Number theory - Vol 3 : Complex #, seq.n series, combinatorics

Lecture-1 (26/Jul) 2.5

1. Complex Numbers (Basics / fundamentals / Mechanical tools)

A. Representations of C.N

A.1 Field ("set")

* Field ≡ a set F w/ +, • (Binary op.) satisfies field axioms
  - closure
  - Associative
  - Commutative
  - ∃ Identity
  - ∃ Inverse
  - distributive

$a, b \in F \longrightarrow a+b \in F$, $a \cdot b \in F$ — closure = "closed sys"

Representations
- Algebraic
  - set / group
  - Numbers
  - Matrices
  - Abstract
  - operators
- geometric
  - Axes
  - origin / coord.

Problems
- $x^2 = 1 \Rightarrow x = \pm 1$
- $x^2 = -1 \Rightarrow x = \pm i$
- $x^2 = 0 \Rightarrow x = 0$
- $x = ?$

Inventions
- -ve #
- sota

Classical number fields

	element	fundamental objects	
• $\mathbb{R}$	a	1	commutative; ← → on a line through 0; ∃ well defined ordering $>$ $<$ $=$
• $\mathbb{C}$	$z = a + ib$ (invented: i), $a, b \in \mathbb{R}$	$1, i$	∄ well defined ordering; ordering is lost; commutative; $\boxed{\text{Problem of } x^2 = -1 \text{ Solved}}$

A.2. vector representation (Dirac's notation)

* $|\psi\rangle = \psi = a + ib \in \mathbb{C}$

  $\langle\psi| = \psi^* = a - ib$ Complex conjugate

* geometrical ∃ Axes / origin

Bra|Ket
$| \rangle$ $\langle |$
$\langle | \rangle$
$[|]$

A.3 Matrix representation (Heisenberg's)

* $\psi = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in \mathbb{C} \iff \psi = a + ib$

set $a = 0$, $b = 1$

$$\boxed{i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}$$

* for Calculation purposes

Always used in "Mixed" for all practical purposes

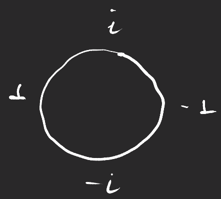
$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

Bridge

3. Algebraic Treatment of C#

3.1 Higher Powers of i computation

* $i = \sqrt{-1}$ $\quad i^2 = -1$, $i^3 = -i$, $\boxed{i^4 = 1}$ => Periodicity in CN



* $\psi = i^n$: $n = \underbrace{4m}_{\text{division by } 4} + r$ $\quad 0 \le r < 4$

$$\boxed{\psi = i^{4m+r} = \underbrace{(i^4)^m}\cdot i^r = i^r}$$

* $i^{-1} = \frac{1}{i} = \frac{i}{i^2} = -i$

$i^{-2} = \frac{1}{i^2} = -1$

$i^{-3} = \frac{1}{i^3} = \frac{1}{-i}\cdot\frac{i}{i} = i$

$i^{-4} = \frac{1}{i^4} = 1$

$i^{-5} = \frac{1}{i^5} = \frac{1}{i}\cdot\frac{i}{i} = -i$ ..... $i^{-n} = \frac{1}{i^{4m+r}} = \frac{1}{i^r}$

Q1. 1. $i^{-999} = i$

2. $\left(i^{18} + \frac{1}{i^{24}}\right)^3 = 0$

3.2 CN as a field (Axioms)

* $\mathbb{C}$: field satisfies axioms $\underbrace{\{+, \bullet\}}$

Composition rule

→ $+$:
- ∃ Identity 0 $\quad \psi + 0 = \psi$
- ∃ Inverse $-\psi$: $\psi + (-\psi) = 0$
- Associative : $\psi_1 + (\psi_2 + \psi_3) = (\psi_1 + \psi_2) + \psi_3$
- Commutative : $\psi_1 + \psi_2 = \psi_2 + \psi_1$

→ $\bullet$:
1. ∃ Identity 1 $\quad \psi \cdot 1 = \psi$
2. $\boxed{\exists \text{ Inverse } \psi^{-1}}$: $\psi \bullet (\psi^{-1}) = 1$ $\quad$ Pair ⟶ needs to relearn
3. Associative : $\psi_1 \cdot (\psi_2 \cdot \psi_3) = (\psi_1 \cdot \psi_2)\psi_3$
4. Commut. : $\psi_1 \cdot \psi_2 = \psi_2 \cdot \psi_1$

* $\forall \psi \;\exists\; \psi^{-1}$: $\psi \circ (\psi^{-1}) = 1$

* given $\psi = a + ib$, $\psi^{-1} = x + iy$ (to be calculated)

$\psi \cdot \psi^{-1} = 1 \Rightarrow (a+ib)(x+iy) = 1 \Rightarrow ax + iay + ibx + \underbrace{i^2}_{-1} by = 1$

$(ax - by) + i(ay + bx) = 1$

Compare $\mathrm{Re}\,\psi$, $\mathrm{Im}\,\psi$

$\mathrm{Re}\,\psi \Rightarrow ax - by = 1$

$\mathrm{Im}\,\psi \Rightarrow bx + ay = 0$

$x = \frac{a}{a^2+b^2} \in \mathrm{Re}\,\psi^{-1}$ $\qquad y = \frac{-b}{a^2+b^2} \in \mathrm{Im}\,\psi^{-1}$

* $\boxed{\psi = \underbrace{a+ib} \longrightarrow \psi^{-1} = \frac{a-ib}{a^2+b^2}}$ • "Inverse computation is hard"

B3. New Concepts Introduced: Modulus, Conjugate

* $\psi = a+ib \in \mathbb{C} \longrightarrow |\psi| = \sqrt{a^2+b^2} = \sqrt{(\mathrm{Re}\,\psi)^2 + (\mathrm{Im}\,\psi)^2} \in \mathbb{R}$ Postulate

* $\psi^{-1} = \frac{a-ib}{a^2+b^2} \Rightarrow \psi^{-1} = \frac{a-ib}{|\psi|^2} \Rightarrow \psi^{-1}|\psi|^2 = a-ib$ } multiply ψ from left

∵ multiplication is endowed

$\underbrace{\psi\psi^{-1}}_{1}|\psi|^2 = \psi(a-ib) \Rightarrow \underbrace{|\psi|^2}_{\text{real}} = \psi\underbrace{(a-ib)}_{\psi^*}$

∵ $\psi\psi^{-1} = \psi^{-1}\psi = 1$

$\boxed{|\psi|^2 = \psi\psi^* \qquad \psi^* = a-ib}$

Complex cojugate

Pen relief (Trick)

* $\psi = a+ib \qquad \psi^{-1} = \frac{1}{a+ib} \times \frac{a-ib}{a-ib} = \frac{a-ib}{a+b^2}$

$\boxed{\psi^{-1} = \frac{1}{\psi} \cdot \frac{\psi^*}{\psi^*} = \frac{\psi^*}{|\psi|^2}}$

B4 Division of 2 C#

* $\psi_1 = a_1+ib_1 \quad , \psi_2 = a_2+ib_2$

$\frac{\psi_1}{\psi_2} = \frac{\psi_1}{\psi_2} \times \frac{\psi_2^*}{\psi_2^*} = \frac{\psi_1\psi_2^*}{|\psi_2|^2} = X+iY$

B5. Properties to manipulate Modulus & Conjugate $\psi = a+ib \;,\; \psi^* = a-ib \;,\; |\psi|^2 = \psi\psi^* = \sqrt{a^2+b^2}$ $i \to -i$

CONJUGATE

1. $(\psi^*)^* = \psi$

∵ $|\psi|^2 = \psi\psi^* \Rightarrow \underbrace{(|\psi|^2)^*}_{|\psi|^2} = (\psi\psi^*)^* = \psi^*(\psi^*)^* \Rightarrow (\psi^*)^* = \frac{|\psi|^2}{\psi^*} = \psi$

2. $\psi + \psi^* = 2\mathrm{Re}(\psi)$

$\psi - \psi^* = 2i\,\mathrm{Im}(\psi)$

3. $\psi = \psi^* \Rightarrow$ Purely real ψ is Hermitian

∵ $a+ib = a-ib \Rightarrow 2ib = 0 \Rightarrow b=0 \Rightarrow \psi = a$

4. $\psi = -\psi^* \Rightarrow$ Purely Img. ψ is Anti-Hermitian

$a+\cancel{ib} = -a+\cancel{ib} \Rightarrow a=0$

5. $\psi\psi^* = |\psi|^2$

6. $(\psi_1 \pm \psi_2)^* = \psi_1^* \pm \psi_2^*$

MODULUS

1. $|\psi| = 0 \Rightarrow \psi = 0$

$\because |\psi| = 0 = \sqrt{a^2+b^2} \Rightarrow a = 0, b = 0$

2. $|\psi^*| = \psi$, $|-\psi| = \psi$

$\because |\psi^*| = \sqrt{(\underbrace{\mathrm{Re}(\psi^*)}_{(\mathrm{Re}\psi)})^2 + (\underbrace{\mathrm{Im}(\psi^*)}_{(\mathrm{Im}\psi)})^2} = |\psi|$

$|-\psi| = \sqrt{(\underbrace{\mathrm{Re}(-\psi)}_{(-\mathrm{Re}\psi)^2})^2 + (\underbrace{\mathrm{Im}(-\psi)}_{(-\mathrm{Im}\psi)^2})^2} = \psi$

