Lecture-13 (22/Jul) 2

Comment on Asymp of ellipse/circle/Parabola :

- Ellipse : $y = mx + c$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $x_{1,2} = \frac{-2a^2mc \pm 2ab\sqrt{a^2m^2 + b^2 - c^2}}{2(b^2 + a^2m^2)}$

Asymp. $\Rightarrow$ $x_{1,2} \to \frac{0}{0}$; $b^2 + a^2m^2 = 0$ $\Rightarrow$ $\boxed{m \in \mathbb{C}}$

$\nexists$ any real Asymptote to ellipse

- circle : $y = mx + c$, $x^2 + y^2 = a^2$ $\Rightarrow$ $x^2 + (mx+c)^2 = a^2$ $\Rightarrow$ $(1+m^2)x^2 + 2mcx + (c^2 - a^2) = 0$

$$x_{1,2} = \frac{-2mc \pm \sqrt{4m^2c^2 - 4(1+m^2)(c^2-a^2)}}{2(1+m^2)} = f(m)$$

Asymp. $\Rightarrow$ $f(m) \to \frac{0}{0}$, $1 + m^2 = 0$ $\Rightarrow$ $\boxed{m = i \in \mathbb{C}}$

$\nexists$ any real asymptote to circle

- Parabola : $y = mx + c$, $y^2 = 4ax$ $\Rightarrow$ $m^2x^2 + (2mc - 4a)x + c^2 = 0$

$\Downarrow$

$$x_{1,2} = \frac{-(2mc-4a) \pm \sqrt{(2mc-4a)^2 - 4m^2c^2}}{2m^2} = \frac{-(2mc-4a) \pm 4\sqrt{a(a-mc)}}{2m^2}$$

Asymptote $\Rightarrow$ $x_{1,2} \to \frac{0}{0}$ $\Rightarrow$ $2mc - 4a = 0$ $\Rightarrow$ $mc - 2(mc) = 0$

$\Downarrow$

$mc = 0$ $\Rightarrow$ $\boxed{c = 0}$

x intercept of line

Zero. = m = 0

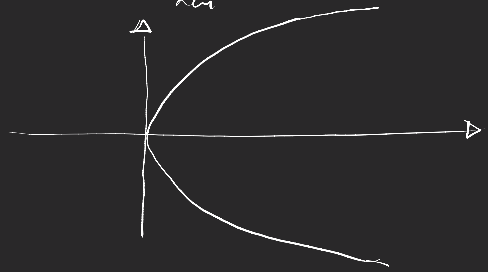
$D = 0$: $\tan\theta$ $\Rightarrow$ $\boxed{a = mc}$

$\underset{0}{m} \cdot \underset{0}{c} = 0$

$\nexists$ any asymptote (no definite slope)

$\downarrow$

In short There is no straight line approaching the Parabola



H.1 Distance from asymptote

$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$

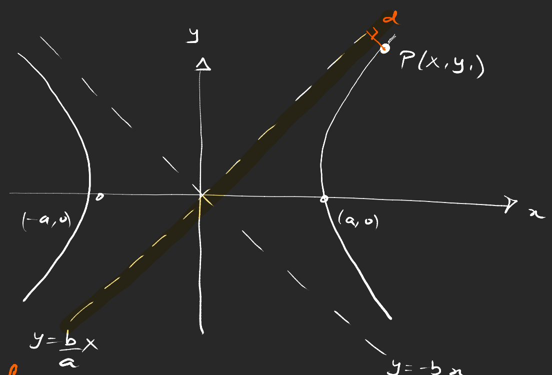
$b^2x_1^2 - a^2y_1^2 = a^2b^2$

$(bx_1 - ay_1) = \frac{a^2b^2}{bx_1 + ay_1}$

* l : $bx - ay = 0$, $P(x_1, y_1)$

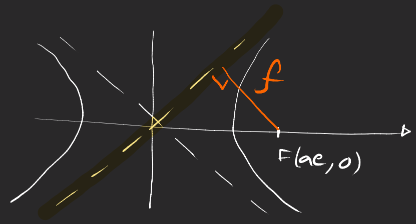
$$\boxed{d = \frac{|bx_1 - ay_1|}{\sqrt{b^2 + a^2}} = \frac{a^2b^2}{\sqrt{a^2+b^2}\,(bx_1 + ay_1)}}$$

$\perp$ dist from a pt. To asym l



extension

* l : $bx - ay = 0$, $P(ae, 0) = F$



$$\boxed{f_{\perp} = \frac{bae}{\sqrt{a^2+b^2}} = b}$$

$\perp$ dist from focus to asymptote

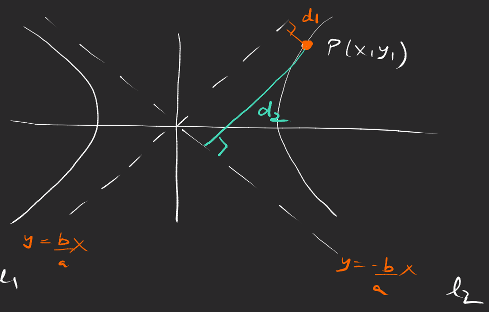
$b^2 = a^2(e^2-1)$

$b^2 = a^2e^2 - a^2$

$b^2 + a^2 = a^2e^2$

H2 Product of Asymptotes

* $d_1 = \dfrac{a^2b^2}{\sqrt{a^2+b^2}\,\underbrace{(bx_1+ay_1)}}$



* l_2 : $bx + ay = 0$, $P(x_1, y_1)$

$$\boxed{d_2 = \frac{|bx_1+ay_1|}{\sqrt{a^2+b^2}} = \frac{a^2b^2}{\sqrt{a^2+b^2}\,\underbrace{(bx_1-ay_1)}}}$$

