $\xrightarrow{\#\ \text{intervals} = 1}$ $f(x) = c$, $\chi_{(-\infty,\infty)} = \begin{cases} 1 & x \in (-\infty,\infty) \\ 0 & x \notin (-\infty,\infty) \end{cases}$

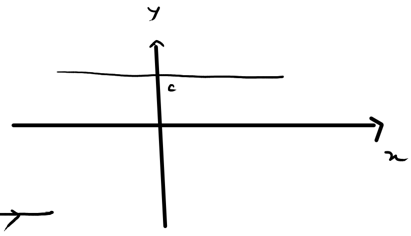
$A_i \cap A_j = \phi$ $\left\{ \begin{array}{l} A_0 = \mathbb{R} \\ \mathbb{R} \cap \phi = \phi \end{array} \right\}$

$\bigcup_{i=1} A_i = \mathbb{R}$

dom = $\mathbb{R}$

Range = c = const

'Continuous' $\longrightarrow$

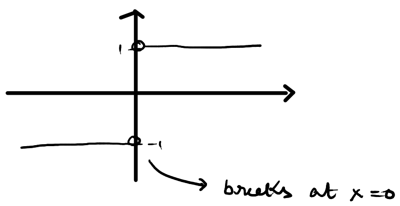


$A \cap \phi = \phi$

A.2 Signum / Sign function

* Returns the sign of a $\mathbb{R}$

* $\text{Sgn}(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ +1 & x > 0 \end{cases}$ $\text{sgn}(0) = 0$



* dom Sgn(x) = $\mathbb{R}$

Range sgn(x) = $\{-1, 0, 1\}$

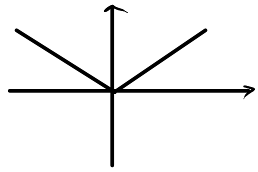
'non Continuous'

* $\boxed{x = |x|\,\text{sgn}(x) \quad \Leftrightarrow \quad x \in \mathbb{R}}$

Any real # can be expressed as pdt of its Absolute value & its sign

$$-\pi = |-\pi| \times (-1) = \pi \times (-1)$$

'Number theory Result'



$$\frac{|x|}{x} = \begin{cases} \frac{x}{x} = 1 & x > 0 \\ \frac{-x}{x} = -1 & x < 0 \end{cases}$$

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Modulus

Continuous

* $$\boxed{\text{sgn}(x) = \begin{cases} \frac{|x|}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} \qquad \left(\frac{0}{0} \neq 0\right)}$$

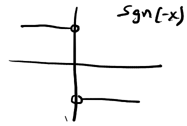
* $$\text{sgn}(x) = \frac{|x|}{x} = \frac{x}{|x|}$$

Lecture-21 (14/oct) 2

* $$\text{sgn}(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases} = \begin{cases} \frac{|x|}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$\downarrow$

$$\text{sgn}(-x) = \begin{cases} -1 & -x < 0 \\ 0 & -x = 0 \\ 1 & -x > 0 \end{cases} = \begin{cases} -1 & x > 0 \\ 0 & x = 0 \\ 1 & x < 0 \end{cases} = -\begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases} = -\text{sgn}(x)$$



$$\text{sgn}(-x) = \begin{cases} \frac{|-x|}{-x} & -x \neq 0 \\ 0 & -x = 0 \end{cases} = \begin{cases} \frac{|x|}{-x} & x \neq 0 \\ 0 & x = 0 \end{cases} = -\begin{cases} \frac{|x|}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} = -\text{sgn}(x)$$

$\boxed{\text{sgn}(-x) = -\text{sgn}(x)}$ $\Rightarrow$ Odd function

"Cool" properties (Heavily used in Adv. Phy. — Decomposition, Limit, Differentiation)

$$\text{sgn}(x) = \begin{cases} \frac{|x|}{x} \\ 0 \end{cases} = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$$

* $x = |x|\,\text{sgn}(x) \qquad x \in \mathbb{R}$

* $\text{Sgn}(x) = 2\underbrace{H(x)}_{\text{Heaviside func}} - 1 \longrightarrow \text{sgn}(0) = 0 = 2H(0) - 1 \Rightarrow \boxed{H(0) = \frac{1}{2}}$

* $$\text{sgn}(x) = \left\lfloor \frac{x}{|x|+1} \right\rfloor - \left\lfloor \frac{-x}{|x|+1} \right\rfloor$$

'floor & Mod. decomp.'

$\hookrightarrow$ Floor fun^n

* $$\text{sgn}\, x = \lim_{n \to \infty} \frac{1 - 2^{-nx}}{1 + 2^{-nx}}$$

$$\text{sgn}\, x = \lim_{n \to \infty} \frac{2}{\pi} \tan^{-1} nx$$

$$\text{sgn}\, x = \lim_{n \to \infty} \tanh nx$$

✓ check : Heaviside function

* $\text{sgn}\, x \approx \tanh kx \qquad k > 1$ ✓

$$\text{sgn}\, x \approx \frac{x}{\sqrt{x^2 + \epsilon^2}} \qquad \epsilon : \text{parameter}$$

* $\text{sgn}\, x = 2H(x) - 1 \longrightarrow \frac{d}{dx}\text{sgn}\, x = 2\frac{d}{dx}H(x) = 2\,\delta(x)$

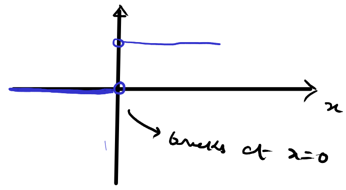
dirac delta function

4.3. Heaviside step fun^n ($H(x)$, $\Theta(x)$) (unit step/theta $func^n$)

Oliver Heaviside (1860s) / rewrote Maxwell eq^n as its present form

Tool : vector calculus / self taught

* $H(x) = \Theta(x) = \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases}$ $= \underbrace{\Theta(x=0)}$ centered at

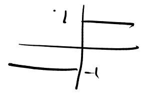


* $H(x) = \frac{1}{2}(1+\text{sgn}(x))$; $H(0) = \frac{1}{2}(1+\cancel{\text{sgn}(0)}) = \frac{1}{2}$ Zero Argument

$\exists$ Rotational symmetry

$H(0)=1/2$, $H(0)=1$, $H(0)=0$ } choices of particular value

* $\underset{\text{odd}}{H(x)-\frac{1}{2}} = \frac{1}{2}\underset{\text{odd}}{\text{sgn}(x)} \Rightarrow 2(H(x)-\frac{1}{2}) = \text{sgn}\,x$



$\text{sgn}(0) = 0$

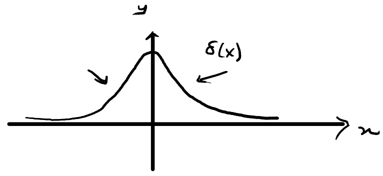
$\frac{1}{2}\underbrace{\text{sgn}(-x)}_{-\text{sgn}\,x \text{ odd}} = (\underbrace{H(-x)}-\frac{1}{2})$ ....

$\boxed{(H(x)-\frac{1}{2}) \text{ is odd } fun^n}$

* $\boxed{\frac{d}{dx}\Theta(x) = \delta(x)}$

Comment on dirac delta (distribution) / Unit Impulse

* to describe a qty. that is 0 everywhere except a $\underbrace{\text{single pt.}}$

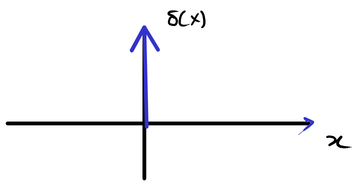


value at this pt is ∞ : $\underbrace{\text{Area under the curve}}$ = finite

* $\delta(x)$: $\delta(x)=0$ $x \neq 0$; Area under $\delta(x) = 1$ } Properties

$\boxed{\delta(x) \simeq \begin{cases} \infty & x=0 \\ 0 & x\neq 0 \end{cases}}$ piecewise

Approximation of the distribution



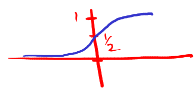
Analytic Approximations

* $H(x) = \Theta(x) \approx \frac{1+\tanh kx}{2} = \underbrace{\frac{1}{1+e^{-2kx}}}$ $H(0) = \frac{1}{2}$

$\tanh x = \frac{e^x - e^{-x}}{e^x+e^{-x}} = \frac{1-e^{-2x}}{1+e^{-2x}}$ +1

$1+\tanh x = \frac{2}{1+e^{-2x}}$

Logistic function / curve / S shaped curve / Sigmoid function



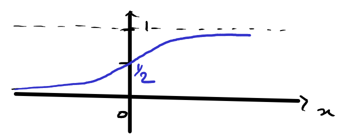
Lecture-22 (15/oct) 2

Comment on logistic function

* $f(x) = \frac{L}{1+e^{-k(x-x_0)}}$ logistic function / S shaped curve

(Pierre François, 1840s) App./origin : Population dynamics / growth

* $L=1$, $k=1$, $x_0=0$ (simplest case) $\longrightarrow$ $f(x) = \frac{1}{1+e^{-x}} \equiv \sigma(x)$ sigmoid function



Properties

* $f(x) = \frac{1}{1+e^{-x}} = \frac{e^x}{1+e^x}$ (1)

$\lim_{x\to\infty} \sigma(x) = \frac{1}{1+0} = 1$

$\lim_{x\to-\infty} \sigma(x) = \frac{1}{1+\infty} = 0$

* $1 - f(x) = 1 - \frac{e^x}{1+e^x} = \underbrace{\frac{1}{1+e^x}}_{f(-x)} \Rightarrow \boxed{1-f(x) = f(-x)}$ (2)

$f(x) = 1 - f(-x)$

$\lim_{x\to 0} \sigma(x) = \frac{1}{1+1} = \frac{1}{2}$

* $T: x \to -x \Rightarrow f(x)$ doesn't care $f(-x) \neq \pm f(x)$

$f(-x) = -f(x) + 1$ can't tell

"odd"

$$\overbrace{\underbrace{f(x)-\tfrac12}_{1-f(-x)}}^{g(x)} = 1 - f(-x) - \tfrac12 = -f(-x) + \tfrac12 = -\underbrace{\left(f(-x)-\tfrac12\right)}_{g(-x)} \longrightarrow \left(f(x)-\tfrac12\right) \text{ is odd function} \quad (3)$$

$g(-x) \Rightarrow g(-x) = -g(x)$

* $f(x) = \frac{1}{1+e^{-x}} = \frac{e^x}{1+e^x}$

$\sinh x = \frac{e^x - e^{-x}}{2}$

$\cosh x = \frac{e^x + e^{-x}}{2}$

$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\cancel{e^x}(1-e^{-2x})}{\cancel{e^x}(1+e^{-2x})} = \frac{1-e^{-2x}}{1+e^{-2x}}$

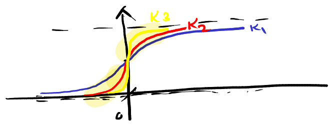
$\tanh x + 1 = \frac{2}{1+e^{-2x}} \Rightarrow \frac12(1+\tanh x) = \underbrace{\frac{1}{1+e^{-2x}}}_{f(2x)} \Rightarrow f(2x) = \frac12(1+\tanh x)$

$\Downarrow$

$f(x) = \frac12\left(1+\tanh\frac{x}{2}\right)$

$$\boxed{\begin{aligned} \tfrac12(1+\tanh x) &= \frac{1}{1+e^{-2x}} \\ \tfrac12\left(1+\tanh\tfrac{x}{2}\right) &= \frac{1}{1+e^{-x}} = \frac{e^x}{1+e^x} \end{aligned}}$$

(4)

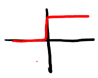


$\frac{1}{1+e^{-kx}}$

$\downarrow k=1$

Sigmoid

* $\frac12(1+\tanh x) = \frac{1}{1+e^{-2x}} \approx H(x)$



by nature

Piecewise

but

$$\boxed{\lim_{k\to\infty} \tfrac12(1+\tanh kx) = \lim_{k\to\infty} \frac{1}{1+e^{-2kx}} = H(x)}$$

$\exists$ Smooth Analytic approximation to Step function / Heaviside

* $f(x) = \frac{1}{1+e^{-x}} = \underbrace{(1+e^{-x})}^{-1}$; $f(1+e^{-x}) = 1 \Rightarrow f + fe^{-x} = 1 \Rightarrow e^{-x} = \frac{1-f}{f}$

$\frac{d}{dx} f(x) = -1\,(1+e^{-x})^{-2} \cdot e^{-x} \cdot (-1) = \frac{e^{-x}}{(1+e^{-x})^2} = \underbrace{\frac{1}{(1+e^{-x})^2}}_{f^2} \cdot e^{-x} = f^2\left(\frac{1-f}{\cancel{f}}\right) = f(1-f)$

$$\boxed{\frac{d}{dx} f(x) = f(x)\,(1-f(x))}$$

1st order Non linear ODE $\longrightarrow$ $f(x) =$ Logistic function

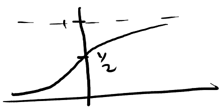
Logistic Diff. Eqn

Solution

* $f(x) + f(\underbrace{-x}) = 1$ $\Rightarrow$ $\exists$ Rotational symmetry about $(0, \frac12)$

reflexion about vertical axis

$1-f(x)$



* $\theta(x) = \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases} = \frac{1}{2}(1+\mathrm{sgn}(x)) = \lim_{k\to\infty} \underbrace{\frac{1}{2}(1+\tanh kx)}_{\frac{1}{1+e^{-2kx}}} \Rightarrow \boxed{\lim_{k\to\infty} \tanh kx = \mathrm{sgn}\, x}$

Choice : $\theta(0) = 1/2$

logistic fun^n

$\mathrm{sgn}\, x \simeq \tanh kx$

Discrete form

* $\Theta : \mathbb{Z} \to \mathbb{R}$

$\Theta[n] = \begin{cases} 1 & n>0 \\ 0 & n<0 \end{cases} = \begin{cases} 1 & n>0 \\ \frac{1}{2} & n=0 \\ 0 & n<0 \end{cases} \quad n \in \mathbb{Z}$