3. $-|\psi| \leq \mathrm{Re}\psi \leq |\psi|$, $-|\psi| \leq \mathrm{Im}\psi \leq |\psi|$

$\because \mathrm{Re}(\psi) = a$, $\mathrm{Im}(\psi) = b$ $\quad |\psi| = \sqrt{a^2+b^2}$, $-|\psi| = -\sqrt{a^2+b^2}$

4. $|\psi_1 \cdot \psi_2| = |\psi_1| \cdot |\psi_2|$

5. $|\psi_1 \pm \psi_2|^2 = |\psi_1|^2 + |\psi_2|^2 \pm 2\,\mathrm{Re}(\psi_1\psi_2^*)$ one of the most imp. "laws" in physics

$\because |\psi_1 \pm \psi_2|^2 \equiv (\psi_1 \pm \psi_2)(\psi_1 \pm \psi_2)^* = (\psi_1 \pm \psi_2)(\psi_1^* \pm \psi_2^*) = \psi_1\psi_1^* \pm \psi_1\psi_2^* \pm \psi_2\psi_1^* + \psi_2\psi_2^*$

$= |\psi_1|^2 + |\psi_2|^2 \pm (\psi_1\psi_2^* + \psi_2\psi_1^*)$

$\psi_1\psi_2^* + (\psi_1\psi_2^*)^* = 2\,\mathrm{Re}(\psi_1\psi_2^*)$

6. $|\psi_1+\psi_2|^2 + |\psi_1-\psi_2|^2 = 2(|\psi_1|^2 + |\psi_2|^2)$

HW
7. $|a\psi_1 - b\psi_2|^2 + |b\psi_1 + a\psi_2|^2 = (a^2+b^2)\{|\psi_1|^2 + |\psi_2|^2\}$

$\psi = a+ib$
$k\psi = ak + ibk$
$|k\psi|^2 = k^2|\psi|^2$

B.6. Square Root of a Complex No.

* $\psi^2 - 1 = 0 \Rightarrow \psi = \pm\sqrt{1}$

$\psi^2 + 1 = 0 \Rightarrow \psi = \pm\sqrt{-1} = \pm i$

* $\psi = a+ib \longrightarrow \sqrt{\psi} = \sqrt{a+ib} = ?$ $\quad \nearrow \overline{a=1, b=0} \rightarrow \sqrt{1}$

* $\sqrt{\psi} = \sqrt{a+ib} = x+iy \in \mathbb{C}$ $\quad$ if $\psi \in \mathbb{C}$ then $\sqrt{\psi} \in \mathbb{C}$

What is $x, y = ?$

$a+ib = (x+iy)^2 = x^2 - y^2 + 2ixy$

$\mathrm{Re} \Rightarrow$ $\boxed{x^2 - y^2 = a}$

$\mathrm{Im} \Rightarrow$ $\boxed{2xy = b}$

quickest route : $(a+b)^2 = (a-b)^2 + 4ab$

* $(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2 = a^2 + b^2 \Rightarrow x^2+y^2 = \sqrt{a^2+b^2}$

$x^2-y^2 = a$

$x^2+y^2 = \sqrt{a^2+b^2}$

$x^2 = \frac{1}{2}\left(a + \sqrt{a^2+b^2}\right) \Rightarrow x$

$y^2 = \frac{1}{2}\left(\sqrt{a^2+b^2} - a\right) \Rightarrow y$

$\Rightarrow \sqrt{z} = \sqrt{a+ib} = x+iy$

Lecture-2 (27/Jul) 2.5

B.7. Cube root of unity

* $z^3-1=0 \Rightarrow z^3 = 1$ # degree = 3 = # of roots

* By inspection, $z_1 = 1 \Rightarrow f(z) = z^3-1$ can be factorized

$$
\begin{array}{r|l}
 & z^2+z+1 \\
\hline
(z-1) & z^3 \quad\quad -1 \\
 & -(z^3 - z^2) \\
\hline
 & z^2 \quad -1 \\
 & -(z^2 - z) \\
\hline
 & z - 1 \\
 & z - 1 \\
\hline
 & \times
\end{array}
$$

$f(z) = (z-1)(z^2+z+1) = 0$

$z = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$

$\to \omega_1 = \frac{-1+i\sqrt{3}}{2} \equiv \omega$

$\to \omega_2 = \frac{-1-i\sqrt{3}}{2} = \omega^*$

* $z = \{1, \omega, \omega^*\}$

Consider, $\omega^2 = \left(\frac{-1+i\sqrt{3}}{2}\right)^2 = \frac{1-3-2i\sqrt{3}}{4} = \frac{-2-2i\sqrt{3}}{4} = \frac{-1-i\sqrt{3}}{2} = \omega^*$

$1+\omega+\omega^2 = 1+\omega+\omega^* = 1 \frac{-1+\cancel{i\sqrt{3}}-1-\cancel{i\sqrt{3}}}{2} = \frac{2-1-1}{2} = 0$

$\boxed{1+\omega+\omega^2 = 0}$ //

Q1. $f(z) = z^4+1$ Roots? By factorization # [redacted] = 4

$z^4+1=0 \Rightarrow (z^2)^2 - i^2 = 0 \Rightarrow (z^2+i)(z^2-i) = 0$ $i^2=-1$

I $z^2-i=0 \Rightarrow \boxed{z = i^{1/2}}$ ✓

II $z^2+i=0 \Rightarrow \boxed{z = (-i)^{1/2}}$ ✓

$\longrightarrow$ . . . . . (LATER) check Q4, 5

Q2. $z = 5+12i$ $\sqrt{z} = \sqrt{5+12i} = ?$

* $\sqrt{5+12i} = x+iy \Rightarrow 5+12i = (x+iy)^2 \Rightarrow 5+12i = x^2-y^2+2ixy$

if $z_1 = z_2$

$\mathrm{Re}\, z_1 = \mathrm{Re}\, z_2$

$\mathrm{Im}\, z_1 = \mathrm{Im}\, z_2$

Re $\Rightarrow x^2-y^2 = 5$

Im $\Rightarrow 2xy = 12$

* $(a+b)^2 = (a-b)^2+4ab \Rightarrow (x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$

$(x^2+y^2)^2 = 25+144 = 169 \Rightarrow \boxed{x^2+y^2 = 13}$ $\quad \because x^2+y^2 > 0$

* $x^2-y^2 = 5$
$x^2+y^2 = 13$

$2x^2 = 18 \Rightarrow \boxed{x = \pm 3}$

$2y^2 = 8 \Rightarrow \boxed{y = \pm 2}$

$\Rightarrow \sqrt{5+2i}$ ⫽ $3+2i$, ⫽ $-3-2i$ $= \pm(3+2i)$

Q3. $z = -15-8i \qquad \sqrt{z} = ? \quad = \pm(1-4i)$

Q4. $z = i\ , \ \sqrt{i} = ?$

* $\sqrt{i} = x+iy \Rightarrow i = x^2-y^2+2ixy \Rightarrow x^2-y^2 = 0$
$2xy = 1$

$\because (x^2+y^2)^2 = (x^2-y^2)^2+4x^2y^2 \Rightarrow (x^2+y^2)^2 = 1 \Rightarrow x^2+y^2 = 1$

$2x^2 = 1 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$ ✓

$2y^2 = 1 \Rightarrow y = \pm\frac{1}{\sqrt{2}}$ ✓

$\boxed{\sqrt{i} = \frac{\pm 1}{\sqrt{2}}(1+i)}$

Q5. $z = -i \qquad \sqrt{-i} = ?$

HW

$\sqrt{-i} = x+iy \Rightarrow -i = x^2-y^2+2ixy \Rightarrow x^2-y^2 = 0$
$2xy = -1$

$\because (x^2+y^2)^2 = (x^2-y^2)^2+4x^2y^2 \Rightarrow (x^2+y^2)^2 = 1 \Rightarrow x^2+y^2 = 1$

? $\boxed{\sqrt{-i} = \frac{\pm 1}{\sqrt{2}}(1-i)}$

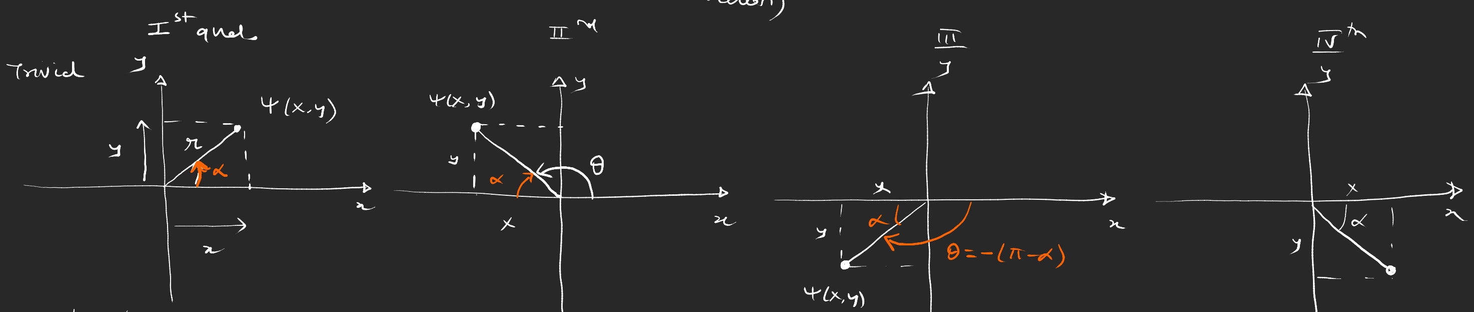
C. Geometric treatment of CN (Basic)