$\perp$ dist from a pt to l_2 asymp

* $d_1 d_2 = \dfrac{a^2b^2}{(a^2+b^2)}\,\dfrac{\cancel{a^2b^2}}{\cancel{(bx_1+ay_1)}(bx_1-ay_1)} = \dfrac{\cancel{a^2}b^2}{\cancel{a^2}e^2} = \dfrac{b^2}{e^2}$

$\dfrac{x_1^2}{a^2} - \dfrac{y_1^2}{b^2} = 1$

$bx_1^2 - ay_1^2 = a^2b^2$

$$\boxed{d_1 d_2 = \frac{b^2}{e^2}}$$

product of 2 asymptotes $l_1 l_2$

$b^2 = a^2(e^2-1)$

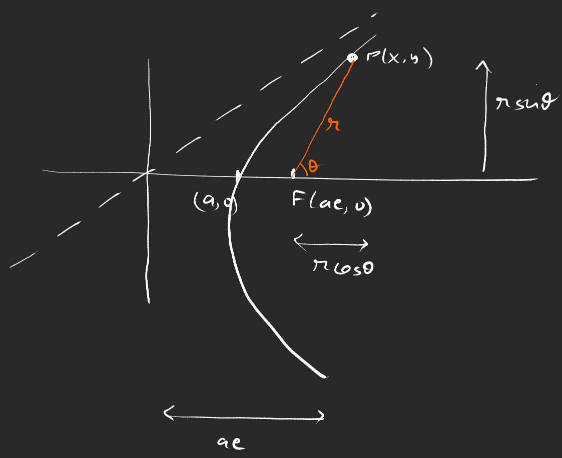
$b^2 + a^2 = a^2e^2$

Product of (distances from a pt. to the) 2 asymptotes

I. Polar forms of Hyperbola

I.1 "Pole" = Focus (Imp.)

* $P(x,y) \xrightarrow{CT} P(r,\theta)$ Coord. Trans^n



$x = ae + r\cos\theta$, $y = r\sin\theta$

$H \equiv \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \Rightarrow \dfrac{(ae+r\cos\theta)^2}{a^2} - \dfrac{r^2\sin^2\theta}{b^2} = 1$

$a^2e^2b^2 + r^2b^2\cos^2\theta + 2aeb^2r\cos\theta - a^2r^2\underbrace{\sin^2\theta}_{1-\cos^2\theta} = a^2b^2$

$b^2 = a^2(e^2-1)$

$b^2 + a^2 = a^2e^2$

$-a^2r^2 + 2are b^2\cos\theta + \underbrace{(a^2+b^2)}_{a^2e^2}r^2\cos\theta = \underbrace{a^2b^2(1-e^2)}_{-b^4}$

$(ar)^2 - 2are b^2\cos\theta - a^2e^2r^2\cos^2\theta = b^4 \Rightarrow (ar)^2 = \underbrace{b^4 + 2aerb^2\cos\theta + a^2e^2r^2\cos^2\theta}_{(b^2+aer\cos\theta)^2}$

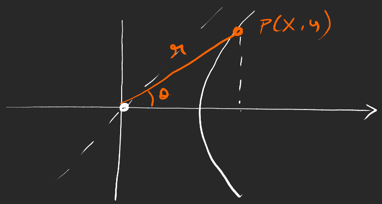
* $ar = b^2 + aer\cos\theta \Rightarrow a(1 - e\cos\theta)r = b^2$

$$\boxed{r = \frac{b^2/a}{1 - e\cos\theta} = \frac{\ell}{1 - e\cos\theta}}$$ $\ell = \frac{LR}{2}$

Polar eqⁿ of Hyperbola

Pole = focus

I 2 "Pole" = Center



* $x = r\cos\theta \;,\; y = r\sin\theta$

* $\frac{(r\cos\theta)^2}{a^2} - \frac{(r\sin\theta)^2}{b^2} = 1 \Rightarrow r^2b^2\cos^2\theta - r^2a^2\underbrace{\sin^2\theta}_{(1-\cos^2\theta)} = a^2b^2$

