Lecture-1 (25/May) 2

1. Orientation (Similarity)

* Congruency = transformation : distances & Angles = invariant ⇒ Area = invariant

  ↙ Translation, ↓ Rotation, ↘ Parity

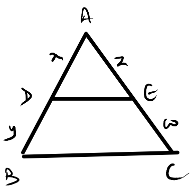
  Angles: Sin θ ∈ Trigonometry

* Similarity = transformation : Angle = invariant ⇒ Distortion in the shape Not allowed

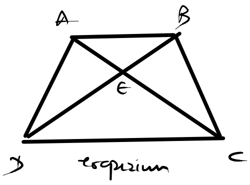
  ↓ Scaling

  Area = ? Relationship ?

*



$\boxed{DE \parallel BC \Rightarrow \frac{x}{y} = \frac{z}{\omega}}$ Thales' thm.



$$\frac{AC}{EC} = \frac{BD}{ED}$$

Componendo

$$\frac{x}{y} = \frac{z}{\omega} \Rightarrow \frac{x}{y} + 1 = \frac{z}{\omega} + 1$$

$$\frac{x+y}{y} = \frac{z+\omega}{\omega}$$

$$\Downarrow$$

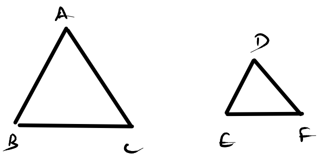
$$\frac{x}{y} - 1 = \frac{z}{\omega} - 1$$

$$\Downarrow$$

$$\frac{x-y}{y} = \frac{z-\omega}{\omega} \quad \text{Dividendo}$$

$$\frac{x+y}{x-y} = \frac{z+\omega}{z-\omega} \quad \{C+D$$

*



$$\boxed{\begin{array}{l} \Delta ABC \sim \Delta DEF \Rightarrow A = D, \; B = E, \; C = F \\ \underbrace{\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}}_{\text{sides are proportions}} \end{array}}$$

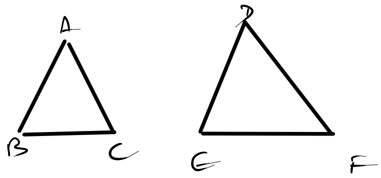
Similarity criteria for Δ

* if $\left.\begin{array}{l} A = D \\ B = E \\ C = F \end{array}\right\} \Rightarrow \Delta ABC \sim \Delta DEF \Rightarrow$ AA simi.

  AAA simil.

* if $\left.\begin{array}{l} \frac{AB}{DE} = \frac{AC}{DF} \\ A = D \end{array}\right\} \Rightarrow \Delta_1 \sim \Delta_2$

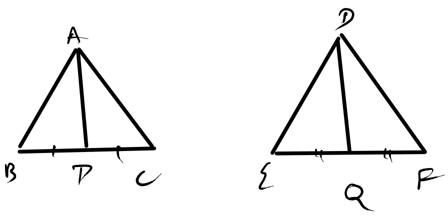
  SAS



* if $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$ SSS simil.

Imp

* thm 1 :



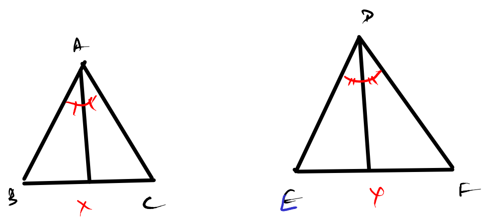
Equiangular Δs

2 Medians : AP, DQ

$$\Downarrow$$

$$\frac{BC}{EF} = \underbrace{\frac{AP}{DQ}}_{\text{ratio of corr. medians}}$$

* thm 2:

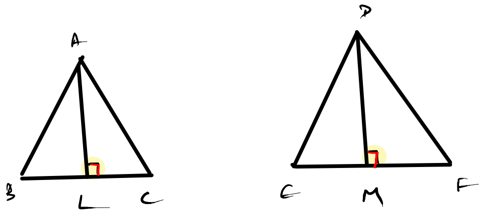


Equiangular Δs

Ang. bisector : AX, DY

$$\frac{BC}{EF} = \underbrace{\frac{AX}{DY}}_{\text{ratio of Ang. bisector segment}}$$

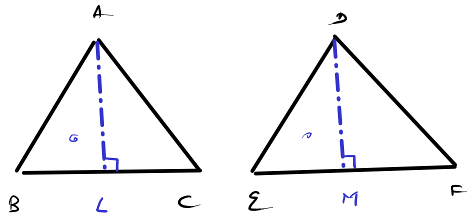
* thm 3 :



Equiangular Δs
$\Downarrow$
$$\frac{BC}{EF} = \frac{AL}{DM}$$

A. Area theorems of sim. Δs

* $$\boxed{\Delta ABC \sim \Delta DEF \Rightarrow \frac{Ar(\Delta ABC)}{Ar(\Delta DEF)} = \frac{AB^2}{DE^2} = \frac{AC^2}{DF^2} = \frac{BC^2}{EF^2}}$$ thm. 1



construct $AL \perp BC$, $DM \perp EF$

* $\underbrace{Ar(\Delta ABC)}_{\Delta_1} = \frac{1}{2}(BC)(AL)$, $Ar(\overbrace{\Delta DEF}^{\Delta_2}) = \frac{1}{2} EF(DM)$

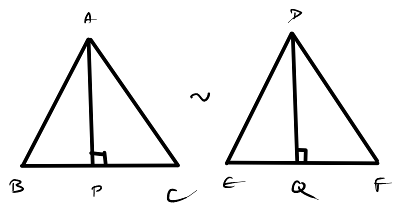
$$\frac{Ar(\Delta_1)}{Ar(\Delta_2)} = \frac{(BC)(\overset{?}{AL})}{(EF)(\underset{?}{DM})}$$

* $\Delta_1 \sim \Delta_2 \Rightarrow A = D,\ B = E,\ C = F$; $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF} \longrightarrow \frac{BC^2}{EF^2} = \frac{AC^2}{DF^2} = \frac{AB^2}{DE^2}$

* ΔALB, ΔDME : $\left.\begin{array}{l}\angle ALB = \angle DME = 90^\circ \\ B = E\end{array}\right\} \Rightarrow \underset{\text{AA sim.}}{\Delta ALB \sim \Delta DME} \rightarrow \frac{AL}{DM} = \frac{LB}{ME} = \boxed{\frac{AB}{DE}}$

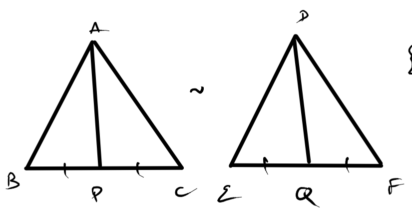
$$\frac{Ar(\Delta_1)}{Ar(\Delta_2)} = \frac{BC}{EF} \cdot \underbrace{\frac{AB}{DE}}_{BC/EF} = \frac{BC^2}{EF^2}$$

* thm 2



$$\frac{Ar(\Delta_1)}{Ar(\Delta_2)} = \frac{AP^2}{DQ^2}$$

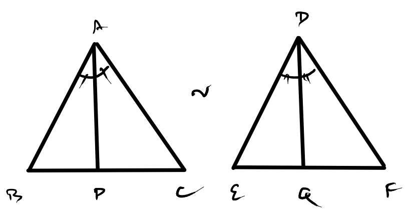
* thm. 3 :



{AP, DQ} Medians

$$\frac{Ar(\Delta_1)}{Ar(\Delta_2)} = \frac{AP^2}{DQ^2}$$

* thm. 4 :



{AP, DQ} Ang-Bisec.

$$\frac{Ar(\Delta_1)}{Ar(\Delta_2)} = \frac{AP^2}{DQ^2}$$

* thm 5 :

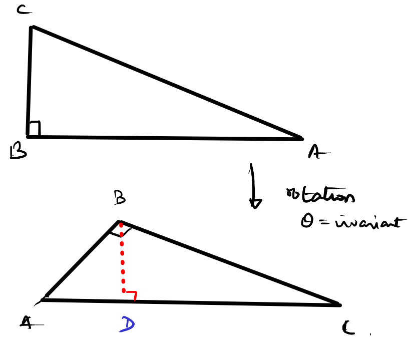
$$\boxed{\underset{\text{scaling}}{\Delta_1 \sim \Delta_2}\ ,\ \underbrace{Ar(\Delta_1) = Ar(\Delta_2)}_{\text{restriction}} \Rightarrow \underset{\text{Congruency}}{\Delta_1 \cong \Delta_2}}$$

B. Pythagoras theorems (The most legit proof)

$\boxed{A_1 = A_4}$, $A_2 = A_5$ (v.p)

$A_1 + A_2 + A_3 = 180^\circ$, $A_4 + A_5 + A_6 = 180^\circ$ $\Rightarrow (A_1 - A_4) + (A_2 - A_5) + (A_3 - A_6) = 0$

* $\boxed{\text{R.A. }\Delta \Rightarrow B^2 + P^2 = H^2}$ thm 1



* BD ⊥ AC 'construct'

AAA sim.

* ΔABC, ΔADB : $\angle ABC = \angle ADB = 90^\circ$, $A = A$ $\Rightarrow$ $\boxed{\text{AA sim}}$ $\Delta ADB \sim \Delta ABC$

$\boxed{\theta_1 + \theta_2 + \theta_3 = 180^\circ}$ Euclidean Assumption (violated for Δ on a sphere)

$\Downarrow$

$$\frac{AD}{AB} = \frac{AB}{AC} \Rightarrow (AC)(AD) = AB^2$$

* ΔABC, ΔBDC : $\angle ABC = \angle BDC = 90^\circ$, $C = C$ $\Rightarrow$ $\Delta ABC \sim \Delta BDC$

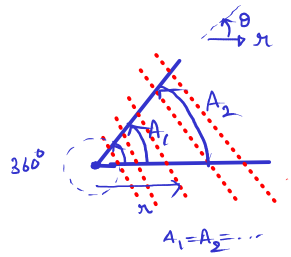
$$\frac{AB}{BD} = \underbrace{\frac{AC}{BC} = \frac{BC}{DC}} \Rightarrow (AC)(DC) = BC^2$$

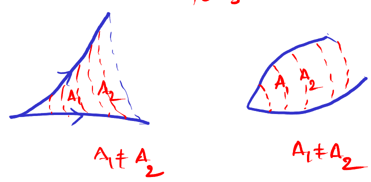
$\boxed{\text{Particular kind of similarity (AA sim. of Δ) is violated in "Non"- Euclidean surfaces}}$

$\Downarrow$

No Pythagoras

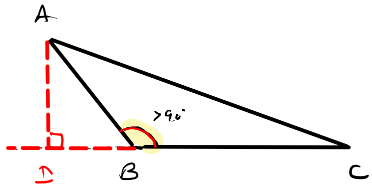
* $$AB^2 + BC^2 = AC\{\underbrace{AD + DC}_{AC}\} = AC^2$$





* ΔABC : obtuse Δ

$\boxed{AD \perp DC \implies AC^2 = AB^2 + BC^2 + 2(BC)(DB)}$ thm 2



* $\Delta ADC\,(R.A) \Rightarrow AC^2 = AD^2 + DC^2 = AD^2 + \underbrace{(DB + BC)^2}_{DB^2 + BC^2 + 2(DB)(BC)} \Rightarrow AC^2 = AB^2 + BC^2 + 2(DB)(BC)$ □

$= AB^2$ (pyth. Δ ADB)

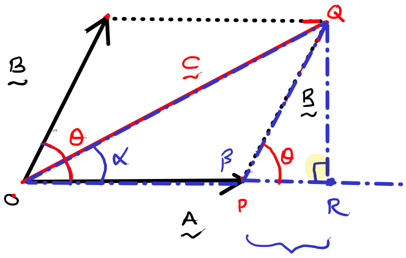
foundations of later cosine formula (vector)

Lecture-2 (26/May) 1.5

comment on parallelogram law of vector addition

* ΔOPQ : obtuse Δ

Naive Pythagoras $\Rightarrow C^2 = A^2 + B^2 + \underbrace{\text{"Correction term"}}_{\sim 1^{st}\text{ order}}$



* $QR \perp OR \Rightarrow \Delta ORQ$ (RA Δ) $\Rightarrow (OQ)^2 = \underbrace{OR^2}_{(OP+PR)^2} + QR^2 = \underbrace{QR^2 + OP^2 + PR^2}_{= PQ^2} + 2(OP)(PR)$

$\sin\theta = \dfrac{QR}{PQ}$

$\cos\theta = \dfrac{PR}{PQ}$

"outside the obtuse Δ"

$$OQ^2 = PQ^2 + OP^2 + 2(OP)\underbrace{(PR)}_{\text{"}(PQ)\cos\theta\text{"}} \longrightarrow OQ = \sqrt{PQ^2 + OP^2 + 2(PQ)(OP)\cos\theta}$$

Black magic

inside the Δ

"outside" (problem)

$\boxed{\text{Trigonometric Inputs}}$

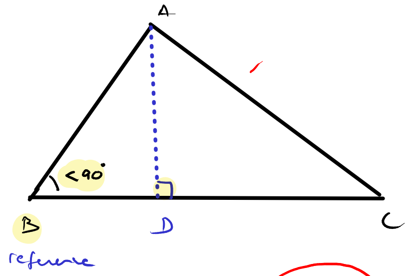
$\cos(180 - \beta) = -\cos\beta$

$$\underbrace{\tan\alpha}_{\text{"inside the Δ"}} = \frac{QR}{OR} = \frac{(PQ)\sin\theta}{OP + \underbrace{(PQ)\cos\theta}_{\text{"outside"}}}$$

* $$OQ = \sqrt{PQ^2 + OP^2 - 2(PQ)(OP)\underbrace{\cos\beta}_{\text{Prob. solved "all inside"}}}$$

* $\triangle ABC$: Acute $\triangle$

Naive pythagoras $\Rightarrow AC^2 = AB^2 + BC^2 +$ "Correction term"



* $\boxed{AD \perp BC \Rightarrow AC^2 = AB^2 + BC^2 - 2(BC)(BD)}$ Thm 3

* $\triangle ADB$ (R.A) $\Rightarrow AB^2 = AD^2 + BD^2$

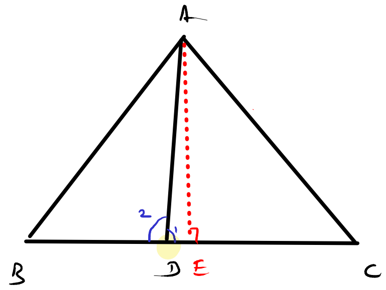
$\triangle ADC$ (R.A) $\Rightarrow AC^2 = AD^2 + \underbrace{DC^2}_{(BC-BD)^2} = AD^2 + (BC - BD)^2 = AD^2 + BC^2 + BD^2 - 2(BC)(BD)$

$= AB^2 + BC^2 - 2(BC)(BD)$

Area $= \frac{1}{2}(BC)(\underbrace{AD}_{\text{Hight}})$

Lecture-3 (27/May) 1.5

* $\triangle ABC$: any $\triangle$, AD : median (BD = DC)



$\boxed{AB^2 + AC^2 = 2\left(\frac{1}{2}BC\right)^2 + 2(AD)^2}$ Thm. 4

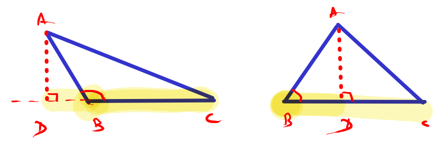
* $AE \perp BC$ 'Construct'

* $\angle AED = 90° \Rightarrow 1 < 90° \Rightarrow \triangle ADC$ acute $\Rightarrow AC^2 = AD^2 + DC^2 - 2\overset{= BD}{(DC)}(DE)$

$2 > 90° \Rightarrow \triangle ADB$ obtuse $\Rightarrow AB^2 = AD^2 + BD^2 + 2(BD)(DE)$

* $AB^2 + AC^2 = 2(AD)^2 + \underbrace{DC^2 + BD^2}_{\left(\frac{1}{2}BC\right)^2 + \left(\frac{1}{2}BC\right)^2} \Rightarrow 2\left(\frac{1}{2}BC\right)^2 + 2(AD)^2$ □

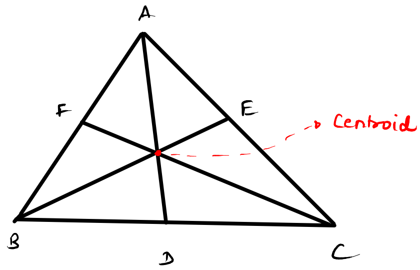
trick to memory



$AC^2 = AB^2 + BC^2 + 2(BC)(DB)$ ↓ from the right angle

$AC^2 = AB^2 + BC^2 - 2(BC)(DB)$

* $\triangle ABC$, {AD, BE, FC} median



* $\boxed{\dfrac{AB^2 + BC^2 + CA^2}{AD^2 + BE^2 + CF^2} = \dfrac{\text{Sum of sq. of sides}}{\text{sum of sq. of median}} = \dfrac{4}{3}}$

Imp! CTM Thm. 5

* AD median $\rightarrow AB^2 + AC^2 = 2\left(\frac{1}{2}BC\right)^2 + 2(AD)^2 = 2\left(\frac{1}{4}BC^2 + AD^2\right)$

BE median $\rightarrow AB^2 + BC^2 = 2\left(\frac{1}{2}AC\right)^2 + 2(BE)^2 = 2\left(\frac{1}{4}AC^2 + BE^2\right)$

CF median $\rightarrow AC^2 + BC^2 = 2\left(\frac{1}{2}AB\right)^2 + 2(FC)^2 = 2\left(\frac{1}{4}AB^2 + FC^2\right)$

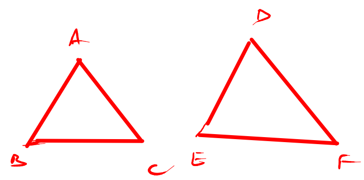
(+)

$\cancel{2}\underbrace{(AB^2 + BC^2 + AC^2)}_{N} = \cancel{2}\left\{\frac{1}{4}\underbrace{(AB^2 + BC^2 + AC^2)}_{N} + \underbrace{AD^2 + BE^2 + FC^2}_{D}\right\} \Rightarrow \frac{3}{4}N = D \Rightarrow \frac{N}{D} = \frac{4}{3}$

Remark

* $\boxed{\Delta_1 \sim \Delta_2 \Rightarrow \dfrac{AB}{DE} = \dfrac{AC}{DF} = \dfrac{BC}{EF} \Rightarrow \dfrac{AB}{DE} = \dfrac{AC}{DF} = \dfrac{BC}{EF} = \dfrac{AB + BC + CA}{DE + EF + FD} = \dfrac{\text{Peri}(\Delta_1)}{\text{Peri}(\Delta_2)}}$

Imp JEE



You will see similar Δs "everywhere" later in Math / Phys! (→ Masters)

2. Geometry with Indices → Index → vector → Section Form. / Ar. Δ → Locus

(A) Index Notation Manipulation (Ref: Trig. vol. 1)

* $x^a = (x^1, x^2, x^3 \ldots x^n)$, a = index runs from 1 to n ; $a \in \mathbb{Z}^+$

1-D list $\underbrace{}_{}$ Components: n - Entries

{ n-Vectors / n-Countable Multiplicity

$n = \max(a)$

* $x^a = (x, y, z)$ '3-vector'

$y^b = (r, \theta)$ b = index runs from 1 to 2 '2-vector'

$p^i = (r, \theta, \phi)$ i = index runs from 1 to 3 '3-vector'

$\uparrow$ 1 length, 2 Angles

Juxtaposition = put "things" side by side

≠

superposition (too wide)

$g^f = \left(17, \frac{4}{5}\sin\theta, i^{\pi}, \ln(1-18e)\right)$ '4-vector'

$p^i = (E, P_1, P_2, P_3)$ '4-vector'

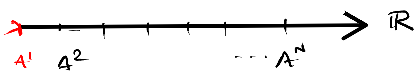
Lecture-4 (28/May) 1.8 + 10

one-on-one Mapping / transformations

Euclidean point

* ϕ = scalar $i \in \mathbb{Z}^+$

* $A^i = (A^1\ A^2 \ldots A^N)_{1\times N}$ Row representation



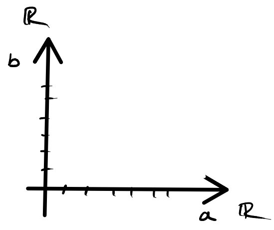
Euclidean 1D space

$= \begin{pmatrix} A^1 \\ A^2 \\ \vdots \\ A^N \end{pmatrix}_{N\times 1}$ Column Rep.

entries = $n(A) = N$

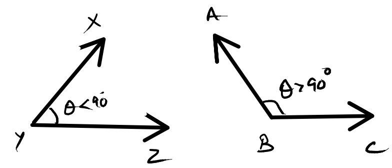
'Number theoretic Notation'

* $M^{ab} = \begin{pmatrix} M^{11} & M^{12} & \ldots & M^{1N} \\ M^{21} & M^{22} & \ldots & M^{2N} \\ \vdots & \vdots & & \vdots \\ M^{N1} & M^{N2} & \ldots & M^{NN} \end{pmatrix}$ 2D-list, Matrix representation



2D space

Lebnizian point



Acute | obtuse {Bigger Angle}

$|XY| \downarrow\downarrow\downarrow$ | $|AB| \downarrow\downarrow\downarrow\downarrow\downarrow$

To Infinitesimal | To infinitely Small

$\overset{X}{Y \cdot Z}$ < $\overset{A}{B \cdot C}$

Point has bigger Size

Entries = $n(M) = N^2$

* $\phi = \lambda \pm \psi$ Scalar "Number" Addition

* $C_i = A_i \pm B_i$, $i \in [0,N]$, $[0,N]$, $[0,N]$ ∃ "Natural" addition = same dim. matrices get added

Machine

* $C_{ab} = A_{ab} \pm B_{ab}$, $\underbrace{N\ N'}_{N\times N'}$, $\underbrace{N\ N'}_{N\times N'}$, $\underbrace{N\ N'}_{N\times N'}$

Richard (Left off)

$\underset{(N+M)\times(N+M)}{C} \equiv \underset{\underbrace{N\times N}_{\text{sq. matrix}}}{A} \oplus \underset{\underbrace{M\times M}_{\text{sq. matrix}}}{B} = \left(\begin{array}{c|c} A_{N\times N} & O_{N\times M} \\ \hline O_{M\times N} & B_{M\times M} \end{array}\right)$

"Kronecker sum"

$A_{ij} = \begin{pmatrix} 1 & 2 \\ 7 & 5 \end{pmatrix}_{2\times 2}$, $B_i = \begin{pmatrix} 1 \\ 3 \end{pmatrix}_{2\times 1}$

N N

Machine

* $\phi = \lambda \cdot \psi$ Scalar multiplication "Normal"

* $B_i = \phi \cdot A_i$ [0-D list × 1-D list] → 1-D list } scaling

* $M_{ij} = \lambda N_{ij}$ [0-D list × 2-D list] → 2-D list

Notation Alert

* $\underbrace{C_{ij}}_{1\times NM} = \underbrace{A_i}_{1\times N} \otimes \underbrace{B_j}_{1\times M}$ "Kronecker product / direct pdt"

$A_i = (1\ 2\ 3)_{1\times 3}$, $B_j = \begin{pmatrix} 7 \\ 5 \end{pmatrix}_{2\times 1}$ ⟶ $\underbrace{A_i \otimes B_j}_{C_{ij}} = (1\ 2\ 3)_{1\times 3} \otimes \begin{pmatrix} 7 \\ 5 \end{pmatrix}_{2\times 1} = \begin{pmatrix} 1\cdot 7 & 2\cdot 7 & 3\cdot 7 \\ 1\cdot 5 & 2\cdot 5 & 3\cdot 5 \end{pmatrix}_{2\times 3}$

* $\underbrace{C^{pq}_{lm}}_{LP\times MQ} \equiv \underbrace{A_{lm}}_{L\times M} \otimes \underbrace{B^{pq}}_{P\times Q}$

$\downarrow$

4-index object (tensor) $\longrightarrow$ "4-D" space

$$\underset{6\times6}{S^{cd}_{ab}} = \underbrace{M_{ab}\otimes N^{cd}}_{M_{ab}N^{cd}} = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}_{3\times2} \otimes \begin{pmatrix} r & s & t \\ x & y & z \end{pmatrix}_{2\times3} \underset{\text{2D rep.}}{=} \begin{pmatrix} ar & as & at & br & bs & bt \\ ax & ay & az & bx & by & bz \\ cr & cs & ct & dr & ds & dt \\ cx & cy & cz & dx & dy & dz \\ er & es & et & fr & fs & ft \\ ex & ey & ez & fx & fy & fz \end{pmatrix}_{6\times6}$$

2D rep of a 4 index obj.

lecture-5 (1/Jan) 2

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \dots + x_n$$

* $K_{ab} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}_{2\times2}$, $a,b = 1,2$; $L_c = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_{2\times1}$, $c = 1,2$; $\alpha,\beta \in \mathbb{C}$

* $S_{ab}{}^{c} \equiv K_{ab}L^c = K_{ab}\otimes L^c = \begin{pmatrix} \alpha & 2\alpha \\ \beta & 2\beta \\ 3\alpha & 4\alpha \\ 3\beta & 4\beta \end{pmatrix}_{4\times2}$

$c=b$

* $\underbrace{\sum_{b=1}^{2} K_{ab}L^b}_{\equiv R_a} = K_{a1}L^1 + K_{a2}L^2 \Rightarrow R_a = K_{a1}L^1 + K_{a2}L^2$

$\equiv R_a$ (1 index obj. / 1D list), $a=1,2$

$a=1 \Rightarrow R_1 = K_{11}L^1 + K_{12}L^2 = \alpha + 2\beta$

$a=2 \Rightarrow R_2 = K_{21}L^1 + K_{22}L^2 = 3\alpha + 4\beta$

$\Rightarrow R_a = \begin{pmatrix} \alpha + 2\beta \\ 3\alpha + 4\beta \end{pmatrix}_{2\times1}$

$c=a$

* $\underbrace{\sum_{a=1}^{2} K_{ab}L^a}_{\equiv G_b} = K_{1b}L^1 + K_{2b}L^2 \Rightarrow G_b = \begin{pmatrix} K_{11}L^1 + K_{21}L^2 \\ K_{12}L^1 + K_{22}L^2 \end{pmatrix} = \begin{pmatrix} \alpha + 3\beta \\ 2\alpha + 4\beta \end{pmatrix}_{2\times1} \Rightarrow \overbrace{\sum_{b=1}^{2} K_{ab}L^b}^{G_a} \neq \overbrace{\sum_{a=1}^{2} K_{ab}L^a}^{R_b}$

$c=b$

* $\underbrace{\sum_{b=1}^{2} L^b K_{ab}}_{\equiv P_a} = L^1K_{a1} + L^2K_{a2} \Rightarrow K_{a1}L^1 + K_{a2}L^2 \Rightarrow$

$$\boxed{\begin{array}{c} \sum_b K_{ab}L^b = \sum_b L^b K_{ab} \\ K_{ab}L^c \neq L^c K_{ab} \ \checkmark \end{array}}$$

$A = (1\ 2)_{1\times2}$

$\downarrow$ transpose

$A^T = \begin{pmatrix} 1 \\ 2 \end{pmatrix}_{2\times1}$

$B = \begin{pmatrix} 1 & 7 \\ 3 & 8 \end{pmatrix}$ $B^T = \begin{pmatrix} 1 & 3 \\ 7 & 8 \end{pmatrix}$

$A \neq A^T$

$A = A^T \Rightarrow a_{ij} = a_{ji}$

symmetric matrix

* $\sum_a K_{ab}L^a = \sum_a L^a K_{ab}$; $\underbrace{\sum K_{ab}L^b}_{X_a} = \underbrace{\sum K_{ab}L^a}_{Y_b}$

$\sum_{a,c,d} T_a L_b M_{cd} G^{acd} = \Pi_b$

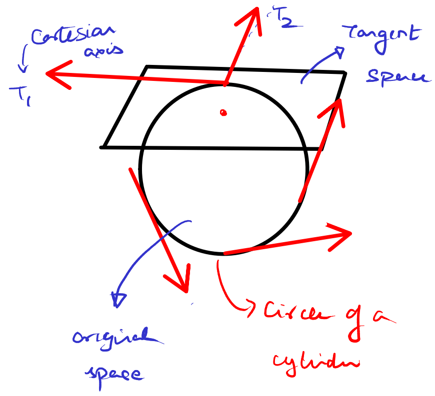
$\sum_c T^b N_c g^c{}_a = P^b{}_a$

$\sum_{i,j} G_{ij} P^i X^j = \rho$ (scalar)

$$* \sum_{a=1}^{2}\sum_{b=1}^{2} K_{ab}L^aL^b = \sum_{a=1}^{2}\Big\{\underbrace{\sum_{b=1}^{2}K_{ab}L^b}_{\equiv X_a}\Big\}L^a = \sum_{a=1}^{2} X_aL^a = X_1L^1+X_2L^2$$

$$X_1 = \sum_{b=1}^{2} K_{1b}L^b = K_{11}L^1 + K_{12}L^2$$

$$X_2 = \sum K_{2b}L^b = K_{21}L^1 + K_{22}L^2$$



$$* \sum_a\sum_b K_{ab}L^aL^b = (K_{11}L^1+K_{12}L^2)L^1 + (K_{21}L^1+K_{22}L^2)L^2$$

$$= K_{11}(L^1)^2 + K_{12}L^2L^1 + K_{21}L^1L^2 + K_{22}(L^2)^2$$

$$= K_{11}(L^1)^2 + K_{22}(L^2)^2 + (K_{12}+K_{21})L^1L^2$$

$$[L^1 L^2] = L^1L^2 - L^2L^1 = \cancel{\alpha\beta} - \cancel{\beta\alpha} = 0$$

Commutator

$$[A,B] \equiv AB - BA$$

$$[L^a, L^b]$$

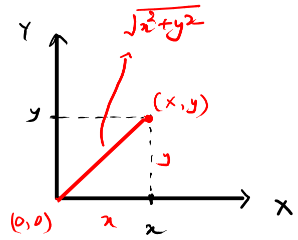
$$* \quad K_{ab} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \delta_{ab}, \quad L^a = \begin{pmatrix}x\\ y\end{pmatrix}$$

$$\boxed{\sum_a\sum_b K_{ab}L^aL^b = (x)^2 + (y)^2 \equiv (\text{distance})^2}$$

'Algebra'

supplements notion of "distance"

Euclidean geometry



$$s = \sqrt{x^2+y^2}$$

notion of coordinates to supplement distance

Lecture-6 (2/Jan) 2.15

Evaluation time!!!

Evaluation : Jan.

Lecture # = 10 - 25

{10-24} =>

1. $\sigma_\alpha = \{\sigma_1, \sigma_2, \sigma_3\}$ index $\alpha = 1, 2, 3$ Set of 3 Matrices

$$\sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}_{2\times2}; \quad \sigma_2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}_{2\times2}; \quad \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}_{2\times2},$$

side remark $\boxed{i^2 = -1}$

Vector Matrix

$$\underset{\sim}{A} = 1\hat{i} + 2\hat{j} + 3\hat{k} = (1 \;\; 2 \;\; 3)$$

Basis $= \begin{pmatrix}1\\2\\3\end{pmatrix}$

$$\underset{\sim}{A} = \{1, 2, 3\}$$

Set

Inner Products / Commutators

$* \quad \sigma_1\sigma_2 = \begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}$ ✓

$* \quad \sigma_2\sigma_1 = \begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}$ ✓

$* \quad [\sigma_1, \sigma_2] = 2i\sigma_3$

Commutator ✓

$* \quad [\sigma_3, \sigma_1] = 2i\sigma_2$

$* \quad [\sigma_\alpha, \sigma_\beta] \overset{?}{=} 2i\sigma_\mu \qquad \alpha, \beta, \mu = 1, 2, 3 \qquad$ 'Naive guess!'

Hints:

$$\delta_{\alpha\beta} = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}, \qquad \epsilon_{\alpha\beta\mu} = \begin{cases} 1 & (1\,2\,3), (2\,3\,1), (3\,1\,2) \\ -1 & (3\,2\,1), (1\,3\,2), (2\,1\,3) \\ 0 & \alpha=\beta \text{ or } \beta=\mu \text{ or } \alpha=\mu \end{cases}$$

Kronecker delta → vector dot

Levi Civita symbol (3 index object) → vector cross

Your Best friends!

Pattern recognition of $\epsilon_{\alpha\beta\mu}$

Regular pattern

$\epsilon_{123} \equiv +1$ Even

* 1 2 3 $+1$

1 3 2

2 1 3 $+1$

2 3 1

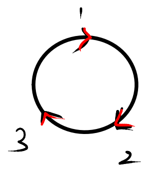
3 1 2 $+1$

3 2 1

$\boxed{1\ 2\ 3}$

2 3 1

3 1 2



Even Permutation

3 2 1

1 3 2

2 1 3

Odd Permut

* $\overset{+1}{(1\ 2\ 3)}$ $\overset{-1}{(1\ 3\ 2)}$ $\overset{+1}{(2\ 3\ 1)}$ $\overset{-1}{(3\ 2\ 1)}$

* $\epsilon_{\alpha\beta\mu} = -\epsilon_{\beta\alpha\mu}$ 'Antisymmetric'

$= -\epsilon_{\mu\beta\alpha}$

* $\underbrace{[\sigma_\alpha, \sigma_\beta]}_{\text{free indices} = \alpha, \beta} \neq 2i\,\underbrace{\sigma_\mu}_{\text{free index} = \mu}$

My Hint

$\boxed{\text{Summed over index}}$

Your Possible Soln

My Hint

$$\sum_j T_{ij} A^{jk} = S_i^{\ k}$$

$$\sum_j A_{ijk} B^j = R_{ik}$$

$$\boxed{[\sigma_\alpha\ \sigma_\beta] = 2i \sum_{\mu=1}^{3} \epsilon_{\alpha\beta\mu}\, \sigma^\mu}$$

$$[\sigma_1, \sigma_2] = 2i \sum_{\mu=1}^{3} \epsilon_{12\mu}\sigma^\mu = 2i \left\{ \underbrace{\cancel{\epsilon_{121}}}_{0}\sigma^1 + \underbrace{\cancel{\epsilon_{122}}}_{0}\sigma^2 + \underbrace{\epsilon_{123}}_{1}\sigma^3 \right\}$$

$$[\sigma_2, \sigma_1] = -2i\sigma_3$$

* $$[\sigma_\beta, \sigma_\alpha] = 2i \sum_\mu \underbrace{\epsilon_{\beta\alpha\mu}}_{-\epsilon_{\alpha\beta\mu}} \sigma^\mu = \underbrace{-2i \sum \epsilon_{\alpha\beta\mu}\sigma^\mu}_{[\sigma_\alpha, \sigma_\beta]} = -[\sigma_\alpha\ \sigma_\beta]$$

Ⓐ Ⓑ Ⓒ

Factorial

$\underset{A \text{ or } B \text{ or } C}{3} \times \underset{B \text{ or } 2}{2} \times \underset{C}{1} \equiv 3!$

No repetition

$\underset{A}{3} \times \underset{A}{3} \times \underset{A}{3} = 3^3$

A or B or C

Repetition

COUNTING !!!

$\downarrow$ Advanced version

$\boxed{\text{Combinatorics}}$

${}^nP_r C$

(later)

$\boxed{\begin{array}{l} \text{AND} \equiv \times \\ \text{OR} \equiv + \end{array}}$

"Derivation" later

$AB = BA$ symmetric

$a_{ij} = a_{ji}$

$AB \neq BA$ usually

$AB - BA = 0 \Rightarrow$ sym.