* $\Theta[n] = \sum_{k=-\infty}^{n} \delta[k]$; $\delta[k] \equiv \delta_{k,0} = \begin{cases} 1 & k=0 \\ 0 & k\neq 0 \end{cases}$ discrete unit impulse function

$\delta_{a,b} = \delta_{ab} = \begin{cases} 1 & a=b \\ 0 & a\neq b \end{cases}$ Kronecker delta

$n=5$

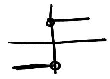
$\Theta[5] = \sum_{k=-\infty}^{5} \delta[k] = \underbrace{\delta[-\infty]}_{\delta_{-\infty,0}=0} + \dots + \underbrace{\delta[0]}_{\delta_{0,0}=1} + \underbrace{\delta[1]}_{\delta_{1,0}=0} + \dots + \underbrace{\delta[5]}_{\delta_{5,0}=0} = 1$

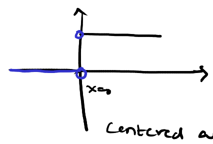
* $\Theta(x) = \frac{x+|x|}{2x}$; $|x| = \begin{cases} x & x\geq 0 \\ -x & x<0 \end{cases} \Rightarrow x+|x| = \begin{cases} 2x & x\geq 0 \\ 0 & x<0 \end{cases} \Rightarrow \frac{x+|x|}{2x} = \begin{cases} 1 & x\geq 0 \\ 0 & x<0 \end{cases} = \theta(x)$

Lecture-23 (16/oct) 2

* $\theta(x) = \theta(\underbrace{x-0}_{\text{centered at}}) = \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases} = \theta(x-0)$

$\mathrm{sgn}\, x = \begin{cases} \frac{|x|}{x} & x\neq 0 \\ 0 & x=0 \end{cases}$





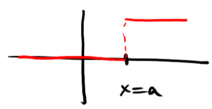
$\theta(0)$

$\theta(x) = \frac{1}{2}(1+\mathrm{sgn}\, x)$: $\theta(0) = \frac{1}{2} \to \theta - \frac{1}{2} = \frac{1}{2}\mathrm{sgn}\, x$

$\mathrm{sgn}(-x) = -\mathrm{sgn}\, x$ odd fun^n

* $\theta(x-a) = \begin{cases} 1 & x>a \\ 0 & x<a \end{cases}$ Shifted Heaviside / Theta fun^n

centered at $x=a$



$x-a=0$

* $\theta(x) = \frac{1}{2}(1+\mathrm{sgn}\, x)$; $\theta(-x) = \frac{1}{2}(1+\overbrace{\mathrm{sgn}(-x)}^{-\mathrm{sgn}\, x})$

$\frac{d}{dx}\theta(x) = \frac{1}{2}\frac{d}{dx}\mathrm{sgn}\, x = \delta(x)$; $\frac{d}{dx}\theta(-x) = -\frac{1}{2}\frac{d}{dx}\mathrm{sgn}\, x = -\frac{1}{2} 2\delta(x) \Rightarrow \boxed{\frac{d}{dx}\theta(-x) = -\delta(x)}$

$\frac{d}{dx}\theta(x) = \delta(x)$

$\delta(x) \simeq \begin{cases} \infty & x=0 \\ 0 & x\neq 0 \end{cases}$

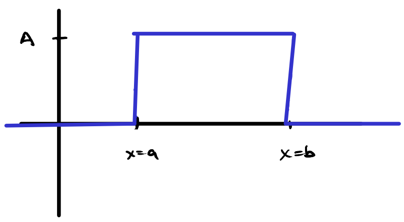
$\delta(x-a) \simeq \begin{cases} \infty & x=a \\ 0 & x\neq a \end{cases}$

A.4 Boxcar function

* $\mathrm{boxcar}(x) = A\{\underbrace{\theta(x-a)}_{} - \underbrace{\theta(x-b)}_{}\}$; $x \in [a,b]$

1 if $x>a$ 1 $x>b$

0 if $x<a$ 0 $x<b$



$\max\{\mathrm{boxcar}(x)\} = A$

$\mathrm{boxcar}(x) = \begin{cases} 0-0 & x<a,\ x<b \\ A(1-0)=A & a<x<b \\ 1-1=0 & x>b,\ x>a \end{cases}$

Appl. : It is a filter to filter out an interval

A.S. Rectangular / Pi / unit pulse / Normalized boxcar

* $\text{Rect}\left(\frac{x}{a}\right) = \Pi\left(\frac{x}{a}\right) = \begin{cases} 0 & |x| > \frac{a}{2} \\ \frac{1}{2} & |x| = \frac{a}{2} \\ 1 & |x| < \frac{a}{2} \end{cases}$

$a = 1$

* $\text{Rect}(x) = \Pi(x) = \begin{cases} 0 & |x| > \frac{1}{2} \\ \frac{1}{2} & |x| = \frac{1}{2} \\ 1 & |x| < \frac{1}{2} \end{cases}$

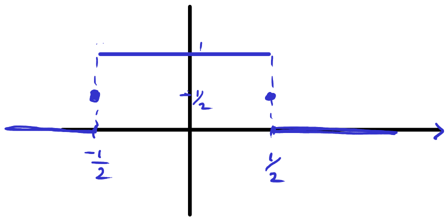
Line of flight

$|x| > \frac{1}{2} \rightarrow x > \frac{1}{2}$ and $x < -\frac{1}{2}$

$|x| = \frac{1}{2} \rightarrow \begin{cases} x = \frac{1}{2} & x \geq 0 \\ x = -\frac{1}{2} & x < 0 \end{cases}$

$|x| < \frac{1}{2} \rightarrow x > -\frac{1}{2}$ and $x < \frac{1}{2}$

$\Pi\left(x < -\frac{1}{2}\right) = 0$



Comment on Mod:

* $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$

* $|x| < 1 \longrightarrow \begin{cases} x \leq 1 & x \geq 0 \\ -x \leq 1 \\ x \geq -1 & x < 0 \end{cases}$

Compact

$x \leq 1$ and $x \geq -1 \Rightarrow x \in [-1, 1]$

Sol^n $x = 1$ point

Analysis $x \geq 1$ line of flight

$(1, \infty)$

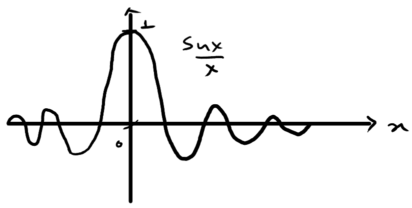
* $|x| > 1 \longrightarrow \begin{cases} x > 1 & x \geq 0 \\ -x > 1 & x < 0 \\ x < -1 \end{cases}$

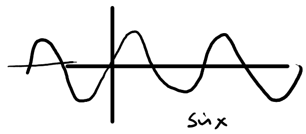
not compact

$x > 1$ and $x < -1 \Rightarrow x \in (-\infty, -1) \cup (1, \infty)$

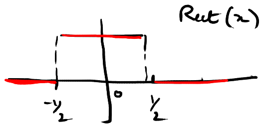
Comment on Sinc Function

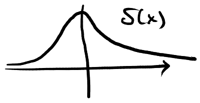
* $\text{Sinc}(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$





* $\boxed{\lim_{a \to 0} \frac{1}{a} \underbrace{\frac{\sin x/a}{x/a}}_{\text{Sinc}\left(\frac{x}{a}\right)} = \delta(x)}$





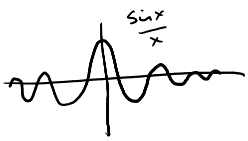
* $\text{Sinc}(-x) = \begin{cases} \frac{\sin(-x)}{-x} = \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases} = \text{Sinc}(x)$ Even fun^n

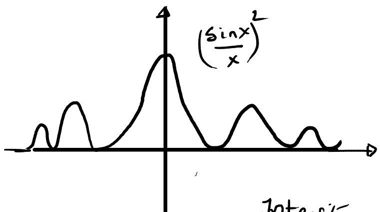
* $\lim_{x \to 0} \frac{\sin x}{x} \xrightarrow[\text{Bernoulli}]{\text{L'Hôpital}} \lim_{x \to 0} \frac{\cos x}{1} = 1$ at $x = 0$ $\text{Sinc}\,x = 1$

$\left(\frac{0}{0}\right)$
'Indeterminate'

* $\text{Rect}(-x) = \begin{cases} 0 & \underbrace{|-x|}_{|x|} > \frac{1}{2} \\ \frac{1}{2} & |-x| = \frac{1}{2} \\ 1 & |-x| < \frac{1}{2} \end{cases} = \text{Rect}(x)$ Even func^n

* $f(x) = \left(\frac{\sin x}{x}\right)^2$ Even fun^n





Intensity $\sim \left(\frac{\sin x}{x}\right)^2$

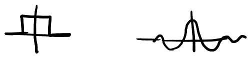
"Kul"

* $\lim_{a \to \infty} \frac{1}{a} \text{Rect}\left(\frac{x}{a}\right) = \delta(x)$; $\lim_{a \to 0} \frac{1}{a} \text{Sinc}\,x = \delta(x)$

$\left(\frac{1}{x} = \infty\right)$ ⟵ - - - - - - ⟶ $(x = 0)$

there is an interplay happening b/w Rect & Sinc

* $\boxed{\text{Fourier transform}\left(\text{Rect}(x)\right) = \text{Sinc}(p)}$



$x \leftrightarrow t$
$x \leftrightarrow p$
$t \leftrightarrow E$
$x \leftrightarrow \vartheta$

Lecture-24 (20/oct) 2

* Step. step = step , step + step = step , α (step) = step.

B. Absolute / Modulus Machine

B.1. Terminological Clarity

* $\underbrace{\text{Modulus}}_{\text{'' unit of measure}}$ function / $\overbrace{\underbrace{\text{Absolute value}}_{|-4| = 4}}^{|x|}$ function (Jean Argand, 1806) abs(x) , |x|

(Math-12)

* $\underbrace{\text{Modulo}}_{\text{'' small measure}}$ operator → Modulo (Gauss, 1801) → Up to / → Congruent to / → Except for

{ Set theory + # theory }

Ex:

1. aRb "a & b are equal up to a relation R"
2. $\int \sin x \, dx = -\cos x + C$ upto an add^n of a constant

"A-B is divisible by C"

* $A \equiv B \pmod C$ "A is the same as B modulo C"
" " " " " " upto C"
" A & B are the same except for the differences accounted for / explained by C"

$\boxed{A \equiv B \pmod C \Leftrightarrow A - B = nC} \rightarrow \frac{A-B}{C} = n$

Ex:

$5 \equiv 1 \pmod 2 \Leftrightarrow 5 - 1 = 4 = 2 \times 2$

$38 \equiv 14 \pmod{12} \Leftrightarrow 38 - 14 = 24 = \underbrace{2}_{n} \times 12$

$25 \equiv 2 \pmod 4 \Leftrightarrow 25 - 2 = 23 = \underbrace{n}_{?} \times 4 \Rightarrow \nexists\, n : n \in \mathbb{Z} \Rightarrow 25 \not\equiv 2 \pmod 4$

$18 \equiv 3 \pmod 5 \Leftrightarrow 18 - 3 = 15 = \underbrace{3}_{n} \times 5$

$13 \equiv 1 \pmod{12} \Leftrightarrow 13 - 1 = 12 = \underbrace{1}_{n} \times 12$

Modulo in Computer Science:

$2\overline{)5}(2$
$\underline{4}$
1

$5 = 2 \times 2 + 1$
⇕
$5 \bmod 2 = 1$ — Remainder

least +ve residue
''
smallest non-ve integer = (Euclid division lemma)

$5 \bmod 2 \equiv \text{Remaind}\left(\frac{5}{2}\right) = 1$

$9 \bmod 3 \equiv \text{Rem.}\left(\frac{9}{3}\right) = 0$

$B \bmod C \equiv \text{Remain}\left(\frac{B}{C}\right)$

Surprise ques:

* Rel^n R on the set $\mathbb{Z}$ is defined as

$R = \{ (x, y) : x - y \text{ is divisible by } n \}$ check if R is

1. Reflexive $(x,x) \in R \ \forall x \in \mathbb{Z}$ — $A = A$
+
2. Symmetric $(x,y) \in R \Rightarrow (y,x) \in R$ "Commutative"
+
3. Transitive $(x,y) \in R, (y,z) \in R \Rightarrow (x,z) \in R$

Proof:

1. if $(x,x) \in R \ \forall x \in \mathbb{Z} \Rightarrow$ Reflexive

then

$x - x = 0 = 0 \times n \Rightarrow x - x$ is divisible by $n \Rightarrow (x,x) \in R \Rightarrow R$ is ref. on $\mathbb{Z}$

2. if $x - y$ is divisible by $n \Leftrightarrow (x,y) \in R$

$x - y = nc \quad c \in \mathbb{Z} \Rightarrow (y - x) = n\underbrace{(-c)}_{d} = nd \quad d \in \mathbb{Z} \Rightarrow (y,x) \in R \Rightarrow R$ is sym. on $\mathbb{Z}$

3. if $(x,y) \in R \Rightarrow x - y$ is divisible by $R \Rightarrow x - y = nd$
& if $(y,z) \in R \Rightarrow y - z$ " " " " $\Rightarrow y - z = ne$
$\Rightarrow x - z = n\overbrace{(d+e)}^{f} = nf \Rightarrow (x,z) \in R \Rightarrow R$ is tran on $\mathbb{Z}$

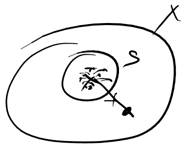
* $\boxed{\text{Equivalence Relation} = \text{Reflexive} + \text{Sym.} + \text{Transitive}}$

* $R = \{ (x,y) : \underbrace{x \equiv y \pmod n}_{x-y \text{ is divisible by } n} \}$ — 2 integer x, y are in Congruence Modulo n

* $\equiv$, $\sim$ Equivalence Relation

$a \equiv b$, $a \sim b$

aRb (Abusive notation)



Homeomorphism

$S^1 \approx I^2 \approx \partial B^2$

* $\{ \text{Set } S, \sim \}$ given $\longrightarrow [a] = \{ x \in S : x \sim a \}$ Equivalence class ✓

* $\{ \text{Set } S, \sim, [a] \}$ given $\longrightarrow S/\sim := \{ [x] : x \in S \}$ Quotient set of S by $\sim$

geometric leap!!