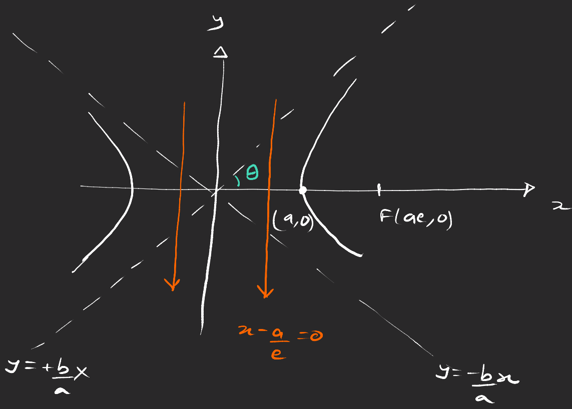
$b^2 = a^2(e^2-1)$

$r^2\underbrace{(a^2+b^2)}_{\cancel{a^2}e^2}\cos^2\theta - r^2\cancel{a^2} = \cancel{a^2}b^2 \Rightarrow r^2(e^2\cos^2\theta - 1) = b^2 \Rightarrow \boxed{r = \frac{b}{\sqrt{e^2\cos^2\theta - 1}}}$

Polar eqⁿ from center

J. Rectangular Hyperbola

J.1 Equation of Rect. hyperbola (Format 1)



* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad b^2 = a^2(e^2-1) \;,\; e > 1$

* $e = \sqrt{2} \Rightarrow \boxed{b = a}$ defⁿ of rect. hyperbola

* $\boxed{x^2 - y^2 = a^2}$ eqⁿ of rect. hyp.

  • Close resemblance with circle

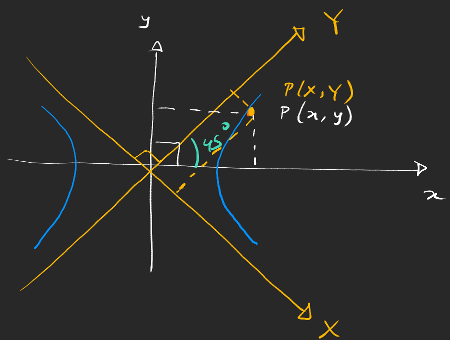
• $V(\pm a, 0)$
• $F(\pm\sqrt{2}a, 0)$
• $LR = \frac{2b^2}{a} = 2a$
• Asymptote $\Rightarrow y = \pm\frac{b}{a}x \xrightarrow{b=a} \boxed{y = \pm x}$ $\quad m = \tan\theta = \pm 1 \Rightarrow \underbrace{\theta = 45^\circ}$

45° lines ("light like" lines)

$m_{l_1} \cdot m_{l_2} = -1 \Rightarrow l_1 \perp l_2 \Rightarrow$ Asymptotes are at 90° $\Rightarrow$ "Rectangular"

Note :
- Asymp. are 45° lines
- Asymp. are ⊥ to each other

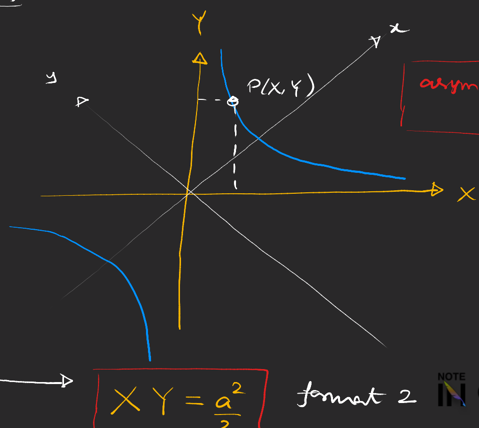
"Problem":



$x^2 - y^2 = a^2$

$\longrightarrow$ Rotate the Axes (x-y) :

Axes = asymptotes



asymptotes treated as axes

$\boxed{XY = \frac{a^2}{2}}$ format 2

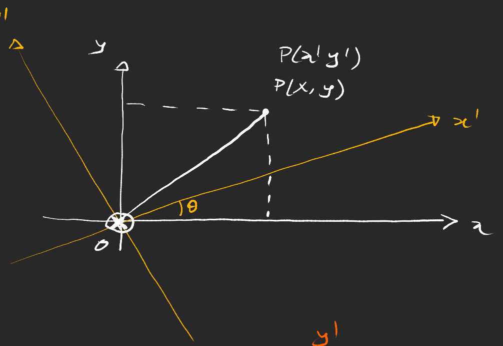
set $a = \sqrt{2}$

$X \cdot Y = 1$

$Y = \frac{1}{X}$

Lecture-14 (23 Jul) 2

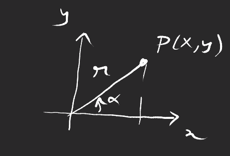
5.2 Concept of Rotation in a plane



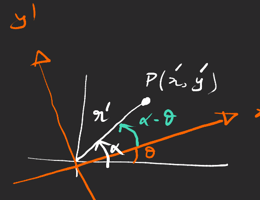
* xy system : $P(x,y) \to P(r,\alpha)$ $\quad x = r\cos\alpha$, $y = r\sin\alpha$

$\downarrow$ R : rot^n operator

x'y' system : $P(x'y') \to P(r',\alpha')$ $\boxed{\begin{matrix} x' = x'(r,\alpha) \\ y' = y'(r,\alpha)\end{matrix}} = ?$



$x^2+y^2=r^2$



$x' = r'\cos(\alpha-\theta)$
$y' = r'\sin(\alpha-\theta)$
$x'^2+y'^2=r'^2$

Transf^n

* $x' = r\cos(\alpha-\theta) = \overbrace{r\cos\alpha}^{x}\cos\theta + \overbrace{r\sin\alpha}^{y}\sin\theta = x\cos\theta + y\sin\theta$