$\underbrace{AB - BA}_{} \neq 0 \rightarrow$

$[A, B]$

$AB = -BA$ Antisym

Lecture 7 (3/Jun) 2.30

$$* \quad \sum_{a=1}^{3}\sum_{b=1}^{3} g_{ab}\underbrace{x^a x^b} = \sum_{a=1}\Big\{\underbrace{\sum_{b=1}^{3} g_{ab}x^b}_{T_a}\Big\}x^a = \sum_{a=1} T_a x^a = \phi \equiv s^2 \in \mathbb{C}$$

$$x^a = (x^1\ x^2\ x^3)_{1\times 3} \in \mathbb{R}$$

$$[x^a, x^b] = \underbrace{x^a x^b} - x^b x^a \overset{?}{=} 0$$

looks like Kroneck-n prdt.

$$x^a \otimes x^b = (x^1\ x^2\ x^3) \otimes (x^1\ x^2\ x^3)$$

$$= (x^1x^1\ x^1x^2\ x^1x^3\ x^2x^1\ x^2x^2\ x^2x^3\ x^3x^1\ x^3x^2\ x^3x^3)_{1\times 9}$$

$$x^b \otimes x^a = x^a \otimes x^b \Rightarrow [x^a, x^b] = 0$$

diagonal Matrices Examples

$$x^b = (x^1\ x^2\ x^3)$$

if $g_{ab} = \delta_{ab} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}_{3\times3} \Rightarrow T_a = g_{a1}x^1 + g_{a2}x^2 + g_{a3}x^3$

Kronecker delta

$T_a = (T_1\ \ T_2\ \ T_3)$ Computed

$= (x^1\ \ x^2\ \ x^3)$ Identity transformation

3 Minkowski metric

if $g_{ab} = \begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}_{3\times3} \Rightarrow T_a = (x^1\ \ -x^2\ \ -x^3)$ Computed

still very Centesian

"small deviation" from Identity

$$x^b = (x^1\ x^2 x^3)$$

$$(1\ 2\ 3\ 4) \rightarrow (1\ -2\ -3\ -4)$$

if $g_{ab} = \begin{pmatrix}1&0&0&0&\cdots\\0&-1&\cdots&&\\0&0&-1&\cdots&\\0&0&0&-1&\cdots\end{pmatrix}_{n\times n} \longrightarrow T_a = \sum_b g_{ab}x^b = (\underbrace{+x^1}\ \underbrace{-x^2\ -x^3\ -x^4 \cdots -x^n})$

Computed

given

Computer = Transformation

Additively Inversed

Entry is left unchanged

$\because g_{11} = +1$

given

$$x^b = (x^1\ \ x^2)$$

$g_{ab} = \begin{pmatrix}-1&0\\0&1\end{pmatrix} \Rightarrow T_a = \sum_b g_{ab}x^b = g_{a1}x^1 + g_{a2}x^2 + \cancel{g_{a3}x^3} \Rightarrow T_a = (T_1\ \ T_2)_{1\times2} = (\overset{\frown}{-x^1}\ \ \underbrace{x^2})$

Computer

DNE

Additively Inversed

Entry is left unchanged

$\because g_{22} = 1$

$g_{ab} = \begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&+1&0\\0&0&0&-1\end{pmatrix} \xrightarrow[\text{transformation}]{T_a = \sum g_{ab}x^b} T_a = (x^1\ \ -x^2\ \ x^3\ \ -x^4)$

Computer

Computed

Mapping

Machine / Computer / Transform^n

given

$$* \quad (1\ \ 2\ \ 3) \xrightarrow[T_a = \sum g_{ab}x^b]{g_{ab} = \delta_{ab}} (1\ \ 2\ \ 3)$$

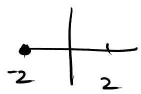
$$(1\ \ 2\ \ 3) \xrightarrow{\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}} (1\ \ \underbrace{-2\ \ -3}_{\text{Mirrored}})$$

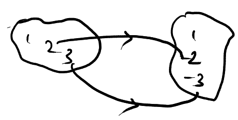
Computed

$$(1\ \ 2\ \ 3) \xrightarrow{\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}} (\underbrace{-1}_{\text{Mirrored}}\ \ 2\ \ 3) \xrightarrow{\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}} (1\ \ 2\ \ 3)$$

diagonal Matrix

$$(1\ \ 2\ \ 3) \xrightarrow{\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}} (1\ \ \underbrace{-2}_{\text{Mirror}}\ \ 3)$$



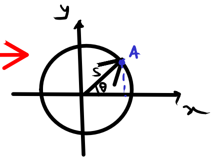


HW

Rule (Algebraic)   's' has a geometrical Interpretation of being DISTANCE $\underset{\sim}{s} = x\hat{i} + y\hat{j}$ Position vector

* $s^2 = \sum_a \sum_b g_{ab} x^a x^b = ?$ Computed / Comment on it

∃ correspondence



1. $g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \delta_{ab}$ (Entries) , $x^a = (x \;\; y)$ (numbers) $\longrightarrow s^2 = (\text{distance})^2 = x^2 + y^2 > 0 \sim$ Pythagoras

Identity Computer   given

2. $g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $x^a = (t, x)$ (x^0, x^1) $\longrightarrow s^2 = t^2 - x^2$
   - $\rightarrow t^2 > x^2 \rightarrow s^2 > 0$ x^a timelike
   - $\rightarrow t^2 < x^2 \rightarrow s^2 < 0$ x^a spacelike
   - $\rightarrow t^2 = x^2 \rightarrow s^2 = 0$ x^a lightlike

3. $g_{ab} = \begin{pmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{pmatrix}$, $x^a = (x \;\; y) \longrightarrow s^2 = x^2 + y^2 + 2xy\cos\theta \sim$ generalized Pythagoras

4. $g_{ab} = \begin{pmatrix} \frac{1}{t^2} & 0 \\ 0 & -1/t^2 \end{pmatrix}$ $x^a = (x \;\; y) \longrightarrow s^2 = \frac{1}{t^2}(x^2 - y^2)$

$\underset{\sim}{s} \cdot \hat{i} = x = |\underset{\sim}{s}|\cos\theta$

$\underset{\sim}{s} \cdot \hat{j} = y = |\underset{\sim}{s}|\sin\theta$

$|\underset{\sim}{s}|^2 = x^2 + y^2$

length of the vector

5. $g_{ab} = \begin{pmatrix} f(x) & 0 \\ 0 & -1/f(x) \end{pmatrix}$

$x^a = (t \;\; x)$

- $\rightarrow f(x) = 1$ Const $\longrightarrow s^2 = t^2 - x^2$ Minkowski space
- $\rightarrow f(x) = 1 + x$ Linear $\longrightarrow s^2 = (1+x)t^2 - \frac{x^2}{(1+x)}$
- $\rightarrow f(x) = 1 - x^2 \longrightarrow s^2 = (1-x^2)t^2 - \frac{x^2}{(1-x^2)}$ ~ DeSitter Space
- $\rightarrow f(x) = (1 + x^2$
  - $\rightarrow s^2 = (1+x^2)t^2 - \frac{x^2}{(1+x^2)}$ ~ Anti DeSitter space

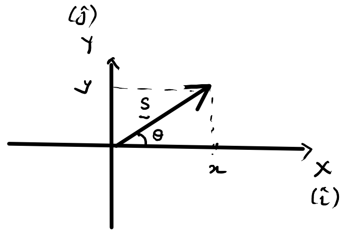
Lecture-8 (4/Jun) 2

Ⓑ Vector Analysis
- → Algebra ✓
- → Calculus (After Calculus)

* $x^a = (x \;\; y) \longrightarrow \{\hat{i} \;\; \hat{j}\} \longrightarrow \underset{\sim}{s} = x\hat{i} + y\hat{j}$ 'Position vector'

1D list   Unit "vector"

Components/ Numbers/Scalars



$\underset{\sim}{s} \cdot \underset{\sim}{s} = |\underset{\sim}{s}|^2 = (x\hat{i} + y\hat{j}) \cdot (x\hat{i} + y\hat{j})$

$= x^2(\hat{i}\cdot\hat{i}) + xy(\hat{i}\cdot\hat{j}) + yx(\hat{j}\cdot\hat{i}) + y^2(\hat{j}\cdot\hat{j}) \overset{?}{=}$ scalar

Presuppositions
- → $x^a = (x \;\; y) \in \mathbb{R} \Rightarrow [x, y] = 0$
- → $\hat{e}_a$

* $\hat{e}_a = \{\hat{e}_1, \hat{e}_2, \hat{e}_3 \dots \hat{e}_n\}$, $M^a = \{M^1, M^2, \dots M^n\}$ index $a = 1$ to n

n-unit vectors (~Quality)   n-scalar components (~Quantity)

1 🍎 = unit

Box of Apples

↓ Quantitatively

Box of Apples ≡ # of Apples in the box

↓

🍎 🍎 🍎 ...

= 50 🍎

Entity = 50 units → Quality

Composite   Component (Quantity)

Composite = 1 unit + 1 unit + ...

* $\boxed{\underset{\sim}{M} = \underline{M} = \bar{M} = \vec{M} = \mathbf{M} \equiv \sum_{a=1}^{n} M^a \hat{e}_a}$ index

n-vector

→ Linear Combination

Max. Power = 1

Contracting & summing over a → Superposition

$M^a \hat{e}_a$: Projection of $\underset{\sim}{M}$ in $\hat{e}_a$ direction

How much "part"/portion"

* $\underset{\sim}{M} = \sum_{a=1}^{3} M^a \hat{e}_a = M^1\hat{e}_1 + M^2\hat{e}_2 + M^3\hat{e}_3$

* $\hat{e}_a = (\hat{e}_1 \;\; \hat{e}_2 \;\; \hat{e}_3)$, $\hat{e}_b = (\hat{e}_1 \;\; \hat{e}_2 \;\; \hat{e}_3) \longrightarrow$
  - $\hat{e}_a \cdot \hat{e}_b$
  - $\hat{e}_a \times \hat{e}_b$
  - $\hat{e}_a \otimes \hat{e}_b$

} Question at hand

* $\hat{e}_a \otimes \hat{e}_b = (\hat{e}_1 \;\; \hat{e}_2 \;\; \hat{e}_3)_{1\times 3} \otimes (\hat{e}_1 \;\; \hat{e}_2 \;\; \hat{e}_3)_{1\times 3} = (\hat{e}_1\hat{e}_1 \;\; \hat{e}_1\hat{e}_2 \;\; \hat{e}_1\hat{e}_3 \;\; \hat{e}_2\hat{e}_1 \;\; \hat{e}_2\hat{e}_2 \;\; \hat{e}_2\hat{e}_3 \;\; \hat{e}_3\hat{e}_1 \;\; \hat{e}_3\hat{e}_2 \;\; \hat{e}_3\hat{e}_3)_{1\times 9}$

Not working.

B.1 **Scalar / Dot product** {Math-12

Scalar/Dot product

* $\underset{\sim}{M} = \sum_{a=1}^{3} M^a \hat{e}_a \longrightarrow \underset{\sim}{M} \cdot \underset{\sim}{M} = \langle \underset{\sim}{M} | \underset{\sim}{M} \rangle = \underbrace{|M|^2}_{} = \sum_{a=1}^{3} M^a \hat{e}_a \sum_{b=1}^{3} M^b \hat{e}_b = \sum_a \sum_b M^a M^b \underbrace{\hat{e}_a \cdot \hat{e}_b}_{} = \text{scalar}$

(Magnitude)^2 of a vector

$\boxed{\exists \text{ a need to impose a condition on } \hat{e}_a \cdot \hat{e}_b \text{ to get a scalar on RHS}}$

to define a scalar product

$\boxed{\hat{e}_a \cdot \hat{e}_b \equiv g_{ab}}$

$\longrightarrow g_{ab} = \delta_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \longrightarrow$ Euclidean vector

$\longrightarrow g_{ab} = \eta_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \longrightarrow$ Minkowski vector

$\vdots$

$$\sum_a \sum_b g_{ab} M^a M^b = \phi = s^2 \in \mathbb{C}$$

$\uparrow$ (distance)^2

* $\underset{\sim}{M} \cdot \underset{\sim}{M} = |M|^2 = \sum_a \sum_b g_{ab} M^a M^b = \sum_a \underbrace{\sum_b g_{ab} M^b}_{N_a} M^a = \sum_a N_a M^a = f$

$$N_a = \sum g_{ab} M^b$$

Lecture-9 (5/Jun) 2

* $M^a = (M^1 \;\; M^2 \;\; \cdots \;\; M^n)_{1\times n}$, $\hat{e}_a = (\hat{e}_1 \;\; \hat{e}_2 \;\; \cdots \;\; \hat{e}_n)_{1\times n}$

Scalar components of n vector
$\downarrow$
Magnitude / scalar

n- Basis unit vectors
$\downarrow$
Direction

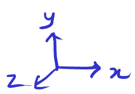
Std notation
$\hat{i} \;\; \hat{j} \;\; \hat{k}$
$\hat{x} \;\; \hat{y} \;\; \hat{z}$
$\hat{e}_x \;\; \hat{e}_y \;\; \hat{e}_z$

$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$

* $\underset{\sim}{M} = \sum_a M^a \hat{e}_a = M^1\hat{e}_1 + M^2\hat{e}_2 + M^3\hat{e}_3 \cdots + M^n\hat{e}_n \in \mathbb{R}^n$

$A \times B = \{\underbrace{(a,b)}_{\text{O.P}} : a \in A, b \in B\}$

{Any arbitrary vector can be decomposed into sums of vectors $\underbrace{\text{along the coordinate directions}}_{\text{Basis vector}}$



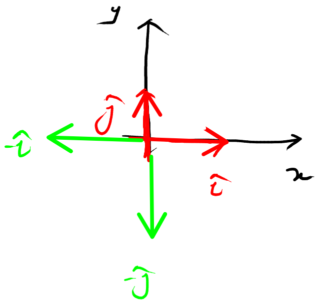
Cylindrical polar — Spherical polar

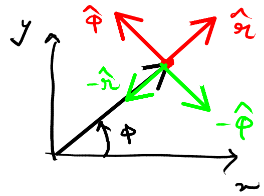
$(\hat{e}_1 \;\; \hat{e}_2 \;\; \hat{e}_3) \longrightarrow (\hat{i} \;\; \hat{j} \;\; \hat{k}) \longrightarrow (\hat{r} \;\; \hat{\phi} \;\; \hat{z}) \longrightarrow (\hat{r} \;\; \hat{\theta} \;\; \hat{\phi})$

$\underbrace{(\hat{x} \;\; \hat{y} \;\; \hat{z})}_{\text{Linear Coord}}$

$\hat{r}$: radial, $\hat{\phi}$: Ang., $\hat{z}$: Linear

$\hat{r}$: radial, $\underbrace{\hat{\theta} \;\; \hat{\phi}}$: Angular





$$\underset{\sim}{M}\cdot\underset{\sim}{M} = \sum\sum M^a M^b \underbrace{\hat{e}_a\cdot\hat{e}_b} \xrightarrow[\text{be a Scalar}]{\text{For it to}} \boxed{\hat{e}_a\cdot\hat{e}_b = g_{ab}}$$

$\underbrace{\underset{\sim}{M}\cdot\underset{\sim}{M}}$: $|M|^2$

Demand !!

Case 1 : Kronecker delta to form basis dot prod

$$g_{ab} = \delta_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad a,b = 1,2,3 \Rightarrow \hat{e}_a\cdot\hat{e}_b = \delta_{ab} = \delta_{ba} = \hat{e}_b\cdot\hat{e}_a \Rightarrow [\hat{e}_a,\hat{e}_b] = 0$$

$\hat{e}_1\cdot\hat{e}_1 = \hat{\imath}\cdot\hat{\imath} = 1$	$\hat{e}_2\cdot\hat{e}_1 = \hat{\jmath}\cdot\hat{\imath} = 0$	$\hat{e}_3\cdot\hat{e}_1 = \hat{k}\cdot\hat{\imath} = 0$
$\hat{e}_1\cdot\hat{e}_2 = \hat{\imath}\cdot\hat{\jmath} = 0$	$\hat{e}_2\cdot\hat{e}_2 = \hat{\jmath}\cdot\hat{\jmath} = 1$	$\hat{e}_3\cdot\hat{e}_2 = \hat{k}\cdot\hat{\jmath} = 0$
$\hat{e}_1\cdot\hat{e}_3 = \hat{\imath}\cdot\hat{k} = 0$	$\hat{e}_2\cdot\hat{e}_3 = \hat{\jmath}\cdot\hat{k} = 0$	$\hat{e}_3\cdot\hat{e}_3 = \hat{k}\cdot\hat{k} = 1$

Rules for unit vector Dot product

Case II : Minkowski to form basis dot prod

$$g_{ab} = \eta_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{4\times4} \qquad a,b = \underbrace{0,1,2,3} \Rightarrow \hat{e}_a\cdot\hat{e}_b = \eta_{ab} \overset{\text{sym.}}{=} \eta_{ba} = \hat{e}_b\cdot\hat{e}_a \Rightarrow [\hat{e}_a,\hat{e}_b] = 0$$

$\hat{e}_0\cdot\hat{e}_0 = 1$	$\hat{e}_1\cdot\hat{e}_0 = 0$	$\hat{e}_2\cdot\hat{e}_0 = 0$	$\hat{e}_3\cdot\hat{e}_0 = 0$
$\hat{e}_0\cdot\hat{e}_1 = 0$	$\hat{e}_1\cdot\hat{e}_1 = -1$	$\hat{e}_2\cdot\hat{e}_1 = 0$	$\hat{e}_3\cdot\hat{e}_1 = 0$
$\hat{e}_0\cdot\hat{e}_2 = 0$	$\hat{e}_1\cdot\hat{e}_2 = 0$	$\hat{e}_2\cdot\hat{e}_2 = -1$	$\hat{e}_3\cdot\hat{e}_2 = 0$
$\hat{e}_0\cdot\hat{e}_3 = 0$	$\hat{e}_1\cdot\hat{e}_3 = 0$	$\hat{e}_2\cdot\hat{e}_3 = 0$	$\hat{e}_3\cdot\hat{e}_3 = -1$

Rules for unit 4-vector dot pdt.

$$\underset{\sim}{A} = \sum_{n=1}^{3} A^n\hat{e}_n = A^1\hat{e}_1 + A^2\hat{e}_2 + A^3\hat{e}_3 \quad , \quad \underset{\sim}{B} = \sum_{m=1}^{3} B^m\hat{e}_m = B^1\hat{e}_1 + B^2\hat{e}_2 + B^3\hat{e}_3$$

$$\underset{\sim}{A}\cdot\underset{\sim}{B} = \sum_n\sum_m A^n B^m \underbrace{\hat{e}_n\cdot\hat{e}_m}_{\equiv g_{nm}} = \sum_n \underbrace{\sum_m} A^n \underbrace{B^m g_{nm}} = \sum_n A^n \underbrace{\{g_{n1}B^1 + g_{n2}B^2 + g_{n3}B^3\}}_{B_n}$$

$$\sum_m g_{nm}B^m \overset{?}{=} (\,?\,)_n$$

Notation Alert

Case 1 : Kronecker delta

$$* \quad g_{nm} = \delta_{nm} \Rightarrow \sum g_{nm} B^m = \sum_m \delta_{nm} B^m = \delta_{n1} B^1 + \delta_{n2} B^2 + \delta_{n3} B^3 = \begin{cases} B^1 & n=1 \\ B^2 & n=2 \\ B^3 & n=3 \end{cases}$$

$$\boxed{\sum_m g_{nm} B^m = B_n}$$

$B_1 = B^1$, $B_2 = B^2$, $B_3 = B^3$

Case 2 : Minkowski

$$* \quad g_{nm} = \eta_{nm} = \overbrace{\sum_m g_{nm} B^m}^{B_n} = \sum_m \eta_{nm} B^m = \eta_{n0} B^0 + \eta_{n1} B^1 + \eta_{n2} B^2 + \eta_{n3} B^3 = \begin{cases} B^0 & n=0 \\ -B^1 & n=1 \\ -B^2 & n=2 \\ -B^3 & n=3 \end{cases}$$

✓ $B^0 = B_0$, $B_1 = -B^1$, $B_2 = -B^2$, $B_3 = -B^3$

$$* \quad \boxed{\underset{\sim}{A} \cdot \underset{\sim}{B} = \sum_n A^n B_n}$$

$$B_n = \sum_m g_{nm} B^m$$

Computable — Computer — given

Case I

$$\underset{\sim}{A} \cdot \underset{\sim}{B} = \sum_n A^n B_n = A^1 B_1 + A^2 B_2 + A^3 B_3$$

$$= A^1 B^1 + A^2 B^2 + A^3 B^3 \ ✓$$

↓

Same as vector Algebra

$$\boxed{\underset{\sim}{A} \cdot \underset{\sim}{B} = A^1 B^1 + A^2 B^2 + A^3 B^3}$$

Case II

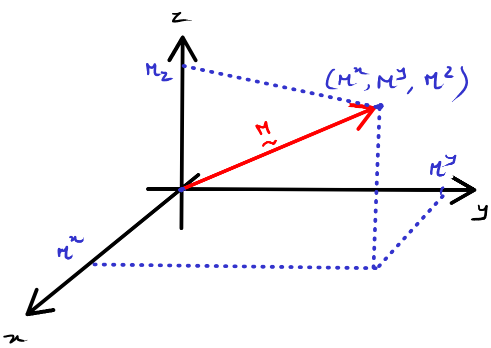
$$\underset{\sim}{A} \cdot \underset{\sim}{B} = \sum A^n B_n = A^0 B_0 + A^1 B_1 + A^2 B_2 + A^3 B_3 \Rightarrow \boxed{\underset{\sim}{A} \cdot \underset{\sim}{B} = A^0 B^0 - A^1 B^1 - A^2 B^2 - A^3 B^3}$$

Lecture-10 (6/Jun) 2

B.2 Direction Cosines of a vector

$$* \quad \underset{\sim}{M} = \sum_a M^a \hat{e}_a \quad , \quad \hat{e}_a \cdot \hat{e}_b = g_{ab} = \delta_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\hat{e}_a = \underbrace{(\hat{e}_x \ \hat{e}_y \ \hat{e}_z)}_{\text{unit vectors along corresp. directions}} \ , \quad M^a = \underbrace{(M^x \ M^y \ M^z)}_{\text{Scalar Comp.}} \ , \quad \left.\begin{matrix} \hat{e}_x \cdot \hat{e}_x = 1 \\ \hat{e}_x \cdot \hat{e}_y = 0 \end{matrix}\right\}$$



* $\underset{\sim}{M}$ Lives completely in a 3-D space

↓

$$\boxed{\begin{array}{l} \text{Can I decompose it in resp. directions ?} \\ \quad \text{"} \\ \text{How much part of the whole } \underset{\sim}{M} \text{ is in each direction ?} \end{array}$$

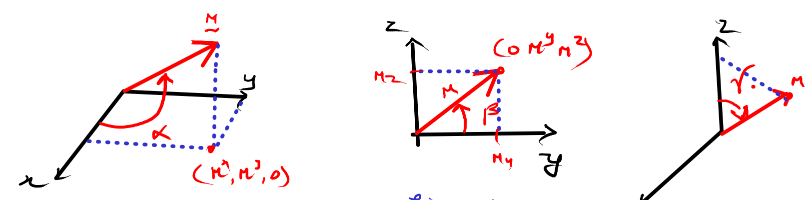


Fig 101

$M^x = M \cos\alpha$ $\quad M^y = M\cos\beta$ $\quad M^z = M\cos\gamma$

* $(\cos\alpha, \cos\beta, \cos\gamma)$: directional Cosines of $\underset{\sim}{M}$

(α, β, γ) - resp. Angles that $\underset{\sim}{M}$ makes with x, y, z axes.

$\boxed{\text{Projection} \equiv \text{a result of collapsing a vector onto } \underbrace{\text{one of the coordinate axes}}_{\text{lines}}}$

* $M^a = (M^x\ M^y\ M^z) = (M\cos\alpha\quad M\cos\beta\quad M\cos\gamma)$

$$\underset{\sim}{M} = \sum_{a=1}^{3} M^a \hat{e}_a = M^x\hat{e}_x + M^y\hat{e}_y + M^z\hat{e}_z = M(\cos\alpha\,\hat{e}_x + \cos\beta\,\hat{e}_y + \cos\gamma\,\hat{e}_z)$$

Contract.

* $$\underset{\sim}{M}\cdot\underset{\sim}{M} = \underbrace{|\underset{\sim}{M}|^2}_{= M^2 \text{ (notation Alert)}} = \sum_a\sum_b M^a M^b \underbrace{\hat{e}_a\cdot\hat{e}_b}_{g_{ab}=\delta_{ab} \text{ demand}} = \sum_{ab} M^a M^b \delta_{ab} = \sum_a M^a \underbrace{\sum_b M^b\delta_{ab}}_{} = \sum_a M^a M_a = M^1M_1 + M^2M_2 + M^3M_3 = (M^x)^2 + (M^y)^2 + (M^z)^2$$

scalar

$$M^1\delta_{a1} + M^2\delta_{a2} + M^3\delta_{a3} = \begin{cases} M_1 & a=1 \\ M_2 & a=2 \\ M_3 & a=3 \end{cases} = M_a$$

$M^2 = |\underset{\sim}{M}|^2 = (M^x)^2 + (M^y)^2 + (M^z)^2$ Magnitude squared of a vector

* $M = \sqrt{M_x^2 + M_y^2 + M_z^2}$

* $$\underset{\sim}{M} = M(\cos\alpha\,\hat{e}_x + \cos\beta\,\hat{e}_y + \cos\gamma\,\hat{e}_z) \Rightarrow \underbrace{|\underset{\sim}{M}|^2}_{} = \underset{\sim}{M}\cdot\underset{\sim}{M} = M^2(\cos\alpha\,\hat{e}_x + \cos\beta\,\hat{e}_y + \cos\gamma\,\hat{e}_z)\cdot(\cos\alpha\,\hat{e}_x + \cos\beta\,\hat{e}_y + \cos\gamma\,\hat{e}_z)$$

$$= M^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma)$$

$$\cancel{M^2} = \cancel{M^2}(\cos^2\alpha + \cos^2\beta + \cos^2\gamma)$$

* $\boxed{\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1}$ direction cosines are not entirely independent

RULE OF INDEX NOTATION

$\boxed{\text{The same index can be seen on the opposite sides of the Eq}^n \text{ (RHS / LHS) not on the same side}}$

(unless they are summed over)

B.3 Dot Product Properties (summarized)

* $\underset{\sim}{A} = \sum_i A^i\hat{e}_i = A^1\hat{e}_1 + A^2\hat{e}_2 + A^3\hat{e}_3$: $\hat{e}_i\cdot\hat{e}_j = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$

How do you SELECT / CHOOSE OUT / EXTRACT OUT a scalar component of a vector?

$\downarrow$ Prescription

$\boxed{\text{Projecting the vector on } \hat{e}_j \text{ direction}}$ = dot product with a unit vector in that direction

Expression

* $$\underset{\sim}{A}\cdot\hat{e}_j \stackrel{?}{=} \left(\sum_i A^i\hat{e}_i\right)\cdot\hat{e}_j = \sum_i \{A^i \underbrace{\hat{e}_i\cdot\hat{e}_j}_{\delta_{ij}}\} = \sum_i A^i \underbrace{\delta_{ij}}_{} = A_j \Rightarrow \boxed{\underset{\sim}{A}\cdot\hat{e}_j = A_j}$$

$j=2 \Rightarrow \underset{\sim}{A}\cdot\hat{e}_2 = (A^1\hat{e}_1 + A^2\hat{e}_2 + A^3\hat{e}_3)\cdot\hat{e}_2 = A_2$

selected out the 2^{nd} component of $\underset{\sim}{A}$

* $\underset{\sim}{A} = \sum A^i\hat{e}_i \rightarrow \underbrace{\underset{\sim}{A}\cdot\hat{e}_i}_{\text{to take out } i^{th} \text{ comp. of } \underset{\sim}{A}} = A_i$

Lecture-11 (7/Jun) 1.5

* $\hat{e}_i = (\hat{x}\ \ \hat{y}\ \ \hat{z})$ given, Rectangular Cart. basis ; $A^i = (a_1\ \ a_2\ \ a_3)$ given, Scalar comp.

1) $\underset{\sim}{A} = \sum_i A^i \hat{e}_i = a_1\hat{x} + a_2\hat{y} + a_3\hat{z}$ (define the vector) ; given $g_{ab} = \delta_{ab}$

2) $\underset{\sim}{A}\cdot\underset{\sim}{A} = \sum_i A^i A_i = \ ?$ (Magnitude of vector)

check

* $\underset{\sim}{A}\cdot\underset{\sim}{A} = \left(\sum_i A^i \hat{e}_i\right)\cdot\left(\sum_j A^j \hat{e}_j\right) = \sum_i \sum_j A^i A^j \underbrace{(\hat{e}_i\cdot\hat{e}_j)}_{\delta_{ij}} = \sum_i A^i \underbrace{\sum_j A^j \delta_{ij}}_{\equiv A_i}$

$A_i = \sum_j g_{ij} A^j$

$\underbrace{A_i = \sum_j \delta_{ij} A^j}$

$A_1 = A^1$

$A_2 = A^2$

$A_3 = A^3$

Calculated

* $A_i = \sum_j \delta_{ij} A^j = \delta_{i1}A^1 + \delta_{i2}A^2 + \delta_{i3}A^3$

$i=1 \Rightarrow A_1 = \delta_{11}A^1 \Rightarrow A_1 = a_1$

$i=2 \Rightarrow A_2 = \delta_{22}A^2 \Rightarrow A_2 = a_2$

$i=3 \Rightarrow A_3 = \delta_{33}A^3 \Rightarrow A_3 = a_3$

$\Rightarrow A_i = (a_1\ \ a_2\ \ a_3) = A^i$

* $\underset{\sim}{A}\cdot\underset{\sim}{A} = \sum A^i A_i = (A^1)^2 + (A^2)^2 + (A^3)^2 = a_1^2 + a_2^2 + a_3^2 = |\underset{\sim}{A}|^2 \Rightarrow |\underset{\sim}{A}| = \sqrt{a_1^2+a_2^2+a_3^2}$ magnitude

3) $\underset{\sim}{A} = a_1\hat{x} + a_2\hat{y} + a_3\hat{z}$, $|\underset{\sim}{A}| = \sqrt{a_1^2+a_2^2+a_3^2}$, $\hat{e}_i = (\hat{x}\ \ \hat{y}\ \ \hat{z})$ unit basis vector, $\hat{e}_i\cdot\hat{e}_j = \delta_{ij}$

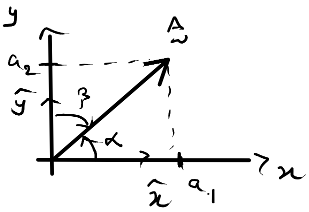
* $\underbrace{\left(\underset{\sim}{A}\cdot\hat{z}\right)} = a_3 \rightarrow \underbrace{\underset{\sim}{A}}\cdot\hat{e}_i = A_i$

$\uparrow$ Projections

$$\boxed{\frac{\underset{\sim}{A} - a_1\hat{x} - a_2\hat{y}}{\hat{z}} \neq a_3}$$

$a_1 = |\underset{\sim}{A}|\cos\alpha$ $\quad a_3 = |\underset{\sim}{A}|\cos\gamma$

$a_2 = |\underset{\sim}{A}|\cos\beta$



$\underset{\sim}{A}\cdot\underset{\sim}{A} \rightarrow$ Magnitude

$\hat{e}_i\cdot\hat{e}_j \rightarrow$ g_{ab} Demand

$\underset{\sim}{A}\cdot\hat{e}_i \rightarrow$??

