Number Theory Vol-2 (Complex Analysis : Part-2)

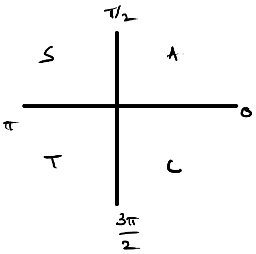
Lecture-1 (12/Nov) 1

1. Transcendental Machines

1.a Trigonometric maps/transformations

* $x \longrightarrow x' \equiv x + \underset{\text{constant}}{P} \Rightarrow f(x) \longrightarrow \underbrace{f(x') = f(x+P)}_{\| \; f(x)} \; ; \; x, P \in \mathbb{R}$

$f(x) \Rightarrow f$ is a periodic fun^n (cyclic) / periodic waveforms



1. Transcendent. fun^n
2. Curve sketching
3. Transf^n – graphical / Algebra
   set/groups
4. Nature of Roots – quad, cubic (using "calculus")
5. App. of cubic Roots

* Periodic fun^n $\Rightarrow f(x+P) = f(x)$

P = period of fun^n

cycle = repeatable part of waveform

Angle

$\sin(x+\pi) = -\sin x$

$\sin(x+2\pi) = \sin x$

$\{x\} = x - \lfloor x \rfloor$

$\{x+1\} = \{x\}$ fractional part

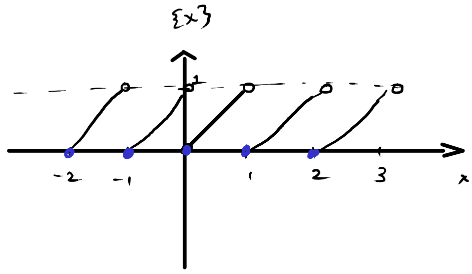
Lecture-2 (13/Nov) 1.5

* $x \xrightarrow{T} x' = x + \epsilon \qquad f(x): f(x+\epsilon) = f(x) \Rightarrow f$ is periodic $\quad$ Period $= \epsilon$

frac(x)

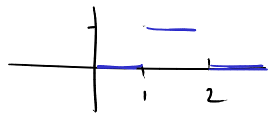
* $\{x\} = x - \lfloor x \rfloor \qquad 0 \le \{x\} < 1$

$\{x+n\} = x + \cancel{n} - \underbrace{\lfloor x+n \rfloor}_{\lfloor x \rfloor + \cancel{n}} = \{x\} \Rightarrow \{x\}$ is periodic $\quad \epsilon = 1$ (period)



* x	$\{x\} = x - \lfloor x \rfloor$
$0 \le x < 1$	x
$1 \le x < 2$	$x-1$
$2 \le x < 3$	$x-2$
$3 \le x < 4$	$x-3$
$-1 \le x < 0$	$x-(-1) = x+1$

* f is periodic $\Rightarrow \exists$ Translational $\underbrace{\text{symmetry}}_{f(x') = f(x)}$ $\quad x' = x + a$



* $f(x) = \sin x \xrightarrow{\hat{T}} f(x+\epsilon) = \sin(x+\epsilon) = f(x) = \sin x$

* $\boxed{\sin(x+\epsilon) = \sin x}$ Trig. Eq^n

General sol^n of T-Eq^n

* $\sin\theta = \sin\alpha \Rightarrow \sin\theta - \sin\alpha = 0 \Rightarrow 2\cos\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right) = 0$

$\cos\left(\frac{\theta+\alpha}{2}\right) = 0 = \sin m\pi = \cos\left(m\pi + \frac{\pi}{2}\right)$

or $\theta = \overset{\text{odd}}{(2m+1)}\pi - \alpha$

$\sin\left(\frac{\theta-\alpha}{2}\right) = 0 = \sin m\pi$

$\Downarrow \qquad m \in \mathbb{Z}$

$\theta = \underset{\text{Even multip}}{2m\pi} + \alpha$

* $\theta = \begin{cases} (2m)\pi + \alpha \\ \text{or} \\ (2m+1)\pi - \alpha \end{cases} = n\pi + (-1)^n\alpha \quad$ Even: $2m\pi + \alpha$; odd: $(2m+1)\pi - \alpha$

C D $\frac{C+D}{2}$ $\frac{C-D}{2}$

S + S = 2 S C

S − S = 2 C S

C + C = 2 C C

C − C = 2 S S*

$A \cdot B = 0$

$\Downarrow$

$A = 0$ or $B = 0$

$\sin n\pi = 0$

$\cos n\pi = (-1)^n$

* $\boxed{\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha \quad n \in \mathbb{Z}}$

$\sin\left(n\pi + (-1)^n\alpha\right) = \sin\alpha \Leftrightarrow \alpha \xrightarrow{\text{Transf}^n} \alpha' = n\pi + (-1)^n\alpha \;;\; \sin\alpha' = \sin\alpha$

$\alpha = 0$

* $\alpha = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = n\pi \longrightarrow \boxed{\sin n\pi = 0}$

(not helpful)

$\cos\left(x + \frac{\epsilon}{2}\right) = 0 = \sin n\pi = \cos\left(\frac{\pi}{2} + n\pi\right) \longrightarrow x$ dependence

* $\sin(x+\epsilon) = \sin x \Rightarrow \sin(x+\epsilon) - \sin x = 0 \Rightarrow 2\cos\left(\frac{2x+\epsilon}{2}\right)\sin\left(\frac{\epsilon}{2}\right) = 0 \Rightarrow$

$\sin\frac{\epsilon}{2} = 0 = \sin n\pi \Rightarrow \epsilon = 2n\pi \xrightarrow{n=1} \boxed{\epsilon = 2\pi}$

fundamental period

$\boxed{\sin(x + 2\pi) = \sin x}$ $\qquad \epsilon = 2\pi$ period

* $\boxed{\begin{array}{l} f(x) = \sin x \longrightarrow \epsilon = 2\pi \\ f(nx) = \sin nx \longrightarrow \epsilon = \frac{2\pi}{n} \end{array}}$

$\sin(x + 2\pi) = \sin x \;,\; \sin(nx + 2\pi) = \sin nx$

$\sin\frac{x}{2} \longrightarrow P = 4\pi$ expand

$\sin 2x \longrightarrow P = \frac{2\pi}{2} = \pi$ contract

$\sin 3x \longrightarrow P = \frac{2\pi}{3}$

demand

* $x \longrightarrow x' = \underbrace{x + \epsilon}_{\sim 2\pi k \text{ (say)}} \Rightarrow f(nx') = \sin n(x + 2\pi k) = \sin(nx + 2\pi nk) = f(nx) = \sin nx$

$k = \frac{1}{n} \Rightarrow \sin(nx + 2\pi) = \sin nx$

Scaling of angle of trig. fn^n changes the Period by $\frac{2\pi}{\text{scale factor}}$

Detour to T-Eq^n

* $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha$

* $\cos\theta = \cos\alpha \Rightarrow \cos\theta - \cos\alpha = 0 \Rightarrow -2\sin\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right) = 0$

$\left.\begin{array}{l} \sin\left(\frac{\theta+\alpha}{2}\right) = 0 = \sin n\pi \Rightarrow \theta = 2n\pi - \alpha \\ \sin\left(\frac{\theta-\alpha}{2}\right) = 0 = \sin n\pi \Rightarrow \theta = 2n\pi + \alpha \end{array}\right\} \Rightarrow \theta = 2n\pi \pm \alpha$

$\boxed{\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha} \longrightarrow \cos(2n\pi \pm \alpha) = \cos\alpha$

* $\alpha = \frac{\pi}{2} \longrightarrow \boxed{\cos\theta = 0 \Rightarrow \theta = (2n+1)\frac{\pi}{2}} \longrightarrow \boxed{\cos\left((2n+1)\frac{\pi}{2}\right) = 0} \longrightarrow \cos\left(n\pi + \frac{\pi}{2}\right) = 0$

$\cos n\pi = (-1)^n$

Transposition

x	*	$2x + 5 = 0$
$\frac{1}{x}$	*	$\frac{1}{x} + 5 = 0$
x, y	*	$x + y = 2$, $x - y = 7$
$\underbrace{\frac{1}{x}}_{t}\ \underbrace{\frac{1}{y}}_{q}$	*	$\frac{1}{x} + \frac{1}{y} = 2,\ \frac{1}{x} - \frac{1}{y} = 7$; $t + q = 2 \quad t - q = 7$

complexity ↑↑

ex:

$\theta = \ldots$ → general sol^n

$\tan\theta + \tan 2\theta + \tan 3\theta = 1$

$x + y + z = 1$

↳ ∞ many sol^n

pattern

Lecture-3 (17/nov) 2

* $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha \quad n \in \mathbb{Z}$ $\quad \underbrace{\text{General sol}^n}$ of T-Eq^n

$\boxed{\text{Periodicity} \Rightarrow \exists \text{ general sol}^n}$

* $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha \quad n \in \mathbb{Z} \quad \xrightarrow[\text{set}]{n = n_0}$ Particular sol^n

* $\tan\theta = \tan\alpha \Rightarrow \frac{\sin\theta}{\cos\theta} - \frac{\sin\alpha}{\cos\alpha} = 0 \Rightarrow \sin(\theta - \alpha) = 0 = \sin n\pi \Rightarrow \boxed{\theta = n\pi + \alpha}$

$\boxed{\text{Algebra (0)}}$

$\nexists$ Periodicity

$\left.\begin{array}{l} 2x + 3y = 1 \\ x + 5y = 2 \end{array}\right\}$

$(x, y) = 1$ sol^n

General sol^n = Particular sol^n = "Just" sol^n

$\boxed{\text{Calculus}}$

$\frac{dy}{dx} = f(x)$ D.E

general sol^n

$y = \int f(x)\,dx = $ dep. on x

+ Boundary Cond^n

$y(x_0) = f(x_0)$

Particular sol^n

ex:

$\frac{dy}{dx} = x$

$\int dy = \int x\,dx$

$y = \frac{x^2}{2} + C$ $\quad C \in \mathbb{R}$ General sol^n

+BC

$\int_a^b x\,dx = \frac{1}{2}(b^2 - a^2)$ indep of x

Particular sol^n

own thing 'n' is acting in the same way as 'C'

$y_1 = \frac{x^2}{2} + 3,\ y_2 = \frac{x^2}{2} + 5,\ y_3 = \frac{x^2}{2} + \pi$ dep. on x

Practice 1

HW

Q1 a. $f(\psi) = 2\sin\psi + 1$ (Machine) $\longrightarrow \boxed{f(\psi) = 0}$ "Machine turns off" $\quad \psi = ?$

b. $g(\chi) = \operatorname{cosec}\chi - 1 \longrightarrow g(\chi) = 0 \quad \chi = ?$

c. $h(\theta) = \cos\theta - \frac{1}{2} \longrightarrow h(\theta) = 0 \quad \theta = ?$

a. $\psi = m\pi + (-1)^{n+1}\frac{\pi}{6} \quad n \in \mathbb{Z}$ $\qquad \pi^{\circ} \to \pi$

b. $\chi = n\pi + (-1)^n \frac{\pi}{2}$

c. $\theta = 2n\pi \pm \frac{\pi}{3}$

Q2 a. $\sqrt{3}\sec 2z = 2 \Rightarrow \sec 2z = \frac{2}{\sqrt{3}} \Rightarrow \cos 2z = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6} \Rightarrow 2z = 2n\pi \pm \frac{\pi}{6} \Rightarrow z = n\pi \pm \frac{\pi}{12}$

$\frac{1}{x} = x^{-1}$

$\frac{1}{\sin} = \sin^{-1}$

b. $f(\psi) \equiv \tan\left(\frac{2\psi}{3}\right) - \sqrt{3} \longrightarrow f(\psi) = 0 \Rightarrow \psi = \psi(n) = ?$

$\tan\frac{2\psi}{3} = \tan\frac{\pi}{3} \Rightarrow \frac{2\psi}{3} = n\pi + \frac{\pi}{3} \Rightarrow \psi(n) = \frac{3n\pi}{2} + \frac{\pi}{2} \quad n \in \mathbb{Z}$

Q3 $h(\alpha) \equiv \sin\alpha + \sin 3\alpha + \sin 5\alpha \longrightarrow h(\alpha) = 0 \Rightarrow \alpha = \alpha(n) = ?$

$h(\alpha) = 2\sin 3\alpha\cos 2\alpha + \sin 3\alpha$

$= \sin 3\alpha\,(2\cos 2\alpha + 1) = 0$

$\to \sin 3\alpha = 0 \Rightarrow \boxed{\alpha = \frac{n\pi}{3}}$

or

$\to \cos 2\alpha = -\frac{1}{2} = \cos\frac{2\pi}{3} \Rightarrow 2\alpha = 2n\pi \pm \frac{2\pi}{3} \Rightarrow \boxed{\alpha = n\pi \pm \frac{\pi}{3}} \quad n \in \mathbb{Z}$

$\sin\theta = \sin\alpha$

$\Downarrow$

$\theta = n\pi \pm (-1)^n\alpha$

$S + S = 2SC$

$-\frac{1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right)$

$= \cos\frac{2\pi}{3}$

HW

Q4. $f(\theta) = \cos\theta + \cos 3\theta - 2\cos 2\theta = 0 \Rightarrow \theta = \theta(n)$ $\quad n \in \mathbb{Z}$