$y' = r'\sin(\alpha-\theta) = r\sin\alpha\cos\theta - r\cos\alpha\sin\theta = y\cos\theta - x\sin\theta$

$$\boxed{\begin{pmatrix} x \\ y\end{pmatrix} \xrightarrow{R} \begin{pmatrix} x' \\ y'\end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}}_{R_{xy} = R(\theta) = \text{rotation matrix}} \begin{pmatrix} x \\ y\end{pmatrix}}$$

In Matrix form

2D

- it describes rotation in xy plane $\overset{\text{only true in 2-D}}{\equiv}$ rotation about z axis
- plane is more fundamental than axis in $D \geq 3$
- $R(\theta) \in SO(2)$ 'rot^n group'

* $x' = x\cos\theta + y\sin\theta$; $y' = -x\sin\theta + y\cos\theta$

$|OP| = \sqrt{x^2+y^2}$; $|OP'| = \sqrt{x'^2+y'^2} = \sqrt{x^2+y^2} = |OP|$

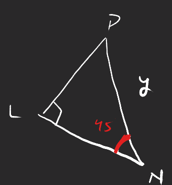
- $\begin{pmatrix} x \\ y\end{pmatrix} \to \begin{pmatrix} x' \\ y'\end{pmatrix} = \underset{\substack{\uparrow \\ \text{change / diff.} \\ \text{(secondary)}}}{R} \begin{pmatrix} x \\ y\end{pmatrix}$: $\underset{\substack{\uparrow \\ \text{const} \\ \text{(primary)}}}{\text{dist b/w 2 pts = invariant}}$

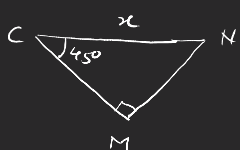
5.3. Application to Rect. Hyperbola (Format 2 of Eq^n)

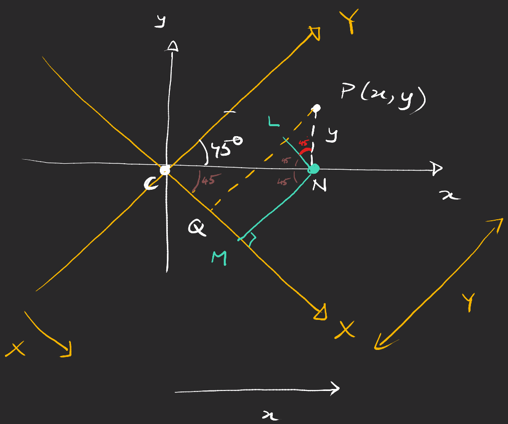
Method 1

* $X = CQ = CM - \underbrace{QM}_{LN} = x\cos 45 - y\cos 45^\circ = \dfrac{x-y}{\sqrt{2}}$

$Y = PQ = PL + \underbrace{LQ}_{NM} = y\sin 45^\circ + x\sin 45^\circ = \dfrac{x+y}{\sqrt{2}}$







* $\boxed{\dfrac{x^2-y^2}{2} = XY}$

* $\mathcal{H} \equiv x^2-y^2 = a^2 \longrightarrow \boxed{XY = \dfrac{a^2}{2} \equiv A^2}$ Eq^n of Rect hyperbola

Asymp are axes

Method 2 (slight right now)

* $(x, y) \xrightarrow{\hat{R}} (X, Y) = \hat{R}(x, y)$ $\quad \hat{R}_{xy}(\theta)$

set

$\boxed{\theta = 45^\circ}$

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ -\frac{1}{\sqrt2} & \frac{1}{\sqrt2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x+y}{\sqrt2} \\ \frac{-x+y}{\sqrt2} \end{pmatrix}$$

* $\mathcal{H} \equiv x^2 - y^2 = a^2 \xrightarrow{\hat{R}} \boxed{XY = A}$

J.4 Mark F, V, dirx, LR

* $\boxed{\mathcal{H} \equiv XY = \frac{a^2}{2} \equiv A^2}$

- Vertex : $V(\pm A, \pm A)$
- Focus : $F(\pm A\sqrt2, \pm A\sqrt2)$ $\qquad b^2 = a^2(1-e^2)$
- Dirx : $x - \frac{a}{e} = 0$ $\qquad e = \sqrt2$