$z = 1+i = r(\cos\theta + i\sin\theta)$

* $z = x+iy = r(\cos\theta+i\sin\theta) = re^{i\theta}$
Cartesian $\qquad$ Polar $\qquad$ Eulerian

$x = r\cos\theta$
$y = r\sin\theta$ $\Rightarrow$

- $|z| = \sqrt{x^2+y^2} = r \in \mathbb{R}$ $\quad$ Magnitude / Modulus of a CN
  How much ?
- $\tan\theta = \frac{y}{x}$ $\quad$ Angle / Argument of a CN
  Where ?
  (direction)

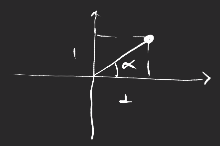


$\tan\alpha = \frac{y}{x} = \tan\theta$ $\qquad \tan\alpha = \left|\frac{y}{x}\right|$ $\quad x<0,\ y>0$ $\qquad \tan\alpha = \left|\frac{y}{x}\right|$ $\quad x<0,\ y<0$ $\qquad \tan\alpha = \left|\frac{y}{x}\right|$ $\quad x>0,\ y<0$

$\boxed{\theta = \alpha}$ $\qquad \boxed{\theta = \pi-\alpha}$ $\qquad \boxed{\theta = -(\pi-\alpha)}$ $\qquad \boxed{\theta = -\alpha}$

$x>0,\ y>0$

* $\psi_1 = 1+i \equiv (1,1) \longrightarrow \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}e^{i\pi/4}$

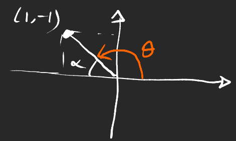


$\tan\alpha = \left|\frac{1}{1}\right| \Rightarrow \alpha = 45^\circ$

$\boxed{\theta = 45^\circ}$

$r = \sqrt{2}$

$\psi_2 = -1+i \equiv (-1,1)$



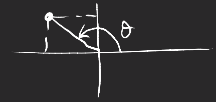
$\tan\alpha = \left\|\frac{1}{-1}\right\| \Rightarrow \alpha = 45^\circ$

$\theta = 180 - 45 = 135^\circ$

$\psi_2 = \sqrt{2}(\cos 135 + i\sin 135^\circ)$

$= \sqrt{2}e^{i\frac{3\pi}{4}}$

"Confusion"



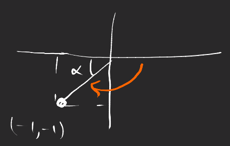
$\tan\theta = \frac{y}{x} = -1 \Rightarrow \tan(180-\theta)$

$\Downarrow$

$-\tan\theta = 1 \Rightarrow \tan(180-\theta) = \tan\frac{\pi}{4}$

$\theta = 180 - 45 = 135^\circ$

* $\psi_3 = -1-i \equiv (-1,-1)$



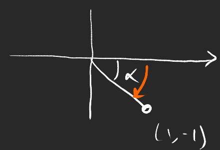
$\tan\alpha = \left|\frac{-1}{-1}\right| = 1 \Rightarrow \alpha = 45^\circ$

$\theta = -(180-45^\circ) = -135^\circ$

$\psi_3 = \sqrt{2}(\cos 135 - i\sin 135^\circ)$

$= \sqrt{2}e^{-i\frac{3\pi}{4}}$

$\psi_4 = +1-i = (1,-1)$

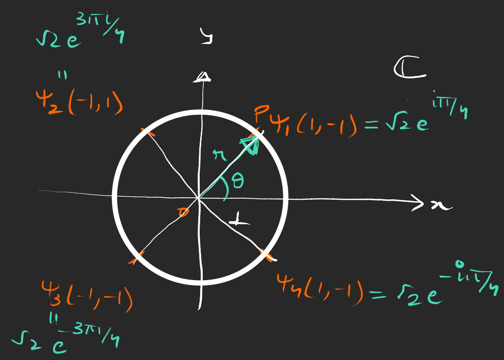


$\tan\alpha = \left|\frac{1}{-1}\right| = \alpha = 45^\circ$

$\theta = -\alpha = -45^\circ$

$\psi_4 = \sqrt{2}(\cos 45 - i\sin 45^\circ)$

$= \sqrt{2}e^{-i\pi/4}$



$r = \sqrt{a^2+b^2} = \text{const} \Rightarrow$ locus = circle

$\tan\theta = \frac{b}{a}$

$\Downarrow$

θ changes (Arg. of CN)

$\Downarrow$

Rotation of vector $\vec{OP} \equiv |\psi\rangle \equiv \psi = z$
$= \underset{\sim}{OP}$

$\left\{ |\psi\rangle = r(\cos\theta + i\sin\theta) \right.$ depicts / captures Rotation in a plane.

Remarks:

- $\forall$ CN $|\psi| =$ fixed

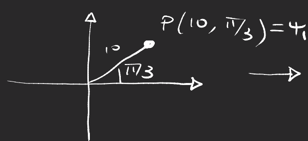
  Argument is not fixed

  $\Downarrow$

  $\exists$ Multiple θ's corresponding to same 'point'

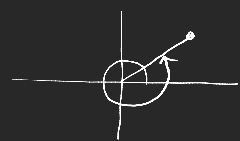
  $\Downarrow$

  $\exists$ more than 1 CN corresponding to same point



$\theta \in [-\pi, \pi]$

$r = 10$



$\theta = \frac{\pi}{3} + 2\pi = \frac{7\pi}{3}$

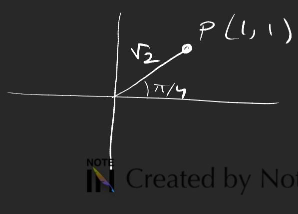
$\theta \notin [-\pi, \pi]$

$r = 10$

$\boxed{\text{if } -\pi \le \theta \le \pi \Rightarrow \text{Principal argument}}$

- $\psi = r(\cos\theta + i\sin\theta) = r(\cos(2n\pi+\theta) + i\sin(2n\pi+\theta))$ $\because$ Periodicity of sin, cos

ex: $\psi_1 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\left(\cos\left(\frac{\pi}{4}+2n\pi\right) + i\sin\left(\frac{\pi}{4}+2n\pi\right)\right)$



$n = 0 \longrightarrow z_1 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\pi/4\right)$

$n = 1 \longrightarrow z_1 = \sqrt{2}\left(\cos\frac{9\pi}{4} + i\sin\frac{9\pi}{4}\right)$ all are equivalent.

$\vdots$

Questions to ponder

$\sin\left(\frac{\pi}{2} + i\ln 2\right) = ?$

$\ln i = ?$

$i^i = ?$

$\cos^{-1}(2) = ?$

$\ln(-4) = ?$

D. Benefits of Eulerian form (entire Trig derivable)

* $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ Eulerian

$z_1 = r_1 e^{i\theta_1}, \quad z_2 = r_2 e^{i\theta_2}$

* $z_1 z_2 = r_1 r_2 e^{i\theta_1} e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)} = r_1 r_2\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)$

$(r_1(\cos\theta_1 + i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2))$

$r_1 r_2\left(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\right)$

$\mathbb{R}e \Rightarrow \cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$

$\Im m \Rightarrow \sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2$

$\Rightarrow$ Trig. is derivable from CN (domain extended)

Q a) $z_1 = \sin\left(\frac{\pi}{2} + i\ln 2\right) = \cos(i\ln 2) = \dfrac{e^{i(i\ln 2)} + e^{-i(i\ln 2)}}{2} = \dfrac{e^{-\ln 2} + e^{\ln 2}}{2}$

$\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

$= \dfrac{e^{\ln 2^{-1}} + e^{\ln 2}}{2} = \dfrac{2^{-1} + 2}{2} = \dfrac{\frac{1}{2} + 2}{2} = \dfrac{5}{4} \in \mathbb{R}$

$a^{\log_a b} = b$

HW

b) $\cos i$

c) $\sin i$

d) $\ln i$

e) i^i

f) $i^{(1+i)}$

g) $\cos^{-1}(2) \in \mathbb{C}$

h) $\sqrt{i}$

Hint

use: • $a^{\log_a b} = b$

• $\ln(ab) = \ln a + \ln b$

extensions
$\theta \to \theta + 2n\pi$

$z = re^{i\theta} \to \ln z = \ln re^{i\theta} = \ln r + \ln e^{i\theta}$
$= \ln r + i\theta$ Principle Branch

$-1 \leq \cos\theta \leq 1$

• $\boxed{e^{z + 2\pi i} = e^z}$ Periodicity $\because e^{2\pi i} = \cos 2\pi + i\sin 2\pi = 1$

