# Cartesian Geometry (Vol-3)

Lecture-1 (22/Jan/24) 15 + 120

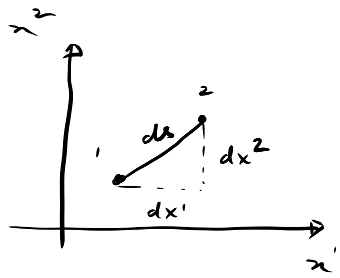
1 Curve sketching

1.1 Concavity

* $\{x^i\}$ basis $\longrightarrow$ $\{\underset{\sim}{\hat{e}}^i\}$ basis unit vectors

$$\hat{e}^i \cdot \hat{e}^j = g^{ij} \xrightarrow[g^{ij} = \delta^{ij}]{\text{Euclidean}} e^i e^j = \delta^{ij}$$

$$ds^2 = g_{ab}\, dx^a dx^b \quad \{\text{distance presumption}$$




* $\tan\theta = \dfrac{dy}{dx}$

Topics

1 Curve sketching → Stability Analysis

2. transformation → graphical; → Coord. (Algebra)

Set theory / Group theory

* $\hat{T}$. $x \to x + a$ transl$^n$

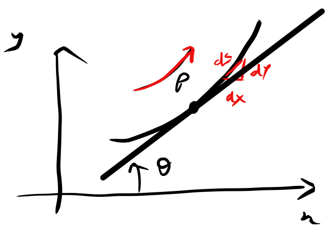
* $\hat{R}$ $v^a = R^a{}_b v^b \to$ rot$^n$ in $\mathbb{R}$

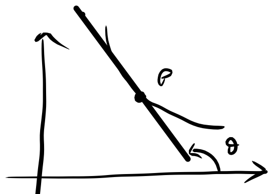
* T + R → in $\mathbb{R}$ " " $\mathbb{C}$; → in $\mathbb{C}$ → LT

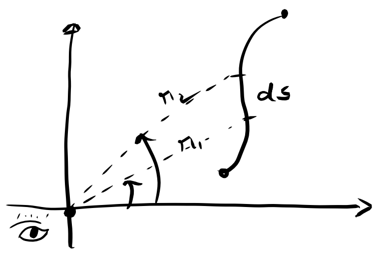
3. Conic Sections → O; → C; → )(; → ⊕

Concave up (CU)

*




$\underset{\sim}{\dfrac{dy}{dx}}$ ↑ as x ↑ $\qquad$ $\dfrac{dy}{dx}$ ↑ as x ↑

tangen

⇓

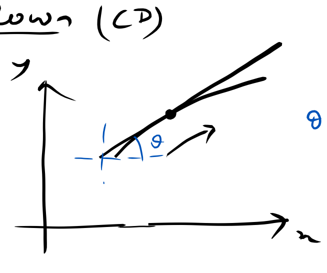
$\boxed{\dfrac{d^2y}{dx^2} > 0}$ $\longrightarrow$ Concave up-num $\equiv$ "minima"

∃ a change in the slope which is +ve

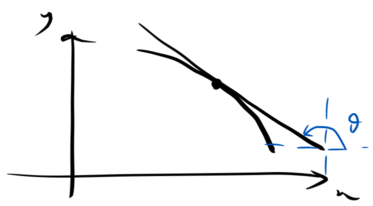
| Note | Curve is above its own tangent |
|---|---|

Concave down (CD)

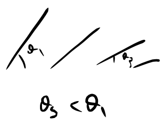
*




θ↓




θ↓




$\underset{\sim}{\dfrac{dy}{dx}}$ ↓ as x ↑ ⇒ $\boxed{\dfrac{d^2y}{dx^2} < 0}$ $\longrightarrow$ Concave down-num $\equiv$ "Maxima"

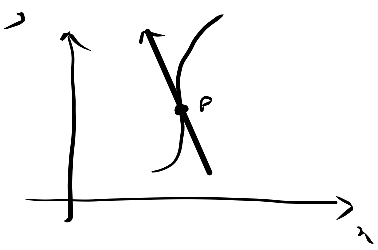
tan θ

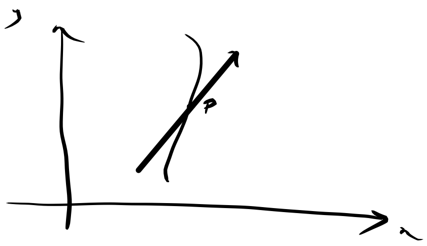
(Convex up)

| Note. | Curve is below its own tangent |
|---|---|

pt of Inflexion

*

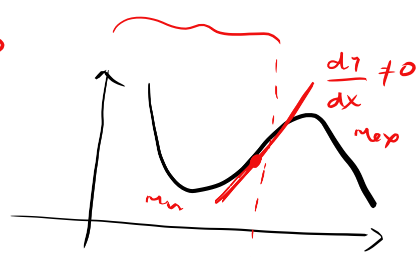



from CU ↔ CD

⇓

∃ pt of inflexion



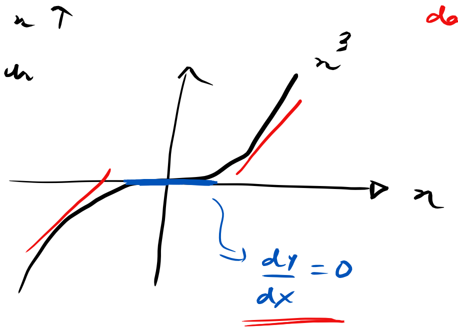

at P it changes its Concavity from up ↔ down (Min ↔ Max.)

at P $\dfrac{dy}{dx} = 0$

$\dfrac{d^2y}{dx^2}$ changes its sign as x ↑ then value at which $\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx}$ → $= 0$ ⇒ Stationary pt of Inflexion

→ $\neq 0$ ⇒ Non-stationary pt of inflexion



1.2 Plots using Concavity (Protocols)

Step 1   check pts. of intersections on $x$ & $y$ axis

$x = 0 \longrightarrow$ $y$ intercept

$y = 0 \longrightarrow$ $x$ intercept

Step 2   pt of Maximum / Minimum values (Equi$^n$ pts / Critical pts)   $\underbrace{\frac{dy}{dx}}_{\text{slope}} = 0 \Rightarrow x = x_1, x_2$

Step 3.   check Maxima & Minima   $\frac{d^2y}{dx^2} = ?$ at $\to x_1$ if $> 0$ minima; $\to x_2 < 0$ maxima

Step 4   At $x_1$ (Minima) $\to$ value of $y \Rightarrow$ Minimum value

$x_2$ (Maxima) $\to$ value of $y \Rightarrow$ Maximum value

Step 5   $\frac{dy}{dx} = f(x) = ?$

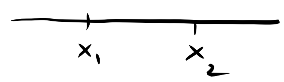
$\left.\frac{dy}{dx}\right|_{x<x_1}$   $>0$ / or $<0$ \

$\left.\frac{dy}{dx}\right|_{x>x_2}$   $>0$ / or $<0$ \

$\left.\frac{dy}{dx}\right|_{x_1<x<x_2}$

⋮

{ check what happens to the slope around the critical points




Step 6   $\frac{d^2y}{dx^2} = 0 \Rightarrow x = \alpha, \beta$,   { A thumb rule

$\left.\frac{d^2y}{dx^2}\right|_{x<\alpha}$   $>0 \longrightarrow$ minima (Concave up)

$<0 \longrightarrow$ Maxima

$\left.\frac{d^2y}{dx^2}\right|_{x>\alpha}$   $>0 \longrightarrow$ min

$<0 \longrightarrow$ max

$\alpha < x < \beta$

If $\left.\frac{dy}{dx}\right|_{x=\alpha}$ $\to = 0$ Stationary pt of Inflexion; $\to \neq 0$ non stationary pt of Inflexion

**Type**   $y = f(x) = (x-\alpha)(x-\beta)$

Eg 1.   $y = (x-1)(x-2) = x^2 - 3x + 2$

* $y = 0 \Rightarrow x = 1, 2$

$x = 0 \Rightarrow y = 2$

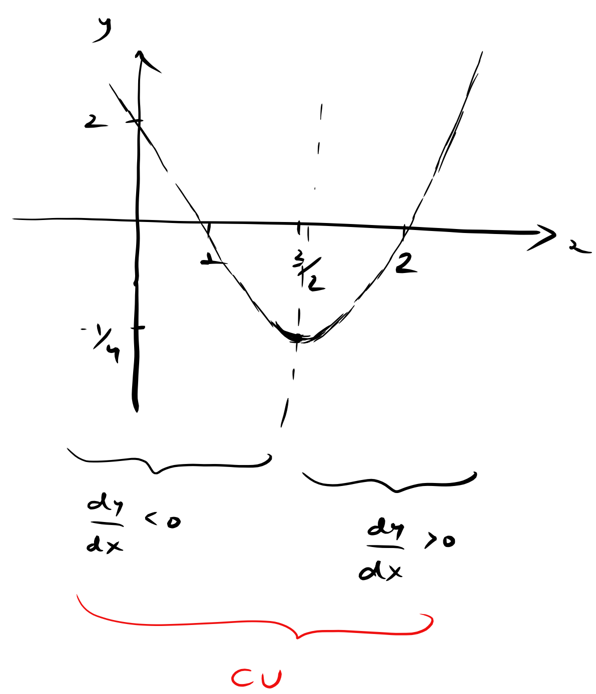
* $\frac{dy}{dx} = 0 \Rightarrow 2x - 3 = 0 \Rightarrow x_0 = 3/2$

$\frac{d^2y}{dx^2} = 2 > 0 \Rightarrow$ Minima at $x_0$

Always Concave up $\Downarrow$ no inflexion point

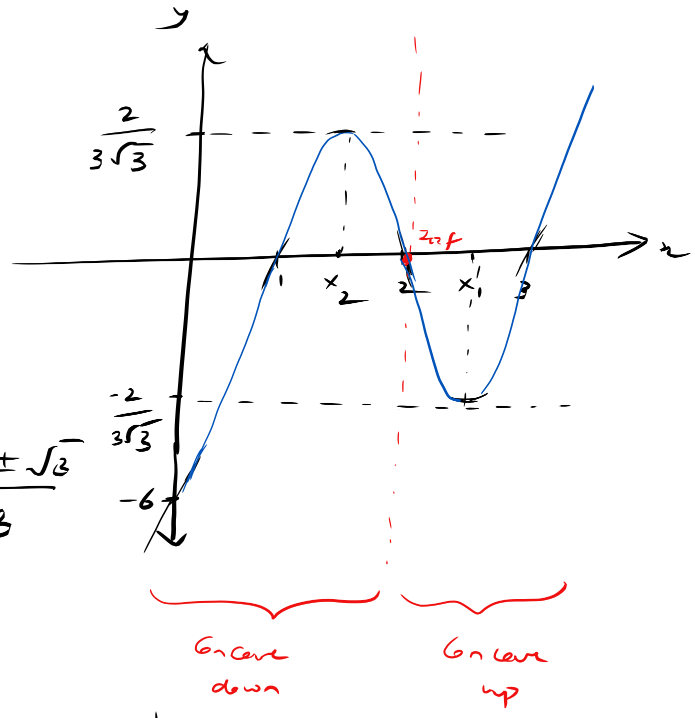
$y|_{x_0} = \left(\frac{3}{2} - 1\right)\left(\frac{3}{2} - 2\right) = \frac{-1}{4}$

* $\left.\frac{dy}{dx}\right|_{x<3/2} = 2(1) - 3 < 0$



$\left.\frac{dy}{dx}\right|_{x>3/2} = 2(2)-3 > 0$

Eg 2 $\quad y = (x-1)(x-2)(x-3)$




* $y = 0 \Rightarrow x = 1, 2, 3$ (Intersection pt)
$\quad x = 0 \Rightarrow y = -6$ ( " )

* $y = (x-1)(x^2-5x+6) = x^3-5x^2+6x-x^2+5x-6$
$\quad = x^3-6x^2+11x-6$

* $\frac{dy}{dx} = 0 \Rightarrow 3x^2-12x+11 = 0 \Rightarrow x_{1,2} = \frac{12 \pm \sqrt{144-4(3)(11)}}{6} = \frac{6 \pm \sqrt{3}}{3}$

$x_{1,2} = 2 \pm \frac{1}{\sqrt{3}}$ (CP)

$x_1 = 2 + \frac{1}{\sqrt{3}} \qquad x_2 = 2 - \frac{1}{\sqrt{3}}$

$x_1 = \frac{6+\sqrt{3}}{3}, \quad x_2 = \frac{6-\sqrt{3}}{3}$

* $\frac{d^2y}{dx^2} = 6x-12 \rightarrow \left.\frac{d^2y}{dx^2}\right|_{x_1} = \cancel{6}\left(\frac{\cancel{6}+\sqrt{3}}{\cancel{3}}\right) - \cancel{12} = 2\sqrt{3} > 0$ "minima"

$\quad \left.\frac{d^2y}{dx^2}\right|_{x_2} = \cancel{6}^{2}\left(\frac{\cancel{6}-\sqrt{3}}{\cancel{3}}\right) - \cancel{12} = -2\sqrt{3} < 0$ "maxima"

* $y|_{x_1} = \left(2+\frac{1}{\sqrt{3}}-1\right)\left(\cancel{2}+\frac{1}{\sqrt{3}}-\cancel{2}\right)\left(2+\frac{1}{\sqrt{3}}-3\right) = \left(1+\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}-1\right)\frac{1}{\sqrt{3}} = \left(\frac{1}{3}-1\right)\frac{1}{\sqrt{3}} = \frac{-2}{3\sqrt{3}}$ minimum value

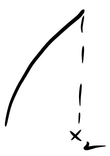
$y|_{x_2} = \left(2-\frac{1}{\sqrt{3}}-1\right)\left(\cancel{2}-\frac{1}{\sqrt{3}}-\cancel{2}\right)\left(2-\frac{1}{\sqrt{3}}-3\right) = \left(1-\frac{1}{\sqrt{3}}\right)(-1)\left(1+\frac{1}{\sqrt{3}}\right)\left(+\frac{1}{\sqrt{3}}\right) = \left(1-\frac{1}{3}\right)\frac{1}{\sqrt{3}} = \frac{2}{3\sqrt{3}}$ Maximum value

Imp Analysis (Concavity)

* $\frac{dy}{dx} = 3x^2-12x+11 = (x-x_1)(x-x_2) = \left(x-\frac{6+\sqrt{3}}{3}\right)\left(x-\frac{6-\sqrt{3}}{3}\right)$

Around the Critical pts.