Q5 $g(\alpha) = \sin m\alpha + \sin n\alpha = 0 \Rightarrow \alpha = \alpha(m, n; \underbrace{p, q}_{\in \mathbb{Z}})$

Q6. $\sin 2\psi + \sin 4\psi + \sin 6\psi = 0 \Rightarrow \psi = ?$

Q7. $\tan^2\chi + (1-\sqrt{3})\tan\chi - \sqrt{3} = 0 \Rightarrow \chi = ?$

Lecture 4 (18/11) 1.40

$\to \theta = 2n\pi + \alpha$ Even

$\to \theta = (2n+1)\pi - \alpha$ odd

4. $\cos 2\theta = 0 \Rightarrow \theta = \left(n\pi \pm \frac{\pi}{4}\right) \longrightarrow 2\theta = (2n+1)\frac{\pi}{2} \Rightarrow \theta = (2n+1)\frac{\pi}{4}$

$\cos\theta = 1 \Rightarrow \theta = 2m\pi$

$\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha$

$\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha$

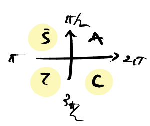
$\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha$

5. $g(\alpha) = \sin m\alpha + \sin n\alpha = 0$

$\sin m\alpha = -\sin n\alpha = \sin(\pi + n\alpha) \Rightarrow m\alpha = 2p\pi + (\pi + n\alpha)$ Even p

or

$m\alpha = (2q+1)\pi - (\pi + n\alpha)$ odd q



$\operatorname{cosec}\left(\frac{3\pi}{2} + \theta\right) = -\sec\theta$

$\sin\theta = 0 \Rightarrow \theta = n\pi$

$\cos\theta = 0 \Rightarrow \theta = 2n\pi \pm \frac{\pi}{2} \to \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

$\Downarrow$

$\theta = (2n+1)\frac{\pi}{2}$

$\tan\theta = 0 \Rightarrow \theta = n\pi$

$\cot\theta = 0 \Rightarrow \theta = (2n+1)\frac{\pi}{2}$

even p

$(m-n)\alpha = (2p+1)\pi \Rightarrow \alpha = \frac{(2p+1)\pi}{m-n}$

or

$(m+n)\alpha = 2q\pi \Rightarrow \alpha = \frac{2q\pi}{m+n}$

q odd

$p, q \in \mathbb{Z}$

6. $\sin 2\psi + \sin 4\psi + \sin 6\psi = 0 \Rightarrow \cos 2\psi = -\frac{1}{2} \Rightarrow \psi = m\pi \pm \frac{\pi}{3}$

$\sin 4\psi = 0 \Rightarrow \psi = \frac{n\pi}{4}$

$S + S = 2SC$

7. $\tan^2 x + (1-\sqrt{3})\tan x - \sqrt{3} = 0 \Rightarrow x = n\pi - \frac{\pi}{4}$ or $m\pi + \frac{\pi}{3}$ $m,n \in \mathbb{Z}$

$\tan x = \dfrac{-(1-\sqrt{3}) \pm \sqrt{(1-\sqrt{3})^2 + 4\sqrt{3}}}{2} = \dfrac{\sqrt{3}-1 \pm \sqrt{1+3-2\sqrt{3}+4\sqrt{3}}}{2} = \dfrac{\sqrt{3}-1 \pm \sqrt{1+3+2\sqrt{3}}}{2} = \dfrac{\sqrt{3}-1 \pm (1+\sqrt{3})}{2}$

HW

$x^2 = a \Rightarrow x = \pm a$

$x^2 + \sqrt{17}x = 0$

8. $\cot^2\beta + 3\operatorname{cosec}\beta + 3 = 0$

9. $2\cos^2\gamma + 3\sin\gamma = 0$

10. $4\cos\theta - 3\sec\theta = \tan\theta$

11. $\tan\delta + \tan 2\delta + \tan\delta\tan 2\delta = 1$

12. $\tan\varepsilon + \tan(\varepsilon + \frac{\pi}{3}) + \tan(\varepsilon + \frac{2\pi}{3}) = 3$

13. $\tan x + \tan 2x + \sqrt{3}\tan x \tan 2x = \sqrt{3}$

Lecture-5 (19/Nov) 2

8. $\beta = n\pi - (-1)^n\frac{\pi}{2} = n\pi + (-1)^{n+1}\frac{\pi}{2}$ $n \in \mathbb{Z}$ ✓

or

$\beta = n\pi + (-1)^{n+1}\frac{\pi}{6}$

9. $\gamma = n\pi + (-1)^{n+1}\frac{\pi}{6}$; $\sin\gamma = 2 \rightarrow$ N.D.

10. $\sin\theta = \dfrac{-1 \pm \sqrt{17}}{8}$
$\rightarrow \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha$; $\alpha = \sin^{-1}\left(\frac{-1+\sqrt{17}}{8}\right)$
or
$\rightarrow \sin\beta \Rightarrow \theta = n\pi + (-1)^n\beta$; $\beta = \sin^{-1}\left(\frac{-1-\sqrt{17}}{8}\right)$

11. $\tan\delta + \tan 2\delta = 1 - \tan\delta\tan 2\delta \Rightarrow \dfrac{\tan\delta + \tan 2\delta}{1 - \tan\delta\tan 2\delta} = 1 \Rightarrow \boxed{\tan 3\delta = 1} = \tan\frac{\pi}{4} \Rightarrow 3\delta = n\pi + \frac{\pi}{4} \Rightarrow \delta = \frac{n\pi}{3} + \frac{\pi}{12}$ $n \in \mathbb{Z}$

Linear		Quadratic	
* $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha$	$\longrightarrow$	$\sin^2\theta = \sin^2\alpha$	
$\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha$ ✓		$\cos^2\theta = \cos^2\alpha$	$\longrightarrow \cos 2\theta = \cos 2\alpha \Rightarrow \theta = n\pi \pm \alpha$
$\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha$		$\tan^2\theta = \tan^2\alpha$	

$\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm (-1)^n\alpha$

$\cos^2\theta = \cos^2\alpha \Rightarrow \theta =$

$\tan^2\theta = \tan^2\alpha \Rightarrow \theta =$

* $\sin^2\theta = \sin^2\alpha \Rightarrow \dfrac{1-\cos 2\theta}{2} = \dfrac{1-\cos 2\alpha}{2} \Rightarrow \cos 2\theta = \cos 2\alpha \Rightarrow 2\theta = 2n\pi \pm 2\alpha \Rightarrow \boxed{\theta = n\pi \pm \alpha}$

✓ $\cos 2\theta = 2\cos^2\theta - 1$
$= 1 - 2\sin^2\theta$

$2\sin^2\theta = 1 - \cos 2\theta$

* $\cos^2\theta = \cos^2\alpha \Rightarrow \dfrac{\cancel{1}+\cos 2\theta}{\cancel{2}} = \dfrac{\cancel{1}+\cos 2\alpha}{\cancel{2}} \Rightarrow \cos 2\theta = \cos 2\alpha \Rightarrow \boxed{\theta = n\pi \pm \alpha}$ $n \in \mathbb{Z}$

* $\tan^2\theta = \tan^2\alpha \Rightarrow \dfrac{\sin^2\theta}{\cos^2\theta} = \dfrac{\sin^2\alpha}{\cos^2\alpha} \Rightarrow \dfrac{1-\cos 2\theta}{1+\cos 2\theta} = \dfrac{1-\cos 2\alpha}{1+\cos 2\alpha} \Rightarrow \dfrac{2}{\cancel{1}+\cancel{2}\cos 2\theta} = \dfrac{\cancel{2}}{\cancel{1}+\cancel{2}\cos 2\alpha} \Rightarrow \cos 2\theta = \cos 2\alpha$
$\Downarrow$
$\boxed{\theta = n\pi \pm \alpha}$

$\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a+b}{a-b} = \frac{c+d}{c-d}$

Practice 2

$\left\{\begin{array}{l}\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha \\ \cos^2\theta = \cos^2\alpha\end{array}\right.$

Q1. $7\cos^2\psi + 3\sin^2\psi = 4$ { Trig. Eqn quad. in sin, cos

$\underbrace{\cos^2\psi}_{(1-\sin^2\psi)}$

1st route

$7 - 7\sin^2\psi + 3\sin^2\psi = 4 \Rightarrow \cancel{-}4\sin^2\psi = \cancel{-}3 \Rightarrow \sin^2\psi = \frac{3}{4} \longrightarrow \sin\psi = \pm\frac{\sqrt{3}}{2} = \pm\sin\frac{\pi}{3} = \sin\pm\frac{\pi}{3} \Rightarrow \psi = n\pi + (-1)^n\left(\pm\frac{\pi}{3}\right)$

$\to n\pi + (-1)^n \pi/3$

$\to n\pi + (-1)^{n+1}\frac{\pi}{3}$

n=1 n=3

$\pi - \pi/3$, $2\pi + \pi/3$

$\pi + \pi/3$, $2\pi - \pi/3$

n=1 n=2

2nd

$\sin^2\psi = \left(\frac{\sqrt{3}}{2}\right)^2 = \left(\sin\frac{\pi}{3}\right)^2 \Rightarrow \boxed{\psi = n\pi \pm \frac{\pi}{3}}$

$-\pi + \frac{\pi}{3}$, $\pi + \frac{\pi}{3}$, $2\pi + \frac{\pi}{3}$ ...

... $-\pi - \pi/3$, $\pi - \pi/3$, $2\pi - \frac{\pi}{3}$ ...

Q2. $2\sin^2\theta + \sin^2 2\theta = 2 \Rightarrow \theta = n\pi \pm \frac{\pi}{2}$ or $\theta = m\pi \pm \frac{\pi}{4}$

$\Downarrow$

$2\sin^2\theta + (2\sin\theta\cos\theta)^2 = 2$

$A \cdot B = 0 \Rightarrow A = 0$ or $B = 0$

HW

Q3. $\sin 3\alpha = 4\sin\alpha\sin(x+\alpha)\sin(x-\alpha)$: $\alpha \neq n\pi$ $n \in \mathbb{Z}$

Q4. $4\sin\psi\sin 2\psi\sin 4\psi = \sin 3\psi$

Q5. $\cos\beta + \operatorname{cosec}\beta = \sqrt{3}$

Lecture-6 (19/Nov) 1.5

HW

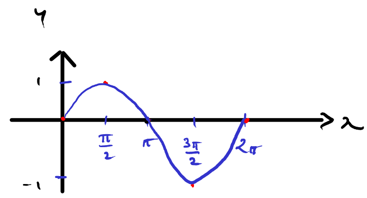
Q6. 1. $\sqrt{3}\cos x + \sin x = \sqrt{2}$

2. $\sqrt{2}\sec\theta + \tan\theta = 1$

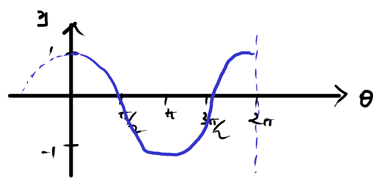
1.a.1. Basic T-curves

* $y = f(\theta) = \sin\theta \in [-1, 1]$

Pendulum / H.O. / wave / Periodic wave



* $y = \cos\theta = \sin\left(\frac{\pi}{2} - \theta\right) \in [-1, 1]$



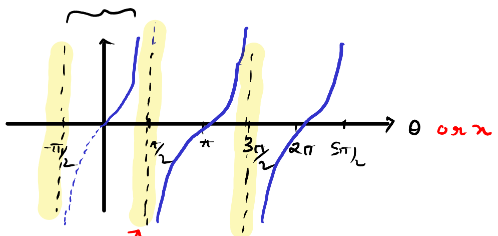
* $y = \tan\theta \in (-\infty, \infty)$

Strictly ↑↑ from $-\infty$ to ∞

$P = \pi$, Fundamental period $= \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\theta \in \mathbb{R} - \left\{(2n+1)\frac{\pi}{2}\right\}$

$y \in \mathbb{R}$



$\boxed{\begin{array}{c}\theta = \pm\frac{(2n+1)\pi}{2} \sim x = \text{const} \\ \text{Asymptotes}\end{array}}$

* $y = \operatorname{cosec}\theta$

0	30	45	60	90
∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

S A
T C

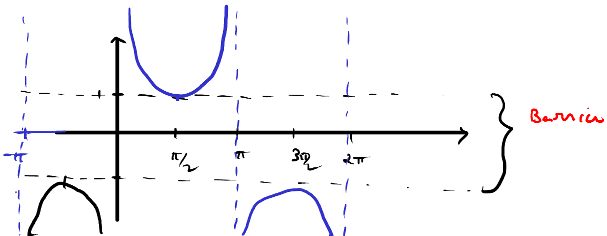
π

$\left.\begin{array}{l}0 < \theta < 90 \\ 90 < \theta < 180\end{array}\right\} \Rightarrow \operatorname{cosec} > 0$

$\pi < \theta < 2\pi \Rightarrow \operatorname{cosec}\theta < 0$

$0 < \theta < 90 \longrightarrow \operatorname{cosec}\theta \downarrow\downarrow$ to 1

$90 < \theta < 180 \longrightarrow \operatorname{cosec}\theta \uparrow\uparrow$ to ∞