$\downarrow$

To project $\underset{\sim}{A}$ in $\hat{e}_i$ direction

$\underset{\sim}{A}\cdot\hat{e}_i = \underbrace{\left(\sum_j A^j \hat{e}_j\right)}\cdot\hat{e}_i = \sum A^j \underbrace{(\hat{e}_i\cdot\hat{e}_j)}_{\delta_{ij}} = \sum_j A^j\delta_{ij} = A^1\delta_{i1} + A^2\delta_{i2} + A^3\delta_{i3}$

$\underset{\sim}{A}\cdot\hat{e}_i = A^1\delta_{i1} + A^2\delta_{i2} + A^3\delta_{i3}$

$i=1$ $\quad \underset{\sim}{A}\cdot\hat{e}_1 = A^1\delta_{11} = A^1 = a_1$

$i=2$ $\quad \underset{\sim}{A}\cdot\hat{e}_2 = A^2\delta_{22} = A^2 = a_2$

$i=3$ $\quad \underbrace{\underset{\sim}{A}\cdot\hat{e}_3} = \underbrace{A^3\delta_{33}} = A^3 = a_3$

* $\cos\alpha = \dfrac{a_1}{|\underset{\sim}{A}|}$

$\cos\beta = \dfrac{a_2}{|\underset{\sim}{A}|}$

$\cos\gamma = \dfrac{a_3}{|\underset{\sim}{A}|}$

Lecture-12 (8 Jan) 1.5 + 1.5

* $\underset{\sim}{A} = \sum_i A^i \hat{e}_i$, $\hat{e}_i = (\hat{e}_1 \dots \hat{e}_n)$ 'Basis unit vector' : $\hat{e}_i \cdot \hat{e}_j = g_{ij} \xrightarrow{\text{Eucl.}} \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

* $\underset{\sim}{A} \cdot (\underset{\sim}{B} + \underset{\sim}{C}) = ?$

Note : (Addition operator)

* $\underset{\sim}{G} = \underset{\sim}{A} - 2\underset{\sim}{B} + 3\underset{\sim}{C}$, $\underset{\sim}{A} = \sum_i A^i e_i$, $\underset{\sim}{B} = \sum_i B^i \hat{e}_i$, $\underset{\sim}{C} = \sum_i C^i e_i$; $\underset{\sim}{G} = \sum G^i \hat{e}_i = ?$

Ex:

$$\left.\begin{array}{lll} \underset{\sim}{A} = \hat{\imath} + \hat{\jmath} + \hat{k} & \Rightarrow & A^i = (1 \;\; 1 \;\; 1) \\ \underset{\sim}{B} = 2\hat{\imath} + 3\hat{\jmath} + \hat{k} & \Rightarrow & B^i = (2 \;\; 3 \;\; 1) \\ \underset{\sim}{C} = 4\hat{\imath} + \hat{\jmath} + 2\hat{k} & \Rightarrow & C^i = (4 \;\; 1 \;\; 2) \end{array}\right\} \quad \hat{e}_i = (\hat{\imath} \;\; \hat{\jmath} \;\; \hat{k})$$

$$\underset{\sim}{G} = \underset{\sim}{A} - 2\underset{\sim}{B} + 3\underset{\sim}{C} = (\hat{\imath} + \hat{\jmath} + \hat{k}) \underbrace{-2(2\hat{\imath} + 3\hat{\jmath} + \hat{k})}_{-4\hat{\imath} - 6\hat{\jmath} - 2\hat{k}} + \underbrace{3(4\hat{\imath} + \hat{\jmath} + 2\hat{k})}_{12\hat{\imath} + 3\hat{\jmath} + 6\hat{k}} = (1 - 4 + 12)\hat{\imath} + (1 - 6 + 3)\hat{\jmath} + (1 - 2 + 6)\hat{k} = 9\hat{\imath} - 2\hat{\jmath} + 5\hat{k}$$

$$\Downarrow$$

$$G^i = (9 \;\; -2 \;\; 5)$$

* $\underset{\sim}{G} = \underset{\sim}{A} - 2\underset{\sim}{B} + 3\underset{\sim}{C} \Rightarrow \sum_i G^i \hat{e}_i = \sum_i A^i \hat{e}_i - 2 \sum_i B^i \hat{e}_i + 3 \sum_i C^i \hat{e}_i$

$$\Downarrow$$

$$G^i = A^i - 2B^i + 3C^i = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - 2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ -2 \\ 5 \end{pmatrix}$$

$\sum_i A_{ijk} B^i = F_{jk}$

must have same # of entries

* $\underset{\sim}{A} + \underset{\sim}{B} = \sum_{i=1}^{n} A^i \hat{e}_i + \sum_{j=1}^{m} B^j \hat{e}_j \underset{m=n}{\overset{i=j}{=}} \sum_i A^i \hat{e}_i + \sum_i B^i \hat{e}_i = \sum_i \underbrace{(A^i + B^i)}_{\in \mathbb{R}} \hat{e}_i = \sum_i (B^i + A^i) \hat{e}_i = \overbrace{\sum B^i \hat{e}_i}^{\underset{\sim}{B}} + \sum A^i \hat{e}_i = \underset{\sim}{B} + \underset{\sim}{A}$

vector addition is Commutative

* $(\underset{\sim}{A} + \underset{\sim}{B}) + \underset{\sim}{C} = \underbrace{\underbrace{\left(\sum_i A^i \hat{e}_i + \sum_i B^i \hat{e}_i \right)}_{\sum_i (A^i + B^i) \hat{e}_i} + \sum_i C^i \hat{e}_i}_{\sum_i \{ \underbrace{(A^i + B^i) + C^i}_{\in \mathbb{R}} \} \hat{e}_i} = \sum_i \{ A^i + (B^i + C^i) \} \hat{e}_i = \sum_i A^i e_i + \underbrace{\sum_i (B^i + C^i) \hat{e}_i}_{\left(\sum_i B^i \hat{e}_i + \sum_i C^i \hat{e}_i \right)} = \underset{\sim}{A} + (\underset{\sim}{B} + \underset{\sim}{C})$

Key pt. of the proof

vect. + is Associative

* $\alpha \in \mathbb{R}$ Scalar, $A^i \in \mathbb{R}$ $\rightarrow$ $\alpha \underset{\sim}{A} = \alpha \sum_i A^i \hat{e}_i = \sum_i \underbrace{(\alpha A^i)}_{B^i} \hat{e}_i = \sum_i B^i e_i = \underset{\sim}{B}$

has no index $\Downarrow$ Can Enter the $\sum$

'vector multi. by a Scalar α'

$\sum x^i = x^1 + x^2 + x^3$

$2 \sum x^i = 2(x^1 + x^2 + x^3)$

$= 2x^1 + 2x^2 + 2x^3$

$= \sum 2x^i$

* $\underset{\sim}{A} - \underset{\sim}{B} = \sum_i A^i \hat{e}_i - \sum_i B^i \hat{e}_i = \sum_i (A^i - B^i) \hat{e}_i = \sum \{ A^i + (-B^i) \} \hat{e}_i = \sum A^i e_i + \sum (-B^i) \hat{e}_i = \underset{\sim}{A} + (-\underset{\sim}{B})$

* $\underset{\sim}{A} \cdot (\underset{\sim}{B} + \underset{\sim}{C}) = \sum_i A^i e_i \cdot \left(\sum_j B^j \hat{e}_j + \sum_j C^j \hat{e}_j \right) = \left(\sum_i A^i \hat{e}_i \cdot \sum_j B^j \hat{e}_j \right) + \left(\sum_i A^i \hat{e}_i \cdot \sum_j C^j \hat{e}_j \right)$

$$\underset{\sim}{A}\cdot(\underset{\sim}{B}+\underset{\sim}{C}) = \left(\sum_i\sum_j A^iB^j \underbrace{\hat{e}_i\cdot\hat{e}_j}_{g_{ij}=\delta_{ij}}\right) + \left(\sum_i\sum_j A^iC^j \underbrace{\hat{e}_i\cdot\hat{e}_j}_{g_{ij}=\delta_{ij}}\right) = \sum_i\sum_j A^iB^j\underbrace{\delta_{ij}} + \sum_i\sum_j A^iC^j\delta_{ij}$$

$$= \underbrace{\sum_i A^iB_i + \sum_i A^iC_i} = \underset{\sim}{A}\cdot\underset{\sim}{B} + \underset{\sim}{A}\cdot\underset{\sim}{C}$$

vector • is distributive under +

$$\underset{\sim}{A}\cdot\underset{\sim}{B} = \sum_i A^iB_i$$

α is indep. of i & j — α is indep of i

$$* \quad \underset{\sim}{A}\cdot(\alpha\underset{\sim}{B}) = \sum_i A^i\hat{e}_i\cdot\left(\alpha\sum_j B^j\hat{e}_j\right) = \underbrace{\alpha\left(\sum_i A^i\hat{e}_i\cdot\sum_j B^j e_j\right)}_{\alpha(\underset{\sim}{A}\cdot\underset{\sim}{B})} = \underbrace{\sum_i(\alpha A^i)\hat{e}_i}_{\alpha\underset{\sim}{A}}\cdot\underbrace{\sum_j B^je_j}_{\underset{\sim}{B}} = (\alpha\underset{\sim}{A})\cdot\underset{\sim}{B}$$

vector • is distributive under •

$$* \quad \underset{\sim}{A}\cdot\underset{\sim}{B} = \sum_i A^i\hat{e}_i\cdot\sum_j B^j\hat{e}_j = \sum_i\sum_j A^iB^j\underbrace{\hat{e}_i\cdot\hat{e}_j}_{\delta_{ij}=\delta_{ji}} = \sum_i\sum_j \underbrace{A^iB^j}\hat{e}_j\cdot\hat{e}_i = \sum_j B^j\hat{e}_j\cdot\sum_i A^ie_j = \underset{\sim}{B}\cdot\underset{\sim}{A}$$

Norm of a vector = Magnitude of vector

$$* \quad \underset{\sim}{A}\cdot\underset{\sim}{A} = \underbrace{|\underset{\sim}{A}|^2}_{\text{"A mod square"}} = \sum_{ij}A^iA^j\underbrace{\hat{e}_i\cdot\hat{e}_j}_{g_{ij}} = \sum_{ij}A^iA^j\underbrace{g_{ij}} = \sum_i A^iA_i \quad , \quad A_i \equiv \sum_j g_{ij}A^j$$

$$A^i \longrightarrow A_i = \sum_j \eta_{ij}A^j$$

Euclidean

$g_{ij} = \delta_{ij}$ $\quad A^i \in \mathbb{R}$

$$\underset{\sim}{A}\cdot\underset{\sim}{A} = |\underset{\sim}{A}|^2 = \sum A^iA_i = \sum A^iA^i$$
$$= (A^1)^2 + (A^2)^2 + (A^3)^2 = 0$$

- $+$ ✓
- 0 → iff $A^i = 0 \ \forall i$ — vector can never be ⊥ to itself
- $-$ → iff $A^i \in \mathbb{C}$

$x^i = (x \quad y \quad z)$

Space

Minkowskian

$g_{ij} = \eta_{ij}$ $\quad i,j = 0,1,2,3$

$$|A|^2 = \sum_i A^iA_i = \sum_i\sum_j \eta_{ij}A^iA^j$$
$$= \eta_{00}(A^0)^2 + \eta_{11}(A^1)^2 + \eta_{22}(A^2)^2 + \eta_{33}(A^3)^2$$
$$|A|^2 = (A^0)^2 - (A^1)^2 - (A^2)^2 - (A^3)^2$$

- $+$: $(A^0)^2 > (A^1)^2 + (A^2)^2 + (A^3)^2$ → A^i is A^0 like — $\underbrace{}_{\text{Timelike}}$
- 0: $(A^0)^2 = (A^1)^2 + (A^2)^2 + (A^3)^2$ → A^i is Null/ light like
- $-$: $(A^0)^2 < (A^1)^2 + (A^2)^2 + (A^3)^2$ → A^i is A^1 like, A^2 like, A^3 like — Spacelike

$$x^i = (\underset{A^0}{t} \quad \underset{A^1}{x} \quad \underset{A^2}{y} \quad \underset{A^3}{z})$$

Spacetime

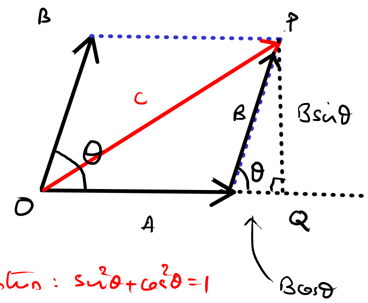
if Mag. of a vector in Minkowski geo. is
- → + vector is called time like
- → 0 vector is called lightlike
- → − vector is called Spacelike

vector Naming

dot of $\underset{\sim}{A}\cdot\underset{\sim}{B}$

* $\underset{\sim}{C} = \underset{\sim}{A} + \underset{\sim}{B} \Rightarrow |\underset{\sim}{C}|^2 = \underset{\sim}{C}\cdot\underset{\sim}{C} = (\underset{\sim}{A}+\underset{\sim}{B})\cdot(\underset{\sim}{A}+\underset{\sim}{B}) = \underset{\sim}{A}\cdot\underset{\sim}{A} + \underset{\sim}{B}\cdot\underset{\sim}{B} + \underset{\sim}{A}\cdot\underset{\sim}{B} + \underbrace{\underset{\sim}{B}\cdot\underset{\sim}{A}}_{\underset{\sim}{A}\cdot\underset{\sim}{B}} = |\underset{\sim}{A}|^2 + |\underset{\sim}{B}|^2 + 2\,\underset{\sim}{A}\cdot\underset{\sim}{B}$ 'Algebraic'

* $$\boxed{\text{def } 1: \quad \underset{\sim}{A}\cdot\underset{\sim}{B} = \sum_i\sum_j g_{ij}A^iB^j = \sum_i\sum_j \delta_{ij}A^iB^j}$$



* geometrically → Pythagoras ⇒ $c^2 = (B\sin\theta)^2 + (A + \overset{\text{distances}}{B\cos\theta})^2$

$= A^2 + B^2\cos^2\theta + 2AB\cos\theta + B^2\sin^2\theta$ {Assumption: $\sin^2\theta + \cos^2\theta = 1$

$c^2 = A^2 + B^2 + 2AB\cos\theta$ 'geometry'

Vector Algebra

$|\underset{\sim}{C}|^2 = |\underset{\sim}{A}|^2 + |\underset{\sim}{B}|^2 + 2\,\underset{\sim}{A}\cdot\underset{\sim}{B}$

vectors

Pythagorean theorem (geometry)

$C^2 = A^2 + B^2 + 2AB\cos\theta$

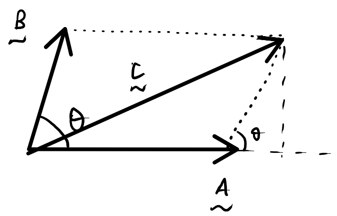
numbers

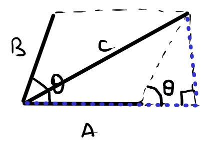
interchangeable

"

Equivalent

'Parallelogram'





iff.

def. 2:

Algebra ← $\boxed{\underset{\sim}{A}\cdot\underset{\sim}{B} = |\underset{\sim}{A}||\underset{\sim}{B}|\cos\theta}$ → Angle

variable → geometry (w Trig.)

Consistency Condition for then a formal system to be Equivalent

Lecture-13 (9/Jun) 2

Euclidean geo.

* $\underset{\sim}{A} = \underbrace{\sum_i A^i\hat{e}_i}_{\text{Linear Comb}^n \text{ of Basis vectors}} \; : \; \hat{e}_i\cdot\hat{e}_j = g_{ij} = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j\end{cases}$ ⇒ $\boxed{\text{Contravariant vectors } A^i \equiv \text{Covariant vector } A_i}$

" the coeff. A^i are the components of $\underset{\sim}{A}$. More often than not we will forget the basis entirely & refer somewhat loosely to "the vector A^i", but keep in mind that this is shorthand "

— Sean Carroll, space & gravity

Scalars
vectors
n-vectors
tensors
Pseudovector
Pseudoscalar
Spinor
p-forms
⋮
Entities

} Matrices as a language to Represent

* $A^i = (A^1\ A^2 \ldots A^n)_{1\times n} \overset{\text{FAPP}}{\equiv} \underset{\sim}{A}$ forget writing $\hat{e}_i$ {Advanced Remark (later)

* $\underset{\sim}{A}\cdot\underset{\sim}{B} = \sum_i\sum_j g_{ij}A^iB^j = \sum_i A^iB_i$ Algebraic def of dot

* $\underset{\sim}{A}\cdot\underset{\sim}{B} = |\underset{\sim}{A}||\underset{\sim}{B}|\cos\theta$ geometrical def of dot, $|\underset{\sim}{A}| = A = \sqrt{\sum_i\sum_j g_{ij}A^iA^j}$

Interpretation of geometric def:

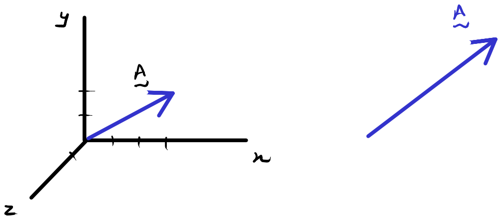
* $\underset{\sim}{A}\cdot\hat{e}_j = A_j$ 'Part of $\underset{\sim}{A}$ in $\hat{e}_j$ direction' is A_j = projection of $\underbrace{\underset{\sim}{A}}_{\text{whole vector}}$ on $\underbrace{\hat{e}_j}_{\text{unit}}$

Idea of unit vectors :

* $\underset{\sim}{A} = \sum A^i \hat{e}_i \; : \; \hat{e}_i \cdot \hat{e}_j = \delta_{ij}$

Ex

$\underset{\sim}{A} = 3\hat{\imath} + 2\hat{\jmath} + \hat{k}$; $|\underset{\sim}{A}| = \sqrt{14}$



* $\underset{\sim}{A} \cdot \hat{\imath} = 3$

$\underset{\sim}{A} \cdot \hat{\jmath} = 2$

$\underset{\sim}{A} \cdot \hat{k} = 1$

* $|\underset{\sim}{A}| = \sqrt{14} \xrightarrow{\text{How}} \boxed{|\underset{\sim}{A}| = 1}$

Magnitude of $\hat{e}_i$ (std. Euclidean basis vector) :

* $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$

$|\hat{e}_a|^2 = \hat{e}_a \cdot \hat{e}_a = \delta_{aa} = 1 \Rightarrow |\hat{e}_a| > 0 \rightarrow \boxed{|\hat{e}_a| = +1}$ ✓

How

$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$

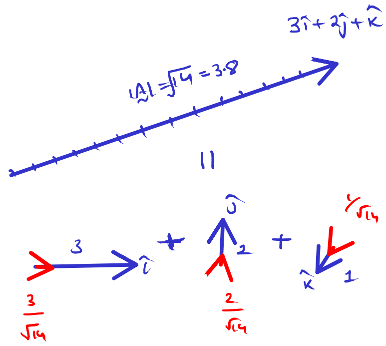
$|\underset{\sim}{\hat{e}_i}|^2 = \sum_{ij} \delta_{ij} \underbrace{\hat{e}^i \cdot \hat{e}^j} = \sum_{ij} \delta_{ij} \delta^{ij}$

$= 3 =$ not correct

$|A|^2 = \sum_{ij} g_{ij} A^i A^j = \sum_i A^i A_i$

* $\underset{\sim}{A} = 3\hat{\imath} + 2\hat{\jmath} + \hat{k} \longrightarrow |\underset{\sim}{A}| = \sqrt{14}$

↓ operation on $\underset{\sim}{A}$ to get $|\underset{\sim}{A}'| = 1$



$|\underset{\sim}{A}| = \sqrt{3^2 + 2^2 + 1^2} = \sqrt{14}$

$\underset{\sim}{A} = 3\hat{\imath} + 2\hat{\jmath} + \hat{k} \longrightarrow \underbrace{\frac{\underset{\sim}{A}}{|\underset{\sim}{A}|}}_{\hat{A}} = \frac{3\hat{\imath} + 2\hat{\jmath} + \hat{k}}{\sqrt{14}} = \underbrace{\frac{3}{\sqrt{14}}}_{0.8}\hat{\imath} + \underbrace{\frac{2}{\sqrt{14}}}_{0.53}\hat{\jmath} + \underbrace{\frac{1}{\sqrt{14}}}_{0.27}\hat{k}$

$\sqrt{(0.8)^2 + (0.53)^2 + (0.26)^2} = 0.997 \sim 1$

$$\boxed{\hat{A} = \frac{\underset{\sim}{A}}{|\underset{\sim}{A}|} = \frac{\sum_i A^i \hat{e}_i \longrightarrow \text{vector}}{\sqrt{\sum_i \sum_j g_{ij} A^i A^j} \longrightarrow \text{Scalar}} \quad \left\{ \text{unit vector of } \underset{\sim}{A} \right.}$$

vector with a length (minimum) of 1 unit in the direction same as $\underset{\sim}{A}$ ⟶ acts as basis vector build out of the vector itself

* $\hat{A} = \frac{3}{\sqrt{14}}\hat{\imath} + \frac{2}{\sqrt{14}}\hat{\jmath} + \frac{\hat{k}}{\sqrt{14}} \longrightarrow \underset{\sim}{G} = \underset{\sim}{A} - 2\underset{\sim}{B} + 3\underset{\sim}{C} = \sum_i G^i \hat{e}_i \longrightarrow$ Any vector can be expanded in terms of Standard Euclidean basis vector

$\{\hat{A}, \hat{B}, \hat{C}\}$ set of basis

$\hat{P}_i$: $\hat{P}_1 = \hat{A}$, $\hat{P}_2 = \hat{B}$, $\hat{P}_3 = \hat{C}$

$\parallel$

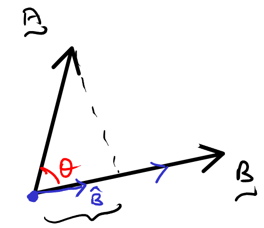
$\sum_i G^i \hat{P}_i = G^1 \hat{A} + G^2 \hat{B} + G^3 \hat{C}$

Any vector can be expanded in terms of Any arbitrary basis vector

* $\underset{\sim}{A} = \sum A^i \hat{e}_i$, $\hat{A} = \dfrac{\underset{\sim}{A}}{|\underset{\sim}{A}|}$, $\underset{\sim}{B} = \sum_j B^j e_j$, $\hat{B} = \dfrac{\underset{\sim}{B}}{|\underset{\sim}{B}|}$, $\underset{\sim}{A} \cdot \underset{\sim}{B} = |A||B|\cos\theta = \sum_{ij} g_{ij} A^i B^j$, $\underset{\sim}{A} \cdot \hat{e}_j = A_j$

* $\underset{\sim}{A} \cdot \hat{B} \equiv \underbrace{A_B}_{\downarrow} = \dfrac{\underset{\sim}{A} \cdot \underset{\sim}{B}}{|\underset{\sim}{B}|} = \dfrac{\underset{\sim}{A} \cdot \underset{\sim}{B}}{|\underset{\sim}{B}|}$ (1^{st} Form)

The Projection of A in the dirn $\hat{B}$



* $\underset{\sim}{A} \cdot \hat{B} = \dfrac{\underset{\sim}{A} \cdot \underset{\sim}{B}}{|\underset{\sim}{B}|} = \dfrac{|\underset{\sim}{A}||\underset{\sim}{B}|\cos\theta}{|\underset{\sim}{B}|} = |\underset{\sim}{A}|\cos\theta$ (2^{nd} form)

$A_B = |A|\cos\theta$

* $(\underbrace{\underset{\sim}{A} \cdot \hat{B}}_{\text{Number}})\hat{B} = A_B \hat{B} = (|A|\cos\theta)\hat{B}$ the projection of A in the $\hat{B}$ dirn

Lecture-14 (10/Jan) 2

* $\underset{\sim}{A} \cdot \underset{\sim}{B} = |\underset{\sim}{A}||\underset{\sim}{B}|\cos\theta = \underbrace{(|A|\cos\theta)}_{\underset{\sim}{A} \cdot \hat{B}}|B| = \underbrace{(|B|\cos\theta)}_{\underset{\sim}{B} \cdot \hat{A}}|A| \Rightarrow A_B|B| = B_A|A|$

$\underset{\sim}{A} = \underset{\sim}{B} \Rightarrow \underset{\sim}{A} \cdot \underset{\sim}{A} = |A|^2$

Schwarz Inequality

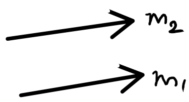
* $-1 \le \cos\theta \le 1 \Rightarrow |\cos\theta| \le 1 \Rightarrow \underbrace{|\underset{\sim}{A}||\underset{\sim}{B}||\cos\theta|}_{} \le |\underset{\sim}{A}||\underset{\sim}{B}| \Rightarrow \boxed{|\underset{\sim}{A} \cdot \underset{\sim}{B}| \le |\underset{\sim}{A}||\underset{\sim}{B}|}$ Cauchy-Schwarz Inequality

(later: Abstract vector Spaces)

Orthogonality

* $\underset{\sim}{A} \cdot \underset{\sim}{B} \overset{\text{def 1}}{=} \sum_i \sum_j A^i B^j \underbrace{\hat{e}_i \cdot \hat{e}_j}_{\delta_{ij}} \overset{\text{def 2}}{=} |\underset{\sim}{A}||\underset{\sim}{B}|\cos\theta$

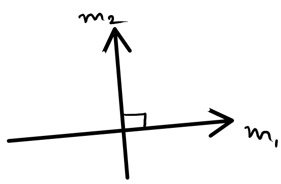
*



geometry

$m_1 = m_2$ Algebra

(Parallelity Condn)



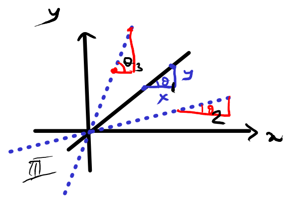
? $m_1 \ne \frac{1}{m_2}$, $m_1 \ne -m_2$?

(Perpendicularity Condn)

HW

$\boxed{m_1 = \dfrac{-1}{m_2}}$ ✓ ⊥ Formula in Trig

Proof (later)



$\tan\theta_1 = \dfrac{y_1}{x_1} = m_1 \Rightarrow y = m_1 x$

$\tan\theta_2 = \dfrac{y_2}{x_2} = m_2 \Rightarrow y = m_2 x$

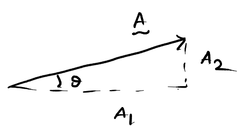
$\tan\theta = m$ (slope)

>0 Acute

<0 obtuse

$=0$ ∥ to x or y axis

* $\underset{\sim}{A} = A_1\hat{i} + A_2\hat{j}$, $\underset{\sim}{B} = B_1\hat{i} + B_2\hat{j}$



$\boxed{\underset{\sim}{A} \cdot \underset{\sim}{B} = 0} \Rightarrow \sum A^i B_i = A_1B_1 + A_2B_2 = 0$

$\Downarrow$

$A_1B_1 = -A_2B_2 \Rightarrow \underbrace{\dfrac{A_2}{A_1}}^{m_1} = -\dfrac{1}{\underbrace{\left(\dfrac{B_2}{B_1}\right)}_{m_2}} \Rightarrow$ Slope of $\underset{\sim}{A} = \dfrac{-1}{\text{Slope of } \underset{\sim}{B}} \Rightarrow \underset{\sim}{A} \perp \underset{\sim}{B}$

$\boxed{\text{orthogonal} = \text{Perpendicular}}$

$\underset{\sim}{A} \cdot \underset{\sim}{B} = 0 \Rightarrow \underset{\sim}{A}$ is orthogonal/Perpendicular to $\underset{\sim}{B}$

* $\hat{e}_i \cdot \hat{e}_j = \delta_{ij} = \begin{cases} 1 & i = j \rightarrow |\hat{e}_1|^2 = |\hat{e}_2|^2 = 1 \rightarrow \text{Std. Basis vectors are Normalized } (|\underset{\sim}{A}| = 1) \\ 0 & i \ne j \rightarrow \text{Std basis vectors are orthogonal} \end{cases} \Rightarrow$ Orthonormal Basis vectors $\hat{e}_i$

Lecture-15 (11/Jun) 2

Practice 1

$\underset{\sim}{A} = \sum_i A^i \hat{e}_i$, $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$, $\underset{\sim}{A} \cdot \underset{\sim}{B} = \sum_{ij} \delta_{ij} A^i B^j$, $|\underset{\sim}{A}|^2 = \underset{\sim}{A} \cdot \underset{\sim}{A} = \sum_{ij} \delta_{ij} A^i A^j$

1. $\underset{\sim}{r} = \{\underset{\sim}{r}_1, \underset{\sim}{r}_2, \underset{\sim}{r}_3\}$, $\hat{e}_i = \{\hat{i}, \hat{j}, \hat{k}\}$: $\hat{e}_i \cdot \hat{e}_j = g_{ij} = \delta_{ij}$, $r_1 = (3 \ -2 \ 1)$, $r_2 = (2 \ -4 \ -3)$, $r_3 = (-1 \ 2 \ 2)$

* $|\underset{\sim}{r}_3| = ?$ $\sqrt{\sum_{ij} \delta_{ij} r_3^i r_3^j} = \sqrt{\sum_i r_3^i (r_3)_i} = \sqrt{(r_3^1)^2 + (r_3^2)^2 + (r_3^3)^2} = \sqrt{(-1)^2+4+4} = 3$

* $\underset{\sim}{R} = \sum_{i=1}^{3} \underset{\sim}{r}_i = ?$, $|\underset{\sim}{R}| = ?$ $= \sqrt{32}$

* $\underset{\sim}{P} \equiv 2\underset{\sim}{r}_1 - 3\underset{\sim}{r}_2 - 5\underset{\sim}{r}_3 = ?$ $|\underset{\sim}{P}| = ? = \sqrt{30}$

2. $\underset{\sim}{r}_1 = 2\hat{i} - \hat{j} + \hat{k}$, $\underset{\sim}{r}_2 = \hat{i} + 3\hat{j} - 2\hat{k}$, $\underset{\sim}{r}_3 = -2\hat{i} + \hat{j} - 3\hat{k}$, $\underset{\sim}{r}_4 = 3\hat{i} + 2\hat{j} + 5\hat{k}$

$\underset{\sim}{r}_4 = a\underset{\sim}{r}_1 + b\underset{\sim}{r}_2 + c\underset{\sim}{r}_3$ find scalars (a, b, c)

$$\begin{pmatrix}3\\2\\5\end{pmatrix} = a\begin{pmatrix}2\\-1\\1\end{pmatrix} + b\begin{pmatrix}1\\3\\-2\end{pmatrix} + c\begin{pmatrix}-2\\1\\-3\end{pmatrix} \Rightarrow \begin{matrix} 2a+b-2c=3 \\ -a+3b+c=2 \\ a-2b-3c=5 \end{matrix} \longleftrightarrow \underbrace{\begin{pmatrix}2&1&-2\\-1&3&1\\1&-2&-3\end{pmatrix}}_{A_{3\times3}} \underbrace{\begin{pmatrix}a\\b\\c\end{pmatrix}}_{X} = \underbrace{\begin{pmatrix}3\\2\\5\end{pmatrix}}_{B} \Rightarrow AX = B$$

$$AX = B \xrightarrow{\exists A^{-1}} \underset{\substack{\text{from the} \\ \text{left}}}{\overrightarrow{A^{-1}}} AX = A^{-1}B \Rightarrow \underset{\sim}{I \cdot X} = A^{-1}B \Rightarrow \boxed{X = A^{-1} \cdot B}$$

$X \cdot I = X$

Matrix method.

Inverse of Matrix (Vol.1 Matrices) ???

$x^{-1}x = 1$

$AA^{-1} = I$

* $a = -2$, $b = 1$, $c = -3$

3. $r_1 = (2 \ 4 \ -5)$ $r_2 = (1 \ 2 \ 3)$ $\longrightarrow$ $\underset{\sim}{R} \equiv \underset{\sim}{r}_1 + \underset{\sim}{r}_2 = ?$ $= 3\hat{i} + 6\hat{j} - 2\hat{k}$

$\longrightarrow$ unit vector $\hat{P}$: $\hat{P} \parallel \underset{\sim}{R}$? $\hat{P} = \frac{3}{7}\hat{i} + \frac{6}{7}\hat{j} - \frac{2}{7}\hat{k}$

$= \left(\frac{3}{7} \ \frac{6}{7} \ \frac{-2}{7}\right)$

4. $A = (2 \ 2 \ -1)$ $B = (6 \ -3 \ 2)$ $\theta = ?$

1. $g_{ab} = \delta_{ab}$

2. $g_{ab} = ?_{ab}$

$\cos\theta = \frac{\underset{\sim}{A} \cdot \underset{\sim}{B}}{|\underset{\sim}{A}||\underset{\sim}{B}|} = \frac{4}{21}$ Angle b/w the 2 vectors via dot product

5. $A = (2 \ \alpha \ 1)$ $B = (4 \ -2 \ -2)$ $\alpha = ?$: $\underset{\sim}{A} \perp \underset{\sim}{B}$

$\alpha = 3$

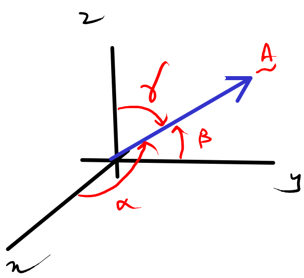
6. $A = (3 \ -6 \ 2)$ Angles made by $\underset{\sim}{A}$ with coordinate axes ? (α, β, γ)

$\boxed{\underset{\sim}{A} \cdot \hat{e}_i = A_i}$ $\rightarrow$

$\left.\begin{matrix} \underset{\sim}{A} \cdot \hat{i} = |A||\hat{i}|\cos\alpha = 3 \Rightarrow \cos\alpha = 3/7 \\ \underset{\sim}{A} \cdot \hat{j} = |A||\hat{j}|\cos\beta = -6 \Rightarrow \cos\beta = -6/7 \\ \underset{\sim}{A} \cdot \hat{k} = |A||\hat{k}|\cos\gamma = 2 \Rightarrow \cos\gamma = 2/7 \end{matrix}\right\}$ $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{9+36+4}{49} = 1$

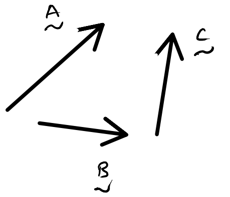
$|\underset{\sim}{A}| = \sqrt{9+36+4} = 7$

Directional Cosines



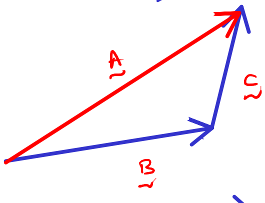
Lecture-16 (12/Jun) 1.5

7- $A^i = (3 \; -2 \; 1)$, $B^i = (1 \; -3 \; 5)$ $C^i = (2 \; 1 \; -4)$, $\hat{e}_i = (\hat{i} \; \hat{j} \; \hat{k})$ Std. Euclid. basis



Comment on the shape of the vectors! (resultant)

Rough Sketch



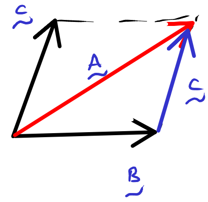
* $A^i = B^i + C^i \rightarrow \underset{\sim}{A} = \underset{\sim}{B} + \underset{\sim}{C}$ (E_q^n 1)

Algebra

geometry

It is a Triangle!

What kind?



* $|\underset{\sim}{A}|^2 = 14$

$|\underset{\sim}{B}|^2 = 35$

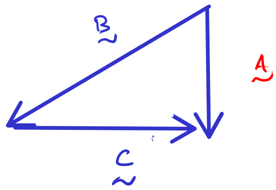
$|\underset{\sim}{C}|^2 = 21$

$|A| \neq |B| \neq |C| \Rightarrow$ ~~Equilateral~~

$\rightarrow |B|^2 = |A|^2 + |C|^2$ Pythagoras thm.

Final diag.

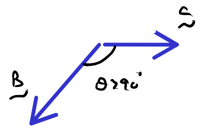
$\underset{\sim}{A} = \underset{\sim}{B} + \underset{\sim}{C}$



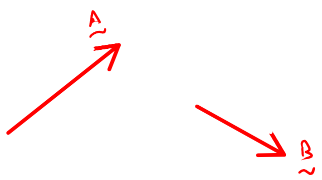
* $\underset{\sim}{A} \cdot \underset{\sim}{B} = \sum_c A^i B^i = A^1B^1 + A^2B^2 + A^3B^3 = 3+6+5 = 14 > 0 \Rightarrow \theta$ b/w $\underset{\sim}{A}$ & $\underset{\sim}{B}$ is Acute

$\underset{\sim}{A} \cdot \underset{\sim}{C} = \sum_c A^i C^i = 6-2-4 = 0 \Rightarrow \underset{\sim}{A} \perp \underset{\sim}{C} \Rightarrow \theta$ b/w $\underset{\sim}{A}$ & $\underset{\sim}{C}$ is $90^\circ \Rightarrow$ R.A.Δ

$\underset{\sim}{B} \cdot \underset{\sim}{C} = \sum B^i C^i = 2-3-20 = -21 < 0 \Rightarrow$ obtuse Angle $\Rightarrow \theta$ b/w $\underset{\sim}{B}$ & $\underset{\sim}{C}$ is obtuse

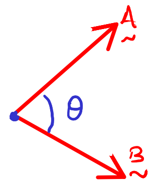


Rule:



$\exists$ an operation

Parallel transport



Join the TAILS of 2 vectors to get the Angle b/w the vector

* $\underset{\sim}{C} = \underset{\sim}{A} + \underset{\sim}{B} \Rightarrow |C|^2 = |A|^2 + |B|^2 + 2\underset{\sim}{A} \cdot \underset{\sim}{B}$; $\boxed{\underset{\sim}{A} \cdot \underset{\sim}{B} = 0} \rightarrow |C|^2 = |A|^2 + |B|^2 \Leftarrow \underset{\sim}{C} = \underset{\sim}{A} + \underset{\sim}{B}$

$\underset{\sim}{C} \equiv \underset{\sim}{A} - \underset{\sim}{B} \Rightarrow |C|^2 = (\underset{\sim}{A} - \underset{\sim}{B}) \cdot (\underset{\sim}{A} - \underset{\sim}{B}) = |A|^2 + |B|^2 - 2\underset{\sim}{A} \cdot \underset{\sim}{B}$ $\rightarrow |C|^2 = |A|^2 + |B|^2 \Leftarrow \underset{\sim}{C} = \underset{\sim}{A} - \underset{\sim}{B}$

In our Question:

* $|B|^2 = |A|^2 + |C|^2 \; \cancel{\pm 2\underset{\sim}{A} \cdot \underset{\sim}{C}}^{\,0}$

$|B|^2 = |A|^2 + |C|^2 + 2\underset{\sim}{A} \cdot \underset{\sim}{C} \Leftarrow \underset{\sim}{B} = \underset{\sim}{A} + \underset{\sim}{C}$ (discarding this solution)

$|B|^2 = |A|^2 + |C|^2 - 2\underset{\sim}{A} \cdot \underset{\sim}{C} \Leftarrow \underset{\sim}{B} = \underset{\sim}{A} - \underset{\sim}{C} \Rightarrow \boxed{\underset{\sim}{A} = \underset{\sim}{B} + \underset{\sim}{C}}$ $\rightarrow$ demand (Already got this from separate Analysis)

$\downarrow$

$A^i = B^i + C^i$

Not $B^i = A^i + C^i$

$\Downarrow$

Hence system is consistent

'Summer Holiday HW'

10th RD {8 weeks} ~ 2H $\longrightarrow$ Deadline = 20th Aug.