$\left.\frac{dy}{dx}\right|_{x<x_2} > 0$




$x=1$

$\left(\frac{3-6-\sqrt{3}}{3}\right)\left(\frac{3-6+\sqrt{3}}{3}\right)$

$\frac{(-3-\sqrt{3})(-3+\sqrt{3})}{3} = \frac{9-3}{3} > 0$

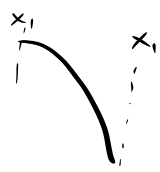
$\left.\frac{dy}{dx}\right|_{x>x_1} > 0$


$x=3$

$\left(\frac{9-6-\sqrt{3}}{3}\right)\left(\frac{9-6+\sqrt{3}}{3}\right) > 0$

$(3-\sqrt{3})(3+\sqrt{3})$

$\left.\frac{dy}{dx}\right|_{x_2<x<x_1} < 0$




$x=2$

$\left(\frac{\cancel{6}-\cancel{6}-\sqrt{3}}{3}\right)\left(\frac{\cancel{6}-\cancel{6}+\sqrt{3}}{3}\right) < 0$

* $\frac{d^2y}{dx^2} = 6x-12 = 0 \Rightarrow x = 2 \Rightarrow$ point of Inflexion, $\left.\frac{dy}{dx}\right|_{x=2} = 3(4) - 12(2)+11 = 23-24 \neq 0$

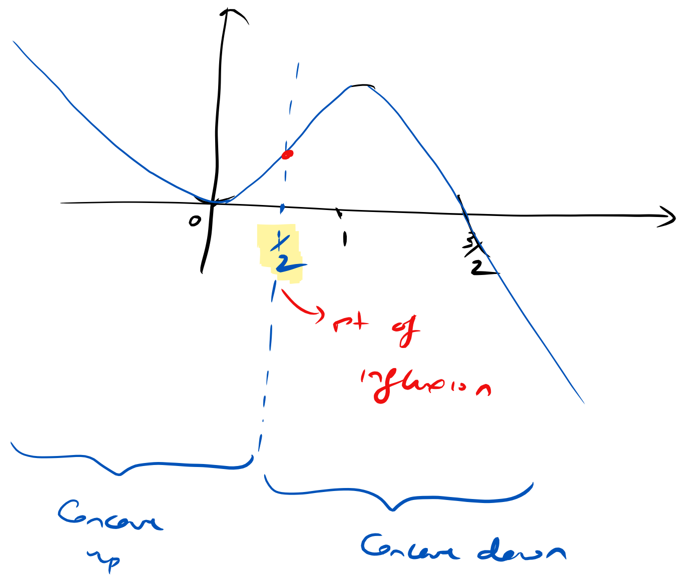
$\Downarrow$

non Stationary pt of Inflexion

$\left.\frac{d^2y}{dx^2}\right|_{x<2} = 6(1)-12 < 0 \Rightarrow$ Maxima / Concave down

$\left.\frac{d^2y}{dx^2}\right|_{x>2} = 6(3)-12 > 0 \Rightarrow$ minima / Concave up

Q3 $y = 3x^2 - 2x^3$




* $y = 0 \Rightarrow x^2(3-2x) = 0 \Rightarrow x = 0, \tfrac{3}{2}$ } Intersection pt

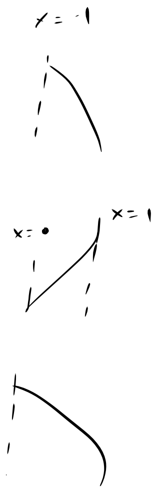
  $x = 0 \Rightarrow y = 0$

* $\frac{dy}{dx} = 6x - 6x^2 = 6x(1-x) = 0 \Rightarrow x = \underset{x_2}{1}, \underset{x_1}{0}$ C.P

* $\frac{d^2y}{dx^2} = 6 - 12x$
  - $\left.\frac{d^2y}{dx^2}\right|_{x=0} > 0 \rightarrow$ Minima
  - $\left.\frac{d^2y}{dx^2}\right|_{x=1} < 0 \rightarrow$ Maxima

Concavity Analysis

* $\frac{dy}{dx} = 6x(1-x)$
  - $\left.\frac{dy}{dx}\right|_{x<0} = 6(-1)(1+1) < 0$
  - $\left.\frac{dy}{dx}\right|_{0<x<1} = 6(\tfrac{1}{2})(1-\tfrac{1}{2}) > 0$
  - $\left.\frac{dy}{dx}\right|_{x>1} = 6(2)(1-2) < 0$




* $\frac{d^2y}{dx^2} = 6 - 12x = 0 \Rightarrow x = \tfrac{1}{2} \Rightarrow$ pt of inflexion

  $\left.\frac{d^2y}{dx^2}\right|_{x<\frac{1}{2}} = 6 - 12(0) > 0 \Rightarrow$ minima $\rightarrow$ Concave up

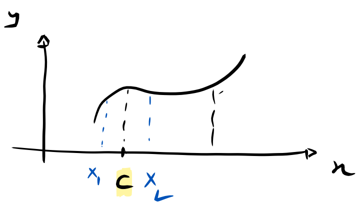
  $\left.\frac{d^2y}{dx^2}\right|_{x>\frac{1}{2}} = 6 - 12(1) < 0 \Rightarrow$ Maxima $\rightarrow$ Concave down

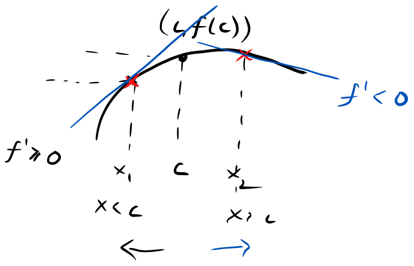
  $\left.\frac{dy}{dx}\right|_{x=\frac{1}{2}} = \overset{3}{6}(\tfrac{1}{2})(1-\tfrac{1}{2}) \neq 0 \Rightarrow$ non stationary pt of inflexion

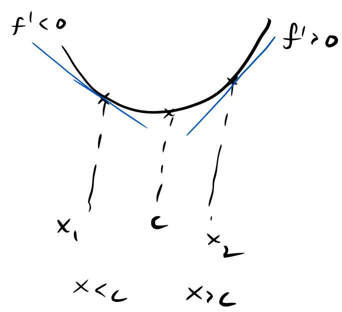
Lecture-2 (28/Jan) 15

**Key points**

* $y = f(x)$
* Critical points (CP) = 'c' in dom $f(x)$ : $f'(c) = 0$
* Monotony $\Rightarrow$ $f'(x) = \begin{cases} >0 \Rightarrow f(x) \text{ increasing} \\ <0 \Rightarrow f(x) \text{ decreasing} \end{cases}$
* Local extrema $\Rightarrow$ appear at points 'c' in dom $f(x)$ : f changes from $\uparrow$ to $\downarrow$ ($f(c)$ maximum)
  or $\downarrow$ to $\uparrow$ ($f(c)$ minimum)
* 1st derivative test $\rightarrow$ let $f'(c) = 0$ $\Rightarrow$ 
  1) $f'(x) > 0$ for $x < c$, $f'(x) < 0$ for $x > c$ $\Rightarrow$ $f(c)$ local Maxima
  2) $f'(x) > 0$ for $x > c$, $f'(x) < 0$ for $x < c$ $\Rightarrow$ $f(c)$ local Minima




* Concavity $\longrightarrow$ $f''(x) = \begin{cases} >0 & f(x) \text{ concave up } \cup \\ <0 & f(x) \text{ concave down } \cap \end{cases}$
* Inflexion pt = $(x_I, f(x_I))$ : $f(x)$ changes its concavity
* IP test $\rightarrow$ let $f''(c) = 0$
  if $f''(x)$ changes sign at $x = c$ then $f(x)$ has an IP at $x = c$
* 2nd derivative test $\rightarrow$ let $f'(c) = 0$
  $f''(c) = \begin{cases} >0 \rightarrow f(c) \text{ is local minimum} \\ <0 \rightarrow f(c) \text{ is " maximum} \end{cases}$

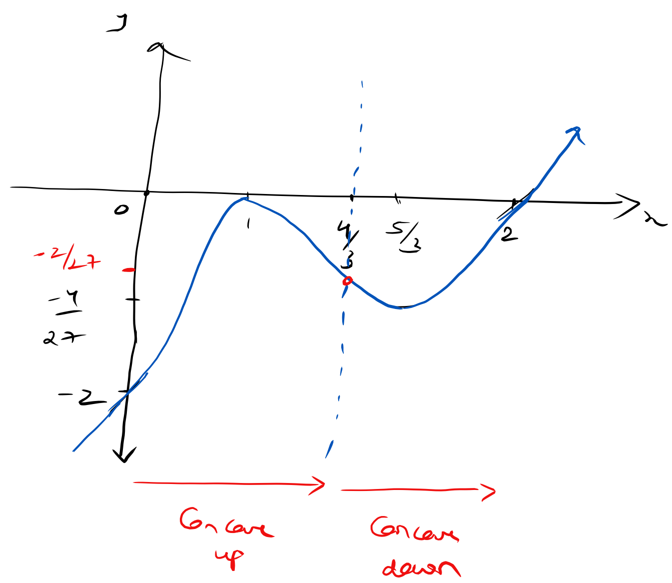
Eg $y = (x-1)^2(x-2)$

* Intersection pt $y = 0 \Rightarrow x = 1, 1, 2$
* $y = (x^2+1-2x)(x-2) = x^3 - 2x^2 + x - 2 - 2x^2 + 4x$
  $= x^3 - 4x^2 + 5x - 2$

$y' = 3x^2 - 8x + 5 = 0 \Rightarrow x = 1, 5/3$

$y'' = 6x - 8$ $\rightarrow$ $y''|_{x=1} < 0$ maxima
$\rightarrow$ $y''|_{x=5/3} > 0$ minima

Max val $y|_{x=1} = 0$, min value $y|_{5/3} = \frac{4}{9}\left(\frac{-1}{3}\right) = -4/27$ ; $x = 0 \rightarrow y = -2$



* $y' = (x-1)(x-5/3)$

$\to y'|_{x<1} = \left(\frac{1}{2}-1\right)\left(\frac{1}{2}-\frac{5}{3}\right) = \left(\frac{-1}{2}\right)\left(\frac{-7}{6}\right) > 0$ ↗

$\to y'|_{1<x<\frac{5}{3}} = \left(\frac{3}{2}-1\right)\left(\frac{3}{2}-\frac{5}{3}\right) = \frac{1}{2}\left(\frac{-1}{6}\right) < 0$ ↘

$\to y'|_{x>5/3} = (2-1)\left(2-\frac{5}{3}\right) = \frac{1}{3} > 0$ ↗

* $y'' = 6x - 8 = 0 \Rightarrow x = \frac{4}{3}$ Point of inflection $\longrightarrow y = -2/27$

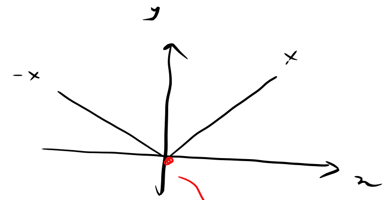
$y''$ → $x > 4/3 \to y'' = 6(2) - 8 > 0$ Concave up

→ $x < \frac{4}{3} \to y'' = 6(1) - 8 < 0$ " down

Type 2

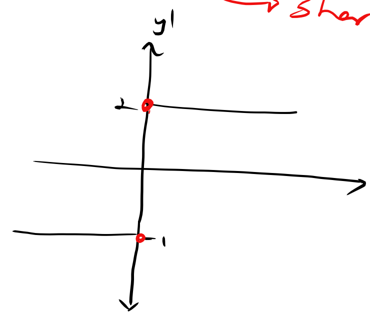
Detour to Continuity & differentiability (Comment on Mod.)

* $y = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$ Mod. fun$^n$




continuity in y

sharp corner in y




discontinuity in y'

Continuous ≡ 'joined at that point'

Differentiable ≡ 'smoothly joined at that point'

"you wouldn't be able to feel the joint"

* $\frac{dy}{dx} = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases} = \frac{x}{|x|}$

$\boxed{\text{differentiability} \Rightarrow \text{Continuity}}$

$\not\Leftarrow$

$|x|$ is not differentiable at $x=0$

but still operationally doable

$\Rightarrow$ It is Cont. but not diff

* $y = |x| = \sqrt{x^2}$

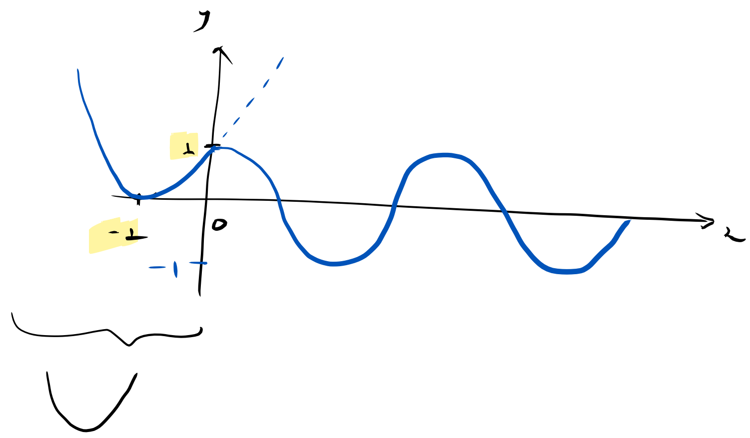
$y' = \frac{1}{\cancel{2}\sqrt{x^2}} \cdot \cancel{2}x = \frac{x}{|x|} = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$

Lecture-3 (28/Jan) 20 min + 10 + 1.5

Ex1 $y = \begin{cases} (x+1)^2 & x \leq 0 \\ \cos x & x > 0 \end{cases}$ LHL / RHL

* $y|_{x=0} = (0+1)^2 = 1$

* $\frac{dy}{dx} = \begin{cases} 2(x+1) & x \leq 0 \\ -\sin x & x \geq 0 \end{cases}$




$x^2+2x+1$

* $\left.\frac{dy}{dx}\right|_{x=0} = 2(x+1)\Big|_{x=0} = 2$ LHS

$\left.\frac{dy}{dx}\right|_{x=0} = -\sin x\Big|_{x=0} = 0$ RHS

$\rightarrow \circ \leftarrow$ } $\Rightarrow$ Handwaving / Physicist's Approach "LHD $\neq$ RHD"

not differentiable at that pt ($x=0$)

Pure Analysis

* $\boxed{x \le 0}$ $y = x^2+2x+1$

$x=0 \quad y=1$

$y=0 \quad \underbrace{x^2+2x+1}_{(x+1)^2} = 0 \Rightarrow x=-1$

$\frac{dy}{dx} = 2x+2 = 0 \Rightarrow x=-1$

$\frac{d^2y}{dx^2} = 2 > 0 \Rightarrow$ minima at $x=-1$ $\cup$

separately the pieces are joined together

the joint $(0,1)$

$\Downarrow$

$y$ is continuous at $x=0$

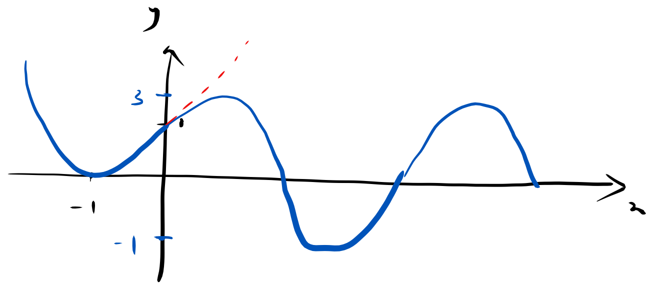
* $\boxed{x > 0}$ $y = \cos x$ $\quad x=0 \quad y=1$