$\lim_{\theta \to \pi} \operatorname{cosec}\theta = \infty$

Dom $= \mathbb{R} - \{ n\pi : n \in \mathbb{Z} \}$, Asymptotes $= \{ \text{lines } L : \theta = n\pi, n \in \mathbb{Z} \}$

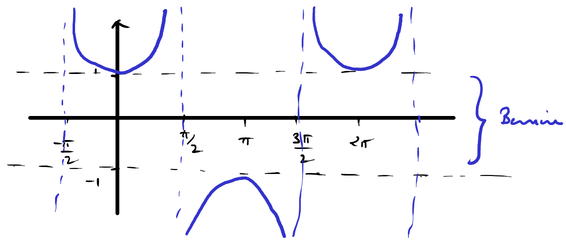
Range $= \mathbb{R} - (-1, 1)$

Period $= 2\pi$

* $y = \sec\theta$

	0	30	45	60	90
$y=\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞

S	A
T	C



$0 < \theta < 90°$ $\sec\theta \uparrow\uparrow$ to ∞

Dom $= \mathbb{R} - \{ (2n+1)\frac{\pi}{2} : n \in \mathbb{Z} \}$

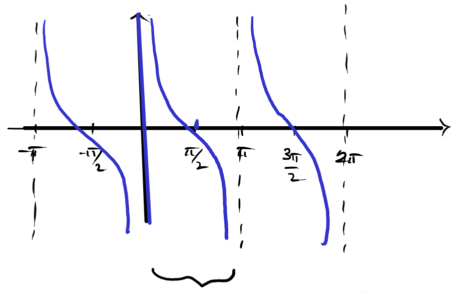
Range $= \mathbb{R} - (-1, 1)$ Asymptote $= \{ L : \theta = (2n+1)\frac{\pi}{2} ; n \in \mathbb{Z} \}$

Period $= 2\pi$

* $y = \cot\theta$

	0	30	45°	60	90
$y=\cot\theta$	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

S	A
T	C



$0 < \theta < 90°$ $\cot\theta \downarrow\downarrow 0$

$90° < \theta < 180°$ $\cot\theta < 0$

Dom $= \mathbb{R} - \{ n\pi : n \in \mathbb{Z} \}$

Range $= \mathbb{R}$ Asympt $= \{ L : \theta = n\pi, n \in \mathbb{Z} \}$

Period $= \pi$

1.a.2 Notion of Periodicity

* Periodicity $\Rightarrow$ $x \xrightarrow{T} x' \equiv x + \epsilon \longrightarrow \psi(x+\epsilon) = \psi(x)$ $\sin(2\pi + \theta) = \sin\theta$ 2π-periodic

* Anti-periodicity $\Rightarrow$ $x \xrightarrow{T} x' \equiv x + \epsilon \longrightarrow \psi(x+\epsilon) = -\psi(x)$ $\sin(\pi + \theta) = -\sin\theta$ π-Antiperiodic

Exponents

$a^0 = 1$

$a^n \cdot a^m = a^{n+m}$

$a^n \cdot b^n = (ab)^n$

Special Case

Example:

* $\psi_1(x) = A_1 e^{ik_x x}$ $\left. \begin{array}{l} x \in (-\infty, \infty) \\ k_x \in (-\infty, \infty) \end{array} \right\}$ no constraint / Continuous values $\longleftrightarrow x$ $\longleftrightarrow k_x$

Periodicity along $\hat{x}$ (Demand)

$\underbrace{\text{Periodicity}}_{\text{Compact}}$

x_1 x_2 x_3

$\psi_1(x+a) = \psi_1(x) \Rightarrow A_1 e^{ik_x(x+a)} = A_1 e^{ik_x x} \Rightarrow e^{ik_x a} = 1 = e^{\pm 2n\pi i} \Rightarrow \boxed{k_x = \pm \frac{2n\pi}{a}}$ $n = 0, 1, 2 \ldots$

$e^{i\theta} = \cos\theta + i\sin\theta = 1$

$e^{i(2n\pi)} = \cos 2n\pi + i\sin 2n\pi = 1$

$\sin 2n\pi = 0$

$\cos n\pi = (-1)^n$

$\cos 2n\pi = (-1)^{2n} = 1$

$\boxed{e^{\pm 2n\pi i} = 1}$

k_x is an integer multiple of $\frac{2\pi}{a}$

k : Quantized / Discritized

x : Periodic

Lecture-7 (24/Nov)

* $\psi(x+\epsilon) = \psi(x) \Rightarrow$ Periodic

* $\psi(\underset{\sim}{r}) = A e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} = A e^{i(k_x x + k_y y)} = A e^{ik_x x} \cdot e^{ik_y y} = A_1 e^{ik_x x} \cdot A_2 e^{ik_y y} \equiv \psi(x)\cdot\psi(y)$ variable separation Method

$\underset{\sim}{r} = x\hat{i} + y\hat{j}$

$\underset{\sim}{k} = k_x\hat{i} + k_y\hat{j}$

$\psi(x) \equiv A_1 e^{ik_x x} \xrightarrow{\psi(x+a) = \psi(x)} A_1 e^{ik_x(x+a)} = A_1 e^{ik_x x} \Rightarrow e^{ik_x a} = 1 = e^{\pm 2n\pi i} \Rightarrow \boxed{k_x = \pm\frac{2n\pi}{a}}$

special pts

$\psi(y) \equiv A_2 e^{ik_y y} \xrightarrow[\text{Periodicity}]{\psi(y+b) = \psi(y)} A_2 e^{ik_y(y+b)} = A_2 e^{ik_y y} \Rightarrow e^{ik_y b} = 1 = e^{\pm 2n\pi i} \Rightarrow \boxed{k_y = \pm\frac{2n\pi}{b}}$

$(x, y) \to -\infty$ to ∞

$(k_x, k_y) \to -\infty$ to ∞

Continuous

$x \leftrightarrow k_x$

$y \leftrightarrow k_y$

$\underset{\sim}{k}\cdot\underset{\sim}{r} = x k_x + y k_y$

$A \equiv A_1 \cdot A_2$

$\boxed{\begin{array}{l}\text{if } x \,\&\, k \text{ are related / connected / "Conjugate variables" then} \\ \text{if } \exists \text{ a symmetry in } x \Rightarrow \exists \text{ discrete values of } k\end{array}}$

$e^{i\theta} = \cos\theta + i\sin\theta$

$\cos n\pi = (-1)^n$

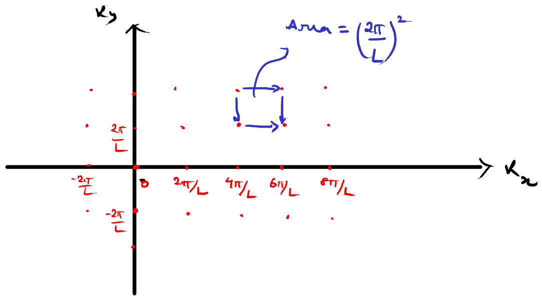
$\sin n\pi = 0$

$\cos 2n\pi = 1$

$\sin 2n\pi = 0$

$e^{\pm 2n\pi i} = 1$

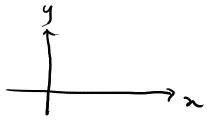
* $a = b = \ell \Rightarrow K_x = K_y = \pm \frac{2n\pi}{\ell}$ ✓



- Plot the variable that has been discretized

in this case (K_x, K_y)-plane

not the (x,y)-plane {Euclidean}



nothing special

except $(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2$

$e^{ik\cdot x}$

units:

x space $[x] = L \longrightarrow [a] = [b] = L$ (length)

k space $[k] = \frac{1}{L} \longrightarrow$

- $\exists$ a unit "square" in the discretized k-space (Area $= \left(\frac{2\pi}{L}\right)^2$)

- periodicity in $x \Rightarrow$ Quantization of k in units of $\frac{2\pi}{L}$

Conjugate Pair

$x \longrightarrow p_x = m\frac{dx}{dt}$

$y \longrightarrow p_y = m\frac{dy}{dt}$

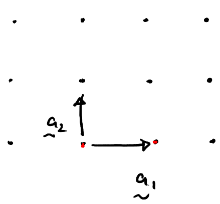
- x-space $\equiv$ Real space / position space

k-space $\equiv$ Fourier space / transformed space / Momentum space / Dual space

* $\boxed{\Lambda = \{ \underset{\sim}{r} \equiv n_1 \underset{\sim}{a}_1 + n_2 \underset{\sim}{a}_2 \; ; \; n_i \in \mathbb{Z} \}}$

Lattice (Real Lattice)

a_i : primitive basis vectors



r-space

$\underset{\sim}{a}_1 = a\hat{x} \quad \underset{\sim}{a}_2 = a\hat{y} \Rightarrow \Lambda = \{ \underset{\sim}{r} \equiv n_1 a\hat{x} + n_2 a\hat{y} \; ; \; n_i \in \mathbb{Z} \}$

Cubic lattice

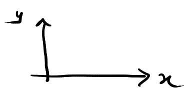
$= \{ \underset{(1,1)}{a\hat{x} + a\hat{y}}, \underset{(2,1)}{2a\hat{x} + a\hat{y}}, \underset{(-1,3)}{-1a\hat{x} + 3a\hat{y}}, \underset{(-7,2)}{-7a\hat{x} + 2a\hat{y}} \dots \}$

$\boxed{\exists \text{ a translation sym. } \begin{array}{l} x \longrightarrow x' \equiv x + a \\ y \longrightarrow y' \equiv y + b \end{array}}$

Property of Real lattice

Lecture-8 (26/11) 1.5 + 10

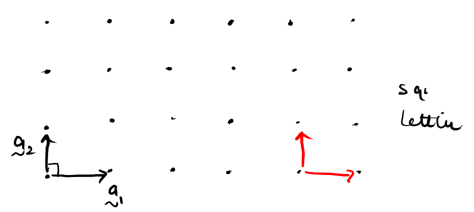
* $\mathbb{R}^2 = \{ (x,y) : x \in \mathbb{R}, y \in \mathbb{R} \}$ $(-\infty, \infty)$ continuous / Euclidean space



* $\Lambda = \{ \underset{\sim}{r} \equiv \sum_{i=1}^{2} n_i \underset{\sim}{a}_i \; : \; n_i \in \mathbb{Z} \}$ Discrete / Bravais Lattice / Direct Lattice / Real lattice

$\underset{\sim}{a}_i$: primitive basis vectors

$\exists$ a minimum separation / distance $\Rightarrow \exists$ $\underbrace{\text{minimum / unit area}}_{\text{unit cell}}$



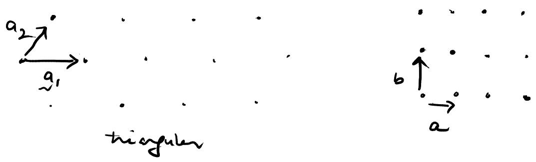
interLattice spacing

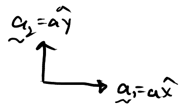
* Square : $\underset{\sim}{a}_1 = a\hat{x}$, $\underset{\sim}{a}_2 = a\hat{y}$ (Given)

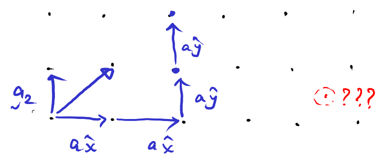
$\Lambda = \{ \underset{\sim}{r} = n_1 \underset{\sim}{a}_1 + n_2 \underset{\sim}{a}_2 \; : \; n_i \in \mathbb{Z} \}$

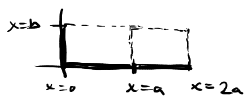
$= \{ \underset{\sim}{r} \equiv n_1 a\hat{x} + n_2 a\hat{y} \; : \; n_i \in \mathbb{Z} \} = \{ \underset{\sim}{r}_1, \underset{\sim}{r}_2, \underset{\sim}{r}_3, \dots \underset{\sim}{r}_n \}$

$= \{ a\hat{x} + a\hat{y}, 2a\hat{x} + 3a\hat{y}, \dots \} \not\ni \{ \frac{a}{2}\hat{x} + \frac{a}{3}\hat{y} \}$









$\ast$ $\forall\ \Lambda\ \exists\ \Lambda^* = \left\{ \underset{\sim}{K} = \sum_{i=1}^{n} n_i \underset{\sim}{b_i} \ : \ n_i \in \mathbb{Z} \right\}$ Fourier space / Momentum space / Dual sp. / Reciprocal space / frequency space

$\boxed{\underset{\sim}{b_i} \ : \ \underset{\sim}{a_i} \cdot \underset{\sim}{b_j} = 2\pi\, \delta_{ij}}$ new primitive basis vector

demand $e^{i\underset{\sim}{K}\cdot\underset{\sim}{r}} = 1 \quad \forall\ \underset{\sim}{r} \in \Lambda,\ \underset{\sim}{K} \in \Lambda^*$

$\hat{e}_j$, $\hat{e}_i$

$\underset{\sim}{\hat{e}_i} \cdot \underset{\sim}{\hat{e}_j} = \delta_{ij}$ ✓

Euclidean space

$\underset{\sim}{A} = \sum_i a_i \underset{\sim}{\hat{e}_i},\ \underset{\sim}{B} = \sum_i b_i \underset{\sim}{\hat{e}_i}$