* Algebra ✓ (2 weeks)
  - R.
  - $f(x) = x^2$
  - $x + y = 7$
  - $2x + 3y = 8$ ✓
  - $x^2 + 2x = 0$ ✓

* Trigonometry ✓ 1 week

* Series — AP $\rightarrow$ 1 week

* Geometry (3 week)
  - $\cdot (x, y)$ ✓
  - $\Delta_1 \sim \Delta_2$ ✓
  - $\bigcirc$, Areas ✓
  - Construction
  - 3D shapes

* $\boxed{\text{Prob. \& statistics}}$ — 1 week

All the ...
1. Examples
2. Exercises

Not Ch-1's

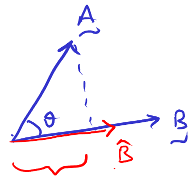
Deal: Pizza

Lecture-17 (13/Jun) 2

$\underset{\sim}{A} = \sum_i A^i \hat{e}_i$, $\hat{e}_i \cdot e_j = \delta_{ij}$, $\underset{\sim}{A} \cdot \hat{e}_j = A_j$, $\underset{\sim}{A} \cdot \underset{\sim}{B} = \sum_{ij} \delta_{ij} A^i B^j = |A||B|\cos\theta$

8. $A^i = (1 \;\; -2 \;\; 1)$, $B^j = (4 \;\; -4 \;\; 7)$, $\hat{e}_i = (\hat{\imath} \;\; \hat{\jmath} \;\; \hat{k})$

find $\underbrace{\text{projection}}_{\text{"shadows"}}$ of the vector $\underset{\sim}{A}$ on $\underset{\sim}{B}$ = ?



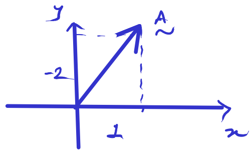
$\hat{B} = \frac{4}{9}\hat{\imath} - \frac{4}{9}\hat{\jmath} + \frac{7}{9}\hat{k}$ $\Rightarrow$ $\underset{\sim}{A} \cdot \hat{B} = \frac{19}{9}$

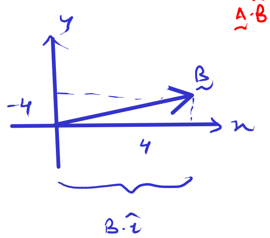
$\underset{\sim}{A} \cdot \hat{B} = \frac{19}{9}$

$\underset{\sim}{A} \cdot \hat{\jmath} = -2$

$\underset{\sim}{A} \cdot \hat{\imath}$

$\underset{\sim}{B} \cdot \hat{\imath} = 4$





* $\underset{\sim}{A} = \sum A^i \hat{e}_i$, $A^i = \underset{\sim}{A} \cdot \hat{e}^i$ $\Rightarrow$ $\boxed{\underset{\sim}{A} = \sum_i (\underset{\sim}{A} \cdot \hat{e}^i)\, \hat{e}_i}$ 'Form 2 of vector'

$\underset{\sim}{A} = A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k} = (\underset{\sim}{A}\cdot\hat{\imath})\hat{\imath} + (\underset{\sim}{A}\cdot\hat{\jmath})\hat{\jmath} + (\underset{\sim}{A}\cdot\hat{k})\hat{k}$

9. determine a unit vector $\perp$ to the plane of $\underset{\sim}{A} = 2\hat{\imath} - 6\hat{\jmath} - 3\hat{k}$ and $\underset{\sim}{B} = 4\hat{\imath} + 3\hat{\jmath} - \hat{k}$ (No need of cross product)

* $\exists\ \underset{\sim}{C} :\ C \perp \underset{\sim}{A}\ \&\ C \perp \underset{\sim}{B} \Rightarrow \underset{\sim}{A}\cdot\underset{\sim}{C} = 0\ ,\ \underset{\sim}{B}\cdot\underset{\sim}{C} = 0$ ✓

* $\underset{\sim}{C} = C_1\hat{\imath} + C_2\hat{\jmath} + C_3\hat{k}$, c^i unknowns : $\sum_i A^i c^i = 0 = \sum_i B^i c^i$

$\underset{\sim}{C}\cdot\underset{\sim}{A} = \sum_i c^i A^i = 2c_1 - 6c_2 - 3c_3 = 0 \Rightarrow 3c_3 = 2c_1 - 6c_2$ — ①

$\underset{\sim}{B}\cdot\underset{\sim}{C} = \sum_i B^i c^i = 4c_1 + 3c_2 - c_3 = 0 \Rightarrow c_3 = 4c_1 + 3c_2$ — ②

② in ① $\Rightarrow 12c_1 + 9c_2 = 2c_1 - 6c_2 \Rightarrow \overset{2}{\cancel{10}}c_1 = -\overset{3}{\cancel{15}}c_2 \Rightarrow c_1 = -\frac{3}{2}c_2$ — ③

2 eqⁿs , 3 variables $c_1\, c_2\, c_3$ $\Rightarrow$ one of 3 var is independent c_2

③ in ② $\Rightarrow c_3 = -6c_2 + 3c_2 = -3c_2 \Rightarrow c_3 = -3c_2$ — ④

* $\underset{\sim}{C} = c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k} = -\frac{3}{2}c_2\hat{\imath} + c_2\hat{\jmath} - 3c_2\hat{k} = c_2\left(-\frac{3}{2}\hat{\imath} + \hat{\jmath} - 3\hat{k}\right)$

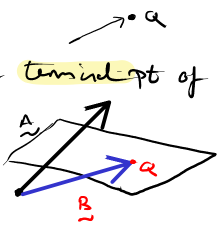
$\hat{\underset{\sim}{C}} = \frac{\underset{\sim}{C}}{|C|} = \frac{\cancel{c_2}\left(-\frac{3}{2}\hat{\imath} + \hat{\jmath} - 3\hat{k}\right)}{\sqrt{c_2^2\left(\frac{9}{4} + \frac{1}{1} + \frac{9}{1}\right)}} = \frac{\left(-\frac{3}{2}\hat{\imath} + \hat{\jmath} - 3\hat{k}\right)}{\sqrt{\frac{49}{4}}} = \frac{\left(-\frac{3}{2}\hat{\imath} + \hat{\jmath} - 3\hat{k}\right)}{\pm\left(\frac{7}{2}\right)} = \pm\left(-\frac{3}{7}\hat{\imath} + \frac{2}{7}\hat{\jmath} - \frac{6}{7}\hat{k}\right)$

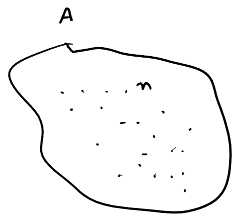
$\to \left(-\frac{3}{7} \;\; \frac{2}{7} \;\; -\frac{6}{7}\right)$

$\to \left(\frac{3}{7} \;\; -\frac{2}{7} \;\; \frac{6}{7}\right)$ ✓

Special Ques. for left

* Find an eqⁿ for the plane $\perp$ to the vector $\underset{\sim}{A} = 2\hat{\imath} + 3\hat{\jmath} + 6\hat{k}$ & passing thru the terminal pt of vector $\underset{\sim}{B} = \hat{\imath} + 5\hat{\jmath} + 3\hat{k}$.





$A = \{1, 2, 3\} \to n(A) = 3$

$P(A) = \{ \{\}, \{1\}\, \{2\}\, \{3\}, \{1,2\}\, \{2,3\}, \{1,3\}\, \{1,2,3\} \}$

$n(P(A)) = 8 = 2^3 = 2^{n(A)}$

this method of counting fails when n is large

↓

Combinatorics allows the counting (even easier!!)

$\binom{n}{r} = {}^nC_r = \frac{n!}{(n-r)!\,r!}$ out of n elements choose r elements (at a time)

total # of subsets $= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$

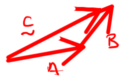
$\binom{n}{1}$: $\{1\}, \{2\}$, $\{3\}\ldots$, $\{n\}$; $\binom{n}{2}$: $\{1,2\}$

$n=8 \Rightarrow A = \{1, 2, 3 \ldots 8\}$ $n(A) = 8$

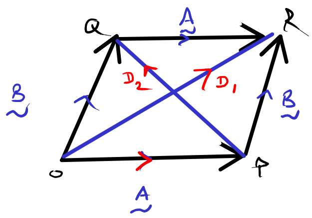
$r=2 \to \{1,2\}, \{3,8\}, \{8,7\}$, $\{8,1\}$ ⋮

$\# = \binom{8}{2} = \frac{8!}{6!\,2!} = \frac{\overset{4}{\cancel{8}} \times 7}{\cancel{2}} = 28$

Lecture-18 (14/3m) 2 | $\underset{\sim}{A}\cdot\hat{e}_j = A_j$ $\hat{e}_i\cdot\hat{e}_j = \delta_{ij}$, $\underset{\sim}{A}\cdot\underset{\sim}{B} = \sum_{ij} \delta_{ij} A^i B^j$, $\underset{\sim}{C} = \underset{\sim}{A} + \underset{\sim}{B}$



10. 2 adjacent sides of a llgm are $\underbrace{2\hat{i}+4\hat{j}-5\hat{k}}_{\underset{\sim}{A}}$ and $\underbrace{\hat{i}+2\hat{j}+3\hat{k}}_{\underset{\sim}{B}}$. find the unit vectors along the diagonals of the llgm.



* $\underset{\sim}{A}, \underset{\sim}{B}$
* parallel transport them
* get $\underbrace{\text{diags}}$. using Addition Rule

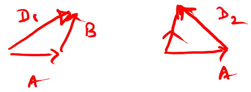
$\{\underset{\sim}{D}_1 \;\; \underset{\sim}{D}_2\}$

$\underset{\sim}{D}_1 = 3\hat{i}+6\hat{j}-2\hat{k}$

$\underset{\sim}{D}_2 = -\hat{i}-2\hat{j}+8\hat{k}$

$\hat{D}_1 = \dfrac{\underset{\sim}{D}_1}{7}$

$\hat{D}_2 = \dfrac{\underset{\sim}{D}_2}{\sqrt{69}}$



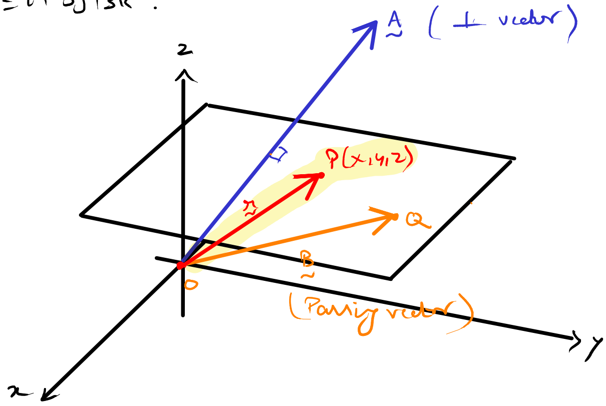
$\underset{\sim}{D}_1 = \underset{\sim}{A}+\underset{\sim}{B}$ $\underset{\sim}{A}+\underset{\sim}{D}_2 = \underset{\sim}{B}$

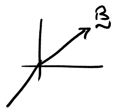
$\underset{\sim}{D}_2 = \underset{\sim}{B}-\underset{\sim}{A}$

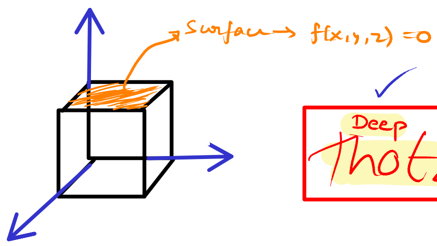
11. find a vector of Mag $\frac{5}{2}$ units $\parallel$ vector $3\hat{i}+4\hat{j}$

$\underset{\sim}{\text{Sandy}} = \frac{3}{2}\hat{i} + 2\hat{j}$

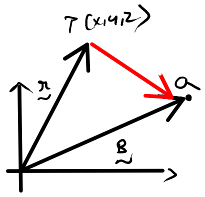
12. Find an eqn for the plane $\perp$ to the vector $\underset{\sim}{A} = 2\hat{i}+3\hat{j}+6\hat{k}$ & passing thru the terminal pt of vector $\underset{\sim}{B} = \hat{i}+5\hat{j}+3\hat{k}$.







Deep Thotz



$\underset{\sim}{r} + \underset{\sim}{PQ} = \underset{\sim}{B}$

* $\underset{\sim}{A} \perp$ Plane , $\underset{\sim}{B}$
* $\underset{\sim}{r}$: Position vector of point P (on the plane) ⟶ P(x,y,z)

$\underset{\sim}{PQ} \equiv \underset{\sim}{B} - \underset{\sim}{r}$ lies on the plane $\underset{\sim}{r} \equiv x\hat{i}+y\hat{j}+z\hat{k}$

* $\underset{\sim}{A} \perp$ Plane $\Rightarrow$ $\boxed{\underset{\sim}{A} \perp \underset{\sim}{PQ}}$ $\Rightarrow$ $\underset{\sim}{A}\cdot\underset{\sim}{PQ} = 0$ $\Rightarrow$ $\boxed{\underset{\sim}{A}\cdot(\underset{\sim}{B}-\underset{\sim}{r}) = 0}$

$\underbrace{\underset{\sim}{A}\cdot\underset{\sim}{B}}_{\text{Known}} = \underset{\sim}{A}\cdot\underset{\sim}{r} \Rightarrow \underbrace{\boxed{2x+3y+6z = 35}}$ Eqn of plane

$f(x) = 0$ $\begin{cases} x+2=0 \\ 2x^2+x+1=0 \\ x^3+2x+1=0 \\ \sum_{i=1}^{n} a_i x^i = 0 \end{cases}$

⇓

Points

$f(x,y) = 0$ $\begin{cases} x+y+2=0 \\ x^2+y+3=0 \\ x^2+y^2=R^2 \end{cases}$

⇓

(lines) Curve

$f(x,y,z) = 0$ $\begin{cases} x+y+z=0 \\ x^2+y^3+z=0 \\ x^2+y^2+z^2=R^2 \end{cases}$

⇓

(Planes) Surface   Sphere

* $\boxed{f(x,y,z) = 2x+3y+6z}$

↑

"Scalar field/function"

Multivariable function

⟶ $f(\overbrace{1,2,4}) = 2+6+24 = 32$ field value

⟶ $f(-1,1,3) = -2+1+18 = 17$

⟶ $f(0,0,0) = 0$

⟶ $f(x,y,z) = 2x+3y+6z = 35$ ⟶ •(x y z)

Scalar field

vector field

tensor fields

Ex: $T(x,y,z) = 2x+3y+6z$ temp field

temp. field inside a Room

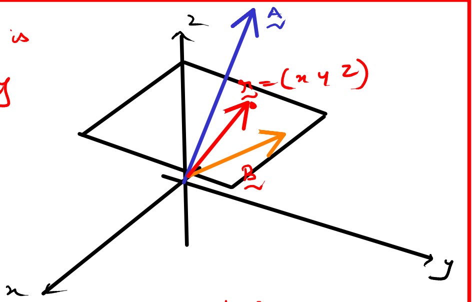
$$\boxed{A \cdot (B - r) = 0}$$

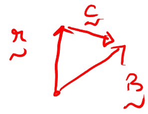
Eq^n of a plane when A is $\Rightarrow$ $\perp$ to it & B is passing thru it

$\perp$ to the plane

Passing thru the plane

arbitrary Position vector of a point on Plane





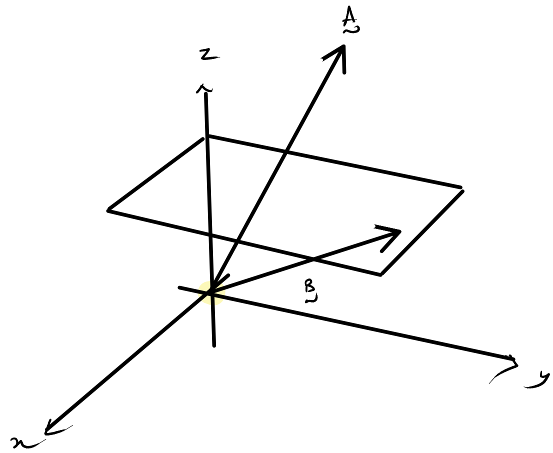
logic

$C = B - r$

$A \perp C \Rightarrow A \cdot C = 0$

HW

13. Distance from the origin to the plane = ?



lecture-19 (15/10/xx) 2

14. $|A| = \sqrt{3}$, $|B| = 2$: $A \cdot B = \sqrt{6}$ $\theta = ?$

15. if $(1\ 1\ 1)$, $(2\ 5\ 1)$, $(3\ 2\ -3)$, $(1\ -6\ -1)$ resp. the pos. vectors of pt. A, B, C, D .

Angle b/w st-line AB & CD = ?

$\cos\theta = -1 \Rightarrow \theta = \pi$

16. $A^i = (4\ 5\ -1)$ $B^i = (1\ -4\ 5)$ $C^i = (3\ 1\ -1)$

$D^i = ?$: $\sum_i D^i A_i = 0$, $\sum_i D^i B_i = 0$, $\sum_i D^i C_i = 21$

$D^i = (7\ -7\ -7)$

17. A , B : $|A| = 2$, $|B| = 1$, $\sum_i A^i B_i = 1$; $\underbrace{(3A - 5B)\cdot(2A + 7B)}$ = ?

$6|A|^2 + \underbrace{21 A\cdot B - 10 B\cdot A}_{11\, A\cdot B \cancel{=1}} - 35|B|^2 = 6(2)^2 + 11 - 35$

$= 24 + 11 - 35 = 0$

Level 2

18. given A, B $\longrightarrow$ $\underbrace{(|A| B + |B| A)}_{\text{Linear Comb}^n\ 1}$, $\underbrace{(|A| B - |B| A)}_{\text{L. Comb}^n\ 2}$; relationships b/w the L.C.1 & L.C 2 ?

* $A + B = C$

$A - B = D$ dependent vectors $\rightarrow$ $f(C, D) = 0$

* $C = |A| B + |B| A$, $D = |A| B - |B| A$

operations

* $C + D = 2|A| B \Rightarrow B = \dfrac{C + D}{2|A|}$
* $C - D = 2|B| A \Rightarrow A =$

* $\boxed{C \cdot D = 0} \Rightarrow$ $C \perp D$

Linear Comb^n s of 2 vectors are orthogonal $(\alpha A \pm \beta B)$

r Imp. in Complex vector spaces $\downarrow$ Hilbert space $\rightarrow$ lang. for QM

19. $\underbrace{A + B + C = 0}$, $|A| = 3$, $|B| = 5$, $|C| = 7$, $\underbrace{\theta \text{ b/w } A \ \& \ B = ?}$

* $A + B = -C \Rightarrow |C|^2 = |A+B|^2 \Rightarrow A \cdot B = \dfrac{|C|^2 - |A|^2 - |B|^2}{2}$

$\underbrace{A \cdot B} = |A||B| \cos\theta \Rightarrow \cos\theta = \dfrac{|C|^2 - |A|^2 - |B|^2}{2|A||B|}$

20. $\{\hat{A}, \hat{B}\}$ unit vectors at an angle $\theta \longrightarrow \sin\dfrac{\theta}{2} = f(|\hat{A} - \hat{B}|) = ?$

* $|\hat{A} - \hat{B}|^2 = |\hat{A}|^2 + |\hat{B}|^2 - 2\underbrace{\hat{A} \cdot \hat{B}}_{|\hat{A}||\hat{B}|\cos\theta} = 2\underbrace{(1 - \cos\theta)}_{2\sin^2\frac{\theta}{2}} = 4\sin^2\dfrac{\theta}{2} \Rightarrow \boxed{\sin\dfrac{\theta}{2} = \dfrac{1}{2}|\hat{A} - \hat{B}|}$

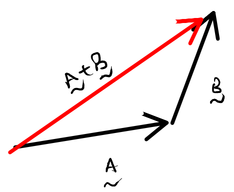
$\dfrac{1}{2}\left\{\sqrt{|\hat{A}|^2 + |\hat{B}|^2 - 2\hat{A} \cdot \hat{B}}\right\}$

$\dfrac{1}{2}\sqrt{2(1 - \hat{A} \cdot \hat{B})} = \sqrt{\dfrac{1 - \hat{A} \cdot \hat{B}}{2}}$

$\cos 2\theta = 2\cos^2\theta - 1$
$= 1 - 2\sin^2\theta$
$\Downarrow$
$\sin\theta = \sqrt{\dfrac{1 - \cos 2\theta}{2}}$

21. $\boxed{A, B : \underbrace{|A + B|}_{\text{whole}} \le \underbrace{|A| + |B|}_{\text{Part}}}$

Triangle Inequality



* $|A + B|^2 = |A|^2 + |B|^2 + 2\underbrace{A \cdot B}_{|A||B|\cos\theta} = |A|^2 + |B|^2 + 2|A||B|\cos\theta$

$\underbrace{|A + B|^2}_{\text{vector}} \neq \underbrace{(|A| + |B|)^2}_{\#}$

* $\boxed{-1 \le \cos\theta \le 1} \Rightarrow |\cos\theta| \le 1 \quad \forall \theta$

$5 < 7$
$5 + 2 < 7 + 2$
$5 - 2 < 7 - 2$
$-5 > -7$

$2|A||B|\cos\theta \le 2|A||B| \Rightarrow \underbrace{|A|^2 + |B|^2 + 2|A||B|\cos\theta}_{|A+B|^2} \le \underbrace{|A|^2 + |B|^2 + 2|A||B|}_{(|A| + |B|)^2}$

* $|A + B|^2 \le (|A| + |B|)^2 \Rightarrow |A + B| \le |A| + |B|$.

Q.E.D
quod erat demonstrandum

Lecture-20 (17 June) 3

22. $\underset{\sim}{A}, \underset{\sim}{B}$; $\underset{\sim}{A}\cdot\underset{\sim}{B} = \sum_{ij} \delta_{ij} A^i B^j \longrightarrow (\underset{\sim}{A}\cdot\underset{\sim}{B})^2 = (\underset{\sim}{A}\cdot\underset{\sim}{B})\overset{\text{Numbr multip.}}{\downarrow}(\underset{\sim}{A}\cdot\underset{\sim}{B}) = \left(\sum_i\sum_j \delta_{ij} A^i B^j\right)\left(\sum_m\sum_n \delta_{mn} A^m B^n\right)$

$$= \sum_i\sum_j\sum_m\sum_n \underbrace{\delta_{ij}}_{i=j} \delta_{mn} A^i B^j A^m B^n$$

$$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

$$\underset{\sim}{A}\cdot\underset{\sim}{B} = \sum_{ij} \underbrace{\hat{e}_i\cdot\hat{e}_j}_{=g_{ij}} A^i B^j$$

$$= \sum_i\sum_m\sum_n \underbrace{\delta_{ii}}\underbrace{\delta_{mn}}_{m=n} A^i B^i A^m B^n = \underbrace{\sum_i\sum_m A^i B_i A^m B_m} = \left(\sum_i A^i B_i\right)\left(\sum_m A^m B_m\right)$$

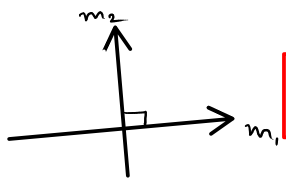
$$|\underset{\sim}{A}|^2|\underset{\sim}{B}|^2 = (\underset{\sim}{A}\cdot\underset{\sim}{A})(\underset{\sim}{B}\cdot\underset{\sim}{B}) = \left(\sum_i A^i A_i\right)\left(\sum_m B^m B_m\right) = \underbrace{\sum_i\sum_m A^i B_m A_i B^m}$$

Look at index matching.

* $\underbrace{(\underset{\sim}{A}\cdot\underset{\sim}{B})^2 \neq |\underset{\sim}{A}|^2|\underset{\sim}{B}|^2}$ 'Cauchy-Schwartz inequality'

$\downarrow$ $\boxed{HW}$

$>$ $<$?



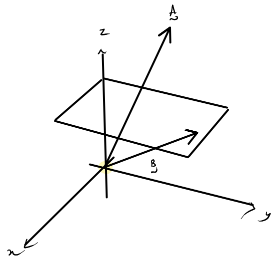
HW

$\boxed{m_1 = \frac{-1}{m_2}}$ ✓ ⊥ Formula in Trig Proof (later)

? $m_1 \neq \frac{1}{m_2}$, $m_1 \neq -m_2$?

(Perpendicularity Condn)

HW 13. Distance from the origin to the plane = ?



23. $\hat{A}, \hat{B}$ find the θ b/w $\underset{\sim}{\hat{A}}, \underset{\sim}{\hat{B}}$: L.C. of $\hat{A}, \hat{B} = \sqrt{3}\hat{A} - \hat{B}$ is a unit vector

* $\hat{C} \equiv \sqrt{3}\hat{A} - \hat{B} \rightarrow |\hat{C}|^2 = 1 \Rightarrow |\sqrt{3}\hat{A} - \hat{B}|^2 = 3 + 1 - 2\sqrt{3}\hat{A}\cdot\hat{B} = 1 \Rightarrow \underbrace{\frac{\sqrt{3}}{2}}_{\cos\frac{\pi}{6}} = \hat{A}\cdot\hat{B} = \cos\theta \Rightarrow \boxed{\theta = \frac{\pi}{6}}$

24. $\underset{\sim}{\hat{A}}$ unit vector $\longrightarrow$ θ with $\hat{i} = \frac{\pi}{4}$; $\hat{A}^i = ?$ Comp. of $\underset{\sim}{\hat{A}}$

θ " $\hat{j} = \frac{\pi}{3}$ $\qquad$ θ with $\hat{k} = ?$

θ " $\hat{k} \in$ Acute Angle

* $\underset{\sim}{\hat{A}} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$: $\underset{\sim}{A}\cdot\hat{e}_j = A_i$

$A_1 = \underset{\sim}{\hat{A}}\cdot\hat{i} = \cos\frac{\pi}{4}$

$A_2 = \underset{\sim}{\hat{A}}\cdot\hat{j} = \cos\frac{\pi}{3}$

$A_3 = \underset{\sim}{\hat{A}}\cdot\hat{k} = \underbrace{\cos\theta}_{\text{D.C}}$

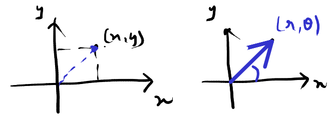
$\Rightarrow |\underset{\sim}{\hat{A}}|^2 = 1 = A_1^2 + A_2^2 + A_3^2 = \cos^2\frac{\pi}{4} + \cos^2\frac{\pi}{3} + \underbrace{\cos^2\theta}$

25. $\underset{\sim}{\hat{A}} = ?$: $\underset{\sim}{\hat{A}} \perp \overbrace{(\hat{i} + 2\hat{j} - \hat{k})}^{\underset{\sim}{B}}$

$\underset{\sim}{\hat{A}} \perp \underbrace{(3\hat{i} - \hat{j} + 2\hat{k})}_{\underset{\sim}{C}}$

* $\underset{\sim}{A} = x\hat{i} + y\hat{j} + z\hat{k}$, $|\underset{\sim}{A}| = x^2 + y^2 + z^2$

Remark on 'circle' in Cartesian geometry



(x, y) $\qquad$ (r, θ)

geometry $\boxed{\text{invariant} = s^2 = x^2 + y^2}$ $\qquad$ const $\qquad$ var.

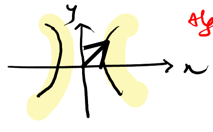
Eqn of circle $\qquad$ $g_{ab} = \delta_{ab}$ Algebra $\qquad$ $x = r\cos\theta$

$\boxed{\sin^2\theta + \cos^2\theta = 1}$ $\qquad$ $y = r\sin\theta$ $\downarrow$ Trigono.

geometry $\boxed{s^2 = x^2 - y^2 \Rightarrow \frac{x^2}{s^2} - \frac{y^2}{s^2} = 1}$ Eqn of Rectangular Hyperbola $a=b$

invariant

$g_{ab} = \eta_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ Algebra



$x = a\sec\theta$

$y = a\tan\theta$

Trigonometry

* $\underset{\sim}{A}\cdot\underset{\sim}{B}=0 \Rightarrow x+2y-z=0$
$\underset{\sim}{A}\cdot\underset{\sim}{C}=0 \Rightarrow 3x-y+2z=0$ } 2 Eqⁿ 3 var

* $$\frac{x}{\begin{vmatrix}2 & -1\\ -1 & 2\end{vmatrix}}=\frac{-y}{\begin{vmatrix}1 & -1\\ 3 & 2\end{vmatrix}}=\frac{z}{\begin{vmatrix}1 & 2\\ 3 & -1\end{vmatrix}} \Rightarrow \frac{x}{3}=\frac{-y}{5}=\frac{z}{-7}$$

Cramer's rule (french)

Proportionality ← trick

* $\frac{x}{3}=\frac{-y}{5}=\frac{-z}{7}\equiv\lambda \rightarrow x=3\lambda,\ y=-5\lambda,\ z=-7\lambda$

* $|A|^2=x^2+y^2+z^2$

$$\hat{\underset{\sim}{A}}=\frac{\underset{\sim}{A}}{|A|}=\frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}=\frac{3\lambda\hat{i}-5\lambda\hat{j}-7\lambda\hat{k}}{\sqrt{9\lambda^2+25\lambda^2+49\lambda^2}}=\frac{3\lambda\hat{i}-5\lambda\hat{j}-7\lambda\hat{k}}{\sqrt{83\lambda^2}}$$

Normalization of a vector ≡ unit vector
↓ QM → Complex vectors
Normalization of a "complex vector"
ψ

* $$\hat{\underset{\sim}{A}}=\pm\left(\frac{3}{\sqrt{83}}\hat{i}-\frac{5}{\sqrt{83}}\hat{j}-\frac{7}{\sqrt{83}}\hat{k}\right)$$
→ $\frac{3}{\sqrt{83}}\hat{i}-\frac{5}{\sqrt{83}}\hat{j}-\frac{7}{\sqrt{83}}\hat{k}$
→ $-\frac{3}{\sqrt{83}}\hat{i}+\frac{5}{\sqrt{83}}\hat{j}+\frac{7}{\sqrt{83}}\hat{k}$

$x^2-y^2=a^2(\sec^2\theta-\tan^2\theta)=a^2$

$a^2=x^2-y^2$

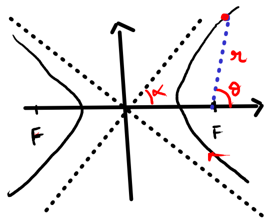
$x=a\cosh\theta',\ y=a\sinh\theta'$

$\cosh^2\theta-\sinh^2\theta=1$ ✓

'Trigonometry for hyperbolas'
↓
∃ Correspondence b/w T-fun & hyp. funⁿ

$\in\mathbb{C}$ ← $i\theta$ → $\in[-2\pi,2\pi]$
e
$\in\mathbb{R}$

Algebra vol. 2
Classical th. of Rots
Trig. vol. 1
last lectures



asymptotes

$\alpha \longrightarrow \alpha=45^\circ \Rightarrow$ Rectangular Hyp.