$y' = -\sin x$

$y'' = -\cos x$

"joined but not smoothly joined"

Ex 2 $y = \begin{cases} (x+1)^2 & x \le 0 \\ 1+2\sin x & x > 0 \end{cases}$




$\frac{dy}{dx} = \begin{cases} 2(x+1)\Big|_{x=0} = 2 \\ 2\cos x\Big|_{x=0} = 2 \end{cases}$ "LHD = RHD"

$\Downarrow$

differentiable $\Rightarrow \exists$ a joint which is smooth

$\boxed{x < 0}$

$y = 1+2\sin x$

$-1 \le \sin x \le 1$

$-2 \le 2\sin x \le 2$

$-1 \le 1+2\sin x \le 3$

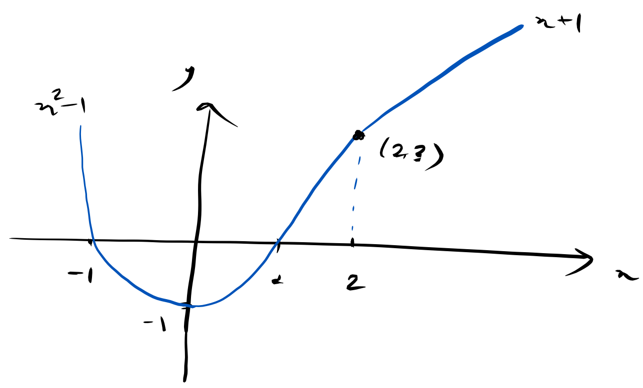
* $y\Big|_{x=0} = \begin{cases} (0+1)^2 = 1 & x \le 0 \\ 1+2\sin(0) = 1 & x > 0 \end{cases}$ $\Rightarrow \exists$ the joint at $(0,1)$

$\Downarrow$

Continuous

illegal notation / desperate move

Ex-3 $y = \begin{cases} x^2-1 & x < 2 \\ x+1 & x \ge 2 \end{cases}$




$y' = \begin{cases} 2x\Big|_{x=2} = 4 & x<2 \\ 1\Big|_{x=2} = 1 & x \ge 2 \end{cases}$

LHD $\neq$ RHD

$\Downarrow$

not smooth joint

$\Downarrow$

not differentiable

$\boxed{x<2}$

$y = x^2-1$

$y' = 2x = 0 \Rightarrow x=0$

$y'' = 2 > 0 \Rightarrow$ minima $\cup$

illegal move

$x=2 \Rightarrow y = (2)^2-1 = 3$

$x = \pm 1 \Rightarrow y = 0$ intersection pts

$x = 0 \Rightarrow y = -1$ ✓

$\boxed{x \geq 2}$

$y = x + 1 \xrightarrow[x=2]{} y = 3$

$\exists$ a point at $(2,3)$

$\Downarrow$

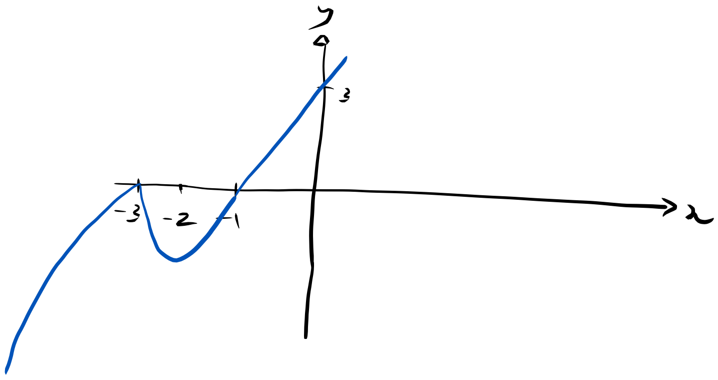
$\exists$ continuity at $x = 2$

Q5 $\quad y = |x+3|(x+1)$

* $y = |x+3|(x+1) = \begin{cases} (x+3)(x+1) = x^2+4x+3 & x+3 \geq 0 \Rightarrow x \geq -3 \\ -(x+3)(x+1) = -x^2-4x-3 & x+3 < 0 \Rightarrow x < -3 \end{cases}$

* $y = 0 \Rightarrow x = -3, -1$

$x = 0 \Rightarrow y = 3$



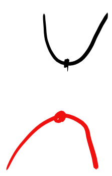


* $\dfrac{dy}{dx} = \begin{cases} 2x+4 & x \geq -3 \\ -2x-4 & x < -3 \end{cases}$

$\dfrac{d^2y}{dx^2} = \begin{cases} 2 & x \geq -3 \\ -2 & x < -3 \end{cases}$ $\quad \boxed{x = -3}$ point of inflexion

* $\dfrac{dy}{dx} = 0 = \begin{cases} 2x+4 \\ -2x-4 \end{cases} \overset{CP}{\Rightarrow} \begin{cases} x = -2 \\ x = -2 \end{cases} \Rightarrow$ CP at $x = -2$

* $\dfrac{d^2y}{dx^2} = \begin{cases} > 0 \\ < 0 \end{cases}$ $\quad$ concave up $x \geq -3$ minima $\quad x < -3$ Maxima



Concavity Analysis

* $\boxed{x < -3}$ $\quad y = -x^2 - 4x - 3 = -(x+3)(x+1)$

$x = 0 \Rightarrow y = -3$

$y = 0 \Rightarrow -x^2 - 4x - 3 = 0$ $\quad \because x < -3$

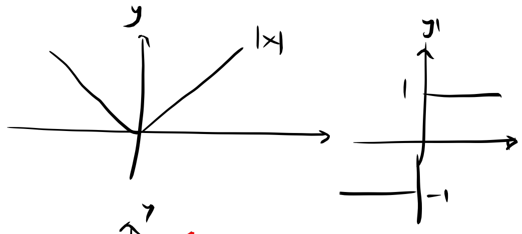
$(x+1)(x+3) = 0 \Rightarrow \cancel{x = -1}, \; x = -3$

* $\dfrac{dy}{dx} = -2x - 4 \Big|_{x=-4} = -2(-4) - 4 = 4 > 0$

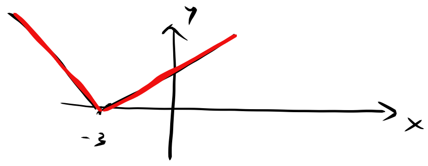
* $x \geq -3 \quad y' = 2x + 4 \Big|_{x=2} = y' > 0$

* $y = \begin{cases} (x+3)(x+1) \\ -(x+3)(x+1) \end{cases}$

$y\Big|_{x=-3} = \begin{cases} 0 \\ 0 \end{cases} \Rightarrow$ at $x = -3$ the curve is continuity







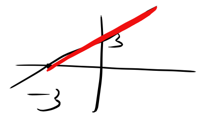


$y = |x+3| = \begin{cases} x+3 & x \geq -3 \\ -(x+3) & x < 3 \end{cases}$

$y = 0 \Rightarrow y = x + 3 \rightarrow x = -3$

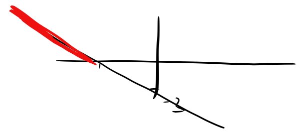
$y = 0 \Rightarrow y = -x - 3 \rightarrow x = -3$

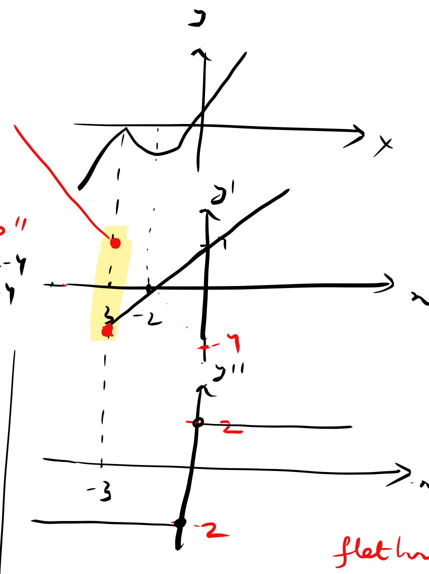
$y = x + 3$





$y = -x - 3$







"Jump"

$y' = -2x - 4$

$0 = -2x - 4$

$x = -2$

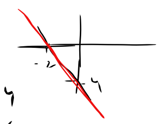
$x = -3$

$y = 2x + 4$

$y = -2x - 4$

$y = -2(-5) - 4$

$10 - 4 = 6$



flatness at $y''$ level

$x = -3$ is the point $\Rightarrow$ continuity

$x = -3$ the point is not smooth

Type 3

Q6 $y = \frac{x+1}{x^2+3}$ $x, y \in \mathbb{R}$

$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z},\ q \neq 0 \right\}$ Rational Numb

* Keep an eye out for Asymptote

Lecture-4 (29/Jan) 2.20

Q6 $y = \frac{x+1}{x^2+3}$ $y, x \in \mathbb{R}$

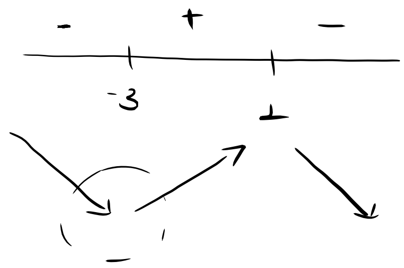
* $x = 0 \rightarrow y = \frac{1}{3}$ 'Intersection pts'

* $y = 0 \rightarrow \frac{x+1}{x^2+3} = 0$ $x = -1$ $x^2+3 \neq 0$

* $y' = \frac{-(x+3)(x-1)}{(x^2+3)^2}$

$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{gf' - fg'}{g^2}$

$\frac{dy}{dx} = 0 \Rightarrow x = -3, 1$ C.P



Analysis of $y'$

$\left.\frac{dy}{dx}\right|_{x<-3} = \frac{-(-)(-)}{+} = - < 0$

$\left.\frac{dy}{dx}\right|_{-3<x<1} = \frac{-(+)(-)}{+} = + > 0$

$\left.\frac{dy}{dx}\right|_{x>1} = \frac{-(+)(+)}{+} < 0$

* $y'' = -\frac{\left\{(x^2+3)^2 \cdot \left[(1)(x-1) + (x+3)(1)\right] - (x+3)(x-1) \cdot 2(x^2+3) \cdot 2x\right\}}{(x^2+3)^4}$

$= -\frac{\left\{(x^2+3)(2x+2) - 4x(x^2+2x-3)\right\}}{(x^2+3)^3} = -\frac{\left\{2x^3+2x^2+6x+6-4x^3-8x^2+12x\right\}}{(x^2+3)^3}$

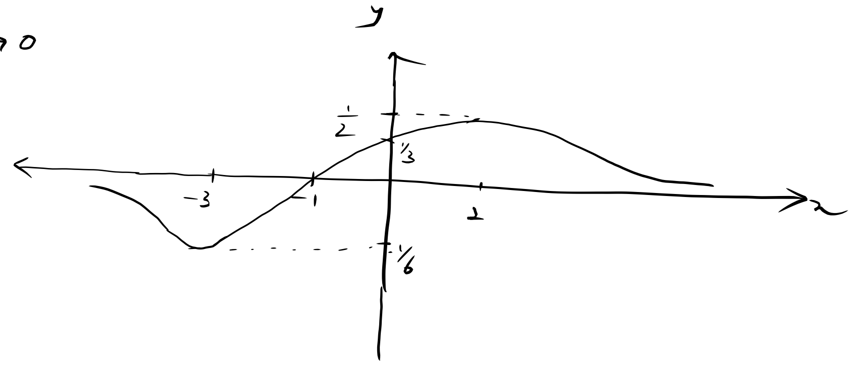
$= -\frac{(-2x^3-6x^2+18x+6)}{(x^2+3)^3} = \frac{2(x^3+3x^2-9x-3)}{(x^2+3)^3}$

* $\left.y''\right|_{x=-3} = \frac{1}{36} > 0 \Rightarrow$ Minima

$\left.y''\right|_{x=1} = -\frac{1}{4} < 0 \Rightarrow$ Maxima

* $\left.y\right|_{x=1} = \frac{1}{2}$, $\left.y\right|_{x=-3} = \frac{-3+1}{9+3} = -\frac{1}{6}$

* $y = \dfrac{x+1}{x^2+3} \longrightarrow x \to \pm\infty \quad y \to 0$




$f(-x) = \dfrac{-x+1}{x^2+3} = \dfrac{-(x-1)}{x^2+3} \neq -f(x)$

Can't check this way

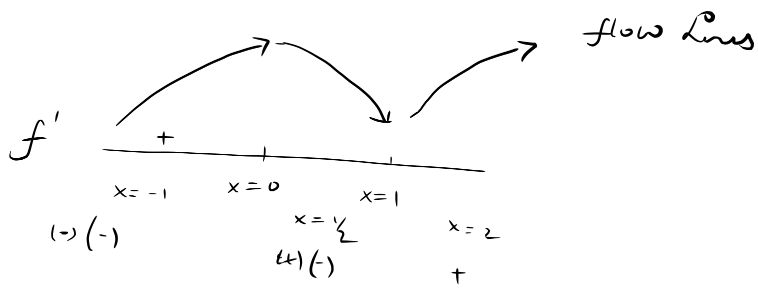
* Inflexion pt $\longrightarrow y'' = 0 = x^3 + 3x^2 - 9x - 3$ Can be done using numerical methods

Eg 7. $y = 2x^5 - 5x^2 + 1$

* $x = 0 \Rightarrow y = 1$

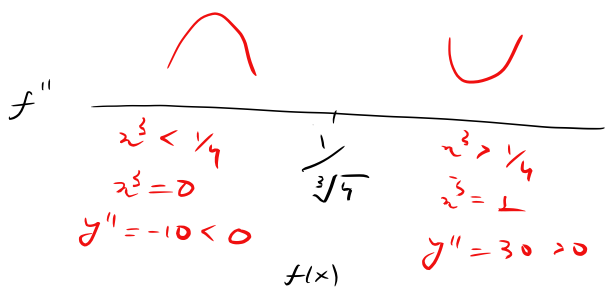
* $y' = 10x^4 - 10x = 0$

$10x(x^3 - 1) = 0 \Rightarrow x = 0, \sqrt[3]{1} = 1 \in \mathbb{R}$



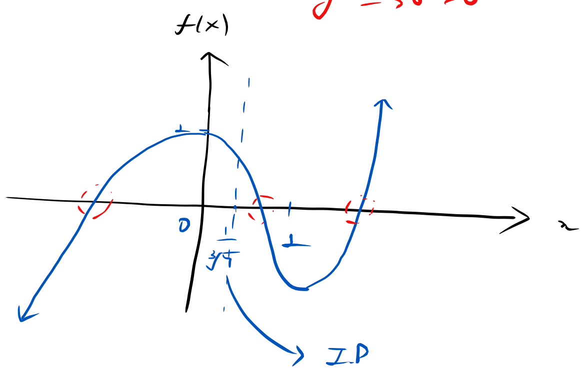

* $y'' = 40x^3 - 10$ $\xrightarrow{x=0} y'' = -10 < 0$ Maxima

$\xrightarrow{x=1} y'' = 30 > 0$ Minima




* $y'' = 0 \Rightarrow \boxed{x^3 = \frac{1}{4}}$ Inflexion point $x = \dfrac{1}{\sqrt[3]{4}}$

