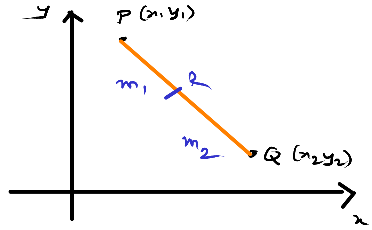
Lecture-40 (Contd. Geo. vol-1)

1. Definition / Forms of St. Line

a. Def. of St. line

Presumption: $\boxed{e_a \cdot e_b = g_{ab} = \delta_{ab} \text{ (Kronecker delta)}}$



* $ax+by+c=0$ 'Lin. Eqn in x, y (1^{st} degree)'

* P, Q satisfy eqn

$ax_1+by_1+c=0$, $ax_2+by_2+c=0$

* R : it divides PQ in $m_1 : m_2$

$m_1 : m_2$

$\frac{m_1}{m_2} = \frac{\lambda}{1}$

$$R\left(\frac{m_1x_2+m_2x_1}{m_1+m_2}, \frac{m_1y_2+m_2y_1}{m_1+m_2}\right) = R\left(\frac{\lambda x_2+x_1}{\lambda+1}, \frac{\lambda y_2+y_1}{\lambda+1}\right)$$

* Does R satisfy the Eqn $ax+by+c=0$

$$LHS = a\left(\frac{\lambda x_2+x_1}{\lambda+1}\right)+b\left(\frac{\lambda y_2+y_1}{\lambda+1}\right)+c = a\lambda x_2+ax_1+b\lambda y_2+by_1+c\lambda+c = \underbrace{(ax_1+by_1+c)}_{0} + \lambda\underbrace{(ax_2+by_2+c)}_{0} = 0$$

* Any pt. on line joining A & B $\underbrace{\text{lies}}_{\text{satisfies}}$ on $ax+by+c=0$

* $\boxed{ax+by+c=0}$ 'Eqn of a straight line' (ESL 1)

"Every 1^{st} degree eqn in x,y represents straight line"

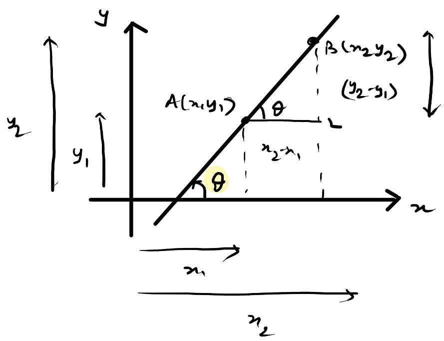
$\boxed{\text{Line} = \text{locus of a pt : pt satisfies } ax+by+c=0}$

Lecture-1 (19/July) 2

b. Slope in terms of Coordinates

* θ : initial angle (+ve) , 2 pts

θ : Angle b/w x axis & the line



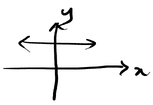
* $\triangle ABL \rightarrow$ $\boxed{\tan\theta = \underbrace{\frac{y_2-y_1}{x_2-x_1}}_{\text{Coord. geo.}} = m}$ → 'Slope'

Trigonometry

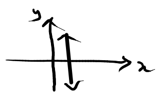
Slope from 2 pts.

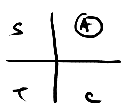
* $-1 \le \sin\theta \le 1$, $-1 \le \cos\theta \le 1$, $-\infty < \underbrace{\tan\theta}_{} < \infty$

$\tan\theta = 0 \Rightarrow m = 0$ line $\parallel$ to x



$\tan\frac{\pi}{2} \to \infty \Rightarrow m \to \infty$ line $\parallel$ to y



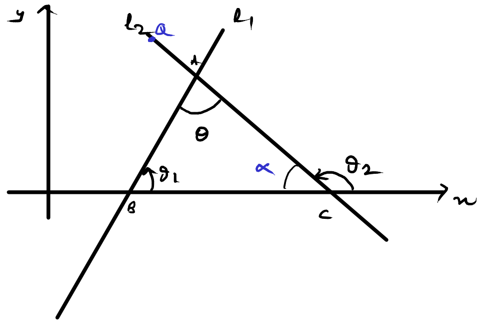


* m
  - $>0 \Rightarrow \tan\theta > 0 \Rightarrow$ line is in $1^{st}/3^{rd}$ quad $\Rightarrow \theta$: Acute →

  - $=0 \Rightarrow \tan\theta = 0 \Rightarrow$ line $\parallel$ to x axis
  - $<0 \Rightarrow \tan\theta < 0 \Rightarrow$ line is in $2^{nd}/4^{th}$ quad $\Rightarrow \theta$: obtuse →

m : −ve

6.1 Quantification of Angle b/w 2 lines



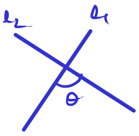
Line $\quad$ Slope

* $l_1 \longrightarrow m_1 = \tan\theta_1$

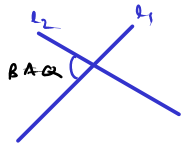
$l_2 \longrightarrow m_2 = \tan\theta_2$

* $\Delta ABC \rightarrow \theta + \theta_1 + \alpha = 180^\circ \Rightarrow \theta + \theta_1 = 180 - \alpha = \theta_2 \Rightarrow \boxed{\theta = \theta_2 - \theta_1}$

* $\tan\theta = \tan(\theta_2 - \theta_1) = \dfrac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1} \Rightarrow \boxed{\tan\theta = \dfrac{m_2 - m_1}{1 + m_1 m_2}}$

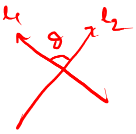


* $\tan \overset{\text{obtuse}}{BAQ} = \tan(\pi - \theta) = -\tan\theta = -\left(\dfrac{m_2 - m_1}{1 + m_1 m_2}\right)$

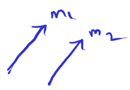


$\tan\theta = \pm\left(\dfrac{m_2 - m_1}{1 + m_1 m_2}\right) \Rightarrow \theta = \tan^{-1}\left(\pm\dfrac{m_2 - m_1}{1 + m_1 m_2}\right)$

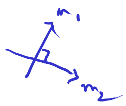
$\boxed{\tan\theta = \left|\dfrac{m_2 - m_1}{1 + m_1 m_2}\right|}$ 'Acute angle b/w 2 lines'



* $\parallel$ Lines $\Rightarrow \theta = 0 \Rightarrow \left|\dfrac{m_2 - m_1}{1 + m_1 m_2}\right| = 0 \Rightarrow m_1 = m_2$



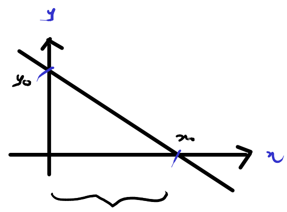
$\perp$ lines $\Rightarrow \theta = \dfrac{\pi}{2} \Rightarrow \left|\dfrac{m_2 - m_1}{1 + m_1 m_2}\right| \to \infty \Rightarrow \boxed{m_1 m_2 = -1}$



6.2 Linear function / Map $\longrightarrow$ slope-intercept form (most renowned)

* $f(x) = ax + b$ Linear fun^n / transformation / Mapping / machine

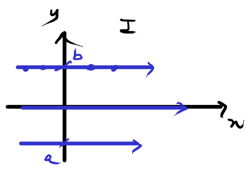
$\exists\ x_0 : f(x) = 0 \Rightarrow x_0$: pts where machine $\underbrace{\text{switches off}}_{\text{Intercepts}}$.

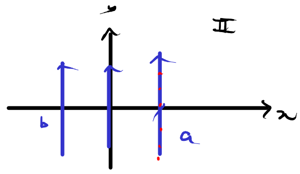


Intercept

* $x_0 : y = 0 \Rightarrow x_0$: x-intercept

$y_0 : x = 0 \Rightarrow y_0$: y-intercept

* 



Special Cases

Constraint eq^n

$y = b$ { Eq^n of line $\parallel x$

x changes

Constraint eq^n

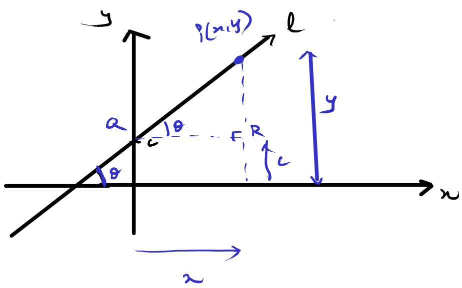
$x = a$ { Eq^n of line $\parallel y$

y changes

"dynamics"

(generates the "curve")

More general case



* l : m = slope, c = y intercept

ΔPRQ : $\tan\theta = \dfrac{y - c}{x} = m \Rightarrow \boxed{y = mx + c}$ (ESL 2)

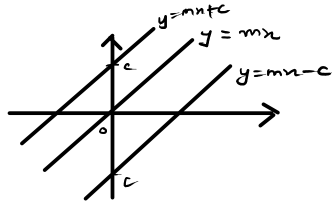
Eq^n of st-line / m-c form

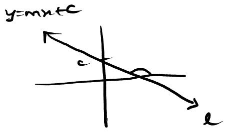
$(m, \underbrace{c}_{y \text{ intercept}})$

* (m,c) : $y = mx + c$

$(m,0)$: $y = mx$ 'passes thru O'

$(0,c)$: $y = c$ (special case I)





given data

* (m,c) : $y = mx + c \xrightarrow[\text{rotation transf}^n]{R}$ $\tan\theta = \dfrac{x-c}{y} = m$

y int

⇓

(m, c_x) $\boxed{x = my + c}$

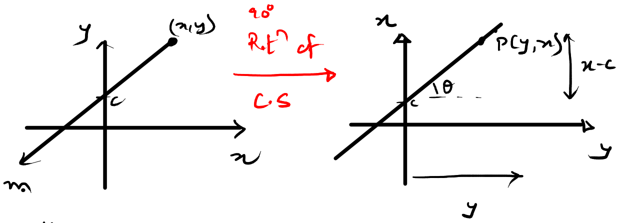
x int.

$\big\downarrow m=0$

$\boxed{x = c}$

line ∥ to y

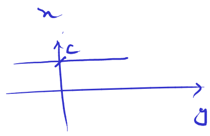
(special case II)



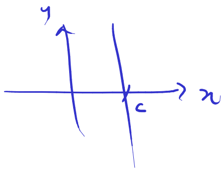
$y = mx + c$

$\big\downarrow m=0$

$y = c$

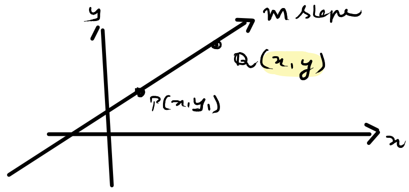


$\big\downarrow$ R (inverse transf^n)



* $(m, 1\,pt) \longrightarrow$ $m = \dfrac{y-y_1}{x-x_1}$ => $\boxed{y - y_1 = m(x - x_1)}$ ESL−III

(m, 1pt) needed

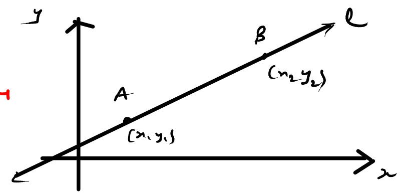


Lecture-2 (20 July) 2

b.3. Further easy forms (pt-pt, int-int)

* $(1\,pt, 2\,pt) \longrightarrow$ $m = \dfrac{y_2-y_1}{x_2-x_1}$, $y - y_1 = m(x - x_1)$ => $\boxed{y - y_1 = \left(\dfrac{y_2-y_1}{x_2-x_1}\right)(x - x_1)}$ ESL−4

(pt 1, pt 2) needed

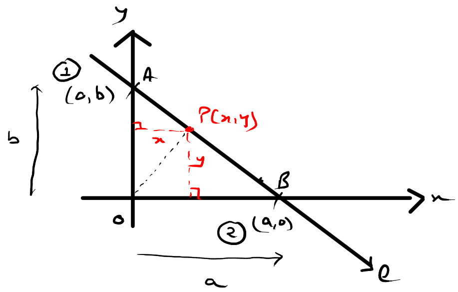


* (x-int, y-int)

Approach 1 : $m = \dfrac{0-b}{a-0} = \dfrac{-b}{a}$ => $y - b = \left(\dfrac{-b}{a}\right)(x)$ => $ay - ba = -bx$

⇓

$\boxed{\dfrac{x}{a} + \dfrac{y}{b} = 1}$



Approach 2 : Ar(AOB) = Ar(POB) + Ar(ΔPOA)

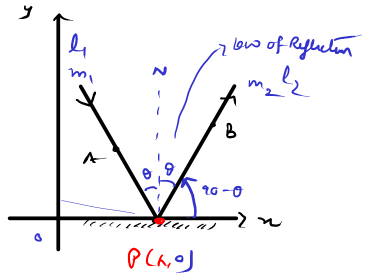
$\cancel{\tfrac{1}{2}}ab = \cancel{\tfrac{1}{2}}ay + \cancel{\tfrac{1}{2}}bx$ => $\boxed{\dfrac{x}{a} + \dfrac{y}{b} = 1}$ ESL−5

(x int, y int) needed

Practice-1

Q1. (l_1, l_2) : $\underbrace{\theta(l_1, l_2)}_{\text{Angle b/w } l_1, l_2} = \frac{\pi}{4}$, $m_{l_1} = \frac{1}{2}$, $m_{l_2} = ?$

* $\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| \longrightarrow m_{l_2} = 3, \frac{-1}{3}$



Q2. A(1,2) : pt on incident ray

B(5,3) : pt on reflected Ray

P(h, k) = ?

* l_1 : $\tan(90+\theta) = \frac{2-0}{1-h} = -\cot\theta$

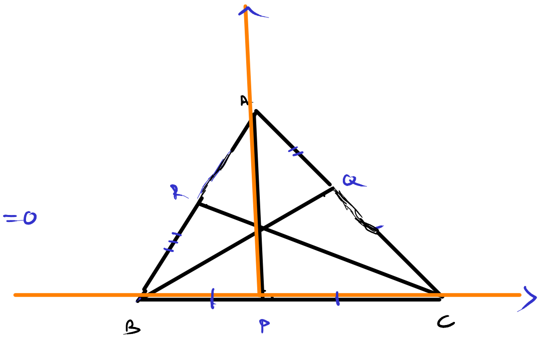
l_2 : $\tan(90-\theta) = \frac{3-0}{5-h} = \cot\theta$

$\Rightarrow h = 13/5$

HW

Q3: Medians of Equal Δ are ⊥ to the corresponding sides (Prove!)

To prove : $\underbrace{m_{AP}} \cdot \underbrace{m_{BC}} = -1 \Rightarrow AP \perp BC$, $\tan\theta = \left|\frac{m_2 - m_1}{1 + m_2 m_1}\right| \xrightarrow{\text{if}} \infty \Rightarrow D = 0$



HW

Q4 : Δ : $A(x_1, x_1\tan\theta_1)$, $B(x_2, x_2\tan\theta_2)$ $C(x_3, x_3\tan\theta_3)$

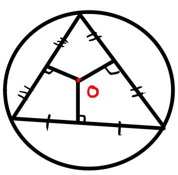
if the circumcenter of ΔABC coincides with the origin & $H(\bar{x}, \bar{y})$ is orthocenter.