$\Downarrow$

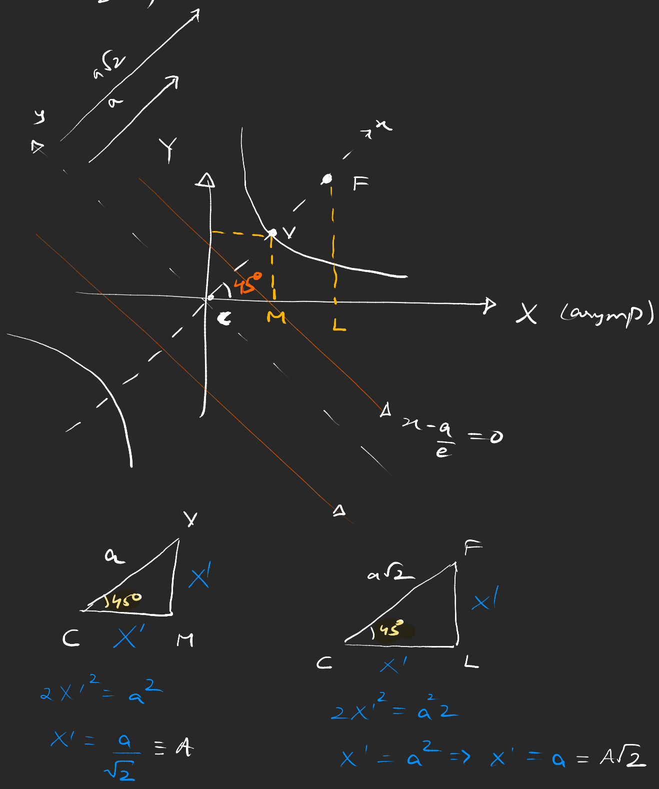
$\boxed{X + Y \pm a = 0}$

$X = \frac{x-y}{\sqrt2}$

$Y = \frac{x+y}{\sqrt2}$

$x = \frac{X+Y}{\sqrt2}$

- $LR = \frac{2b^2}{a} = 2a = 2\sqrt2 A$



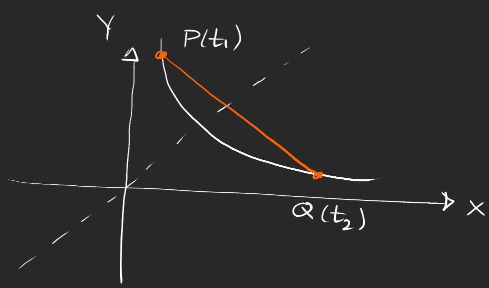
J.5 Parametric form of rect hyp

* $\boxed{\begin{matrix} X = At \\ Y = \frac{A}{t} \end{matrix}} \Rightarrow XY = A^2$ $\qquad$ t : parameter

J.6 Eq^n of chord

* $P(X_1, Y_1) \longrightarrow P\left(At_1, \frac{A}{t_1}\right) = P(t_1)$ shorthand

$Q(X_2, Y_2) \longrightarrow Q\left(At_2, \frac{A}{t_2}\right)$



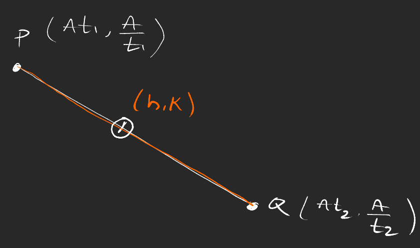
Eq^n PQ : $Y - \frac{A}{t_1} = \frac{\cancel{A}\left(\frac{1}{t_2} - \frac{1}{t_1}\right)}{\cancel{A}(t_2 - t_1)}(X - At_1) = -\frac{1}{t_1t_2}(X - At_1)$

$t_1t_2Y - At_2 = -X + At_1 \Rightarrow \boxed{X + t_1t_2Y = A(t_1+t_2)}$ $\qquad$ • $m_{chord} = -\frac{1}{t_1t_2}$

eq^n of chord

J7 Eq^n of chord with (h, k) as mid pt

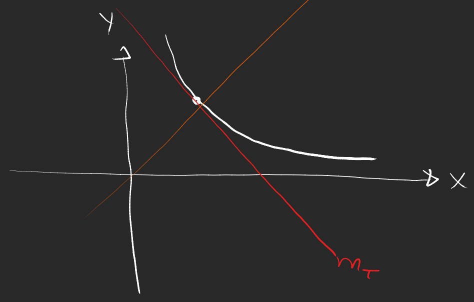
* $(h, k) = \left(\frac{A(t_1+t_2)}{2}, \frac{A}{2}\left(\frac{1}{t_1} + \frac{1}{t_2}\right)\right)$



$t_1 + t_2 = \frac{2h}{A}$, $k = \frac{A}{2}\frac{t_1+t_2}{t_1t_2} = \frac{\cancel{A}}{\cancel{2}} \cdot \frac{\cancel{2}h}{\cancel{A}} \frac{1}{t_1t_2} = \frac{h}{t_1t_2}$

Eq^n of ch : $X + t_1t_2Y = A(t_1+t_2) \Rightarrow \boxed{kX + hY = 2hk}$

$\frac{h}{K}$ $\frac{2h}{A}$



J8. Tangency cond^n

* $XY = A^2$, $Y = mX + C$

$\Downarrow$

$X(mX+C) = A^2 \Rightarrow mX^2 + CX - A^2 = 0$

$$X_{1,2} = \frac{-C \pm \sqrt{C^2+4mA^2}}{2m}$$

→ D=0 2 same root
→ D>0 2 distinct root
→ D<0 2 img. root

* $D=0 \Rightarrow C^2+4mA^2=0 \rightarrow \boxed{m = -\frac{C^2}{4A^2}}$ tangency cond^n

J9 Tangent to Rect. hyp.