* $y\big|_{x=0} = 1$, $y\big|_{x=1} = 2 - 5 + 1 = -2$




* at $x = 0$ Max. $\Rightarrow$ 1 cut on $-\hat{x}$ & 1 on $+\hat{x}$

$x = 1$ Min $\Rightarrow$ 1 cut on $+\hat{x}$

no asymptotes

$\exists$ 3 real roots (Conclusion via graphical method)

doubt

Eg $I = \displaystyle\int \frac{e^{2x}-1}{e^{2x}+1}\,dx = \int \frac{\cancel{e^x}(e^x - e^{-x})}{\cancel{e^x}(e^x + e^{-x})}\,dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$ $\Bigg|$ $e^x + e^{-x} = t$, $(e^x - e^{-x})dx = dt$

$= \displaystyle\int \frac{1}{t}\,dt$

$= \ln t = \ln(e^x + e^{-x}) + \ln\frac{1}{2} = \ln\left(\frac{e^x + e^{-x}}{2}\right) = \ln\cosh x + c$

$\left.\begin{aligned} \frac{e^x + e^{-x}}{2} &= \cosh x \\ \frac{e^x - e^{-x}}{2} &= \sinh x \end{aligned}\right\} \Rightarrow \tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} \longrightarrow \boxed{\int \tanh x\,dx = \ln(\cosh x) + c}$

Method 2

* $\displaystyle\int \frac{e^{2x}-1}{e^{2x}+1} = \int \frac{e^{2x}+1-1-1}{e^{2x}+1} = \int \frac{\cancel{e^{2x}+1}}{\cancel{e^{2x}+1}} - \frac{2}{e^{2x}+1}\,dx = \int 1 - \frac{2}{e^{2x}+1}\,dx$

$= x - 2\displaystyle\int \frac{dx}{e^{2x}+1}$

$e^{2x} + 1 = t$

$2e^{2x}dx = dt$

$$x - 2\int \frac{dt}{2(t-1)t} = x - \underbrace{\int \frac{dt}{t(t-1)}}_{\text{thru Partial fraction}} = x - \int\left\{\frac{1}{t} - \frac{1}{(t-1)}\right\}dt = \ln(\cosh x)$$

HW

simplest version of partial fraction

$$\boxed{\frac{1}{x(x+a)} = \frac{?}{x} + \frac{?}{(x+a)}}$$

of choice of m & n

$$\frac{1}{\cancel{x(x+a)}} = \frac{m}{x} + \frac{n}{(x+a)} = \frac{mx + an + nx}{\cancel{x(x+a)}}$$

$$1 = (m+n)x + am \xrightarrow{\text{Coeff. company}} m+n = 0 \Rightarrow n = -1/a$$

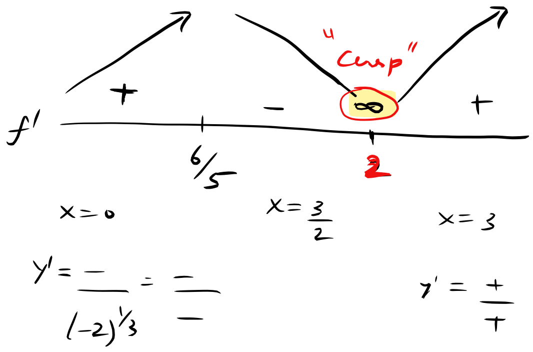
$$am = 1 \Rightarrow m = 1/a$$

$$\frac{1}{x(x+a)} = \frac{1/a}{x} - \frac{1/a}{(x+a)} = \frac{1}{xa} - \frac{1}{a(x+a)}$$

Lecture-5 (30/Jan)

Q8 $y = x(x-2)^{2/3}$

$$y' = x\,\frac{2}{3}(x-2)^{-1/3} + (x-2)^{2/3} = \frac{2x}{3(x-2)^{1/3}} + (x-2)^{2/3} = \frac{5x-6}{3(x-2)^{1/3}}$$



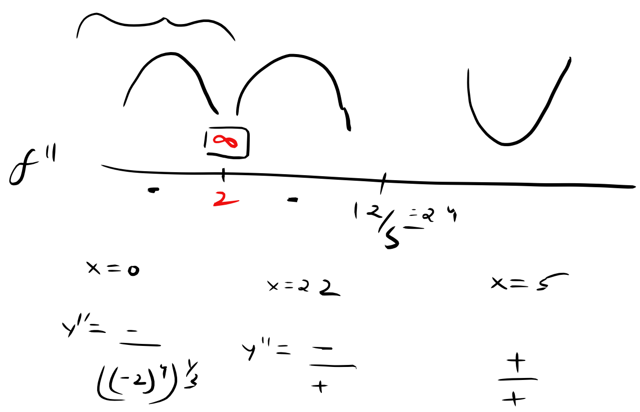

$$y' = 0 = \frac{5x-6}{3(x-2)^{1/3}} \longrightarrow x = 6/5$$

$\longrightarrow$ $x = 2$ $\quad y' \to \infty$ DNE

$15x - 30 - 5x + 6 = 10x - 24$

$$y'' = \frac{3(x-2)^{1/3}\,5 - (5x-6)\cancel{3}\,\frac{1}{\cancel{3}}(x-2)^{-2/3}}{9(x-2)^{2/3}} = \frac{(x-2)^{-2/3}\left\{15(x-2) - (5x-6)\right\}}{9(x-2)^{2/3}}$$

$$= \frac{10x-24}{9(x-2)^{4/3}}$$




$y'' = 0 \Rightarrow \boxed{x = \frac{12}{5}}$ pt of inflection

$x = 2 \Rightarrow y'' \to \infty$ DNE

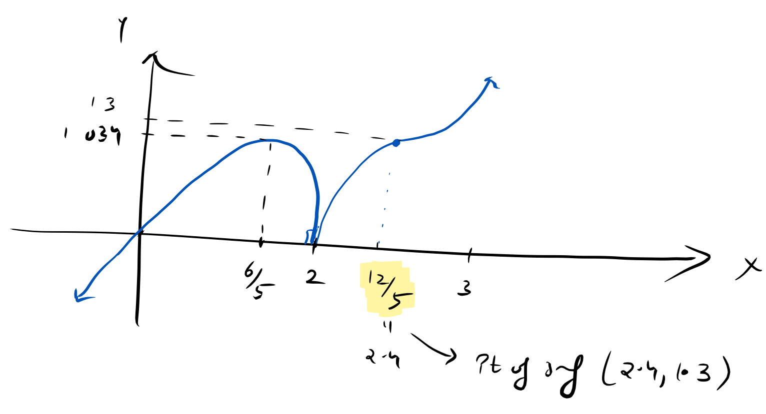
* $x = 0 \Rightarrow y = 0$

* $y'' = \dfrac{10x-24}{9(x-2)^{4/3}}$

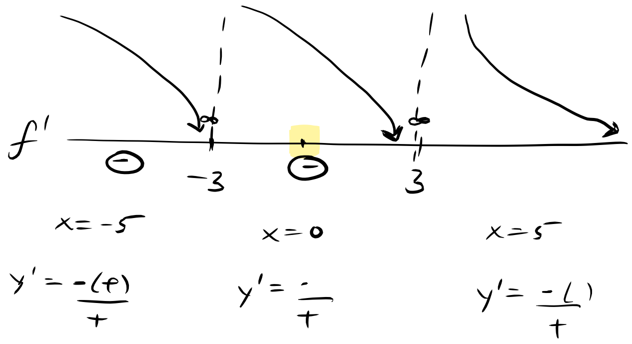
$x = 6/5 \rightarrow y'' < 0 \Rightarrow$ Max

$x > \frac{12}{5} \rightarrow y'' > 0 \Rightarrow$ Min

$y\big|_{x=6/5} = 1.034$



Q9. $y = \dfrac{x}{x^2-9} \qquad x \in \mathbb{R}$




* $y' = \dfrac{(x^2-9)(1) - x(2x)}{(x^2-9)^2} = \dfrac{-x^2-9}{(x^2-9)^2} = \dfrac{-(x^2+9)}{(x^2-9)^2}$

$y' = 0 \Rightarrow$ no critical pt in $\mathbb{R}$

$x = \pm 3 \longrightarrow y' \to \infty$ DNE

* $y'' = -\left\{ \dfrac{(x^2-9)^2(2x) - (x^2+9)\cdot 2(x^2-9)2x}{(x^2-9)^4} \right\}$

$y''$ $\to -3<x<0 \longrightarrow y'' = \dfrac{(-)(+)}{(-)} > 0$ Min. ($x=-2$)

$\to 0<x<3 \longrightarrow y'' = \dfrac{(+)(+)}{-} < 0$ Max ($x=2$)

$= \dfrac{-x(x^2-9)}{(x^2-9)^4}\left\{ 2(x^2-9) - 4(x^2+9) \right\} = \dfrac{-x}{(x^2-9)^3}\left\{ -2x^2 - 54 \right\} = \dfrac{2x(x^2+27)}{(x^2-9)^3}$

$y'' = \dfrac{2x\,\overbrace{(x^2+27)}}{(x^2-9)^3} = 0 \Rightarrow \boxed{x=0}$ Inflection pt

$\in \mathbb{R}$

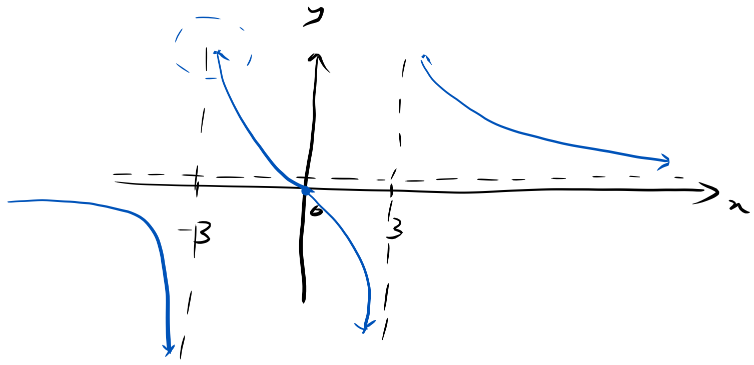

* Asymp $\lim\limits_{x\to\pm 3} \dfrac{x}{x^2-9} \to \pm\infty \qquad \boxed{x = \pm 3}$ vertical asymp

$\lim\limits_{x\to\pm\infty} \left(\dfrac{x}{x^2-9}\right) = \lim\limits_{x\to\pm\infty} \left(\dfrac{1}{x - 9/x}\right) = \left(\dfrac{1}{\infty - \underbrace{9/\infty}_{0}}\right) = \dfrac{1}{\infty} = 0 \qquad \boxed{x = \pm\infty}$

Horizontal Asymp

$Y = \dfrac{x}{x^2-9}$

$Y(-x) = -\dfrac{(x)}{x^2-9} = -y(x) \Rightarrow$ odd



Lecture-6 (21/feb)

1

* $A = \sqrt{a_1} + \sqrt{a_2} \dots + \sqrt{a_n} = \sum_i \sqrt{a_n}$

$B = b_1\sqrt{a_1} \quad + b_n\sqrt{a_n} = \sum_i b_i\sqrt{a_i}$

$C = b_1 m_1 \sqrt{a_1} + \dots + \quad = \sum_i b_i m_i \sqrt{a_i}$

* $\beta = \underbrace{\sqrt{3}}_{IR} + \sqrt{5} + \sqrt{7} = \underbrace{\sum(\text{parts})}$

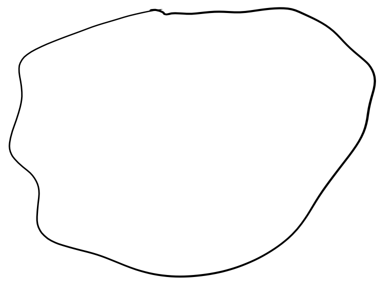
$\sqrt{a_1} = \alpha\sqrt{a_3} + \beta\sqrt{a_4}$ L.D

* $P(n) = \sum_{i=0}^{n} b_i \sqrt{a_i}$ L I $\Rightarrow \exists\, b_i : \sum b_i\sqrt{a_i} = 0$ iff $b_1 = \dots = b_n = 0$

Assumptions

$n = \frac{p}{q}$ $q \neq 0$, $p, q \in \mathbb{Z}$

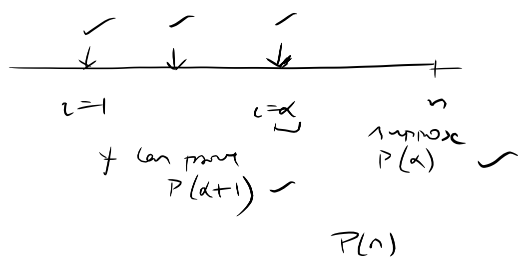
space of states



$\frac{p}{q} = \sum b_i \sqrt{a_i} \Rightarrow \boxed{\frac{p}{q} - \sum_{i=0}^{n} b_i\sqrt{a_i} = 0}$

$(b_1\sqrt{a_1} + b_2\sqrt{a_2} \quad )$

$p = q\sum b_i\sqrt{a_i}$




* To prove $\sqrt{a_1} + \sqrt{a_2} \quad + \sqrt{a_n}$ is IRR

Case 1 $\sqrt{a_1}, \quad , \sqrt{a_n}$ are all irrational

Proof

$P(n) : \sqrt{a_1} + \sqrt{a_2} + \quad \sqrt{a_n}$ is irrational

$P(1) = \sqrt{a_1}$ is true

$P(2) = \sqrt{a_2} + \sqrt{a_2}$ ..