Quotient space of I^2 by $\sim$ Equivalence Rel^n

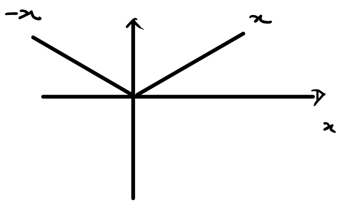
ex: $I^2 / \underbrace{(0,y) \sim (1,y) \ \forall y, \ (x,0) \sim (x,1) \ \forall x} = T^2$

$\square$

Lecture-25 (21/oct) 1.45

B.2 Absolute /Modulus function

* $|x| = \text{abs}(x) = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$



(Gauss 1801)

Modular Arithmetic

$a = b + nc$

$a \equiv b \pmod c \iff a - b = nc$

$a \equiv b \pmod 2 \iff a - b = 2n$

$\frac{a-b}{n} = 2$

$a = b + 2n$

$5\overline{)20}(4$
20
$\overline{0}$

$20 = 5 \times 4$

$21 = 1 + 5 \cdot 4$

Euclid's division Lemma

Modulo = Equivalence rel^n

Modulus = function

$a, b, c \in \mathbb{R}$

* $|a| \geq 0$ Non-negativity

* $|a| = 0 \Rightarrow a = 0$ +ve definiteness

* $|ab| = |a||b|$

$|a+b| \leq |a| + |b|$ Triangle inequality

* $||a|| = |a|$ Idempotence

* $|-a| = |a|$ Evenness.

* $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

S , $x \in S$, $\cdot$ (Binary op.)

$x \cdot x = x$ (op. is Idempotent)

Imp. properties , $a, b \in \mathbb{R}$

* $\boxed{|x| = +\sqrt{x^2}}$

$x = |x| \, \text{sgn}(x)$

$-2 = |2| (-1)$

ex:

$\sqrt{\cos^2 x} = |\cos x| = \begin{cases} \cos x & 0 < x \leq \frac{\pi}{2} \\ -\cos x & \frac{\pi}{2} < x \leq \pi \end{cases}$

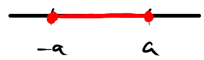
* $\boxed{|x| \leq a \Rightarrow -a \leq x \leq a}$ I. Most Imp. results for Modulus inequality

check:

* $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$

* $x \geq 0 \longrightarrow |x| \leq a \Rightarrow x \leq a$
$x < 0 \longrightarrow |x| \leq a \Rightarrow -x \leq a \Rightarrow x \geq -a$

$\Bigg\} \Rightarrow -a \leq x \leq a \Rightarrow x \in [-a, a]$



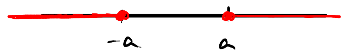
* $\boxed{|x| \geq a \Rightarrow x \geq a \text{ or } x \leq -a}$ II

check

* $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$

* $x \geq 0 \longrightarrow |x| \geq a \Rightarrow x \geq a$
$x < 0 \longrightarrow |x| \geq a \Rightarrow -x \geq a \Rightarrow x \leq -a$

$\Bigg\} \Rightarrow x \geq a$ or $x \leq -a$



* $\boxed{|x-b| \leq a \Rightarrow x \in [b-a, b+a]}$

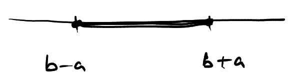
$f(x) = |x|$
$\downarrow x \to x-b$
$f(x-b) = |x-b|$

check 1 (1st Prin.)

* $|x-b| = \begin{cases} x-b & x-b \geq 0 \\ -(x-b) & x-b < 0 \end{cases}$

* $x-b \geq 0 \Rightarrow x \geq b \longrightarrow |x-b| \leq a \Rightarrow x-b \leq a \Rightarrow x \leq a+b$
$x-b < 0 \Rightarrow x < b \longrightarrow |x-b| \leq a \Rightarrow -(x-b) \leq a \Rightarrow x-b \geq -a \Rightarrow x \geq b-a$

$\Bigg\} \Rightarrow$



check 2

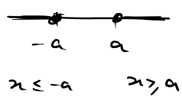
* $|x-b| \leq a \overset{\text{via I.}}{\Rightarrow} -a \leq x-b \leq a \Rightarrow b-a \leq x \leq b+a$

* $\boxed{|x-b| \geq a \Rightarrow x \leq b-a \text{ or } x \geq b+a}$

Check

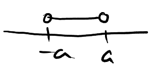
* $x-b \leq -a$ or $x-b \geq a$
$x \leq b-a$ $\quad$ $x \geq b+a$

$|x| \geq a \Rightarrow x \notin (-a, a)$
$x \in [-a, a]^c$

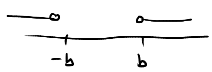


More interesting results (derivable)

* $|x| \geq a \Rightarrow -a \leq x \leq a$



$|x| \leq b \Rightarrow x \leq -b$ or $x \geq b$



$x \in (-\infty, -b] \cup [b, \infty)$

1. $a \leq |x| \leq b \Rightarrow x \in [a, b] \cup [-b, -a]$

Check 1

* $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$

$x \geq 0 \longrightarrow a \leq |x| \leq b \Rightarrow a \leq x \leq b \Rightarrow x \in [a, b]$
$x < 0 \longrightarrow a \leq |x| \leq b \Rightarrow a \leq -x \leq b \Rightarrow -a \geq x \geq -b \Rightarrow x \in [-b, -a]$

$\Bigg\} \Rightarrow x \in [a, b] \cup [-b, -a]$

check 2

* $a \leq |x| \leq b \Rightarrow |x| \geq a$ and $|x| \leq b \Rightarrow (x \geq a$ or $x \leq -a)$ and $(-b \leq x \leq b) \Rightarrow a \leq x \leq b$ and $-b \leq x \leq -a$
$\Downarrow$
$x \in [a, b] \cup [-b, -a]$

2. $a \leq |x-c| \leq b \Rightarrow |x-c| \geq a$ and $|x-c| \leq b \Rightarrow x \in [a+c, b+c] \cup [c-b, c-a]$

HW

* $|3x-2| \leq \frac{1}{2}$ $\qquad |x| \geq a, \ |x| \leq a$

* $|x-2| \geq 5$

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Practice 2 (Modulus fun + Inequalities) : $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$ only pre requisite

Q1 $|x-2| \geq 5$

unconventional/convinient notation

$|x-2| \geq 5 \Rightarrow \begin{cases} (x-2) \geq 5 \Rightarrow x \geq 7 \\ -(x-2) \geq 5 \Rightarrow x-2 \leq -5 \Rightarrow x \leq -3 \end{cases} \Rightarrow x \in (-\infty, -3] \cup [7, \infty)$

given

$|x| \leq a \Rightarrow -a \leq x \leq a$

$|x| \geq a \Rightarrow x \leq -a$ or $x \geq a$

Q2 $1 \leq |x-2| \leq 3$ doubly bounded Modulus $\Rightarrow x \in$?

$1 \leq |x-2| \leq 3 \Rightarrow \begin{cases} 1 \leq (x-2) \leq 3 \Rightarrow 3 \leq x \leq 5 \\ 1 \leq -(x-2) \leq 3 \Rightarrow -1 \leq x \leq 1 \end{cases} \Rightarrow x \in [-1, 1] \cup [3, 5]$

Before (vol 1 inequality)

$\left.\begin{matrix} x-1 \geq 5 \\ x \geq 2 \end{matrix}\right\}$ sys.

Q3 $|\alpha-1| \leq 5$, $|\alpha| \geq 2$ system of Inequalities

$\downarrow \qquad \downarrow$

Analysis 1 $\quad$ Analysis 2 $\longrightarrow$?

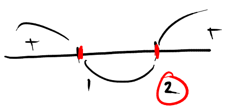
$\alpha \in [-4, -2] \cup [2, 6]$

Q4 $\dfrac{|\psi| - 1}{|\psi| - 2} \geq 0$, $\psi \in \mathbb{R}$, $\underbrace{\psi \neq \pm 2}$

Hint : if no Modulus

$\underbrace{\dfrac{x-1}{x-2}} \geq 0 \iff \boxed{f(x) \geq 0}$

$f(x) = \dfrac{x-1}{x-2}$ $\qquad$ C.P : $\left.\begin{matrix} x-1=0 \\ x-2=0 \end{matrix}\right\} \Rightarrow x_{1,2} = 1, 2$



$f(x>2) = +$

$f(1<x<2) = -$

$f(x<1) = +$

$\xrightarrow[x \neq 2]{\because f(x) \geq 0}$ $x \in (-\infty, 1] \cup (2, \infty)$

$\uparrow$

$f(x) = 0$

Q5 $f(|\psi|) = \dfrac{|\psi|-1}{|\psi|-2}$ $\qquad \boxed{|\psi| \equiv z}$ trick

$\downarrow$

$f(z) = \dfrac{z-1}{z-2} \xrightarrow{f(z) \geq 0} \begin{cases} z \leq 1 \\ z > 2 \end{cases} \xrightarrow{z = |\psi|}$ $\left.\begin{matrix} |\psi| \leq 1 \Rightarrow \psi \in [-1, 1] \\ |\psi| > 2 \Rightarrow \psi \in (-\infty, -2) \cup (2, \infty) \end{matrix}\right\} \Rightarrow \psi \in [-1, 1] \cup (-\infty, -2) \cup (2, \infty)$

Q6 $\dfrac{-1}{|\varphi|-2} \geq 1$

* Standard form : $\boxed{f(x) \lesseqgtr 0}$ $\Rightarrow$ Applicability of wavy curve.

* Non-std. form : $f(x) \gtreqless a \Rightarrow \underbrace{f(x) - a}_{g(x)} \gtreqless 0 \Rightarrow g(x) \gtreqless 0 \Rightarrow$ Applicability of wavy curve

More annoying !!

$f(x) > a \Rightarrow \dfrac{f(x)}{a} > 1$

$\dfrac{f(x)}{a} - 1 > 0$

$\dfrac{f(x) - a}{a} > 0 \Rightarrow f(x) - a > 0$

Do Not divide

* $\dfrac{-1}{|\varphi|-2} - 1 \geq 0 \Rightarrow \dfrac{-1-(|\varphi|-2)}{|\varphi|-2} \geq 0 \Rightarrow \dfrac{1-|\varphi|}{|\varphi|-2} \geq 0 \Rightarrow \dfrac{|\varphi|-1}{|\varphi|-2} \leq 0$

$z \equiv |\varphi|$

$\dfrac{z-1}{z-2} \leq 0 \Rightarrow \varphi \in (-2,-1] \cup [1,2)$

Q7 $\left|\dfrac{2}{\alpha-4}\right| > 1$, $\alpha \neq 4$

$\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$

$\dfrac{2}{|\alpha-4|} > 1 \Rightarrow 2 > |\alpha-4| \Rightarrow |\alpha-4| < 2 \rightarrow \alpha \in (2,6)$ $\alpha \neq 4$.

HW

* $\dfrac{|x+3|+x}{x+2} > 1$

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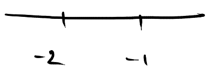
Q8. $\dfrac{|x+3|+x}{x+2} > 1$

* $\underbrace{\dfrac{|x+3|-2}{x+2}}_{f(x)} > 0$

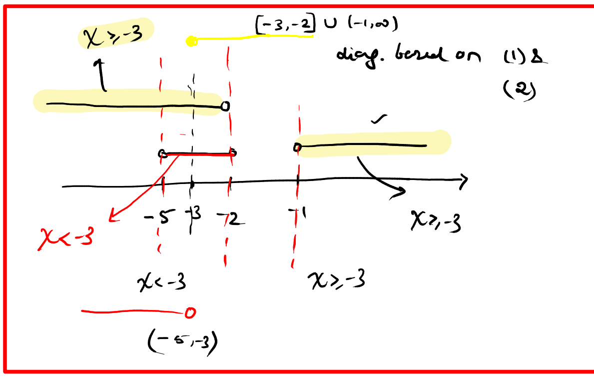
* Case 1 :

$|x+3| = (x+3)$ $\quad x+3 \geq 0 \Rightarrow x \geq -3$

$f(x) = \dfrac{x+3-2}{x+2} = \dfrac{x+1}{x+2} > 0$



$x \in (-\infty,-2) \cup (-1,\infty)$ $\quad$ (1) $\xrightarrow{x \geq -3}$ $\boxed{x \in [-3,-2) \cup (-1,\infty)}$ — (1')

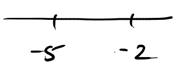


* Case 2 :

$|x+3| = -(x+3)$ $\quad x+3 < 0 \Rightarrow x < -3$

$f(x) = \dfrac{-x-5}{x+2} \xrightarrow{\hspace{2em}} \dfrac{x+5}{x+2} < 0$

$f(x) > 0$

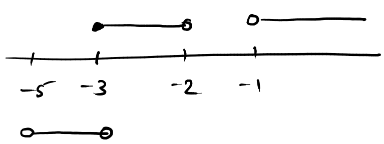


$x \in (-5,-2)$ (2)

$\downarrow x < -3$

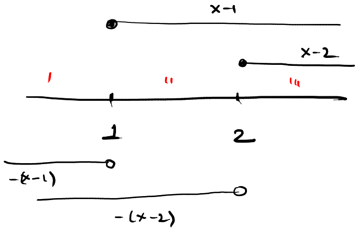
$\boxed{x \in (-5,-3)}$ — (2'')

diag. based on 1' & 2''



$x \in (-5,-2) \cup (-1,\infty)$

Q9.[7] $|x-1| + |x-2| \geqslant 4$



* $|x-1| = \begin{cases} x-1 & x \geqslant 1 \\ -(x-1) & x < 1 \end{cases}$, $|x-2| = \begin{cases} x-2 & x \geqslant 2 \\ -(x-2) & x < 2 \end{cases}$

Method 2:

* $-\infty < x < 1$

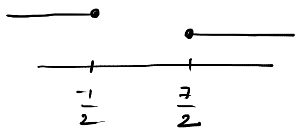
$$f(x) = \overset{M_1}{|x-1|} + \overset{M_2}{|x-2|} = -(x-1) - (x-2) = -2x+3 \xrightarrow{f \geqslant 4} -2x+3 \geqslant 4 \Rightarrow -2x \geqslant 1 \Rightarrow \boxed{x \leq \frac{-1}{2}}$$

* $1 \leq x < 2$

$$f(x) = +(\cancel{x}-1) - (\cancel{x}-2) = 1 \xrightarrow{f \geqslant 4} 1 \geqslant 4$$ Absurd !! ($\nexists$ solution)

* $2 \leq x < \infty$

$$f(x) = (x-1) + (x-2) = 2x-3 \xrightarrow{f \geqslant 4} 2x-3 \geqslant 4 \Rightarrow \boxed{x \geqslant \frac{7}{2}}$$



$$\boxed{x \in \left(-\infty, -\tfrac{1}{2}\right] \cup \left[\tfrac{7}{2}, \infty\right)}$$

Q10[2] $\frac{|\beta-1|}{\beta+2} < 1$

* $\beta \in (-\infty, -2) \bigcup \left(-\frac{1}{2}, \infty\right)$ ★★ Good job

HW

* $|\alpha-1| + |\alpha-2| + |\alpha-3| \geqslant 6$ $\alpha \in (-\infty, 0] \cup [4, \infty)$ ✓

Lecture-28 (26/10) 1:45

Detour to Series

A. Sum of Series

* $\underset{\sim}{A} = \sum_{i=1} A_i \hat{\underset{\sim}{e}}^i$: $\hat{\underset{\sim}{e}}_i \cdot \hat{\underset{\sim}{e}}_j = g_{ij} = \delta_{ij} \Rightarrow$ orthonormal basis vectors

* $A = \{A_1, A_2 \dots A_n\} \in \mathbb{C}$ Components

* A : $A_{i+1} = A_i + \alpha$ α = constant = invariant = Common difference $\Rightarrow$ Arithmetic Progression (A.P)

* $S_n \equiv \sum_{i=1}^{n} A_i$ 'sum of n terms'

$1+2+3 = 3+2+1$

A.i A.P

* $S_n = A_1 + A_2 + A_3 + \dots + A_n$

"variables"

$S_n = A_1 + (A_1+\alpha) + (A_1+2\alpha) + (A_1+3\alpha) + \dots + (A_1+(n-1)\alpha)$

1 eq^n $\left.\begin{matrix} A_1 \\ \alpha \end{matrix}\right\} 1$, $S_n \} 1$

$A_3 = A_2 + \alpha = A_1 + \alpha + \alpha$
$= A_1 + 2\alpha$

$S_n = (A_1 + (n-1)\alpha) + (A_1 + (n-2)\alpha) \dots + (A_1+\alpha) + A_1$

General Prescription

$\boxed{\text{Trick: write every term in the sum in the reverse order}}$

(+)

$\alpha + (n-2)\alpha$

Add add the 2 series.
Subtr.
Mult. / divide ...

* $2S_n = (2A_1 + (n-1)\alpha) + (2A_1 + (n-1)\alpha) + \dots$ $2A_1 + (n-1)\alpha$

$= n(2A_1 + (n-1)\alpha)$

* $\boxed{S_n = \frac{n}{2}\{2A_1 + (n-1)\alpha\}}$ $\underbrace{\{A_1, \alpha, n\}}_{\text{given}} \longrightarrow \sum (AP)_n$

* $A_1 = 1$, $\alpha = 1$, n $\longrightarrow \boxed{S_n = \frac{n(n+1)}{2}} \longrightarrow \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ ex: $\sum_{i=1}^{n} 1 = n$

$1+2+3+\dots n$

* $S_n = \sum_{i=1}^{n} (A_1 + (i-1)\alpha) = \sum_{i=1}^{n} A_1 + \sum_{i=1}^{n} (i-1)\alpha = A_1 \underbrace{\sum_{i=1}^{n} 1}_{n} + \alpha \underbrace{\sum_{i=1}^{n} (i-1)} = \frac{n}{2}\{2A_1 + (n-1)\alpha\}$

$\alpha\left(\sum i - \sum 1\right) = \alpha\left(\frac{n(n+1)}{2} - n\right) = \alpha\left(\frac{n(n-1)}{2}\right)$

* $S_n = \frac{n}{2}\{A_1 + \underbrace{A_1 + (n-1)\alpha}_{A_n}\} = \frac{n}{2}\{A_1 + A_n\}$ given $\{A_1, A_n, \#\text{ terms}\}$

* $S_n - S_{n-1} = A_n = A_1 + (n-1)\alpha$

A.2 G.P

* $A : \frac{A_{i+1}}{A_i} = \beta \Rightarrow A_{i+1} = \beta A_i$ β = invariant = Common Ratio $\Rightarrow$ Geometric progression

* $S_n = \sum_{i=1}^{n} A_i = A_1 + \dots A_n$

$S_n = A_1 + A_1\beta + A_1\beta^2 + A_1\beta^3 + \dots + A_1\beta^{n-2} + \overbrace{\cancel{A_1\beta^{n-1}}}^{A_n}$

$\beta S_n = A_1\beta + A_1\beta^2 + A_1\beta^3 + \dots + \cancel{A_1\beta^{n-1}} + A_1\beta^n$

(-)

$S_n - \beta S_n = A_1 - A_1\beta^n \Rightarrow \boxed{S_n = \frac{A_1(1-\beta^n)}{1-\beta}}$

General - General prescription

Manipulate terms in the series & try to get rid of as many terms as possible

$\beta \neq 1$

$S_n = 1+1+1+\dots 1 = n$ AP $\alpha = 0 \to \exists$ Formula ; GP $\beta = 1 \to$ DNE

* $\lim S_n = \infty$
$\beta > 1$

* $S_n = \begin{cases} \frac{A_1(1-\beta^n)}{1-\beta} & \beta < 1 \\ \frac{A_1(\beta^n - 1)}{\beta - 1} & \beta > 1 \end{cases}$: $S_n > 0$

supercalifragilisticexpialidocious

Lecture - 29 (27/oct) 1:45

* $S_n = a + a+\alpha + a+2\alpha + \dots \overbrace{(a+(n-1)\alpha)}^{a_n} = \frac{n}{2}[2a + (n-1)\alpha] = \frac{n}{2}(a_1 + a_n)$ n : finite

$\boxed{\lim_{n\to\infty} S_n \to \infty}$ in $\mathbb{R}$

* $1+2+3 \dots \infty = \frac{-1}{12} \in [-1, 0]$ in $\mathbb{C}$ (example!) Answer is finite. It's just a fact for you.

* $S_n = a + a\beta + a\beta^2 + \dots \underbrace{a\beta^{n-1}}_{a_n} = \frac{a(1-\beta^n)}{1-\beta} = \frac{a - a\beta^n}{1-\beta} = \frac{a - \overbrace{a\beta^{n-1}}\cdot\beta}{1-\beta} = \frac{a - a_n\beta}{1-\beta}$ $\beta < 1$ n : finite

$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left\{\frac{a}{1-\beta} - \frac{a\beta^n}{1-\beta}\right\} = \frac{a}{1-\beta} - \frac{a \cancelto{0}{\lim_{n\to\infty}\beta^n}}{1-\beta} = \frac{a}{1-\beta}$ $|\beta| < 1$

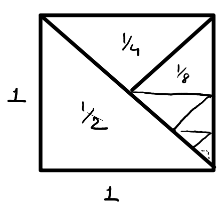
$\lim_{n\to\infty} x^n = \begin{cases} \infty & x > 1 \\ 0 & x < 1 \end{cases}$

* $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$ to $\infty \neq 0$ $\longrightarrow$ $S_\infty = \frac{1/2}{1 - 1/2} = 1$

(~Zeno paradox)

1 grain of Rice (Produces 0 sound) + 1 grain + ... 10^5 times (0 sound) = $\exists$ Production of Sound (Illusion)

$\boxed{0 + 0 + \dots + 0 = 0}$



$Ar = 1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \infty$

* $\lim_{n\to\infty} S_n = \begin{cases} \dfrac{a}{1-\beta} & |\beta| < 1 \\ \infty & |\beta| > 1 \end{cases}$

B. Product of series

B.1 G.P

* $P_n \equiv \prod_{i=0}^{n} a\beta^i = a \cdot a\beta \cdot a\beta^2 \cdots a\beta^{n-1} \cdot a\beta^n = a^{n+1} \cdot \beta^{1+2+3+\cdots(n-1)+n} = a^{n+1}\beta^{\frac{n(n+1)}{2}} = (a\beta^{n/2})^{n+1} = (a\sqrt{\beta^n})^{n+1} = (\sqrt{a^2\beta^n})^{n+1}$

B.2 AP

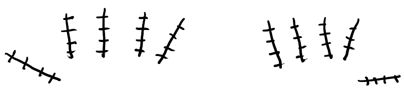
* $P_n \equiv \prod_{k=0}^{n-1}(a_1 + k\alpha) = a_1 \cdot (a_1 + d) \cdots (a_1 + (n-1)\alpha) = ?$

* $\prod_{k=0}^{n-1}(a_1 + k\alpha) = \prod_{k=0}^{n-1} \alpha\left(\frac{a_1}{\alpha} + k\right) = \alpha\left(\frac{a_1}{\alpha}\right) \cdot \alpha\left(\frac{a_1}{\alpha}+1\right) \cdot \alpha\left(\frac{a_1}{\alpha}+2\right) \cdots \alpha\left(\frac{a_1}{\alpha} + (n-1)\right) = \alpha^n \underbrace{\prod_{k=0}^{n-1}\left(\frac{a_1}{\alpha} + k\right)} = ?$ HW

Lecture-30 (29/oct) 2

Digression to Combinatorics

1. Basic Principles of Counting (Ref: PnC)

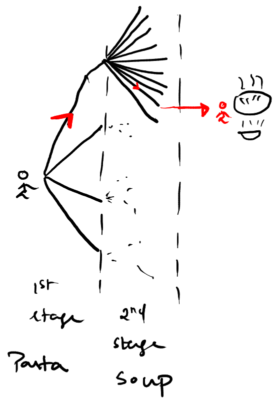


* Indistinguishable markers / fingers / label

* 1 finger — 4 markers

10 fingers — $\underbrace{4}_{\text{\# ways to choose a marker}} \times \underbrace{10}_{\text{\# ways to choose a finger}} = 40$

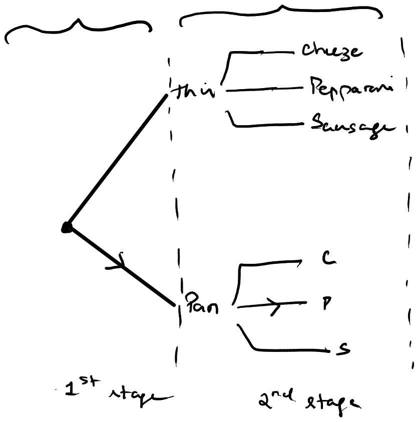
*



$n_1 = \overset{1+1+1+1}{4}$, $n_2 = 10$

# of Possible levels = $4 \times 10 = 40$

Sequential description / Tree diagram



Either T or P & Either C or P or S

$(1+1) \times (1+1+1)$

$2 \times 3 = 6$

Total # of possible comb^n of ordering a Pizza

* r = stages / levels ; n_i : choices at stage i ; $i = 1, 2, \ldots r$

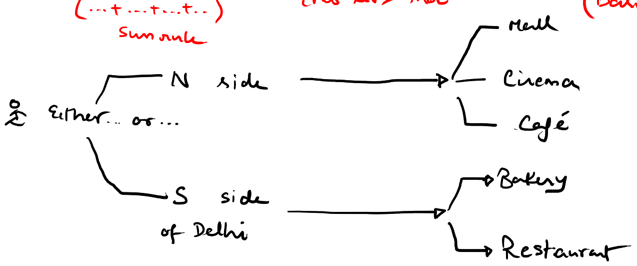
$\boxed{\#\text{ choices} = n_1 \times n_2 \times \cdots \times n_r = \prod_{i=1}^{r} n_i}$

Product rule of Counting (Basic principle of Counting)

$(\ldots + \ldots + \ldots + \ldots)$ Sum rule

This AND that

*



$\begin{Bmatrix} \text{decided to visit} \\ \text{1 side} \end{Bmatrix}$

if N: (Either ...or...) $1+1+1$ = 3

OR +

if S: (Either ... or) $1+1$ = 2

$3 + 2 = 5$ → # ways you end up visiting 1 side of town

Sum Rule

$$\boxed{\# \text{ choices} = \alpha_1 + \alpha_2 + \alpha_3 \dots \alpha_n}$$

Either ... or...

A

* $n(A) = n$ → n

ordered list = arrangement = Permutation

$n \cdot (n-1)(n-2) \cdot \dots \dots 3 \cdot 2 \cdot 1 \equiv n! = {}^nP_n$ (No rep.)

" # of ways to create an ordered list of n "

" " " " arrange n objects taken n at time

$A = \{\alpha, \beta, \gamma, \delta\}$: $n(A) = 4$

$4 \times 3 \times 2 \times 1 = 24$ arrangements (No rep^n)

ex: $\delta \ \alpha \ \beta \ \gamma$

Either ... or ...