* $\underbrace{X_{1,2}}_{X_1} = \frac{-c}{2m}$, $Y_1 = mX_1 + C = -\frac{c^2}{4A^2}\left(\frac{-c}{2m}\right) + c = \frac{c^3}{8A^2\left(-\frac{c^2}{4A^2}\right)} + c = \frac{c}{2}$

$\boxed{\text{P.O.C } (X_1, Y_1) = \left(\frac{-c}{2m}, \frac{c}{2}\right)}$

$X_1 = \frac{-c}{2m}$, $Y_1 = \frac{c}{2}$ $\Rightarrow \frac{X_1}{Y_1} = -\frac{1}{m} \Rightarrow \boxed{m = -\frac{Y_1}{X_1}}$

1 pt

* $T \equiv Y - Y_1 = m(X - X_1) \Rightarrow Y - Y_1 = -\frac{Y_1}{X_1}(X - X_1) \Rightarrow X_1Y - X_1Y_1 = -Y_1X + X_1Y_1$

$Y_1X + X_1Y = 2X_1Y_1 \Rightarrow \boxed{\frac{X}{X_1} + \frac{Y}{Y_1} = 2}$ EOT 1pt form (X_1, Y_1) needed

Param form

* $(X_1, Y_1) = \left(At, \frac{A}{t}\right)$

$T \equiv \frac{X}{X_1} + \frac{Y}{Y_1} = 2 \Rightarrow \boxed{\frac{X}{t} + tY = 2A}$ EOT parameter t needed

J.10 Normal to Rect. hyperbola

* $m_T = -\frac{Y_1}{X_1} \Rightarrow \boxed{m_N = \frac{X_1}{Y_1}}$, $(X_1, Y_1) = \left(At, \frac{A}{t}\right)$

$\downarrow$

$m_N = \frac{At}{A/t} = t^2$

$N \equiv Y - Y_1 = m_N(X - X_1) \Rightarrow \boxed{Y - \frac{A}{t} = t^2(X - At)}$ EON t needed

HW

Q1. $H \equiv x^2 - 4y^2 = 36$

Eq^n of tangent to H which is $\perp$ to line $x - y + 4 = 0$

Q2. $H \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ The locus of pt of intersection of 2 tangents if the pdt. of their slopes is c^2 will be ?

Q3. Asymptotes $A_1 \equiv 2x - y = 3$
$A_2 \equiv 3x + y - 7 = 0$

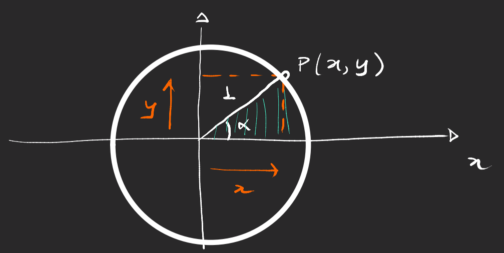
find eqⁿ of $\mathcal{H}$ whose asymptotes pass thru. point $(1,1)$ = ?

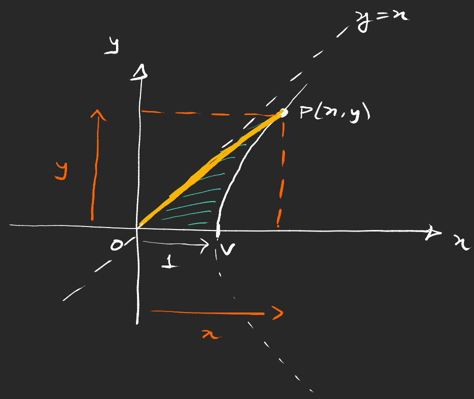
Q4. $x^2 - 2y^2 - 2x + 8y - 1 = 0$ show that it's a hyperbola

- F, V, dirx, LR, ... ?

Lecture-15 (24/Jul) 2:15

K. Birth of hyperbolic T fuⁿ (Area under the curve)





$C : x^2 + y^2 = 1$ $\quad r = 1$

$(2\pi) \to \pi r^2$ Area $\Rightarrow \boxed{Ar = \frac{1}{2}\alpha}$ $\xrightarrow{\text{Analogy}}$ $\mathcal{H} = x^2 - y^2 = 1$

$\boxed{\text{Area} \equiv \frac{\alpha}{2}}$
$\downarrow$
$\boxed{\begin{array}{l} x = ? \\ y = ? \end{array}}$

$\boxed{\begin{array}{l} x = \cos\alpha \\ y = \sin\alpha \end{array}}$

Circular transfⁿ

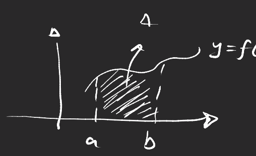
Task : for the given area $\underbrace{\text{area}}_{OPV}$ $A = \frac{\alpha}{2}$ compute the coordinates of P

given : • $x^2 - y^2 = 1 \Rightarrow \boxed{y = \sqrt{x^2 - 1}}$

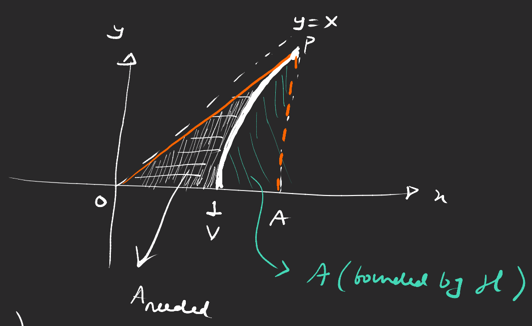
• $A = \frac{\alpha}{2}$

Computation :

formulae needed : • $A = \int_a^b f(x)\,dx$



Most general method to compute Area



• $A_{\text{needed}} = Ar(\Delta OAP) - Ar(\text{bounded by hyperbola})$

Tricky bit : $Ar(\Delta OAP) = \frac{1}{2}\,\text{base} \times \text{hgt} = \frac{1}{2}xy$

$Ar(\text{bound. by } \mathcal{H}) = \int_{x=1}^{x=X} y\,dx$

{ Power of Asymptotes / Imagination
P pt can go as close as you want to $y = x$ asymp.