Assume

$P(K) = \sqrt{a_1} + \quad + \sqrt{a_K}$ is true — (1)

assumptions

$P(K+$

To prove

$P(K+1) = \sqrt{a_1} + \quad + \sqrt{a_K} + \sqrt{a_{K+1}}$

Assumptions . $P(K+1)$ is not true

$\underbrace{\sqrt{a_1} + \quad . \quad + \sqrt{a_K}} + \sqrt{a_{K+1}} = \frac{p}{q}$

$\sqrt{a_{K+1}} = \frac{p}{q} - \sum_{i=1}^{K} \sqrt{a_i} \Rightarrow a_{K+1} = \left(\frac{p}{q}\right)^2 + \left(\sum_{i=1}^{n} \sqrt{a_i}\right)^2 + 2\left(\frac{p}{q}\right)\sum_{i=1}^{K} \sqrt{a_i}$

$a_{K+1} - \frac{p^2}{q^2} = \underbrace{\left(\sum_1^K \sqrt{a_i}\right)^2}_{*} + 2\left(\frac{p}{q}\right) \underbrace{\sum_{i=1}^{K} \sqrt{a_i}}_{\text{irr } (1)}$

Case * $\to \sqrt{a_1}, \sqrt{a_2} \quad \sqrt{a_n}$ rational (1) Easy

$\to$ // , irrational (2)

2)

$$\alpha = \left(\sum_{i=1}^{k} \sqrt{a_i}\right)^2 = \left(\sqrt{a_1} + \sqrt{a_2} + \quad \sqrt{a_k}\right)^2$$

$$= \underbrace{\sum_{i=1}^{k} (\sqrt{a_i})^2}_{\sum_{i=1}^{k} a_i} + 2\sum_{j=1}^{k}\left\{\sum_{i=1}^{j-1}\sqrt{a_i}\sqrt{a_j}\right\}$$

rational

$$a_{k+1} - \frac{p^2}{q^2} - 2\left(\frac{p}{q}\right)\sum_{i=1}^{k} a_i = 4\frac{p}{q}\sum_{j=1}^{k}\left\{\sum_{i=1}^{j-1}\sqrt{a_i}\right\}\sqrt{a_j}$$ ??

(2)

$x \in \mathbb{Z}^+$

$x.\quad x \rightarrow 3x$

$\frac{3x}{2}$

5

15

$\frac{15}{2}$ 7.5 $\longrightarrow$ 7

$7 \cdot 3 = 21$

$\frac{21}{2}$ 10.5 $\longrightarrow$ 10

$\frac{\cancel{10}^5\, 3}{\cancel{2}}$ 15 $\longrightarrow$ No

0 1 2 3 4 5 6 7 8 9 10

11 12

23

$23 \times 3 = 69$

$\frac{69}{2} = 34.5$

34 yes

$34 \times 3$ 102 no

$\frac{102}{2}$ 51

$9 \,\overline{)\,51}$

27

$27 \times 3$ 81

$\frac{81}{2} = 40.5$

40 yes

$40 \times 3$ 120 no

$\frac{120}{2}$ 60

$9 \,\overline{)\,60}$

19

$19 \times 3$ 57

$\frac{57}{2}$ 26.5

26 Yes

$26 \times 3$ 78 no

$\frac{78}{2}$ 39

$9 \,\overline{)\,39}$

1.) $\{0, 2, 4, 6, 8, \ldots\}$ $\frac{3}{2}$ once

2) $\{3, 7, 11, 15, 19, \quad\}$ $\frac{3}{2}$ twice

3) $\{5, 9, 13, \quad\}$ $\frac{3}{2}$ 3rd

1

1 $\frac{3}{2}$ $\infty$ times

2) $\{3, 7, 11, 15, 19 \quad\}$ Comm diff $= 4$

$\downarrow \left(\frac{3}{2}\right)^2$

$\{ \quad \}$ Com diff $= 9$ → Quotient with 9

1) $\{0, 2, 4, 6, 8, \quad\}$ Com diff $= 2$

$\downarrow \frac{3}{2}$

$\{ \quad \}$ Comm diff $= 3$ → Quotient by 3

3) $\{5, 9, 13, \quad\}$ Com diff $= 4$

$\downarrow \left(\frac{3}{2}\right)^3$

$\{5(\;), 9(\;) \quad\}$ $CD = \left(\frac{3}{2}\right)^3 \cdot 4 = \frac{27 \cdot \cancel{4}}{\cancel{8}\,2} = \frac{27}{2}$ → Multiply by 2 Quotient with 27

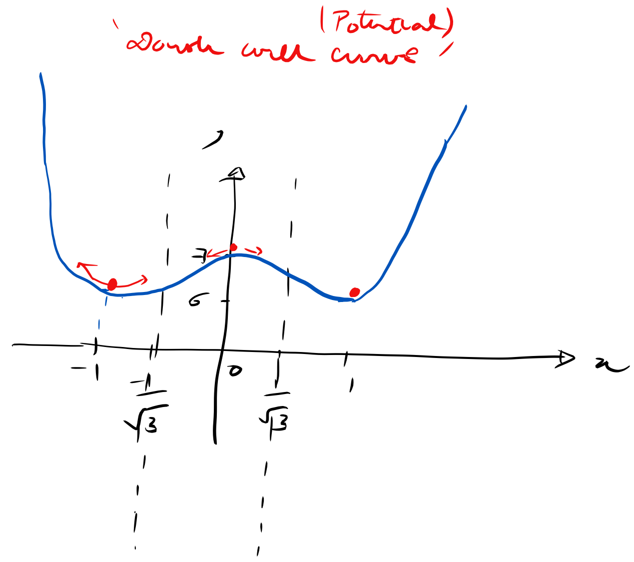
Lecture-7 (4 feb) 1.17

Q10 $y = x^4 - 2x^2 + 7$

* $x^4 - 2x^2 + 7 = 0$

$x = 0 \Rightarrow y = 7$

* $\frac{dy}{dx} = 4x^3 - 4x = 0 \Rightarrow 4x(x^2-1) = 0 \rightarrow x = 0, \; x = \pm 1$ } CP




* $\frac{d^2y}{dx^2} = 12x^2 - 4$ ; $x=0 \rightarrow y'' = -4 < 0 \Rightarrow$ Max ; $x = \pm 1 \rightarrow y'' = 8 > 0 \Rightarrow$ min

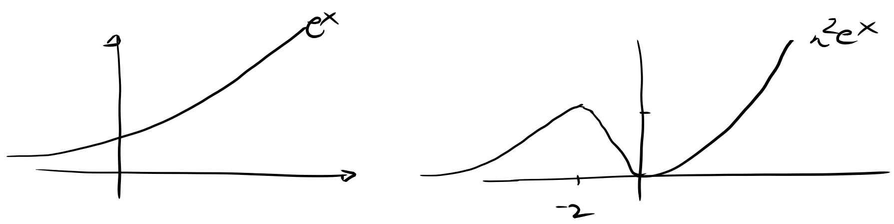
$y\big|_{x=0} = 7$ Maximum val

$y\big|_{x=\pm 1} = 1 - 2 + 7 = 6$ min. val

$\frac{d^2y}{dx^2} = 0 \Rightarrow \boxed{x = \pm \frac{1}{\sqrt{3}}}$ Inflexion pt $x^2 = 1/3$

* $\frac{d^2y}{dx^2} = 12x^2 - 4$
  - $x^2 < -\frac{1}{3} \longrightarrow y'' = 12(-1)^2 - 4 = 8 > 0$ min
  - $-\frac{1}{3} < x^2 < 0 \longrightarrow y'' = 12(-0.1)^2 - 4 < 0$ Max
  - $0 < x^2 < 1/\sqrt{3} \longrightarrow y'' = 12(0.1)^2 - 4 < 0$ Max
  - $x^2 > \frac{1}{3} \longrightarrow y'' = 12(1)^2 - 4 = 8 > 0$ min

Q11 $\overline{\text{2mins}}$ $y = x^2 e^x$




$e^{-\infty} = 0$ $\quad x^2 e^x \xrightarrow{x \to -\infty} 0$

## 1.3 Stability Analysis using Calculus (App. of curve sketching)

* In any dynamical system, stability analysis is done to find equi$^n$ positions & to check if it is stable or unstable

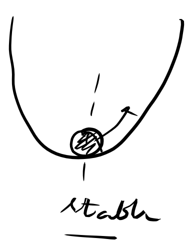
* Introduce the idea of a $\underbrace{\text{'Potential' field}}_{\text{Scalar}}$ instead of Force

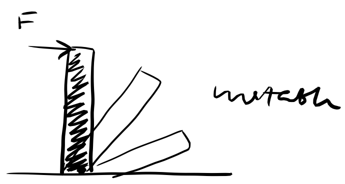
$W_{D_F} = \underset{\sim}{F} \cdot \underset{\sim}{dx} \xrightarrow[\text{Force}]{\text{Conservative}} \underbrace{W_{D_c}}_{F\cdot dx} = -\underset{\sim}{\Delta} U \longrightarrow \underset{\sim}{F} = -\frac{dU}{dx}\hat{n}$ $\quad \underset{\sim}{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

$\boxed{\underset{\sim}{F} = -\underset{\sim}{\nabla} U}$

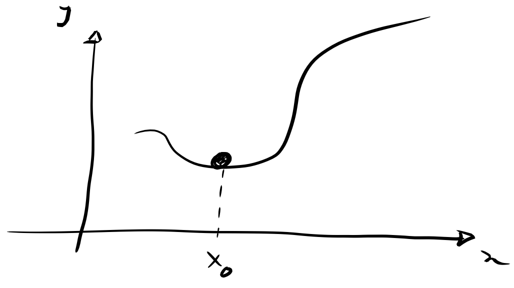
* $\underset{\sim}{F} = 0 = -\frac{dU}{dx} \Rightarrow U(x) = \text{const}$ Equi$^n$ point

$\frac{d^2U}{dx^2}\Big|_{x_0} > 0$ min. (stable point) $\qquad \frac{d^2U}{dx^2}\Big|_{x_0} < 0$ Max (unstable pt)



Case I   Particle of mass $m$ under the influence of a potential $V(x)$

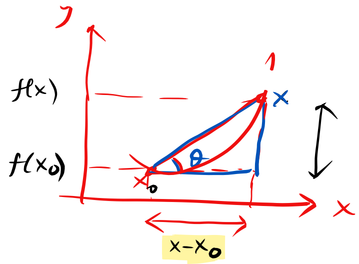


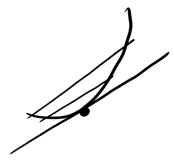

* At $x=x_0$ . $\left.\frac{dV}{dx}\right|_{x_0} = 0$   stationary point

Recap

* $\frac{df}{dx} \overset{\text{def 1}}{=} \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$\frac{df}{dx} \overset{\text{def 2}}{=} \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$



$\tan\theta = \frac{\Delta y}{\Delta x} = \frac{f(x)-f(x_0)}{x-x_0}$

$f(x) \to V(x)$

$(x-x_0)\frac{dV}{dx} = V(x)-V(x_0) \Rightarrow V(x) = V(x_0) + (x-x_0)\left.\frac{dV}{dx}\right|_{x_0} + \frac{(x-x_0)^2}{2!}\left.\frac{d^2V}{dx^2}\right|_{x_0} + \cdots$ (by def 2) ✓

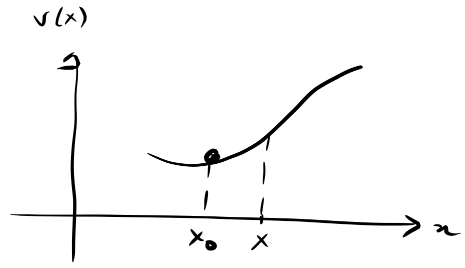
$h\frac{dV}{dx} = V(x+h) - V(x) \Rightarrow V(x+h) = V(x) + h\left.\frac{dV}{dx}\right|_{x=0} + \frac{h^2}{2!}\left.\frac{d^2V}{dx^2}\right| + \cdots$ (by def 1)

{ Taylor series

Lecture-8 (5/feb)   2

* $V(x)$ scalar fun$^n$ $\longrightarrow$ $\underset{\sim}{F} \equiv -\frac{dV}{dx}\hat{n}$

* $x_0$ : $\left.\frac{dV}{dx}\right|_{x_0} = 0$   stationary point (Equi$^m$)




* $V(x) = V(x_0) + (x-x_0)\cancel{\left.\frac{dV}{dx}\right|_{x_0}}^{\,0} + \frac{(x-x_0)^2}{2!}\left.\frac{d^2V}{dx^2}\right|_{x} + \frac{(x-x_0)^3}{3!}\left.\frac{d^3V}{dx^3}\right|_{x_0}$ .

$V(x) \simeq V(x_0) + \frac{(x-x_0)^2}{2}\left.\frac{d^2V}{dx^2}\right|_{x_0}$   { neglecting cubic & higher order terms : Case I

if $x_0$ is a stable Equi$^m$ pt (Minima) $\Rightarrow$ $\left.\frac{d^2V}{dx^2}\right|_{x_0} > 0$

$$\boxed{V(x) \simeq V(x_0) + \frac{1}{2}k(x-x_0)^2} \quad \text{— (1)} \qquad \boxed{k \equiv \left.\frac{d^2V}{dx^2}\right|_{x_0} > 0}$$

Potential near stable equi$^m$ pt

* $$\boxed{\underset{\sim}{F} = -\frac{dV}{dx} = -k(x-x_0)} \quad \text{— (2)}$$ this eq$^n$ is an oscillating Eq$^n$ / SHO

$\omega \equiv \sqrt{\frac{k}{m}}$ ,  $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$

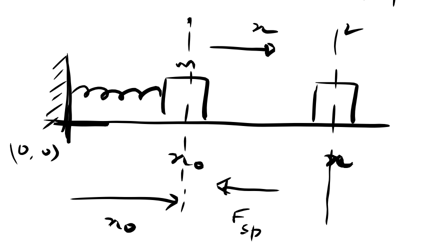
$$\boxed{V(x) = V(x_0) + \frac{1}{2}m\omega^2(x-x_0)^2} \quad \text{— (3)}$$

Potential Eq$^n$ for SHO

* $\omega = \sqrt{\frac{k}{m}}$ = indep of $x$ –

if SHO is not near eq$^m$ $\Rightarrow$ $\omega$ depends on $x$ (Amplitude)

Physical situation (spring mass sys)




stretch it to $x$ & release

$\underset{\sim}{F}_{spring} = -k\underset{\sim}{x}$

$W_{spring} = \underset{\sim}{F}\,d\underset{\sim}{x}$

$= |\underset{\sim}{F}||d\underset{\sim}{x}|\cos\pi$

$= -|\underset{\sim}{F}||d\underset{\sim}{x}|$

$-\Delta U = \cancel{-}\int kx\,dx$

$\Delta U = \int_{x_0}^{x} kx\,dx = \frac{1}{2}k(x^2 - x_0^2)$

$U_2 - U_1 = \frac{1}{2}kx^2 - \frac{1}{2}kx_0^2$

Case II  start pt $x_0$  $\left.\frac{dV}{dx}\right|_{x_0} = 0$ , $\left.\frac{d^2V}{dx^2}\right|_{x_0} = 0$