$1+1+1+1$ $1+1+1$ $1+1$ 1 (Sum rule)

$\underline{4} \times \underline{4} \times \underline{4} \times \underline{4} = 4^4$ arrangements

$1+1+1+1$ $1+1+1+1$ ... (Rep^n)

1 comb^n (rep) : $\alpha\,\alpha\,\alpha\,\alpha$

1 comb^n (ex.) : $\delta\,\alpha\,\beta\,\gamma$

2^nd comb^n (ex.) : $\alpha\,\delta\,\beta\,\gamma$

2. Alternative / Double factorial

* $$n!! = \begin{cases} \prod_{k=1}^{n/2} (2k) = n \cdot (n-2)(n-4) \dots 4 \cdot 2 & n = \text{even} \\ \prod_{k=1}^{\frac{n+1}{2}} (2k-1) = n(n-2)(n-4) \dots 3 \cdot 1 & n = \text{odd} \end{cases}$$

$n!! \neq (n!)!$

* $9!! = 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1$

* $(2n)!! = 2n \dots 6 \cdot 4 \cdot 2$

$(2n-1)!! = (2n-1) \dots 5 \cdot 3 \cdot 1$

* $$\boxed{n!!\ (n-1)!! = n!}$$

HW

$\int_0^{\pi/2} \sin^n x \, dx =$

{ to simply the notation (T-citygrids) Arthur Schuster (1904) }

$\int_0^{\pi/2} \sin x \, dx = 1$

$\int_0^{\pi/2} \sin^2 x \, dx = \frac{\pi}{4}$

$\int_0^{\pi/2} \sin^3 x \, dx = ?$ HW

$\int_0^{\pi/2} \sin^5 x \, dx = ?$ HW

Lecture-31 (30/oct) 2

* $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$

* $n!! \equiv \begin{cases} \prod_{k=1}^{n/2} 2k & n = \text{Even} \\ \prod_{k=1}^{n+1/2} 2k-1 & n = \text{odd} \end{cases}$

* $(2n)!! = 2\cdot 4\cdot 6\cdot 8\cdots 2n$

$(2n-1)!! = 1\cdot 3\cdot 5\cdot 7\cdots 2n-1$

* $\boxed{n! = n!!\cdot(n-1)!!}$; $2n! = 2n!!\,(2n-1)!!$

* $(2n)!! = 2\cdot 4\cdot 6\cdot 8\cdots 2n = 2^n\cdot(1\cdot 2\cdot 3\cdots n) = 2^n n!$

* $2n! = \underbrace{(2n)!!}_{2^n\cdot n!}(2n-1)!! \Rightarrow (2n-1)!! = \dfrac{(2n)!}{2^n\cdot n!}$ ■

Method 1 ★★

* $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$

* $(2n)! = 2n(2n-1)(2n-2)(2n-3)\cdots 4\cdot 3\cdot 2\cdot 1$

$= 2^n\underbrace{(n(n-1)(n-2)\cdots)}_{n!}\underbrace{((2n-1)(2n-3)\cdots 3\cdot 1)}_{\therefore (2n-1)!!} \Rightarrow (2n-1)!! = \dfrac{(2n)!}{2^n\cdot n!}$

Method 2

* $(2n-1)!!\,2^n\cdot n! = \{(2n-1)(2n-3)\cdots 1\}\underbrace{2\cdot 2\cdot 2\cdots 2}_{n \text{ times}}\{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1\}$

$= \{(2n-1)(2n-3)\cdots 1\}\{2n(2n-2)(2n-4)\cdots 6\cdot 4\cdot 2\}$

$= \{2n\cdot(2n-1)(2n-2)(2n-3)\cdots 4\cdot 3\cdot 2\cdot 1\} = (2n)!$

3. Pochhammer's symbols / Stirling Numbers

* $(x)_n \equiv x^{\underline{n}} = x(x-1)(x-2)\cdots(x-n+1) = \prod_{k=1}^{n}(x-k+1) = \prod_{k=0}^{n-1}(x-k)$ Falling Factorial

Leo Pochhammer 1870

* $x^{(n)} \equiv x^{\overline{n}} = x(x+1)(x+2)\cdots(x+n-1) = \prod_{k=1}^{n}(x+k-1) = \prod_{k=0}^{n-1}(x+k)$ Raising Factorial

Donald Knuth 1970

dummy variables

$\prod_{k=1}^{n}(x-k+1) = \prod_{\alpha=0}^{n-1}(x-\alpha)$

$k-1\equiv\alpha$ $\quad \alpha\neq k$

$k=1$ $\quad \alpha=0$

$k=n$ $\quad \alpha=n-1$

$\sum_{a=1}^{3} x^a x_a = \sum_{b=1}^{3} x^b x_b$

* $(x)_0 = 1$ $\qquad x^{(0)} = 1$

$(x)_1 = x$ $\qquad x^{(1)} = x$

$(x)_2 = x(x-1) = x^2 - x$ $\qquad x^{(2)} = x(x+1) = x^2+x$ $\qquad \left[{2 \atop 1}\right] = 1!$

$(x)_3 = x(x-1)(x-2) = x^3 - 3x^2 + 2x$ $\qquad x^{(3)} = x(x+1)(x+2) = x^3+3x^2+2x$ $\qquad \left[{3 \atop 1}\right] = 2!$

$(x)_4 = x(x-1)(x-2)(x-3) = x^4 - 6x^3 + 11x^2 - 6x$ $\qquad x^{(4)} = x(x+1)(x+2)(x+3) = x^4+6x^3+11x^2+6x$ $\qquad \left[{4 \atop 1}\right] = 3! = 3\times 2 = 6$

* $\boxed{(x)_n = \sum_{k=0}^{n} s(n,k)\,x^k}$

Signed Stirling's # of 1st kind

$s(n,k)$ = Coeff. of falling factorials

} James Stirling, 1730s — Stirling # / Stirling Approx. v.imp

Stat. Mech.
QM
QFT

* $(x)_3 = x^3 - 3x^2 + 2x = \sum_{k=0}^{3} s(3,k)\,x^k = s(3,0) + s(3,1)x + s(3,2)x^2 + s(3,3)x^3$

$s(3,0) = 0$

$s(3,1) = 2$

$s(3,2) = -3$

$s(3,3) = 1$

* $\boxed{x^{(n)} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k}$ unsigned sterling # of 1^{st} kind

* $s(n,k) = (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}$ ex: $s(3,2) = (-1)^{3-2} \underbrace{\begin{bmatrix} 3 \\ 2 \end{bmatrix}}_{3} = -3$

$s(3,3) = (-1)^{3-3} \underbrace{\begin{bmatrix} 3 \\ 3 \end{bmatrix}}_{1} = 1$

Lecture-32 (1/Nov.) 1:15

* $n! \longrightarrow n!!$ (i E) $n \in \mathbb{Z}^+$

* $(x)_n$; $x^{(n)}$ Pochhammer symbol

* $(x)_n = \sum_{k=0}^{n} s(n,k)\, x^k \longrightarrow s(n,k) = (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}$

$x^{(n)} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k$

* $x^{(3)} \equiv \underbrace{x(x+1)(x+2)}_{} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} x^0 + \begin{bmatrix} 3 \\ 1 \end{bmatrix} x^1 + \begin{bmatrix} 3 \\ 2 \end{bmatrix} x^2 + \begin{bmatrix} 3 \\ 3 \end{bmatrix} x^3 + \begin{bmatrix} 3 \\ 4 \end{bmatrix} x^4$

0 2 3 1 0

$x(x^2+3x+2)$

$x^3 + 3x^2 + 2x + 0x^4$

Properties

1. $\begin{bmatrix} n \\ 0 \end{bmatrix} = 0$

2. $\begin{bmatrix} 0 \\ k \end{bmatrix} = 0$ $k > 0$

3. $\begin{bmatrix} n \\ k \end{bmatrix} = 0$ $k > n$

4. $\begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)!$

5. $\begin{bmatrix} n \\ n \end{bmatrix} = 1$

6. $\begin{bmatrix} n \\ n-1 \end{bmatrix} = \frac{n!}{(n-2)!\,2!} \equiv \binom{n}{2} = {}^nC_2$ "n choose 2"

$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = 3 = \frac{3!}{1!\,2!} = \binom{3}{2} = {}^3C_2$

$\begin{bmatrix} 4 \\ 3 \end{bmatrix} = 6 = \frac{4!}{2!\,2!} = \binom{4}{2} = {}^4C_2$

4. Gamma function

* Euler's Integral $\rightarrow \int_0^\infty x^{n-1} e^{-x} dx = (n-1)!$

* $\boxed{\Gamma(z) = \overline{|z} \equiv \int_0^\infty x^{z-1} e^{-x} dx}$ Gamma function (def 1)

ex: $\int_0^\infty e^{-x} dx = 1 = (1-1)! = 0!$

$\rightarrow \int_0^\infty e^{-x} x\, dx = 1 = (2-1)! = 1!$

HW

$\rightarrow \int_0^\infty e^{-x} x^2 dx = 2 = (3-1)! = 2!$

old Problematic

$(-3)!$ $\left(\frac{3}{2}\right)!$

Need Extension of !

Dan. Bernoulli

↓ notation due to Legendre, 1786

(Absolutely Convergent)

* $\Gamma n = (n-1)!$; $n! = n(n-1)! \Rightarrow (n-1)! = \frac{n!}{n} = \Gamma n$ is a non integral value for $n \notin \mathbb{Z}$

* $\boxed{\Gamma z = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3 \cdots n}{z(z+1)(z+2)\cdots(z+n)} n^{z}}$ $z \neq 0, -1, -2, \ldots$ $z \in \mathbb{R}$ or $\mathbb{C}$ "analytical continuation" of $n!$

(def 2) Weierstrass's infinite Pdt.

$$\int_0^\infty e^{-x}dx \quad x=-t \quad dx=-dt$$

$$-\int_0^\infty e^{t}dt = -e^{-x}$$

$$-e^{-x}\Big|_0^\infty = -\{e^{-\infty} - e^{0}\} = -(0-1) = 1$$

$$\int_0^\infty e^{x}dx \to \text{divergent}$$

lecture-33 (21 Nov.) 1.45

* $\Gamma(z) = \Gamma z$

$$\Gamma z+1 = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3 \cdots n}{(z+1)(z+2)(z+3)\cdots(z+n)(z+n+1)} n^{z+1} = \lim_{n\to\infty} \overbrace{\frac{n z}{(z+n+1)}}^{f} \overbrace{\left\{\frac{1\cdot 2\cdot 3 \cdots n}{z(z+1)(z+2)\cdots(z+n)} n^{z}\right\}}^{g}$$

$$= \underbrace{\left(\lim_{n\to\infty}\frac{nz}{z+n+1}\right)}_{z}\underbrace{\left(\lim_{n\to\infty}\frac{1\cdot 2\cdots n}{z(z+1)\cdots(z+n)}\cdot n^{z}\right)}_{\Gamma z}$$

Legendre's not^n ∂ d

$\lim_{x\to a} f(x)g(x) = (\lim f(x)) \lim g$

$n! = n(n-1)!$

$n! \propto (n-1)!$

Recursive fun^n

M. Imp. prop. 1

* $\boxed{\Gamma z+1 = z\Gamma z} \longrightarrow n! = n(n-1)!$ ✓

$$\lim_{n\to\infty} \frac{nz}{n+z+1} \quad \left(\frac{\infty}{\infty}\right)$$

$$\lim_{n\to\infty} \frac{\cancel{n}z}{\cancel{n}\left(1+\frac{z}{n}+\frac{1}{n}\right)} = z$$

Couple of values of Γz (Integral values)

* $\Gamma 1 = \lim_{n\to\infty} \frac{(\cancel{1\cdot 2 \cdots n})}{\cancel{1\cdot 2\cdot 3 \cdots n}(1+n)} n = \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}(1+1/n)} = 1$

* $\Gamma 2 = \Gamma 1+1 = 1\Gamma 1 = 1$

$\Gamma 3 = \Gamma 2+1 = 2\Gamma 2 = 2$

$\Gamma 4 = \Gamma 3+1 = 3\Gamma 3 = 3\cdot 2\cdot 1 = 6$

$\vdots$

$\Gamma n = (n-1)(n-2)\cdots 3\cdot 2\cdot 1 = (n-1)!$

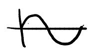
$\boxed{\Gamma n = (n-1)!}$

x $\quad \sin x$ $\quad \frac{d}{dx}$ $\quad e^x$ $\quad \ln x$

x^2 $\quad |x|$ $\quad \Sigma \quad \int \quad \Pi$

x^n $\quad \sinh x$ $\quad \lim$

$y = \sin x$



graph. rep. 1

Algebraic rep. 2

↓

$y = x - \frac{x^3}{3!} \ldots$ taylor's rep. 3

.... Fourier rep 4

* $\Gamma z = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3 \cdots n}{z(z+1)\cdots(z+n)} n^{z} = \lim_{n\to\infty} \frac{n^z}{z} \underbrace{\left\{\frac{1}{\frac{(z+1)}{1}\frac{(z+2)}{2}\frac{(z+3)}{3}\cdots\frac{(z+n)}{n}}\right\}}$

Weierstrass ∞ pdt (Form 1)

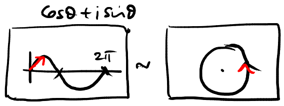
$$\frac{1}{\left(\frac{z}{1}+1\right)\left(\frac{z}{2}+1\right)\left(\frac{z}{3}+1\right)\cdots\left(\frac{z}{n}+1\right)} = \left(\frac{z}{1}+1\right)^{-1}\left(\frac{z}{2}+1\right)^{-1}\cdots\left(\frac{z}{n}+1\right)^{-1}$$

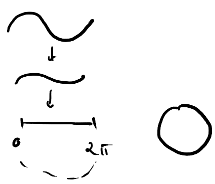
$$\| \quad \prod_{k=1}^{n}\left(\frac{z}{k}+1\right)^{-1}$$

$$\Gamma z = \lim_{n\to\infty} \frac{1}{z}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)^{-1} n^{z}$$ Weierstrass (form 2)

$e^{i\theta}$ ~ Rotation

$\cos\theta + i\sin\theta$





$\Gamma z+1 = z\Gamma z$

$\Gamma -z+1 = -z\Gamma -z$

* $$\frac{1}{\Gamma 1-z\,\Gamma z} = \frac{1}{-z\Gamma -z\,\Gamma z} = \lim_{n\to\infty} \frac{1}{\cancel{-z}\left(\left(\frac{1}{\cancel{-z}}\right)\prod_{k=1}^{n}\left(1-\frac{z}{k}\right)^{-1}\cancel{n^{-z}}\right)\left(\frac{1}{z}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)^{-1}\cancel{n^{z}}\right)}$$

$$= z\prod_{k=1}^{\infty}\left(1-\frac{z}{k}\right)\left(1+\frac{z}{k}\right) = z\prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2}\right) = \cancel{z}\cdot\frac{\sin\pi z}{\pi \cancel{z}}$$

Imp

$$\boxed{\prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2}\right) = \frac{\sin\pi z}{\pi z}}$$

— using partial fraction expansion in $\mathbb{C}$ Analysis

— " Fourier series.

$$\frac{1}{\sum_{i=1} x^i} \neq \sum x^i$$

$$\frac{1}{\prod x^i} = \prod x^{-i}$$

* $\boxed{\Gamma z\, \Gamma 1-z = \dfrac{\pi}{\sin \pi z}}$ Euler's Reflection Formula

$$\left(\int_0^\infty t^{z-1} e^{-t} dt\right)\cdot\left(\int_0^\infty t^{-z} \cdot e^{-t} dt\right) = \frac{\pi}{\sin \pi z}$$

$\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} \cdots = \sum$

why so powerful result!