Computation : $A = \frac{1}{2}xy - \int_{x=1}^{x=x} y\,dx = \frac{1}{2}x\underbrace{\sqrt{x^2-1}}_{y} - \underbrace{\int_{x=1}^{x=X} \sqrt{x^2-1}\,dx}$

Std. Integral

Std. Integral : $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln_e\left(x+\sqrt{x^2-a^2}\right)$

* $A = \frac{1}{2}xy - \left\{ \frac{x}{2}\sqrt{x^2-1} - \frac{1}{2}\ln x+\sqrt{x^2-1} \Big|_{x=1}^{x=x} \right\}$

$= \frac{1}{2}xy - \left\{ \frac{x}{2}\sqrt{x^2-1} - \frac{1}{2}\ln\, x+\sqrt{x^2-1}) - (0-0) \right\}$

$A = \frac{1}{\cancel{2}}xy - \frac{x}{\cancel{2}}\sqrt{x^2-1} + \frac{1}{\cancel{2}}\ln|x+\sqrt{x^2-1}| = \frac{\alpha}{\cancel{2}}$

$x^2-y^2=1$

* $\cancel{xy} - \cancel{xy} + \ln_e|x+\sqrt{x^2-1}| = \alpha \Rightarrow e^\alpha = x+\sqrt{x^2-1}$

$e^{2\alpha} = x^2 + x^2 - 1 + 2x\sqrt{x^2-1} = 2x^2 + 2x\sqrt{x^2-1} - 1 = 2x\underbrace{\left(x+\sqrt{x^2-1}\right)}_{e^\alpha} - 1$

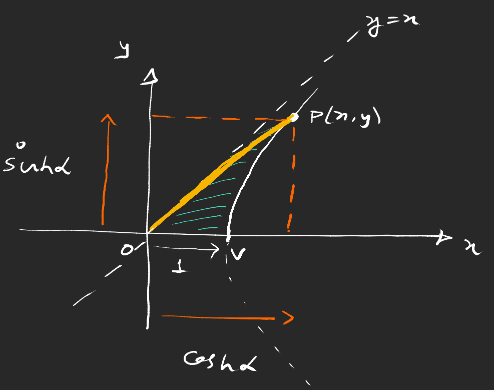
$e^{2\alpha} = 2xe^\alpha - 1 \Rightarrow e^{2\alpha} + 1 = 2e^\alpha \cdot x \Rightarrow x = \frac{e^{2\alpha}+1}{2e^\alpha} = \frac{e^\alpha + e^{-\alpha}}{2}$

$\boxed{x = \frac{e^\alpha+e^{-\alpha}}{2} \equiv \cosh\alpha} \in \mathbb{R}$

* $y = \sqrt{x^2-1} = \sqrt{\left(\frac{e^\alpha+e^{-\alpha}}{2}\right)^2 - 1} = \frac{1}{2}\sqrt{e^{2\alpha}+e^{-2\alpha}+2-4} = \frac{1}{2}\sqrt{e^{2\alpha}+e^{-2\alpha}-2} = \frac{\sqrt{(e^\alpha-e^{-\alpha})^2}}{2} = \frac{e^\alpha-e^{-\alpha}}{2}$

$\boxed{y = \frac{e^\alpha - e^{-\alpha}}{2} \equiv \sinh\alpha} \in \mathbb{R}$

Birth of Hyperbolic Trig.



Note :

- Hyp. T fun^n are not based on traditional notion of angle
  they are defined on the basis of the area intercepted arc on the hyp.
- Size of the hyp angle is twice the area of its hyp sector $\boxed{2A_{hyp} = \alpha}$

Tending To Complex

"Flat" Spaces

Real planes

Complex planes

Complex geometry

- $\mathbb{R}^2 = \mathbb{R}\times\mathbb{R} = \{(x,y) : x,y\in\mathbb{R}\}$

$\vdots$

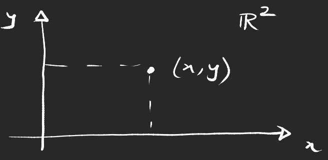
$\mathbb{R}^n = \mathbb{R}\times\mathbb{R}\times\cdots\times\mathbb{R}$ n-D real space

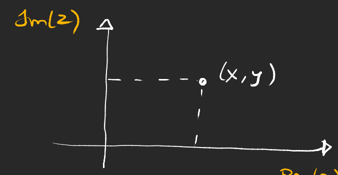
$\mathbb{C} = \{ z : x+iy,\ x,y\in\mathbb{R}\}$ "2D real space"