* $V(x) = V(x_0) + (x-x_0)\underbrace{\cancel{\left.\frac{dV}{dx}\right|_{x_0}}}_{0} + \frac{(x-x_0)^2}{2!}\underbrace{\cancel{\frac{d^2V}{dx^2}}}_{0} + \frac{(x-x_0)^3}{3!}\underbrace{\left.\frac{d^3V}{dx^3}\right|_{x_0}}_{\equiv k'} +$

$V(x) = V(x_0) + k'\frac{(x-x_0)^3}{3!}$

$\underset{\sim}{F} = -\frac{dV}{dx} = -\frac{k'}{2!}(x-x_0)^2 \Rightarrow \boxed{\underset{\sim}{F} \propto -(x-x_0)^2}$ ✓

Not at all a simple harmonic oscillation

$\not\exists$ stability Analysis

$U(x) - U(x_0) = \frac{1}{2}kx^2 - \frac{1}{2}kx_0^2$

choice of coord. depends on me

$x_0 = 0$  $U(x_0) = 0$

$U(x) = \frac{1}{2}kx^2$ ✓

more generally

$\underset{\sim}{F} = -k(x-x_0)$

$m\ddot{x} = -k(x-x_0)$

$\ddot{x} = -\underbrace{\frac{k}{m}}(x-x_0) = -\omega^2(x-x_0)$

$\ddot{x} + \omega^2(x-x_0) = 0$

$\omega = \sqrt{\frac{k}{m}}$  Angular frequency

$T = \frac{2\pi}{\omega}$  time period of oscillation (SHO)

k = SHM const = Force const = Spring const

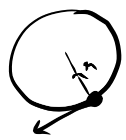
Comment on Angular variable

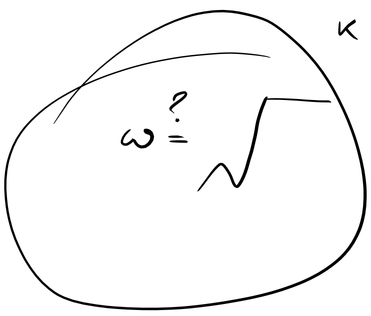
$x = r\theta$
$v = r\omega$
$a = r\alpha$

* $x \longrightarrow \theta$

$V(\theta) = V(\theta_0) + \frac{1}{2}k(\theta-\theta_0)^2$

$k = \left.\frac{d^2V}{d\theta^2}\right|_{\theta_0} > 0$



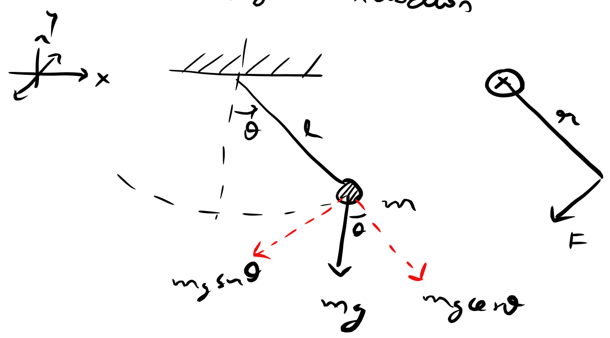



Linear  Rotation

$\underset{\sim}{F} = -k\underset{\sim}{x} \longrightarrow \underset{\sim}{\tau} = -k\underset{\sim}{\theta}$

$\underset{\sim}{\tau} = \underset{\sim}{r}\times\underset{\sim}{F}$
$= \underset{\sim}{r}\times(ma_r)$
$= r(mr\alpha)$
$= \underbrace{mr^2}\alpha$
$\underset{\sim}{\tau} = I\underset{\sim}{\alpha}$

Simple physical situation




$\underset{\sim}{\tau}_F = \underset{\sim}{r}\times\underset{\sim}{F}$
$= l(mg\sin\theta)$ ⊗

not SHM  $\boxed{\tau = -mgl\sin\theta}$

$\tau = -\underbrace{mgl}_{k}\theta$

1st order
$\sin\theta \sim \theta$
SHO

$\boxed{k_{SHM} = mgl}$

Lecture-9 (10/feb) 15

* $\underset{\sim}{F} = -\frac{dV}{dx}\hat{x}$ , $V(x) \simeq V(x_0) + \frac{1}{2}k(x-x_0)^2$ : $\left.\frac{dV}{dx}\right|_{x_0} = 0$ , $k = \left.\frac{d^2V}{dx^2}\right|_{x_0} > 0$  case I

$\underset{\sim}{F}(x) = -k(x-x_0)$

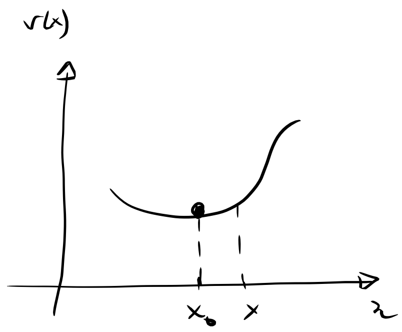
* $\underbrace{F_{SHM}}_{m\ddot{x}} = -k(x-x_0) \Rightarrow \ddot{x} = -\omega^2(x-x_0)$ , $\omega = \sqrt{\frac{k}{m}} \rightarrow T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$

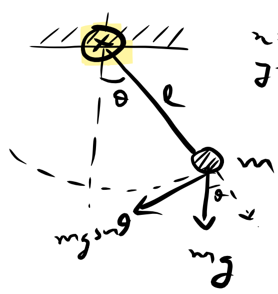
* $x \longrightarrow \theta$  $V(\theta) \simeq V(\theta_0) + \frac{1}{2}k(\theta-\theta_0)^2$  $\left.\frac{dV}{d\theta}\right|_{\theta_0} = 0$ , $k = \left.\frac{d^2V}{d\theta^2}\right|_{\theta_0} > 0$

"$\underset{\sim}{F}$" $= -k(\theta-\theta_0)$

* $\underset{\sim}{F}_{SHM} = -k\underset{\sim}{x}$  $\omega = \sqrt{\frac{k}{m}}$  $\xrightarrow{x\to\theta}$  $\underset{\sim}{\tau} = -k\theta$  $\omega = \sqrt{\frac{k}{I}}$  $T = \frac{2\pi}{\omega}$

$I = \sum_i m_i r_i^2$




$x = r\cos\theta$
$y = r\sin\theta$

r = const

$$\boxed{\underset{\sim}{F} = -k\underset{\sim}{x}}$$

$$m\ddot{x} = -kx$$

$$\ddot{x} = -\omega^2 x$$

$$\underset{\sim}{\tau} = \underset{\sim}{r} \times \underset{\sim}{F} = mgl\sin\theta \quad \otimes$$

$$\underset{\sim}{\tau} = -mgl\sin\theta$$

$$\tau \simeq -(mgl)\theta \qquad \sin\theta \sim \theta$$

$$\boxed{\underset{\sim}{\tau} = -k\underset{\sim}{\theta}}$$ ✓ small oscill

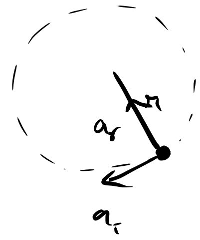
$$\underset{\sim}{\tau} = \underset{\sim}{r} \times \underset{\sim}{F_r}$$

$$= r\, m (a_r)$$

$$= r\, m (r\alpha)$$

$$= \underbrace{mr^2}_{I}\alpha$$

$$\boxed{\underset{\sim}{\tau} = I\underset{\sim}{\alpha}}$$ ✓




$$x = r\theta$$

$$v = r\omega$$

$$a = r\alpha$$

$$I\alpha = -k\theta$$

$$I\frac{d^2\theta}{dt^2} = -k\theta \Rightarrow \ddot{\theta} = -\frac{k}{I}\theta$$

$$\boxed{\ddot{\theta} = -\omega^2\theta}$$

Q1 Consider motion of a classical particle of mass unity in 1D double well

Potential given by $V(x) = \frac{1}{4}(x^2-2)^2$

If particle is displaced infinitesimally from minimum of the axis

Compute Angular freq of oscillation ?

$$\boxed{\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{\left.\frac{d^2V}{dx^2}\right|_{x_0}}{m}}}$$

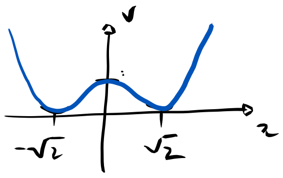
Calculus

Physics borrowed qty

Stable equ$^n$ pt

* $V(x) = \frac{1}{4}(x^2-2)^2$ , ∃ small oscillations ( ∃ minima $x_0$ )

$$V'(x) = \frac{\cancel{2}}{\cancel{4}}(x^2-2)\,\cancel{2}x = 0 \Rightarrow \boxed{x_0 = 0, \pm\sqrt{2}}$$




$$V'' = (x^2-2)(1) + (2x)(x) = -x^2-2$$

$x_0 = 0 \rightarrow \left.\frac{d^2V}{dx^2}\right|_{x_0} = -2 < 0$ max

$x_0 = \pm\sqrt{2} \rightarrow \left.\frac{d^2V}{dx^2}\right|_{x_0} = 4 > 0$ min

$$\boxed{\omega = \sqrt{4} = 2}$$

$$\frac{dV}{dx} = x(x^2-2) \qquad \frac{d^3V}{dx^3} = -2x$$

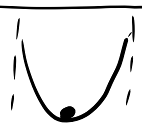
$$V'\big|_{\sqrt{2}} = 0 \qquad V'''\big|_{\sqrt{2}} = -2\sqrt{2}$$

$$V'''' = -2 \qquad V^{(5)} = 0$$

Analysis

$$V(x) = \frac{1}{4}(x^2-2)^2 = \frac{1}{4}(x^4+4-2x^2) = \cancel{V(\sqrt{2})}^{0} + \cancel{\left.\frac{dV}{dx}\right|_{x=\sqrt{2}}}^{0}(x-\sqrt{2}) + \frac{1}{2!}(x-\sqrt{2})^2\underbrace{\left.\frac{d^2V}{dx^2}\right|_{x=\sqrt{2}}}_{4} + \frac{1}{3!}(x-\sqrt{2})^3\left.\frac{d^3V}{dx^3}\right|_{\sqrt{2}}$$

$$\boxed{\underbrace{\frac{1}{4}(x^4-2x^2+4)}_{=\frac{1}{4}(x^2-2)^2} \simeq 2(x-\sqrt{2})^2}$$

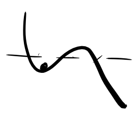


$$\frac{1}{4}(x^2-2)^2 = 2(x-\sqrt{2})^2 - \frac{\sqrt{2}}{3}(x-\sqrt{2})^3$$

$$= (x-\sqrt{2})^2\left\{\frac{2}{1} - \frac{\sqrt{2}}{3}(x-\sqrt{2})\right\}$$

$$\boxed{\frac{1}{4}(x^2-2)^2 \simeq (x-\sqrt{2})^2\left(\frac{8-x\sqrt{2}}{3}\right)}$$

$$\frac{1}{4}(x^2-2)^2 \simeq 2(x-\sqrt{2})^2 - \frac{\sqrt{2}}{3}(x-\sqrt{2})^3 - \frac{1}{4!}(x-\sqrt{2})^4(\cancel{-2})$$



$$= (x-\sqrt{2})^2\left\{\frac{2}{1} - \frac{\sqrt{2}}{3}(x-\sqrt{2}) - \frac{1}{12}(x-\sqrt{2})^2\right\} = (x-\sqrt{2})^2\left\{\frac{24 - 4\sqrt{2}(x-\sqrt{2}) - (x^2+2-2\sqrt{2})}{12}\right\}$$

$$= (x-\sqrt{2})^2\left\{\frac{24 - 4\sqrt{2}x - 8 - x^2 - 2 + 2\sqrt{2}}{12}\right\} = \frac{(x-\sqrt{2})^2}{12}\left\{-x^2 - 2\sqrt{2}x + 16\right\}$$

HW

HW

Q $V(x) = ax^3 - bx$ frq of oscillation about stable equm pot = ?

Q $V(x) = 2\left[e^{-3/2 x} - 2e^{-3/4 x}\right]$ if particle oscillates under the influence of this pot & does small osc around min of $V(x)$ $m=4$

frq, $T = ?$ (sec)

Q Tot Energy of a particle of mass m oscillating about origin is $E = \frac{1}{2}mv^2 + V_0 \cosh\left(\frac{x}{\ell}\right)$

T of oscill = ?

Lecture-10 (11/feb) 2

Soln

Q2 $V(x) = ax^3 - bx$ , $\omega = \sqrt{\frac{K}{m}} = \sqrt{\frac{\left.\frac{d^2V}{dx^2}\right|_{x_0}}{m}}$

$V'(x) = 3ax^2 - b = 0 \Rightarrow x^2 = \frac{b}{3a}$ , $x_{\pm} = \pm\sqrt{\frac{b}{3a}}$

$V''(x) = 6ax$

$x_+ \rightarrow V'' = 6a\sqrt{\frac{b}{3a}} > 0 \rightarrow \min \rightarrow K = 6a\sqrt{\frac{b}{3a}} = \sqrt{12ab}$

$x_- \rightarrow V'' = -6a\sqrt{\frac{b}{3a}} < 0 \rightarrow \max$

$\omega = \left(\frac{(12ab)^{1/2}}{m}\right)^{1/2} = \left(\frac{12ab}{m^2}\right)^{1/4}$

Q3 $V(x) = 2\left[e^{-3/2 x} - 2e^{-3/4 x}\right]$ $K = \left.\frac{d^2V}{dx^2}\right|_{x_0}$

* $V' = 2\left[-\frac{3}{2}e^{-3/2 x} - 2\left(-\frac{3}{4}\right)e^{-3/4 x}\right] = 0$

$-3e^{-3/2 x_0} + 3e^{-3/4 x_0} = 0 \Rightarrow e^{-\frac{3}{2}x_0} = e^{-\frac{3}{4}x_0} \Rightarrow \cancel{-\frac{3}{2}}x_0 = \cancel{-\frac{3}{4}}x_0 \Rightarrow \left(\frac{1}{2} - \frac{1}{4}\right)x_0 = 0 \Rightarrow \boxed{x_0 = 0}$

* $V'' = 2\left[\frac{9}{4}e^{-3/2 x} - 2\,\frac{9}{16}e^{-3/4 x}\right] = \frac{\cancel{2}\times 9}{\cancel{4}_2}\left[e^{-\frac{3}{2}x} - \frac{2}{4}e^{-\frac{3}{4}x}\right]$

$V''\Big|_{x_0} = \frac{9}{2}\left[1 - \frac{1}{2}\right] = \frac{9}{4} = K \Rightarrow T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{K}{m}}} = \frac{2\pi}{\sqrt{\frac{9}{4\cdot 4}}} = \frac{2\times 3.14}{\frac{3}{4}} = 8.37$ sec