* "Factorials" of non integer value

* $z = \frac{1}{2} \longrightarrow \Gamma\frac{1}{2}\, \Gamma 1-\frac{1}{2} = \dfrac{\pi}{\sin\frac{\pi}{2}} = \pi \Rightarrow \left(\Gamma\frac{1}{2}\right)^2 = \pi \Rightarrow \boxed{\Gamma\frac{1}{2} = \sqrt{\pi}}$

* $\Gamma z+1 = z \Gamma z$

$\Gamma\frac{3}{2} = \Gamma\frac{1}{2}+1 = \frac{1}{2}\Gamma\frac{1}{2} = \frac{\sqrt{\pi}}{2}$

$\Gamma\frac{5}{2} = \Gamma\frac{3}{2}+1 = \frac{3}{2}\Gamma\frac{3}{2} = \frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}$

$\Gamma\frac{7}{2} = \Gamma\frac{5}{2}+1 = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$; $\boxed{\Gamma n = (n-1)!}$ Abusive notation

$\Gamma\frac{7}{2} = \left(\frac{7}{2}-1\right)! = \left(\frac{5}{2}\right)! = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\underbrace{\left(\frac{-1}{2}\right)!}_{\sqrt{\pi}}$

Abuse of notn

$\Gamma\frac{1}{2} = \left(\frac{1}{2}-1\right)! = \left(\frac{-1}{2}\right)! = \sqrt{\pi}$

$n!$ $n \in \mathbb{Z}^+$

Lecture 34 (3 Nov) 2:15

* $n! = n(n-1)!$ $n \in \mathbb{Z}^+$

* $n!!$; $n! = \underbrace{n!!}_{\text{"Even"}}\ \underbrace{(n-1)!!}_{\text{"odd"}}$ $n \in ?$

* $(2n)!! = 2^n n!$; $(2n-1)!! = \dfrac{(2n)!}{2^n n!}$

* $(x)_n = x^{\underline{n}} = x(x-1)(x-2)\cdots(x-n+1) = \prod_{k=1}^{n}(x-k+1) = \prod_{k=0}^{n-1}(x-k) = \sum_{k=0}^{n} s(n,k)\, x^k$ signed stirling #

$x^{(n)} = x^{\overline{n}} = x(x+1)\cdots\cdots(x+n+1) = \prod_{k=0}^{n+1}(x+k) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k$ unsigned

$s(n,k) = (-1)^{n-k}\begin{bmatrix} n \\ k \end{bmatrix}$

weirstrass ∞ pdt

* $\Gamma z = \lim_{n\to\infty} \dfrac{n!}{z(z+1)\cdots(z+n)} n^z$: $\Gamma z+1 = z\Gamma z$, Reflection formula $\Gamma z\, \Gamma 1-z = \dfrac{\pi}{\sin \pi z} \Rightarrow \Gamma\frac{1}{2} = \sqrt{\pi}$

Multiple integration

$L = \int dl$

$A = \int dl \int dl$

$= \iint dl_1 dl_2$

$V = \int dl \int dl \int dl$

$= \iiint dl_1 dl_2 dl_3$

* $\Gamma z = \int_0^\infty e^{-x} x^{z-1} dx$, $\Gamma 1-z = \int_0^\infty e^{-t} t^{-z} dt$ $z \in \mathbb{C}$

$\Gamma z\, \Gamma 1-z = \int_0^\infty e^{-x} x^{z-1} dx \cdot \int_0^\infty e^{-x} x^{-z} dx \neq \int_0^\infty\int_0^\infty e^{-2x}\cdot x^{-1} dx_1 = \text{Indep. of } z \neq \dfrac{\pi}{\sin \pi z}$ (Spot the fallacy!)

* $\Gamma z\, \Gamma 1-z = \int_0^\infty e^{-t} t^{z-1} dt \underbrace{\int_0^\infty e^{-u} \cdot u^{-z} du}$ $= \int_0^\infty \int_0^\infty \underbrace{e^{-t}\cdot e^{-vt}}_{e^{-(v+1)t}} v^{-z}\, dv\, dt = \int_0^\infty\int_0^\infty e^{-(v+1)t} v^{-z} dv\, dt$

$u = vt \Rightarrow du = t\,dv$

$\int_{v=0}^\infty e^{-vt}\cdot (vt)^{-z}\, t\,dv = t^{1-z}\int_0^\infty e^{-vt} v^{-z} dv$

$= \int_0^\infty \left\{\int_0^\infty e^{-(v+1)t} dt\right\} v^{-z} dv$, with $\int_0^\infty e^{-(v+1)t} dt = \dfrac{1}{1+v}$

$\dfrac{dS\cos\alpha}{r} = d\Omega$

$\int e^x dx = x e^x$

$\int_0^\infty e^{-\alpha x} dx = \frac{1}{\alpha} e^{-\alpha x}$

$\Gamma z\, \Gamma 1-z = \boxed{\int_0^\infty \dfrac{v^{-z}}{1+v} dv = \dfrac{\pi}{\sin \pi z}}$ dep. on z std. Integral

* $\prod_{m=1}^{\infty}\left(1-\frac{z^2}{m^2}\right) = \frac{\sin \pi z}{\pi z}$, $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots = \sum_{m=1}^{\infty} \frac{x^{2m-1}}{(2m-1)!}\cdot(-1)^{m-1} = \sum_{m=0}^{\infty} \frac{x^{2m+1}}{(2m+1)!}(-1)^m$

Pdt. rep. of Sine — Summ. rep. of Sine

Sinous Business

* $n! = n!!(n-1)!! \Rightarrow n!! = \frac{n!}{(n-1)!!} = \frac{\overbrace{n!}^{n}}{(n-1)!}(n-2)!! = n(n-2)!!$

$(n-1)! = (n-1)!!\,(n-2)!!$

$(-3)!!$? , $(-2)!!$?

$\cancel{(-3)!}$ $\cancel{(-2)!}$

$(-4)!!$ $(-5)!!$

$(-3)!! =$

* $\boxed{n!! = n(n-2)!!}$ Recursive rel^n

$(n+2)!! = (n+2)\,n!! \Rightarrow n!! = \frac{(n+2)!!}{n+2}$

$(-1)!! = \frac{1!!}{1} = 1$

$(-2)!! = \frac{0}{0}$ (Indeterminate)

$(-3)!! = \frac{(-1)!!}{-1} = -1$

$(-5)!! = \frac{(-3)!!}{-3} = \frac{-1}{3}(-1) = \frac{1}{3}$

Calculations for -ve odd Integers

$\nexists$ -ve even integer $(\ )!!$ not defined

-ve Extension of !!

$\boxed{n!! \ : \ n \in \mathbb{Z}^+ \cup \mathbb{Z}^-_{odd}}$

* $\boxed{n!!\,(-n)!! = (-1)^{\frac{n-1}{2}}\, n}$

* $3!!\,(-3)!! = 3\cdot 1\cdot(-1) = -3$

$5!!\,(-5)!! = 5\cdot 3\cdot 1\left(\frac{1}{3}\right) = 5$

$7!!\,(-7)!! = 7\cdot 5\cdot 3\cdot 1\left(\frac{(-5)!!}{-5}\right) = 7\cdot\cancel{5}\cdot\cancel{3}\cdot 1\cdot\frac{1}{-\cancel{5}}\cdot\frac{1}{\cancel{3}} = -7$

$\mathbb{C}$ extensions

$\Gamma(z+1) = z\Gamma(z)$: $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$\Gamma(z) = (z-1)! \rightarrow \Gamma(z+1) = z!$

* $\psi!! = \psi(\psi-2)(\psi-4)\dots 5\cdot 3\cdot 1$ $\quad \psi$: odd

$= \underbrace{2^{\frac{\psi-1}{2}}}_{\frac{2^{\psi/2}}{\sqrt{2}}} \underbrace{\frac{\psi}{2}\,\frac{\psi-2}{2}\,\frac{(\psi-4)}{2}\dots\frac{5}{2}\cdot\frac{3}{2}}$

$= \frac{\Gamma\left(\frac{\psi}{2}+1\right)}{\Gamma\left(\frac{3}{2}\right)} = \frac{\Gamma\left(\frac{\psi}{2}+1\right)}{\frac{1}{2}\sqrt{\pi}} = \frac{2}{\sqrt{\pi}}\Gamma\left(\frac{\psi}{2}+1\right)$

$7!! = \underbrace{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}}\cdot 2^3$

$\Gamma\left(\frac{9}{2}\right) = \Gamma\left(\frac{7}{2}+1\right) = \frac{7}{2}\Gamma\left(\frac{7}{2}\right) = \frac{7}{2}\Gamma\left(\frac{5}{2}+1\right) = \frac{7}{2}\cdot\frac{5}{2}\Gamma\left(\frac{5}{2}\right)$

$= \frac{7}{2}\cdot\frac{5}{2}\Gamma\left(\frac{3}{2}+1\right) = \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\Gamma\left(\frac{1}{2}+1\right) = \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$

$\boxed{\psi!! = \sqrt{\frac{2}{\pi}}\cdot 2^{\psi/2}\,\Gamma\left(\frac{\psi}{2}+1\right)} \longrightarrow \Gamma\left(\frac{\psi}{2}+1\right) = \sqrt{\frac{\pi}{2}}\,\frac{\psi!!}{2^{\psi/2}}$ ✓

$\frac{\psi}{2}+1 = \frac{-1}{2} \Rightarrow \frac{\psi}{2} = \frac{-3}{2}$

$\Gamma\left(\frac{-1}{2}\right) = \sqrt{\frac{\pi}{2}}\cdot\frac{(-3)!!}{2^{-3/2}} = \frac{\pi}{\sqrt{2}}(-1)\sqrt{2}\cdot 2^{3/2}$

$7!! = \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\underbrace{\frac{\Gamma\left(\frac{7}{2}+1\right)}{\Gamma\left(\frac{1}{2}+1\right)}} = \frac{\Gamma\left(\frac{7}{2}+1\right)}{\Gamma\left(\frac{3}{2}\right)}$

* $(2\psi)!! = \overbrace{2\psi\,(2\psi-2)\,(2\psi-4)\dots}^{\psi \text{ terms}}\ 8\cdot 6\cdot 4\cdot 2$ $\quad \psi$: Even

$= 2^{\psi}\,\underbrace{\psi(\psi-1)(\psi-2)\dots 3\cdot 2\cdot 1} = 2^{\psi}\underbrace{\psi!}_{\Gamma(\psi+1)} \Rightarrow \boxed{(2\psi)!! = 2^{\psi}\,\Gamma(\psi+1)}$ ✓

* $\boxed{\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2}\cos^n x\,dx = \frac{(n-1)!!}{n!!} \times \begin{cases} 1 & n \text{ odd} \\ \frac{\pi}{2} & n \text{ even}\end{cases}}$

"Just a Fact" — Beautiful result

$\cos n\pi = (-1)^n$

$\sin n\pi = 0 \quad \forall n$

Lecture-35 (4/Nov) 2

Practice-3 (Particular values of gamma fun)

$n! = n(n-1)!$ $n \in \mathbb{Z}^+$: $0! = 1$

$n!! = n(n-2)!!$ $n \in \mathbb{Z}^+ \cup \mathbb{Z}^-_{odd}$

$\Gamma n = (n-1)!$ $n \in \mathbb{Z}^+$

$\Gamma z+1 = z\Gamma z$ $z \in \mathbb{C}$

$\Gamma z\,\Gamma 1-z = \dfrac{\pi}{\sin \pi z}$ reflection formula

Q1. a. $\Gamma 5 = 24$

b. $\Gamma 3 = 2$

c. $\Gamma 1 = 1$

Q2.

a. $\Gamma \frac{1}{2} = \sqrt{\pi}$ ✓

b. $\Gamma \frac{3}{2} = \dfrac{\sqrt{\pi}}{2}$

c. $\Gamma \frac{5}{2} = \frac{3}{4}\sqrt{\pi}$

d. $\Gamma \frac{7}{2} = \frac{15}{8}\sqrt{\pi}$

$\Gamma \frac{1}{2} = \sqrt{\pi}$

Q3.

a. $\Gamma \frac{-1}{2} = -2\sqrt{\pi}$

b. $\Gamma \frac{-3}{2} = \frac{4}{3}\sqrt{\pi}$

c. $\Gamma \frac{-5}{2} = \frac{-8}{15}\sqrt{\pi}$

Euler integral

$$\int_0^\infty e^{-x} x^{n-1} dx = \Gamma n$$

+

Method of substitution

Q4.

a. $$\int_0^\infty e^{-\alpha x} x^n dx = \int_0^\infty e^{-t}\left(\frac{t}{\alpha}\right)^n \frac{dt}{\alpha} = \frac{1}{\alpha^{n+1}}\int_0^\infty e^{-t} t^{n+1-1} dt = \frac{\Gamma n+1}{\alpha^{n+1}}$$