Lecture-3 (28/Jul) 2

HW

b) $\cos i = \frac{e^2+1}{2e} \in \mathbb{R}$

c) $\sin i = \frac{e^{-1}-e}{2i} = \frac{1-e^2}{2e} = i\left(\frac{e^2-1}{2e}\right) \in \mathbb{C}$

d) $\ln i = \frac{i\pi}{2} \in \mathbb{C}$

e) $i^i = e^{-\pi/2} \in \mathbb{R}$

f) $i^{(1+i)} = e^{\ln_e i^{(1+i)}} = e^{(1+i)\ln i} = e^{(1+i)\frac{i\pi}{2}} = e^{(i-1)\frac{\pi}{2}}$

g) $\cos^{-1}(2) \in \mathbb{C} \longrightarrow \cos^{-1}(2) = \psi \Rightarrow \cos\psi = 2$

$\frac{e^{i\psi}+e^{-i\psi}}{2} = 2 \rightarrow (e^{i\psi})^2 \ldots$

$\Downarrow$

$\psi = \cos^{-1}(2) = \frac{1}{i}\ln(2\pm\sqrt{3}) + 2n\pi \in \mathbb{C}$

- $\cos^{-1}(2) \in \mathbb{C}$
- It is a multivalued "fun"

$\exists \infty$ values

$\cos^{-1}(1) = 0$ {only 1 value

$\ln\psi = \ln re^{i\theta} = \ln re^{i(\theta+2n\pi)}$

$\ln\psi = \ln r + i(\theta+2n\pi)$

$\ln i = \ln e^{i(\frac{\pi}{2}+2n\pi)} = i(\frac{\pi}{2}+2n\pi)$ $\to \frac{i\pi}{2}$, $\to \frac{5i\pi}{2}$, $\to \frac{9i\pi}{2}$, $\vdots$, ∞ #

$\boxed{-\pi < \theta < \pi}$

Principle value

h) $\sqrt{i} = \frac{1}{\sqrt{2}}(1+i)$

$\sqrt{i} = e^{\ln_e \sqrt{i}} = e^{\frac{1}{2}\ln i} = e^{\frac{i}{2}(\frac{\pi}{2}+2n\pi)}$

$= e^{\frac{i\pi}{4}+n\pi i}$

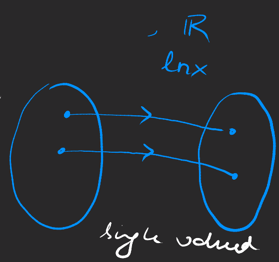
$n=0$: $\omega_1 = e^{i\pi/4} = \frac{1}{\sqrt{2}}(1+i)$

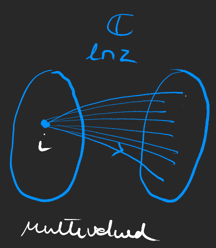
$n=1$: $\omega_2 = e^{\frac{5\pi}{4}i} = \frac{-1}{\sqrt{2}}(1+i)$

$n=2$: $\omega_3 = e^{i(\frac{\pi}{4}+2\pi)} = e^{i\pi/4}$ (repeated)

$\Downarrow$

we get 2 values for $\sqrt{i}$





② Sequence & Series

* $\mathbb{R}$, Arrangements of Numbers

- 1-D linear
- # of ways immaterial right now

$\to$ No rule $\equiv$ sequence

$\to$ $\exists$ Rule $\equiv$ progression
- AP
- GP
- H.P
- AGP arithmetico-geometric
- special

A. AP

* $a_1 \ldots a_n$ AP iff $a_n - a_{n-1} = \text{const} = d \longrightarrow \boxed{AP(a_1, d)}$

$\boxed{a_n = a_1 + (n-1)d}$ general term (nth term from start)

* $a_1 = a$, $CD = d$, # terms $= n$ $\quad 1, 2 \ldots \overset{m}{\downarrow} \; n$

$T_m \equiv m^{th}$ term from last $= (n-m+1)^{th}$ term from start

$= a_{n-m+1}$

$= a + (n-m+\cancel{1}-\cancel{1})d$

$= a + (n-m)d$

example

$P \equiv 1, 2, 3, 4, 5, 6, 7$ $\quad n = 7$, $d = 1$

5^{th} term from start $= 3^{rd}$ term from end

$= (7 - 3 + 1)^{th}$

Total $\quad 3^{rd}$ from last

Reversed P :

$P' \equiv 7, 6, 5, 4, 3, 2, 1$ $\quad n = 7$, $d = -1$

3^{rd} from start $= a + (n-1)(d)$

$\boxed{T_m = a + (n-m)d}$ m^{th} from last

Trick : $P \equiv a_1, a_2 \cdots a_{n-m} \cdots a_{n-1}\, a_n \longrightarrow P' \equiv a_n\, a_{n-1} \cdots a_2\, a_1$

P': 1^{st} term $= a_n$, $CD = -d$

$\boxed{T_m = a_n + (m-1)(-d)}$

$T_m = a + (n-1)d - md + d$
$= a + (n-1)d + (m-1)(-d)$
$= a_n + (m-1)(-d)$

$S_n = a_1 + a_2 \cdots a_{n-1} + a_n$
$+ \; S_n = a_n + a_{n-1} + \cdots a_2 + a_1$
$S_n =$ formula for AG

* $S_n = \frac{n}{2}(a + a_n) = \frac{n}{2}(2a + (n-1)d)$ sum of n terms

$d > 0 \Rightarrow S_\infty \to +\infty$
$d < 0 \Rightarrow S_\infty \to -\infty$ divergent

$S_\infty = \lim_{n\to\infty} S_n$

A.1 Properties of AP

* a, b, c : AP $\Rightarrow$ $\boxed{b = \frac{a+c}{2}}$

* AP : $a_1\, a_2 \cdots a_n$, $CD = d = a_n - a_{n-1}$

- Add K / subtract K : $a_1 \pm K, a_2 \pm K \cdots a_n \pm K$ — AP w/ $CD = d$
- multiply K : $Ka_1, Ka_2 \cdots Ka_n$ — AP w/ $CD = Kd$
- divide K : $\frac{a_1}{K} \cdots \frac{a_n}{K}$ — AP w/ $CD = \frac{1}{K}d$
- Power K / exponentiate : $a_1^K \cdots a_n^K$ — Not AP $K \in \mathbb{Z}$

$1\ 2\ 3\ 4\ 5$ AP
$1^{-1}\ 2^{-1}\ 3^{-1}\ 4^{-1}\ 5^{-1}$ not

* $AP_1(a_1, d_1)$, $AP_2(b_1, d_2)$

- $AP_1 \pm AP_2$: $(a_1 \pm b_1), (a_2 \pm b_2) \cdots (a_n \pm b_n)$ AP

$CD = a_n + b_n - a_{n-1} - b_{n-1} = (a_n - a_{n-1}) + (b_n - b_{n-1})$
$= d_1 + d_2 \equiv d$ (const

- $AP_1 \cdot AP_2$: $(a_1 b_1), (a_2 b_2), \cdots (a_n b_n)$ Not AP.

$CD = a_n b_n - a_{n-1} b_{n-1} \neq d_1 \cdot d_2$

- $K(AP_1) \pm \ell(AP_2)$: $(Ka_1 \pm \ell b_1), (Ka_2 \pm \ell b_2) \cdots (Ka_n \pm \ell b_n)$ AP

$CD = Ka_n \pm \ell b_n - (Ka_{n-1} \pm \ell b_{n-1})$
$= K(a_n - a_{n-1}) \pm \ell(b_n - b_{n-1}) = Kd_1 \pm \ell d_2 = \text{const}$

A.2 Special selections (V Helpful ! → TRICK)

* $a-d \;,\; a \;,\; a+d$ CD: d # of terms: 3

$\sigma = 3a$
$P = a(a^2-d^2)$ } ⇒ (a, d) : Calculable

ex: The sum of 3 #s in AP is -3, Product is 8 } find Numbers ?

$a=-1, \; d = \pm 3$ #: $-4, -1, 2$

2 −1 −4
−4 2 −1 } How many such arrangement ? Combinatories

* $(a-3d)\;(a-d)\;(a+d)\;(a+3d)$ CD: $2d$ # terms: 4

$\sigma = 4a$
$P = (a^2-d^2)(a^2-9d^2)$ } (a, d)

* $a-2d \quad a-d \quad a \quad a+d \quad a+2d$ d 5

$\sigma = 5a$

* $(a-5d)\,(a-3d)\,(a-d)\,(a+d)\,(a+3d)\,(a+5d)$ $2d$ 6

# of terms	Middle terms	CD
odd	a	d
Even	$(a-d), (a+d)$	$2d$

Try it out

Bad choice

$a, \; a+d, \; a+2d$: AP

$\sigma = 3a + \cancel{3d} = A$

$p = \underbrace{a}_{\frac{A}{3}}(a+d)(a+2d) = B$

Lecture-4 (29/Jul) 1.5

today: GP, AGP, HP
Sat.: Means, Special series

B. G.P

* $a_1, a_2 \ldots a_n$: GP iff $\frac{a_2}{a_1} = \frac{a_3}{a_2} = \ldots \frac{a_n}{a_{n-1}} = r$ Common ratio

$\boxed{GP(a, r) \; : \; a_n = ar^{n-1}}$ general term, n^{th} term (from the start)

$P \equiv 1, 2, 2^2, 2^3, 2^4, 2^5, 2^6$ $n=7$, $r=2$

5^{th} from start $= 3^{rd}$ from last
$\quad\parallel$
$(7-3+1)^{th}$ from start

* $T_m \equiv m^{th}$ term from last $= (n-m+1)^{th}$ term from start

$= a_{n-m+1} = ar^{n-m}$

$\boxed{T_m = ar^{n-m}}$

Trick : Consider reverse series $\Rightarrow$ 1st term $= a_n$, $CR = \frac{1}{r}$ $\boxed{T_n = a_n\left(\frac{1}{r}\right)^{n-1}}$

* $S_n = \sum_{n=1}^{n} a_n = \sum_{n=1} ar^{n-1} = a + \cancel{ar} + \cancel{ar^2} + \dots \cancel{ar^{n-1}}$

$rS_n = \sum_{n=1} ar^n = \cancel{ar} + \cancel{ar^2} + \dots \cancel{ar^{n-1}} + ar^n$

$-$ $\qquad -$

$S_n(1-r) = \sum a(r^{n-1} - r^n) = a(1-r^n) \Rightarrow \boxed{S_n = \frac{a(1-r^n)}{1-r}}$ $\quad r \neq 1$

* $S_n = \underbrace{a + a + a + \dots + a}_{n \text{ terms}} = na$ $\qquad r = 1 \Rightarrow$ AP not a GP.