$\frac{\bar{y}}{\bar{x}} = ?$

Hint : centroid

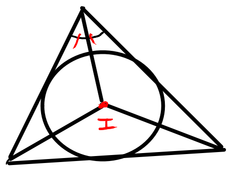
Centers of Triangles

circumcenter



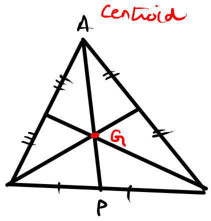
⊥ bisector

Incenter



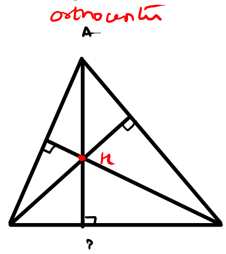
Angle bisector

Centroid



AP : Median

orthocenter



AP : perpendicular / Altitude

Judgment ground

1. ⊥ bisector
2. ∠ "
3. Alt.
4. Median

⋮

"Peanut Butter Cookies

Are Best In

Milk Chocolate

And Ovaltine"

$$\boxed{\text{* Centers remain invariant under similarity trans}^n \begin{cases} \to \text{transl}^n \\ \to \text{Rot}^n \\ \to \text{Dilation} \\ \to \text{Parity / Reflection} \end{cases}}$$

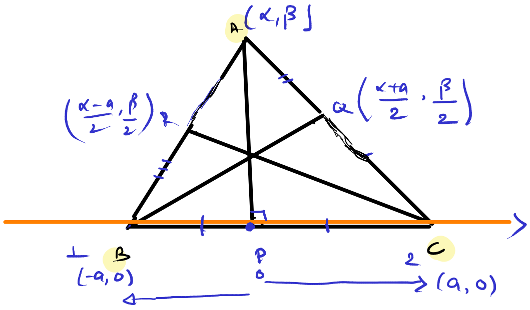
* $\{H, G, O\}$: Collinear.

* Equilateral Δ $\Rightarrow$ O = H = I = G (centroid)

Lecture-3 (21/July) 2

Sol^n

Q3 Medians of Equil Δ are $\perp$ to the corresponding sides (Prove!)



* To prove : $m_{AP}\cdot m_{BC}=-1 \Rightarrow \underbrace{AP}\perp\underbrace{BC}$; $\tan\theta=\left|\frac{m_2-m_1}{1+m_2m_1}\right| \longrightarrow \infty \Rightarrow D=0$

* $A(\alpha,\beta)$, $B(-a,0)$, $C(a,0)$, $P(0,0)$

* $m_{AP}=\frac{\beta-0}{\alpha-0}$, $m_{BC}=\frac{0-0}{2a}=0$ (not useful) $\rightarrow$ try $AC\perp BQ$

* $m_{AC}=\frac{\beta-0}{\alpha-a}=\frac{\beta}{\alpha-a}$, $m_{BQ}=\frac{0-\beta/2}{-a-\left(\frac{\alpha+a}{2}\right)}=\frac{\beta/2}{2a+\alpha+a}=\frac{\beta/2}{3a+\alpha}$

* Equil $\Delta \Rightarrow AB=BC \Rightarrow \beta^2+(\alpha+a)^2=4a^2=\beta^2+(\alpha-a)^2 \Rightarrow 2\alpha a=-2\alpha a \Rightarrow 4\alpha a=0 \Rightarrow \boxed{\alpha=0}$

|| AC

$\boxed{\beta=a\sqrt{3}}$

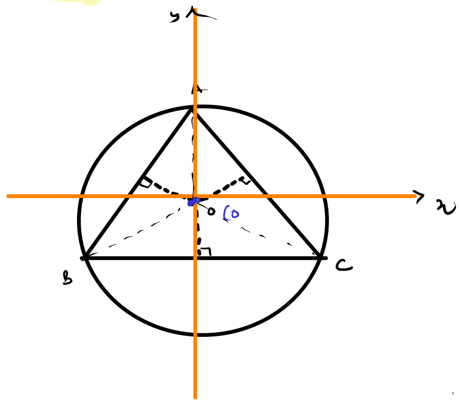
* $m_{AC}=\frac{\beta}{a}=-\sqrt{3}$, $m_{BQ}=\frac{\beta}{3a}=\frac{1}{\sqrt{3}} \longrightarrow m_{AC}\cdot m_{BQ}=-1 \Rightarrow AC\perp BQ$ $\square$

HW

Q4 : Δ : $A(x_1,x_1\tan\theta_1)$, $B(x_2,x_2\tan\theta_2)$ $C(x_3,x_3\tan\theta_3)$

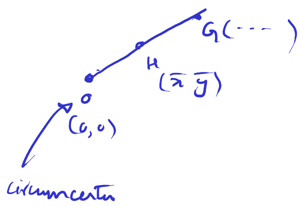
if the circumcenter of ΔABC coincides with the origin & $H(\bar{x},\bar{y})$ is orthocenter.

$\frac{\bar{y}}{\bar{x}}=?$



* $OA=OB=OC=\alpha \Rightarrow \underbrace{x_1^2+x_1^2\tan^2\theta_1}_{x_1^2\sec^2\theta_1}=\underbrace{x_2^2+x_2^2\tan^2\theta_2}_{x_2^2\sec^2\theta_2}=\underbrace{x_3^2+x_3^2\tan^2\theta_3}_{x_3^2\sec^2\theta_3}=\alpha^2$

$$\left.\begin{aligned} x_1&=\alpha\cos\theta_1 \\ x_2&=\alpha\cos\theta_2 \\ x_3&=\alpha\cos\theta_3 \end{aligned}\right\} \rightarrow \begin{aligned} &A(\alpha\cos\theta_1,\alpha\sin\theta_1) \\ &B(\alpha\cos\theta_2,\alpha\sin\theta_2) \\ &C(\alpha\cos\theta_3,\alpha\sin\theta_3) \end{aligned}$$



* $G=\left(\frac{\alpha(\cos\theta_1+\cos\theta_2+\cos\theta_3)}{3},\frac{\alpha(\sin\theta_1+\sin\theta_2+\sin\theta_3)}{3}\right)$ 'centroid'

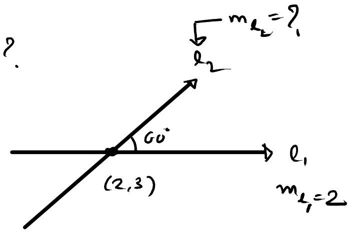
{circum, centroid, orthocen} collinear $\Rightarrow m_{centroid}=\underbrace{m_{orthocenter}}_{\frac{\bar{y}}{\bar{x}}}$

* $$\boxed{\frac{\bar{y}}{\bar{x}}=\frac{\sin\theta_1+\sin\theta_2+\sin\theta_3}{\cos\theta_1+\cos\theta_2+\cos\theta_3}}$$

Q5. ℓ : $\underbrace{\tan^{-1}\sqrt{2}}_{m=\tan(\tan^{-1}\sqrt{2})}$ with x ; $\underbrace{-\frac{\sqrt{3}}{2}}_{c}$ cuts at y axis EOL ? $\quad y=\sqrt{2}x-\frac{\sqrt{3}}{2}$ □

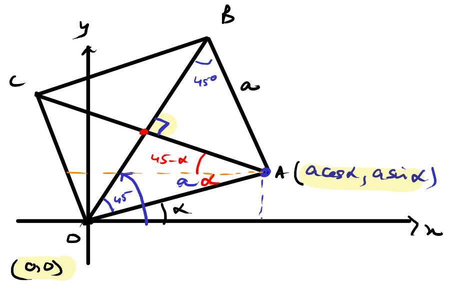
Q6. The $\perp$ from O to a line meets at $(-2,9)$, EOL ? $\quad 9y-2x=85$

Q7. $\{\ell_1, \ell_2\}$ passing thru $(2,3)$ intersecting each other at $60°$: $m_{\ell_1}=2$; $EOL_{\ell_2}=?$



$$y-3=\frac{2+\sqrt{3}}{1-2\sqrt{3}}(x-2)$$ □

Q8. Eq of {diag. 1, diag. 2} ?



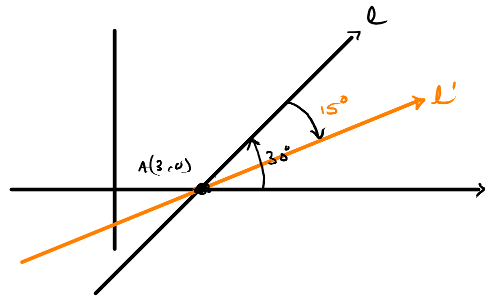
* OB : → $y=\frac{(\cos\alpha+\sin\alpha)}{\cos\alpha-\sin\alpha}x$ $\qquad m_{AB}=\tan\left(\frac{\pi}{4}+\alpha\right)$

* AC → $y-a\sin\alpha=-\left(\frac{\cos\alpha-\sin\alpha}{\cos\alpha+\sin\alpha}\right)(x-a\cos\alpha)$ $\qquad m_{CA}=\frac{-1}{\tan\left(\frac{\pi}{4}+\alpha\right)}$

Q9. ℓ : passes thru $A(3,0)$, $\underbrace{30°}_{\text{+ve dir}^n}$ with x axis

↓

$\ell \xrightarrow{R} \ell'$: $15°$ rotated ℓ in $\underbrace{\text{cw}}_{\text{clockwise}}$ dirⁿ , A being the pivot of Rotation

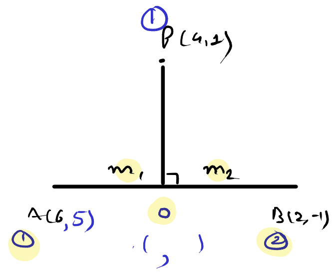


$\underbrace{\text{EOL } \ell' = ?}_{(pt,\ angle)}$

* $x=(2-\sqrt{3})y+3$ □

Lecture-4 (22/July) 2

Q10. ℓ : $A(6,5)$, $B(2,-1)$; $P(4,1)$; draw $\perp$ from P to AB ; Ratio it divides AB = ? $(m_1:m_2)$



* $O \equiv \left(\frac{m_1 2+m_2 6}{m_1+m_2}, \frac{m_1(-1)+m_2 5}{m_1+m_2}\right)$ → talks about AB. — ①

* $PO \perp AB \Rightarrow \underbrace{m_{PO}}\cdot m_{AB}=-1 \Rightarrow m_{PO}=\frac{-1}{\frac{-1-5}{2-6}}=\frac{-4}{6}=\frac{-2}{3}$

* Eqⁿ of line PO → $y-1=\frac{-2}{3}(x-4) \Rightarrow 3y-3=-2x+8 \Rightarrow \boxed{3y+2x-11=0}$ is satisfied by O — ②

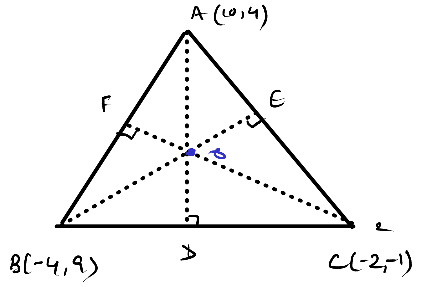
* ① in ② $\Rightarrow 3\left(\frac{-m_1+5m_2}{m_1+m_2}\right)+2\left(\frac{2m_1+6m_2}{m_1+m_2}\right)-11=0$

$$3\left(\frac{-\lambda+5}{\lambda+1}\right)+4\left(\frac{\lambda+3}{\lambda+1}\right)-11=0 \Rightarrow -3\lambda+15+4\lambda+12-11\lambda-11=0$$ $\qquad \lambda=\frac{m_1}{m_2}$

⇓

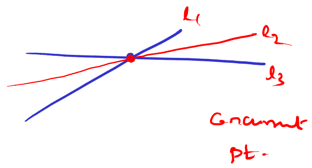
$$-10\lambda+16=0 \Rightarrow \lambda=\frac{16}{10}\ \frac{8}{5}=\frac{8}{5}$$

Q 11. Eq^n of altitudes ; O = orthocentre , coordinates = ?



* $m_{AD} \cdot m_{BC} = -1 \Rightarrow m_{AD} = \frac{1}{5} \Rightarrow (y-4) = m_{AD}(x-10) \Rightarrow 5y - x - 10 = 0$
* $m_{BE} \cdot m_{AC} = -1 \Rightarrow m_{BE} = \frac{-12}{5} \Rightarrow (y-9) \quad m_{BE}(x+4) \Rightarrow 5y + 12x + 3 = 0$
* $m_{CF} \cdot m_{AB} = -1 \Rightarrow m_{CF} = \frac{14}{5} \Rightarrow (y+1) \quad m_{CF}(x+2) \Rightarrow 5y - 14x - 23 = 0$

* $$\begin{vmatrix} x & y & 1 \\ -1 & 5 & -10 \\ 12 & 5 & 3 \end{vmatrix}$$



$$\frac{x}{15+50} = \frac{-y}{-3+120} = \frac{1}{-5-60}$$

$$\frac{x}{65} = \frac{y}{-117} = \frac{1}{-65} \Rightarrow x = -1 \checkmark$$

$$y = \frac{117}{65} = \frac{9}{5} \checkmark$$

Law of Radioactive decay.

$$\frac{dy}{dt} = -ky$$

$$\int \frac{dy}{y} = \int -k\,dt$$

$$\int \frac{1}{y} dy = \int -k\,dt$$

$$\ln_e y = -kt + \ln y_0$$

$$\underbrace{\ln y - \ln y_0}_{} = -kt$$

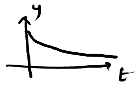
$$\ln \frac{y}{y_0} = -kt$$

$$\rightarrow y(t) = y_0 e^{-kt}$$

$y(0) = y_0$

$y(\infty) = e^{-\infty}$

$= \frac{1}{\infty} = 0$



Q 12. opp. vertices of ▭ (1,2), (5,5)

if other vertices lie on a line $x=3$; Eq^n of the sides of ▭ = ?