Spherical coordinates $r\ \theta\ \varphi$

Least Action → g_{ab} ⇓ EOM

$$g_{ab}=\begin{pmatrix}\left(1-\frac{r_s}{r}\right) & 0 & 0 & 0\\ 0 & \frac{-1}{\left(1-\frac{r_s}{r}\right)} & 0 & 0\\ 0 & 0 & -r^2 & 0\\ 0 & 0 & 0 & -r^2\sin^2\theta\end{pmatrix}$$

S.C metric variable

$$g_{ab}=\eta_{ab}=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix},\ \delta_{ab}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}$$

Minkowski (Const Entries)   Kronecker

26. $\underset{\sim}{A}=\hat{i}+\hat{j}+\hat{k}$. The scalar pdt of $\underset{\sim}{A}$ with a unit vector along the sum of the vectors $\underset{\sim}{B}=2\hat{i}+4\hat{j}-5\hat{k}$ & $\underset{\sim}{C}=\lambda\hat{i}+2\hat{j}+3\hat{k}$ is 1. $\lambda=?$
unit vector along $\underset{\sim}{B}+\underset{\sim}{C}=?$

* $\underset{\sim}{D}=\underset{\sim}{B}+\underset{\sim}{C}$

* $$\hat{\underset{\sim}{D}}=\frac{\underset{\sim}{D}}{|\underset{\sim}{D}|}=\frac{\underset{\sim}{B}+\underset{\sim}{C}}{|\underset{\sim}{B}+\underset{\sim}{C}|}=\frac{1}{7}(3\hat{i}+6\hat{j}-2\hat{k})$$

* $\underset{\sim}{A}\cdot\hat{\underset{\sim}{D}}=1$

Lecture-21 (18/Jun) 2

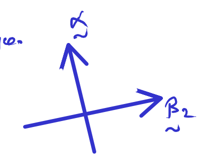
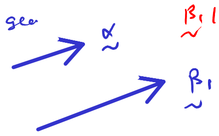
27. $\hat{e}_i$ = Euclidean/std. basis unit vectors , $\hat{e}_i \cdot \hat{e}_j = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$ Normalization orthogonal } orthonormal vectors ' JEE/MIT/ Boards/ NCERT ,

$\underset{\sim}{\alpha} = \sum_i \alpha^i \hat{e}_i$, $\alpha^i = (3 \;\; -1)$ $\Rightarrow$ $\underset{\sim}{\alpha} = 3\hat{i} - \hat{j}$

$\underset{\sim}{\beta} = \sum_i \beta^i \hat{e}_i$, $\beta^i = (2 \;\; 1 \;\; -3)$ $\Rightarrow$ $\underset{\sim}{\beta} = 2\hat{i} + \hat{j} - 3\hat{k}$

Express $\underset{\sim}{\beta} = \underset{\sim}{\beta_1} + \underset{\sim}{\beta_2}$: $\underbrace{\underset{\sim}{\beta_1} \parallel \underset{\sim}{\alpha} \;,\; \underset{\sim}{\beta_2} \perp \underset{\sim}{\alpha}}_{\text{Constraint Eq}^n}$ 'Decomposition of a vector into $\overbrace{\text{Linear Comb}^n}^{\text{superposition}}$ of 2 diff. vectors'

* $\underset{\sim}{\beta} = \underset{\sim}{\beta_1} + \underset{\sim}{\beta_2} \Rightarrow \underset{\sim}{\alpha} \cdot \underset{\sim}{\beta} = \underset{\sim}{\alpha} \cdot \underset{\sim}{\beta_1} + \cancel{\underset{\sim}{\alpha} \cdot \underset{\sim}{\beta_2}}^{\,0} \Rightarrow \underbrace{\underset{\sim}{\alpha} \cdot \underset{\sim}{\beta_1}}_{|\alpha||\beta_1|\cos\theta} = \underbrace{\underset{\sim}{\alpha} \cdot \underset{\sim}{\beta}}_{\text{known}}$ ✓

geo.  geo  $\underset{\sim}{\beta_1} \parallel \underset{\sim}{\alpha}$

Alge. $\underset{\sim}{\beta_2} \cdot \underset{\sim}{\alpha} = 0$ Alge. $\underset{\sim}{\beta_1} = \overset{\text{any scalar}}{\lambda}\, \underset{\sim}{\alpha}$

New Approach

* $\boxed{\underset{\sim}{A} \parallel \underset{\sim}{B} \Rightarrow \underset{\sim}{A} \propto \underset{\sim}{B} \Rightarrow \underset{\sim}{A} = \lambda \underset{\sim}{B}}$ $\not\Rightarrow$ $\overset{\text{scalar}}{\lambda} = \dfrac{\underset{\sim}{A}}{\underset{\sim}{B}}$ "division of vectors" → Comment later

$\Downarrow$

* $\underset{\sim}{\beta_1} \parallel \underset{\sim}{\alpha} \Rightarrow \underset{\sim}{\beta_1} = \overset{\text{unknown}}{\lambda}\, \underset{\sim}{\alpha}$ → Eq^n to solve for $\underset{\sim}{\beta_1}$

* $\underset{\sim}{\beta} = \underset{\sim}{\beta_1} + \underset{\sim}{\beta_2} \Rightarrow \underset{\sim}{\beta_2} = \underset{\sim}{\beta} - \underset{\sim}{\beta_1} = \overset{\text{known}}{\underset{\sim}{\beta} - \lambda \underset{\sim}{\alpha}}$ → Eq^n to solve for $\underset{\sim}{\beta_2}$

* $\underset{\sim}{\beta_2} \perp \underset{\sim}{\alpha} \Rightarrow \underset{\sim}{\beta_2} \cdot \underset{\sim}{\alpha} = 0 \Rightarrow (\underset{\sim}{\beta} - \lambda \underset{\sim}{\alpha}) \cdot \underset{\sim}{\alpha} = 0 \Rightarrow \underset{\sim}{\alpha} \cdot \underset{\sim}{\beta} - \lambda\, \underset{\sim}{\alpha} \cdot \underset{\sim}{\alpha} \Rightarrow \boxed{\overset{\text{scalar}}{\lambda} = \underbrace{\dfrac{\underset{\sim}{\alpha} \cdot \underset{\sim}{\beta}}{\underset{\sim}{\alpha} \cdot \underset{\sim}{\alpha}}}_{\text{known}}} = \dfrac{1}{2}$

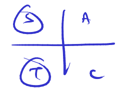
* $\underset{\sim}{\beta_1} = \lambda \underset{\sim}{\alpha} = \left(\dfrac{\underset{\sim}{\alpha} \cdot \underset{\sim}{\beta}}{\underset{\sim}{\alpha} \cdot \underset{\sim}{\alpha}}\right) \underset{\sim}{\alpha}$

$\underset{\sim}{\beta_2} = \underset{\sim}{\beta} - \lambda \underset{\sim}{\alpha} = \underset{\sim}{\beta} - \left(\dfrac{\underset{\sim}{\alpha} \cdot \underset{\sim}{\beta}}{\underset{\sim}{\alpha} \cdot \underset{\sim}{\alpha}}\right) \underset{\sim}{\alpha}$ } $\Rightarrow \underbrace{\underset{\sim}{\beta_1} + \underset{\sim}{\beta_2}}_{\text{Decomposition}} = \underset{\sim}{\beta}$

$\boxed{\text{= thats no bad !}}$

* $\underset{\sim}{\beta_1} = \frac{3}{2}\hat{i} - \frac{1}{2}\hat{j}$, $\underset{\sim}{\beta_2} = \frac{1}{2}\hat{i} + \frac{3}{2}\hat{j} - 3\hat{k}$

28. $A^i = (2x^2 \;\; 4x \;\; 1)$ $B^i = (7 \;\; -2 \;\; x)$ find $\underset{?}{x}$: θ b/w $\underset{\sim}{A}$ & $\underset{\sim}{B}$ is $\underbrace{\text{obtuse}}_{\theta > 90^\circ}$ ' EZ MIT '



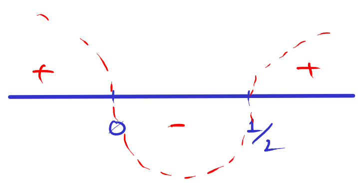
* $\underset{\sim}{A} \cdot \underset{\sim}{B} = \sum_{ij} \delta_{ij} A^i B^j = \sum A^i B_i = |\underset{\sim}{A}||\underset{\sim}{B}| \cos\theta \Rightarrow \cos\theta = \dfrac{\underset{\sim}{A} \cdot \underset{\sim}{B}}{|A||B|}$

$\theta > 90^\circ \Rightarrow \cos\theta < 0 \Rightarrow \dfrac{\underset{\sim}{A} \cdot \underset{\sim}{B}}{|\underset{\sim}{A}||\underset{\sim}{B}|} < 0$ Vector $\underbrace{\text{Inequalities}}$

$\underset{\text{+ve}}{|A|, |B| > 0} \Rightarrow \boxed{\underset{\sim}{A} \cdot \underset{\sim}{B} < 0} \Rightarrow \overbrace{14x^2 - 8x + x < 0}^{\text{Algebraic Inequalities}} \Rightarrow \underbrace{x(2x-1)}_{f(x)} < 0$

Wavy curve Method for Inequality Analysis

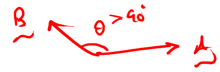
* $f(x) = x(2x-1) = 0 \Rightarrow$ Critical points : $x_1 = 0$, $x_2 = 1/2$



$f(x > \frac{1}{2}) = 2(4-1) > 0$

$f(0 < x < \frac{1}{2}) = \frac{1}{4}\left(\frac{1}{2}-1\right) < 0$ $\qquad \boxed{f(x) < 0} \Rightarrow 0 < x < \frac{1}{2} \rightarrow \infty$ many comb^n of $\underset{\sim}{A}, \underset{\sim}{B}$

$f(x < 0) = -1(-2-1) > 0$



Lecture-22 (19/6) 2

29. $\{\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}\}$ vectors, $\{l, m, n\}$ Scalars

* $|l\underset{\sim}{A} + m\underset{\sim}{B} + n\underset{\sim}{C}|^2 = \begin{cases} ? & \text{In general} \\ ? & \{\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}\} \text{ Mutually } \perp \end{cases}$

* $\underset{\sim}{M} \equiv l\underset{\sim}{A} + m\underset{\sim}{B} + n\underset{\sim}{C} \Rightarrow |\underset{\sim}{M}|^2 = \underset{\sim}{M}\cdot\underset{\sim}{M} = (l\underset{\sim}{A} + m\underset{\sim}{B} + n\underset{\sim}{C})\cdot(l\underset{\sim}{A} + m\underset{\sim}{B} + n\underset{\sim}{C})$

$$\boxed{|l\underset{\sim}{A} + m\underset{\sim}{B} + n\underset{\sim}{C}|^2 = l^2|\underset{\sim}{A}|^2 + m^2|\underset{\sim}{B}|^2 + n^2|\underset{\sim}{C}|^2 + 2\{lm\,\underset{\sim}{A}\cdot\underset{\sim}{B} + mn\,\underset{\sim}{B}\cdot\underset{\sim}{C} + ln\,\underset{\sim}{A}\cdot\underset{\sim}{C}\}}$$

$$\Big\downarrow \; \underset{\sim}{A} \perp \underset{\sim}{B} \perp \underset{\sim}{C}$$

$$|l\underset{\sim}{A} + m\underset{\sim}{B} + n\underset{\sim}{C}|^2 = l^2|\underset{\sim}{A}|^2 + m^2|\underset{\sim}{B}|^2 + n^2|\underset{\sim}{C}|^2$$

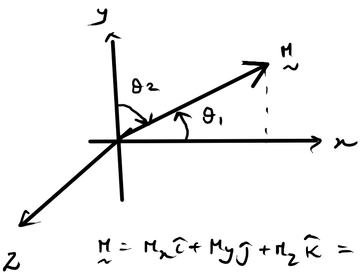
30. $\{\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}\}$ Mutually $\perp$ of Equal magnitude

JEE / MIT / Board

P.T : L.C $\underset{\sim}{A} + \underset{\sim}{B} + \underset{\sim}{C}$ is equally inclined with $\underset{\sim}{A}, \underset{\sim}{B}$ & $\underset{\sim}{C}$ $\rightarrow$ Calc. Direction Cosine

find the Angle ?

*



$\underset{\sim}{M} = M_x\hat{i} + M_y\hat{j} + M_z\hat{k} = \sum_i M^i \hat{e}_i \Rightarrow M^i = \underset{\sim}{M}\cdot\hat{e}_i = \left(\sum_j M^j \hat{e}_j\right)\cdot\hat{e}_i$

$M_x = |\underset{\sim}{M}| \cos\theta_1$

$M_y = |\underset{\sim}{M}| \cos\theta_2$

$$\left.\begin{aligned} \cos\theta_1 &= \frac{M_x}{|\underset{\sim}{M}|} = \frac{\underset{\sim}{M}\cdot\hat{i}}{|\underset{\sim}{M}|} \\ \cos\theta_2 &= \frac{M_y}{|\underset{\sim}{M}|} = \frac{\underset{\sim}{M}\cdot\hat{j}}{|\underset{\sim}{M}|} \end{aligned}\right\} \rightarrow \boxed{\cos\theta_i = \frac{M_i}{\sum_{ab}\delta_{ab}M^a M^b} = \frac{\underset{\sim}{M}\cdot\hat{e}_i}{\sum \delta_{ab}M^a M^b} = \frac{\left(\sum_j M^j e_j\right)\cdot\hat{e}_i}{\sum_{ab}\delta_{ab}M^a M^b}}$$

* $\underset{\sim}{M} = \underset{\sim}{A} + \underset{\sim}{B} + \underset{\sim}{C}$: $\underset{\sim}{A}\cdot\underset{\sim}{B} = \underset{\sim}{B}\cdot\underset{\sim}{C} = \underset{\sim}{C}\cdot\underset{\sim}{A} = 0$, $|\underset{\sim}{A}| = |\underset{\sim}{B}| = |\underset{\sim}{C}| = k$

* $|\underset{\sim}{M}|^2 = |\underset{\sim}{A} + \underset{\sim}{B} + \underset{\sim}{C}|^2 = |\underset{\sim}{A}|^2 + |\underset{\sim}{B}|^2 + |\underset{\sim}{C}|^2 + 2(\cancel{\underset{\sim}{A}\cdot\underset{\sim}{B} + \underset{\sim}{B}\cdot\underset{\sim}{C} + \underset{\sim}{C}\cdot\underset{\sim}{A}})^{0} = 3k^2 \Rightarrow |\underset{\sim}{A} + \underset{\sim}{B} + \underset{\sim}{C}| = k\sqrt{3}$

$\cos\theta_1 = \frac{\underset{\sim}{M}\cdot\underset{\sim}{A}}{|\underset{\sim}{M}||\underset{\sim}{A}|} = \frac{1}{\sqrt{3}} = \cos\theta_2 = \cos\theta_3$

31. $\{\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}\}$ $\longrightarrow$ Mag. 3, 4, 5 resp.

if Each is $\perp$ to sum of other two.

find Mag of L.C $\underset{\sim}{A}+\underset{\sim}{B}+\underset{\sim}{C}$?

* $|\underset{\sim}{A}|=3$, $|\underset{\sim}{B}|=4$, $|\underset{\sim}{C}|=5$ $\Rightarrow$ $|\underset{\sim}{A}+\underset{\sim}{B}+\underset{\sim}{C}|^2 = |\underset{\sim}{A}|^2+|\underset{\sim}{B}|^2+|\underset{\sim}{C}|^2 + 2\underbrace{(\underset{\sim}{A}\cdot\underset{\sim}{B}+\underset{\sim}{B}\cdot\underset{\sim}{C}+\underset{\sim}{C}\cdot\underset{\sim}{A})} = 50$

$$\underbrace{\underset{\sim}{A}\cdot\underset{\sim}{B}+\underset{\sim}{B}\cdot\underset{\sim}{C}}_{(\underset{\sim}{A}+\underset{\sim}{C})\cdot\underset{\sim}{B}\,=\,0}+\underbrace{\underset{\sim}{C}\cdot\underset{\sim}{A}+\underset{\sim}{A}\cdot\underset{\sim}{B}}_{(\underset{\sim}{B}+\underset{\sim}{C})\cdot\underset{\sim}{A}\,=\,0}+\underbrace{\underset{\sim}{B}\cdot\underset{\sim}{C}+\underset{\sim}{C}\cdot\underset{\sim}{A}}_{(\underset{\sim}{B}+\underset{\sim}{A})\cdot\underset{\sim}{C}\,=\,0}$$

new

32. $\underset{\sim}{A} = c\log_2 x\,\hat{i} - 6\hat{j} + 3\hat{k}$, $\underset{\sim}{B} = (\log_2 x)\hat{i} + 2\hat{j} + (2c\log_2 x)\hat{k}$

$c = ?$: Angle Made by $\underset{\sim}{A}$ & $\underset{\sim}{B}$ is obtuse $\forall\ x \in (0,\infty)$

sum.
vectors
Matrices
algebraic
log.
$\mathbb{R}$
$\mathbb{C}$

* $\theta > 90^\circ \Rightarrow \cos\theta < 0 \Rightarrow \dfrac{\underset{\sim}{A}\cdot\underset{\sim}{B}}{|\underset{\sim}{A}||\underset{\sim}{B}|} < 0 \Rightarrow \boxed{\underset{\sim}{A}\cdot\underset{\sim}{B} < 0}$

'Alg. vol. 2 lec-5/6,7

Quad. Inequality Prob. 1

* $c(\log x)^2 - 12 + 6c\log x < 0$ log. Inequation $\Rightarrow$ $\boxed{cy^2+6cy-12<0}$ in y

* $y = \log_2 x \Leftrightarrow 2^y = x$ Prob. 2 $\Rightarrow x \in (0,\infty) \Rightarrow y = \log x > 0$ $\because y \in \mathbb{R}$

To handle "big" powers you create a Machine called logarithm.

Quad. Eqn
Alg. vol 1

Prob. 1

* $cy^2+6cy-12<0 \Rightarrow (y-\alpha)(y-\beta)<0$

$y = f(x) = ax^2+bx+C$

$x = \dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ α, β

$\alpha,\beta = \dfrac{-6c\pm\sqrt{36c^2+48c}}{2c} = \dfrac{-6c\pm2\sqrt{9c^2+12c}}{2c} = \dfrac{-3c\pm\sqrt{9c^2+12c}}{c}$

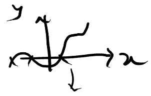
$D = 36c^2+48c$

⋮

To be cont.

Lecture-23 (20/Jun) 2

$y(x)=\sum_i a_i x^i$ Machine ; $y=f(x)=0 \Rightarrow$ Becoming-0 of a Poly. Machine $=$ cuts at x axis



switching off

* $\overbrace{ax^2+bx+c}^{y(x)}=0 \Rightarrow x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-b\pm\sqrt{D}}{2a}$ $\to x_1=\alpha$, $\to x_2=\beta$, $D\equiv b^2-4ac$

$\underbrace{ax^2+bx+c}_{(x-\alpha)(x-\beta)}$

* $\alpha+\beta=\frac{-b}{a}$, $\alpha\beta=\frac{c}{a}$, $y=ax^2+bx+c$ geometry : Parabola $\cup$ $\cap$ $x\in\mathbb{R}$

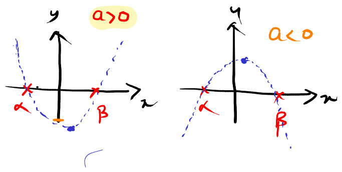
$D\equiv b^2-4ac$

$D>0$

$x_1=\alpha=\frac{-b+\sqrt{D}}{2a}$

$x_2=\beta=\frac{-b-\sqrt{D}}{2a}$

cuts $=2$ $\cup$ $\cap$



$x=\frac{\alpha+\beta}{2}=\frac{-b}{2a}$ → Mid val ⇐ Symmetry of parabola

$y=a\left(\frac{-b}{2a}\right)^2+b\left(\frac{-b}{2a}\right)+c=\frac{-D}{4a}$ Min. val of y

$\boxed{D>0}$

$a>0 \to y<0$

$a<0 \to y>0$

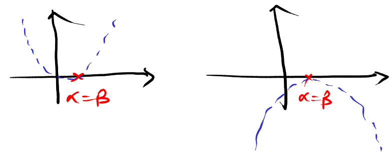
$\boxed{\text{vertex } \left(\frac{-b}{2a},\frac{-D}{4a}\right)}$

Statement about Parabolas

$D=0$

$x_{1,2}=\frac{-b}{2a}=\alpha=\beta$

cuts $=1$

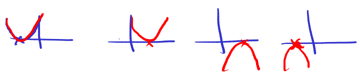


$x=\frac{\alpha+\beta}{2}=\alpha=\beta=\frac{-b}{2a}$

$y=0$ cuts at x axis $\to \alpha,\beta$

$\boxed{x=\frac{-b}{2a}}$

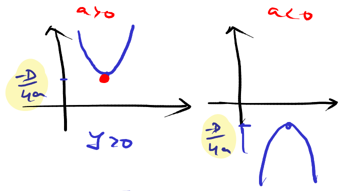
$a>0$, $b>0$	$a>0$, $b<0$	$a<0$, $b>0$	$a<0$, $b<0$
↓	↓	↓	↓
$x<0$	$x>0$	$x>0$	$x<0$



$D<0$

⇓

No cuts at x axis

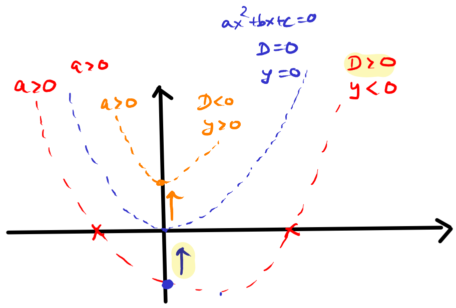


$\mathbb{R}$

$ax^2+bx+c=0 \not\Rightarrow (x-\alpha)(x-\beta)=0$

$\boxed{\text{Not factorizable in } \mathbb{R}}$

$\left(\frac{-b}{2a},\frac{-D}{4a}\right)$



$\left(\frac{-b}{2a},\frac{-D}{4a}\right)$

$\uparrow \Rightarrow D\to 0 \Rightarrow y\to 0$

⇓

$\alpha\to\beta$

$\uparrow \Rightarrow D<0 \Rightarrow y>0$

No cuts at x

32. $\underset{\sim}{A} = c\log_2 x\,\hat{i} - 6\hat{j} + 3\hat{k}$, $\underset{\sim}{B} = (\log_2 x)\hat{i} + 2\hat{j} + (2c\log_2 x)\hat{k}$

$c = ?$: Angle made by $\underset{\sim}{A}$ & $\underset{\sim}{B}$ is obtuse $\forall\ x \in (0, \infty)$

* $\theta > 90^\circ \Rightarrow \cos\theta < 0 \Rightarrow \boxed{\underset{\sim}{A}\cdot\underset{\sim}{B} < 0}$ $\forall\ x \in (0,\infty)$

* $c(\log x)^2 + 6c\log x - 12 < 0$ log. quad inequation

* $y = f(x) = c(\log x)^2 + 6c(\log x) - 12 = cA^2 + 6cA - 12$, $A = \log_2 x \Rightarrow x = 2^A$

$\log_2 2 = 1$
$\Updownarrow$
$2^1 = 2$

$A = \log_2 x \Leftrightarrow x = 2^A$

$\because x \in (0,\infty) \Rightarrow x > 0 \Rightarrow (+ve) = 2^{(A)}$ — $+ \Rightarrow 2^5 > 0$; $- \Rightarrow 2^{-8} = \frac{1}{2^8} > 0$ $\Big\} \Rightarrow A = \log_2 x \gtrless 0$

* $cA^2 + 6cA - 12 < 0 \Rightarrow y < 0$

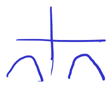
$\boxed{D = 36c^2 + 48c}$

$$\boxed{\begin{array}{c} y = ax^2 + bx + c = 0 \Rightarrow (x-\alpha)(x-\beta) = 0 \\ x_{1,2} = \dfrac{-b \pm \sqrt{D}}{2a} \in \mathbb{R} \end{array}}$$

✓ Roots: $D \geq 0$; ✗ Roots: $\boxed{D < 0}$

* if $c > 0 \Rightarrow D > 0$, $y < 0$ (Not apt.)

* $\boxed{\text{if } c < 0 \Rightarrow D < 0}$

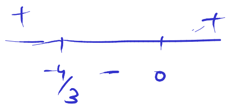


$\Downarrow$

$c < 0$, $\underbrace{c(3c+4)}_{g(c)} < 0$

$\downarrow$ CP (Critical pt)

$c = 0$, $c = -4/3$



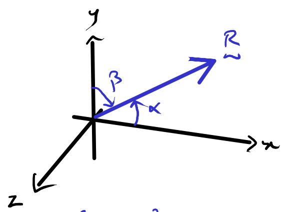
$g(c) = c(3c+4)$

$c = 1 \Rightarrow g(c) > 0$
$c = -1 \Rightarrow g(c) = -1(1) < 0$
$c = -2 \Rightarrow g(c) = -2(-6+4) > 0$ $\Big\}$ $g(c) < 0 \Rightarrow$ $\boxed{-\frac{4}{3} < c < 0 \Rightarrow c \in \left(-\frac{4}{3}, 0\right)}$ Ans.

33. $\underset{\sim}{R} = (a^2-4)\hat{i} + 2\hat{j} - (a^2-9)\hat{k}$

find a : $\underset{\sim}{R}$ makes an acute Angles with Cord. Axes. ?

JEE /MIT

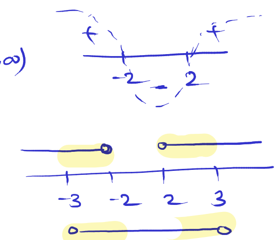


$\{\alpha, \beta, \gamma\}$

* $\alpha < 90^\circ \Rightarrow \cos\alpha > 0 \Rightarrow \frac{\underset{\sim}{R}\cdot\hat{i}}{|R|} > 0 \Rightarrow \boxed{\underset{\sim}{R}\cdot\hat{i} > 0}$ $\because |R| > 0$

$\beta < 90^\circ \Rightarrow \cos\beta > 0$
$\gamma < 90^\circ \Rightarrow \cos\gamma > 0$ $\Big\} \Rightarrow \underset{\sim}{R}\cdot\hat{j} = 2 > 0$, $\boxed{\underset{\sim}{R}\cdot\hat{k} > 0}$

* $\underset{\sim}{R}\cdot\hat{i} > 0 \Rightarrow (a^2-4) > 0 \Rightarrow (a-2)(a+2) > 0 \Rightarrow a \in (-\infty,-2) \cup (2,\infty)$
and — and — and — and
$\underset{\sim}{R}\cdot\hat{k} > 0 \Rightarrow -(a^2-9) > 0 \Rightarrow (a-3)(a+3) < 0 \Rightarrow a \in (-3,3)$



$\boxed{a \in (-3,-2) \cup (2,3)}$

$|R|\cos\alpha = R_x$

$\cos\alpha = \frac{R_x}{|R|} = \frac{\underset{\sim}{R}\cdot\hat{i}}{|R|}$

Lecture-24 (21/Jan) 2

34. $\{\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}\}$ unit vectors , $\mathbb{R}^3$

$$\underbrace{|\underset{\sim}{A}-\underset{\sim}{B}|^2 + |\underset{\sim}{B}-\underset{\sim}{C}|^2 + |\underset{\sim}{C}-\underset{\sim}{A}|^2}_{X} \leq \lambda \qquad \lambda = ?$$

$(x+y+z)^2 = x^2+y^2+z^2+2xy+2yz+2zx$

$(x-y-z)^2 = (\ \ ) + 2(-xy+yz-2x)$

* $X = \{ |A|^2+|B|^2 - 2A\cdot B + |B|^2+|C|^2-2B\cdot C + |C|^2+|A|^2 - 2A\cdot C \}$

$X = \{ 6 - 2(A\cdot B + B\cdot C + A\cdot C)\} \Rightarrow 2(A\cdot B + B\cdot C + A\cdot C) = 6 - X$ Equality 1

* To introduce inequality, $\exists$ 2 ways
  - $-1 \leq \cos\theta \leq 1$
  - $|\underset{\sim}{B}|^2 \geq 0$ $\quad \underset{\sim}{B} \in \mathbb{R}$

* $$|\underset{\sim}{A}+\underset{\sim}{B}+\underset{\sim}{C}|^2 \geq 0 \Rightarrow \underbrace{|A|^2+|B|^2+|C|^2}_{3} + \underbrace{2(A\cdot B + B\cdot C + C\cdot A)}_{= 6-X} \geq 0 \Rightarrow 9 - X \geq 0 \Rightarrow \underbrace{X}_{= |A-B|^2+|B-C|^2+|C-A|^2} \leq 9 \Rightarrow \boxed{\lambda = 9}$$

$$\boxed{|\underset{\sim}{A}-\underset{\sim}{B}|^2 + |\underset{\sim}{B}-\underset{\sim}{C}|^2 + |\underset{\sim}{C}-\underset{\sim}{A}|^2 \leq 9}$$

$\{\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}\}$ unit vector $\Rightarrow$ $\underbrace{|A|^2=1}_{\text{Normalization}}$, $\underbrace{A\cdot B \neq 0}_{\text{Non-orthogonal}}$

35. $\{\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}\}$: $|\underset{\sim}{A}| = 1$, $|\underset{\sim}{B}| = 2$, $|\underset{\sim}{C}| = 3$

if the projection of $\underset{\sim}{B}$ along $\underset{\sim}{A}$ = proj. of $\underset{\sim}{C}$ along $\underset{\sim}{A}$

$\underset{\sim}{B}$ & $\underset{\sim}{C}$ are perp. to each other

$\Big\}$ $|3\underset{\sim}{A} - 2\underset{\sim}{B} + 2\underset{\sim}{C}| = ? = \sqrt{61}$

Expand it

$\underset{\sim}{A} = \sum A^i \hat{e}_i$

$\Downarrow$

$\underset{\sim}{A}\cdot\hat{e}_i = A_i$

$\underset{\sim}{A}\cdot\hat{B}$ = proj. of $\underset{\sim}{A}$ along $\underset{\sim}{B}$

* $\underset{\sim}{B}\cdot\hat{A} = \underset{\sim}{C}\cdot\hat{A} \Rightarrow (\underset{\sim}{B}-\underset{\sim}{C})\cdot\hat{A} = 0 \Rightarrow (\underset{\sim}{B}-\underset{\sim}{C})\cdot\underset{\sim}{A} = 0$
* $\underset{\sim}{B}\cdot\underset{\sim}{C} = 0$
* $|3\underset{\sim}{A} - 2\underset{\sim}{B} + 2\underset{\sim}{C}|^2 = \ldots\ldots$

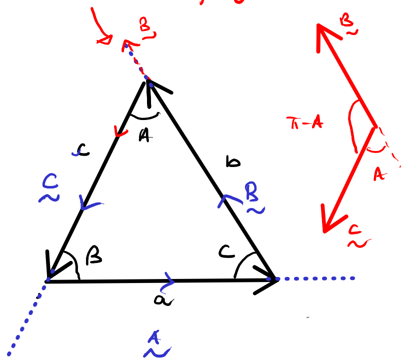
$\boxed{\text{Level-3}}$ (Geometry proofs via vectors : Dot pdt. app)

* if a, b, c lengths of the sides opp. respectively to the Angles A, B, C of Δ

I. $$\boxed{\cos A = \frac{b^2+c^2-a^2}{2bc} , \quad \cos B = \frac{c^2+a^2-b^2}{2ca}}$$

"Purely geometrical statement"

Cosine formulas



check Intro. of vectors , $\underset{\sim}{A} = \sum A^i \hat{e}_i$

* $BC = \underset{\sim}{A}$, $AC = \underset{\sim}{B}$, $AB = \underset{\sim}{C}$ 'sides are given directions / orientations'

$|\underset{\sim}{A}| = a$, $|\underset{\sim}{B}| = b$, $|\underset{\sim}{C}| = c$

$\underset{\sim}{A} \rightarrow$ "$|\underset{\sim}{A}|$"

$\hat{\underset{\sim}{A}}$

* $\underset{\sim}{A}+\underset{\sim}{B} = -\underset{\sim}{C}$ Triangle law

$\Downarrow$

$\underbrace{\underset{\sim}{A}+\underset{\sim}{B}+\underset{\sim}{C}}_{\text{vector}} = \underset{\text{vector}}{\underset{\sim}{0}} \Rightarrow \underset{\sim}{B}+\underset{\sim}{C} = -\underset{\sim}{A} \Rightarrow |\underset{\sim}{B}+\underset{\sim}{C}|^2 = |-\underset{\sim}{A}|^2 \Rightarrow |\underset{\sim}{B}|^2 + |\underset{\sim}{C}|^2 + 2\underbrace{\underset{\sim}{B}\cdot\underset{\sim}{C}}_{|B||C|\cos(\pi-A) = -\cos A} = |\underset{\sim}{A}|^2 \Rightarrow$

includes actual sides of a Δ

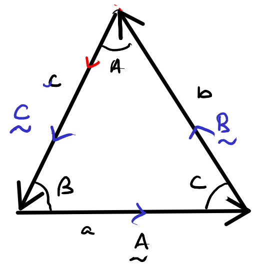
$\overbrace{b^2+c^2} - 2bc\cos A = a^2$

$\Downarrow$

$\cos A = \dfrac{b^2+c^2-a^2}{2bc}$ $\Big\}$ $\square$

II. $\boxed{a = b\cos C + c\cos B \quad , \quad b = c\cos A + a\cos C \quad , \quad c = b\cos A + a\cos B}$ Projection formula

Reverse engineer.



* $a = b\cos C + c\cos B$

$|\underset{\sim}{A}| = |\underset{\sim}{B}|\cos C + |\underset{\sim}{C}|\cos B$

Std. Sol^n

$$\underset{\sim}{A}+\underset{\sim}{B}+\underset{\sim}{C}=0$$

$$\Downarrow$$

$$\boxed{\underset{\sim}{A} = -(\underset{\sim}{B}+\underset{\sim}{C})}$$

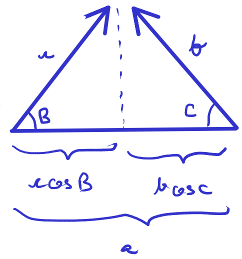
$$\underset{\sim}{A}\cdot\underset{\sim}{A} = |\underset{\sim}{A}|^2 = -\underset{\sim}{A}\cdot(\underset{\sim}{B}+\underset{\sim}{C})$$

$$= -(\underset{\sim}{A}\cdot\underset{\sim}{B} + \underset{\sim}{A}\cdot\underset{\sim}{C})$$

$$= -\{|A||B|\cos(\pi-C) + |A||C|\cos(\pi-B)\}$$

$$a^2 = ab\cos C + ac\cos B$$

$$a = b\cos C + c\cos B$$

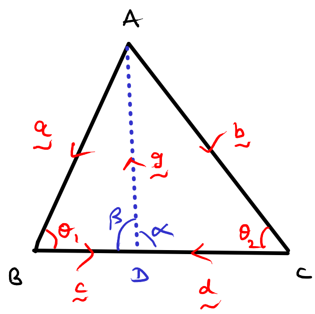


36. Isosceles Δ $\rightarrow$ P.T median to the base is $\perp$ to the base

Meth. 1

* D : median $\Rightarrow$ BD = BC , AB = AC (isosceles) $\Rightarrow$ $\theta_1=\theta_2=\theta$

* P.T : $\underset{\sim}{AD}\cdot\underset{\sim}{BC}=0 \Rightarrow \underset{\sim}{g}\cdot(\underset{\sim}{BC})=0$



translation of the problem into vectors (Big deal)

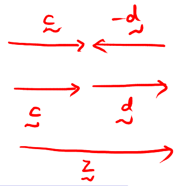
* $\underset{\sim}{a}+\underset{\sim}{c}+\underset{\sim}{g}=0 \Rightarrow \underset{\sim}{a}+\underset{\sim}{c}=-\underset{\sim}{g}$ | $|\underset{\sim}{a}| = AB$, $|\underset{\sim}{c}| = BD$ $\Big\}\Rightarrow |\underset{\sim}{a}|=|\underset{\sim}{b}|$

$\underset{\sim}{b}+\underset{\sim}{d}+\underset{\sim}{g}=0 \Rightarrow \underset{\sim}{b}+\underset{\sim}{d}=-\underset{\sim}{g}$ | $|\underset{\sim}{b}| = AC$, $|\underset{\sim}{d}| = DC$ $\Rightarrow |\underset{\sim}{d}|=|\underset{\sim}{c}|$

* $\underset{\sim}{g}\cdot\underset{\sim}{c} = -(\underset{\sim}{a}\cdot\underset{\sim}{c}+|\underset{\sim}{c}|^2) = -\{|a||c|\cos\theta + |c|^2\}$

$\underset{\sim}{g}\cdot\underset{\sim}{d} = -(\underset{\sim}{b}\cdot\underset{\sim}{d}+|\underset{\sim}{d}|^2) = -\{|d||b|\cos\theta + |d|^2\}$

$\underset{\sim}{g}\cdot(\underset{\sim}{c}-\underset{\sim}{d}) = 0 \Rightarrow \boxed{\underset{\sim}{g}\cdot\underset{\sim}{z}=0}$, $\underset{\sim}{g}\perp\underset{\sim}{z}$



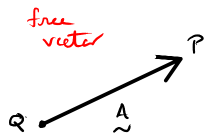
* $\underset{\sim}{g}\cdot\underset{\sim}{c} - \underset{\sim}{g}\cdot\underset{\sim}{d} = -\cancel{|a||c|\cos\theta} + \cancel{|b||d|\cos\theta} - \cancel{|c|^2} + \cancel{|d|^2} = 0 \Rightarrow \underset{\sim}{g}\cdot\underset{\sim}{c} = \underset{\sim}{g}\cdot\underset{\sim}{d} \Rightarrow \cancel{|g|}\cancel{|c|}\cos\alpha = \cancel{|g|}\cancel{|d|}\cos\beta \Rightarrow \boxed{\cos\alpha = \cos\beta}$

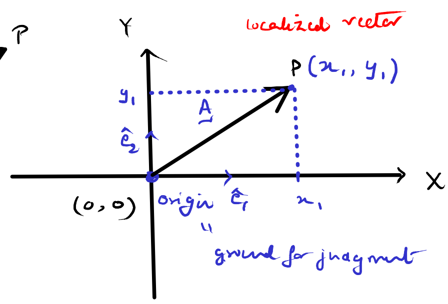
T. Eq^n

* $\alpha+\beta=\pi \Rightarrow \cos\alpha = \underbrace{\cos(\pi-\alpha)}_{-\cos\alpha} \Rightarrow 2\cos\alpha = 0 \Rightarrow \cos\alpha = \cos\frac{\pi}{2} \Rightarrow \alpha = \pi/2$, $\beta = \frac{\pi}{2} \Rightarrow AD \perp BC$

Lecture-25 (22/6) 2

B.4 Position vector of a point $\rightarrow$ Absolute / $\rightarrow$ Relative





* $\underset{\sim}{A} = \sum_i A^i \hat{e}_i$: $\hat{e}_i\cdot\hat{e}_j = g_{ij} \longrightarrow \delta_{ij} \Rightarrow$ Euclidean geo

Case I : $\exists$ 1 non-origin pt P

* Parallel transport the $\underset{\sim}{A}$ to $\underbrace{\text{Cartesian plane}}_{\mathbb{R}^3}$ $\Rightarrow$ tail is known $\downarrow$ Q(0,0,0) $\Big\}$ Descartes' trick