$\underset{\sim}{A}\cdot\underset{\sim}{B} = \sum_{ij} a_i b_j \underbrace{\underset{\sim}{\hat{e}_i}\cdot\underset{\sim}{\hat{e}_j}}_{\delta_{ij}} = a_i b^i$

$\underset{\sim}{a_1}\cdot\underset{\sim}{b_1} = 2\pi$

$a\hat{x}\cdot(\underset{\sim}{b_1}) = 2\pi$, $\underset{\sim}{b_1} = \frac{2\pi}{a}\hat{x}$

ex: cubic : $\underset{\sim}{a_1} = a\hat{x},\ \underset{\sim}{a_2} = a\hat{y},\ \underset{\sim}{a_3} = a\hat{z}$ Real lattice

$\underset{\sim}{b_1} = \frac{2\pi}{a}\hat{x} \quad \underset{\sim}{b_2} = \frac{2\pi}{a}\hat{y} \quad \underset{\sim}{b_3} = \frac{2\pi}{a}\hat{z}$ Fourier space

$x \longleftrightarrow p_x$

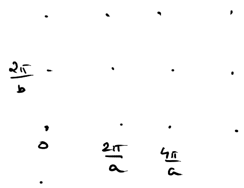
$t \longleftrightarrow \omega$

Real var. Fourier variables

$\ast$ $\psi(x) = A_1 e^{iK_x x} \xrightarrow{\psi(x+a) = \psi(x)} \cancel{A_1} e^{iK_x(x+a)} = \cancel{A_1} e^{\cancel{iK_x x}} \Rightarrow \overbrace{e^{iK_x a}} = 1 = e^{\pm 2n\pi i} \Rightarrow K_x = \pm\frac{2\pi}{a} n$

$\psi(y) = A_2 e^{iK_y y} \xrightarrow[\psi(y+b) = \psi(y)]{} K_y = \pm \frac{2\pi}{b} n$

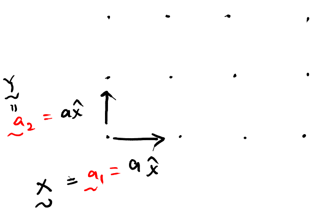
continuous



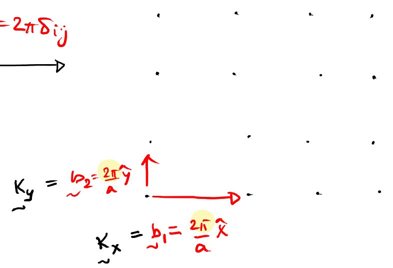
discrete

Real lattice / sp

Fourier / Reciprocal lattice / space



$\xrightarrow{a_i \cdot b_j = 2\pi\delta_{ij}}$



$\underset{\sim}{x} \to \underset{\sim}{x} + \underset{\sim}{a} \ : \ \psi(\underset{\sim}{x}+\underset{\sim}{a}) = \psi(\underset{\sim}{x})$

$[\underset{\sim}{a}] \sim L$

$[\underset{\sim}{b}] \sim \frac{1}{L} = L^{-1}$ reciprocal

$\boxed{\text{transform the problem from Real space} \longrightarrow \text{fourier space} \ \because\ \exists \text{ periodicity}}$

$\downarrow$ More Rigorous

Fourier transform

$\ast$ $\psi(x) \ : \ \psi(x+\epsilon) = e^{iK\epsilon}\psi(x)$ $\begin{cases} K = 0 \Rightarrow \psi(x+\epsilon) = \psi(x) & \text{Periodic} \\ K = \frac{\pi}{\epsilon} \Rightarrow \psi(x+\epsilon) = -\psi(x) & \text{Antiperiodic} \end{cases}$ Block periodic function (Condensed matter Physics)

Felix block, 1929

Lecture - 9 (26/nov) 1.5+10

$\ast$ $x \xrightarrow{T} x' = x+\epsilon \ : \ \psi(x+\epsilon) = e^{\widetilde{iK\epsilon}}\psi(x)$ {Block periodic fn It is usually Complex fn

$\psi(x+a) = \psi(x)$ Period.

$\psi(x+a) = -\psi(x)$ Anti.

$\psi(x+a) = \underset{\sim}{e^{iKa}}\psi(x)$

$(\cos Ka + i\sin Ka)$

ex: $Ka = \frac{\pi}{6} \Rightarrow K = \frac{\pi}{6a}$

$\psi(x+a) = \left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)\psi(x)$

$\leftarrow$ CW $\rightarrow$

$\ast$ $\psi(\underset{\sim}{r}) = e^{i\underset{\sim}{K}\cdot\underset{\sim}{r}}\, U(\underset{\sim}{x})$: $U(\underset{\sim}{r}) = U(\underset{\sim}{r}+\underset{\sim}{a})$ lattice periodicity

check $\Downarrow$

$U(\underset{\sim}{r}) = e^{-i\underset{\sim}{K}\cdot\underset{\sim}{r}}\psi(\underset{\sim}{r}) \xrightarrow[\underset{\sim}{r} \to \underset{\sim}{r}+\underset{\sim}{a}]{\hat{T}} U(\underset{\sim}{r}+\underset{\sim}{a}) = e^{-i\underset{\sim}{K}\cdot(\underset{\sim}{r}+\cancel{\underset{\sim}{a}})}\underbrace{\psi(\underset{\sim}{r}+\underset{\sim}{a})}_{e^{i\underset{\sim}{K}\cdot\underset{\sim}{a}}\psi(\underset{\sim}{r})} = e^{-i\underset{\sim}{K}\cdot\underset{\sim}{x}}\psi(\underset{\sim}{r}) = U(\underset{\sim}{r})$

$\ast$ $\boxed{\psi(r) = e^{iKr}U(x) \xrightarrow{\psi(r+a) = e^{iKa}\psi(r)} U(r+a) = U(r)}$

1.b. Exponential Mechanics

$\ast$ $f(x) = \sum_{i=0}^{n} a_i x^i = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$ Polynomial

$\ast$ $f(x) = a^x$ $a > 0$, $a \neq 1$ 'Exponential fn^n'; $x \in \mathbb{R}$, $a \in \mathbb{R}^+ - \{1\}$

x^2

$x^{3\times10^8} <<<< exp.$

Polynomial growth | exp. growth

Case I $a > 1$

$\ast$ $f(x) = a^x = \begin{cases} > 1 & x > 0 \\ = 1 & x = 0 \\ < 1 & x < 0 \end{cases}$

$\lim_{x \to \infty} a^x = a^\infty = \infty$

$\lim_{x \to -\infty} a^x = a^{-\infty} = \frac{1}{\infty} = 0$

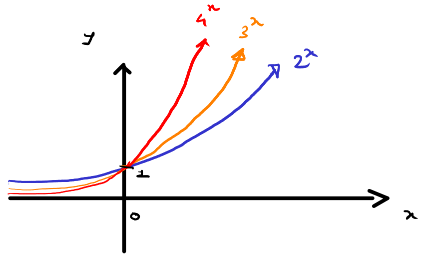
($\exists$ an asymptote $L \equiv y = 0$)

$x > 0$ $\quad 2^x < 3^x < 4^x < 5^x$

$x = 3$ $\quad 2^3 \quad 3^3 \quad 4^3 \quad 5^3$

$x < 0$ $\quad 2^x > 3^x > 4^x > 5^x$

$x = -1$ $\quad \frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \frac{1}{5}$



Monotonically Increasing curve
(exp. growth)

Case II : $0 < a < 1$

$\ast$ $f(x) = a^x = \begin{cases} < 1 & x > 0 \\ 1 & x = 0 \\ > 1 & x < 0 \end{cases}$

$\lim_{x \to \infty} a^x = a^\infty \sim \left(\frac{1}{b}\right)^\infty \sim \frac{1}{\infty} = 0$

$\lim_{x \to -\infty} a^x = a^{-\infty} = \frac{1}{a^\infty} \sim \frac{1}{\left(\frac{1}{b}\right)^\infty} = b^\infty \sim \infty$

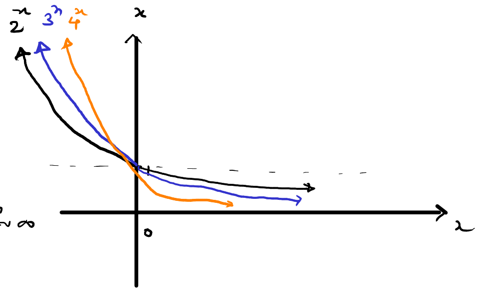
$\exists$ an asymptote $y = 0$

$x > 0$ $\quad \frac{1}{2^x} > \frac{1}{3^x} > \frac{1}{4^x} > \frac{1}{5^x}$

$x = 3$ $\quad \frac{1}{2^3} > \frac{1}{3^3} \quad \frac{1}{4^3} \quad \frac{1}{5^3}$

$x < 0$ $\quad \frac{1}{2^x} < \frac{1}{3^x} < \frac{1}{4^x} < \frac{1}{5^x}$

$x = -1$ $\quad 2 \quad 3 \quad 4 \quad 5$



Monotonically decreasing curve
(exp. decay)

dom $f(x) = \mathbb{R}$

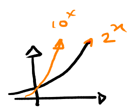
Range $f(x) = (0, \infty)$

Transcendental functions are able to capture 'This World' more accurately.
& hence they are transcendent to Algebra but immanent to this world (physical systems)

Imp/ Interesting result.

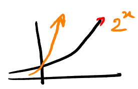
$\ast$ $f(x) = a^x$ (exponent: x; Base: a)

Base
- base-10 (decimal) $\longrightarrow f(x) = 10^x$ $\quad a > 1$



greater growth

- Natural / Eulerian base = e $\Big\}$ e^x, e^{-x} $\longrightarrow e^x + e^{-x} = ? \longrightarrow ??$
- base-2 (binary) $\longrightarrow f(x) = 2^x$ $\quad a > 1$



smaller growth

Lecture-10 (27/Nov) 2

Detour to AP and GP (part 2)

2.b.1 Simple Interest / growth

* Problems of growth / decline
  - → Calculus $\frac{dy}{dx}$, $\lim_{h \to 0}$
  - → Progression / Series $a_1, a_2 \ldots a_n$

* P — init amt of substance → P changes by a certain Rate (r) → After t time what is the amount of the substance? — Problem in physics!

* P — init amt of Money → P changes by a certain Rate of Interest (r) → After t years/annual what is the amt / current balance? — Problem in finances / Economics?

* Sample 1: $P = 10$ € $\xrightarrow{\text{To the Bank}}$ $r = 2\%$, $t = 10$ years (Policies)

$10 \times 2\% = 0.2$ € = Interest / growth rate (Fixed)

$(10 \times 2\%) \times 10$ years $= 2$ € = final growth achieved

Amt left after 10 years = Balance = $10 + 2 = 12$ €

(10: Init amt; 2: growth)

P = init amt, r = rate of interest (%), t = # of years

$$\boxed{SI(t) = Prt}$$

$$Amt(t) = P + Prt \quad \text{`At the End'}$$

∃ Remembrance (memory) of the history of the sys

'Simple Interest' (Simplest Model of growth)

(withdrawal) experiment ends: 0 ——— $t-1$ ——— t

* t^{th} term of series: $SI(t) = Prt$, $SI(t-1) = Pr(t-1)$

$SI(1) = Pr$
$SI(2) = Pr(2)$
$SI(3) = Pr(3)$
⋮

} series. Arithmetic Progression

$SI(t) - SI(t-1) = Prt - (Pr(t-1)) = Pr = \text{const} \Rightarrow$ $\boxed{SI \text{ forms an AP}}$

$$S_n(n) = \sum_{n=0}^{\infty} (2n^2+1)$$

$a_n = 2n^2 + 1$

$a_{n-1} = 2(n-1)^2 + 1$

$a_n - a_{n-1} = \begin{cases} \text{indep of } n & \text{AP} \\ \text{dep on } n & \text{not AP} \end{cases}$

$= 2(n^2 - (n-1)^2)$

$2(2n-1) = \text{dep. on } n$

Sample 2: System's init Energy = 10 J, r = Energy added to the sys/sec $= 5\%$ (% of init Energy)

t = Energy in the system after t time = 30 sec.

a. Energy added $= 10 \times 5\% = 10 \times \frac{5}{100} = 0.5$ J $= x$

b. Energy added after t time $= \underbrace{(10 \times 5\%)}_{x} \times 30 = 0.5 \times 30 = 15$ J

10 J: 1s → $10 \times 5\% = x$; 2s → x; 3s → x; ⋮ ts

Entire History is unforgotten

Static History

c. total Energy acquired after t time $= 10 + (15) = 25\,J$

1.b.2. Compound Interest ($n = 1$: Compounding once in t time)

static history

$I \longrightarrow F$

$\exists$ a inverse relⁿ

$\Downarrow$

going from F to I is a possibility

$\Downarrow$

sys has not forgot the init state

(NM)

{unitarity Principle}

till here story is same as that of

$CI(1) = SI(1)$

* $CI(1) = \underset{\text{init amt}}{p}\, r \cdot \underset{\text{end time}}{1}$ $\Rightarrow$ $\underbrace{Amt(1)}_{P(t=1) \equiv P(1)} = p + pr = p(1+r)$

orig history

* $CI(2) = P(1) \cdot r$ $\Rightarrow$ $\underbrace{Amt(2)}_{\equiv P(2)} = P(1) + CI(2)$

original history Forgotten

$= p(1+r) + p(1+r)\cdot r$

$= p(1+r)(1+r)$

$= p(1+r)^2$

$$p \xrightarrow[\text{once in } t^{th} \text{ yr}]{\uparrow} \underset{\substack{P(t^{th}) \\ \shortparallel}}{Amt(t^{th})}$$

$\underbrace{\text{once in } t^{th}\text{ year}}_{CI(t^{th})}$

$\uparrow$ once in t^{th} yr

$P(t')$

$\uparrow$ in t'

$P(n) = P(n-1) + CI(n)$

* $CI(3) = P(2) \cdot r$ $\Rightarrow$ $Amt(3) = P(2) + CI(3)$

$= p(1+r)^3$

⋮

$o \longrightarrow t$

$I \qquad F$

$\nexists$ a time reversed sys.