Approach 1

* $m_{AB} \cdot m_{BC} = -1 \Rightarrow \left(\frac{2-y_1}{1-3}\right)\left(\frac{5-y_1}{5-3}\right) = -1 \Rightarrow 10 - 7y_1 + y_1^2 = 4 \Rightarrow \boxed{y_1^2 - 7y_1 + 6 \quad 0}$

$\Downarrow$

$y_1 = 1, 6$

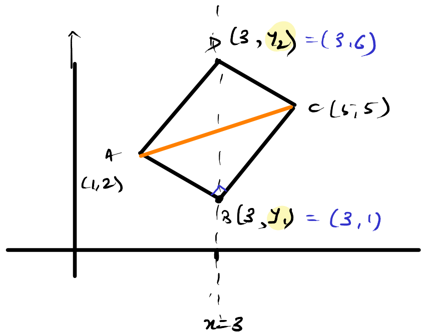
* $m_{DC} \cdot m_{AD} = -1 \Rightarrow \left(\frac{y_2-5}{3-5}\right)\left(\frac{y_2-2}{3-1}\right) = -1 \Rightarrow y_2^2 - 7y_2 + 6 = 0$

(Redundant)

$\Downarrow$

$y_2 = 1, 6$

* $m_{AD} = m_{BC} \Rightarrow \boxed{y_1 + y_2 = 7}$

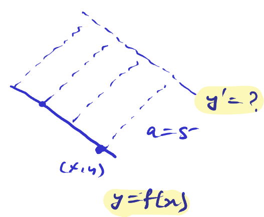


* AB : $y - 1 = \underbrace{\left(\frac{2-1}{1-3}\right)}_{\frac{-1}{2}}(x-3) \Rightarrow 2y - 2 = -x + 3 \Rightarrow x + 2y - 5 = 0$

$$\boxed{y = \frac{-1}{2}x + \frac{5}{2}}$$

$m = -\frac{1}{2}$

$c = \frac{5}{2} + (5)$



BC : $y = 2x - 5 \rightarrow m = 2, \; c = -5 + (5)$

DC : $2y = -x + 15 \Rightarrow \boxed{y = \frac{-1}{2}x + \frac{15}{2}}$

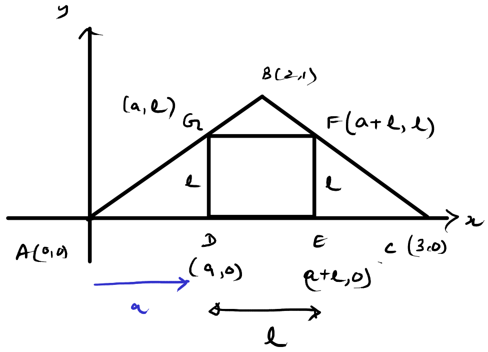
$m = -\frac{1}{2}$

$c = \frac{15}{2}$

AD : $y = 2x \quad m = 2, \; c = 0$

Lecture-5 (23 July) 2

Q13 ΔABC : $A(0,0)$, $B(2,1)$, $C(3,0)$
Sq. is inscribed in the Δ : 2 vertices of $\square$ lie on AC } Coord. of $\square$ = ?



* $\{a, l\}$?

* AB : $y-0 = \left(\frac{1-0}{2-0}\right)(x-0) \Rightarrow y = \frac{1}{2}x \to G(a,l)$ satisfies it
$\downarrow$
AC : $y-0 = 0 \Rightarrow y=0$ $\qquad l = \frac{1}{2}a$

BC : $y-0 = \left(\frac{1-0}{2-3}\right)(x-3) \Rightarrow y = -(x-3) \Rightarrow x+y=3 \to F(a+l, l)$ satisfies it.
$\downarrow$
$a+l+l=3 \Rightarrow a+2l=3 \Rightarrow 4l=3 \Rightarrow l = 3/4$
$a = 3/2$

* D $(3/2, 0)$
E $(9/4, 0)$
F $(3/2, 3/4)$
G $(9/4, 3/4)$

Q14. $K_1 = 273$: $F_1 = 32$
$K_2 = 373$: $F_2 = 212$ } $\xrightarrow[\text{Linear}]{K \,\&\, F}$ 1. Eq^n $(K, F) = ?$ $\to K = K(F)$
2. $F = ?$: $K = 0$

* $K = K(F) = \sum_{i=0}^{n} a_i F^i = a_0 + a_1 F + a_2 F^2 + a_3 F^3 + \dots$ Series Expansion of function

$K \sim a_0 + a_1 F$

* $K - 273 = \frac{373-273}{212-32}(F-32) \Rightarrow K = \underbrace{\frac{5}{9}}_{\text{Linear}}(F-32) + 273 \Rightarrow K = K(F) = \underbrace{\frac{2325}{9}}_{a_0} + \underbrace{\frac{5}{9}}_{a_1}F + \underbrace{0\,(F^2)}_{\text{Higher order terms dropped}}$

Detour to Calculus

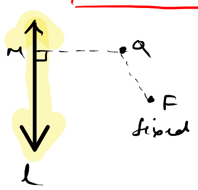
pts. $\Delta s^2 = \Delta x^2 + \Delta y^2$ (finite diff) (Infinitely small diff)

$(x_1, y_1), (x_2, y_2) \xrightarrow{\text{ESL}} y - y_1 = \frac{y_2-y_1}{x_2-x_1}(x-x_1) \Rightarrow y = y(x)$ Linear fun^n

$\Delta x = x_2 - x_1$
$\Delta y = y_2 - y_1$

$y_1 = a_0 + a_1 x \to f_1(x) = \underbrace{f(x=0)}_{\text{const}} + \left.\frac{df}{dx}\right|_{x=0} \underbrace{x}_{\text{Linear}}$

$\to y = ax^2 + bx + c$? Quadr fun^n $\cup \cap$ $\qquad y_2 = \underbrace{(a_0 + a_1 x)}_{\text{Linear}} + a_2 x^2 \to f_2(x) = \underbrace{f_1(x)}_{} + \left.\frac{d^2 f}{dx^2}\right|_{x=0} x^2$



$y = ax^3 + bx^2 + cx + d$ Cubic fun^n $\qquad$ algebraic analog of directrix $\qquad f(x=0) + \left.\frac{df}{dx}\right|_x x$

(fixed) $ax + by + c = 0$ (Linear i/p to generate 2nd order quad. terms)
Directrix

Intuition to Taylor expansion (2nd Order)
$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots = \sum \frac{d^n f}{dx^n} \frac{x^n}{n!}$ later

* $y = ax^2 + bx + c \to \frac{dy}{dx} = 2ax + b = 0 \Rightarrow x = \frac{-b}{2a}$ 'Extrema pt'
$\downarrow$
if $\boxed{y=0} \Rightarrow x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \to x = \left.\frac{dy}{dx}\right|_{x=0} \pm \frac{1}{2a}\sqrt{b^2-4ac}$

$f \xrightarrow{\text{i/p}} \frac{df}{dx} \xrightarrow{\text{i/p}} \frac{d}{dx}\left(\frac{df}{dx}\right) \xrightarrow{n^{th}} \frac{d}{dx}\left(\frac{d^{n-1} f}{dx^{n-1}}\right)$

{ successive derivatives }

$\frac{d}{dx}$ = Machine / transform^n / Map / special operator
$\{+ - \div\}$

* $\frac{d^2x}{dt^2} = a_0$ 'EOM' $\longrightarrow$ $x(t) = x_0 + v_0 t + \frac{1}{2} a_0 t^2 = \sum_{i=0}^{2} a_i t^i$

$\frac{d^2x}{dt^2} = a$

$x = x_0 + \left.\frac{dx}{dt}\right|_0 t + \left.\frac{d^2x}{dt^2}\right|_0 \frac{t^2}{2} + \frac{1}{3!}\frac{d^3x}{dt^3} t^3$

* $\frac{d^2y}{dx^2} = \underbrace{2a}$ "$\underbrace{\text{EOM}}_{\text{D.E}}$" of Quad. "EOS." $\equiv$ Newton's 2nd Law.

Must be Endowed with B.C (Most difficult part) otherwise no unique Sol^n

$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt}$

$\frac{dy}{dx} = 2ax + b \rightarrow \int dy = \int(2ax+b)\,dx \Rightarrow y(x) = ax^2 + bx + c = \sum_{n=0}^{2} a_n x^n$

$v \rightarrow t$ $\quad v \rightarrow x \rightarrow t$

chain rule.

* $$\boxed{y(x) = y_0 + \underbrace{\left.\frac{dy}{dx}\right|_{x=0}}_{2ax+b} x + \frac{1}{2}\underbrace{\left.\frac{d^2y}{dx^2}\right|_{x=0}}_{a} x^2}$$ $\longrightarrow$ $y(0) = y_0 = a\cancel{(0)}^2 + b\cancel{(0)} + c = c$

"Any fun^n can be expanded in terms of Powers of x"
— Power series

* $$\boxed{\underbrace{\frac{d^2y}{dx^2} = 2a}_{\text{D.E}} \; ; \; \underbrace{\overset{\text{"displacement"}}{y(0)} = c \; , \; \overset{\text{"velocity"}}{y'(0)} = b}_{\text{BC}}}$$

(time)

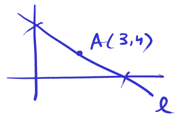
Lecture-6 (25/July) 2

Any Sept Oct Nov. } St. Lines / Conic Sections / T-Eq^n / ITF

Dec – Calculus

Q15. A(3,4) ; l passing thru A ; $\sum$(intercepts on axes) = 14 , $l \equiv$? EOL

* $a + b = 14$ —① ; EOL : $\frac{x}{a} + \frac{y}{b} = 1$ passing thru A(3,4)



* $\frac{3}{a} + \frac{4}{b} = 1 \Rightarrow 3b + 4a = ab$ — ②

$3b + 4(14-b) = (14-b)b \Rightarrow b^2 - 15b + 56 = 0$

$b^2 - 7b - 8b + 56 = 0 \Rightarrow b(b-7) - 8(b-7) = 0 \Rightarrow b_{1,2} = 7, 8$

$\Downarrow$

$a_{1,2} = 7, 6$

$\left\{ \frac{x}{7} + \frac{y}{7} = 1 \; , \; \frac{x}{6} + \frac{y}{8} = 1 \right\} \Rightarrow x + y = 7 \; , \; \cancel{8x + 6y = 48}$

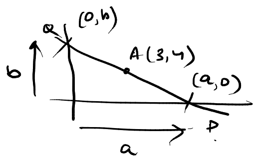
$4x + 3y = 24$

* Pradeep ✓ theory
* Resnick Halliday.
  - fundamental of phy
* HC verma – Ques.
* SL Arora ✓ theory

1. Kin. Fluids
2. NLM SHM
3. WPE Waves
4. COM Thermody.

Sat. → Rot^n gravit^n

Appr 2



* $y - y_1 = \underbrace{\frac{y_2 - y_1}{x_2 - x_1}}_{-\frac{b}{a}} (x - x_1)$; $m_{QA} = m_{PA} \Rightarrow \frac{b-4}{-3} = \frac{4}{a-3}$ — ①

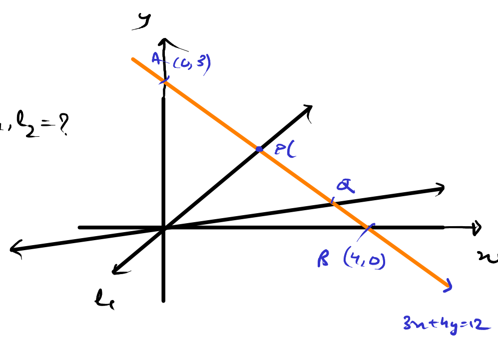
$ab - 3b - 4a + 12 = 12 \Rightarrow 4a + 3b = ab \Rightarrow \boxed{\frac{4}{b} + \frac{3}{a} = 1}$ —①

* $a + b = 14$ —②

Q16. $\{l_1, l_2\}$: passing thru O ; $l_3 = 3x + 4y = 12$ $\overset{\text{Equal parts}}{\text{trisected}}$ by $\{l_1, l_2\}$, EOL $l_1, l_2 = ?$

Giw the Axes

the Boundary to make it a segment



* $\frac{x}{4} + \frac{y}{3} = 1 \longrightarrow a = 4, \; b = 3$

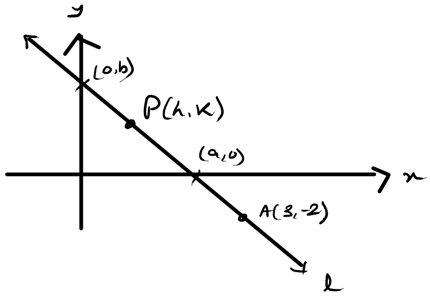
* $P\left(\frac{4}{3}, 2\right), \; Q\left(\frac{8}{3}, 1\right) \Rightarrow l_1 \equiv 3x - 2y = 0$

$l_2 \equiv 3x - 8y = 0$

Q17 $\quad l$: passing thru $A(3,-2)$

find the locus of the middle pt. of the portion of the line intercepted b/w the axes.

$m_1 : m_2 = 1:1$ $\quad$ Boundary & line segment



* $P(h,k) = \left(\frac{0+a}{2}, \frac{0+b}{2}\right) = \left(\frac{a}{2}, \frac{b}{2}\right) \Rightarrow$ $a = 2h$, $b = 2k$ $\quad$ 'Section formula'

* $l: \frac{x}{a} + \frac{y}{b} = 1$ $\quad$ satisfies $\rightarrow$ $\frac{3}{2h} - \frac{2}{2k} = 1$ $\xrightarrow{(h,k)\to(x,y)}$ $3y - 2x = 2xy$

6.4 Not So Easy forms

1. Normal form.

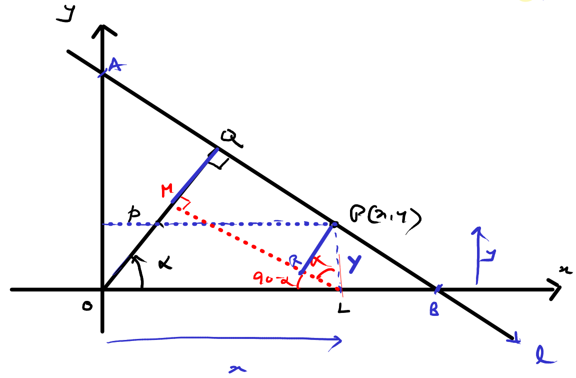
$y = mx + c = x\tan\theta + c$

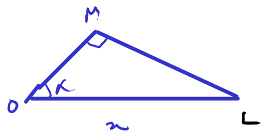
* Perpendicular distance $= p$
* Angle that $\perp$ makes with $\hat{x} = \alpha$ $\quad \}$ given

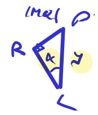
* $x(..\alpha) \pm y(...\alpha) .. \pm p = 0$ $\quad$ Aim / Format

* $p = \underbrace{OM}_{x\cos\alpha} + \underbrace{\overset{\overset{RP}{||}}{MQ}}_{y\sin\alpha} \Rightarrow$ $\boxed{x\cos\alpha + y\sin\alpha = p}$

Normal form / Perpendicular form (ESL 6)







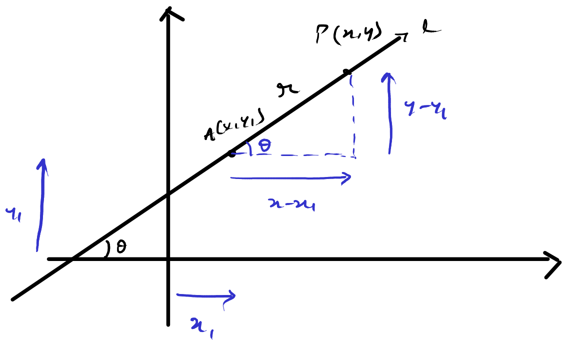
$\alpha + 90 + x = 180 \Rightarrow x = 90 - \alpha$

2. Distance form / Parametric form

* fixed pt : $A(x_1, y_1)$

* Dist of a $\underbrace{\text{generic pt}}_{\text{Moving}}$ on the line from $\underbrace{A}_{\text{fixed pt}} = r \Rightarrow$ $\boxed{|AP| = r}$ $\quad \}$ given