Q4 $E = \frac{1}{2}mv^2 + \underbrace{v_0\cosh\left(\frac{x}{\ell}\right)}_{P.E = U}$ $\omega = \sqrt{\frac{\left.\frac{d^2V}{dx^2}\right|_{x_0}}{m}}$ given $x_0 = 0$

* $U = V_0\cosh\frac{x}{\ell} \Rightarrow V(x) = \frac{V_0\cosh\frac{x}{\ell}}{m}$

$K = \left.\frac{d^2V}{dx^2}\right|_{x_0} = \frac{v_0}{m}\frac{1}{\ell^2}\cosh\frac{x}{\ell}\Big|_{x_0} = \frac{v_0}{2m\ell^2} \Rightarrow \omega = \sqrt{\frac{v_0}{2m\ell^2}}$

$T = 2\pi\sqrt{\frac{2m\ell^2}{v_0}}$

$V = \frac{U}{m}$

$\cosh x = \frac{e^x + e^{-x}}{2}$

$(\cosh x)' = \frac{e^x - e^{-x}}{2} = \sinh x$

$(\cosh x)'' = \frac{e^x + e^{-x}}{2} = \cosh x$

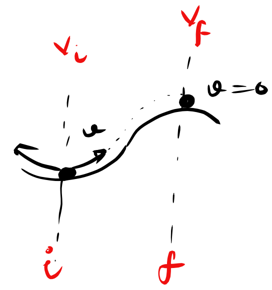
$\cosh 0 = \frac{1}{2}$

Q5 $V(x) = ax - bx^3$ particle of mass $m$ moving under $V(x)$

init it is at rest at stable pt $\rightarrow \frac{d^2V}{dx^2} > 0$

What min speed should be given to it:

Motion becomes unstable $\rightarrow \frac{d^2V}{dx^2} < 0$



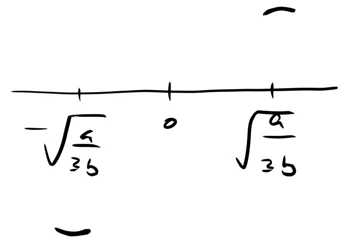

Sys is Conservative

$E_i = E_f$

$K_i + V_i = \cancel{K_f}^{\,0} + V_f$

$K_i = \frac{1}{2}mv^2 = \underbrace{V_f - V_i}_{>0}$

$*$ $V' = a - 3bx^2 = 0 \Rightarrow x^2 = \frac{a}{3b} \Rightarrow x_0 = \pm\sqrt{\frac{a}{3b}}$




$V'' = -6bx$

$x_+ \rightarrow V'' = -6a\sqrt{\frac{a}{3b}} < 0$ max $\longrightarrow x_+$

$x_- \rightarrow V'' = +6a\sqrt{\frac{a}{3b}} > 0$ min $\longrightarrow x_-$

$$V_f = V\big|_{x_+} = a\sqrt{\frac{a}{3b}} - b\left(\frac{a}{3b}\right)^{3/2}$$

$$V_i = V\big|_{x_-} = -a\sqrt{\frac{a}{3b}} - b\left(-\sqrt{\frac{a}{3b}}\right)^3 = -a\sqrt{\frac{a}{3b}} + b\left(\sqrt{\frac{a}{3b}}\right)^3$$

$$\frac{1}{2}mv^2 = V_f - V_i = a\sqrt{\frac{a}{3b}} - b\left(\frac{a}{3b}\right)^{3/2} - \left\{-a\sqrt{\frac{a}{3b}} + b\left(\frac{a}{3b}\right)^{3/2}\right\}$$

$$\frac{1}{2}mv^2 = 2a\sqrt{\frac{a}{3b}} - 2b\left(\frac{a}{3b}\right)^{3/2} = 2\sqrt{\frac{a}{3b}}\left\{a - \cancel{b}\left(\frac{a}{3\cancel{b}}\right)\right\} = 2\sqrt{\frac{a}{3b}}\,\frac{2}{3}a = \frac{4a}{3}\sqrt{\frac{a}{3b}}$$

$$v^2 = \frac{8a}{3m}\sqrt{\frac{a}{3b}} = \sqrt{\frac{64a^3}{27m^2b}} \Rightarrow \boxed{v_{min} = \left(\frac{64a^3}{27m^2b}\right)^{1/4}}$$

Q6 The potential corresponding to the force b/w atoms of diatomic molecule AB

$V(x) = \frac{a}{x^{12}} - \frac{b}{x^6}$ $a, b$ const, $x$ = separation b/w atoms


1 Bond length for stable config?

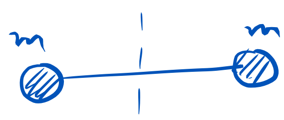
2 freq of oscillation of atoms of mass of each is $m$?

$*$ $\frac{dV}{dx} = -\frac{12a}{x^{13}} + \frac{6b}{x^7} = 0 \Rightarrow \frac{2a}{x_0^6} = b \Rightarrow x_0 = \left(\frac{2a}{b}\right)^{1/6}$ ✓ $x_0^6 = \frac{2a}{b}$

$*$ $k = \frac{d^2V}{dx^2}\Big|_{x_0} = \frac{12\cdot13a}{x^{14}} - \frac{6\cdot7b}{x^8}\Big|_{x_0} = \frac{6}{x^8}\left(\frac{2\cdot13a}{x^6} - 7b\right)\Big|_{x_0}$

$$= \frac{6}{\left(\frac{2a}{b}\right)^{8/6}}\left\{\frac{\overset{13}{\cancel{26}}\,\cancel{a}}{\cancel{2a}}\,b - 7b\right\} = 6\cdot6b\left(\frac{b}{2a}\right)^{8/6} = 36b\,\frac{(b)^{4/3}}{(2a)^{4/3}} = \frac{\overset{18}{\cancel{36}}\,(b)^{7/3}}{\cancel{2}\,2^{1/3}a^{4/3}} \Rightarrow \boxed{k = \frac{18}{2^{1/3}}\,\frac{b^{7/3}}{a^{4/3}}}$$

$$\omega = \sqrt{\frac{k}{\mu}} = \left(\frac{18}{2^{1/3}}\,\frac{b^{7/3}}{a^{4/3}}\,\frac{2}{m}\right)^{1/2} = \text{simplify HW} \left(\quad\right)^{1/2}$$




$\mu = \frac{m_1m_2}{m_1+m_2} = \frac{m}{2}$

HW

$U(x) = \frac{\alpha}{x^2} - \frac{\beta}{x}$ 'm' is constrained by this $U(x)$ $T = 4\pi\sqrt{\frac{2\alpha^3 m}{\beta^4}}$

$V(x) = ax^2 + \frac{b}{x^2}$ unit mass → small oscill' about min of potential $\omega = ?$

Lecture-11 (17/feb) 1:50

2. Transformations

2.1 Graphical transformations $(D=2, \mathbb{R}^2)$

[A] transf' "after" the orig fun'

A1 $f(x) \longrightarrow f'(x) = \begin{cases} f(x)+d \\ f(x)-d \end{cases}$ $d>0$

$\boxed{y = f(x)}$

fiducial curve / Reference / Basis curve

✓ $f(x)+d = (a,b) \longmapsto (a, b+d)$ leaving x value unchanged

✓ $f(x)-d = (a,b) \longmapsto (a, b-d)$ " " "

$y = f(x)$

$f: x \to y$

$f: A \to B$

$f: (x,y) \to (x', y')$

* $f'(x) = f(x)+d \Rightarrow f'(x) - f(x) = d$

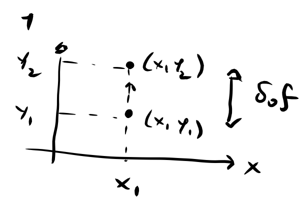
$f'(x) = f(x) - d \Rightarrow f'(x) - f(x) = -d$

→ $\boxed{\delta_0 f(x) \equiv f'(x) - f(x)}$

change in the function at a constant x

{ Notation

$\delta_0 f(x) = \begin{cases} +d = \text{const} \\ -d = \text{const} \end{cases}$



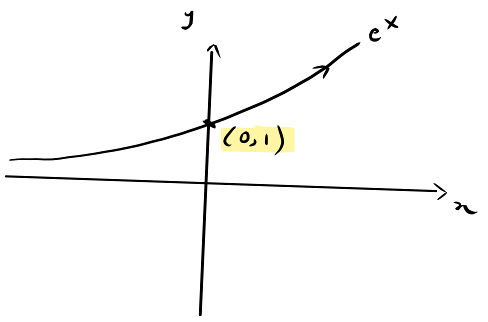
Ex1 $y = e^x$

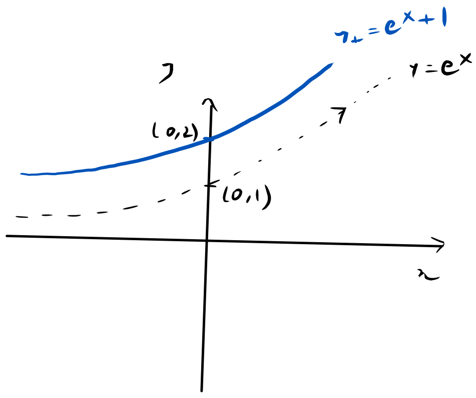
$y_+ = e^x + 1$

$y_- = e^x - 1$

$\{x, y\}$

$d = 1 = \text{const}$





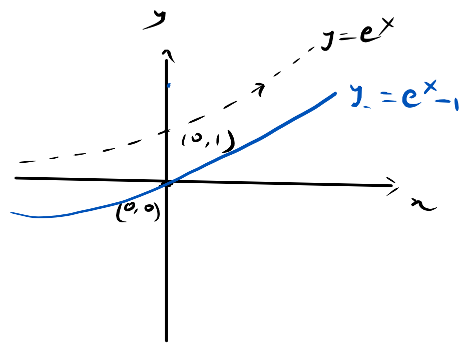
$y_+$ $(a,b) \longmapsto (a, b+1)$

$\boxed{(0,1) \longrightarrow (0,2)}$

shifted upwards by 1

$y_+ = e^x + 1$

$y_+|_0 = 2$



$y_-$ $(a,b) \to (a, b-1)$

$\boxed{(0,1) \to (0,0)}$

shifted downwards by 1

$y_-$ $e^x - 1$

$y_-|_{x=0} = 0$

Calculus,

$y = f(x) \Rightarrow \frac{dy}{dx} = \frac{df}{dx}$

$y' = f(x) \pm d \Rightarrow \frac{dy'}{dx} = \frac{df}{dx} = \frac{dy}{dx}$

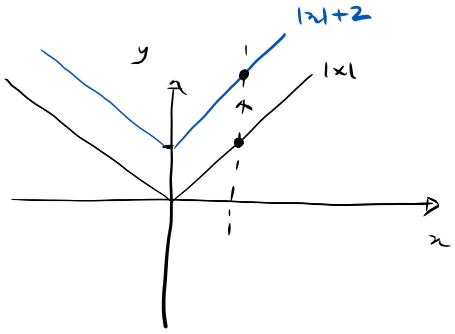
$\boxed{\text{slope}(f(x)) = \text{slope}(f(x) \pm d)}$

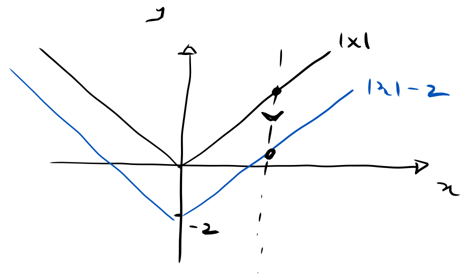
* $f(x) \longrightarrow f(x) + a$ graph is shifted upwards by 'd' units

$f(x) \longrightarrow f(x) - a$ " " " downwards " 'd' "

Ex 2   $y = |x| \longrightarrow y_+ = |x| + 2$   $y_- = |x| - 2$

$$y = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$




$y_+\Big|_{x=0} = 2$   $y_-\Big|_{x=0} = -2$

slopes are unchanged
$m_1 = m_2$ ||

Q2   $f(x) \longrightarrow f'(x) = c\,f(x)$ ← dilation / scale factor of y

$y \longrightarrow y'$

$f \longrightarrow f' : (a, b) \longmapsto (a, cb)$

$$\frac{df'}{dx} = \frac{d}{dx} c f(x) = c \frac{df}{dx}$$

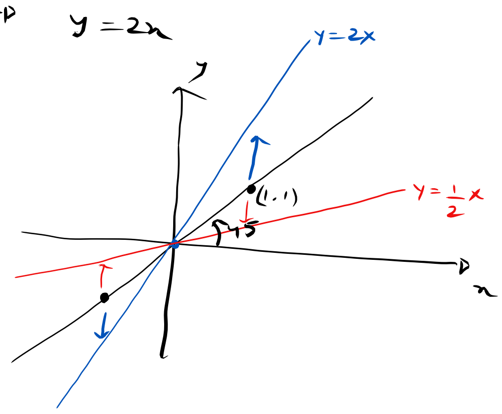
$\boxed{\text{slope}(y') = c \text{ slope}(y)}$ → stretched 'c' times along y

$f(x) \longrightarrow f'(x) = \frac{1}{c} f(x)$

$f \longrightarrow f'$   $(a,b) \longmapsto (a, \frac{b}{c})$

$\boxed{\text{slope}(y') = \frac{1}{c} \text{slope}(y)}$
↓
shrunk 'c' times along y

Ex 3   $\boxed{y = x} \longrightarrow y = 2x$   $y = \frac{1}{2}x$



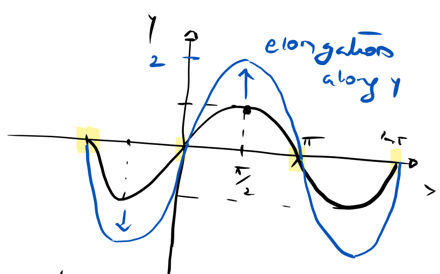

$(a, b) \longmapsto (a, 2b)$
$(1, 1) \longrightarrow (1, 2)$
$(-1, -1) \longrightarrow (-1, -2)$

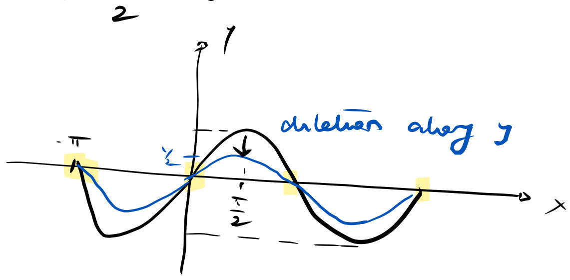
$\frac{dy}{dx} = 2 = \tan\theta_1$

$(a, b) \longrightarrow (a, \frac{b}{2})$
$(1, 1) \longrightarrow (1, \frac{1}{2})$