* $\underbrace{|OP|^2}_{|\underset{\sim}{A}|^2} = (x_2-x_1)^2 + (y_2-y_1)^2 = x^2+y^2 \Rightarrow |\underset{\sim}{A}| = \sqrt{x^2+y^2}$

distance formula

* Aim : to give Coord. of pt Q & pt P

* $\boxed{\nexists \text{ pos. vec. of 1 isolated point}}$ Rule

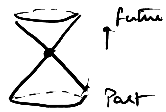
Real lives don't matter to Math

* $\exists$ a "Hidden" point / Reference point / origin $\Rightarrow$ position value assignment

$\underbrace{\text{"Hidden"}}_{O}$ ... $\underbrace{\text{origin}}_{\text{"Biblical"}}$

* $\exists$ a "Hidden" point / Reference point / origin $\Rightarrow$ Time-value assignment $\longrightarrow$ SR $\Rightarrow$ "all" is relative WRT light Cone

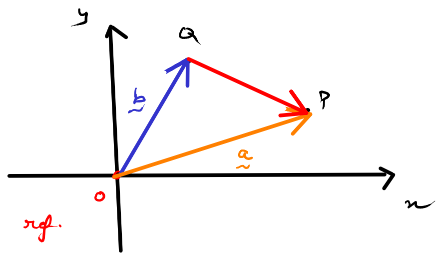
$\underbrace{\text{"Hidden"}}_{\text{Heart beat 4 clock}}$ ... $\underbrace{\text{origin}}_{\text{"Biblical"}}$



$\boxed{\text{if } \exists \text{ atleast two Entities} \Rightarrow \text{ ... -value assignment}}$ $\longrightarrow$ Ethics / Morality

* $\boxed{\underset{\sim}{R} \equiv \text{vector : } \exists \text{ the tail at a reference origin}}$

Absolute Position vector



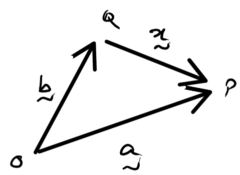
Case II : $\exists$ 2 non-origin points P & Q

* $\exists$ absolute pos. vector for Q = $\underset{\sim}{b}$ $\rightarrow$ Q connects O
* $\exists$ absolute pos. vector for P = $\underset{\sim}{a}$ $\rightarrow$ P connects O

$\Rightarrow$ $\exists$ Connection b/w $\underbrace{\text{P \& Q}}_{\text{Relative Position Vector}}$

Notation Alert

$\underset{\sim}{QP} = \underset{\rightarrow}{QP} = \overrightarrow{QP}$



* triangle law $\Rightarrow$ $\underset{\sim}{OQ} + \underset{\sim}{QP} = \underset{\sim}{OP}$ $\Rightarrow$ $\underset{\sim}{QP} = \underset{\sim}{OP} - \underset{\sim}{OQ} = \underset{\sim}{a} - \underset{\sim}{b}$

$\underset{\sim}{QP}$ = $\underset{\sim}{x}$

$\boxed{\underset{\sim}{QP} = (\text{Pos. vector of P}) - (\text{pos. vector of Q}) = \text{pos. vector of Head} - \text{Pos. vec. of tail}}$

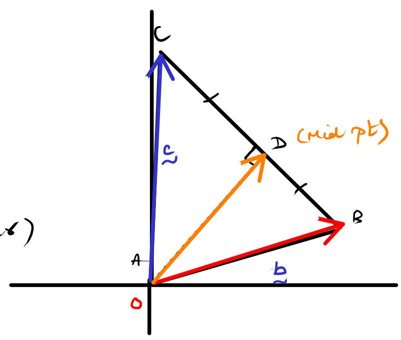
Pos. vector relative to Q

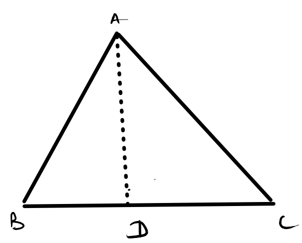
a vector is always relative to its tail

36. Isosceles Δ $\rightarrow$ P.T median to the base is $\perp$ to the base

Meth. 2

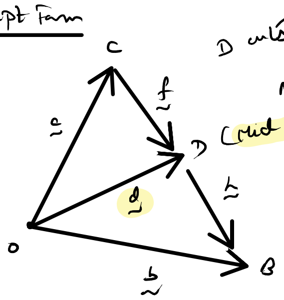
* $\{A, B, C\} \rightarrow$ Triangle
* A marked as origin
* $\{\underset{\sim}{AB}, \underset{\sim}{AC}, \underset{\sim}{AD}\}$: absolute pos. vectors





Detour to Midpt Form

Internal div. of CB via D



D cuts CB in 1:1 $\leftarrow$ (later) m n

(mid point) $\Rightarrow$ CD = DB $\Rightarrow$ $\frac{CD}{DB} = \frac{1}{1}$ (Ratio)

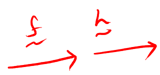
$\hookrightarrow |\underset{\sim}{f}| = |\underset{\sim}{h}|$ $\Rightarrow$ $\underset{\sim}{f} = \underset{\sim}{h}$ $\Rightarrow$ $\underset{\sim}{d} - \underset{\sim}{c} = \underset{\sim}{b} - \underset{\sim}{d}$

$\downarrow$

Mid-point formula

* $\underset{\sim}{c} + \underset{\sim}{f} = \underset{\sim}{d}$
  $\underset{\sim}{d} + \underset{\sim}{h} = \underset{\sim}{b}$ $\Rightarrow$ $\underset{\sim}{d} = \underset{\sim}{b} - \underset{\sim}{h}$

$\Bigg\}$ $\Rightarrow$ $2\underset{\sim}{d} = \underset{\sim}{c} + \cancel{\underset{\sim}{f}} + \underset{\sim}{b} - \cancel{\underset{\sim}{h}}$ $\Rightarrow$ $\boxed{\underset{\sim}{d} = \frac{\underset{\sim}{b} + \underset{\sim}{c}}{2}}$



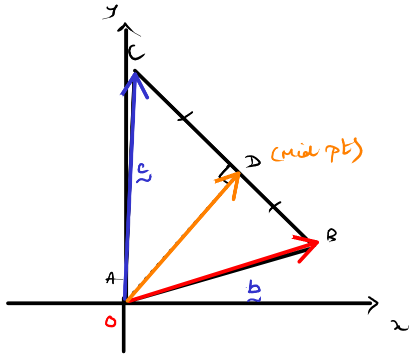
$\underset{\sim}{f} \longrightarrow$
$-\underset{\sim}{h} \longleftarrow$
$|\underset{\sim}{f}| = |\underset{\sim}{h}|$ $\Rightarrow$ $\underset{\sim}{f} = \underset{\sim}{h}$?

$\underset{\sim}{f} = f_1\hat{i} + f_2\hat{j}$
$\underset{\sim}{h} = h_1\hat{i} + h_2\hat{j}$
$\Bigg\}$ $f_1^2 + f_2^2 = h_1^2 + h_2^2$

$\underset{\sim}{f} = \underset{\sim}{h}$ $\Rightarrow$ $f_1 = h_1$, $f_2 = h_2$

Lecture-26 (23/Jan) 2

* T.P: $\underset{\sim}{AD} \perp \underset{\sim}{BC} \Rightarrow \underset{\sim}{AD}\cdot\underset{\sim}{BC} = 0$, Δ is isosceles $\Rightarrow |\underset{\sim}{c}| = |\underset{\sim}{b}|$



* pos. vector of pt. C = $\underset{\sim}{c}$
  " " " " B = $\underset{\sim}{b}$
  " " " " D = $\dfrac{\underset{\sim}{b}+\underset{\sim}{c}}{2}$

* $\underset{\sim}{AD} = \vec{AD} = \underbrace{\text{pos. vect of } D}_{\frac{b+c}{2}} - \underbrace{\text{pos vector of } A}_{\underset{\sim}{0}} = \dfrac{\underset{\sim}{b}+\underset{\sim}{c}}{2}$

* $\underset{\sim}{BC} = \vec{BC}$ = pos. vect. of C − pos. vect. of B $= \underset{\sim}{c} - \underset{\sim}{b}$

* $\underset{\sim}{AD}\cdot\underset{\sim}{BC} = \dfrac{(\underset{\sim}{b}+\underset{\sim}{c})}{2}\cdot(\underset{\sim}{c}-\underset{\sim}{b}) = \dfrac{|\underset{\sim}{c}|^2-|\underset{\sim}{b}|^2}{2} = 0$

vectors, vector dot

$(\underset{\sim}{A}+\underset{\sim}{B})\cdot(\underset{\sim}{A}-\underset{\sim}{B}) = \underset{\sim}{A}\cdot\underset{\sim}{A} - \underset{\sim}{B}\cdot\underset{\sim}{B} - \underset{\sim}{A}\cdot\underset{\sim}{B} + \underset{\sim}{B}\cdot\underset{\sim}{A}$

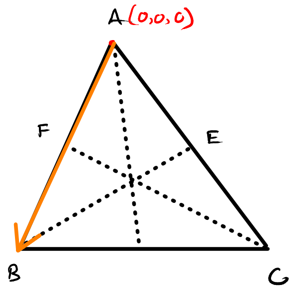
$\Vert$

$\underbrace{(|A|+|B|)}_{\text{numbers}}\cdot(|A|-|B|)$ normal # multip.

$\underbrace{\alpha \underset{\sim}{A}}_{\text{scaling}}$ $\alpha = 2$, $\alpha = 1/2$

37. If 2 medians of a Δ are $\underbrace{\text{equal}}$, P.T $\underbrace{\Delta \text{ is isosceles}}$

$|AB| = |AC|$
$\Downarrow$
$|\underset{\sim}{b}| = |\underset{\sim}{c}|$



intuition

* Set up the origin → A (0,0,0) Step 1

* pos. vect. of B pt = $\underset{\sim}{b}$ Step 2
  " " " C " = $\underset{\sim}{c}$

* Pos. vect. of F (mid pt of AB) = $\dfrac{\underset{\sim}{b}}{2}$ step 3
  ↑ halves $\underset{\sim}{b}$

  " " " E (" " " AC) = $\dfrac{\underset{\sim}{c}}{2}$ ✓

* Medians $\{\vec{BE}, \vec{CF}\}$ $\xrightarrow{\text{given}}$ $\boxed{|\vec{BE}| = |\vec{CF}|}$ Equal length step 4

$\vec{BE}$ = pos. vector of E − pos. vect of B $= \dfrac{\underset{\sim}{c}}{2} - \underset{\sim}{b}$ ✓

$\vec{CF}$ = " " F − " " C $= \dfrac{\underset{\sim}{b}}{2} - \underset{\sim}{c}$ ✓

Algebra

* $|\underset{\sim}{BE}| = |\underset{\sim}{CF}| \Rightarrow |\underset{\sim}{BE}|^2 = |\underset{\sim}{CF}|^2 \Rightarrow \left|\dfrac{\underset{\sim}{c}}{2}-\underset{\sim}{b}\right|^2 = \left|\dfrac{\underset{\sim}{b}}{2}-\underset{\sim}{c}\right|^2 = \dfrac{1}{4}|\underset{\sim}{c}|^2 + |\underset{\sim}{b}|^2 - \cancel{2\cdot\dfrac{1}{2}\underset{\sim}{c}\cdot\underset{\sim}{b}} = \dfrac{1}{4}|\underset{\sim}{b}|^2 + |\underset{\sim}{c}|^2 - \cancel{2\dfrac{1}{2}\underset{\sim}{b}\cdot\underset{\sim}{c}} \Rightarrow \cancel{\tfrac{3}{4}}|\underset{\sim}{b}|^2 = \cancel{\tfrac{3}{4}}|\underset{\sim}{c}|^2$

$\Downarrow$

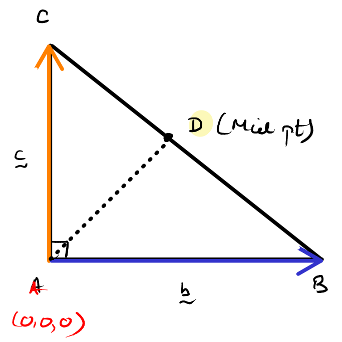
$\boxed{|\underset{\sim}{b}| = |\underset{\sim}{c}|} \Leftarrow |\underset{\sim}{b}|^2 = |\underset{\sim}{c}|^2$

isosceles Δ

38. P.T the $\underbrace{\text{mid pts of Hyp. of RA}\Delta}_{CD = DB}$ is $\underbrace{\text{equidistant from its vertices.}}$

$\Downarrow$

CD = DB = AD



* Set up an origin → A step 1

* pos. vect. of C pt = $\underset{\sim}{c}$ Step 2
  " " " B " = $\underset{\sim}{b}$
  " " " D (mid pt of BC) = $\dfrac{\underset{\sim}{b}+\underset{\sim}{c}}{2}$

* $AC \perp AB \Rightarrow \underset{\sim}{b}\cdot\underset{\sim}{c} = 0$

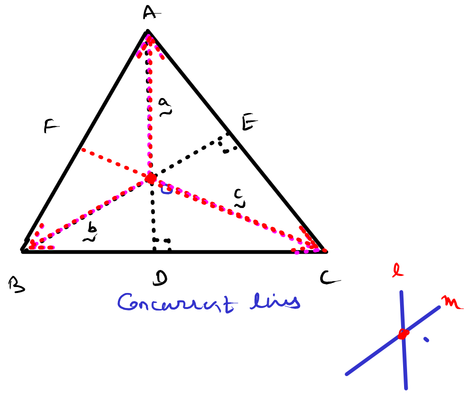
* $\vec{AD}$ = pos. vec. D − pos. vect. of A $= \dfrac{\underset{\sim}{b}+\underset{\sim}{c}}{2}$ ⟶ $|\vec{AD}|$ = length of side AD

* $|\underset{\sim}{AD}|^2 = \left(\frac{\underset{\sim}{b}+\underset{\sim}{c}}{2}\right)\cdot\left(\frac{\underset{\sim}{b}+\underset{\sim}{c}}{2}\right) = \frac{1}{4}\left\{\underbrace{|b|^2+|c|^2}_{|AB|^2+|AC|^2=|CB|^2} + \cancel{2\,\underset{\sim}{b}\cdot\underset{\sim}{c}}^{\,0}\right\} = \frac{1}{4}|CB|^2 = \frac{1}{4}\,4\,|CD|^2 = |CD|^2 \Rightarrow |AD| = |CD| = |DB|$

39. $\underbrace{\text{Altitudes}}_{\perp}$ of the triangle are $\underbrace{\text{concurrent}}$ (P.T)

⇓

all passes thru O



Intuition

2 lines are concurrent ⟶ T. Prove $\underbrace{\text{lines are altitudes}}_{AD\perp BC,\ BE\perp AC}$

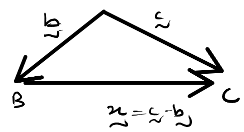
if ∃ any 3rd line from the remaining vertex & it has to be concurrent

⇓

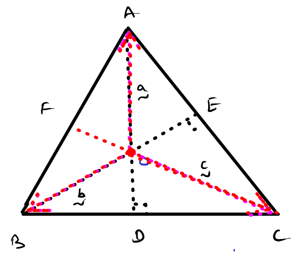
line is Altitude

translate it into vector space

* $\boxed{\text{Set up the origin}}$ ⟶ O (0,0,0) Intersection pt



* pos. vector of pt. A = $\underset{\sim}{a}$
  " " " " B = $\underset{\sim}{b}$
  " " " " C = $\underset{\sim}{c}$



* $AD \perp BC \Rightarrow \underset{\sim}{AD}\cdot\underset{\sim}{BC} = 0 \Rightarrow \underset{\sim}{OA}\cdot\underset{\sim}{BC} = 0 \Rightarrow \underset{\sim}{a}\cdot(\underset{\sim}{c}-\underset{\sim}{b}) = 0 \Rightarrow \underset{\sim}{a}\cdot\underset{\sim}{c} - \underset{\sim}{a}\cdot\underset{\sim}{b} = 0$
* $BE \perp AC \Rightarrow \underset{\sim}{BE}\cdot\underset{\sim}{AC} = 0 \Rightarrow \underset{\sim}{OB}\cdot\underset{\sim}{AC} = 0 \Rightarrow \underset{\sim}{b}\cdot(\underset{\sim}{c}-\underset{\sim}{a}) = 0 \Rightarrow \underset{\sim}{b}\cdot\underset{\sim}{c} - \underset{\sim}{a}\cdot\underset{\sim}{b} = 0$

$-$

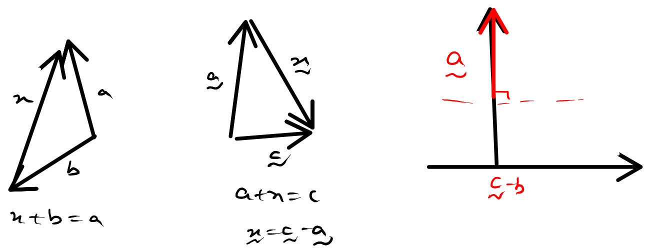
$\underset{\sim}{a}\cdot\underset{\sim}{c} - \underset{\sim}{b}\cdot\underset{\sim}{c} = 0$

* $\underset{\sim}{c}\cdot\underbrace{(\underset{\sim}{a}-\underset{\sim}{b})}_{\underset{\sim}{AB}} = 0 \Rightarrow \underset{\sim}{OF}\cdot\underset{\sim}{AB} = 0 \Rightarrow \underset{\sim}{CF}\cdot\underset{\sim}{AB} = 0$

⇓

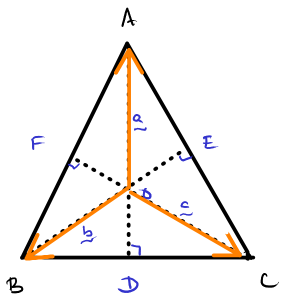
$CF \perp AB$

3 $\underbrace{\text{Concurrent line}}_{\text{Passes thru O}}$ is Altitude



Lecture-27 (24/6m) 2

40. Perp. bisector of a the sides of Δ are concurrent Prove it!



* 2 lines can be concurrent & Perp. bisector (Assumption)
* 3rd line to be concurrent also has to be a $\perp$ (to prove)

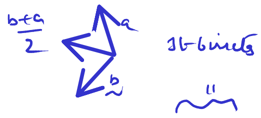
* Set up the origin.
* Pos. vect. of pt A = $\underset{\sim}{a}$
  " B = $\underset{\sim}{b}$
  " C = $\underset{\sim}{c}$

* Pos vector of pt E (Mid pt) $= \frac{\underset{\sim}{a}+\underset{\sim}{c}}{2} = \underset{\sim}{OE}$

  F (") $= \frac{\underset{\sim}{a}+\underset{\sim}{b}}{2} =$

$D\;(\;\text{"}\;) = \dfrac{\underset{\sim}{b}+\underset{\sim}{c}}{2} = \underset{\sim}{OD}$

2 lines are ⊥ bisector (are concurrent)



* $OE \perp AC \Rightarrow \underset{\sim}{OE}\cdot\underset{\sim}{AC}=0 \quad \left(\dfrac{\underset{\sim}{a}+\underset{\sim}{c}}{2}\right)\cdot(\underset{\sim}{c}-\underset{\sim}{a})=0 \Rightarrow |c|^2-|a|^2=0$

$OD \perp BC \Rightarrow \underset{\sim}{OD}\cdot\underset{\sim}{BC}=0 \quad \left(\dfrac{\underset{\sim}{b}+\underset{\sim}{c}}{2}\right)\cdot(\underset{\sim}{b}-\underset{\sim}{c})=0 \Rightarrow |b|^2-|c|^2=0$

$\Rightarrow |b|^2-|a|^2=0 \Rightarrow \underbrace{(\underset{\sim}{b}-\underset{\sim}{a})}_{\underset{\sim}{AB}}\cdot(\underset{\sim}{b}+\underset{\sim}{a})=0 \Rightarrow \underset{\sim}{AB}\cdot\underbrace{\overbrace{\dfrac{(\underset{\sim}{b}+\underset{\sim}{a})}{2}}}_{\underset{\sim}{OF}}=0$

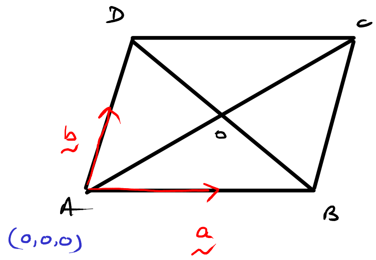
$\underset{\sim}{AB}\cdot\underset{\sim}{OF}=0 \Rightarrow \underset{\sim}{AB}\perp\underset{\sim}{OF} \Rightarrow$ OF is a ⊥ bisector of $\underset{\sim}{AB}$ / The point it passes from is O ⇒ all 3 lines are concurrent

41. if $\underbrace{\text{the diags.}}$ of ||gm are equal ⇒ ||gm is $\underbrace{\text{a rect.}}$ (to prove)

$|AC|^2 = |BD|^2$

⇑

$\underset{\sim}{a}\cdot\underset{\sim}{b}=0 \quad (\underset{\sim}{a}\perp\underset{\sim}{b})$



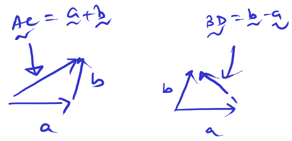
* $|AC|^2 = |BD|^2 \Rightarrow |\underset{\sim}{a}+\underset{\sim}{b}|^2 = |\underset{\sim}{b}-\underset{\sim}{a}|^2 \Rightarrow \cancel{|a|^2}+\cancel{|b|^2}+2\underset{\sim}{a}\cdot\underset{\sim}{b} = \cancel{|b|^2}+\cancel{|a|^2}-2\underset{\sim}{b}\cdot\underset{\sim}{a} \Rightarrow 4\underset{\sim}{a}\cdot\underset{\sim}{b}=0$

⇓

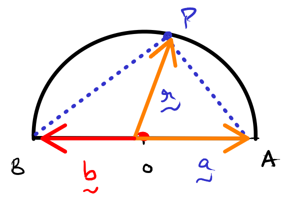
$\underset{\sim}{a}\cdot\underset{\sim}{b}=0$

⇓

$\underset{\sim}{a}\perp\underset{\sim}{b}$



42. Angle in a semi-circle is a R.A (P.T)

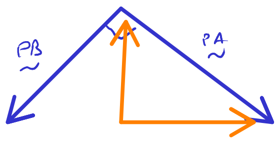


* take any arbitrary pt on circumference

* $\underbrace{\angle BPA = 90^\circ}$ (to prove)

$BP \perp AP \Rightarrow \underset{\sim}{BP}\cdot\underset{\sim}{AP}=0$

* Set up the origin : O



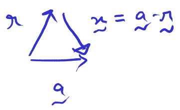
pos. vect of A = $\underset{\sim}{a}$

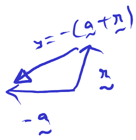
B = $\underset{\sim}{b}$

P = $\underset{\sim}{r}$

$\Rightarrow |\underset{\sim}{a}| = |\underset{\sim}{b}| = |\underset{\sim}{r}|$

$\boxed{\underset{\sim}{b} = -\underset{\sim}{a}}$





* $\underset{\sim}{PA}$ = pos. vec. of A − pos. vect of P = $\underset{\sim}{a}-\underset{\sim}{r}$

$\underset{\sim}{PB}$ = pos vect of B − pos vect of P = $\underset{\sim}{b}-\underset{\sim}{r} = -\underset{\sim}{a}-\underset{\sim}{r} = -(\underset{\sim}{a}+\underset{\sim}{r})$

$\underset{\sim}{r}+\underset{\sim}{y} = -\underset{\sim}{a}$

$\underset{\sim}{y} = -\underset{\sim}{a}-\underset{\sim}{r}$

$= -(\underset{\sim}{a}+\underset{\sim}{r})$

* $\underset{\sim}{PA}\cdot\underset{\sim}{PB} = (\underset{\sim}{a}-\underset{\sim}{r})\cdot(\underset{\sim}{b}-\underset{\sim}{r}) = \underset{\sim}{a}\cdot\underset{\sim}{b}-\underset{\sim}{a}\cdot\underset{\sim}{r}-\underset{\sim}{b}\cdot\underset{\sim}{r}+\underset{\sim}{r}\cdot\underset{\sim}{r}$ Lot of work

↓

$(\underset{\sim}{a}-\underset{\sim}{r})\cdot(-\underset{\sim}{a}-\underset{\sim}{r}) = -(\underset{\sim}{a}-\underset{\sim}{r})\cdot(\underset{\sim}{a}+\underset{\sim}{r}) = -\{\cancel{|a|^2}-\cancel{|r|^2}\} = 0$

$\underset{\sim}{A} = \sum A^i e_i = A^1\hat{e}_1 + A^2\hat{e}_2$

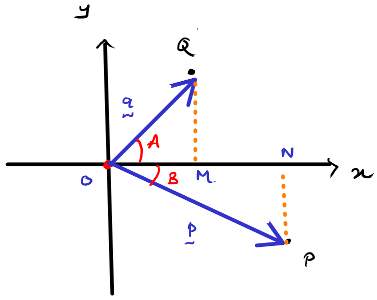
$\underset{\sim}{\nabla} = \dfrac{\partial}{\partial x}\hat{i} + \dfrac{\partial}{\partial y}\hat{j} =$

Lecture-28 (25/6/21) 2

Step 1	:	Set up the origin (Most difficult)
2	:	Triangle Law $\underset{\sim}{C}=\underset{\sim}{A}+\underset{\sim}{B}$ ∋ Position vector
3	:	Dot product whenever necessary

prescription to translate a geometry prob into vector → Trigonometry → Calculus

* Most fundamental Id. of Trig. $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$ (Trig.)



* Set up the Origin : O

* Pos. vect of Q pt $= \underset{\sim}{q}$

  P pt $= \underset{\sim}{p}$

* $\underset{\sim}{q} \cdot \underset{\sim}{p} = |\underset{\sim}{q}||\underset{\sim}{p}| \cos(A+B)$ — ①

Eval. of dot prodt

Approach 2

* $\underset{\sim}{OQ} = \underset{\sim}{q} = \sum_{i=1}^{2} q^i \hat{e}_i = q^x \hat{\imath} + q^y \hat{\jmath} = |\underset{\sim}{q}| \cos A \, \hat{\imath} + |\underset{\sim}{q}| \sin A \, \hat{\jmath}$

  $= |\underset{\sim}{q}| \{ \cos A \hat{\imath} + \sin A \hat{\jmath} \}$

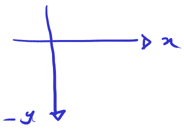
basis unit vect.

$\hat{e}_i = (\hat{\imath} \quad \hat{\jmath})$

* $\underset{\sim}{OP} = \underset{\sim}{p} = \sum_{i=1}^{2} p^i \hat{e}_i = p^x \hat{\imath} + p^y (-\hat{\jmath})$

  $= |\underset{\sim}{p}| \cos B \, \hat{\imath} + |\underset{\sim}{p}| \sin B (-\hat{\jmath})$

  $= |\underset{\sim}{p}| \{ \cos B \hat{\imath} - \sin B \hat{\jmath} \}$



$\{ \hat{e}_i = (\hat{\imath} \quad -\hat{\jmath}) \}$

* $\underset{\sim}{q} \cdot \underset{\sim}{p} = |\underset{\sim}{p}||\underset{\sim}{q}| \{ \cos A \cos B \underbrace{\hat{\imath} \cdot \hat{\imath}}_{1} - \sin A \sin B \underbrace{\hat{\jmath} \cdot \hat{\jmath}}_{1} - \cos A \sin B \underbrace{\cancel{\hat{\imath} \cdot \hat{\jmath}}}_{0} - \sin A \cos B \underbrace{\cancel{\hat{\imath} \cdot \hat{\jmath}}}_{0} \}$

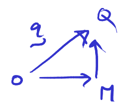
$\underset{\sim}{q} \cdot \underset{\sim}{p} = |\underset{\sim}{q}||\underset{\sim}{p}| \{ \cos A \cos B - \sin A \sin B \}$ — ②

$= |\underset{\sim}{q}||\underset{\sim}{p}| \cos(A+B)$

Eq ①

* $\boxed{\cos(A+B) = \cos A \cos B - \sin A \sin B}$

Approach 1



triangle law

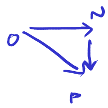
$\underset{\sim}{OM} + \underset{\sim}{MQ} = \underset{\sim}{OQ}$

$MQ = OQ \sin A$

$\underset{\sim}{MQ} = (|\underset{\sim}{OQ}| \sin A) \hat{\jmath}$

$\underset{\sim}{OM} = (|\underset{\sim}{OQ}| \cos A) \hat{\imath}$

$\underset{\sim}{OQ} = |\underset{\sim}{OQ}| \{ \cos A \hat{\imath} + \sin A \hat{\jmath} \}$



$\underset{\sim}{ON} + \underset{\sim}{NP} = \underset{\sim}{OP}$ triangle law

$\underset{\sim}{OP} = |\underset{\sim}{OP}| \{ \cos B \hat{\imath} - \sin B \hat{\jmath} \}$

Remark

* $\underset{\sim}{q} = \sum_i q^i \hat{e}_i$

  $\underset{\sim}{p} = \sum_j p^j \hat{e}_j$

* $\underset{\sim}{q} \cdot \underset{\sim}{p} = \sum_i \sum_j q^i p^j \underbrace{(\hat{e}_i \cdot \hat{e}_j)} = \sum_i \sum_j q^i p^j g_{ij} = \sum_i \sum_j g_{ij} q^i p^j$

Euclidean

$g_{ij} = \delta_{ij}$

$\underset{\sim}{q} \cdot \underset{\sim}{p} = q^\circ p^\circ + q^1 p^1$

* $g_{ij} = \eta_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ 'Minkowski metric'

* $\underset{\sim}{q} \cdot \underset{\sim}{p} = \sum_{ij} \eta_{ij} q^i p^j = \underbrace{\eta_{00}}_{1} q^\circ p^\circ + \underbrace{\eta_{11}}_{-1} q^1 p^1 = q^\circ p^\circ - q^1 p^1$

$\{ q^\circ, p^\circ, q^1, p^1 \}$? Think what would happen ?!

geo.

$\sin^2\theta + \cos^2\theta = 1 \Rightarrow C^2 = A^2 + B^2 + 2AB \cos\theta$

vector form equivalence $\Downarrow$

$\underset{\sim}{A} \cdot \underset{\sim}{B} = |A||B| \cos\theta$

$\Downarrow$

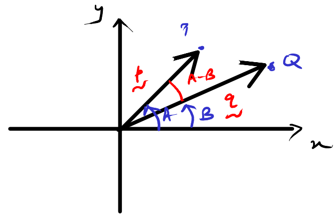
$\cos(A+B) = \cos A \cos B - \sin A \sin B$

Euclidean spell

Lecture-29 (26 Nov) 2

* $\cos(A-B) = \cos A\cos B + \sin A\sin B$

* $\underset{\sim}{q}\cdot\underset{\sim}{p} = |\underset{\sim}{q}||\underset{\sim}{p}|\cos(A-B)$

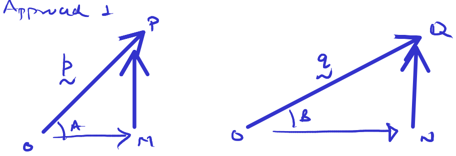


Approach 2

* $\underset{\sim}{p} = \sum_i p^i \hat{e}_i = p^x\hat{i} + p^y\hat{j} = (|\underset{\sim}{p}|\cos A)\hat{i} + (|\underset{\sim}{p}|\sin A)\hat{j}$

⋮

Approach 1

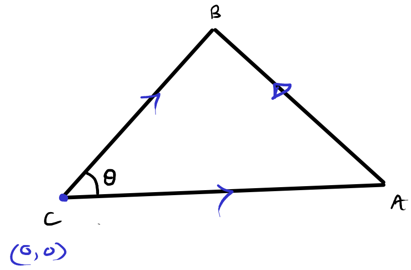


Triangle Law

$\underset{\sim}{OP} = \underset{\sim}{OM} + \underset{\sim}{MP}$ $\qquad$ $\underset{\sim}{OQ} = \underset{\sim}{ON} + \underset{\sim}{NQ}$

$\underset{\sim}{OM} = (|\underset{\sim}{p}|\cos A)\,\hat{i}$

* Cosine Form. of triangle $\sim$ dot product of vector



* $\cos\theta = \dfrac{CB^2 + AC^2 - AB^2}{2(AC)(CB)} \Rightarrow AB^2 = BC^2 + AC^2 - 2(AC)(BC)\cos\theta$ — ①

$AB^2 = |\underset{\sim}{AB}|^2 = |\underset{\sim}{CB} - \underset{\sim}{AC}|^2$

$= |\underset{\sim}{BC}|^2 + |\underset{\sim}{AC}|^2 - 2\,\underset{\sim}{AC}\cdot\underset{\sim}{BC}$ — ②

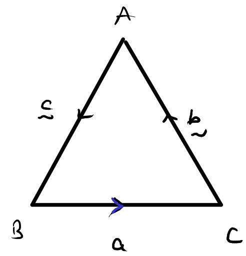
$\underset{\sim}{AC} + \underset{\sim}{AB} = \underset{\sim}{CB}$

$\underset{\sim}{AB} = \underset{\sim}{CB} - \underset{\sim}{AC}$

① = ② $\Rightarrow$ $\underset{\sim}{AC}\cdot\underset{\sim}{BC} = |\underset{\sim}{AC}||\underset{\sim}{BC}|\cos\theta$ □

Level-4 Geometry proofs via vectors (Basics such as triangle law)

1. $\{\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}\} \longrightarrow$ represent sides of a Δ

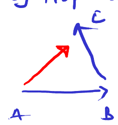


* orientation to the sides (say Anticlockwise)

* $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} = \vec{BC} + \vec{CA} + \vec{AB} = \vec{CA} + \underbrace{\vec{AB} + \vec{BC}}_{=\vec{AC}} = \underbrace{\vec{CA} + \vec{AC}}_{} = \vec{0}$

$(\text{P.V of } A - \cancel{\text{P.V of } C}) + (\cancel{\text{P.V of } C} - \text{P.V of } A)$

Try "Proof" of triangle law



$\vec{AB} + \vec{BC} = \vec{AC}$

$(\cancel{\text{pos. vect of } B} - \text{p.v. of } A) + (\text{pos. vect of } C - \cancel{\text{pos. vect of } B})$

||

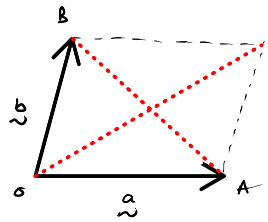
P.V of C − P.V. of A ≡ $\vec{AC}$

*

closed Δ orientation

$$\boxed{\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} = 0}$$

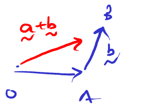
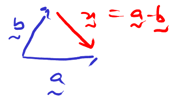
2. $\{\underset{\sim}{a}, \underset{\sim}{b}\}$; $\boxed{|\underset{\sim}{a}+\underset{\sim}{b}| = |\underset{\sim}{a}-\underset{\sim}{b}|}$ $\Rightarrow$? geometrically



* $\Delta\, OCA \Rightarrow \underset{\sim}{OA} + \underset{\sim}{AC} = \underset{\sim}{OC} \Rightarrow \underset{\sim}{OC} = \underset{\sim}{a} + \underset{\sim}{b}$

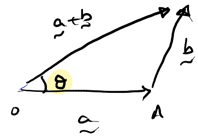
* $\Delta\, OBA \Rightarrow \underset{\sim}{OB} + \underset{\sim}{BA} = \underset{\sim}{OA} \Rightarrow \underset{\sim}{BA} = \underset{\sim}{a} - \underset{\sim}{b}$

* $|OC|$ = length of diag of "115m"
$|BA|$ = " " " 2 " } Equal diag

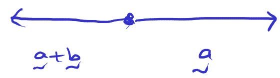
* $|\underset{\sim}{a}+\underset{\sim}{b}|^2 = |\underset{\sim}{a}-\underset{\sim}{b}|^2 \Rightarrow \underset{\sim}{a}\cdot\underset{\sim}{b} = 0 \Rightarrow \underset{\sim}{a} \perp \underset{\sim}{b}$ Rectang.