Sum of ∞ GP

* $S_\infty \equiv \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \underbrace{\lim_{n\to\infty} \frac{a}{1-r}}_{\frac{a}{1-r}} - \underbrace{\lim_{n\to\infty} \frac{ar^n}{1-r}}$

$\longrightarrow 0$ iff $|r| < 1$

$\longrightarrow \infty$ iff $|r| > 1$

$\lim_{n\to\infty} r^n = \infty$

$\lim_{n\to\infty} \frac{1}{r^n} = 0$

* $S_\infty = \begin{cases} \frac{a}{1-r} & |r| < 1 \\ \infty & |r| > 1 \end{cases}$ $\Longrightarrow$ resolution of Zeno's paradox

convergent

divergent

* $S_n = 1 + 2 + 2^2 \dots \infty = \infty$

$R_n = 1 + 1 + 1 + \dots \infty = \infty$

$T_n = 1 + \frac{1}{2} + \frac{1}{4} + \dots \infty = 2$

3.1 Properties of GP

* a, b, c GP $\Rightarrow$ $\boxed{b^2 = ac}$ test for GP

* $GP(a_1, r)$

Multiply/divide k : $ka_1 \dots ka_n$ GP $CR = r$

Add/subtract k $\quad a_1 \pm k, \dots a_n \pm k$ $\quad$ { Not GP, Not AP

Power k $\quad a_1^k \dots a_n^k$ $\quad$ GP $\quad CR = r^k$

log base k $\quad \log_k a_1 \dots \log_k a_n$ $\quad$ Not GP, AP $\quad CD = \log_k a_2 - \log_k a_1 = \log_k \frac{a_2}{a_1} = \log_k r = \text{const}$

* $AP(a, d)$

Power k : $k^{a_1}, k^{a_2} \dots k^{a_n}$ $\quad$ GP $\quad CR = \frac{k^{a_2}}{k^{a_1}} = k^{a_2 - a_1} = k^d = \text{const}$

* $GP(a_1, r_1)$, $GP(b, r_2)$

 - $GP_1 \times GP_2 \equiv a_1 b_1, a_2 b_2 \ldots a_n b_n$ GP $CR = \frac{a_2 b_2}{a_1 b_1} = r_1 r_2 = \text{const}$

 - $GP_1 \pm GP_2 \equiv a_1+b_1, a_2+b_2, \ldots a_n+b_n$ neither AP nor GP

B2 Special Selection

* $a, ar, ar^2, \ldots ar^{n-1}$ => General selection

Special Selection

					CR	# terms
	$\frac{a}{r}$	a	ar		r	3
$\frac{a}{r^3}$	$\frac{a}{r}$	ar	ar^3		r^2	4
$\frac{a}{r^2}$	$\frac{a}{r}$	a	ar	ar^2	r	5

# terms	Middle term	CR
odd	a	r
Even	ar	r^2

FAPP { for all practical purposes

C. AGP (arithmetico-geometric prog.)

C.1 Setup & general term

* AP (a, d) → $a_n = a + (n-1)d$ GP(a, r) → $a_n = ar^{n-1}$ $d = a_n - a_{n-1}$ $r = \frac{a_n}{a_{n-1}}$

* Every term's 1st term governed by AP, 2nd term governed by GP } => AGP

ex: $S \equiv 1 + 3x + 5x^2 + 7x^3 + \ldots$ AGP

$R \equiv 3\cdot2, 4\cdot2^2, 5\cdot2^3, 6\cdot2^4 \ldots$ AGP

or

$6, 16, 40, 96 \ldots$ very hard to spot

$P \equiv a, (a+d)r, \ldots\ldots$

* $\boxed{T_n = [a+(n-1)d] \cdot r^{n-1}}$

C 2 Sum of n terms

* $S_n = a + (a+d)r + (a+2d)r^2 + \ldots \ldots (a+(n-1)d)r^{n-1}$

$rS_n = \quad ar + (a+d)r^2 + \ldots \ldots (a+(n-2)d)r^{n-1} + (a+(n-1)d)r^n$

$S_n(1-r) = a + \{rd + r^2d + r^3d + \ldots\ldots r^{n-1}d\} - (a+(n-1)d)r^n$

$$S_n(1-r) = a + \frac{rd(1-r^{n-1})}{1-r} - (a+(n-1)d)r^n$$

$$S_n = \begin{cases} \dfrac{a}{1-r} + \dfrac{rd(1-r^{n-1})}{(1-r)^2} - \dfrac{(a+(n-1)d)r^n}{1-r} & r \neq 1 \\[2ex] \dfrac{n}{2}\left(2a+(n-1)d\right) & r = 1 \\[2ex] \dfrac{a}{1-r} + \dfrac{rd}{(1-r)^2} & n \to \infty,\ |r| < 1 \\[2ex] \infty & n \to \infty,\ |r| > 1 \end{cases}$$

$$\lim_{n \to \infty} \frac{1}{r^n} = 0$$

D. H P (Harmonic progression)

* $AP(a_1, d)$: $d = a_n - a_{n-1}$

$a_n = a + (n-1)d$

$$\boxed{HP \equiv \text{Inverse}(AP)}$$

$$HP \equiv \frac{1}{a_1}, \frac{1}{a_2}, \ldots \frac{1}{a+(n-1)d}$$

$$\boxed{T_n \equiv \frac{1}{a_n} = \frac{1}{a+(n-1)d}}$$ n^{th} Term

* No direct formula for S_n so Brute force it. (See : special series)

Lecture-5 (31 Aug) 2.5

E. Means

E.1 Arithmetic mean (AM)

* Single AM of 2 # : a , A , b : AP => $A = \frac{a+b}{2}$ 1 AM b/w 2 #s ≡ "Average"

* m AM of 2 # : $a, \underbrace{A_1, A_2 \dots A_m}_{m\ AM}, b$: AP   # terms = m+2

$\underbrace{a_{m+2}}_{b} = a + (m+2-1)d \Rightarrow b = a + (m+1)d \Rightarrow \boxed{d = \frac{b-a}{m+1}}$

AP(a, d)

$A_1 = a + d = a + \frac{b-a}{m+1}$

$A_2 = a + 2d = a + 2\frac{(b-a)}{m+1}$

1 ≤ m ≤ n $\boxed{A_m = a + md = a + m\frac{(b-a)}{m+1}}$ $\xrightarrow{m=1}$ $A_1 = \frac{a+b}{2}$

situation 1

ex: S_1: 3, 7, 11, 15, 19

avg

$A_1 = \frac{a+b}{2} = \frac{3+19}{2} = \frac{22}{2} = 11$

$A_2 = \frac{3+11}{2} = \frac{14}{2} = 7$

$A_3 = \frac{11+19}{2} = \frac{30}{2} = 15$

situation 2

AP   3 , 7 11 15 19   3 AMs b/w 3 & 19

m = 3

$d = \frac{b-a}{m+1} = \frac{19-3}{4} = \frac{16}{4} = 4$

$A_1 = a + d = 3 + 4 = 7$

$A_2 = a + 2d = A_1 + d = 11$

$A_3 = a + 3d = 15$

Note : AM of n real # ≡ $A = \frac{a_1 + a_2 + \dots a_n}{n} = \frac{\sum_{i=1}^{n} a_i}{n}$

E.2 Geometric Mean (GM)

* Single G.M b/w 2 # : a, G, b : GP => $G = \sqrt{ab}$

* m GM b/w 2 # : $a, G_1, G_2 \dots G_m, b$: GP   # terms = m+2

$a_{m+2} = b = ar^{m+2-1} = ar^{m+1} \Rightarrow \boxed{r = \left(\frac{b}{a}\right)^{\frac{1}{m+1}}}$ $a_n = ar^{n-1}$

Common ratio when you need to insert m GM b/w 2 #s a, b

$G_1 = ar$

$G_2 = ar^2$ .....

* GM of n real # : $G = \sqrt{ab}$ $n=2$

$$\boxed{G = (a_1 \cdot a_2 \cdots a_n)^{1/n}}$$

E.3 Harmonic Mean (HM)

* Single HM $\boxed{a, H, b \;:\; HP \Rightarrow \frac{1}{a}, \frac{1}{H}, \frac{1}{b} \;:\; AP}$ trick

No direct way.