* θ : slope of line (conventions following) ACW : +ve

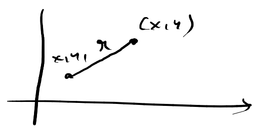


* $\cos\theta = \frac{x-x_1}{r}$, $\sin\theta = \frac{y-y_1}{r}$

$\Downarrow$

$\boxed{\frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta} = r}$ $\quad$ Distance Form (ESL-7)

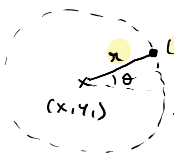
* $y - y_1 = r\sin\theta \Rightarrow$
$x - x_1 = r\cos\theta \Rightarrow$ $\boxed{\begin{matrix} y = y_1 + r\sin\theta \\ x = x_1 + r\cos\theta \end{matrix}}$ $\quad$ Parametric eq^n / symmetric form (ESL-8)



$x = x(r,\theta)$
$y = y(r,\theta)$
parameters

$\downarrow$

$(y-y_1)^2 + (x-x_1)^2 = r^2$ $\quad$ (θ goes for a ride)



$(x,y) = (x_1 + r\cos\theta, y_1 + r\cos\theta)$

$\rightarrow \theta : [0, 2\pi]$

var. constantly changing

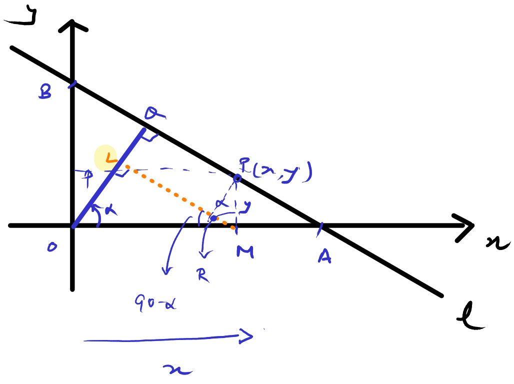
$\Downarrow$

locus (circle) generated

Lecture-7 (26/5uly) 1:5

Practice-2

Q1. l : perp. dist $= 3\sqrt{2}$, perp. Angle $= 75^\circ$, $l \equiv ?$



* $p = x\cos\alpha + y\sin\alpha$; $l \equiv x\cos 75 + y\sin 75 = 3\sqrt{2}$

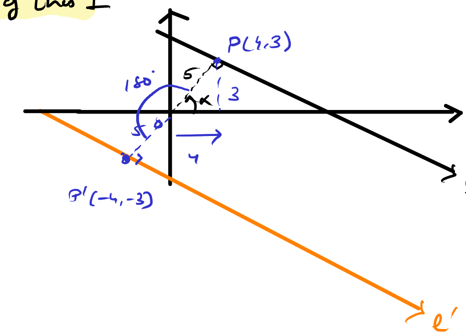
$(\sqrt{3}-1)x + (\sqrt{3}+1)y = 12$

$OL = x\cos\alpha$, $LQ = y\sin\alpha$ ✓

$p = x\cos\alpha + y\sin\alpha$

Deep think

Q2 find Eq^n of line upon which the perp. length (from O) is $\underbrace{5}_{p}$ & the slope of this $\perp$ is $3/4$



* $\tan\alpha = m = \frac{3}{4} \rightarrow \sin\alpha = \frac{3}{5}$, $\cos\alpha = \frac{4}{5}$; $x\cos\alpha + y\sin\alpha = p$

from +ve x ACW

2 possibilities

$\tan\alpha = \frac{+3}{+4}$ $\quad$ $\tan\alpha = \frac{-3}{-4}$

$P(3,4)$ $\quad$ $P(-3,-4)$

I quad $\quad$ III quad

$\tan\alpha = \tan(180+\alpha)$

$m = \tan\alpha = \tan(180+\alpha)$

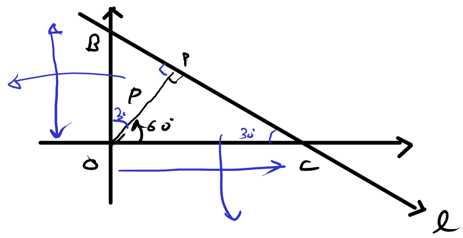
$x\cos\alpha + y\sin\alpha = p$

$x\cos(180+\alpha) + y\sin(180+\alpha) = p \Rightarrow -x\cos\alpha - y\sin\alpha = p$

$x\cos\alpha + y\sin\alpha = p$

$\Rightarrow 4x + 3y \pm 25 = 0$

Q3. l : $Ar(\Delta BOC) = 54\sqrt{3}$, $\alpha = 60^\circ$ $\quad l = ?$



* $Ar = \frac{1}{2}(OC)(OB) = 54\sqrt{3} \Rightarrow \frac{1}{2}(2p)\left(\frac{2p}{\sqrt{3}}\right) = 54\sqrt{3}$

$\Downarrow$

$p = 9$

$\frac{\sqrt{3}}{2} = \cos 30 = \frac{p}{OB}$

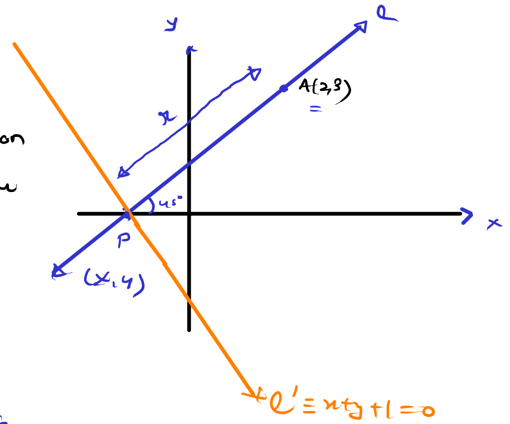
$\frac{1}{2} = \sin 30 = \frac{p}{OC}$

* $x\underbrace{\cos 60}_{\frac{1}{2}} + y\underbrace{\sin 60}_{\frac{\sqrt{3}}{2}} = 9 \Rightarrow x + y\sqrt{3} = 18 \equiv l$

HW

Q4. l : passes from A makes 45° with x axis $\rightarrow$ EoL $= ?$

$\rightarrow$ length of intercept on l b/w A & the line $x+y+1=0 = ?$



* $\frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta} = r$

* l : $\frac{x-2}{\cos 45^\circ} = \frac{y-3}{\sin 45^\circ} \Rightarrow x - y + 1 = 0$ ✓

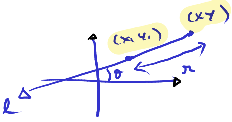
* $\frac{x-2}{\cos 45} = \frac{y-3}{\sin 45} = r \Rightarrow P(X,Y) = P\left(2+\frac{r}{\sqrt{2}}, 3+\frac{r}{\sqrt{2}}\right)$ lies on l'

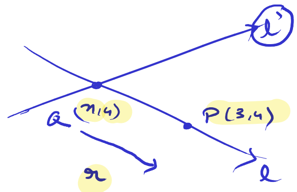
* $x+y+1=0$ satisfies $P \Rightarrow 2+\frac{r}{\sqrt{2}} + 3+\frac{r}{\sqrt{2}} + 1 = 0 \Rightarrow \sqrt{2}r + 6 = 0 \Rightarrow |r| = 3\sqrt{2}$

Lecture-8 (27/July) 2

Q5. l: passes thru. $P(3,4)$, makes $\frac{\pi}{6}$ with $\hat{x}$; l meets $l' \equiv 12x+5y+10=0$ at Q ; $|PQ|$

"dist (of l' from P) along l"





* $\frac{x-3}{\cos\pi/6} = \frac{y-4}{\sin\pi/6} = r$

$\Downarrow$

$x = 3 + r\frac{\sqrt{3}}{2}$, $y = 4 + \frac{r}{2}$ $\xrightarrow[Q(x,y)]{l' \equiv 12x+5y+10=0}$ $12\left(3+r\frac{\sqrt{3}}{2}\right) + 5\left(4+\frac{r}{2}\right) + 10 = 0$

$x = x(r)$
$y = y(r)$

l & l' meets at Q

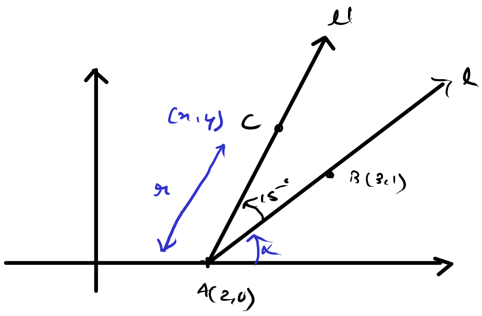
$\boxed{Q \text{ satisfies } l'}$

$\Downarrow$

$36 + r6\sqrt{3} + 20 + \frac{5}{2}r + 10 = 0 \Rightarrow r = \frac{132}{12\sqrt{3}+5}$ □

Q6. l: from $A(2,0)$ & $B(3,1)$

$l \xrightarrow[\text{rot}^n \text{ trans}^n]{R} l'$: $\theta_{rot} = 15°$ in Anticw dir^n . Pivot = A



1. EOL l' = ?

2. R $\underbrace{\text{"rot}^n\text{ matrix"}}$ trans^n Every pt. on $l \rightarrow l'$ (uniquely) 1-1 map $B \rightarrow C = ?$

* $\frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta} = r \Rightarrow l' \equiv \frac{x-2}{\cos 60} = \frac{y-0}{\sin 60} \Rightarrow x\sqrt{3} - 2\sqrt{3} = y \Rightarrow \boxed{y - x\sqrt{3} + 2\sqrt{3} = 0}$ l' (Part I)

* $A, B \rightarrow \tan\alpha = \frac{1-0}{3-2} \Rightarrow \alpha = 45° \longrightarrow \theta = 60°$

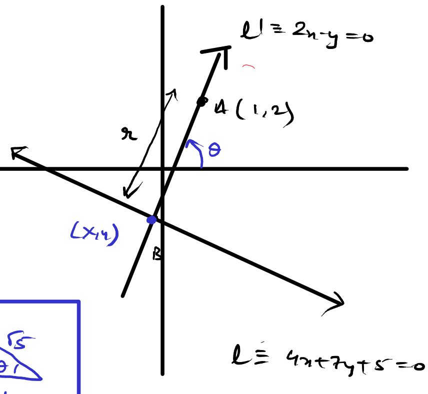
* R : Pt$(l) \longrightarrow$ Pt(l') $\Rightarrow R: B \longrightarrow C$: dist is invariant
$(|AB| = |AC| = r)$

* $r = |AC| = |AB| = \sqrt{2} \longrightarrow \frac{x-2}{\cos 60} = \frac{y-0}{\sin 60} = \sqrt{2} \Rightarrow C(x,y) \equiv \left(2+\frac{1}{\sqrt{2}}, \frac{\sqrt{6}}{2}\right)$ (Part 2)

Q7. $l \equiv 4x+7y+5=0$
$A \equiv A(1,2)$
$l' \equiv 2x-y=0$

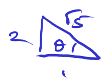
dist of l from A along l' = ?

dist (of l from A) along l'



* $\frac{x-1}{\cos\theta} = \frac{y-2}{\sin\theta} = r \Rightarrow \frac{x-1}{\frac{1}{\sqrt{5}}} = \frac{y-2}{\frac{2}{\sqrt{5}}} = r \Rightarrow \sqrt{5}(x-1) = \frac{\sqrt{5}}{2}(y-2) = r$

$\sqrt{5}x - \sqrt{5} = \frac{\sqrt{5}}{2}y - \sqrt{5} = r$



$\tan\theta = 2$
$\sin\theta = 2/\sqrt{5}$
$\cos\theta = 1/\sqrt{5}$

B satisfies l

$\Downarrow$

$x = \frac{\sqrt{5}+r}{\sqrt{5}} = 1 + \frac{r}{\sqrt{5}}$

$y = 2\left(\frac{\sqrt{5}+r}{\sqrt{5}}\right) = 2 + \frac{2r}{\sqrt{5}}$

$4\left(1+\frac{r}{\sqrt{5}}\right) + 7\left(2+\frac{2}{\sqrt{5}}r\right) + 5 = 0$

$\Downarrow$

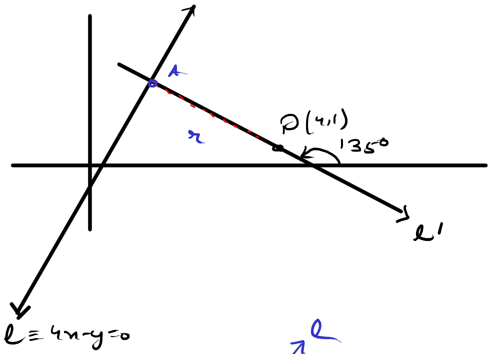
$r = \frac{23\sqrt{5}}{18}$

Q8 $l \equiv 4x - y = 0$

$P(4, 1)$

l' : makes 135° with $\hat{x}$

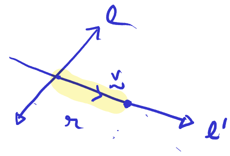
dist. of l from P measured along the l' = ?



so till this pt on l'

"dist (of l from P) measuring along l'"

vector connotation later



* $\dfrac{x-4}{\cos 135} = \dfrac{y-1}{\sin 135} = r \longrightarrow \begin{cases} x = 4 + r\cos 135 \\ y = 1 + r\sin 135 \end{cases}$ (x, y) satisfies $4x - y = 0 \Rightarrow 4(4 + r\cos 135) - (1 + r\sin 135) = 0$

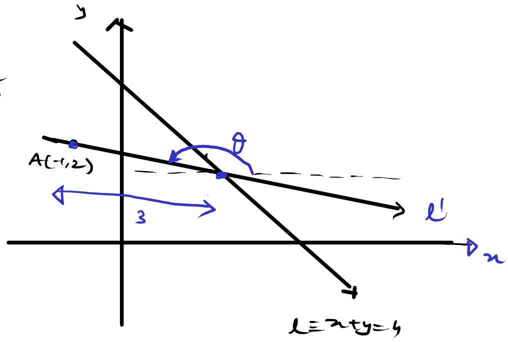
$\Downarrow$

$r = 3\sqrt{2}$

Q9. $A(-1, 2)$

$l \equiv x + y = 4$

find the $\overbrace{\text{dir}^n}^{\theta}$ in which a l' must be drawn thru. A such that its point of intersection with l is at dist $\underbrace{3}_{r}$ units from the point. ? (A)



* $\dfrac{x-(-1)}{\cos\theta} = \dfrac{y-2}{\sin\theta} = r \Rightarrow x = -1 + 3\cos\theta, \; y = 2 + 3\sin\theta$ ✓

$\Downarrow$ $x + y = 4$ satisfy (x, y)

$-1 + 3\cos\theta + 2 + 3\sin\theta = 4 \Rightarrow 3\cos\theta + 3\sin\theta = 3 \Rightarrow (\sin\theta + \cos\theta = 1)^2$

$\cancel{1} + \underbrace{2\sin\theta\cos\theta}_{} = \cancel{1} \Rightarrow \sin 2\theta = \sin n\pi$

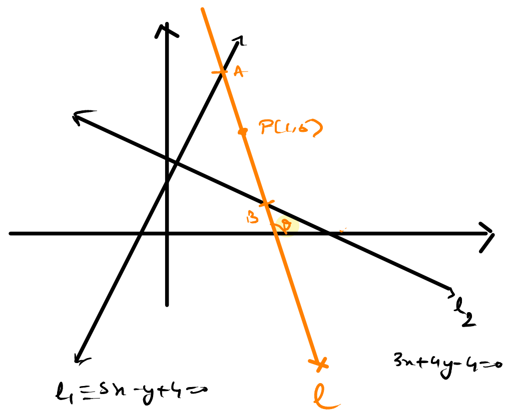
$\Downarrow$

$\boxed{\theta = \dfrac{n\pi}{2}}$ $\begin{cases} \theta = 0 \\ \theta = \dfrac{\pi}{2} \end{cases}$

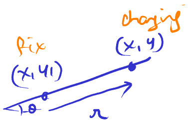
Lecture-9 (28 July) 2

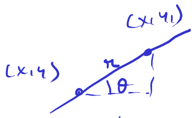
Q10. l : its segment b/w the $l_1 \equiv 5x - y + 4 = 0$ & $l_2 \equiv 3x + 4y - 4 = 0$ is bisected at the point $(1, 5)$, EOL $l \equiv$?