Ex : $|\underset{\sim}{a}+\underset{\sim}{b}| = |\underset{\sim}{a}|$



$|a||b|\cos\theta$

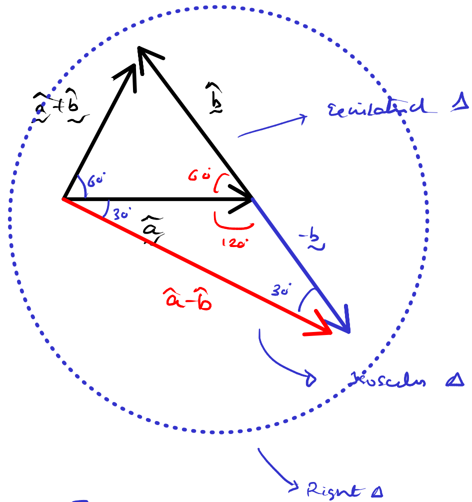
$|\underset{\sim}{a}+\underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2 \Rightarrow |\underset{\sim}{b}|^2 + 2\underset{\sim}{a}\cdot\underset{\sim}{b} = 0 \Rightarrow \boxed{\underset{\sim}{a}\cdot\underset{\sim}{b} = -\frac{|\underset{\sim}{b}|^2}{2}}$, $\underset{\sim}{a}\cdot\underset{\sim}{b} \leq 0$



3. if sum of 2 unit vectors is a unit vector

$|$diff of unit vectors$| = ?$ JEE / Boards !

translate vector into geometry :



* $|\hat{a}| = 1 = |\hat{b}| \Rightarrow |\hat{a}+\hat{b}| = 1 \Rightarrow \{\hat{a}, \hat{b}, \hat{a}+\hat{b}\}$ represent sides of Equilateral Triangle

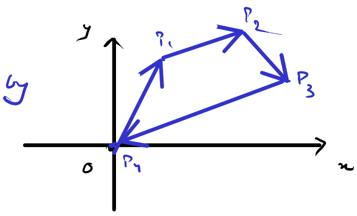
* $|\hat{a}| = 1$, $|-\hat{b}| = |\hat{b}| = 1 \Rightarrow \{\hat{a}, -\hat{b}, \hat{a}-\hat{b}\}$ rep. sides of Isosceles Δ

* $\{\hat{a}+\hat{b}, \hat{a}-\hat{b}, 2\hat{b}\}$ rep. sides of RAΔ $\Rightarrow |2\hat{b}|^2 = \underbrace{|\hat{a}+\hat{b}|^2}_{1} + |\hat{a}-\hat{b}|^2 \Rightarrow |\hat{a}-\hat{b}| = \sqrt{3}$ □

lecture-30 (27/Jan) 2

4. $\{\underbrace{P_1 P_2 P_3 P_4}_{\text{pts in a plane}}, O\}$ ↑ origin of the vector → P_4 coincides with O iff. $\boxed{\vec{OP_1} + \vec{P_1P_2} + \vec{P_2P_3} + \vec{P_3P_4} = x}$, $x = ?$

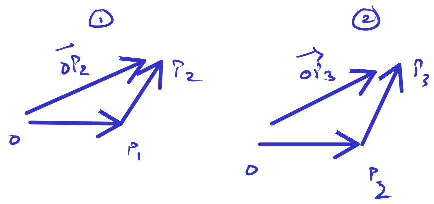
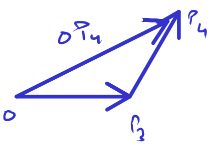
geometry spanned by the eqn is a closed geom



* $\underbrace{\underbrace{\underbrace{\vec{OP_1} + \vec{P_1P_2}}_{\vec{OP_2}} + \vec{P_2P_3}}_{\vec{OP_3}} + \vec{P_3P_4}}_{\vec{OP_4}} = \vec{OP_4}$ = pos. vect. of pt P_4 − p.v. of pt O

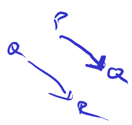
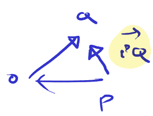
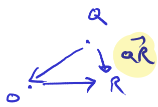
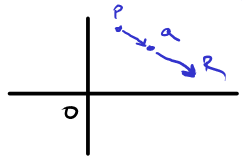
= pos " " " P_4 − " " " P_4 ($\because$ P_4 coincides O)

$= \vec{0}$

5. $\{P, Q, R, O\}$ if $\underbrace{\vec{PO} + \vec{OQ}}_{\overset{\shortparallel}{\vec{PQ}}} = \underbrace{\vec{QO} + \vec{OR}}_{\vec{QR}} \Rightarrow ?$

* $\vec{QR} = \vec{PQ} \Rightarrow$ they are in same dirn → ∥ (discarded)
↳ on same line
⇓
$\{P, Q, R\}$ Collinear

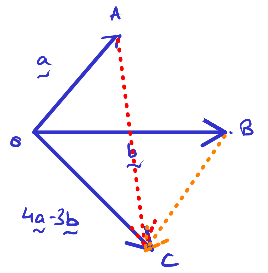





6. $\{A, B, C, O\} \longrightarrow \{\underset{\sim}{a}, \underset{\sim}{b}, 4\underset{\sim}{a} - 3\underset{\sim}{b}\}$ vectors drawn from O to the pt. A, B, C respectively

$\{\vec{AC}, \vec{BC}\} = ?$

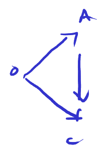
Part 1 (Algebra)

Part 2 (geom.)



* $\vec{AC}$ = P.V of C − P.V of A pt $= 3(\underset{\sim}{a} - \underset{\sim}{b})$

$\vec{BC}$ = " " " − " " B " $= 4(\underset{\sim}{a} - \underset{\sim}{b})$



$\underset{\sim}{OA} + \underset{\sim}{AC} = \underset{\sim}{OC}$

$\Downarrow$

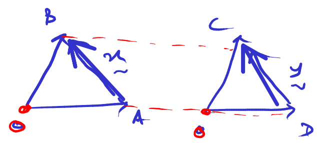
$\underset{\sim}{AC} = 4\underset{\sim}{a} - 3\underset{\sim}{b} - \underset{\sim}{a} = 3(\underset{\sim}{a} - \underset{\sim}{b})$

$\underset{\sim}{OB} + \underset{\sim}{BC} = \underset{\sim}{OC} \Rightarrow \underset{\sim}{BC} = \underset{\sim}{OC} - \underset{\sim}{OB}$

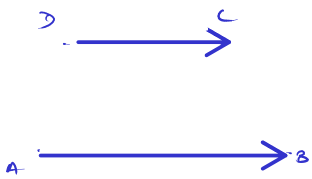
7. $\{\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}, \underset{\sim}{d}\}$ P.V → Non null vectors / Distinct

vectors from O to pts A, B, C, D resp.

if $\underset{\sim}{b} - \underset{\sim}{a} = \underset{\sim}{c} - \underset{\sim}{d} \Rightarrow$ ABCD is a .....



* $\underset{\sim}{b} - \underset{\sim}{a} = \underset{\sim}{c} - \underset{\sim}{d} \Rightarrow \underbrace{\vec{OB} - \vec{OA}}_{\vec{AB}} = \underbrace{\vec{OC} - \vec{OD}}_{\vec{DC}}$



* $\vec{AB} = \vec{DC} \Rightarrow$ they are in the same dim → $\parallel \Rightarrow AB \parallel DC$

→ Collinear (discarded) $\because$ Distinct

$\downarrow$

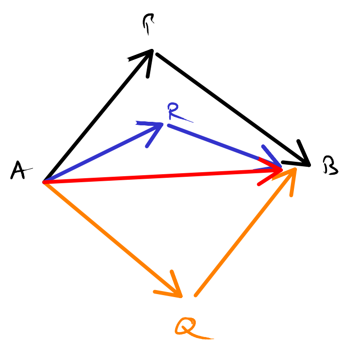
$|AB| = |DC|$

* ABCD : $\parallel^{gm}$

8. $\{A, B, P, Q, R\}$ pts in a plane

$\underbrace{\vec{AP} + \vec{AQ} + \vec{AR} + \vec{PB} + \vec{QB} + \vec{RB}}_{\vec{AB} + \vec{AB} + \vec{AB}} = ? = \underset{\sim}{3\vec{AB}}$

\# terms = 6 = 3 (Pairs)



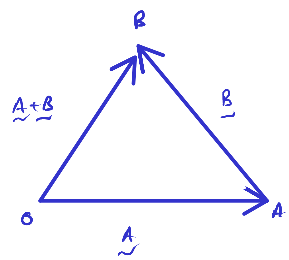
* $\vec{A_1A_2} + \vec{A_2A_3} + \vec{A_3A_4} + \vec{A_4A_5} \cdots \vec{A_{n-1}A_n} = \vec{A_1A_n}$

* $\underbrace{\vec{A_1A_2} + \vec{A_2A_3}}_{A_1A_3} + \underbrace{\vec{A_1A_4} + \vec{A_4A_3}}_{A_1A_3} + \underbrace{\vec{A_1A_5} + \vec{A_5A_3}}_{\vec{A_1A_3}} \cdots + \underbrace{\vec{A_1A_n} + \vec{A_nA_3}}_{\vec{A_1A_3}} = n(\vec{A_1A_3})$

9. $\{\underset{\sim}{A}, \underset{\sim}{B}\}$

* $|\underset{\sim}{A} + \underset{\sim}{B}| \le |\underset{\sim}{A}| + |\underset{\sim}{B}|$ Prove that!

Non-Collinear : $\{\underset{\sim}{A}, \underset{\sim}{B}\}$



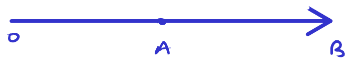
* ΔOAB : $OA + AB > OB$ 'Triangle sum of sides inequality'

$\Downarrow$

$|OA| + |AB| > |OB| \Rightarrow |\underset{\sim}{A} + \underset{\sim}{B}| < |\underset{\sim}{A}| + |\underset{\sim}{B}|$

Given $\{A, B\}$

* $OB = OA + AB \Rightarrow |OB| = |OA| + |AB| \Rightarrow |A+B| = |A|+|B|$



Master inequality

* $\boxed{|A+B| \le |A|+|B|} \xrightarrow{B \to -B} \boxed{|A-B| \le |A|+|B|}$ *

$A = (A-B)+B \Rightarrow |A| = |A-B+B| \le |A-B|+|B| \Rightarrow \boxed{|A-B| \ge |A|-|B|}$ *

10. $\{a, b\} \longrightarrow c = 3a+4b \;,\; 2c = a-3b$

linear comb^n 1   L.C.2

* if c & a same dir^n $\Rightarrow |c| \overset{>}{?} |a| \Rightarrow |c| = \frac{13}{11}|a|$ ✓

if c & b opp. dir^n $\Rightarrow |c| \underset{>}{?} |b| \Rightarrow |c| = \frac{13}{5}|b|$ ✓

* $c = 3a + 4b$

$2c = a - 3b \;\}\times 3 \Rightarrow 6c = 3a - 9b$

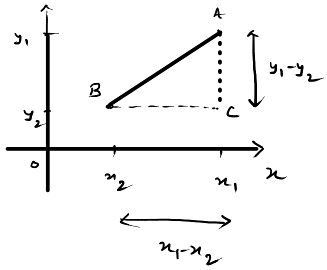
$-5c = 13b \Rightarrow c = -\frac{13}{5}b$ ✓

Lecture-31 / 8(July) 2

B.5 Non vector Coord. geometry $\{$ Ref : Grade 10/11 (Coord. geometry) Prelims $)$

* $A(x_1, y_1)$, $B(x_2, y_2)$; $\hat{e}_i \cdot \hat{e}_j = g_{ij} = \delta_{ij}$

Pythagoras $\Rightarrow$ $\boxed{(\text{dist}(AB))^2 = (x_1-x_2)^2 + (y_1-y_2)^2 \equiv |AB|^2}$ 'Distance formula'

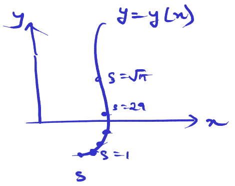


Ex1 : $A(-1,y)$, $B(5,7)$ lie on a circle $O(2,-3y)$. $y = ?$, $R = ?$

$$\sqrt{(2+1)^2 + (-3y-y)^2} = \sqrt{(2-5)^2 + (-3y-7)^2} \Rightarrow y = -1 \rightarrow R = 5$$
$$7 \rightarrow R = \sqrt{793}$$

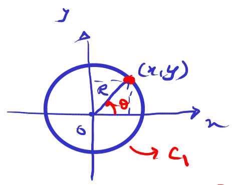
Ex2 : $P(at^2, 2at)$, $Q\left(\frac{a}{t^2}, \frac{2a}{t}\right)$, $S(a, 0)$ $\longrightarrow$ $\frac{1}{SP} + \frac{1}{SQ} = f(?) = \frac{1}{a}$ Indep. of t

$\left.\begin{array}{l} x_1 = at^2 \\ y_1 = 2at \end{array}\right\}$ Parametric eqn of Parabola



* Parametric eqn $x = x(s)$, $y = y(s)$, s = Parameter.

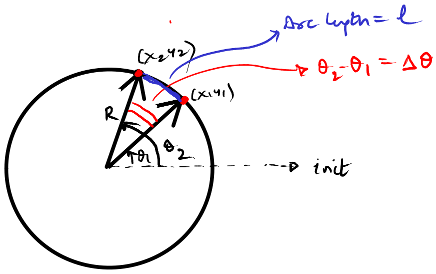
* Eqn of circle : $x^2 + y^2 = R^2 \Rightarrow y = \underbrace{\sqrt{R^2 - x^2}} \Rightarrow \boxed{y = f(x)}$



rad $= R =$ const

$y = y(x) \equiv \begin{array}{l} x = x(s) \\ y = y(s) \end{array}$

$\boxed{\text{Parameter = quantity that generates the whole curve if kept continuously/smoothly changing}}$

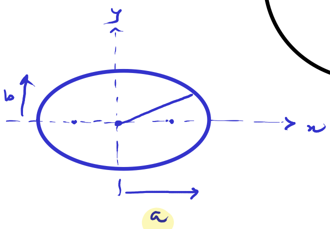


* Angle (θ) = Parameter

Param. eqn of circle : $\left.\begin{array}{l} x = R\cos\theta = x(\theta) \\ y = R\sin\theta = y(\theta) \end{array}\right\}$ $\quad \frac{x^2}{R^2} + \frac{y^2}{R^2} = 1$ ✓ $\quad \Downarrow \quad x^2 + y^2 = R^2$

* Param eqn of Ellipse : $x = a\cos\theta = x(\theta)$, $y = b\sin\theta = y(\theta)$ $\quad a = b = R$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad ✓$$



(Circum $= 2\pi R$) $\longrightarrow$ (2π)
Length $\quad$ Angle

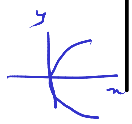
$\frac{2\pi R}{2\pi} \longrightarrow 1^c$ Angle

$\Delta\theta$ Angle $\longrightarrow \Delta\theta \cdot R = \Delta l$

$\Downarrow$

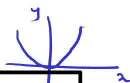
$\boxed{\Delta\theta = \frac{\Delta l}{R}}$

* Parametric eqn of Parabola : $\left.\begin{array}{l} x = at^2 = x(t) \\ y = 2at = y(t) \end{array}\right\}$ $\quad y^2 = 4ax \Rightarrow x = \frac{1}{4a} y^2$



OR

$\left.\begin{array}{l} x = 2at = x(t) \\ y = at^2 = y(t) \end{array}\right\}$ $\quad x^2 = 4ay \Rightarrow y = \frac{1}{4a} x^2 + 0x + 0$



Quadratic / Parabola

$\boxed{\text{Elimination of the parameter : will not be Easy Always! Sometimes not doable!}}$ $\Rightarrow$ usefulness of Parametric Eqn $\{$ Requires Calculus

Ex3 : Δ Equilateral $A(0,0)$, $B(3, \sqrt{3})$, $C(?)$

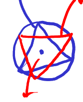
* $x^2 + y^2 = 12$, $x^2 + y^2 = 6x + 2\sqrt{3}\,y$ $\longrightarrow$ $\left.\begin{array}{l} x = 0 \Rightarrow y = 2\sqrt{3} \\ x = 3 \Rightarrow y = -\sqrt{3} \end{array}\right\}$ $C(0, 2\sqrt{3})$ or $C(3, -\sqrt{3})$

(underbrace under x^2+y^2: 12)

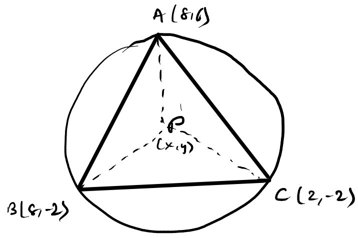
Lecture-32 (9 July) 2

Ex-4 $\Delta\{(8,6),(8,-2),(2,-2)\} \longrightarrow$ Circumcenter of Δ $(x,y) = ?$

$C_2 \neq C_1$



Circumcenter



* $AP = BP = CP = r$
* $\boxed{\text{Circumcenter of } \Delta \equiv \text{Center of Circle in which the } \Delta \text{ is embedded}}$
* Incircle = Inscribed Circle = Circle embedded in a polygon

  ॥

  tangents of a circle are sides of polygon

unique

Incenter

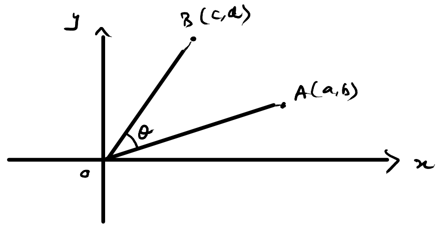
* $\overset{①}{(x-8)^2+(y-6)^2} = \overset{②}{(x-8)^2+(y+2)^2} = \overset{③}{(x-2)^2+(y+2)^2}$

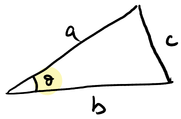
* ① = ② $\Rightarrow$ $y = 2$

  ② = ③ $\Rightarrow$ $x = 5$

  } Circumcenter $(5,2)$, Circumradius $= 5$

Ex: 5 $\cos\theta = ? = \dfrac{ac+bd}{\sqrt{a^2+b^2}\cdot\sqrt{c^2+d^2}}$





$c^2 = a^2+b^2-2ab\cos\theta$

a. Section Formula ┌ Internal
└ External

* $g_{ij} = \delta_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\hat{e}_i = (\hat{i}\ \hat{j})$

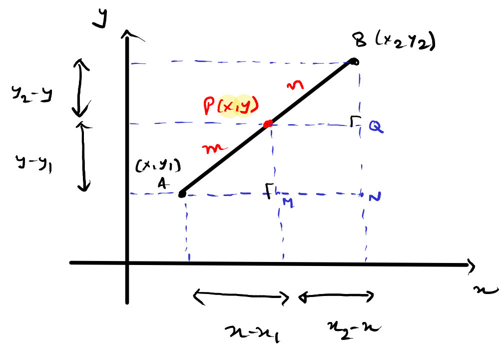
* $s^2 = \delta_{ij}x^i x^j \longrightarrow s^2 = (x_2-x_1)^2+(y_2-y_1)^2$

* ΔPMA, ΔBQP $\begin{cases} \angle PMA = \angle PQB = 90° \\ \angle PAM = \angle BPQ \end{cases}$ $\Rightarrow$ $\Delta_1 \sim \Delta_2$ $\because$ AA sim.

$\Downarrow$

$\dfrac{AM}{PQ} = \dfrac{AP}{PB} = \dfrac{PM}{BQ}$ } Ratios

$\dfrac{\sqrt{(x-x_1)^2+(y-y_1)^2}}{\sqrt{(x-x_2)^2+(y-y_2)^2}}$ Not nice !



$\dfrac{AP}{BP} = \dfrac{m}{n}$

$\Uparrow$

* $\boxed{P(x,y) \text{ cuts } AB \text{ in a ratio of } m:n}$ Demand

* $\dfrac{x-x_1}{x_2-x} = \dfrac{y-y_1}{y_2-y} = \dfrac{m}{n}$ $\Rightarrow (x-x_1)n = (x_2-x)m \Rightarrow xn - x_1 n = x_2 m - xm$

$\Downarrow$

$x = \dfrac{mx_2+nx_1}{m+n}$

$\Downarrow$

$yn - y_1 n = my_2 - ym$

$\Downarrow$

$y = \dfrac{my_2+ny_1}{m+n}$

$F = c$ Const.

$F = -kx$ Linear in x

$x^2+y^2+z^2 = R^2$

$\uparrow$

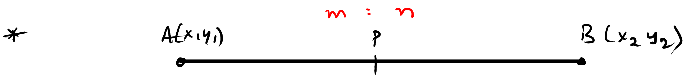
3D geometry

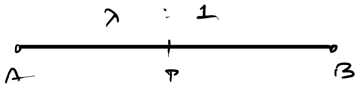
$\begin{cases} F_g = \dfrac{Gm_1m_2}{x^2} \\ F_e = \dfrac{Kq_1q_2}{x^2} \end{cases}$ Inverse squares

$F_{(weak)} = g^2 \dfrac{e^{-\lambda x}}{x}$ Exponential

$F = -k\sin x$ Sinusoidal in x

$\sum a_n x^n$

* 

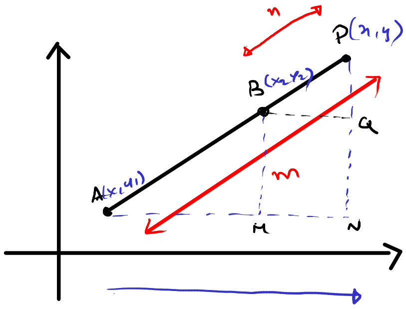


$$\boxed{P(x,y) = \left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right) = \left(\frac{\lambda x_2+x_1}{\lambda+1}, \frac{\lambda y_2+y_1}{\lambda+1}\right) \quad \lambda \equiv \frac{m}{n}}$$

Section formula "Internal"

$\lambda = 1 = \frac{m}{n} \Rightarrow P\left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$ Mid pt formula

$\frac{AB}{BP} = \frac{m}{n} \Rightarrow$



Case 2:

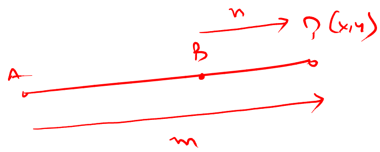
* $\boxed{\frac{AP}{BP} = \frac{m}{n}}$ P cuts AB "Externally" in m : n

demand

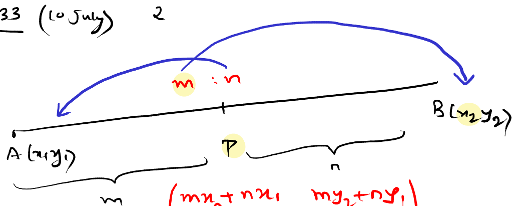
* $\Delta AMB \sim \Delta BQP \Rightarrow \frac{AM}{BQ} = \frac{BM}{PQ} = \underbrace{\frac{AB}{BP}}_{\frac{m-n}{n}}$ } + ⊥ Componendo Approach 2

Approach 1

$\frac{AM+BQ}{BQ} = \frac{BM+PQ}{PQ} = \underbrace{\frac{AP}{BP}}_{\frac{m}{n} \text{ given}} \Rightarrow \frac{x-x_1}{x-x_2} = \frac{y-y_1}{y-y_2} = \frac{m}{n} \Rightarrow \boxed{P(x,y) = \left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}\right)}$



Lecture-33 (10 July) 2

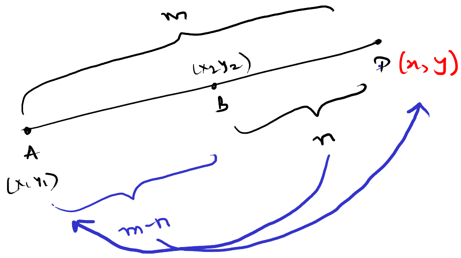
* 

$\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$

||

$\left(\frac{\lambda x_2+x_1}{\lambda+1}, \frac{\lambda y_2+y_1}{\lambda+1}\right) \quad \lambda \equiv \frac{m}{n}$

Internal "cut"



$(x_2 y_2) = \left(\frac{(m-n)x + nx_1}{m}, \frac{(m-n)y + ny_1}{m}\right)$

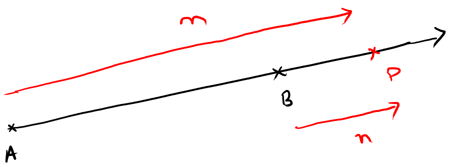
⇓

$P\left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}\right)$

||

$\left(\frac{\lambda x_2-x_1}{\lambda-1}, \frac{\lambda y_2-y_1}{\lambda-1}\right) \quad \lambda \equiv \frac{m}{n}$

External "cut"



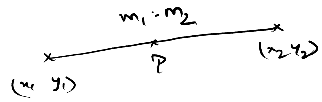
Trick:

P is the point on the ray AB : it is at 'm' from A
at 'n' from B

I owe the Mario to Rohan - Dir

Practice 2

1. $\overline{AB}$: $A(6,3)$, $B(-4,5)$, $\exists\ P(x,y)$: P cuts $\overline{AB}$ in $3:2$ internally. ?

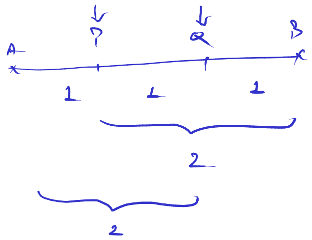


$$P\left(\frac{m_1x_2+m_2x_1}{m_1+m_2}, \frac{m_1y_2+m_2y_1}{m_1+m_2}\right)$$

$P\left(0, \frac{21}{5}\right)$

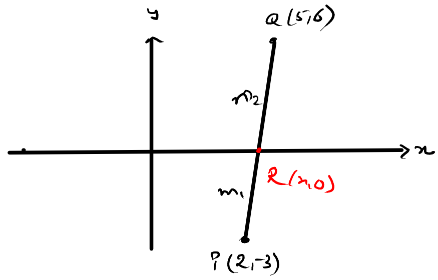
2. $\overline{AB}$: $A(1,-2)$, $B(-3,4)$, if $\overline{AB}$ is trisected $\to$ $P, Q = ?$

$P\left(\frac{-1}{3}, 0\right)$, $Q\left(\frac{-5}{3}, 2\right)$



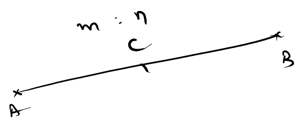
3. $\overline{PQ}$: $P(2,-3)$, $Q(5,6)$ $(m:n)$ in which $\overline{PQ}$ cuts x axis ?

$$R(x,0) = R\left(\underbrace{\frac{m_1 5 + m_2 2}{m_1+m_2}}, \frac{m_1 6 - m_2 3}{m_1+m_2}\right)$$



4. $\overline{AB}$: $A(3,5)$, $B(-3,-2)$, $C\left(\frac{3}{5}, \frac{11}{5}\right)$ cuts AB in what Ratio ?

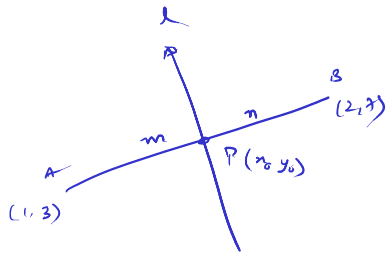
$m:n = 2:3$



5. $\overline{AB}$: $A(2,5)$, B, $C(-1,2)$ cuts int. AB in $3:4$ $B = ?$

$B(-5,-2)$

6. l : $3x+y-9=0$ line, $\overline{AB}$: $A(1,3)$, $B(2,7)$; l divides $\overline{AB}$ ratio ?

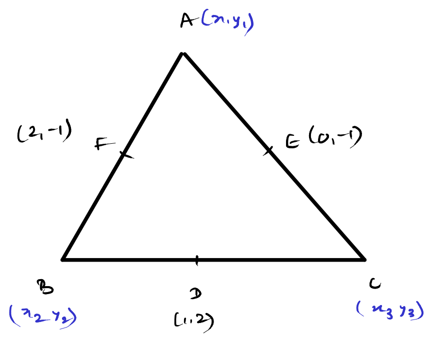


* $P(x_0,y_0)$ satisfies l : $\boxed{3x_0+y_0-9=0}$

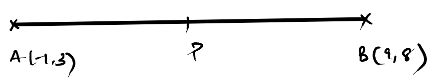
↑ unknown m, n → $\frac{3}{4}$ simplest

7. ΔABC : $\{D,E,F\}$ Midpts $\to$ A, B, C ?

* $A(1,-4)$
$B(3,2)$
$C(-1,2)$



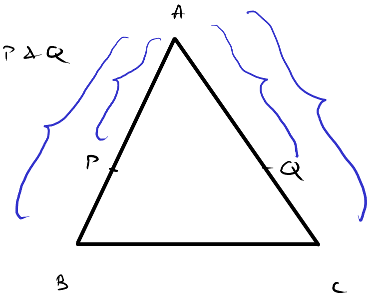
8. $\frac{AP}{BP} = \frac{K}{1}$; P is a pt lies on $x-y+2=0$ $K = ?$



Lecture-34 (11/July) 2

9. ΔABC : $A(5,5)$, $B(1,5)$, $C(9,1)$, A line is drawn to intersect AB & AC at P & Q

such that $\frac{AP}{AB} = \frac{AQ}{AC} = \frac{3}{4}$. $|PQ| = ?$



* $\frac{AP}{AB} = \frac{AQ}{AC} = \frac{3}{4} \Rightarrow \frac{AP}{AP+PB} = \frac{AQ}{AQ+QC} = \frac{3}{4} \Rightarrow 1+\frac{PB}{AP} = 1+\frac{QC}{AQ} = \frac{4}{3}$

$\frac{PB}{AP} = \frac{QC}{AQ} = \frac{1}{3} \Rightarrow \frac{AP}{PB} = \frac{AQ}{QC} = \frac{3}{1} \longrightarrow P(2,5)$

$Q(8,2)$

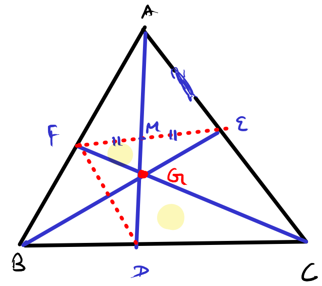
$|PQ| = \sqrt{(2-8)^2+(5-2)^2} = \sqrt{36+9} = \sqrt{45}$

6. Applications of Section Formula :

Comment (Centroid Property)

* ΔABC ; $\{AD, BE, CF\}$ medians

* $\frac{AG}{GD} = ?$



* $\boxed{FE \parallel BC \; , \; FD \parallel AC \quad \text{'Construct'}}$

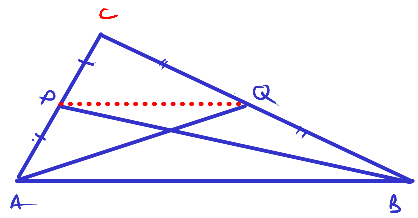
EFDC : $\parallel^{gm} \Rightarrow EF = DC$

* ΔAFE : AM is median $\Rightarrow \boxed{FM = ME}$

($\because$ AD med)

$FE = \underbrace{FM}_{FM} + \underbrace{ME}_{} \Rightarrow \underbrace{FE}_{DC} = 2(FM) \Rightarrow \boxed{DC = 2(FM)}$

HW



if P/Q is mid pt (PB, AQ median)

$\Downarrow$

$PQ \parallel AB$

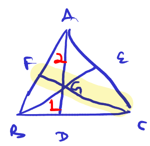
* ΔFMG , ΔCGD : $\left.\begin{array}{l} \angle FGM = \angle DGC \\ \angle GFM = \angle GCD \end{array}\right\}$ AA sim $\Rightarrow \Delta FMG \sim \Delta CDG$

$\Downarrow$

$\underbrace{\frac{FM}{CD}}_{\frac{1}{2}} = \frac{MG}{DG} = \frac{FG}{CG}$

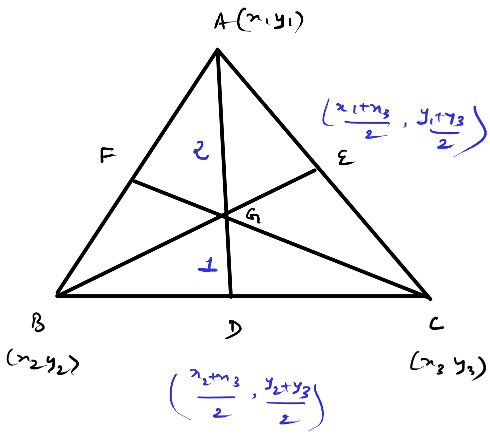
* $\frac{FG}{CG} = \frac{1}{2} \Rightarrow \boxed{\frac{CG}{FG} = \frac{AG}{GD} = \frac{BG}{GE} = \frac{2}{1}}$

'Centroid property'

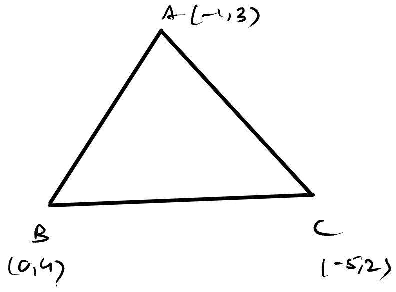


Coordinate of the centroid

* $\boxed{G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)}$



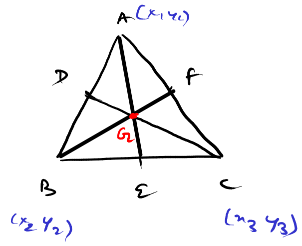
Ex1 $l : x-2y+k=0$ is a median of Δ ; $k=?$
line



$G\left(\frac{-6}{3}, \frac{9}{3}\right) = G(-2,3)$

G satisfies $l \Rightarrow -2-6+k=0 \Rightarrow \boxed{k=8}$

Ex2 : $\{D(1,1), E(2,-3), F(3,4)\}$ mid pts. - $G=?$



* $G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

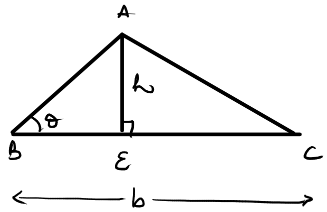
* $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = (1,1)$; $\left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}\right) = (3,4)$

$\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right) = (2,-3)$

c. Area of Triangle

1. $Ar(\Delta) = \frac{1}{2} \times b \times h$

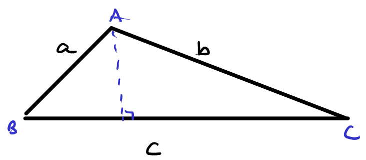
$Ar(\Delta) = \frac{1}{2}(BC)(AB\sin\theta)$



$\tan\theta = \frac{h}{BE}$

$\sin\theta = \frac{h}{AB}$

2. $\boxed{Ar(\Delta) = \sqrt{s(s-a)(s-b)(s-c)}}$ $s = \frac{a+b+c}{2}$



Derivation → Trig. : $\Delta = \frac{1}{2}(BC)(AB\sin\theta) = \frac{1}{2} ca \sin\theta$

→ pythagorean

Lecture-35 (12/July) 2

* Euclid (Elements) : 300 bce
* Archimedes : 290 bce
* Brahmagupta : 600 ce → BSS = bhrama-sputa-siddhanta = Correctly established doctrine of bhrama.