$\Downarrow$

$$\frac{1}{H} = \frac{\frac{1}{a}+\frac{1}{b}}{2} = \frac{a+b}{2ab} \Rightarrow \boxed{H = \frac{2ab}{a+b}}$$

* m HM b/w 2 # : $a, H_1, H_2 \cdots H_m, b$: HP $\Rightarrow \frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2} \cdots \frac{1}{H_m}, \frac{1}{b}$: AP # terms $= m+2$

$\Downarrow$

$$\frac{1}{b} - \frac{1}{a} = (m+1)D \Rightarrow \boxed{D = \frac{\frac{1}{b}-\frac{1}{a}}{m+1}}$$

D of an AP

$$\underbrace{T_{m+2}}_{\frac{1}{b}} = \frac{1}{a} + (m+2-1)D \qquad \left\{ \because a_n = a+(n-1)d \right.$$

$\Downarrow$

$$\frac{1}{H_1} = \frac{1}{a} + D$$

$$\frac{1}{H_2} = \frac{1}{a} + 2D \quad \cdots \quad \frac{1}{H_m} = \frac{1}{a} + mD$$

ex: if H_1, H_2, H_3 3 HM b/w 9 and α : $\frac{1}{H_1} + \frac{1}{H_2} + \frac{1}{H_3} = \frac{5}{3}$ $\alpha = ?$

$\downarrow$

$9, H_1, H_2, H_3, \alpha$: HP $\Rightarrow \boxed{\frac{1}{9}, \underbrace{\frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}}, \frac{1}{\alpha} \;:\; AP}$

1st 5th

$m = 3$

$$A = \frac{a+b}{2}$$

$$d - \frac{b-a}{m+1} = \frac{\frac{1}{\alpha} - \frac{1}{9}}{4}$$

$$\frac{1}{H_2} = \frac{\frac{1}{9}+\frac{1}{\alpha}}{2} \quad \text{— (1)}$$

$$\frac{1}{H_2} = \frac{\frac{1}{H_1}+\frac{1}{H_3}}{2} \quad \text{— (2)}$$

$$\because \frac{1}{H_1} + \frac{1}{H_2} + \frac{1}{H_3} = \frac{5}{3} \Rightarrow \underbrace{\left(\frac{1}{H_1} + \frac{1}{H_3}\right)}_{\frac{2}{H_2}} + \frac{1}{H_2} = \frac{5}{3} \Rightarrow \frac{3}{H_2} = \frac{5}{3}$$

$\Downarrow$

$$\frac{1}{2}\left(\frac{9+\alpha}{9\alpha}\right) \cdot \cancel{3} = \frac{5}{\cancel{3}} \Rightarrow 9 + \alpha = 10\alpha$$

$\Downarrow$

$$\boxed{\alpha = 1}$$

E.4. Relationship b/w AM, GM, HM

a, G, b GP? $\Updownarrow$ $G = \sqrt{ab}$

* 2 real # a, b : $A = \frac{a+b}{2}$, $G = \sqrt{ab}$, $H = \frac{2ab}{a+b}$

* $H = \frac{2ab}{a+b} = \frac{ab}{\frac{a+b}{2}} \Rightarrow H = \frac{G^2}{A} \Rightarrow \boxed{G = \sqrt{AH}} \Leftrightarrow$ A, G, H are in GP

{ GM of 2 real # is also GM of their AM & HM.

Consider

* $A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2} = \frac{(\sqrt{a}-\sqrt{b})^2}{2} \geq 0 \Rightarrow \boxed{A \geq G}$

* $A \geq G \Rightarrow AH \geq GH \Rightarrow G^2 \geq GH \Rightarrow \boxed{G \geq H}$

$\boxed{A \geq G \geq H}$ Arithmetic-geometric-Harmonic Mean inequality

* $A = \frac{a+b}{2} \Rightarrow 2A = a+b = \sum$ roots

$G = \sqrt{ab} \Rightarrow G^2 = ab = \prod$ roots

$ax^2+bx+c=0 \Rightarrow x_{1,2} = \alpha, \beta$

$\Downarrow$

$\alpha+\beta = -\frac{b}{a}$

$\alpha\beta = \frac{c}{a}$

$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

$x^2 - (\alpha+\beta)x + \alpha\beta = 0$

Consider $x^2 - 2Ax + G^2 = 0$ is satisfied by a, b

* $\boxed{x = A \pm \sqrt{A^2 - G^2}}$ 2 nos. a, b

$A \geq G \Rightarrow x \in \mathbb{R}$

$A < G \Rightarrow x \in \mathbb{C}$

F. Special Series & Miscellaneous stuff

F.1 Intro to special series / Method of difference

* $S_n(AP) = \frac{n}{2}[2a + (n-1)d]$

$S_n(GP) = \frac{a(1-r^n)}{1-r}$, $S_\infty = \frac{a}{1-r}$ $|r| < 1$

* $S_1 \equiv 9 + 99 + 999 + \ldots$ n terms $= f(n)$ they want you to do this!

$= (10-1) + (100-1) + (1000-1) + \ldots$

$= (\underbrace{10 + 10^2 + 10^3 + \ldots}_{n \text{ terms}}) - (\underbrace{1+1+1+\ldots}_{n \text{ terms}}) = \frac{10(1-10^n)}{1-10} - n = \frac{10 - 10^{n+1} + 9n}{-9} = \frac{1}{9}(10^{n+1} - 9n - 10)$

$S_1 = f(n) = 9 + 99 + 999 + \ldots = GP - AP$

* $S_2 \equiv 8 + 88 + 888 + \ldots$ n times

$= (10-2) + (100-2^{(1)}) + (1000-2^{(1)}) \ldots$ Ain't gonna work!

$S_2 = 8(1 + 11 + 111 + \ldots) = \frac{8}{9}(9 + 99 + 999 + \ldots) = \frac{8}{9}S_1$

$S_2 = g(n) \neq GP-AP$ directly

$= R(8+88+\dots) = K(GP-AP)$

Method of Difference

* $M_n \equiv 1\cdot2 + 2\cdot4 + 3\cdot8 + 4\cdot16 + \dots (n-1)2^{n-1} + n\cdot2^n$ (n terms)

$T_n = [a+(n-1)d]r^{n-1}$ AGP

$2M_n \equiv 1\cdot4 + 2\cdot8 + 3\cdot16 + \dots + (n-1)2^n + n\cdot2^{n+1}$

(−)

$-M_n = (2 + 4 + 8 + 16 + \dots + 2^n) - n2^{n+1}$

$-M_n = \frac{2(1-2^n)}{(1-2)} - n2^{n+1} \Rightarrow -M_n = -2 + 2^{n+1} - n2^{n+1} \Rightarrow \boxed{M_n = (n-1)2^{n+1} + 2}$

$\Downarrow$

$M_n = (n-1)2^n\cdot2 + 2 = [(n-1)2^n + 1]\cdot2$

HW

* $N_n \equiv 2\cdot3 + 3\cdot9 + 4\cdot27 + \dots (n+1)3^n = \sum_{i=1}^{n}(i+1)3^i = ?$

* $P_n \equiv 1^2 + 2^2\cdot\frac{1}{3} + 3^2\cdot\frac{1}{3^2} + 4^2\cdot\frac{1}{3^3} + \dots \infty$

* $f(n) = 1 + 3n + 5n^2 + 7n^3 + \dots \infty = 2$ Solve for $n = ?$

F2 Sigma notation

* $\sum_{r=1}^{n} f(r) = f(1) + f(2) + f(3) + \dots + f(n)$ r = index / dummy var.

$\sum_{r=1}^{n} a f(r) = a\sum f(r)$

$\sum (f(r) \pm g(r)) = \sum f(r) \pm \sum g(r)$

* $\sum f\cdot g \neq \sum f \cdot \sum g$, $\sum \frac{f}{g} \neq \frac{\sum f}{\sum g}$ Beware!

* $\sum_{r=1}^{n} r = 1+2+3+\dots+n = \frac{n(n+1)}{2}$ Imp. result 1

* $\sum_{r=1}^{n} r^2 = 1^2+2^2+3^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$ Imp results 2

* $\sum_{r=1}^{n} r^3 = 1^3+2^3+3^3+\dots+n^3 = \left(\frac{n(n+1)}{2}\right)^2$ Imp result 3.