$\alpha x = t$

$\alpha dx = dt$

b. $$\int_0^\infty e^{-\beta x^2} x^n dx = \int_0^\infty e^{-t}\left(\sqrt{\frac{t}{\beta}}\right)^n \frac{dt}{2\sqrt{\beta t}} = \frac{1}{2\beta^{\frac{n+1}{2}}}\int_0^\infty e^{-t}\, t^{n/2 - 1/2} dt$$

$$= \frac{1}{2\beta^{\frac{n+1}{2}}}\int_0^\infty e^{-t}\, t^{\frac{n+1}{2}-1} dt$$

$\frac{n-1}{2} = \overbrace{\frac{n}{2}-\frac{1}{2}+1-1}$

$= \frac{n+1}{2} - 1$

$\hookrightarrow x = \sqrt{t/\beta}$

$\beta x^2 = t$

$2\beta x\, dx = dt$

$\Downarrow$

$dx = \dfrac{dt}{2\beta\sqrt{\frac{t}{\beta}}} = \dfrac{dt}{2\sqrt{\beta t}}$

$2dt = d(2t)$

$2t = \alpha$

$2dt = d\alpha$

$2dt = d(2t)$

$$\int_0^\infty e^{-\beta x^2} x^n dx = \frac{\Gamma \frac{n+1}{2}}{2\beta^{\frac{n+1}{2}}}$$

$$* \boxed{\int_0^\infty e^{-t} t^{n/2} \frac{(dt)^{1/2}}{\beta^{\frac{n+1}{2}}} = \left\{\frac{1}{2^{n+1}}\int_0^\infty e^{-2t}(2t)^n \frac{d(2t)}{\beta^{n+1}}\right\}^{1/2} = \left(\frac{1}{(2\beta)^{n+1}}\,\Gamma n+1\right)^{1/2}}$$

$$\sqrt{\frac{\Gamma n+1}{2}} \neq \Gamma \frac{n+1}{2}$$

$$\left(\int f(x)dx\right)^2 \neq \int f^2(x)\, dx$$

"Genius" method

(Rohan's method)

c. $$\int_0^\infty e^{-\alpha x} x^3 dx = \frac{\Gamma 4}{\alpha^4}$$

$$* \int_0^\infty e^{-\alpha x} dx = \int_0^\infty e^{-\alpha x} x^0 dx = \frac{1}{\alpha}$$

d. $$\int_0^\infty x^{11} e^{-x/\alpha} dx = \Gamma 12\, \alpha^{12}$$

$$* \int_0^\infty e^{-\beta x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{\beta}}$$

e. $$\int_0^\infty e^{-x^2/\alpha} x^2 dx = \frac{\sqrt{\pi}}{4}\alpha^{3/2}$$

$\Gamma ib = ?$ HW

$i^n - q^n = ?$

f. $$\int_0^\infty e^{-\alpha x^2} x^3 dx = \frac{1}{2\alpha^2}$$

g. $$\int_0^\infty e^{-\frac{i}{2}ax^2} dx = \frac{\Gamma \frac{1}{2}}{2\cdot\left(\frac{i}{2}a\right)^{1/2}} = \sqrt{\frac{\pi}{2ia}}$$

v.v important

$$\int_0^\infty e^{-\alpha x} x^n dx = \frac{\Gamma n+1}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\beta x^2} x^n dx = \frac{\Gamma \frac{n+1}{2}}{2\beta^{\frac{n+1}{2}}}$$

$\Gamma \frac{3}{2} = \frac{1}{2}\sqrt{\pi}$

Lecture-36 (5|11) 2

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ $\quad (-1)!! = 1$

Qs. a. $\boxed{\Gamma\left(\frac{n}{2}\right) = \sqrt{\pi}\,\frac{(n-2)!!}{2^{\frac{n-1}{2}}}} \longrightarrow \Gamma\left(\frac{3}{2}\right)$

$n!! = n(n-2)!!$

$n!! = \sqrt{\frac{2}{\pi}}\, 2^{n/2}\, \underbrace{\Gamma\left(\frac{n}{2}+1\right)}$ $\quad n:\text{odd}$

$\frac{n}{2}+1 = \frac{\alpha}{2}$, $\quad \frac{n}{2} = \left(\frac{\alpha}{2}-1\right)$

$= \underbrace{\sqrt{\frac{2}{\pi}}}\, 2^{\frac{\alpha-2}{2}}\, \Gamma\left(\frac{\alpha}{2}\right)$

$\frac{2^{\frac{\alpha-1}{2}}}{\sqrt{\pi}}$

$\Gamma(z+1) = z\Gamma(z)$

$\Gamma(-z+1) = -z\Gamma(-z)$

b. $\psi = ib \;:\; i^2 = -1 \longrightarrow \Gamma(ib) = ?$; $\quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$

* $\Gamma(ib)\underbrace{\Gamma(1-ib)} = \Gamma(ib)\cdot\left(-ib\,\Gamma(-ib)\right) = -ib\,\Gamma(ib)\Gamma(-ib) = \frac{\pi}{\sin ib\pi} = \frac{\pi}{-\frac{1}{i}\sinh b\pi}$

$\Gamma(ib)\underbrace{\Gamma(-ib)}_{(\Gamma(ib))^*} = \frac{\pi}{b\sinh b\pi} \Rightarrow \boxed{|\Gamma(ib)|^2 = \frac{\pi}{b\sinh b\pi}} \longrightarrow |\Gamma(i)|^2 = \frac{\pi}{\sinh \pi}$

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

$i\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}$

$\downarrow \quad \theta \to i\theta$

$i\sin i\theta = \frac{e^{-\theta} - e^{\theta}}{2}$

$-i\sin i\theta = \frac{e^{\theta} - e^{-\theta}}{2} = \sinh\theta \in \mathbb{R}$

$(z_1+z_2)^* = z_1^* + z_2^*$ $\quad$ intuitively!

$(z_1 z_2)^* = z_1^* z_2^*$

Check (Property)

$\Gamma(z) = \lim_{n\to\infty} \frac{n!}{(z+1)(z+2)\cdots(z+n)}\, n^z$ $\quad n \in \mathbb{C}$ $\quad \underbrace{z = a+ib \longrightarrow \exists\, z^* = a-ib}$

$\Gamma(z^*) = \lim_{n\to\infty}\left(\frac{n!}{[(z^*+1)(z^*+2)\cdots(z^*+n)]}\,(n^{z^*})\right)$ $\quad |z|^2 = zz^* = a^2+b^2 \in \mathbb{R}$

$= \lim_{n\to\infty}\left(\frac{n!}{(z+1)^*(z+2)^*\cdots(z+n)^*}\, n^{z^*}\right) = \lim_{n\to\infty}\left(\frac{n!}{((z+1)(z+2)\cdots(z+n))^*}\,(n^z)^*\right)$

$= \left(\lim_{n\to\infty}\frac{n!}{(z+1)\cdots(z+n)}\, n^z\right)^* = (\Gamma(z))^*$

* $\boxed{(\Gamma(z))^* = \Gamma(z^*) \Rightarrow \Gamma(z)\underbrace{\Gamma(z^*)}_{(\Gamma(z))^*} \in \mathbb{R}}$

c. $\Gamma\left(\frac{1}{2}+ib\right) = ?$

$\Gamma\left(\frac{1}{2}+ib\right)\Gamma\left(\frac{1}{2}-ib\right) = \left|\Gamma\left(\frac{1}{2}+ib\right)\right|^2 = \frac{\pi}{\sin\left(\frac{1}{2}+ib\right)\pi} = \frac{\pi}{\cos ib\pi} = \frac{\pi}{\cosh \pi b}$ ✓

$\sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta$

$-i\sin i\theta = \sinh\theta$

$\cos i\theta = \cosh\theta$

B. Pdt of series (contd.)

B.2 A.P.

* $\underbrace{a_1\cdot(a_1+d)(a_1+2d)\cdots(a_1+(n-1)d)}_{n-\text{terms}} = \prod_{k=0}^{n-1} a_1 + kd = \prod_{k=0}^{n-1} d\left(\frac{a_1}{d}+k\right) = d^n \underbrace{\prod_{k=0}^{n-1}\left(\frac{a_1}{d}+k\right)}$

$x^{(n)} = \prod_{k=0}^{n-1}(x+k)$

$\prod_{k=0}^{n-1}\left(\frac{a_1}{d}+k\right) = x^{(n)} = x^{\bar{n}}$ $\quad$ Form-1

$\prod_{k=0}^{n-1}\left(\frac{a_1}{d}+k\right) = \frac{\Gamma\left(\frac{a_1}{d}+k\right)}{\Gamma\left(\frac{a_1}{d}\right)}$ $\quad$ Form 2

in terms of Raising factorial / Gamma fun

* $\boxed{P = a_1\cdot(a_1+d)\cdots(a_1+(n-1)d) = d^n \prod_{k=0}^{n-1}\left(\frac{a_1}{d}+k\right) = d^n x^{(n)} = d^n \frac{\Gamma\left(\frac{a_1}{d}+k\right)}{\Gamma\left(\frac{a_1}{d}\right)}}$

$\Gamma(z+1) = z\Gamma(z)$

$\Gamma(z+2) = (z+1)\Gamma(z+1) = (z+1)z\Gamma(z)$

$\Gamma(z+3) = (z+2)\Gamma(z+2) = \underbrace{(z+2)(z+1)z}\,\Gamma(z)$

$\vdots$

$\Gamma(z+m) = \left(\prod_{k=0}^{m-1}(z+k)\right)\Gamma(z)$

$\frac{\Gamma(z+m)}{\Gamma(z)} = \prod_{k=0}^{m-1}(z+k)$

c. Special Identity (that they won't tell you in Algebra) GP.

Method 1

$$S_n = a + ar + ar^2 + \dots ar^{n-1} = \frac{a(1-r^n)}{1-r}$$

* $S = 1 + x + x^2 + \dots \quad x^{n-1} = \dfrac{1-x^n}{1-x}$ ✓ useful result.

* $1 - x^n = (1-x)\left(1 + x + x^2 + \dots + x^{n-1}\right) \qquad x \equiv \dfrac{b}{a}$

* $1 - \dfrac{b^n}{a^n} = \left(1 - \dfrac{b}{a}\right)\left(1 + \dfrac{b}{a} + \left(\dfrac{b}{a}\right)^2 + \left(\dfrac{b}{a}\right)^3 + \dots \left(\dfrac{b}{a}\right)^{n-1}\right)$

$$\frac{a^n - b^n}{a^n} = \frac{(a-b)}{a}\left(1 + \frac{b}{a} + \frac{b^2}{a^2} + \frac{b^3}{a^3} \dots \frac{b^{n-1}}{a^{n-1}}\right) \Rightarrow a^n - b^n = (a-b)\, a^{n-1}\left(1 + \frac{b}{a} + \frac{b^2}{a^2} + \dots \frac{b^{n-1}}{a^{n-1}}\right)$$

$$\boxed{a^n - b^n = (a-b)\left(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \dots + ab^{n-2} + b^{n-1}\right)}$$

Lecture-37 (6/11) 1.45

Method 2

* $1 - x^n = (1-x)\underbrace{\left(1 + x + x^2 + \dots x^{n-1}\right)}_{\sum_{i=0}^{n-1} x^i} = (1-x)\sum_{i=0}^{n-1} x^i$

$$\sum_{i=1}^{n-1} x^i \equiv 1 + x + x^2 + \dots x^{n-1} = \frac{(1-x^n)}{1-x} \quad \text{GP}$$

$\Big\downarrow x = \dfrac{b}{a}$

* $1 - \dfrac{b^n}{a^n} = \left(1 - \dfrac{b}{a}\right)\sum_{i=0}^{n-1}\left(\dfrac{b}{a}\right)^i \Rightarrow a^n - b^n = \dfrac{a^n (a-b)}{a}\sum_{i=0}^{n-1}\left(\dfrac{b}{a}\right)^i = a^{n-1}(a-b)\sum_{i=0}^{n-1}\left(\dfrac{b}{a}\right)^i$

$$\boxed{a^n - b^n = (a-b)\underbrace{\sum_{i=0}^{n-1} a^{n-i-1} b^i}_{a^{n-1} + a^{n-2}b + \dots + ab^{n-2} + b^{n-1}}}$$

Reverse Eng.

* $(a-b)\sum_{i=0}^{n-1} a^{n-i-1} b^i \longrightarrow a^n - b^n$ HW

Method 3 (Ruffini's Rule of division of poly.)

HW

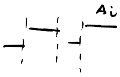
* $x^n - y^n = (x-y)\left(x^{n-1}y^0 + x^{n-2}y + x^{n-3}y^2 + \dots x^1 y^{n-2} + y^{n-1}\right)$ Prove that!

$$(x-y)\overline{\big)\; x^n - y^n \;\big(}^{\;?}$$

Caution!!

2.1.d. Piecewise Linear (PL) / Segmented / Hybrid Machines (Contd.)