1^{st} 2^{nd}

* $CI(t) = P(t-1)\,r$ $\Rightarrow$ $\underbrace{Amt(t)}_{P(t)} = P(t-1) + CI(t)$

$P(t) = p(1+r)^t$ — (withdrawal time) time of End of Experiment

Amt. of substance generated at the withdrawal time after being compounded once/yr

init/original history

Rate of interest / Increment / decline

Calculated / Compounded once a yr./annually

$p \to CI(1) \to P(1)$ ✓

$p \to CI(1_a) \to P(1_a)$, $\to CI(1_b) \to P(1_b)$

Compounding twice in a yr.

* $\boxed{P(t) = p(1+r)^t = p\,a^t}$ $\qquad a \equiv 1+r > 1$

Lecture-11 (1/dec) 2

* $\underset{\text{orig.}}{p_0} \xrightarrow[\text{once in } t^{th}\text{ yr/min}]{\uparrow} \underset{\substack{P(t^{th}) \\ \shortparallel}}{Amt(t^{th})}$

$\uparrow$ on orig. history: $SI(t^{th}) = \underset{\text{const}}{p_0} \cdot r \cdot t$

$\boxed{Amt(t) = p_0 + SI(t)}$

Intermediary states are not imp.

p_0 ———— $P(t)$

$0 \qquad t$

$\uparrow$ once in t'th year/min/sec on Realtime history: $CI(t) = \underbrace{P(t-1)}_{\text{function if known}} r$ ✓ $\Rightarrow$ $\underset{\substack{\shortparallel \\ P(t)\ ✓}}{Amt(t)} = P(t-1) + CI(t)$ } recursive rel. of $P(t)$

withdrawal time / End of experiment

$\boxed{P(t) = p(1+r)^t \equiv p_0 a^t}$ $\qquad a > 1$

Compound Interest (not linear but exponential)

* $P(t) = p_0(1+r)^t$

$P(t-1) = p_0(1+r)^{t-1}$

* $\dfrac{P(t)}{P(t-1)} = (1+r) = \text{const}$ $\Rightarrow$ C.I forms a G.P

* $P(t) - P(t-1) = p_0(1+r)^t\left\{1 - \dfrac{1}{1+r}\right\} = p_0(1+r)^t \dfrac{r}{1+r} = \underbrace{\dfrac{p_0 r}{1+r}}_{\text{const}} \underbrace{(1+r)^t}_{\text{dep on } t}$ $\Rightarrow$ Not an A.P.

Simple growth

$10 \times 2\% = \underbrace{0.2}_{\text{Every year a fix. Number}}\,€ = SI(1)$

after 10 yrs : $SI(10) = 0.2 \times 10 = 2\,€$ growth

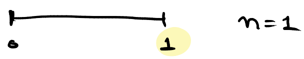
" " " : $Amt(10^{th}) = p_0 + SI(10)$

$= 10 + 2 = 12$

1.6.3. Compounding k times in a year/min/sec/t

* $(p_0, r) \longrightarrow CI(t=1) = p_0 r$ — Calculation/Measurement done once in t time (1 time compounded)

init amt, Rate

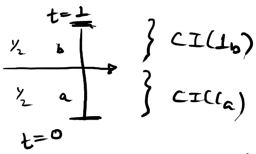


* $(p_0\ r) \rightarrow CI(1_a) = \frac{p_0 r}{2} \Rightarrow \underbrace{Amt(1_a)}_{P(1_a) = \text{amt after half a yr}} = p_0 + \frac{p_0 r}{2} = p_0\left(1+\frac{r}{2}\right)$

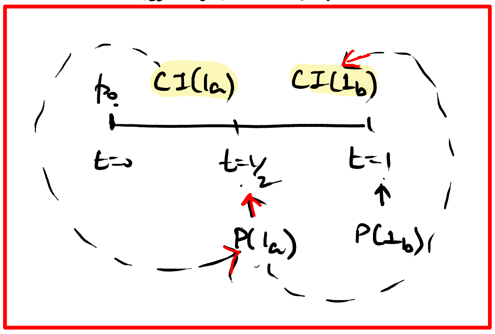
$\rightarrow CI(1_b) = P(1_a)\frac{r}{2} \Rightarrow \underbrace{Amt(1_b)}_{P(1_b)} = P(1_a) + CI(1_b)$

$= p_0\left(1+\frac{r}{2}\right) + p_0\left(1+\frac{r}{2}\right)\frac{r}{2}$

$= p_0\left(1+\frac{r}{2}\right)\left\{1+\frac{r}{2}\right\} = p_0\left(1+\frac{r}{2}\right)^2$



Scenario at the end of 1st yr



* $CI(2_a) = P(1_b)\frac{r}{2} \Rightarrow \underbrace{Amt(2_a)}_{P(2_a)} = P(1_b) + CI(2_a)$

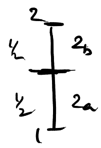
growth in 1st half of year 2

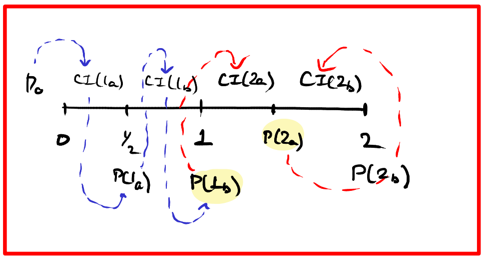
$= p_0\left(1+\frac{r}{2}\right)^2 + p_0\left(1+\frac{r}{2}\right)^2\frac{r}{2} = p_0\left(1+\frac{r}{2}\right)^3$

$CI(2_b) = P(2_a)\frac{r}{2} \Rightarrow \underbrace{Amt(2_b)}_{P(2_b)} = P(2_a) + CI(2_b)$

growth in 2nd half of yr. 2

$= p_0\left(1+\frac{r}{2}\right)^3 + p_0\left(1+\frac{r}{2}\right)^3\frac{r}{2} = p_0\left(1+\frac{r}{2}\right)^{2\cdot 2}$



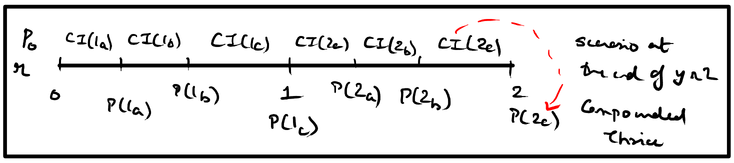


Scenario at the end of 2nd yr.

$Amt(2_b) = p_0\left(1+\frac{r}{2}\right)^{2\cdot 2}$

at the End of 2nd yr

Compounded twice a yr.
n=2



$CI(1_a) = p_0\frac{r}{3}$

$P(1_a) = p_0\left(1+\frac{r}{3}\right)$

$P(2_c) = p_0\left(1+\frac{r}{3}\right)^{2\cdot 3}$

* $$\boxed{P(t_k) = p_0\left(1+\frac{r}{k}\right)^{t\cdot k}}$$

amt at the end of t^{th} yr./min/sec.

Compounded k Times a yr./min/sec

"Cool" Analysis/observation

* $p_0 = 1$ € ; $r = 100\%$ very generous $t=1$

Case 1

* $P(1_1) = 1+1 = 2$ € $t=1$ / once a yr

good returns

Case II

* $r = 100\%$ interest/annum → 50% 1st semester; → 50% 2nd sem.

$P(1_2) = \left(1+\frac{1}{2}\right)^2 = 2.25$ €

Case III

* $r = 100\%$ inter. Calc. Every month → 1/12, → 1/12, ⋮, → 1/12 $\longrightarrow P(1_{12}) = \left(1+\frac{1}{12}\right)^{12} = 2.6130$ €

great returns

Case IV

* Calc interest every day $\to \frac{1}{365}$, $\to \frac{1}{365}$, ⋮, $\to \frac{1}{365}$

$$\longrightarrow P(1_{365}) = \left(1+\frac{1}{365}\right)^{365} = 2.714567\ldots \text{€}$$

Case V

* Calc. interest every min $\to \frac{1}{365\times24\times60}$, $\to$, $\to$ ⋮, $\to$

$$\longrightarrow P(1_{365\times24\times60}) = \left(1+\frac{1}{365\times24\times60}\right)^{365\times24\times60} = 2.7182\ldots \text{€}$$

Pathetic returns / gains

Case 10^{23}

* $P(1_{10^{23}}) = \left(1+\frac{1}{10^{23}}\right)^{10^{23}} = 2.71828\ldots$ €

$P(1_\infty) = \left(1+\frac{1}{\infty}\right)^\infty \longrightarrow e \sim 2.7182 \Rightarrow \exists$ a limiting value of the Amt

$P(t_K) = P_0\left(1+\frac{r}{K}\right)^{t\cdot K} \xrightarrow{r=1} \lim_{K\to\infty} P(t_K) = P_0 e^t \qquad 1 < e < 3 \qquad \exists$ an upper bound

$P(t) = P_0(1+r)^t \longrightarrow P(t) = P_0 a^t \qquad a = 1+r > 1 \qquad \nexists$ any upper bound

exponential

Lecture 12 (2/dec) 1.15

J. Bernoulli 1680s

1.6.4 Continuously Compounded ($K\to\infty$ / consistency check of e as a limit)

* P_0 = init amt, r = rate, K = # of times compounded, t = time of withdrawal

$0 \quad K^{th} \quad t$

K = No. of divisions $\to\infty$

$P(t_K) = P_0\left(1+\frac{r}{K}\right)^{tK}$ = Amt after t^{th} yr compounded K times / yr

* $$P(t_\infty) = \lim_{K\to\infty} P_0\left(1+\frac{r}{K}\right)^{tK} = \lim_{K\to\infty} P_0\left[\left(1+\frac{r}{K}\right)^K\right]^t = P_0\left[\underbrace{\lim_{K\to\infty}\left(1+\frac{r}{K}\right)^K}_{\equiv S}\right]^t$$

* $S = \left(1+\frac{r}{K}\right)^K \Rightarrow \ln_e S = \ln\left(1+\frac{r}{K}\right)^K = K\ln\left(1+\frac{r}{K}\right)$

Powers $\rightleftarrows$ log.

$\Downarrow$

$\lim_{K\to\infty} S = \lim_{K\to\infty}\left(1+\frac{r}{K}\right)^K = ?$

$\log a^b = b\log a$

$$\lim_{K\to\infty}(\ln S) = \lim_{K\to\infty} \overset{(\infty\cdot 0)}{K\ln\left(1+\frac{r}{K}\right)} = \lim_{K\to\infty}\left[\frac{\ln_e\left(1+\frac{r}{K}\right)}{\frac{1}{K}}\right] \xrightarrow[\text{Rule}]{\text{L'Hôpital}} \lim_{K\to\infty} \frac{\frac{1}{\left(1+\frac{r}{K}\right)}\cdot\left(\cancel{-}\frac{r}{\cancel{K^2}}\right)}{\left(\frac{\cancel{-1}}{\cancel{K^2}}\right)} = \lim_{K\to\infty}\frac{r}{\left(1+\frac{r}{K}\right)}$$

differentiable

$1^\infty \neq 1$

$\uparrow$

Indeterminate form

$$\underbrace{\lim_{K\to\infty}(\ln S)}_{=\ln\left(\lim_{K\to\infty} S\right)} = r \Rightarrow \lim_{K\to\infty} S = e^r \Rightarrow \boxed{\lim_{K\to\infty}\left(1+\frac{r}{K}\right)^K = e^r}$$

V. imp. property of lim, log.

Digression to Mathematical Induction

* $P(n)$ = any statement $\to$ T ✓ to check, $\to$ F

<u>Algorithm</u>

* Step 1 (Base case) : prove $P(1)$ is True $\Rightarrow$ $P(n)$ is true for $n = 1$

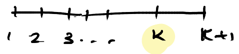
  Step 2 : Let / Assume $P(K)$ is true $n = K$ (Premise)

  Step 3 : use $P(K)$ to prove $P(K+1)$ is true
  $P(K)$ is T $\Rightarrow$ $P(K+1)$ is true } Inductive Case

  Step 4 : Principle says $P(n)$ is true $\forall\ n \in \mathbb{N}$

<u>Lecture-13</u> (3/12) 1:45

Q1. $P(n) = 1+2+3+\dots+n = \dfrac{n(n+1)}{2}$



* 1. $n=1$ $\quad P(1) = 1 = \dfrac{1(1+1)}{2}$ $\Rightarrow P(1)$ is True

  2. $n = K$ Let / Assume $P(K)$ is True $\Rightarrow$ $P(K) = 1+2+3+\dots+K = \dfrac{K(K+1)}{2}$ —— ① (Premise)

  3. To prove $P(K+1)$ is true { ie : $1+2+\dots\ K+1 = \dfrac{(K+1)(K+2)}{2}$

  4. Proof : $P(K+1) = \underbrace{1+2+\dots+K}_{\frac{K(K+1)}{2} \text{ from ①}} + K+1 = \dfrac{K(K+1)}{2} + K+1 = \dfrac{K(K+1)+2(K+1)}{2} = \dfrac{(K+1)(K+2)}{2} = RHS \Rightarrow P(K+1)$ is True

  5. if $P(K)$ is T $\Rightarrow$ $P(K+1)$ is T $\Rightarrow$ $P(n)$ holds true $\forall\ n \in \mathbb{N}$ Q.E.D.