Lecture-6 (4 Aug.) 2+2

HW

$P_n \equiv 1^2 + 2^2 \cdot \frac{1}{3} + 3^2 \cdot \frac{1}{3^2} + 4^2 \cdot \frac{1}{3^3} + \ldots \infty$

* $P_n \equiv 1 + 4 \cdot \frac{1}{3} + 9 \cdot \frac{1}{3^2} + 16 \cdot \frac{1}{3^3} \ldots$

$3 \quad 5 \quad 7 \ldots \} \sim AP$ method of diff. check

$\frac{1}{3} P_n \equiv 1 \cdot \frac{1}{3} + 4 \cdot \frac{1}{3^2} + 9 \cdot \frac{1}{3^3} \ldots$

$(-)$

$\frac{2P}{3} = 1 + (3)\frac{1}{3} + 5\left(\frac{1}{3^2}\right) + 7\left(\frac{1}{3^3}\right) \ldots \infty$

$\frac{1}{3}\frac{2P}{3} = \frac{1}{3} + 3 \cdot \frac{1}{3^2} + 5 \cdot \frac{1}{3^3} \ldots$

$(-)$

$\frac{2}{3}\frac{2P}{3} = 1 + 2 \cdot \frac{1}{3} + 2 \cdot \frac{1}{3^2} + 2 \cdot \frac{1}{3^3} \ldots \Rightarrow \frac{2 \cdot \cancel{2}P}{3 \cdot 3} = 1 + \cancel{2}\underbrace{\left(\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots\right)}_{\frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{\cancel{2}}} = \cancel{2}$

$\boxed{P = \frac{9}{2}}$

Direct formula:

AGP $S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2} \quad |r| < 1$

$\frac{2P}{3} - 1 = 3\left(\frac{1}{3}\right) + 5\left(\frac{1}{3^2}\right) + \ldots$

$= \frac{\frac{1}{3}}{1-\frac{1}{3}} + \frac{2 \cdot \frac{1}{3}}{\left(1-\frac{1}{3}\right)^2} = \frac{1}{2} + \frac{3}{2} = 2$

$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$

$P = 9/2$

Imp results

* $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$

* $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$

Proof:

* $(x+1)^3 = x^3 + 1 + 3x(x+1) \Rightarrow \underbrace{(x+1)^3 - x^3 = 3x^2 + 3x + 1}$

Plug some #

- $x = 1 \qquad \cancel{2^3} - 1^3 = 3 \cdot 1^2 + 3 \cdot 1 + 1$

2

⋮

diagonal cancellation

$n-1 \qquad \cancel{n^3} - \cancel{(n-1)^3} = 3(n-1)^2 + 3(n-1) + 1$

$n \qquad (n+1)^3 - \cancel{n^3} = 3n^2 + 3n + 1$

$(+)$

* $(n+1)^3 - 1^3 = 3(1^2 + 2^2 + 3^2 + \ldots n^2) + 3(1+2+3+\ldots n) + \underbrace{(1+1+1+\ldots 1)}_{n \text{ times}}$

$(n+1)^3 - 1^3 = 3\sum_{K=1}^{n} K^2 + 3\underbrace{\sum K}_{\frac{n(n+1)}{2}} + n$

Cleaning up

* $n^3 + \cancel{1} + 3n(n+1) - \cancel{1} = 3\sum K^2 + \frac{3n(n+1)}{2} + n \Rightarrow 3\sum K^2 = n^3 + 3n^2 + 3n - \frac{3n^2 - 3n}{2} - n$

$$= \frac{2n^3+6n^2+6n-3n^2-3n-2n}{2} = \frac{2n^3+3n^2+4n}{2}$$

$$\boxed{\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}}$$

$$= \frac{n(2n^2+3n+4)}{2} = \frac{n(n+1)(2n+1)}{2}$$

* $\sum_{r=1}^{n} r^3 = \left(\frac{n(n+1)}{2}\right)^2$

proof: $(x+1)^4 = (x^2+1+2x)^2 = x^4+1+4x^2+2x^2+4x+4x^3 \Rightarrow (x+1)^4 - x^4 = 4x^3+6x^2+4x+1$

$x = 1 \quad \cancel{2^4} - 1^4 = 4\cdot1^3+6\cdot1^2+4\cdot1+1$

$2 \quad 3^4 - \cancel{2^4} = 4\cdot2^3+6\cdot2^2+4\cdot2+1$

$\vdots$

$n-1 \quad \cancel{n^4} - (n-1)^4 = 4(n-1)^3+6(n-1)^2+4(n-1)+1$

$n \quad (n+1)^4 - \cancel{(n)^4} = 4n^3+6n^2+4n+1$

(+)

$$(n+1)^4 - 1^4 = 4\sum k^3 + 6\sum k^2 + 4\sum k + n$$

$$n^4+\cancel{1}+6n^2+4n+4n^3-\cancel{1} = 4\sum k^3 + \underbrace{6\sum k^2}_{\frac{n(n+1)(2n+1)}{6}} + 4\overbrace{\sum k}^{\frac{n(n+1)}{2}} + n$$

* $4\sum k^3 = n^4+4n^3+6n^2+4n - n(n+1)(2n+1) - 2n(n+1) - n$

$= n^4+4n^3+6n^2+\cancel{4n} - 2n^3 - n^2 - 2n^2 - \cancel{n} - 2n^2 - \cancel{2n} - \cancel{n}$

$= n^4+2n^3+n^2 = n^2(n^2+2n+1) = [n(n+1)]^2 \Rightarrow \sum k^3 = \left(\frac{n(n+1)}{2}\right)^2$

Try:

* $\sum k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

Q1. $T_n = n^2-n+3 \qquad \sum_{n=1}^{p} T_n = ?$

* $\sum T_n = \sum(n^2-n+3) \Rightarrow \sum T_n = \sum n^2 - \sum n + 3\sum_{n=1}^{p} 1 = \frac{p(p+1)(2p+1)}{6} - \frac{p(p+1)}{2} + 3p = \frac{p(p^2+8)}{3}$ ✓

HW

Q2. $T_p = (p-1)(p+2) \qquad S_n = \sum_{p=1}^{n} T_p = ?$

Q3. $T_r = 3r^2 - 2^r + 1 \qquad S_n = \sum T_r = ?$

Q4. $T_m = 2m^2+2n^2+3\cdot2^m+3\cdot2^n \qquad S_n = \sum T_m = ?$

Q5. $P = 1\cdot3+2\cdot5+3\cdot7+\ldots = ?$

Q6. $Q \equiv 1^2\cdot 1 + 2^2\cdot 4 + 3^2\cdot 7 + 4^2\cdot 10 + \dots = ?$

Q7. $T_p = p^2 - n + 2^n + p$ $\qquad S_n = \sum_{p=1}^{n} T_p = ?$ sum of n terms of a sequence w/ general term T_p

7) $S_n = \sum T_p = \sum_{p=1}^{n} p^2 - n + 2^n + p = \sum_{p=1} p^2 - n\sum_{p=1} 1 + 2^n\sum_{p=1} 1 + \sum_{p=1} p$

$$= \frac{n(n+1)(2n+1)}{6} - n\cdot n + 2^n n + \frac{n(n+1)}{2}$$

F.3 Method of difference (to calculate general term)

1 2 3 4 (AP)

* $S_n \equiv 1 + 2 + 4 + 7 + 11 + 16 + \dots\dots T_n$

$S_n \equiv \quad 1 + 2 + 4 + 7 + 11 + \dots\dots T_{n-1} + T_n$

(−)

$$0 = 1 + \underbrace{\left(1 + 2 + 3 + 4 + \dots \quad + (T_n - T_{n-1})\right)}_{\frac{(n-1)n}{2}} - T_n \Rightarrow \boxed{T_n = \frac{n(n-1)+2}{2}}$$

terms = n−1

$\because \sum n = \frac{n(n+1)}{2}$

HW

Q8 $R_n \equiv 1 + 3 + 7 + 15 + 31 + \dots$ $\qquad T_n = ?$ $\qquad R_n = ?$ $\qquad \frac{1}{2} + 1$

Q9 $S_n \equiv 2\cdot 1 + 2^2\cdot 3 + 2^3\cdot 5 + 2^4\cdot 7 \dots = ?$

Q10 $S_n = 1 + 3 + 7 + 13 + \dots + T_n$ $\qquad T_n = ?$ $\qquad$ MFSS

Q11. $\boxed{S_n = 1 + 2 + 5 + 11 + 21 + \dots}$ $\qquad T_n = ?$

***

F.4 Alternative methods to compute T_n, S_n (in case of method of difference)

1. Sequence : AP $\Rightarrow$ $T_n = an + b = f(n)$ linear in n

$\Downarrow$

$S_n = An^2 + Bn$ quadratic in n

A, B, a, b arbitrary const. wrt 1st or 2nd term

AP: $a_n = a + (n-1)d = \underbrace{d}_{\text{"a"}} n + \underbrace{(a-d)}_{\text{"b"}}$

$S_n = \sum T_n = a\sum n + b\sum 1$

$= \frac{an(n+1)}{2} + bn$

$= \underbrace{\frac{a}{2}}_{A} n^2 + \underbrace{\left(b + \frac{a}{2}\right)}_{B} n$

2. if difference of consecutive terms : AP $\Rightarrow$ $T_n = an^2 + bn + c$

$S_n = An^3 + Bn^2 + Cn$

1 3 6 11 ....