$|AP| = |BP|$



* $\dfrac{x - x_1}{\cos\theta} = \dfrac{y - y_1}{\sin\theta} = r$





$\dfrac{\sin\theta}{\cos\theta} = \dfrac{y_1 - y}{x_1 - x}$

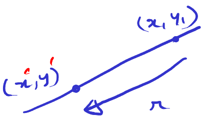
$\dfrac{x_1 - x}{\cos\theta} = \dfrac{y_1 - y}{\sin\theta} = r$

* $\dfrac{x-1}{\cos\theta} = \dfrac{y-5}{\sin\theta} = r \rightarrow \underbrace{(x, y)}_{} = (1 + r\cos\theta, 5 + r\sin\theta)$

satisfies l_1

"misguided"

* $\dfrac{x' - x_1}{\cos\theta} = \dfrac{y' - y_1}{\sin\theta} = -r$

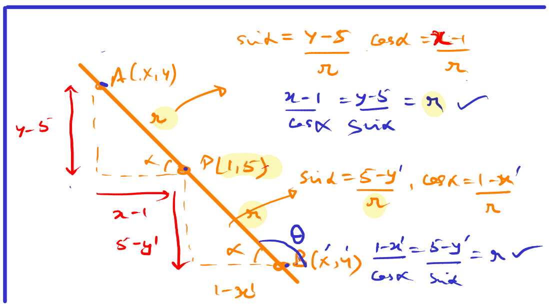


V. Imp.

$\Downarrow$

$\dfrac{x' - 1}{\cos\theta} = \dfrac{y - 5'}{\sin\theta} = -r \Rightarrow \underbrace{(x', y')}_{} = (1 - r\cos\theta, 5 - r\sin\theta)$

satisfy l_2



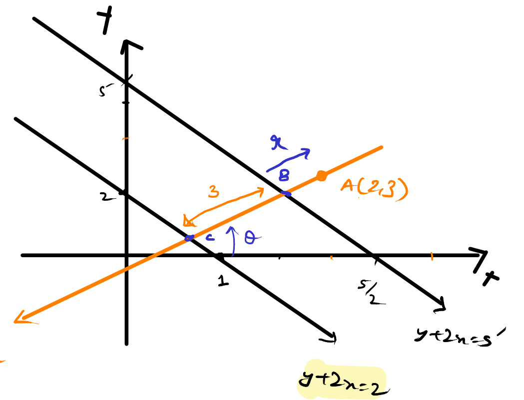
$* \quad 5(1+r\cos\theta) - (5+r\sin\theta) + 4 = 0$
$3(1-r\cos\theta) + 4(5-r\sin\theta) - 4 = 0$ $\Bigg\} \Rightarrow \tan\theta = \frac{107}{3}$ ✓

$* \quad \boxed{L \equiv y-5 = \tan\theta(x-1)} \Rightarrow 107x - 3y - 92 = 0$ ✓

5.

Q11. l : passes thru $(2,3)$ & makes intercept of 3 units b/w $l_1 \equiv y+2x=2$
& $l_2 \equiv y+2x=5$

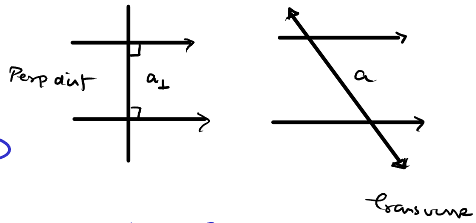
EOL $l \equiv$?



$* \quad \frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta} = r$

$* \quad AB : \frac{x-2}{\cos\theta} = \frac{y-3}{\sin\theta} = r \Rightarrow B(2+r\cos\theta, 3+r\sin\theta)$

$AC : \frac{x-2}{\cos\theta} = \frac{y-3}{\sin\theta} = r+3 \Rightarrow C(2+(r+3)\cos\theta, 3+(r+3)\sin\theta)$



$* \quad B$ satisfies $l_2 \Rightarrow 3+r\sin\theta+4+2r\cos\theta = 5 \Rightarrow r(\sin\theta+2\cos\theta) = -2$ — ①

C, " $l_1 \Rightarrow 3+r\sin\theta+3\sin\theta+4+2r\cos\theta+6\cos\theta = 2 \Rightarrow r(\sin\theta+2\cos\theta)+3(\sin\theta+2\cos\theta) = -5$

$(r+3)(\sin\theta+2\cos\theta) = -5$ — ②

$* \quad (\sin\theta+2\cos\theta)(-3) = -2+5 \Rightarrow \boxed{\sin\theta+2\cos\theta = -1}$ Trig. Eqn

$* \quad 2\cos\theta = -1-\sin\theta \rightarrow 4\underbrace{\cos^2\theta}_{1-\sin^2\theta} = 1+\sin^2\theta+2\sin\theta \Rightarrow 4-4\sin^2\theta = 1+\sin^2\theta+2\sin\theta$

$\Downarrow$

$\boxed{5\sin^2\theta+2\sin\theta-3=0}$

$\Downarrow$

$\sin\theta = -1, \ 3/5$

$\cos\theta = 0, \ -4/5 \rightarrow \tan\theta = -3/4$ ✓

$\sin^2\theta+\cos^2\theta=1$

$(\sqrt{x}+\sqrt{y}=1)^2$

$\Downarrow$

$x+y+2\sqrt{xy}$

$\sqrt{x} = 1-\sqrt{y}$

$x = 1+y-2\sqrt{y}$

Reduction of Complx.

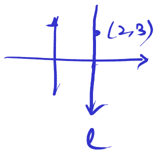
Illegal Algebraically

$l \equiv \frac{x-2}{0} = \frac{y-3}{-1}$

$-1(x-2) = 0(y-3)$

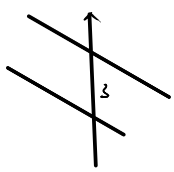
$\boxed{x-2=0} \Rightarrow x=2$

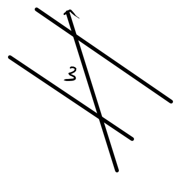
$\tan\theta = \infty$



$l \equiv \frac{x-2}{-4/5} = \frac{y-3}{3/5}$

$\frac{3}{5}(x-2) = -\frac{4}{5}(y-3) \Rightarrow \boxed{4y+3x=18}$



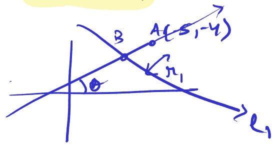


Lecture-10 (29/July) 1-5

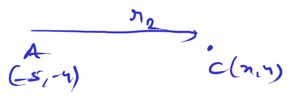
Q12. l : thru. $A(-5,-4)$ meets $l_1 \equiv x+3y+2=0$ at B
$l_2 \equiv 2x+y+4=0$ at C
$l_3 \equiv x-y-5=0$ at D

: $\left(\frac{15}{AB}\right)^2 + \left(\frac{10}{AC}\right)^2 = \left(\frac{6}{AD}\right)^2$, with $AB = r_1$, $AC = r_2$, $AD = r_3$

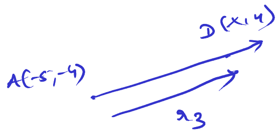
EOL $l \equiv$?



$\frac{x+5}{\cos\theta} = \frac{y+4}{\sin\theta} = r_1$



$\frac{x+5}{\cos\theta} = \frac{y+4}{\sin\theta} = r_2$



$\frac{x+5}{\cos\theta} = \frac{y+4}{\sin\theta} = r_3$

$(-5+r_1\cos\theta,\ -4+r_1\sin\theta)$ $\downarrow$ satisfies l_1

$(-5+r_2\cos\theta,\ -4+r_2\sin\theta)$ $\downarrow$ satisfies l_2

$(-5+r_3\cos\theta,\ -4+r_3\sin\theta)$ $\downarrow$ satisfies l_3

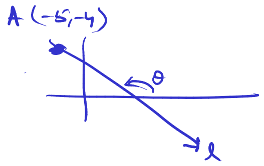
$-5+r_1\cos\theta-12+3r_1\sin\theta+2=0$ $\qquad$ $-10+2r_2\cos\theta-\cancel{4}+r_2\sin\theta+\cancel{4}=0$ $\qquad$ $-5+r_3\cos\theta+4-r_3\sin\theta-5=0$

$r_1=\dfrac{15}{\cos\theta+3\sin\theta}$ $\qquad$ $r_2=\dfrac{10}{2\cos\theta+\sin\theta}$ $\qquad$ $r_3=\dfrac{6}{\cos\theta-\sin\theta}$

* $\left(\dfrac{15}{r_1}\right)^2+\left(\dfrac{10}{r_2}\right)^2=\left(\dfrac{6}{r_3}\right)^2 \Rightarrow (\cos\theta+3\sin\theta)^2+(2\cos\theta+\sin\theta)^2=(\cos\theta-\sin\theta)^2$

$\cancel{\cos^2\theta}+9\sin^2\theta+6\cos\theta\sin\theta+4\cos^2\theta+\cancel{\sin^2\theta}+4\cos\theta\sin\theta=\cancel{\cos^2\theta}+\cancel{\sin^2\theta}-2\cos\theta\sin\theta$

* $(3\sin\theta)^2+(2\cos\theta)^2+2\cdot 6\cos\theta\sin\theta=0 \Rightarrow (3\sin\theta+2\cos\theta)^2=0 \Rightarrow \boxed{\tan\theta=-2/3}$

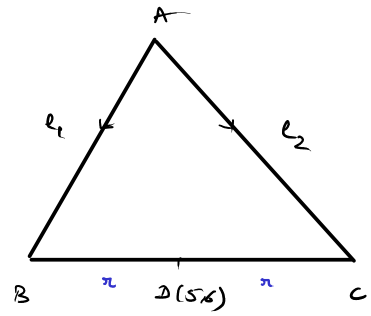


* $l \equiv y+4=\underbrace{\tan\theta}_{-\frac{2}{3}}(x+5) \Rightarrow 3y+12=-2x-10 \Rightarrow \boxed{3y+2x+22=0}$ EOL

Q13. $\triangle ABC$: AB $\rightarrow$ $2x+3y=29\equiv l_1$; $\exists$ $D(5,6)$: mid pt of BC

AC $\rightarrow$ $x+2y=16\equiv l_2$

EOL BC = ?



* BC : $\dfrac{x-5}{\cos\theta}=\dfrac{y-6}{\sin\theta}$

* B (x,y) satisfies l_1 $\Rightarrow$ use Δ top $\rightarrow$ $2(5+r\cos\theta)+3(6+r\sin\theta)=29$ — ①

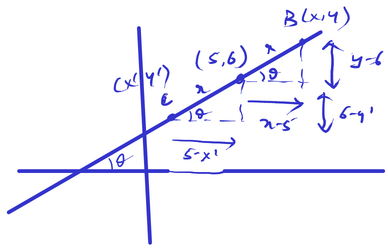
$10+2r\cos\theta+18+3r\sin\theta=29$

$r(2\cos\theta+3\sin\theta)=1 \Rightarrow r=\dfrac{1}{2\cos\theta+3\sin\theta}$

* C (x',y') satisfies l_2 $\Rightarrow$ use Δ down $\rightarrow$ $(5-r\cos\theta)+2(6-r\sin\theta)=16$. — ②

$5-r\cos\theta+12-2r\sin\theta=16$

$-r(\cos\theta+2\sin\theta)=-1 \Rightarrow r=\dfrac{1}{\cos\theta+2\sin\theta}$



$2\cos\theta+3\sin\theta=\cos\theta+2\sin\theta \Rightarrow \sin\theta=-\cos\theta \Rightarrow \tan\theta=-\dfrac{1}{1} \Rightarrow \theta=\dfrac{3\pi}{4}$ ✓

Δ top : $\sin\theta=\dfrac{y-6}{r}$, $\cos\theta=\dfrac{x-5}{r}$

$\dfrac{x-5}{\cos\theta}=\dfrac{y-6}{\sin\theta}=r$ ✓

* BC : $y-6=\tan\theta(x-5) \Rightarrow y-6=-x+5 \Rightarrow \boxed{y+x-11=0}$

Δ down : $\sin\theta=\dfrac{6-y'}{r}$, $\cos\theta=\dfrac{5-x'}{r}$

$\dfrac{5-x'}{\cos\theta}=\dfrac{6-y'}{\sin\theta}=r$ ✓

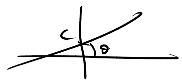
6.5. Transformation of forms. (Easy bit!)

1. Slope form / Int-Int form

* $ax+by+c=0 \Rightarrow by=-ax-c \Rightarrow y=-\dfrac{a}{b}x-\dfrac{c}{b}=mx+c$ (Std form $\longrightarrow$ slope-int form)

$\downarrow$

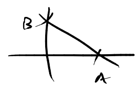
$m=-\dfrac{a}{b}$, $c=-\dfrac{c}{b}$



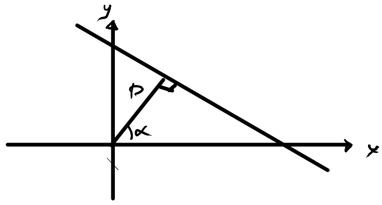
* $ax+by+c=0 \Rightarrow ax+by=-c \Rightarrow \dfrac{x}{(-c/a)}+\dfrac{y}{(-c/b)}=1 \Rightarrow \dfrac{x}{A}+\dfrac{y}{B}=1$ (Std. form $\longrightarrow$ Int-Int form)

$\downarrow$

$A=-\dfrac{c}{a}$, $B=-c/b$



2. Normal form



* $ax+by+c=0 \longrightarrow x\cos\alpha + y\sin\alpha - p = 0$

  (a,b,c) given

  $a=\cos\alpha$, $b=\sin\alpha$, $c=-p$

* $\dfrac{a}{\cos\alpha} = \dfrac{b}{\sin\alpha} = \dfrac{c}{-p} \Rightarrow \cos\alpha = \dfrac{-ap}{c}$, $\sin\alpha = \dfrac{-bp}{c} \Rightarrow 1 = \dfrac{a^2p^2}{c^2} + \dfrac{b^2p^2}{c^2} = \dfrac{p^2}{c^2}(b^2+a^2) \Rightarrow \boxed{p = \dfrac{c}{\sqrt{a^2+b^2}}}$ $p>0$

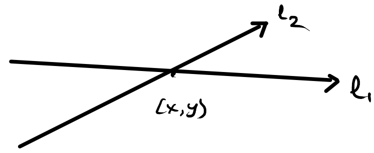
* $\cos\alpha = \dfrac{-a}{\sqrt{a^2+b^2}}$, $\sin\alpha = \dfrac{-b}{\sqrt{a^2+b^2}}$

* $ax+by+c=0 \longrightarrow \dfrac{1}{\sqrt{a^2+b^2}}\{ax+by+c\} = 0 \Rightarrow \dfrac{-a}{\sqrt{a^2+b^2}}x - \dfrac{b}{\sqrt{a^2+b^2}}y - \dfrac{c}{\sqrt{a^2+b^2}} = 0 \Rightarrow \underbrace{\dfrac{-a}{\sqrt{a^2+b^2}}}_{\cos\alpha}x - \underbrace{\dfrac{b}{\sqrt{a^2+b^2}}}_{\sin\alpha}y = \underbrace{\dfrac{c}{\sqrt{a^2+b^2}}}_{p}$ Normal form.