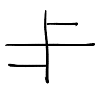
A. Step functions

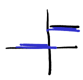
* $f(x) = \sum_{i=0}^{n} \alpha_i \chi_{A_i}(x)$  $\bigcup_{i=1}^{n} A_i = \mathbb{R}$

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

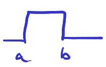
interval

* $f(x) = \alpha$

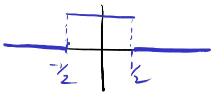
* $\text{Sgn}(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases} = \begin{cases} \dfrac{|x|}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$  $x = |x| \,\text{sgn}\, x$

* $\theta(x-a) = \begin{cases} 1 & x > a \\ 0 & x < a \end{cases}$ 

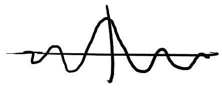
$\dfrac{d}{dx}\theta(x) = \delta(x)$, logistic $\theta(x) \approx \dfrac{1}{1+e^{-2x}}$, sigmoid $f(x) = \dfrac{1}{1+e^{-x}}$, $\theta(x) = \dfrac{x + |x|}{2x}$

* $f(x) = A\left[\theta(x-a) + \theta(x-b)\right]$ 

* $\text{Rect}\,x = \begin{cases} 0 & |x| > \frac{1}{2} \\ \frac{1}{2} & |x| = \frac{1}{2} \\ 1 & |x| < \frac{1}{2} \end{cases}$



$\text{sinc}\,x = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$

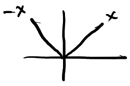


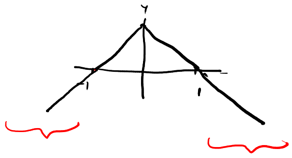
B. Modulus $|x|$, $|x| \geq a \Rightarrow x \geq a,\ x \leq -a$

$|x| \leq a \Rightarrow -a \leq x \leq a$; $x \in [-a, a]$

C. Triangular function / Tent / Hat fun^n

* $|x| = \begin{cases} x & x > 0 \\ -x & x < 0 \end{cases}$ $\xrightarrow{\text{transf}}$ $1 - |x| = \begin{cases} 1 - x & x \geq 0 \\ 1 + x & x < 0 \end{cases}$

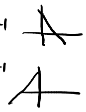




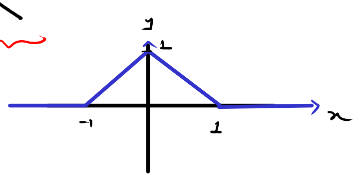
$y = mx + c$

$y = -x + 1$

$y = x + 1$



* $\text{Tri}(x) = \begin{cases} 1 - |x| & |x| < 1 \\ 0 & \text{otherwise} \end{cases}$



$\text{tri}(2x) = \begin{cases} 1 - 2|x| & |x| < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$

D. Floor, Ceiling, Fractional part function (by Legendre 1798.)

* $\text{Floor}(x) = \lfloor x \rfloor = \text{Max}\{m \in \mathbb{Z} : m \leq x\}$

HW sketch! all three

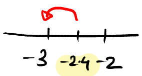
* $\lfloor 3.27 \rfloor = \text{Max.}\{m \in \mathbb{Z} : m \leq 3.27\}$

$= \text{Max}\{\ldots -2, -1, 0, 1, 2, 3\} = 3$

$\lfloor 3.0001 \rfloor = 3$

* $\lfloor -2.4 \rfloor = \text{Max.}\{m \in \mathbb{Z} : m \leq -2.4\}$

$= \text{Max.}\{-5, -4, -3\} = -3$



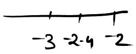
* $\text{Ceiling}(x) = \lceil x \rceil = \text{Min}\{m \in \mathbb{Z} : m \geq x\}$

$\lceil 2.4 \rceil = \text{Min}\{m \in \mathbb{Z} : m \geq 2.4\}$

$= \text{Min}\{3, 4, \ldots \infty\} = 3$

$\lceil -2.4 \rceil = \text{Min}\{m \in \mathbb{Z} : m \geq -2.4\}$

$= \text{Min}\{-2, -1, 0, 1, \ldots\} = -2$



Fractional part

* $\text{Frac}(x) = \{x\} = x - \lfloor x \rfloor$ $x \geq 0$ $\rightarrow$ $\boxed{0 \leq \{x\} < 1}$

$\lfloor 2.00001 \rfloor = 2$

$\{2.00001\} = 2.00001 - 2 = 0.00001 \in [0, 1)$

$\lfloor -2.01 \rfloor = -3 \longrightarrow \lceil -2.01 \rceil = -2$

$\{-2.01\} = -2 - (-2.01) = 0.01$

ex:

x	$\lfloor x \rfloor$	$\lceil x \rceil$	$\{x\}$
2	2	2	0
2.01	2	3	0.01
2.4	2	3	0.4
2.999	2	3	0.999
-2.7	-3	-2	0.7
-2	-2	-2	0

* $\text{frac}(x) = \begin{cases} x - \lfloor x \rfloor & x \geq 0 \\ \lceil x \rceil - x & x < 0 \end{cases}$

Lecture-38 (10/Nov) 2

Geometry
P.C n!
Prob. th.
Series/C
Pre-degree

* $\lfloor x \rfloor = \text{Max}\{m \in \mathbb{Z} : m \leq x\}$ $x \in \mathbb{R}$ aka Greatest Int. fun^n

$\lceil x \rceil = \text{Min}\{m \in \mathbb{Z} : m \geq x\}$ smallest Int. fun^n

$\text{Frac}(x) = \{x\} = x - \lfloor x \rfloor \; : \; 0 \leq \{x\} < 1$

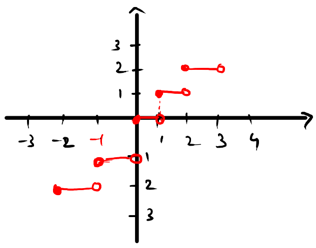
$\lfloor 1 \rfloor =$

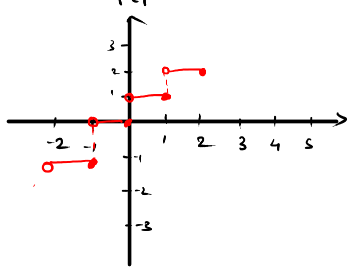
$\lfloor 0.5 \rfloor = 0$

$\lfloor 2.4 \rfloor = 2$ $\lfloor -2.4 \rfloor = -3$

$\lceil 2.4 \rceil = 3$

$\lceil -2.4 \rceil = -2$





Properties

* $\lfloor 1 \rfloor = \text{Max}\{m \in \mathbb{Z} : m \leq 1\} = \text{Max}.\{\cdots -3, \ldots 0, 1\} = 1$

$\lfloor 1.5 \rfloor = 1$

* $\lfloor x+n \rfloor = \text{Max}\{m \in \mathbb{Z} : \underbrace{m \leq x+n}\}$ $n \in \mathbb{Z}, \; x \in \mathbb{R}$

$= \text{Max}\{m \in \mathbb{Z} : m \leq x\} + n$

$\text{Max}\{1, 2, \ldots x+n\} = \lfloor x+n \rfloor = \underbrace{\text{Max}\{1, 2 \ldots x\}}_{\lfloor x \rfloor} + n$

* $\lfloor x+n \rfloor = \lfloor x \rfloor + n$

* $\lceil x+n \rceil = \lceil x \rceil + n$

* $\{x+n\} = x + \cancel{n} - \underbrace{\lfloor x+n \rfloor}_{\lfloor x \rfloor + \cancel{n}} = x - \lfloor x \rfloor = \{x\}$

$\{2.001\} = 0.001$

$\{2.001 + 2\} = \{4.001\} = 0.001$

* $\{x\} \equiv x - \lfloor x \rfloor \longrightarrow x - \lfloor x \rfloor \equiv \{x\} \equiv x \; (\text{Mod } 1)$

$\{x\} \equiv x \; (\text{Mod } 1)$ $\dfrac{\{x\} - x}{1} = n$

$c \equiv a \; (\text{Mod } b)$

$\Updownarrow$

$c - a \simeq 0$ $c \equiv a$

$\dfrac{c-a}{b} = n$

Later!

Modular Arithmetic

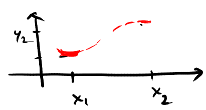
(Riemannian)

* $x_1 \leq x_2 \Rightarrow \lfloor x_1 \rfloor \leq \lfloor x_2 \rfloor$, $\lceil x_1 \rceil \leq \lceil x_2 \rceil$

$(2.5 < 5.3)$ $\lfloor 2.5 \rfloor$ 2 $\lfloor 5.3 \rfloor$ 5 $\lceil 2.5 \rceil$ $\lceil 5.3 \rceil$

Monotonically Increasing fun^n

$x_1 < x_2 \Rightarrow \overbrace{f(x_1)}^{y_1} < \overbrace{f(x_2)}^{y_2}$



Montonically $\underbrace{\text{Increasing}}$ Fun^n

Non-decreasing

* $\lceil x \rceil - \lfloor x \rfloor = \text{Min}\{m \in \mathbb{Z} : m \geq x\} - \text{Max}\{m \in \mathbb{Z} : m \leq x\}$

$= \text{Min}\{x, x+1 \ldots\} - \text{Max}\{\cdots 0, 1, 2, \ldots x\}$ $x \in \mathbb{Z}$

$= x - x = 0$

* $\lceil x \rceil - \lfloor x \rfloor = 1$ iff $x \notin \mathbb{Z}$

$\lceil 2.4 \rceil - \lfloor 2.4 \rfloor = \text{Min}\{3, 4, \ldots\} - \text{Max}.\{\cdots -1, 0, 1, 2\} = 3 - 2 = 1$

$$\boxed{\lceil x \rceil - \lfloor x \rfloor = \begin{cases} 0 & x \in \mathbb{Z} \\ 1 & x \notin \mathbb{Z} \end{cases}}$$

* $\lfloor n \rfloor = \lceil n \rceil = n$ $n \in \mathbb{Z}$

* $\lfloor x \rfloor + \lceil -x \rceil = \text{Max}\{m \in \mathbb{Z} : m \leq x\} + \text{Min}\{m \in \mathbb{Z} : m \geq -x\} = 0$

$\lfloor x \rfloor + \lceil -x \rceil = 0 \Rightarrow \lfloor x \rfloor = -\lceil -x \rceil$ or $\lceil x \rceil = -\lfloor -x \rfloor$

HW

* $\lfloor x\rfloor + \lfloor x+\frac{1}{n}\rfloor + \lfloor x+\frac{2}{n}\rfloor \cdots + \lfloor x+\frac{n-1}{n}\rfloor = \sum_{k=0}^{n-1} \lfloor x+\frac{k}{n}\rfloor = \lfloor nx\rfloor$ Hermite's Identity

$\lceil x\rceil + \lceil x-\frac{1}{m}\rceil + \lceil x-\frac{2}{m}\rceil + \cdots + \lceil x-\frac{m-1}{m}\rceil = \sum_{k=0}^{m-1} \lceil x-\frac{k}{m}\rceil = \lceil mx\rceil$

* $\left\lfloor \frac{x+m}{n}\right\rfloor \overset{?}{=} \left\lfloor \frac{\lfloor x\rfloor+m}{n}\right\rfloor$ Prove!

$\left\lceil \frac{x+m}{n}\right\rceil \overset{?}{=} \left\lceil \frac{\lceil x\rceil+m}{n}\right\rceil$

* $\left\lceil\frac{n}{m}\right\rceil + \left\lceil\frac{n-1}{m}\right\rceil + \cdots + \left\lceil\frac{n-m+1}{m}\right\rceil = n$

$\left\lfloor\frac{n}{m}\right\rfloor + \left\lfloor\frac{n+1}{m}\right\rfloor \cdots + \left\lfloor\frac{n+m-1}{m}\right\rfloor = n$

* $\lfloor x\rfloor - \lfloor -x\rfloor = \begin{cases} 2\lfloor x\rfloor+1 & x\in\mathbb{Z} \\ 2\lfloor x\rfloor & x\notin\mathbb{Z}\end{cases}$

$\lceil x\rceil - \lceil -x\rceil = \begin{cases} 2\lceil x\rceil-1 & x\in\mathbb{Z} \\ 2\lceil x\rceil & x\notin\mathbb{Z}\end{cases}$

Comment on Iverson Bracket (1960)

* Kenneth Iverson : developer of APL $\longrightarrow$ C++ / MatLab / Python

* P : $[P] \equiv \begin{cases} 1 & \text{if } P \text{ is true} \\ 0 & \text{otherwise}\end{cases}$ "It maps statement to a function"

* $\exists$ correspondence b/w logic / set th. $\longrightarrow$ Iverson Bracket

* $\neg$: $A \longrightarrow \neg A = U - A \longrightarrow [\neg P] = 1 - \underbrace{[P]}_{\begin{cases}1 & P \text{ true}\\ 0 & \text{false}\end{cases}}$

Not
Negation

Homeomorphism

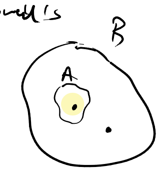
$[\triangle \approx \square] = 1$

$[\underset{\sim}{F} = m\underset{\sim}{\ddot{x}}] = 0$

$[\text{the name is Rohan}] = 1$

$[1 = 2-1] = 1$

$[\underbrace{\partial_a F^{ab} = 4\pi j^a}_{\text{Maxwell's}}] = 0$



$[P\wedge Q] = [P][Q]$

Ex: Common function

* $\delta_{ij} \equiv \begin{cases} 1 & i=j \\ 0 & i\neq j\end{cases} \longrightarrow \delta_{ij} = [i=j]$

* $\chi_A(x) = \begin{cases} 1 & x\in A \\ 0 & x\notin A\end{cases} \longrightarrow \chi_A(x) = [x\in A]$

characteristic fun

* $\Theta(x) = \begin{cases} 1 & x>0 \\ 0 & x<0\end{cases} \longrightarrow \Theta(x) = [x>0]$

Heaviside

* $\operatorname{sgn}(x) = \begin{cases} 1 & x>0 \\ 0 & x=0 \\ -1 & x<0\end{cases} \longrightarrow \operatorname{sgn}(x) = \underbrace{[x>0]}_{1} - \underbrace{[x<0]}_{1}$

* $|x| = \begin{cases} x & x\geq 0 \\ -x & x<0\end{cases} \longrightarrow |x| = x[x\geq 0] - x[x<0]$

x $\quad$ x

HW

* $\lfloor x\rfloor = \sum_n n\cdot[n\leq x<n+1]$

$\lceil x\rceil = \sum_n n\cdot[n-1<x\leq n]$

$\lfloor 2.5\rfloor = \sum_n n\cdot[n\leq 2.5<n+1] = 1\cdot\overbrace{[1\leq 2.5<2]}^{0}_{\text{False}} + 2\overbrace{[2\leq 2.5<3]}^{1}_{\text{True}} + 3\overbrace{[3\leq 2.5<4]}^{0} + 0 + \ldots = 2$

$\lceil 2.5\rceil = \sum_n n\cdot[n-1<2.5\leq n] = 1\cdot\underbrace{[0<2.5\leq 1]}_{0} + 2\underbrace{[1<2.5\leq 2]}_{0} + 3\underbrace{[2<2.5\leq 3]}_{1} + 0 + \ldots = 3$