Q.2. $P(n)$: " n^2+n is Even Natural # "

* $P(1)$: $1+1 = 2 \in \mathbb{N}_{even}$ $\Rightarrow P(1)$ T

* $P(K)$: $K^2+K \in \mathbb{N}_{even}$ Premise (Let)

$P(K+1)$

$\Downarrow$

$\boxed{K^2+K = 2\alpha}$ $\quad \alpha \in \mathbb{N}$ —— ①

* To prove $P(K+1)$ is T

  $P(K+1) : (K+1)^2 + (K+1) = K^2+1+2K+K+1 = \underbrace{K^2+K}_{=2\alpha \text{ eq①}} + 2(K+1) = 2\{\alpha + K+1\} \equiv 2\beta$ $\quad \beta \equiv \alpha+K+1 \in \mathbb{N}$

* $P(K)$ is T $\Rightarrow$ $P(K+1)$ is T $\longrightarrow$ $P(n)$ is true $\forall\ n \in \mathbb{N}$

HW
Q3. $P(n)$: $n(n+1)(2n+1)$ is divisible by 6 $\quad n \in \mathbb{N}$

HW
Q4. $P(n) = 1+4+7+\dots\ (3n-2) = \dfrac{n}{2}(3n-1)$

Q5. $\boxed{P(n) : \ 1^2+2^2+3^2+\dots+n^2 = \dfrac{n(n+1)(2n+1)}{6}}$ sum of sq. of 1st n Natural #

Imp. for Series

<u>AP</u> (a, d)

$a_n = a+(n-1)d$

$S_n = \dfrac{n}{2}(2a+(n-1)d)$

$= \dfrac{n}{2}(a+a_n)$

* P(1): $1^2 = \frac{1(1+1)(2\times1+1)}{6} = \frac{1\times\cancel{2}\times\cancel{3}}{\cancel{6}} = 1 \Rightarrow P(1)$ is T

* Assumption P(K) is T $\Rightarrow 1^2+2^2+3^2+\dots+K^2 = \frac{K(K+1)(2K+1)}{6}$ —— (1)

* To prove P(K+1) is T $\left\{\text{ie: } 1^2+\dots+(K+1)^2 = \frac{(K+1)(K+2)(2K+3)}{6}\right.$

$$P(K+1) = \underbrace{1^2+2^2+3^2+\dots+K^2}_{\text{from (1)}}+(K+1)^2 = \frac{K(K+1)(2K+1)}{6}+(K+1)^2 = \frac{1}{6}(K+1)\{K(2K+1)+6(K+1)\} = \frac{1}{6}(K+1)(K+2)(2K+3)$$

* P(K) is T $\Rightarrow$ P(K+1) is T $\Rightarrow$ P(n) is T $\forall\ n \in \mathbb{N}$

HW

Q6. $\boxed{P(n):\ 1^3+2^3+3^3+\dots+n^3 = \left(\frac{n(n+1)}{2}\right)^2}$ Sum of cubes of 1^{st} n terms

Imp for series.

closed sys.

Q7 P(n): $1\cdot3+2\cdot3^2+3\cdot3^3+\dots+n\cdot3^n = \frac{(2n-1)3^{n+1}+3}{4}$ $n \in \mathbb{N}$.

* $P(1) = 1\cdot3^1 = \frac{(2\times1-1)3^{1+1}+3}{4} = \frac{1\times3^2+3}{4} = \frac{9+3}{4} = 3 \Rightarrow$ P(1) is True

* Assumption (Premise): P(K) is T $\Rightarrow 1\cdot3+2\cdot3^2+3\cdot3^3\dots+K\cdot3^K = \frac{(2K-1)3^{K+1}+3}{4}$ —— (1)

* to prove: P(K+1) is T

$$1\cdot3+2\cdot3^2+3\cdot3^3+\dots K\cdot3^K+(K+1)\cdot3^{K+1} = \dots\dots\dots = \frac{(2(K+1)-1)\,3^{(K+1)+1}+3}{4} \Rightarrow P(n) \text{ is T } \forall\ n \in \mathbb{N}$$

closed sys.

this is stone is black
that is .... .. } Aristotle

All stones are black

Inductive logic

Q8. P(n): $|\sin nx| \le n|\sin x|$ $x \in \mathbb{R},\ n \in \mathbb{N}$

* P(1): $|\sin x| = 1|\sin x| \Rightarrow$ P(1) is T

* Assume P(K) is T $\Rightarrow |\sin Kx| \le K|\sin x|$ —— (1)

* To prove P(K+1) is T $\{$ ie: $|\sin(K+1)x| \le (K+1)|\sin x|$ ✓

P(K+1). $|\sin(K+1)x| = |\sin Kx\cos x+\cos Kx\sin x| \le \underbrace{|\sin Kx|}_{\le K|\sin x| \text{ (from (1))}}|\cos x|+|\cos Kx||\sin x|$

$|A+B| \le |A|+|B|$

$|AB| = |A||B|$

$|\sin(K+1)x| \le K|\sin x||\cos x|+|\cos Kx||\sin x| \le K|\sin x|\cdot1+1\cdot|\sin x| \le (K+1)|\sin x|$

$-1 \le \cos x \le 1$

P(n) is T $\forall\ n \in \mathbb{N}$

$\boxed{|\cos x| \le 1}$

$K|\sin x||\cos x| \le K|\sin x|$

HW

Q9 P(n): $\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\dots+\frac{1}{n(n+1)} = \frac{n}{n+1}$

HW

Q10. P(n): $1+\frac{1}{1+2}+\frac{1}{1+2+3}+\frac{1}{1+2+3+4}\dots+\frac{1}{1+2+\dots+n} = \frac{2n}{n+1}$ $\forall\ n \in \mathbb{N}$

HW

Q11. Q(n): $\frac{1}{1\cdot3}+\frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\dots+\frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

$\boxed{\exists\ 6 \text{ HW ques. for tom.}}$

Lecture-14 (41dec) 1.5

$\sum n = \frac{n(n+1)}{2}$
$\sum n^2 = \frac{n(n+1)(2n+1)}{6}$
$\sum n^3 = \left(\frac{n(n+1)}{2}\right)^2$

Q12. $Q(n)$: $1\cdot2\cdot3 + 2\cdot3\cdot4 + 3\cdot4\cdot5 \cdots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$ $\forall\, n \in \mathbb{N}$

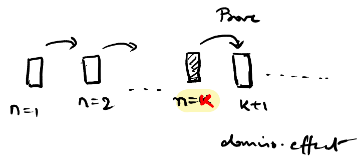
* $P(1)$: $1\cdot2\cdot3 = \frac{1\cdot(1+1)(1+2)(1+3)}{4} = \frac{1\cdot2\cdot3\cdot\cancel{4}}{\cancel{4}}$ $\Rightarrow$ $P(1)$ is T

* Assume $P(K)$ is T $\Rightarrow$ $1\cdot2\cdot3 \cdots + K(K+1)(K+2) = \frac{K(K+1)(K+2)(K+3)}{4}$ —— ①

* To prove $P(K+1)$ is T $\Big\{$ ie: $1\cdot2\cdot3 \cdots + (K+1)(K+2)(K+3) \overset{?}{=} \frac{(K+1)(K+2)(K+3)(K+4)}{4}$

$P(K+1)$: $1\cdot2\cdot3 \cdots + (K+1)(K+2)(K+3) = \cdots\cdots$ $\Rightarrow$ $P(n)$ is true $\forall\, n \in \mathbb{N}$

Q13. $S(n) = n^3 + 3n^2 + 5n + 3$ is divisible by 3 $\forall\, n \in \mathbb{N}$



* $S(1) = 1 + 3(1)^2 + 5 + 3 = 12$ divisible by 3 $\Rightarrow$ $S(1)$ is T

* $S(K) = K^3 + 3K^2 + 5K + 3$ (Assumption)

* $S(K+1) = (K+1)^3 + 3(K+1)^2 + 5(K+1) + 3 = ?$ $\cdots$ $\Rightarrow$ $P(n)$ is T $\forall\, n \in \mathbb{N}$

Q14. $P(n)$: $x^n - y^n$ is divisible by $(x-y)$ $\{x, y\} \in \mathbb{Z}$, $n \in \mathbb{N}$

* $P(1)$: $x^1 - y^1 = x - y$ $\Rightarrow$ divisible $\Rightarrow$ $P(1)$ T

* $P(K)$: $x^K - y^K$ is divisible by $x-y$ $\Rightarrow$ $\frac{x^K - y^K}{x-y} = \lambda = \text{const}$ $\Rightarrow$ $x^K - y^K = (x-y)\lambda$ —— ①
'Assumption'

* $P(K+1)$: $x^{K+1} - y^{K+1}$ is divisible by $(x-y)$ 'To prove' $\Big\{$ ie: $x^{K+1} - y^{K+1} = (x-y)\{\cdots\cdots\}$

$P(K+1)$: $x^{K+1} - y^{K+1} = x^K \cdot x - y^K y = x^K x - x^K y + x^K y - y^K y = x^K(x-y) + y\underbrace{(x^K - y^K)}_{=(x-y)\lambda \text{ eq.①}} = (x-y)\{x^K + \lambda y\}$

$P(K+1)$ is T $\Rightarrow$ $P(n)$ is T $\forall\, n$

Q15. $P(n)$: $x^{2n} - y^{2n}$ is divisible by $(x+y)$ $\{x, y\} \in \mathbb{Z}$

* $P(1)$: $x^2 - y^2 = x^2 - y^2 = (x+y)(x-y)$ $\Rightarrow$ $x^2 - y^2$ divisible by $(x+y)$

$P(K+1)$: $x^{2K+2} - y^{2K+2} = (x+y)\{x^{2K} - yK\}$ $\cdots$ HW TBD

HW

Q16. 1. $P(n)$: $41^n - 14^n$ is multiple of 27

2. $Q(n)$: $7^n - 3^n$ is divisible by 7

Lecture-15 (8/dec) 2

hw
Q17 a. $P(n): (2^{3n}-1)$ divisible by 7 $\forall\, n \in \mathbb{N}$

b. $Q(n): (10^{2n-1}+1)$ " " " $\forall\, n \in \mathbb{N}$

Type 2: trig. + inequalities

Q18. $P(n): 1^2+2^2+3^2 \cdots + n^2 > \frac{n^3}{3} \quad \forall\, n \in \mathbb{N}$

* $P(1): 1^2 > \frac{1^3}{3} \Rightarrow P(1)$ is T

$P(k): 1^2+2^2+\dots k^2 > \frac{k^3}{3}$ — (1) Assumption

* $P(k+1): 1^2+2^2+\dots (k+1)^2 > \frac{(k+1)^3}{3}$ (to prove!) $\Rightarrow$

$(k+1)^3 = k^3+1+3k(k+1)$
$= k^3+1+3k^2+3k$
$(k+1)^3+3k+2 = k^3+3k^2+6k+3$

$1^2+2^2+3^2+\dots k^2 > \frac{k^3}{3} \Rightarrow 1^2+2^2+\dots k^2+(k+1)^2 > \frac{k^3}{3}+(k+1)^2 = \frac{1}{3}\left\{k^3+3(k^2+1+2k)\right\} = \frac{1}{3}\left\{\underbrace{k^3+3k^2+6k+3}_{(k+1)^3+(3k+2)}\right\}$

$1^2+2^2+\dots (k+1)^2 > \frac{1}{3}\left\{(k+1)^3+(3k+2)\right\} > \frac{1}{3}(k+1)^3 \Rightarrow P(n)$ is T $\forall\, n \in \mathbb{N}$

$A > B+C$
$A > B$

hw
Q19 $P(n): 1+2+\dots+n < \frac{(2n+1)^2}{8} \quad \forall\, n \in \mathbb{N}$

Q20 a) $Q(n): \left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{n}\right) = n+1$

hw b) $R(n): \left(1+\frac{1}{3}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\cdots\left(1+\frac{2n+1}{n^2}\right) = (n+1)^2 \quad \forall\, n \in \mathbb{N}$

Q21. a) $P(n): \sin\theta+\sin 2\theta+\sin 3\theta+\dots+\sin n\theta = \dfrac{\sin\left(\frac{n+1}{2}\right)\theta \,\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}$