Lecture-11 (30/Jul) 1.5

2. St. Lines & solutions of linear eqn in 2 var

2.1 Pt of intersection of straight curves

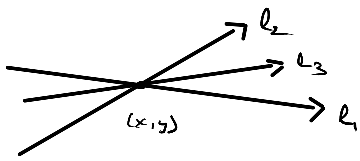


* $\left.\begin{array}{l} l_1 \equiv a_1x+b_1y+c_1=0 \\ l_2 \equiv a_2x+b_2y+c_2=0 \end{array}\right\}$ 2 simultaneous linear eqn

$$\begin{vmatrix} x & y & 1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \longrightarrow x = \frac{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} , \quad y = -\frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$ Cramer's rule

$$\boxed{\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}} \longrightarrow (x,y) - \text{pt of intersection of } l_1 \,\&\, l_2$$

2.2. Concurrent Condn for 3 lines



* $l_1 \equiv a_1x+b_1y+c_1=0$

  $l_2 \equiv a_2x+b_2y+c_2=0$ (a_i, b_i, c_i) constants

  $l_3 \equiv a_3x+b_3y+c_3=0$

* $l_1, l_2 \Rightarrow x = \dfrac{\Delta_1}{\Delta}$, $y = \dfrac{-\Delta_2}{\Delta}$, $\Delta_1 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$, $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

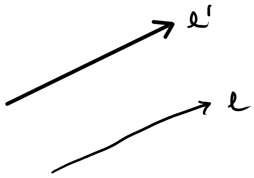
(x,y) satisfy $l_3 \Rightarrow a_3\dfrac{\Delta_1}{\Delta} + b_3\left(\dfrac{-\Delta_2}{\Delta}\right) + c_3 = 0 \Rightarrow a_3\Delta_1 - b_3\Delta_2 + c_3\Delta = 0$

$$a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - b_3\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + c_3\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$$

Eigenvalues
Eng vol.1
Lec-33

$$\boxed{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0} \Rightarrow \text{3 lines are Concurrent}$$

2.3. EOL $\parallel$ & $\perp$ to a given



* $l \equiv ax+by+c=0 \longrightarrow m_l = \frac{-a}{b}$

Case I

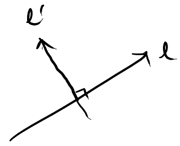
* $l \parallel l' \Rightarrow m_{l'} = m_l = \frac{-a}{b} \Longrightarrow l' \equiv y = m_{l'}x + c' = -\frac{a}{b}x + c' \Rightarrow by = -ax + c'b$

$$\Downarrow$$

$$l' \equiv ax + by - c'b = 0$$

$l' \equiv ax + by + \lambda = 0$; $\lambda \equiv -c'b$ {EOL $\parallel$ to given line l

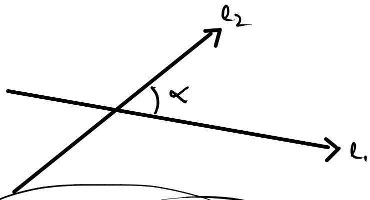
Case II



* $l \perp l' \Rightarrow m_{l'} \cdot m_l = -1 \Rightarrow m_{l'} = \frac{b}{a} \Rightarrow l' \equiv y = m_{l'}x + c' = \frac{b}{a}x + c' \Rightarrow l' \equiv ay - bx - c'b = 0$

$l' \equiv bx - ay + \lambda$, $\lambda \equiv c'b$ { EOL $\perp$ to given line l

2.4. Angle b/w 2 lines in terms of Coeff (l_1, l_2)



* $l_1 \equiv a_1x + b_1y + c_1 = 0 \longrightarrow m_{l_1} = \frac{-a_1}{b_1} \equiv m_1$

$l_2 \equiv a_2x + b_2y + c_2 = 0 \longrightarrow m_{l_2} = -\frac{a_2}{b_2} \equiv m_2$

* $\underbrace{\tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|}_{\alpha(l_1, l_2) \text{ in terms of slopes}} = \left|\frac{\frac{-a_1}{b_1} + \frac{a_2}{b_2}}{1 + \frac{a_1 a_2}{b_1 b_2}}\right| \Rightarrow \boxed{\tan\alpha = \left|\frac{a_2 b_1 - a_1 b_2}{a_1 a_2 + b_1 b_2}\right|}$

Angle b/w the lines

* $\parallel \Rightarrow m_1 = m_2 \Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} \Rightarrow a_1 b_2 = a_2 b_1 \Rightarrow \tan\alpha = 0 \longrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

* $\perp \Rightarrow m_1 \cdot m_2 = -1 \Rightarrow \frac{a_1}{b_1} \cdot \frac{a_2}{b_2} = -1 \Rightarrow a_1 a_2 + b_1 b_2 = 0 \Rightarrow \tan\alpha \to \infty$

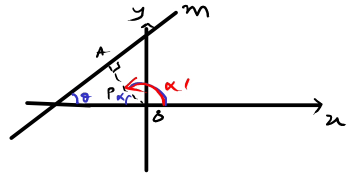
* l_1, l_2 are coincident $\Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

* $l_1 l_2$ intersect $\Rightarrow \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

Lecture-12 (3 Aug) 1.5

Practice-3

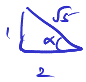
Q1. $m_l = 2$, $p_\perp$ (perp. dist from origin) $= \sqrt{5}$, EOL $l \equiv$?



Appr. 1

* $\tan\theta = 2 \Rightarrow \tan(90-\alpha) = 2 \Rightarrow \cot\alpha = 2 \Rightarrow \tan\alpha = \frac{1}{2}$

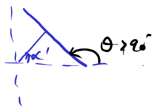
$\theta + \alpha + 90 = 180°$



$\sin\alpha = \frac{1}{\sqrt{5}}$, $\cos\alpha = \frac{2}{\sqrt{5}}$

* α = Angle made by +ve x axis & Normal in ACW

Normal form: $x\cos\alpha' + y\sin\alpha' = \sqrt{5}$

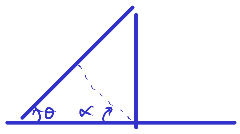


$x\cos(180-\alpha) + y\sin(180-\alpha) = \sqrt{5}$

$\theta \to \alpha \to \alpha'$

$-x\cos\alpha + y\sin\alpha = \sqrt{5}$

$-x\frac{2}{\sqrt{5}} + y\frac{1}{\sqrt{5}} = \sqrt{5} \Rightarrow \boxed{y = 2x + 5}$

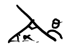


$ax + by + c = 0$

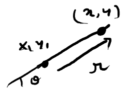
$y = mx + c$

$\frac{x}{a} + \frac{y}{b} = 1$

$x\cos\alpha + y\sin\alpha = p$



$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$



$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$

Appr. 2

* m : given $\Rightarrow y = 2x + \underset{\substack{\text{(y-intercept)}\\ \text{unknown}}}{c} \Rightarrow \underbrace{-2x + y = c}_{ax+by=c} \longrightarrow$ Normal form $\frac{-2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}y = \frac{c}{\sqrt{5}}$

divide $\sqrt{a^2+b^2}$

$x\cos\alpha + y\sin\alpha = p = \sqrt{5}$ (given)

$\left|\frac{c}{\sqrt{5}}\right| = \sqrt{5} \Rightarrow c = \pm 5$

$-2x + y = \pm 5 \Rightarrow y = 2x \pm 5$

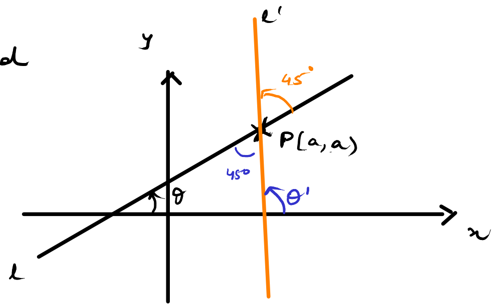
Q2. $l : (k-3)x - (4-k^2)y + k^2 - 7k + 6 = 0$

k
- $l \parallel x$ axis ? $k = 3$
- $l \parallel y$ axis ? $k = \pm 2$
- l passing thru O ? $k = 1, 6$

Q3. $l : 2x - y = 5$; $l \xrightarrow[\substack{T\\ \text{(Rotation transf}^n)}]{\text{transf}^n} l'$: Pivot is where x & y are equal

Rotation is of $45°$ ACW

EOL $l' \equiv$?



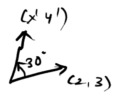
* $m_l = \tan\theta = 2$
* l (satisfy P) $\Rightarrow a = 5 \Rightarrow P(5,5)$ Pivot ; $\theta + 45 = \theta'$
* l' : $y - 5 = \tan\theta'(x - 5)$, $\tan(\theta+45) = \frac{\tan\theta + \tan 45}{1 - \tan\theta\tan 45} = \frac{1+2}{1-2} = -3$

$y - 5 = -3(x - 5) \Rightarrow \boxed{3x + y = 20}$

Comment on transf^n Matrix

$$\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}_{2\times2}\begin{pmatrix}x\\ y\end{pmatrix}_{2\times1} = \begin{pmatrix}x'\\ y'\end{pmatrix}$$

$(2,3) \xrightarrow{30°} (x', y')$



$$T = \begin{pmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}_{2\times2}$$

$$\begin{pmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}2\\ 3\end{pmatrix} = \begin{pmatrix}x'\\ y'\end{pmatrix}$$

$x' = \sqrt{3} - \frac{3}{2}$

$y' = \frac{1 + 3\sqrt{3}}{2}$

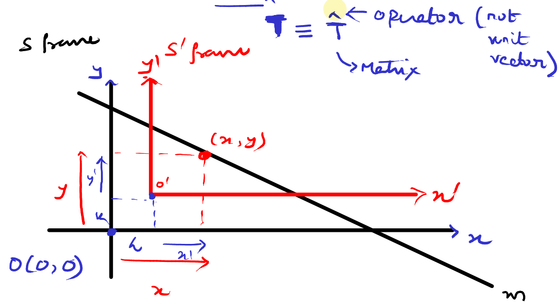
$$\begin{pmatrix}2\\ 3\end{pmatrix} \xrightarrow{T} \begin{pmatrix}\sqrt{3} - \frac{3}{2}\\ \frac{1+3\sqrt{3}}{2}\end{pmatrix}$$

Lecture-13 (4/Aug) 1.5

Noether's thms (1880-35)
Space transl^n $x \to x+a$: $\underset{\sim}{p}$ = inv.
time transf^n $t \to t+\lambda$: E = inv.

Q4 P.T : translation of Axes (Shifting of origin) - later leaves the slope of a line invariant

notation: $\mathsf{T} \equiv \hat{T}$ ← operator (not unit vector) → Matrix



* $l : ax+by+c=0 \Rightarrow m_l = -\frac{a}{b}$

$$\begin{array}{ccc} l = l(\underbrace{x,y}) & \xrightarrow{T} & l' = l'(\underbrace{x',y'}) \\ \downarrow \text{S frame} & & \downarrow \text{S' frame} \\ \text{origin } (0,0) & \xrightarrow{T} & \text{origin } O'(h,k) \end{array}$$

* $(x,y) \xrightarrow{T} (x',y')$: $x = x'+h$
$y = y'+k$

* $l' = l'(x',y') \equiv a(x'+h)+b(y'+k)+c=0 \Rightarrow ax'+by'+\underbrace{(ah+bk+c)}_{g}=0 \Rightarrow ax'+by'+g=0 \Rightarrow \boxed{m_{l'} = -\frac{a}{b}}$
$by' = -ax'-g$

$$\boxed{ax+by+c=0 \xrightarrow{T} ax'+by'+g=0 \quad : \quad m = \text{invariant}}$$

Q5. $\left.\begin{array}{l} l_1 \equiv mx+(2m+3)y+m+6=0 \\ l_2 \equiv (2m+1)x+(m-1)y+m-9=0 \end{array}\right\}$ $m = ?$: l_1 intersect l_2 at $\underbrace{\text{a pt on Y axis}}_{P(0,y)}$.

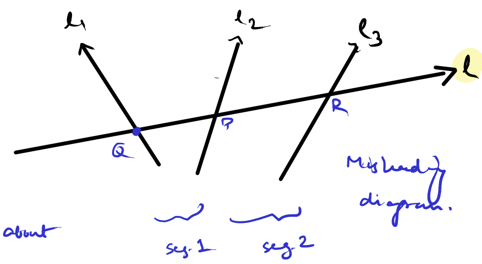
* $m = -1, 21$

Q6. $\left.\begin{array}{l} l_1 \equiv y = m_1x+c_1 \\ l_2 \equiv y = m_2x+c_2 \\ l_3 \equiv x=0 \end{array}\right\}$ $Ar(\Delta) = ?$

* $A(0, c_1)$, $B(0, c_2)$, $C\left(\underbrace{\frac{c_1-c_2}{m_2-m_1}}, \frac{m_2c_1-m_1c_2}{m_2-m_1}\right) \Rightarrow Ar(\Delta) = \frac{1}{2}\left|\frac{(c_1-c_2)^2}{m_1-m_2}\right|$

Lecture-14 (5/Aug) 1.5

Q7. $\left.\begin{array}{l} l_1 \equiv 4x+y-9=0 \\ l_2 \equiv x-2y+3=0 \\ l_3 \equiv 5x-y-6=0 \end{array}\right\}$ PT : make equal intercepts on $\underbrace{\text{any line of gradient 2}}$
$l \equiv y = 2x+c$
No worries about this



* $\boxed{\text{Equal Intercept} \equiv P \text{ is the midpt of } QR \Rightarrow QP = PR}$ To prove

$\exists$ line l $\Rightarrow$ Existence of $\underbrace{\text{finite distance}}_{QR}$

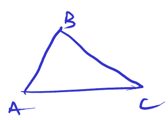
* l_1 intersects l $\Rightarrow Q\left(\frac{9-c}{6}, \frac{9+2c}{3}\right)$

l_2 " l $\Rightarrow P\left(\frac{3-2c}{3}, \frac{6-c}{3}\right)$

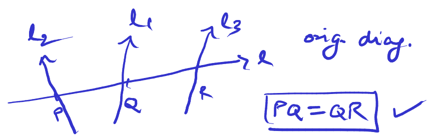
l_3 " l $\Rightarrow R\left(\frac{6+c}{3}, \frac{12+5c}{3}\right)$

* Distance formula Approach

$AB = BC \Rightarrow$ B lies on AC (not mid pt) $\Rightarrow \boxed{Ar\ \Delta = 0} = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

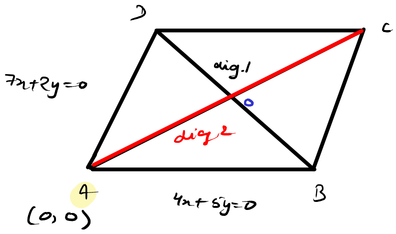


$\exists$ finite, non 0 Area of Δ



* Section formula Approach → Mid pt of P, R $= Q\left(\frac{9-c}{6}, \frac{9+2c}{3}\right)$ $\boxed{PQ = QR}$ ✓

Q8. ▱ (||gm) 2 sides $\ulcorner l_1 \equiv 4x+5y=0$
$\hookrightarrow l_2 \equiv 7x+2y=0$ (consecutive sides) ; diag 1 $\equiv 11x+7y=9$; diag.2 $\equiv$?