2 3 5

3. if difference of differences of consecutive terms : AP $\Rightarrow$ $T_n = an^3 + bn^2 + cn + d$

$n = An^4 + Bn^3 + Cn^2 + Dn$

4. if terms are in GP $\Rightarrow$ $T_n = ar^n = f(n)$ Polynomial

$S_n = Ar^n + B$

GP: $a_n = ar^{n-1}$

$= \underbrace{ar}_{\text{"a"}} r^n$

$S_n = \dfrac{a(1-r^n)}{1-r}$

$= \underbrace{\dfrac{a}{1-r}}_{B} - \underbrace{\dfrac{a}{1-r}}_{A} r^n$

5. if diff. of consecutive terms: GP $\Rightarrow$ $T_n = ar^n + b$

$S_n = Ar^n + Bn + C$

6. if diff of difference of consecutive terms: GP $\Rightarrow$ $T_n = ar^n + bn + c$

$S_n = Ar^n + Bn^2 + Cn + D$

Q12. $S_n = \overbrace{1 + 3 + 7 + 15 + 31 + \ldots}^{n \text{ terms}}$ No pattern

2 4 8 16

GP $\Rightarrow$ $S_n = Ar^n + Bn + C$

$\boxed{S_n = 2^n A + nB + C}$

$r = 2$

unknowns = 3

$\Downarrow$

Eqn = 3

$S_1 = 1 = 2A + B + C$ — ①

$S_2 = 4 = 4A + 2B + C$ — ②

$S_3 = 11 = 8A + 3B + C$ — ③

② − ① $\Rightarrow$ $3 = 2A + B$ — ④

③ − ② $\Rightarrow$ $7 = 4A + B$ — ⑤

⑤ − ④ $\Rightarrow$ $4 = 2A$ $\Rightarrow$ $\boxed{A = 2}$, $B = -1$, $C = -2$ $\Rightarrow$ $\boxed{S_n = 2^{n+1} - n - 2}$

Q13. $R_n = 1 + 2 + 5 + 11 + 21 + \ldots$

1 3 6 10 — 1st level of diff.

2 3 4 — 2nd level of diff. (or 3rd level) AP $\Rightarrow$

$T_n = an^3 + bn^2 + cn + d$

$S_n = An^4 + Bn^3 + Cn^2 + Dn$

unknowns = 4

Eqn = 4

$S_1 = 1 = A + B + C + D$

$S_2 = 3 = 16A + 8B + 4C + 2D$

$S_3 = 8 = 81A + 27B + 9C + 3D$

$S_4 = 19 = 256A + 64B + 16C + 4D$

HW

$T_1 = 1 = a + b + c + d$ — ①

$T_2 = 2 = 8a + 4b + 2c + d$ — ②

$T_3 = 5 = 27a + 9b + 3c + d$ — ③

$T_4 = 11 = 64a + 16b + 4c + d$ — ④

②−① $\Rightarrow$ $1 = 7a + 3b + c$ — ⑤

③−② $\Rightarrow$ $3 = 19a + 5b + c$ — ⑥

④−③ $\Rightarrow$ $6 = 37a + 7b + c$ — ⑦

⑥ − ⑤ $\Rightarrow$ $2 = 12a + 2b$ — ⑧

⑦ − ⑥ $\Rightarrow$ $3 = 18a + 2b$ — ⑨

$⑦ - ⑧ \Rightarrow 1 = 6a \Rightarrow a = \frac{1}{6}$ ✓

$b = 0, \quad c = -\frac{1}{6} \quad d = 1$

$$\boxed{T_n = \frac{1}{6}n^3 - \frac{1}{6}n + 1}$$

5.5 Sequence $\sum \left(\frac{1}{a_1 a_2 \dots a_n}\right)$

* Sequence " $\sum \left(\frac{1}{a_1 a_2 \dots a_n}\right)$ " : $a_1, a_2 \dots a_n$ AP (a, d)

$$\sum \frac{1}{a_1 a_2 \dots a_n} = \frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \frac{1}{a_3 \cdot a_4} + \dots \frac{1}{a_{n-1} a_n}$$

* $\underbrace{\frac{1}{a_1 a_2} = \frac{1}{a(a+d)}}$ $= \frac{A}{a} + \frac{B}{a+d}$ Method of partial fraction

$$\frac{1}{\cancel{a(a+d)}} = \frac{Aa + Ad + Ba}{\cancel{a(a+d)}} \Rightarrow 1 = (A+B)a + Ad$$

$\to A + B = 0$ Compare coeff of a

$\to Ad = 1$ coeff of constant

$\Downarrow$

$\boxed{A = \frac{1}{d}}$

$B = \frac{-1}{d}$

$$\frac{1}{a(a+d)} = \frac{1}{d}\left(\frac{1}{a} - \cancel{\frac{1}{a+d}}\right)$$

check

* $\frac{1}{a_2 a_3} = \frac{1}{(a+d)(a+2d)} \overset{\downarrow}{=} \frac{1}{d}\left(\cancel{\frac{1}{a+d}} - \cancel{\frac{1}{a+2d}}\right)$

$$\frac{1}{a_3 a_4} = \frac{1}{(a+2d)(a+3d)} = \frac{1}{d}\left(\cancel{\frac{1}{a+2d}} - \cancel{\frac{1}{a+3d}}\right)$$

$$\frac{1}{a_{n-1} a_n} = \frac{1}{(a+(n-2)d)(a+(n-1)d)} = \frac{1}{d}\left(\cancel{\frac{1}{a+(n-2)d}} - \frac{1}{a+(n-1)d}\right)$$

* $\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \dots + \frac{1}{a_{n-1} a_n} = \frac{1}{d}\left(\frac{1}{a} - \frac{1}{a+(n-1)d}\right) = \frac{(n-1)}{a(a+(n-1)d)}$

Q 17 $S_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots \frac{1}{n(n+1)}$

$T_r = \frac{1}{r(r+1)} \quad r = 1, \dots n$

Partial fraction (Format 1)

$$\frac{1}{r(r+1)} = \frac{A}{r} + \frac{B}{r+1}$$

$$S_n = \sum_{r=1}^{n} T_r = \sum \frac{1}{r(r+1)} = \sum_{r=1}^{n} \left(\frac{1}{r} - \frac{1}{(r+1)}\right)$$

$$= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) \qquad = 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

$$+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)$$

$$\left(\cancel{\frac{1}{n}}-\frac{1}{n+1}\right)$$

Q15. $$\lim_{n\to\infty}\left(\sum_{n=1}^{n}\frac{1}{(2r+1)(2r+4)}\right)$$

$$\sum_{n=1}^{\infty}\frac{1}{(2r+1)(2r+4)}=\frac{1}{3\cdot 6}+\frac{1}{5\cdot 8}+\frac{1}{7\cdot 10}+\frac{1}{9\cdot 12}+\cdots\infty$$

$$\frac{1}{3\cdot 6}=\frac{1}{3}\left(\frac{1}{3}-\frac{1}{6}\right)$$

$$\frac{1}{5\cdot 8}=\frac{1}{3}\left(\frac{1}{5}-\frac{1}{8}\right)$$

Partial fraction: Key

usually

$$\sum_{n} f(n)\cdot g(n)\neq\sum f(n)\cdot\sum g(n)$$

$$\sum\frac{1}{(2r+1)(2r+4)}=\frac{1}{3}\left\{\left(\frac{1}{3}-\frac{1}{6}\right)+\left(\frac{1}{5}-\frac{1}{8}\right)+\left(\frac{1}{7}-\frac{1}{10}\right)+\left(\frac{1}{9}-\frac{1}{12}\right)+\cdots\right\}$$

$$=\frac{1}{3}\left\{\left(\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots\infty\right)-\left(\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\frac{1}{12}\cdots\infty\right)\right\}$$

$$=\frac{1}{3}\left\{\underbrace{\sum_{n=1}^{\infty}\frac{1}{2r+1}}_{\leftarrow a\rightarrow}-\underbrace{\sum_{n=1}^{\infty}\frac{1}{2r+4}}_{\leftarrow b\rightarrow}\right\}=\frac{1}{3}(a-b)$$

Q16. $$S_n=\frac{1}{1\cdot 2\cdot 3\cdot 4}+\frac{1}{2\cdot 3\cdot 4\cdot 5}+\frac{1}{3\cdot 4\cdot 5\cdot 6}+\cdots\frac{1}{n(n+1)(n+2)(n+3)}$$

$$\frac{1}{1\cdot 2\cdot 3\cdot 4}=\frac{1}{3}\,\frac{4-1}{1\cdot 2\cdot 3\cdot 4}=\frac{1}{3}\left\{\frac{\cancel{4}}{1\cdot 2\cdot 3\cdot\cancel{4}}-\cancel{\frac{1}{\cancel{1}\cdot 2\cdot 3\cdot 4}}\right\}$$

$$\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}$$

$$\frac{1}{2\cdot 3\cdot 4\cdot 5}=\frac{1}{3}\,\frac{5-2}{2\cdot 3\cdot 4\cdot 5}=\frac{1}{3}\left\{\cancel{\frac{\cancel{5}}{2\cdot 3\cdot 4\cdot\cancel{5}}}-\cancel{\frac{\cancel{2}}{\cancel{2}\cdot 3\cdot 4\cdot 5}}\right\}$$

$$\vdots$$

$$\frac{1}{n(n+1)(n+2)(n+3)}=\frac{1}{3}\,\frac{n+3-n}{n(n+1)(n+2)(n+3)}=\frac{1}{3}\left\{\cancel{\frac{\cancel{(n+3)}}{n(n+1)(n+2)\cancel{(n+3)}}}-\frac{\cancel{n}}{\cancel{n}(n+1)(n+2)(n+3)}\right\}$$

$$* \quad S_n=\frac{1}{3}\left\{\frac{1}{1\cdot 2\cdot 3}-\frac{1}{(n+1)(n+2)(n+3)}\right\}$$

Friday

- Special series 2 exercise

RD