1. Role of 0
2. +/-ve #
3. $\sqrt{\ }$
4. Linear Eqn / Quadratic Eqn
5. Series

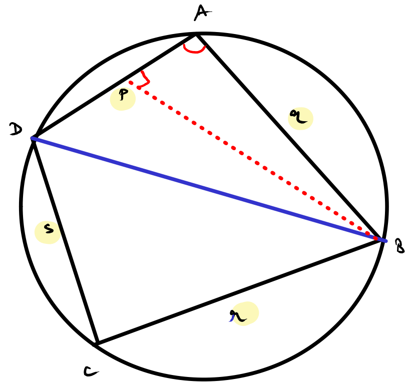
- written in verses / not symbol

2023 CE

* Sridacharya : 800 ce : $ax^2+bx+c=0 \Rightarrow x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

* Heron : 60 ce (Engineer)

3. Bhramagupta's formula (Ar of cyc. quad.)



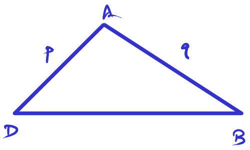
* Ar of a $\underbrace{\text{cyclic quadrilateral}}_{\text{quad. inscribed in circle}}$ (ABC) = Ar(ADB) + Ar(BDC)

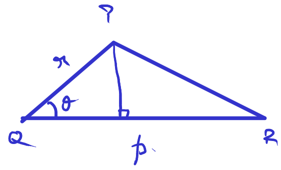
$$= \frac{1}{2}p(q\sin A) + \frac{1}{2}s(r\sin C)$$

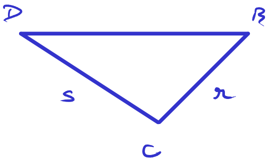
* ∵ cyclic ⇒ $\angle DAB + \angle DCB = 180°$ ⇒ $\angle A = 180 - C$ ⇒ $\sin A = \sin(180-C) = \sin C$

* $\Delta = \frac{1}{2}(pq+rs)\sin A$ ⇒ $\Delta^2 = \frac{1}{4}(pq+rs)^2 \underbrace{\sin^2 A}_{(1-\cos^2 A)}$

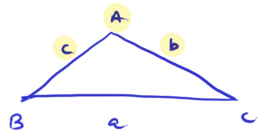
$$4\Delta^2 = (pq+rs)^2(1-\cos^2 A) = (pq+rs)^2 - (\underbrace{(pq+rs)\cos A}})^2$$







$Ar = \frac{1}{2}p\, r\sin\theta$



$a^2 = b^2 + c^2 - 2bc\cos A$

* ΔADB : $DB^2 = p^2 + q^2 - 2pq\cos A$

ΔDBC : $DB^2 = s^2 + r^2 - 2sr\cos C$ 'Cosine formula'

⇓

$$p^2 + q^2 - 2pq\cos A = s^2 + r^2 - 2sr\underbrace{\cos C}_{-\cos A}$$

{ ∵ $A = 180 - C$

$\cos A = \cos(180-C) = -\cos C$ }

$$p^2 + q^2 - s^2 - r^2 = \underbrace{2(sr+pq)\cos A}$$

Simplification of RHS needed

* $4\Delta^2 = (pq+rs)^2 - \frac{1}{4}(p^2+q^2-s^2-r^2)^2$ ⇒ $\boxed{16\Delta^2 = 4(pq+rs)^2 - (p^2+q^2-s^2-r^2)^2}$

$$(2(pq+rs))^2 - (p^2+q^2-s^2-r^2)^2 = [2(pq+rs) + (p^2+q^2-r^2-s^2)][2(pq+rs) - (p^2+q^2-r^2-s^2)]$$

$$= [p^2+q^2+2pq - (r^2+s^2-2rs)][(r^2+s^2+2rs) - (p^2+q^2-2pq)]$$

$$= [(p+q)^2 - (r-s)^2][(r+s)^2 - (p-q)^2]$$

$$= (p+q+r-s)(p+q-r+s)(r+s+p-q)(r+s-p+q)$$

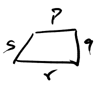
* $$\Delta^2 = \underbrace{\frac{(p+q+r-s)}{2}}_{(S-s)} \underbrace{\frac{(p+q-r+s)}{2}}_{(S-r)} \underbrace{\frac{(r+s+p-q)}{2}}_{S-q} \underbrace{\frac{(r+s-p+q)}{2}}_{(S-p)}$$

Semi perimeter

$S = \frac{p+q+r+s}{2} - p$

$S - p = \frac{q+r+s-p}{2}$

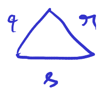
* $\boxed{\text{Ar of cyc. quad} = \sqrt{(S-p)(S-q)(S-r)(S-s)}}$



Bhramagupta's formula (general formula)

↓ $p \to 0$

$\boxed{\text{Ar of cycl. } \Delta = \sqrt{S(S-q)(S-r)(S-s)}}$

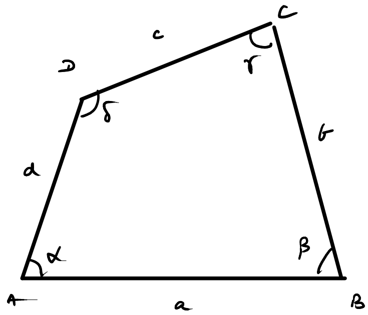


all Δs are cyclic

Heron's

HW (End of this week)

* Bretschneider's formula (1800) : Ar. of a non cyclic quad. = ?
(Carl Anton B.S.)



$$\underbrace{\text{non cyclic}}$$

$\alpha+\gamma \neq 180°$

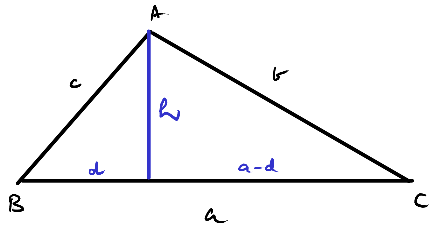
$\beta+\delta \neq 180°$

$\alpha+\beta+\gamma+\delta = 360°$

$$\left\{ \begin{aligned} \cos(\alpha+\beta) &= \cos(360-(\gamma+\delta)) \\ &= \cos(\gamma+\delta) \end{aligned} \right\}$$

4. Heron's formula

* Ar. of ΔABC = ?



Pythagorean method

* $\left.\begin{aligned} b^2 &= h^2+(a-d)^2 \\ c^2 &= h^2+d^2 \end{aligned}\right\} \Rightarrow b^2 = a^2+c^2-2ad \Rightarrow \boxed{d = \dfrac{a^2+c^2-b^2}{2a}}$

* $\Delta = \dfrac{1}{2}ah$

$$h^2 = c^2-d^2 = c^2-\left(\frac{a^2+c^2-b^2}{2a}\right)^2 = \left(\frac{c}{1}-\frac{a^2+c^2-b^2}{2a}\right)\left(c+\frac{a^2+c^2-b^2}{2a}\right)$$

$$= \frac{(2ac-a^2-c^2+b^2)(2ac+a^2+c^2-b^2)}{4a^2} = \frac{1}{4a^2}\left\{-(a^2+c^2-2ac-b^2)(a^2+c^2+2ac-b^2)\right\}$$

$$= \frac{-1}{4a^2}\left((a-c)^2-b^2\right)\left((a+c)^2-b^2\right) = \frac{1}{4a^2}\left\{\left((a+c)^2-b^2\right)\left(b^2-(a-c)^2\right)\right\}$$

$$= \frac{1}{4a^2}\left\{\underbrace{(a+c+b)}_{2s}\underbrace{(a+c-b)}_{2(s-b)}\underbrace{(b+a-c)}_{2(s-c)}\underbrace{(b-a+c)}_{2(s-a)}\right\}$$

$s = \dfrac{a+b+c}{2} - a$

$s-a = \dfrac{b+c-a}{2}$

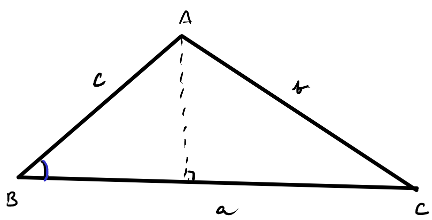
$$h^2 = \frac{4}{a^2}s(s-a)(s-b)(s-c) \qquad h = \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$$

* $\boxed{\text{Ar. of } \Delta = \sqrt{s(s-a)(s-b)(s-c)}}$

Lecture-36 (13/July) 2

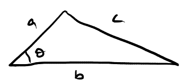
Trigonometric Method: HW

$\left\{ \begin{array}{ll} \text{NCERT: } & 9^{th} \text{ grade} \\ \text{RD} & \text{Heron's formula} \end{array} \right.$



* Ar of $\Delta = \dfrac{1}{2}ac\underbrace{\sin B}$, $b^2 = c^2+a^2-2ca\cos B$

$$\sin B = \sqrt{1-\cos^2 B} = \sqrt{1-\left(\frac{c^2+a^2-b^2}{2ac}\right)^2} = \frac{\sqrt{4a^2c^2-(a^2+c^2-b^2)^2}}{2ac}$$



$c^2 = a^2+b^2-2ab\cos\theta$

* $$\Delta = \frac{1}{4}\sqrt{4a^2c^2-(a^2+c^2-b^2)^2} = \frac{1}{4}\sqrt{(2ac+a^2+c^2-b^2)(\underbrace{2ac-a^2-c^2}_{-(a-c)^2}+b^2)}$$

$$= \frac{1}{4}\sqrt{\left((a+c)^2-b^2\right)\left(-(a-c)^2+b^2\right)}$$

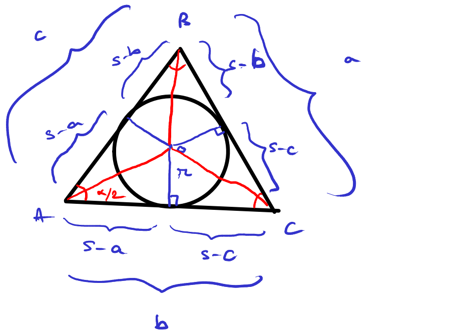
$$= \frac{1}{4}\sqrt{\underbrace{(a+c+b)}_{2s}\underbrace{(a+c-b)}_{2(s-b)}\underbrace{(b+a-c)}_{2(s-c)}\underbrace{(b-a+c)}_{2(s-a)}} , \quad s = \frac{a+b+c}{2} - a$$

$s-a = \dfrac{b+c-a}{2}$

* $\boxed{Ar. \text{ of } \Delta = \sqrt{s(s-a)(s-b)(s-c)}}$; $s = \dfrac{a+b+c}{2}$

Trick of splitting the side

Trig. Method 2 (Law of Cots)



* O : Incentre of Δ

* $s \equiv \dfrac{a+b+c}{2}$ → $2s-a-c=b \Rightarrow (s-a)+(s-c)=b$
→ $2s-a-b=c \Rightarrow (s-a)+(s-b)=c$
→ $2s-b-c=a \Rightarrow (s-b)+(s-c)=a$

* $A = \frac{1}{2}br + \frac{1}{2}ar + \frac{1}{2}cr = \frac{1}{2}(a+b+c)r = sr \Rightarrow r = \dfrac{A}{s}$

$= 3sr - 2rs = \underbrace{3rs}_{rs+rs+rs} - r(a+b+c) = r(s-a)+r(s-b)+r(s-c)$

$= r^2\left\{\underbrace{\frac{s-a}{r}}_{\cot\alpha/2} + \frac{s-b}{r} + \frac{s-c}{r}\right\} = r^2\left(\cot\frac{\alpha}{2} + \cot\frac{\beta}{2} + \cot\frac{\gamma}{2}\right)$

$P = 6s - 2\overbrace{(a+c+b)}^{2s}$
$= 2s$

$\cot\alpha/2 = \dfrac{s-a}{r}$

$\cot\beta/2 = \dfrac{s-b}{r}$

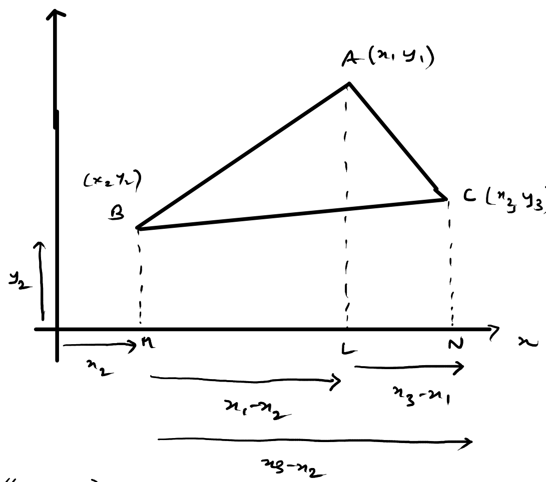
$\cot\gamma/2 = \dfrac{s-c}{r}$

* $\boxed{\alpha+\beta+\gamma = \pi \Rightarrow \underbrace{\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2} = \frac{\pi}{2}}_{\text{Constraint}}}$ → $\underbrace{\cot\frac{\alpha}{2} + \cot\frac{\beta}{2} + \cot\frac{\gamma}{2}}_{\text{Sums}} = \underbrace{\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}}_{\text{products.}}$ Trig. HW

* $A = r^2\left(\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}\right) = r^2\left\{\frac{(s-a)(s-b)(s-c)}{r^3}\right\} = \frac{(s-a)(s-b)(s-c)}{r} = \frac{(s-a)(s-b)(s-c)}{A/s}$

$A^2 = s(s-a)(s-b)(s-c) \Rightarrow \boxed{A = \sqrt{s(s-a)(s-b)(s-c)}}$

5. Descartes' formula / Determinant formula



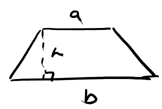
* $Ar(\Delta ABC) = Ar(ABML) + Ar(ACNL) - Ar(BCNM)$

$= \frac{1}{2}\left\{\underbrace{(BM+AL)}_{y_2+y_1} ML + \underbrace{(AL+CN)}_{y_1+y_3} LN - \underbrace{(BM+CN)}_{y_2+y_3} MN\right\}$

$= \frac{1}{2}\left\{(y_1+y_2)(x_1-x_2) + (y_1+y_3)(x_3-x_1) - (y_2+y_3)(x_3-x_2)\right\}$

$= \frac{1}{2}\left\{\cancel{y_1x_1} - y_1x_2 + y_2x_1 - \cancel{y_2x_2} + y_1x_3 - \cancel{y_1x_1} + \cancel{y_3x_3} - y_3x_1 - y_2x_3 + \cancel{y_2x_2} - \cancel{y_3x_3} + y_3x_2\right\}$

$= \frac{1}{2}\left\{x_1(y_2-y_3) + x_2(y_3-y_1) + x_3(y_1-y_2)\right\}$



$A = \frac{1}{2}(a+b)h$

* $\boxed{\Delta = \frac{1}{2}\left| x_1(y_2-y_3) + x_2(y_3-y_1) + x_3(y_1-y_2)\right|}$ 'Cartesian formula for Δ'

Absolute value
‖
Mod. ⇒ scalar comp. of the Area

Area is actually vector

$\underset{\sim}{A} = A^i \underset{\sim}{e}_i$

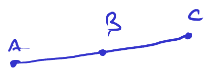
Caution!?

* $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\left\{x_1(y_2-y_3) - \underbrace{y_1(x_2-x_3)}_{-y_1x_2+y_1x_3} + (x_2y_3 - y_2x_3)\right\} = \frac{1}{2}\left\{x_1(y_2-y_3) + x_2(y_3-y_1) + x_3(y_1-y_2)\right\}$

App. of determinant

* A, B, C : Collinear ⇒ $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \Leftrightarrow Ar(\Delta) = 0 \leftrightarrow \boxed{AC = AB + BC}$

if they lie on the same line

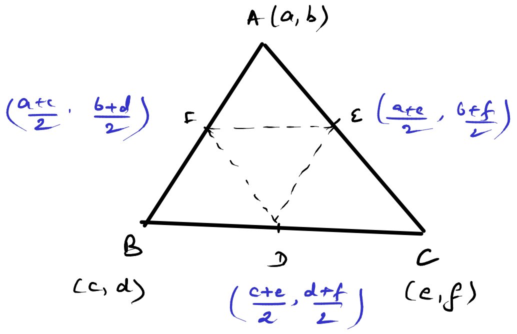


Lecture-37 (14|July) 2

Ex1: $\triangle ABC$: $A(t, t-2)$, $B(t+2, t+2)$, $C(t+3, t) \longrightarrow Ar(\Delta) = ?$

$$* \quad \Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} t & t-2 & 1 \\ t+2 & t+2 & 1 \\ t+3 & t & 1 \end{vmatrix} = |-4| = 4$$

Ex:2 $\dfrac{Ar(\Delta ABC)}{Ar(\Delta DEF)} = ?$: $\{D, E, F\}$ mid pts



$$* \quad \Delta_1 = \frac{\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}}{\begin{vmatrix} \frac{a+c}{2} & \frac{b+d}{2} & 1 \\ \frac{a+e}{2} & \frac{b+f}{2} & 1 \\ \frac{c+e}{2} & \frac{d+f}{2} & 1 \end{vmatrix}} = \frac{N}{D}$$

D → Simplification needed.

Detour to Determinants :

$$* \quad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei-fh) - b(di-fg) + c(dh-eg)$$

$\downarrow$ $R_i \to kR_i$ / $C_i \to kC_i$

$$\boxed{\begin{vmatrix} ka & kb & kc \\ d & e & f \\ g & h & i \end{vmatrix} = k\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$$

1.

$$* \quad \begin{vmatrix} \frac{a+c}{2} & \frac{b+d}{2} & 1 \\ \frac{a+e}{2} & \frac{b+f}{2} & 1 \\ \frac{c+e}{2} & \frac{d+f}{2} & 1 \end{vmatrix} \longrightarrow \frac{1}{2}\cdot\frac{1}{2}\begin{vmatrix} a+c & b+d & 1 \\ a+e & b+f & 1 \\ c+e & d+f & 1 \end{vmatrix}$$

Need more simplification

$$* \quad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei-fh) - b(di-fg) + c(dh-eg)$$

$\downarrow$

$$\begin{vmatrix} a+\alpha & b+\beta & c+\gamma \\ d & e & f \\ g & h & i \end{vmatrix} = (a+\alpha)(\quad) - (b+\beta)(\quad) + (c+\gamma)(\quad)$$

$$= \underbrace{a(\quad) - b(\quad) + c(\quad)}_{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} + \underbrace{\alpha(\quad) - \beta(\quad) + \gamma(\quad)}_{\begin{vmatrix} \alpha & \beta & \gamma \\ d & e & f \\ g & h & i \end{vmatrix}}$$

2.

* $\begin{vmatrix} a+kb & b & c \\ d+ke & e & f \\ g+kh & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + k \begin{vmatrix} b & b & c \\ e & e & f \\ h & h & i \end{vmatrix}$ 3.

* $\begin{vmatrix} a & b & c \\ a & b & c \\ d & e & f \end{vmatrix} = a(\quad) - b(\quad) + c(\quad) = 0$ 'can expand from any corner!' 4.

$= d(\cancel{bc}-\cancel{bc}) - a(\cancel{ec}-\cancel{bf}) + a(\cancel{ec}-\cancel{bf}) = 0$

* A matrix $\longrightarrow$ $|A|$ $\longrightarrow$ $|kA| = k^n|A|$ one k from each row. 5

* $\begin{vmatrix} \frac{a+c}{2} & \frac{b+d}{2} & 1 \\ \frac{a+e}{2} & \frac{b+f}{2} & 1 \\ \frac{c+e}{2} & \frac{d+f}{2} & 1 \end{vmatrix} = \frac{1}{2}\cdot\frac{1}{2}\left\{ \begin{vmatrix} a & b+d & 1 \\ a & b+f & 1 \\ c & d+f & 1 \end{vmatrix} + \begin{vmatrix} c & b+d & 1 \\ e & b+f & 1 \\ e & d+f & 1 \end{vmatrix} \right\}$

$= \frac{1}{4}\left\{ \underbrace{\begin{vmatrix} a & b & 1 \\ a & b & 1 \\ c & d & 1 \end{vmatrix}}_{R_1=R_2 \;\leftarrow 0 \rightarrow} + \underbrace{\begin{vmatrix} a & d & 1 \\ a & f & 1 \\ c & f & 1 \end{vmatrix}}_{(fc-\cancel{af})-(dc-\cancel{af})+(da-af)} + \underbrace{\begin{vmatrix} c & b & 1 \\ e & b & 1 \\ e & d & 1 \end{vmatrix}}_{(be-cd)-(b\cancel{e}-cd)+(\cancel{be}-cb) \leftarrow 0 \rightarrow} + \begin{vmatrix} c & d & 1 \\ e & f & 1 \\ e & f & 1 \end{vmatrix} \right\}$

* $\dfrac{\Delta = \begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}}{D} = \dfrac{(cf-ed)-\cancel{(be-af)+(ad-cb)}}{\frac{1}{4}\left(cf-\cancel{dc}\cancel{+af+ad}\right)+\left(be-ed+\cancel{cd}-cb\right)} = \frac{4}{1}$

HW

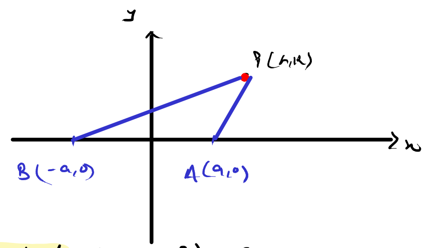
* $A(3,4)$, $B(5,-2)$; find P ? : $PA = PB$ & Ar$(PAB) = 10$ $\left\{ \begin{matrix} P(7,2) \\ P(1,0) \end{matrix} \right.$ $\quad |x| = 30 \Rightarrow x = \pm 30$

* $A(6,3)$, $B(-3,5)$, $C(4,-2)$; $P(x,y)$ $\longrightarrow$ $\dfrac{Ar(\Delta PBC)}{Ar(\Delta ABC)} = \left|\dfrac{x+y-2}{7}\right|$

Lecture-38 (15 July) 2

C. Locus

* $P(h,k)$ moving pt. in $\mathbb{R}^2$ $\exists$ ∞ Possible paths/trajectories/configurations/tracing
* $P(h,k)$ moving pt : $\exists$ $\underbrace{\text{'Constraint Eq}^n\text{'}}_{\text{Rel}^n \text{ b/w coordinates}}$ $\Rightarrow$ $\boxed{\exists \text{ fixed singular path} \equiv \text{LOCUS (Particular) "Location"}}$



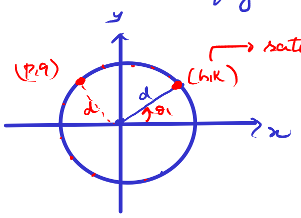
Ex1 $P(h,k)$: sum of the squares of the distances of P from 2 fixed pts $A(a,0)$, $B(-a,0)$ is constant qty $(=2c^2)$. Eq^n of locus ?

* $(PA)^2 + (PB)^2 = 2c^2$. 'Constraint Eq^n'

$\rightarrow \{(h-a)^2+k^2\}+\{(h+a)^2+k^2\} = 2c^2 \Rightarrow h^2+k^2+a^2=c^2 \Rightarrow h^2+k^2 = c^2-a^2 = d^2 \Rightarrow \boxed{h^2+k^2=d^2}$ at particular instant when at (h,k)

$\boxed{(h,k) \xrightarrow{\text{replaced}} (x,y) \Rightarrow \text{Eq}^n \text{ of locus}}$ Prescription for Eq^n of locus.

* $x^2+y^2=d^2=\text{const}$ Eq^n of circle

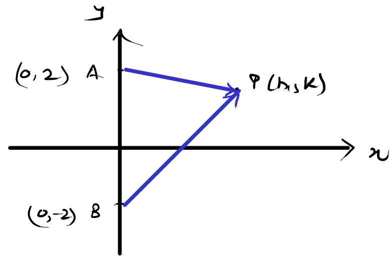


$\rightarrow$ satisfies $h^2+k^2=d^2$ (At θ_1)

$p^2+q^2=d^2$ (At θ_2)

$\vdots$

Ex-2 $P(h,k)$: $A(0,2)$, $B(0,-2)$ $\longrightarrow$ sum of the distances from P to A&B is Const $(=6)$. Eq^n of locus ?



* $PA+PB=6 \Rightarrow \sqrt{h^2+(k-2)^2}+\sqrt{h^2+(k+2)^2}=6$

$\cancel{h^2}+(k-2)^2 = 36+\cancel{h^2}+(k+2)^2-2\cdot 6\cdot\sqrt{h^2+(k+2)^2}$

$\cancel{k^2}+\cancel{4}-4k = 36+\cancel{k^2}+\cancel{4}+4k-12\sqrt{h^2+(k+2)^2}$

$\cancel{12}^{3}\sqrt{h^2+(k+2)^2} = \cancel{36}^{9}+\cancel{8}^{2}k \Rightarrow 9h^2+5k^2=45 \longrightarrow 9x^2+5y^2=45$

$\frac{x^2}{5}+\frac{y^2}{9}=1 \Rightarrow \boxed{\frac{x^2}{(\sqrt5)^2}+\frac{y^2}{3^2}=1}$ Eq^n of locus = Ellipse

$\sqrt{x}+\sqrt{y}=3$ sep. of variables

$x+y+2\sqrt{xy}=9$

Complexity is still there

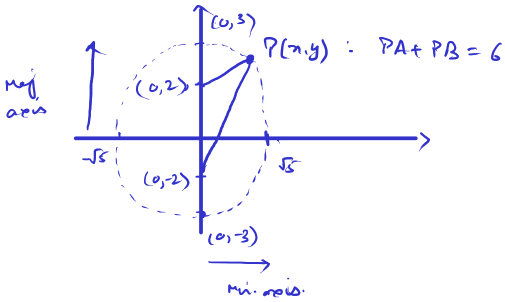
$\sqrt{x}=3-\sqrt{y}$

$x=9+y-6\sqrt{y}$

$\sqrt{y}=\frac{y-x+9}{6}$

$y=\left(\frac{y-x+9}{6}\right)^2$

Curve sketching: $\sqrt5 = 2.2$



Lecture-39 (16 July) 2

Comment on Ancient view of Locus :

* Kaṇāda (6 bce-2 bce) - physicist

Main goal = what exists ?

* Creator of school of Thought = Vaiśeshika

Dharma = Property/quality — Inheres in the Dharmin

Dharmin = Property-possessor/ Supporter/ Locus ≡ WRT/ Reference frame

Property Red

Apple possessor

* fire on a mountain ≡ Mountain possesses the fire

$\underbrace{\text{fire}}_{\text{Dharma}}$ on a $\underbrace{\text{mountain}}_{\text{Dharmin}}$

* Pen in Hand ≡ Hand possesses the pen

$\underbrace{\text{Pen}}_{\text{Dharma}}$ in $\underbrace{\text{Hand}}_{\text{Dharmin}}$

* $\underset{\text{Dharma}}{\underbrace{\text{Blue}}}$ $\underset{\text{Dharmin}}{\underbrace{\text{pen}}}$ ≡ Color blue is supported by the pen
  (Inheres in)

* Red ball rolling on the ground
  → $\overset{\text{Dharma}}{\overbrace{\text{Red color}}}$ is supported by $\overset{\text{Dharmin}}{\overbrace{\text{the ball rolling}}}$ on the ground
  → $\underset{\text{Dharma}}{\underbrace{\text{Rolling}}}$ on the ground is supported by the $\underset{\text{Dharmin}}{\underbrace{\text{Red ball}}}$

* Sound is impermanent (Dogmatic eg.) = there is impermanence in sound (Dog. Eng.) = $\underset{\text{Dharmin}}{\underbrace{\text{Sound}}}$ is a possessor of $\underset{\text{Dharmin}}{\underbrace{\text{impermanence}}}$
  (Kanada)
  ||
  Impermanence inheres in the sound

---

* $P(h,k)$: moving pt : ∃ Constraint Eq^n (= Property) ⇒ Path of P = Locus (= Property-possessor)

  Eg: $|PO| = r$ — Eg: $\underbrace{\text{Circle}}$

  {Relationship b/w the point & locus} — property possessor

Ex 3:- $P(h,k)$; $A(ae,0)$, $B(-ae,0)$ ⟶ $PA + PB = 2a$ ⟶ locus (Eq^n)

Property = (Complex relationship) — property possessor = ?

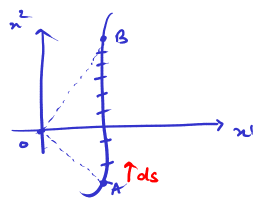
* if property = distance function
  ⇓
  property possessor = trajectory / locus / path
  $\underbrace{\qquad}$ ~ dynamics

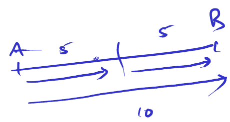
$\underbrace{\text{Distance function}}$ = const

$s^2 = \sum_a \sum_b g_{ab} x^a x^b$ ⟶ $g_{ab} = \delta_{ab} = \begin{cases} 1 & a=b \\ 0 & a \neq b \end{cases}$

distance prescription
⇓
$s = |AB| = \sqrt{\sum_{ab} g_{ab} x^a x^b}$ ⟶ $\left(\frac{s}{n}\right)$: dist b/w A & n^{th} div.

total dist b/w A & B





$$\underbrace{ds + ds + \ldots \infty}_{\infty \text{ sums.}} = s$$
↑ finite

$\boxed{\lim_{n\to\infty} \left(\frac{s}{n}\right) \to 0}$ → distance can't be zero

Infinitely small
||
∃ infinitesimal = ds
(later Calculus!)

⇓
$\sum_{i=1}^{\infty} ds_i = s$

$\Delta s = s_2 - s_1$
finite difference

* $\sqrt{(h-ae)^2 + k^2} + \sqrt{(h+ae)^2 + k^2} = 2a$

$(h-ae)^2 + \cancel{k^2} = 4a^2 + (h+ae)^2 + \cancel{k^2} - 4a\sqrt{(h+ae)^2 + k^2}$

$\cancel{h^2} + \cancel{a^2e^2} - 2hae$ $\quad$ $\cancel{h^2} + \cancel{a^2e^2} + 2hae$

$4a\sqrt{(h+ae)^2 + k^2} = 4a^2 + 4hae$

$\sqrt{(h+ae)^2 + k^2} = a + he$

$(h+ae)^2 + k^2 = a^2 + h^2e^2 + \cancel{2hae}$

$h^2 + a^2e^2 + \cancel{2hae}$

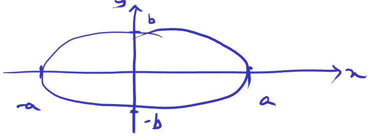
$h^2(1-e^2) + k^2 = a^2(1-e^2)$ ⇒ $h^2 + \frac{k^2}{1-e^2} = a^2$ ⇒ $\frac{h^2}{a^2} + \frac{k^2}{a^2(1-e^2)} = 1$ $\xrightarrow[b^2 = a^2(1-e^2)]{}$ $\frac{h^2}{a^2} + \frac{k^2}{b^2} = 1$

$b^2 = a^2(1-e^2)$ ⇒ $b^2 = a^2 - a^2e^2$ ⇒ $b^2 + a^2e^2 = a^2$ ⇒ $a^2e^2 = a^2 - b^2$

$$\boxed{e = \sqrt{1 - \frac{b^2}{a^2}}}$$

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$ Ellipse = Property possessor

* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $b^2=a^2(1-e^2)$



$e=\sqrt{1-\frac{b^2}{a^2}}$ $\quad 1>\frac{b^2}{a^2} \Rightarrow a^2>b^2$

$\downarrow$

Eccentricity of the ellipse

$e \not> 1$

$e<1$

$e=1 \longrightarrow \frac{b^2}{a^2}=0$ contradicting.

$e=0 \longrightarrow a=b$ (Circle)

Ex:4 $P(h,k)$: $PA=PB$ Constraint eqn $\longrightarrow$ Locus ? $A(1,3)$ $B(-2,1)$

$6x+4y=5$ {St. line

HW

* $l_1: x\cos\alpha+y\sin\alpha=a$
  $l_2: x\sin\alpha-y\cos\alpha=b$ } find the locus of the pt. of intersection of l_1, l_2
  α: variable

* Rod length l, slides b/w 2 $\perp$ lines. find the locus of its mid pt ?

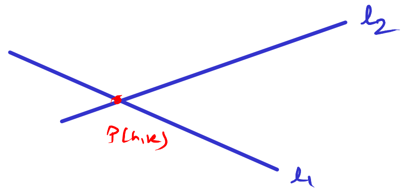
* AB is variable line sliding b/w axes : A lies on x axis
  B " " y axis

  P is a var. pt on AB : $PA=b$, $PB=a$, $AB=a+b$. find locus of P

* Q is a var. pt on $x^2=4y$, O : origin find the locus of mid pt of OQ ?

Lecture-40 (17/July) 1.5

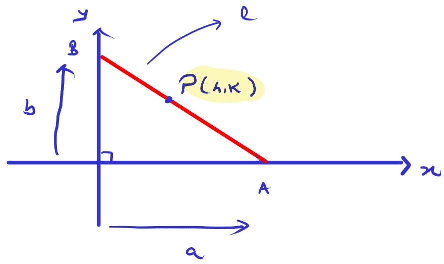
* $l_1: x\cos\alpha+y\sin\alpha=a$
  $l_2: x\sin\alpha-y\cos\alpha=b$ } (h,k) satisfies both eqn



$\Downarrow$

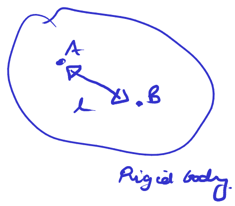
$h\cos\alpha+k\sin\alpha=a$
$h\sin\alpha-k\cos\alpha=b$ } $\longrightarrow h^2+k^2=a^2+b^2 \longrightarrow x^2+y^2=\underbrace{a^2+b^2}_{=c^2}$ Eqn of a $\underbrace{\text{circle}}_{\text{locus}}$

* Rod length, $l=|AB|$ ✓
  sliding $\rightarrow A(a,0)$, $B(0,b)$
  $P\left(\frac{a}{2},\frac{b}{2}\right)=P(h,k)$



* $\underbrace{AB^2}_{\text{Const}}=OB^2+OA^2 \Rightarrow a^2+b^2=l^2 \longrightarrow (2h)^2+(2k)^2=l^2 \longrightarrow x^2+y^2=\underbrace{\frac{l^2}{4}}_{c^2}$

Eqn of Circle $|AB|=l$

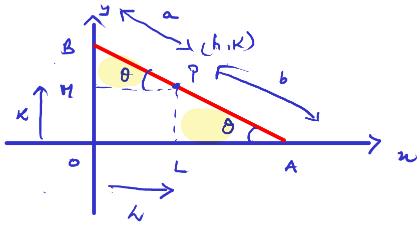


* AB is variable line sliding b/w axes : A lies on x axis
  B " " y axis

  P is a var. pt on AB : $PA=b$, $PB=a$, $AB=a+b$. find locus of P

* $\sin\theta = \frac{k}{b}$, $\cos\theta = \frac{h}{a}$ $\rightarrow$ $P(a\cos\theta,\ b\sin\theta)$

Locus.



$$\underbrace{\sin\theta = \frac{k}{b}, \quad \cos\theta = \frac{h}{a}}$$

$$\downarrow$$

$$\frac{h^2}{a^2} + \frac{k^2}{b^2} = 1 \longrightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Eqⁿ of Ellipse

Contd. in math 7-12 cartesian geometry volume 2