Note : l_1 & l_2 satisfy origin $(0,0)$ => $A = A(0,0)$

* l_2 intersects diag.1 => $D\left(\frac{-2}{3}, \frac{7}{3}\right)$

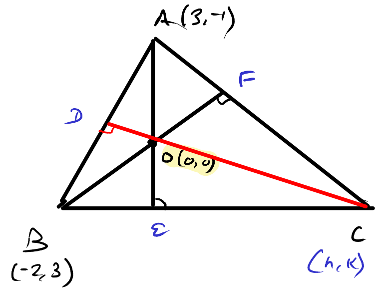
l_1 " diag 1 => $B\left(\frac{5}{3}, \frac{-4}{3}\right)$

* diag. of ||gm bisect => O mid pt of DB => $O\left(\frac{1}{2}, \frac{1}{2}\right)$

* diag.2 = $y - 0 = \frac{y_2 - 0}{x_2 - 0}(x-0)$ => $y = x$ ✓45

Lecture-15 (6/Aug) 2

Q9. ΔABC : $A(3,-1)$, $B(-2,3)$, Its orthocentre is at the origin ; $C \equiv$?



* $(m_{BO}, m_{AO}, m_{AC}, m_{BC}) \rightarrow$ $\left.\begin{aligned} m_{BO} \cdot m_{AC} &= -1 \\ m_{AO} \cdot m_{BC} &= -1 \end{aligned}\right\}$ => $(h,k) = \left(\frac{-36}{7}, \frac{-45}{7}\right)$

$m_{AO} = m_{AE}$

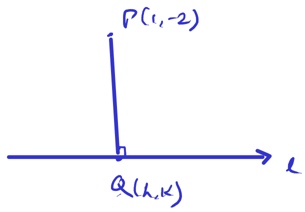
$m_{BO} = m_{BF}$

Q10. $\left.\begin{aligned} l_1 &\equiv ax+y+1=0 \\ l_2 &\equiv x+by+1=0 \\ l_3 &\equiv x+y+c=0 \end{aligned}\right\}$ Concurrent => $\overbrace{\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c}}^{f(a,b,c)} = ?$

* $\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$ => $abc + 2 = a+b+c$ □

* $f(a,b,c) = \frac{(1-b)(1-c) + (1-a)(1-c) + (1-a)(1-b)}{(1-a)(1-b)(1-c)} = \frac{1-c-b+bc+1-c-a+ac+1-b-a+ab}{(1-a)(1-b)(1-c)} = \frac{3-2(a+b+c)+ab+bc+ca}{(1-a)(1-b)(1-c)}$ ....=....= 1

Q11. $P(1,-2)$, $l \equiv y = 2x+1$; a ⊥ is drawn from P to l , Coord. of the foot of the ⊥ = ? (Q)



* $l \equiv 2x - y + 1 = 0$ => $m_l = 2$

$m_{PQ} \cdot m_l = -1$ => $m_{PQ} = -\frac{1}{2}$, $l' \equiv y + 2 = m_{PQ}(x-1)$ => $2y + x + 3 = 0$ ✓

$Q(-1,-1)$

Q12 $l(\theta) \equiv x\sec\theta + y\operatorname{cosec}\theta = a$

$P(a\cos^3\theta, a\sin^3\theta)$; $l'(\theta) \equiv$? : l' passes thru P
$l' \perp l$

* $x\cos\theta - y\sin\theta = a\cos 2\theta \equiv l'(\theta)$

HW

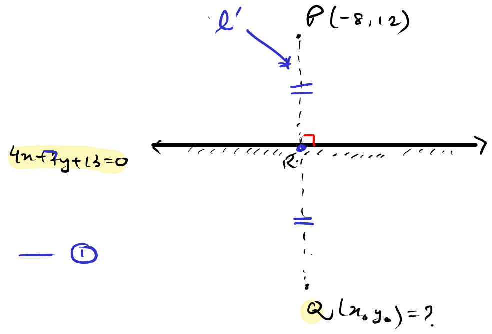
* Line Mirror $\equiv 4x + 7y + 13 = 0$, Img $\left(P(-8,12)\right) = ?$

* ΔABC : $l_1 = 3x - 2y + 6 = 0$, $l_2 \equiv 4x + 5y - 20 = 0$, Orthocentre $= (1,1)$, $l_3 \equiv ?$ (3rd side of Δ)

Lecture-16 (10/Aug.) 1.5

* $l_M \Rightarrow m_l = \frac{-4}{7} \longrightarrow m_{l'} = \frac{7}{4}$

$l' \equiv y-12 = \frac{7}{4}(x+8) \Rightarrow -7x+4y-104=0 \Rightarrow 7x-4y+104=0$



* $R\left(\frac{-8+x_0}{2}, \frac{12+y_0}{2}\right) \xrightarrow{\text{sat. } l} \frac{-32+4x_0+84+7y_0}{2}+13=0 \Rightarrow 4x_0+7y_0+78=0$ — ①

* $Q(x_0,y_0) \xrightarrow{\text{sat. } l'} 7x_0-4y_0+104=0$ — ②

$$\left.\begin{array}{l} 4x_0+7y_0+78=0 \\ 7x_0-4y_0+104=0 \end{array}\right\} \Rightarrow (x_0, y_0) = (-16, -2) \checkmark$$

* ΔABC : $l_1 = 3x-2y+6=0$, $l_2 \equiv 4x+5y-20=0$, Orthocentre $= (1,1)$, $l_3 \equiv ?$ (3rd side of Δ)

* $m_{l_1} = \frac{3}{2}$, $m_{l_2} = \frac{-4}{5}$

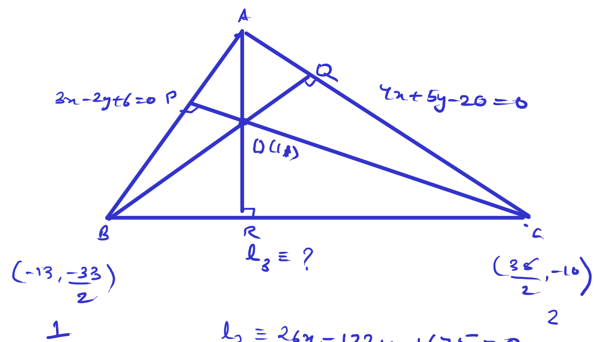
$\downarrow$ $\quad\downarrow$

$m_{PC} = \frac{-2}{3}$, $m_{BQ} = \frac{5}{4}$

$l_{PC} \equiv y-1 = \frac{-2}{3}(x-1)$ $\qquad$ $l_{BQ} = y-1 = \frac{5}{4}(x-1)$

$3y-3 = -2x+2$ $\qquad$ $4y-4 = 5x-5$

$2x+3y-5=0 \equiv l_{PC}$ $\qquad$ $5x-4y-1=0 \equiv l_{BQ}$



* $(l_{PC}, l_{AC}) \Rightarrow \left.\begin{array}{l} 2x+3y-5=0 \\ 4x+5y-20=0 \end{array}\right\} \quad \frac{x}{-60+25} = \frac{-y}{-40+20} = \frac{1}{10-12} \Rightarrow x = \frac{35}{2}, y=-10 \Rightarrow C\left(\frac{35}{2}, -10\right)$

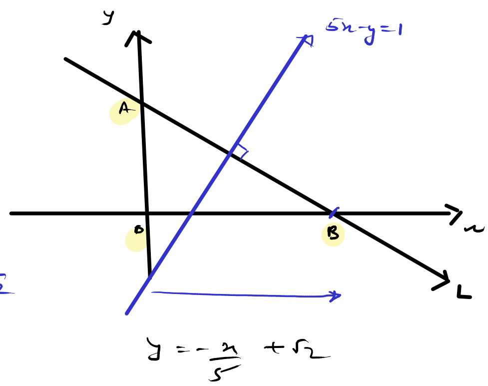
* $(l_{BQ}, l_{AB}) \Rightarrow \left.\begin{array}{l} 5x-4y-1=0 \\ 3x-2y+6=0 \end{array}\right\} \quad \frac{x}{-24-2} = \frac{-y}{30+3} = \frac{1}{-10+12} \Rightarrow x = \frac{-26}{2}, y = \frac{-33}{2} \quad B\left(-13, \frac{-33}{2}\right)$

* $l_3 \equiv \left(y+\frac{33}{2}\right) = \frac{-10+\frac{33}{2}}{\frac{35}{2}+13}(x+13) \Rightarrow 2y+33 = \frac{13}{61}(x+13) \Rightarrow 26x-122y-1675=0 \checkmark$

Q13. $l \equiv 5x-y=1$, $L \perp l$ $\qquad$ EOL = ?

Δ formed by L & coordinate axes, $\underbrace{Ar(\Delta)}$ = 5

$Ar(\Delta AOB) = \frac{1}{2} OA(OB) = 5$



$m_l = 5$

$L \equiv y = m_L x + c = \frac{-1}{5}x + c$; $Ar \Rightarrow \frac{1}{2}c(5c) = 5 \Rightarrow c^2 = 2 \Rightarrow c = \pm\sqrt{2}$

$y = \frac{-1}{5}x \pm \sqrt{2} \Rightarrow x+5y \pm \sqrt{2} = 0 \checkmark$

new

$l \equiv ax+by+c=0$, $P(x_1, y_1)$

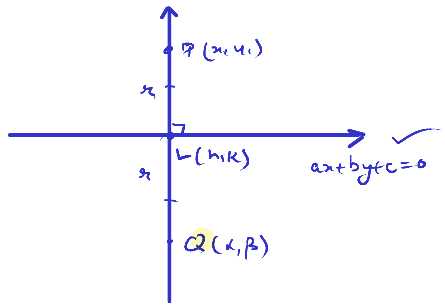
$L(h,k)$ is the foot of $\perp$ from P to l

$Q(\alpha, \beta) = \text{Img.}(P)$

1. $\frac{h-x_1}{a} = \frac{k-y_1}{b} = ?$ (P.T)

2. $\frac{\alpha-x_1}{a} = \frac{\beta-y_1}{b} = ?$ (P.T)

Lecture-17 (22/Aug.) 1.5

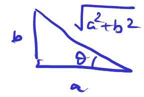


* $|PL| = |LQ| = r$
* line PQ

$\to \dfrac{h-x_1}{\cos\theta} = \dfrac{k-y_1}{\sin\theta} = r$ (L, P) — ①

$\to \dfrac{\alpha-x_1}{\cos\theta} = \dfrac{\beta-y_1}{\sin\theta} = 2r$ (Q, P) — ②

* $PQ \perp l \Rightarrow m_{PQ} = \tan\theta = \dfrac{-1}{m_l} = \dfrac{-1}{\frac{-a}{b}} = \dfrac{b}{a}$



{note: diag is not to scale}

$\sin\theta = \dfrac{b}{\sqrt{a^2+b^2}}$

$\cos\theta = \dfrac{a}{\sqrt{a^2+b^2}}$

* ① $\Rightarrow L(\underbrace{x_1 + r\cos\theta}_{h}, \underbrace{y_1 + r\sin\theta}_{k})$ satisfies $ax+by+c=0$

$a(x_1 + r\cos\theta) + b(y_1 + r\sin\theta) + c = 0 \Rightarrow ax_1 + ar\cos\theta + by_1 + br\sin\theta + c = 0$

$\Downarrow$

$r = -\dfrac{(ax_1+by_1+c)}{a\cos\theta + b\sin\theta} = -\dfrac{(ax_1+by_1+c)}{\sqrt{a^2+b^2}}$

* ① $\Rightarrow \dfrac{h-x_1}{a} = \dfrac{k-y_1}{b} = -\dfrac{ax_1+by_1+c}{a^2+b^2}$

② $\Rightarrow \dfrac{\alpha-x_1}{a} = \dfrac{\beta-y_1}{b} = -2\left(\dfrac{ax_1+by_1+c}{a^2+b^2}\right)$

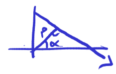
$y = mx+c$

$y - y_1 = m(x-x_1)$

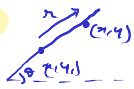
$y - y_1 = \dfrac{y_2-y_1}{x_2-x_1}(x-x_1)$

$\dfrac{x}{a} + \dfrac{y}{b} = 1$

* $x\cos\alpha + y\sin\alpha = p$



* $\dfrac{x-x_1}{\cos\theta} = \dfrac{y-y_1}{\sin\theta} = r$



Q14. a) $l_1 = x - y\sqrt{3} - 5 = 0$
$l_2 = \sqrt{3}x + y - 7 = 0$

$\theta(l_1, l_2) = ?$ $= 90^\circ \Rightarrow l_1 \perp l_2$

b) Lines whose intercepts on the axes are $\underbrace{a, -b}_{l_1}$ & $\underbrace{b, -a}_{l_2}$ $\theta(l_1, l_2) = ?$ tangent

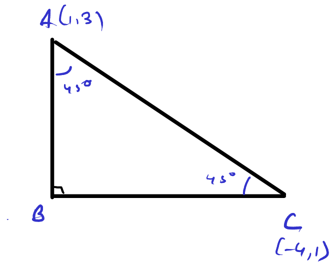
$\tan\theta = \pm\left(\dfrac{b^2-a^2}{2ab}\right)$

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

hw

Q15 Right Isosceles Δ : hypo. has its ends at (1, 3) & (−4, 1)

Eq^n of legs of Δ = ?



* $m_{AC}, m_{BC} \to \tan\theta = \left|\dfrac{m_{BC} - m_{AC}}{1 + m_{BC} m_{AC}}\right|$

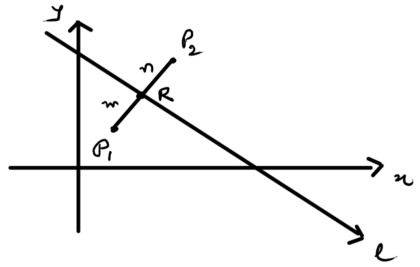
3. The Most Interesting Part

3.1. Relative Position of 2 pts wrt line / line segment

* $l \equiv ax+by+c=0$, $(\underbrace{P_1(x_1,y_1),\ P_2(x_2,y_2)})$ given

* $R\left(\dfrac{mx_2+nx_1}{m+n},\ \dfrac{my_2+ny_1}{m+n}\right)$ satisfies l

* $a\left(\dfrac{mx_2+nx_1}{m+n}\right)+b\left(\dfrac{my_2+ny_1}{m+n}\right)+c=0$



ratio in which line cuts a line seg.