* $P(1): \sin\theta = \dfrac{\sin\left(\frac{1+1}{2}\right)\theta\, \cancel{\sin\frac{\theta}{2}}}{\cancel{\sin\theta/2}} = \sin\frac{2\theta}{2} \Rightarrow P(1)$ is True

* $P(k): \sin\theta+\sin 2\theta+\dots+\sin k\theta = \dfrac{\sin\left(\frac{k+1}{2}\right)\theta\, \sin\frac{k\theta}{2}}{\sin\frac{\theta}{2}}$ — (1) Assumption

* $P(k+1): \sin\theta+\sin 2\theta+\dots+\sin k\theta+\sin(k+1)\theta = \dfrac{\sin\left(\frac{k+2}{2}\right)\theta\, \sin\frac{(k+1)\theta}{2}}{\sin\frac{\theta}{2}}$ (to prove)

$\sin 2\theta = 2\sin\theta\cos\theta$
$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$LHS = \underbrace{\sin\theta+\sin 2\theta+\dots+\sin k\theta}_{eq.(1)}+\sin(k+1)\theta = \dfrac{\sin\frac{(k+1)}{2}\theta\,\sin\frac{k\theta}{2}}{\sin\frac{\theta}{2}} + \sin(k+1)\theta = \dfrac{\sin\frac{(k+1)}{2}\theta\,\sin\frac{k\theta}{2}+\overbrace{\sin(k+1)\theta}^{=2\sin\frac{(k+1)}{2}\theta\cos\frac{(k+1)}{2}\theta}\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}}$

$= \dfrac{\sin\frac{(k+1)}{2}\theta}{\sin\frac{\theta}{2}}\left\{\cancel{\sin\frac{k\theta}{2}} + \underbrace{2\cos\frac{(k+1)}{2}\theta\,\sin\frac{\theta}{2}}_{=\sin\frac{(k+2)}{2}\theta-\cancel{\sin\frac{k\theta}{2}}}\right\} = \dfrac{\sin\left(\frac{k+2}{2}\right)\theta\,\sin\frac{(k+1)}{2}\theta}{\sin\frac{\theta}{2}} \Rightarrow P(k+1)$ is T

$\Downarrow$

$P(n)$ is T $\forall\, n \in \mathbb{N}$

$\frac{C+D}{2} \quad \frac{C-D}{2} \quad C \quad D$
$S+S = 2SC$
$S-S = 2CS$
$2\sin C\cos D = \sin\frac{C+D}{2}+\sin\frac{C-D}{2}$
$C \leftrightarrow D$
$\underbrace{2\sin D\cos C}_{2\cos C\sin D} = \sin\frac{C+D}{2} + \sin\frac{D-C}{2}$
$= \sin\left(\frac{C+D}{2}\right) - \sin\left(\frac{C-D}{2}\right)$

HW

b) $P(n): \cos\beta \cos 2\beta \cos 4\beta \ldots \cos 2^{n-1}\beta = \dfrac{\sin 2^n\beta}{2^n \sin\beta} \quad \forall\ n \in \mathbb{N}.$

Q22. Given : $a_1 = 1$, $a_2 = 1$, $a_{n+2} = a_{n+1} + a_n \quad n \geq 1$ 1 1 2 3 5 8 13 . . .

Recursive rel^n

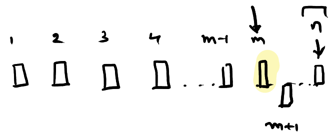
$$P(n): a_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\} \quad \forall\ n \geq 1$$

'Fibonacci's relation'

* $P(1)$: $a_1 = \frac{1}{\sqrt{5}}\left\{\left(\frac{\cancel{1}+\sqrt{5}}{2}\right)^1 - \left(\frac{\cancel{1}-\sqrt{5}}{2}\right)^1\right\} = \frac{1}{\cancel{\sqrt{5}}}\cdot\cancel{\sqrt{5}} = 1 = a_1 \Rightarrow P(1)$ is T

* $P(k)$: $a_k = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right\}$ — 1 Assumption

* $P(k+1)$: $a_{k+1} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right\}$ to prove



$\boxed{P(m) \text{ is True TILL } m \text{ VALUE}}$

$$a_{k+1} = a_k + a_{k-1} \quad \forall \underbrace{(n-1) \geq 1}_{n \geq 2}$$

$$= \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k + \left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right\}$$

$$= \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\underbrace{\left[1 + \frac{1+\sqrt{5}}{2}\right]}_{\frac{3+\sqrt{5}}{2}} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\underbrace{\left[1 + \frac{1-\sqrt{5}}{2}\right]}_{\frac{3-\sqrt{5}}{2}}\right\}$$

verify

surds

$$1 + \frac{1+\sqrt{5}}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{3+\sqrt{5}}{2}$$

$$1 + \frac{1-\sqrt{5}}{2} = \left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{3-\sqrt{5}}{2}$$

$$= \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right\} \Rightarrow P(n) \text{ is T}$$

HW

Q23. a, $P(n)$: $4^n + 15n - 1$ divisible by 9

b $Q(n)$: $10^n + 3\cdot 4^{n+2} + 5$,, ,, 9

c $R(n)$: $2\cdot 7^n + 3\cdot 5^n - 5$,, ,, 24

Lecture-16 (9/dec.) 1:15

Q24. $P(n)$: $(2n+7) < (n+3)^2 \quad \forall\ n \in \mathbb{N}$ using this prove $\underbrace{(n+3)^2 \leq 2^{n+3}}_{Q(n)} \quad \forall\ n \in \mathbb{N}$

* $P(1)$: $(2\cdot 1 + 7) < (1+3)^2 \Rightarrow 9 < 16 \Rightarrow P(1)$ is True

* $P(k)$: $2k + 7 < (k+3)^2$ Assumption —— 1

* $P(k+1)$: $2(k+1) + 7 < (k+4)^2$ to prove

* $2k + 7 < (k+3)^2$ . . . . QED.

Q25. Bernoulli's inequality :

$$\boxed{(1+a)^n > (1+na) \qquad a > -1,\ a \neq 0,\ n > 2}$$

$\boxed{\text{out of dignmus to induction}}$

1.6.5. Convergence test of series (using Bernoulli inequality) / lim as $e = 2.718\ldots$

* $a_n = \left(1+\frac{1}{n}\right)^n$ $\qquad a_{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1}$

* $$\frac{a_{n+1}}{a_n} = \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^n} = \frac{\left(\frac{n+2}{n+1}\right)^{n+1}}{\left(\frac{n+1}{n}\right)^n} = \left(\frac{n+2}{n+1}\right)^{n+1}\left(\frac{n}{n+1}\right)^n = \left(\frac{n+2}{n+1}\right)^{n+1}\underbrace{\left(\frac{n}{n+1}\right)^n\cdot\frac{n}{n+1}}_{\left(\frac{n}{n+1}\right)^{n+1}}\cdot\frac{n+1}{n} = \left(\frac{(n+2)n}{(n+1)^2}\right)^{n+1}\cdot\frac{n+1}{n}$$

$$= \left(\frac{n^2+2n}{n^2+2n+1}\right)^{n+1}\cdot\left(\frac{n+1}{n}\right) = \left(\frac{n^2+2n+1-1}{n^2+2n+1}\right)^{n+1}\cdot\left(\frac{n+1}{n}\right) = \left(1-\frac{1}{(n+1)^2}\right)^{n+1}\cdot\left(\frac{n+1}{n}\right)$$

Bernoulli's

$(1+a)^n > (1+na)$

$a = \frac{-1}{(n+1)^2}$, $n = n+1$ set

$$\left(1-\frac{1}{(n+1)^2}\right)^{n+1} > \underbrace{\left(1+\cancel{(n+1)}\left(\frac{-1}{\cancel{(n+1)^2}}\right)\right)}_{\underbrace{1-\frac{1}{n+1}}_{\frac{n}{n+1}}}$$

Caution:

$\because \left(1-\frac{1}{(n+1)^2}\right)^{n+1} > \frac{n}{n+1}$ (via Bernoulli) ✓

$$\underbrace{\left(1-\frac{1}{(n+1)^2}\right)^{n+1}\cdot\frac{n+1}{n}}_{\frac{a_{n+1}}{a_n}} > \cancel{\frac{n}{n+1}}\cdot\cancel{\frac{n+1}{n}} > 1 \Rightarrow \boxed{a_{n+1} > a_n}$$ sequence is increasing ↑↑

$\frac{a_{n+1}}{a_n} > 1$

* $b_n = \left(1+\frac{1}{n}\right)^{n+1}$ $\qquad b_{n+1} = \left(1+\frac{1}{n+1}\right)^{n+2}$

$$\frac{b_n}{b_{n+1}} = \frac{\left(1+\frac{1}{n}\right)^{n+1}}{\left(1+\frac{1}{n+1}\right)^{n+2}} = \frac{\left(\frac{n+1}{n}\right)^{n+1}}{\left(\frac{n+2}{n+1}\right)^{n+2}} = \left(\frac{n+1}{n}\right)^{n+1}\cdot\left(\frac{n+1}{n+2}\right)^{n+2} = \left(\frac{n+1}{n}\right)^{n+1}\cdot\left(\frac{n+1}{n+2}\right)^{n+2}\cdot\left(\frac{n+1}{n}\right)\frac{n}{n+1} = \left(\frac{(n+1)^2}{n(n+2)}\right)^{n+2}\cdot\frac{n}{n+1}$$

$$= \left(\frac{n^2+1+2n}{n^2+2n}\right)^{n+2}\frac{n}{n+1} = \left(1+\frac{1}{n^2+2n}\right)^{n+2}\cdot\frac{n}{n+1} > 1$$

check

$(1+a)^n > (1+na)$

$$\left(1+\frac{1}{n^2+2n}\right)^{n+2} > \underbrace{\left(1+\cancel{(n+2)}\frac{1}{n\cancel{(n+2)}}\right)}_{\underbrace{1+\frac{1}{n}}_{\frac{n+1}{n}}}$$

$$\left(1+\frac{1}{n^2+2n}\right)^{n+2}\cdot\frac{n}{n+1} > \underbrace{\frac{n+1}{\cancel{n}}\cdot\frac{\cancel{n}}{n+1}}_{1}$$

$\frac{b_n}{b_{n+1}} > 1 \Rightarrow b_n > b_{n+1}$ sequence is decreasing

* $a_n < a_{n+1}$, $b_n > b_{n+1}$

$b_n = \left(1+\frac{1}{n}\right)^{n+1} = \left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right) > \left(1+\frac{1}{n}\right)^n = a_n \Rightarrow \boxed{b_n > a_n}$

$$\boxed{a_1 < a_2 < a_3 \cdots < a_n < b_n < b_{n-1} \cdots < b_2 < b_1}$$

∃ upper bound $\qquad$ ∃ lower bound

$L_1 \equiv \lim_{n\to\infty} a_n$ $\qquad L_2 \equiv \lim_{n\to\infty} b_n$

1+2+ - - - = n(n)

* $b_n - a_n = \left(1+\frac{1}{n}\right)^{n+1} - \left(1+\frac{1}{n}\right)^n = \underbrace{\left(1+\frac{1}{n}\right)^n}_{a_n}\left(\cancel{1}+\frac{1}{n}-\cancel{1}\right) = \frac{1}{n}a_n$ $\qquad \boxed{b_n - a_n = \frac{1}{n}a_n}$

$b_n = \left(1+\frac{1}{n}\right)^{n+1}$

$b_1 = (1+1)^2 = 4$

* $a_n < b_n \Rightarrow a_n < b_1 \Rightarrow \underbrace{\frac{a_n}{n}}_{b_n - a_n} < \frac{b_1}{n}$, $\qquad b_n - a_n = \frac{1}{n}a_n$

$b_n - a_n < \frac{b_1}{n} \Rightarrow \boxed{b_n - a_n < \frac{4}{n}}$.