$\frac{dy}{dx} = \frac{1}{2} = \tan\theta_2$

Ex 4   $y = \sin x \longrightarrow y = 2\sin x$   $y = \frac{1}{2}\sin x$




* $y\Big|_{x=n\pi} = \sin n\pi = 0$
↓
$y = 2\sin x\Big|_{n\pi} = 2 \times 0 = 0$

"Critical points" are untouched

$(a, b) \longmapsto (a, 2b)$
$(\frac{\pi}{2}, 1) \longmapsto (\frac{\pi}{2}, 2)$
$(\pi, 0) \longmapsto (\pi, 0)$

$(a, b) \longmapsto (a, \frac{b}{2})$
$(\frac{\pi}{2}, 1) \longmapsto (\frac{\pi}{2}, \frac{1}{2})$

* $-1 \leq \sin x \leq 1 \longrightarrow \boxed{-2 \leq 2\sin x \leq 2}$ ,

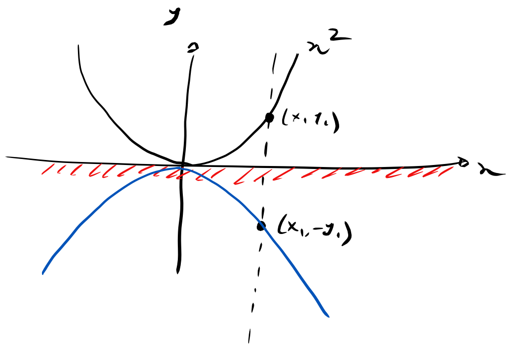
$P(\sin x) = 2\pi$
$P(2\sin x) = 2\pi$ } Period remains intact

Comment $\boxed{c=-1}$

* $f(x) \longrightarrow f'(x) = -f(x)$

$f \longrightarrow f' : (a,b) \longmapsto (a,-b)$ flip over the $x$ axis { mirror transformation

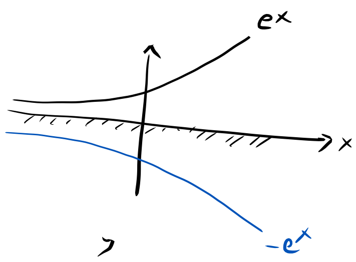
Ex5 $f(x) = -x^2$



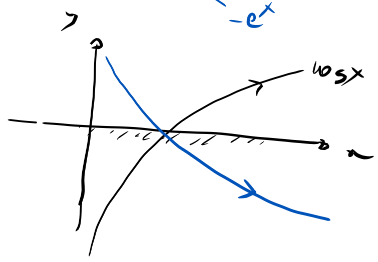

$$\frac{df'}{dx} = \frac{d}{dx}(-f(x)) = -\frac{df}{dx}$$

$$\boxed{\text{slope}(y') = -\text{slope}(y)}$$

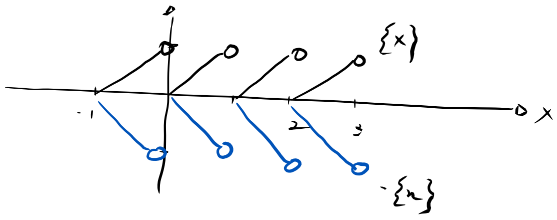
Ex6 $f(x) = -e^x$




$f(x) = -\log x$



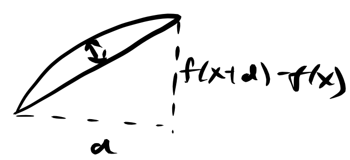

$f(x) = -\{x\}$




Lecture-12 (24/feb) 2

(B) Transf$^n$ "before" the orig func ('could do its job ie transf$^n$')

B1 $x \to x \pm d$ Translation

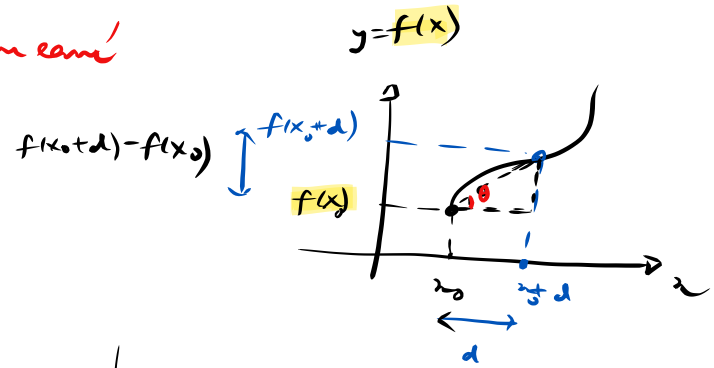



* $f(x) \longrightarrow f(x')$

$f(x')$ $(a,b) \longrightarrow (a \pm d, b)$ '$y$ remains the same'

$$\underbrace{f(x') - f(x)}_{\delta f(x)} = f(x+d) - f(x) = d\left(\frac{df}{dx}\right)$$

change in functions at different space points

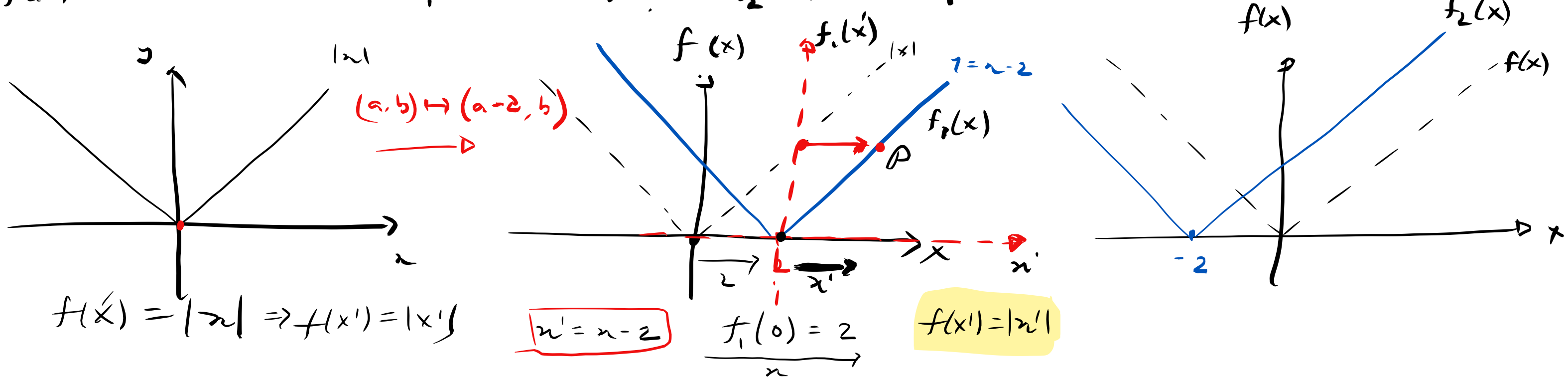



$$\frac{\Delta y}{\Delta x} = \tan\theta = \frac{f(x_0+d) - f(x_0)}{d}$$

$\frac{dy}{dx}$ "vanishing"

$x \to x' = x \pm 2$

Ex7 $y = f(x) = |x| \longrightarrow y = f_1(x') = |x-2|$ $f_2(x') = |x+2|$

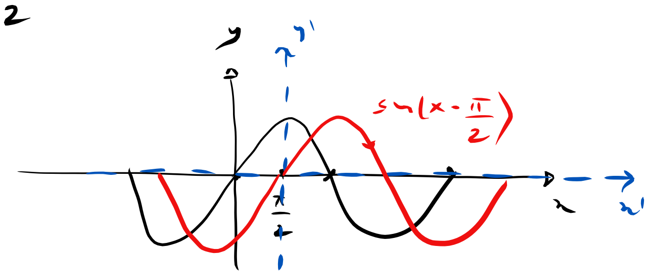


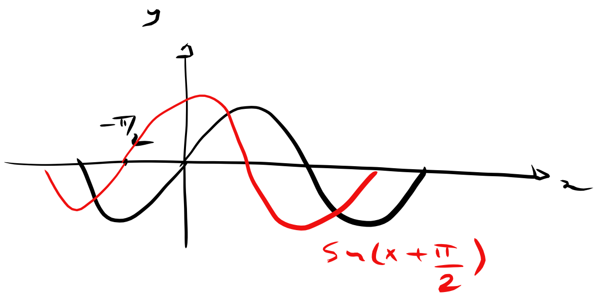
* $y = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$ $\qquad$ $y = |x-2| = \begin{cases} x-2 & x \geq 2 \\ -(x-2) & x < 2 \end{cases}$

$$\underbrace{\frac{dy}{dx} = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases} \qquad \frac{dy}{dx} = \begin{cases} 1 & x \geq 2 \\ -1 & x < 0 \end{cases}}_{\text{slope remains the same}}$$

**Ex8** $\quad y = \sin x \xrightarrow[x \to x - \frac{\pi}{2}]{} y_1 = \sin\left(x - \frac{\pi}{2}\right)$ $\qquad$ $y_2 = \sin\left(x + \frac{\pi}{2}\right)$

$x' = x - \frac{\pi}{2}$



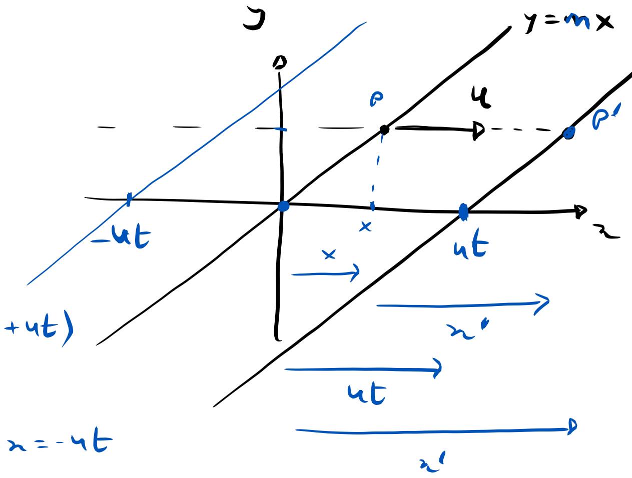




$y = \sin\left(x - \frac{\pi}{2}\right) = 0 = \sin n\pi$ $\qquad$ $\sin\left(x + \frac{\pi}{2}\right) = 0$

$x = \frac{\pi}{2}$ $\qquad$ $x = -\frac{\pi}{2}$

$y' = \sin x'$ in $(x'y')$ plane

**Ex9**




① $y' = m(x - ut)$
$= m(x')$

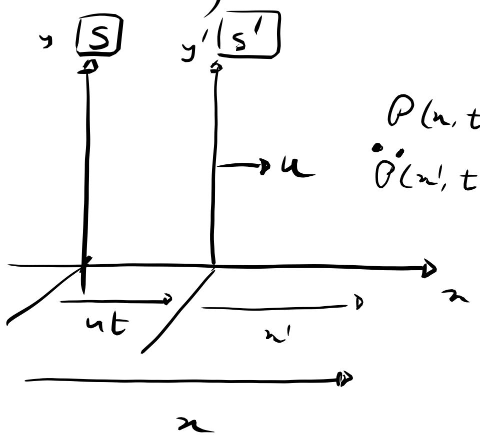
$y = m(x + ut)$
$\downarrow$
$y|_0 \Rightarrow x = -ut$

⊛ $\boxed{x' = x - ut}$ $\Rightarrow$ 'curve' has shifted to the right

EOL ① $\quad y - 0 = m(x - ut)$

$\boxed{y = m(x - ut)}$

Another way to represent the situation,

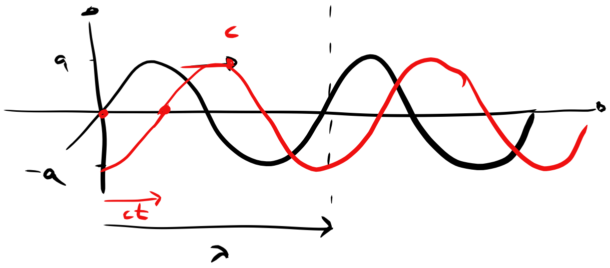



$P(x, t)$
$P'(x', t')$

$\left.\begin{aligned} x' &= x - ut \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}\right\}$ Galilean transformation

$x \longrightarrow x' = x - ut$

**Ex10**




Simple Harmonic oscillation

$y = a \sin\theta = a \sin \omega t = a \sin Kx$ $\qquad$ $Kc = \omega$

$\downarrow \; x \to x' = x - ct$

$x = ct \;|\; y = 0$

$y = a \sin(K(x - ct))$

$= a \sin(Kx - Kct)$

fix $x$ in wave eq^n $x = x_1$ (say)

$\left.\begin{aligned} y &= a\sin(Kx_1 - \omega t) \\ &= a \sin(\omega t + \Phi) = A \sin(\omega t + \Phi) \end{aligned}\right\}$

wave eq^n $\longrightarrow$ $\boxed{y = a \sin(Kx - \omega t)}$

Comment

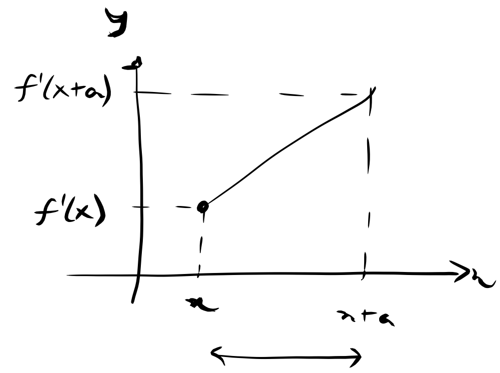
$*$ $f(x) \longrightarrow f'(x') = f(x) + \delta f(x)$ $\qquad x \to x' = x + a$

$\underbrace{f'(x+a)} = f(x) + \delta f(x)$

$f'(x) + a\dfrac{df}{dx} = f(x) + \delta f(x)$

$\delta f(x) = f'(x) - f(x) + a\dfrac{df}{dx}$

$$\boxed{\underbrace{\delta f(x)} = \underbrace{\delta_0 f(x)}_{\substack{= f'(x) - f(x) \\ \text{change in fun at const } x}} + \underbrace{a\frac{df}{dx}}_{\text{amt of change in fun wrt } x}}$$




$\dfrac{f'(x+a) - f'(x)}{a} = \dfrac{df}{dx}$

$\underbrace{\underbrace{f'(x-a)} - f'(x)}_{\text{change the } x} = a\dfrac{df}{dx}$

ex $\quad \boxed{f(x) = x^2}$, $\quad x \to x' = x + a$

$f'(x') = f(x)$

$f'(x+a) = f(x)$

$f'(x) = f(x-a) = (x-a)^2 \Rightarrow \boxed{f'(x) = (x-a)^2}$

$\delta_0 f(x) = f'(x) - f(x) = (x-a)^2 - x^2 = \underbrace{a^2 - 2ax}$

$f(x) = x^2 \xrightarrow{x \to x+a} f(x+a) = (x+a)^2 = f'(x')$

$f'(x') = f(x)$