$\mathbb{C}^2 = \mathbb{C}\times\mathbb{C}$

$\vdots$

$\mathbb{C}^n = \mathbb{C}\times\cdots\times\mathbb{C}$





$\mathbb{C}^1$

2D real space representation (Argand's idea)

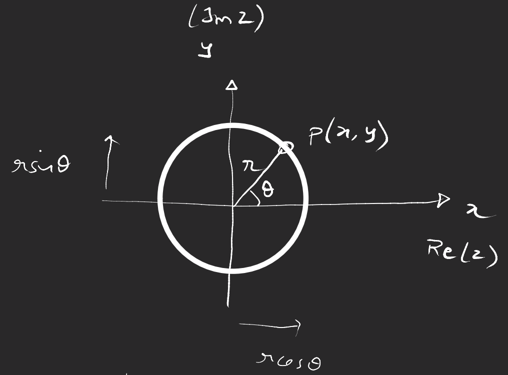
- $\boxed{\mathbb{R}^{2n} \longrightarrow \mathbb{C}^n}$

Naively

- $\boxed{\mathbb{C}^n \sim \mathbb{R}^{2n} + \text{'special structure'}}$ V. Advanced

later

- $\mathbb{C}^n$: uses $i = \sqrt{-1}$



* $z \equiv (x,y) = x+iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$\forall z \quad \exists z^* : z^* = x - iy = r(\cos\theta - i\sin\theta) = re^{-i\theta}$

Conjugate

* $e^{i\theta} = \cos\theta + i\sin\theta$
$e^{-i\theta} = \cos\theta - i\sin\theta$ $\Big\} \Rightarrow$ $\boxed{\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}}$

$\mathbb{R} \longrightarrow \mathbb{C}$

$\cos\theta + i\sin\theta = e^{i\theta}$ Euler's formula

* $\boxed{\begin{aligned}\mathrm{Re}(z) &= x \\ \mathrm{Im}(z) &= y\end{aligned}}$

* $r^2 = x^2 + y^2$

$\forall z \quad \exists |z| : |z|^2 = z^* z \in \mathbb{R} \qquad |z|^2 = (x+iy)(x-iy) = x^2+y^2 = r^2$

Modulus of a complex is a real qty. in so far as it is built out of Complex No.

{ not real in a strict sense }

$x = r\cos\theta$
$y = r\sin\theta$

$r^2 = x^2+y^2$
$\tan\theta = \frac{y}{x}$ $\Big\}$ V. imp

* $|z|^2 = (re^{i\theta})(re^{-i\theta}) = r^2$

$|e^{i\theta}|^2 = (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = \cos^2\theta + \sin^2\theta = 1 \Rightarrow \boxed{|e^{i\theta}|^2 = 1}$

* $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$
$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

$\mathbb{R} \longrightarrow \mathbb{C}$

$\boxed{\theta \to i\theta}$ $\longrightarrow$

$\cos i\theta = \frac{e^{-\theta}+e^{\theta}}{2} \equiv \cosh\theta$

$\sin i\theta = \frac{e^{-\theta}-e^{\theta}}{2i} = \frac{-1}{i}\,\frac{e^{\theta}-e^{-\theta}}{2} \equiv i\sinh\theta$

$\boxed{\begin{aligned}\cos i\theta &= \cosh\theta \\ \sin i\theta &= i\sinh\theta\end{aligned}}$ $\begin{aligned}&\in \mathbb{R} \\ &\in \mathbb{C}\end{aligned}$

* $\cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2}$
$\sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}$ $\Bigg\} \Rightarrow$ $\begin{aligned}e^{\theta} &= \cosh\theta + \sinh\theta \\ e^{-\theta} &= \cosh\theta - \sinh\theta\end{aligned} \Rightarrow \boxed{\cosh^2\theta - \sinh^2\theta = 1}$

fundamental Id.

- Osborn's Rule $\quad$ T. id $\longrightarrow$ Hyp id

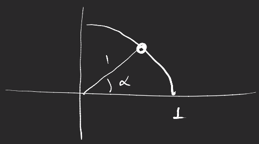
$\sin^2\theta + \cos^2\theta = 1$

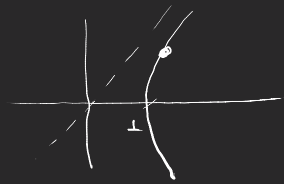
$1 + \tan^2\theta = \sec^2\theta$

$1 + \cot^2\theta = \operatorname{cosec}^2\theta$

- Product of sin will $\xrightarrow{\hspace{4em}}$ flip sign
- put h everywhere

$$\boxed{\begin{aligned} -\sinh^2\theta + \cosh^2\theta &= 1 \\ 1 - \tanh^2\theta &= \operatorname{sech}^2\theta \\ 1 - \coth^2\theta &= -\operatorname{cosech}^2\theta \end{aligned}}$$





$x^2 + y^2 = 1$ $\xrightarrow{r=1}$ $x^2 - y^2 = 1$

unit circle $\quad$ unit Rect Hyp. $\quad a = 1$

$x = \cos\alpha$

$y = \sin\alpha$

Circular fun^n

$x = \sec\theta$

$y = \tan\theta$

$\boxed{\begin{aligned} x &= \cosh\theta \\ y &= \sinh\theta \end{aligned}}$

Hyperbolic