* $b_n - a_n = \frac{1}{n} a_n \Rightarrow \lim_{n\to\infty} (b_n - a_n) = \underbrace{\lim_{n\to\infty} \frac{1}{n} a_n}_{=0} \Rightarrow \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = 0 \Rightarrow \boxed{\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = e}$

Lecture-17 (10/dec) 2

1730s

Comment on D'Alembert Ratio test for series

* a_n given $\longrightarrow$ $\boxed{\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \begin{cases} >1 & \text{divergent} \\ <1 & \text{Convergent} \\ =1 & \text{Indecisive} \end{cases}}$

Ex 1. $S = \sum_{n=0}^{\infty} \frac{n}{2^n}$

* $a_n = \frac{n}{2^n}$, $a_{n+1} = \frac{n+1}{2^{n+1}} \longrightarrow \frac{a_{n+1}}{a_n} = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{n+1}{2n} = \frac{1}{2} + \frac{1}{2n}$

Say $n=2 \Rightarrow \frac{a_{n+1}}{a_n} = \frac{3}{4}$

$n=3 \Rightarrow \frac{a_{n+1}}{a_n} = \frac{4}{6} < \frac{3}{4} \Rightarrow \boxed{\frac{a_{n+1}}{a_n} \leq \frac{3}{4} \quad \forall\ n \geq 2}$ Result 1

$\lim_{n\to\infty} \left| \frac{1}{2} + \frac{1}{2n} \right| = \frac{1}{2} < 1 \Rightarrow$ Series Converges

Power series

$f(x) = \sum a_n x^n$

Ex2 $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{R} \rightarrow$

* $a_n = \frac{(-1)^{n+1} (x-1)^n}{n}$, $a_{n+1} = \frac{(-1)^{n+2}}{n+1} (x-1)^{n+1}$

$\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+2} (x-1)^{n+1}}{n+1} \cdot \frac{n}{(-1)^{n+1} (x-1)^n} = (-1)(x-1) \frac{n}{n+1}$

$\lim_{n\to\infty} \left| \frac{1-x}{1+\frac{1}{n}} \right| = \underbrace{|x-1| \lim_{n\to\infty} \left| \frac{1}{1+\frac{1}{n}} \right|}_{1 = \text{Rad. of Conv.}} = |x-1|$

$\rightarrow |x-1| > 1$ divergent

$\rightarrow |x-1| < 1$ Convergent

$\Downarrow$

$|x-0| = \begin{cases} x & x>0 \\ -x & x<0 \end{cases}$

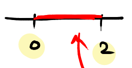
$|x-1| = \begin{cases} x-1 & (x-1) \geq 0 \\ -(x-1) & (x-1) < 0 \end{cases}$

$-1 < (x-1) < 1$

$\begin{cases} x-1 > 0 \Rightarrow x > 1 \longrightarrow (x-1) < 1 \Rightarrow x < 2 \\ x-1 < 0 \Rightarrow x < 1 \longrightarrow -(x-1) < 1 \Rightarrow x-1 > -1 \Rightarrow x > 0 \end{cases}$

End pt. check:

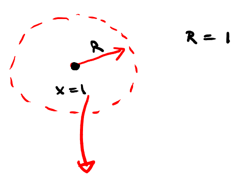
* Test the End pts separately using different test



$\boxed{0 < x < 2}$

Region of Convergence

Interval (IoC)



Convergence Region

$S(x) = \sum \frac{(-1)^{n+1}}{n} (x-1)^n$

$S(x=0) = \sum \frac{(-1)^{n+1}}{n} (-1)^n = \sum \frac{(-1)^{2n+1}}{n} = \sum \frac{\overbrace{(-1)}^{2n} (-1)}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$ 'Harmonic Series'

$S(x=2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

$\boxed{\text{Can't go further than this on the End pt check}}$

interval of x val. that can be plugged into the series : it converges

AP:

$S_n = \frac{n}{2}(2a + (n-1)d)$

$\lim_{n\to\infty} S_n \to \infty$

GP:

$S_n = \frac{a(1-r^n)}{1-r} \xrightarrow{n\to\infty} \frac{a}{1-r}$ finite

1.6.6 Applications of exponential function

1. Difference eq^n of compounded interest (once compounded)

$$* \; p_0 \xrightarrow{CI(t_1) = p_0 r} P(t_1) \xrightarrow{CI(t_2) = P(t_1) r} P(t_2) \; \cdots \cdots$$

$P(t_1) = p_0 + p_0 r = p_0(1+r)$, $P(t_2) = P(t_1) + P(t_1) r$

r rate — discrete values

$n-1$ $\quad n$

$P(t_n) = P(t_{n-1}) + P(t_{n-1}) r$

$$** \; \boxed{P(t_n) - P(t_{n-1}) = r\, P(t_{n-1})}$$ Difference Eq^n — Recursive eq^n

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ (continuous fn)

$P(t_n) = P(t_{n-1})(1+r)$

$= p_0 (1+r)^{n-1} (1+r)$

$P(t_n) = p_0 (1+r)^n$ — { Compounded interest once in t time

Lecture-18 (11/dec) 1.5

2. Constant differential eq^n (Simple interest) e^0

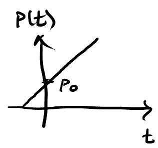
* $P(t)$ = Amt of subs. at t time (umbrella term) ~ "dynamical variable / state variable"

* $\boxed{\frac{dP}{dt} = \text{const}}$ ~ "Newton's 1st law" (dynamical law) { Law of "growth" → incline, ↳ decline

$$\int_{p_0}^{P(t)} dP = \int_0^t \alpha\, dt \Rightarrow P(t) - p_0 = \alpha t \Rightarrow P(t) = p_0 + \underbrace{\alpha t}_{\updownarrow \, SI(t)} \Rightarrow \boxed{P(t) = p_0(1 + rt)}$$

$\exists$ correspondence : $\alpha \equiv p_0 r$

$P(t) = f(\alpha) = g(p_0, r)$



$y = f(x)$

$f' = \frac{dy}{dx} = g(x)$ → >0, <0, $=0$

for every infinitesimal change in x $\exists$ infinitesimal change in $f(x)$ ↑↓

3. Variable Differential Law of growth (Compounded ∞ times ($n=\infty$) → e^t)

* p_0 = init amt. at $t_0 (t=0)$

r = const rate of increment

* $\boxed{\frac{dP}{dt} \propto P(t)}$ ↳ amt present at time { Law of growth

$x^n = a \Leftrightarrow n = \log_x a$

$\log a - \log b = \log \frac{a}{b}$

⇓

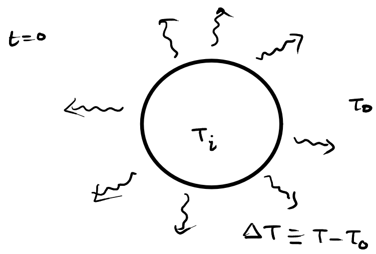
$$\frac{dP}{dt} = r P(t) \Rightarrow \int_{p_0}^{P(t)} \frac{dP}{P} = \int_0^t r\, dt \Rightarrow \ln P(t) \Big|_{p_0}^{P(t)} = rt \Rightarrow \ln P(t) - \ln p_0 = rt \Rightarrow \ln_e \left(\frac{P(t)}{p_0} \right) = rt \Rightarrow \frac{P(t)}{p_0} = e^{rt}$$

$\boxed{P(t) = p_0 e^{rt}}$ $\quad r$ = const of proportionality $\quad$ exponential growth

t, $P(t)$, p_0, dt, dP

$$* \; \lim_{k \to \infty} P(t_k) = \lim_{k \to \infty} p_0 \left(1 + \frac{r}{k}\right)^{kt} = p_0 e^{rt}$$

4. Variable differential Law of decay (Newton's Law of Cooling : e^{-t})



* $T(t)$ = temp of the body at time t

* $\boxed{\frac{dT}{dt} \propto -\Delta T(t)}$ $\quad$ Newton's Law of Cooling

$\Delta T \equiv T - T_0$

$$\frac{dT}{dt} = -K \Delta T \Rightarrow \frac{dT}{dt} = -K(T - T_0) \Rightarrow \int_{T_i}^{T(t)} \frac{dT}{T - T_0} = \int_0^t -K\, dt \Rightarrow \left[\ln T - T_0\right]_{T_i}^{T(t)} = -Kt$$

$\ln\left(\frac{T(t)-T_0}{T_i-T_0}\right) = -kt \Rightarrow T(t)-T_0 = e^{-kt}(T_i-T_0)$

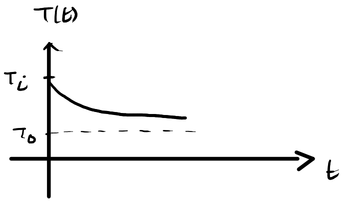
$\int \frac{dx}{x-a} = \int \frac{dy}{y} = \ln y = \ln(x-a)$

$x-a = y$

$dx = dy$

* $\boxed{T(t) = T_0 + (T_i - T_0)\, e^{-kt}}$ Exponential decay

* $\lim_{t\to\infty} T(t) = T_0$ Equi^m is attained

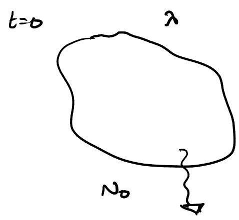


lecture-19 (15/dec) 2

5. Variable differential law of decay (Radioactive decay : e^{-t})

physical sys.

* N_0 = # of radioactive particle/subst. at $t=0$

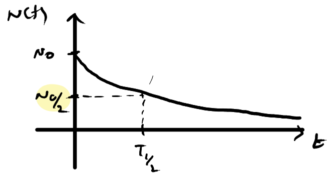


* $\boxed{R \equiv \frac{dN}{dt} \propto -N(t)}$ Law of Radioactive decay. (Typical!)

$\frac{dN}{dt} = -\lambda N \Rightarrow \int_{N_0}^{N(t)} \frac{dN}{N} = -\int_0^t \lambda\, dt \Rightarrow \ln N \Big|_{N_0}^{N(t)} = -\lambda t \Rightarrow \ln\frac{N}{N_0} = -\lambda t \Rightarrow \frac{N}{N_0} = e^{-\lambda t}$

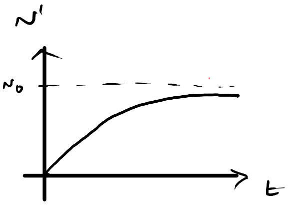
$\boxed{N(t) = N_0\, e^{-\lambda t}}$ # of active subst. at t time

Radioactivity decay Eq^n



$\lim_{t\to\infty} N(t) = 0$

* # of subs. decayed $(N') = N_0 - N(t) = N_0(1-e^{-\lambda t})$



* if $N = \frac{N_0}{2} \Rightarrow \frac{\cancel{N_0}}{2} = \cancel{N_0}\, e^{-\lambda T} \Rightarrow e^{\lambda T} = 2 \Rightarrow$ $\boxed{T_{1/2} = \frac{\ln 2}{\lambda} = \frac{0.693}{\lambda}}$ Half life of the subst. (λ)

Radioactive Const.

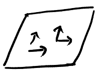
$\lim_{t\to\infty} N'(t) = N_0$

4 Models in 1 glance

I

$\frac{dP}{dt} = \text{const}$

" uniform / Homogenous field " / Neutral

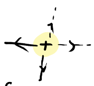


ii

$\frac{dP}{dt} \propto P$

$\underbrace{\frac{dP}{dt} > 0}$

$\left.\begin{array}{l}\forall \uparrow \text{ in } dt \\ \exists \uparrow \text{ in } P\end{array}\right\} \Rightarrow$ "divergence"



iii

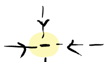
$\frac{dT}{dt} \propto -\Delta T$

iv

$\frac{dN}{dt} \propto -N$

$\underbrace{\frac{dP}{dt} < 0}$

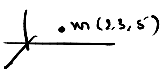
$\left.\begin{array}{l}\uparrow \text{ in } dt \\ \exists \downarrow dP\end{array}\right\} \Rightarrow$ Convergence

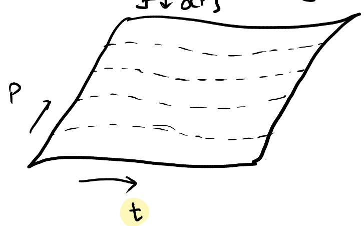


* field ≡ that which fills space ⇒ non local

It has particular value at each pt in space

Particles are local





1.c Logarithmic Maps

* $y = e^x$ exp. growth / the values can't fit within a compact area (Problem of exp.)

$\lim_{x\to\infty} e^x = \infty$, $\lim_{n\to\infty} e^{-x} = 0$

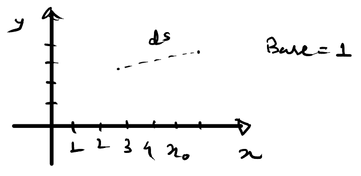
"Here focus is more on the value of e^x"

x	$\left(1+\frac{1}{x}\right)^x$
10	2.5937
100	
1000	
⋮	⋮
1000000	e

* $\boxed{N = a^x \iff x = \log_a N}$

to display numerical data over a wide range of values in a compact way $\Big\}$ ⇒ logarithmic Scale

"the focus has shifted onto the $\underbrace{\text{powers of } a^x}_{\text{"logarithms"}}$"

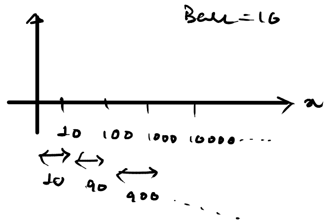


* $\{\hat{e}_i\}$: $\hat{e}_i \cdot \hat{e}_j = g_{ij}$ $\quad ds^2 = g_{ij}\, dx^i dx^j$ distance prescription

Linear Number line ⇒ ∃ unit of distance & it's fixed → linear increment

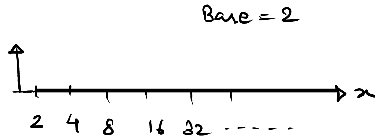
⇓

Spacing = invariant



* $\boxed{\text{Logarithmic scale} \Rightarrow \text{Spacing} \neq \text{invariant} \Rightarrow \text{Non Linear}}$

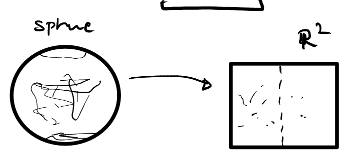
"To reduce the quantities to smaller scopes/scales" → Logarithmic units



* Examples

1. Signal levels → dB (decibel) → Sound wave
2. Acidity / Basicity → pH → chemicals
3. Information theory → Shannon → Entropy / bit or byte → $2^8 = 256$
4. Earthquakes → Richter magnitude
5. Euclidean fractal → use log. for scaling

Atlas



tricky!