$(ax_2+by_2+c)m+(ax_1+by_1+c)n=0 \Rightarrow \boxed{\dfrac{m}{n}=-\dfrac{(ax_1+by_1+c)}{(ax_2+by_2+c)}}$

$\dfrac{m}{n}>0 \Rightarrow \dfrac{(\ )_1}{(\ )_2}<0$

$\Downarrow$

$(\ )_1>0$, $(\ )_2<0$ or $(\ )_1<0$, $(\ )_2>0$

P_1 & P_2 are on the opp. sides

$\dfrac{m}{n}<0 \Rightarrow \dfrac{(\ )_1}{(\ )_2}>0$

$\Downarrow$

$(\ )_1>0$, $(\ )_2>0$ or $(\ )_1<0$, $(\ )_2<0$

P_1 & P_2 are on the same sides

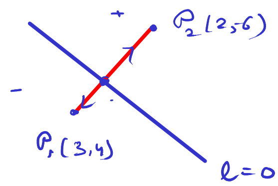
$\boxed{\text{same sign} \Rightarrow \text{same side}}$

Ex: $P_1(3,4)$ & $P_2(2,-6)$ $l \equiv 3x-4y=8$ Are P_1 & P_2 on same/opp. side of l ?

* $l(x,y) \equiv 3x-4y-8 \rightarrow$ st. line

$l(P_1)=-15$, $l(P_2)=22$

$l(P_1)<0$, $l(P_2)>0 \Rightarrow$ opp sign

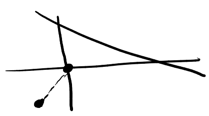


Comment about Origin :

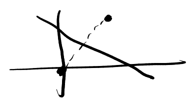
* $P_1(x_1,y_1)$, $P_2(0,0)$; $l \equiv ax+by+c=0$

$l(P_1)=ax_1+by_1+c$
$l(P_2)=c$

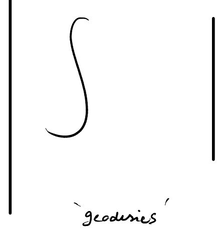
→ $l(P_1)$ & $l(P_2)$ same sign — P_1 lie on the side of the origin



→ $l(P_1)$ & $l(P_2)$ opp. sign — P_1 lie on the opp. side of the origin



Big Question HW
What is a straight line ?

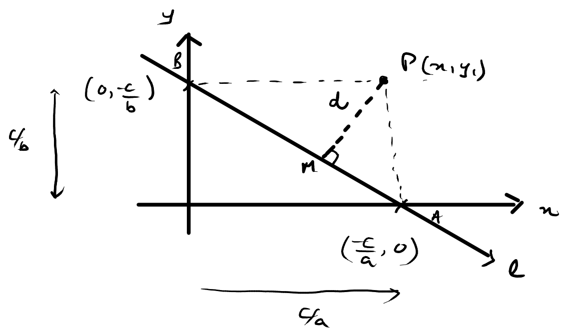


'geodesics'

3.2. Normal dist of a pt. from a line (Unique)

* $l(x,y) \equiv ax+by+c=0$, $P(x_1,y_1)$ given

* $d \equiv \perp$ dist from l to P = ?



* $Ar(BPA)=\frac{1}{2}(AB)d=\frac{1}{2}\sqrt{\frac{c^2}{b^2}+\frac{c^2}{a^2}}\,d=\frac{c}{2ab}\sqrt{a^2+b^2}\cdot d$ — ①

$Ar(\Delta ABC)=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ -\frac{c}{a} & 0 & 1 \\ 0 & -\frac{c}{b} & 1 \end{vmatrix}=\frac{1}{2}\left|x_1\left(\frac{c}{b}\right)-y_1\left(\frac{-c}{a}\right)+1\left(\frac{c^2}{ab}\right)\right|$

$=\frac{c}{2ab}\left|ax_1+by_1+c\right|$ — ②

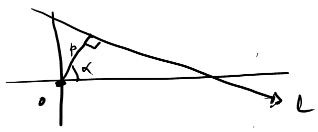
* $Ar(\Delta) = \frac{\cancel{c}}{\cancel{2ab}}\sqrt{a^2+b^2}\, d = \frac{\cancel{c}}{\cancel{2ab}} | ax_1+by_1+c | \Rightarrow$ $\boxed{d = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}}$ $\perp$ dist of a pt from a line

$[\ell, P(x_1,y_1)]$ needed

* $\boxed{d = \frac{|c|}{\sqrt{a^2+b^2}}}$ $[\ell, P(0,0)]$

$\perp$ dist from a line to the origin.

$r = x\hat{i} + y\hat{j}$

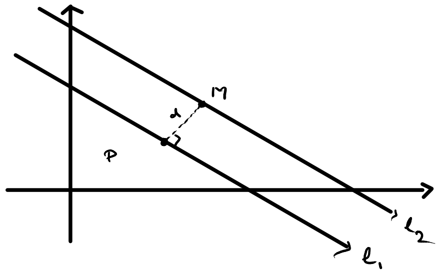


$\ell \equiv x\cos\alpha + y\sin\alpha = p$

normal form

Lecture-19 (13/Aug) 1.5

3.3. Normal dist. b/w 2 || lines



* $\ell_1 \parallel \ell_2 \Rightarrow a_1 = a_2, \; b_1 = b_2$

$\ell_1 \equiv a_1x + b_1y + c_1 = 0 \Rightarrow \ell_1 \equiv ax + by + c_1 = 0$

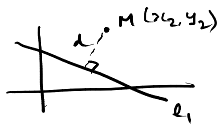
$\ell_2 \equiv a_2x + b_2y + c_2 = 0 \Rightarrow \ell_2 \equiv ax + by + c_2 = 0$

indep. variables (a, b, c_1, c_2)

* $d(\ell_1, \ell_2) = ?$

* $d = \frac{|ax_2 + by_2 + c_1|}{\sqrt{a^2+b^2}}$

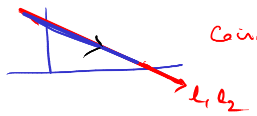
Note: (x_2, y_2) can be translated on the line ℓ_2 anywhere $\Rightarrow$ (x_2,y_2) is not imp.



* M lies on $\ell_2 \Rightarrow$ M satisfies $\ell_2 \Rightarrow ax_2 + by_2 + c_2 = 0 \Rightarrow \boxed{ax_2 + by_2 = -c_2}$

perp. dist.

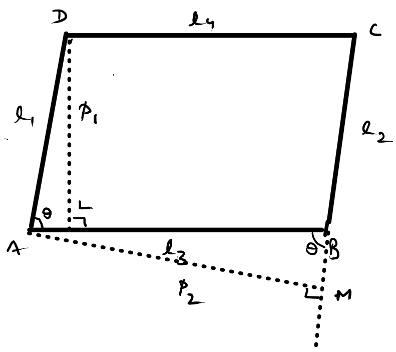
$\boxed{d(\ell_1 \parallel \ell_2) = \frac{|c_1 - c_2|}{\sqrt{a^2+b^2}}}$ $\xleftrightarrow{c_1 \to c_2}$ $d(\ell_1 \parallel \ell_2) \to 0$



Coincident lines $c_1 = c_2$

2 || lines (c_1, c_2, a, b) needed

3.4 Determinant formula for Ar ||gm



given

* $\ell_1 \equiv a_1x + b_1y + c_1 = 0$

$\ell_2 \equiv a_1x + b_1y + c_2 = 0$

$\ell_3 \equiv a_2x + b_2y + d_1 = 0$

$\ell_4 \equiv a_2x + b_2y + d_2 = 0$

$\longrightarrow$ Indep. vars $= \{a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2\}$

Ar ||gm = ?

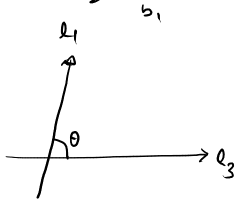
* $Ar(\text{||gm}) = AB \times p_1$; $P_1 = AD\sin\theta$; $P_2 = AB\sin\theta$

$\boxed{Ar = \frac{P_1 P_2}{\sin\theta}}$

* $p_1 = \perp$ dist b/w $\ell_3, \ell_4 = \frac{|d_1 - d_2|}{\sqrt{a_2^2 + b_2^2}}$

$P_2 = \perp$ " " $\ell_1, \ell_2 = \frac{|c_1 - c_2|}{\sqrt{a_1^2 + b_1^2}}$

$m_2 = -\frac{a_1}{b_1}$

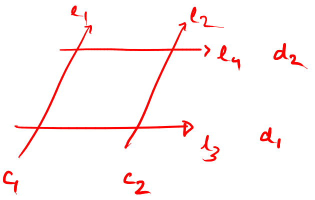


$m_1 = -\frac{a_2}{b_2}$

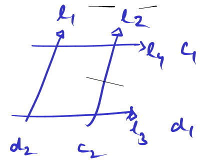
* $\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{-\frac{a_2}{b_2} + \frac{a_1}{b_1}}{1 + \frac{a_1 a_2}{b_1 b_2}}\right| = \left|\frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_1 b_2}\right|$

* $\sin\theta = \dfrac{a_1b_2 - a_2b_1}{\sqrt{(a_1^2+b_1^2)(a_2^2+b_2^2)}}$

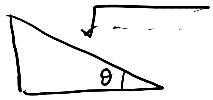
* $\boxed{A_r \;\; ||^{gm} = \left| \dfrac{\overbrace{(c_1-c_2)}^{\text{diff b/w } || \text{ lines}}(d_1-d_2)}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \right|}$



$(a_1b_2 - a_2b_1)$



Std textbook notation



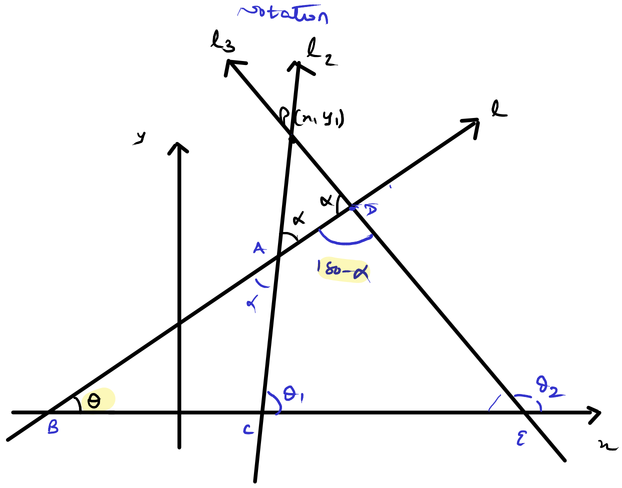
$(a_1a_2 + b_1b_2)$

3.5. Lines passing thru a pt. making α with given line of slope m

* $l \equiv y = mx + c$ (not given), $P(x_1,y_1)$, α (given)

$m = \tan\theta$
given

$m_2 = \tan\theta_1$
$m_3 = \tan\theta_2$



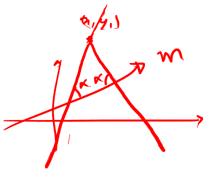
* Line passing thru $P(x_1,y_1)$ making α with l → $l_2 \equiv y - y_1 = \tan\theta_1(x-x_1)$ ✓
↳ $l_3 \equiv y - y_1 = \tan\theta_2 (x-x_1)$ ✓

$\theta_1 = \theta + \alpha$ (ΔABC) → $\tan\theta_1 = \dfrac{\tan\theta + \tan\alpha}{1-\tan\theta\tan\alpha} = \dfrac{m+\tan\alpha}{1-m\tan\alpha}$

$\theta_2 = 180 - \alpha + \theta$ (ΔDBE)

↳ $\tan\theta_2 = \tan(180+\theta-\alpha) = \tan(\theta-\alpha) = \dfrac{\tan\theta - \tan\alpha}{1+\tan\theta\tan\alpha} = \dfrac{m-\tan\alpha}{1+m\tan\alpha}$

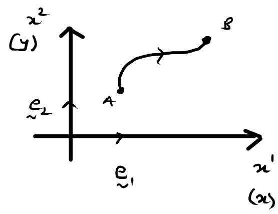
$$\boxed{y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x-x_1)}$$



(m, 1 pt, α) needed

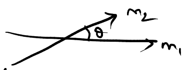
Lecture-20 (17/Aug.) 2

* $s^2 = \sum_{a=1}^{3}\sum_{b=1}^{3} g_{ab}x^ax^b \longrightarrow g_{ab} = \delta_{ab} = \begin{cases} 1 & a=b \\ 0 & a\neq b \end{cases}$ Rectangular Cartesian sys.



* $\boxed{\text{St line in Cartesian system} \to \text{Linear functions}}$ ⇒ $f(x) = c + ax = y$ (rect.)

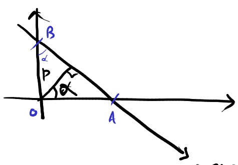
* $ax + by + c = 0$



* $y = mx + c$, $m = \tan\theta = \left|\dfrac{m_2-m_1}{1+m_1m_2}\right| = \dfrac{\Delta y}{\Delta x} = \dfrac{y_2-y_1}{x_2-x_1}$

traj: $x = x(y)$
lin.
quad
cubic

* $\dfrac{x}{a} + \dfrac{y}{b} = 1$;



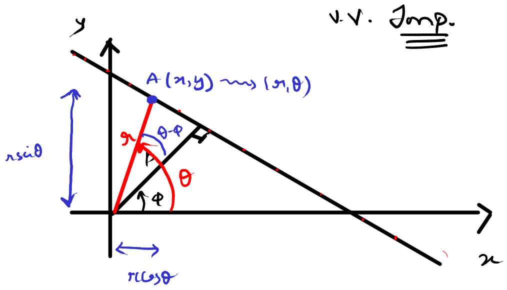
$\dfrac{x}{OA} + \dfrac{y}{OB} = 1$, $\cos\alpha = \dfrac{p}{OA}$, $\sin\alpha = \dfrac{p}{OB}$

$x\cos\alpha + y\sin\alpha = p$ ⇒ $y = f(x) = -\underbrace{\dfrac{\cos\alpha}{\sin\alpha}}_{\cot\alpha}x + \underbrace{\dfrac{p}{\sin\alpha}}_{p\,\operatorname{cosec}\alpha} = x\cot\alpha + p\operatorname{cosec}\alpha$

Normal Form.

Eq^n of Straight Line in Polar Coordinates

V.V. Imp.



EOL (r,θ) — unique Normal to this line

$\boxed{r\cos(\theta-\varphi) = p}$ ⟶ p : $\theta = \varphi$

$r\{\cos\theta\cos\varphi + \sin\theta\sin\varphi\} = \underbrace{(r\cos\theta)}_{x}\cos\varphi + \underbrace{r\sin\theta}_{y}\sin\varphi = p \Rightarrow x\cos\varphi + y\sin\varphi = p$ (x,y)

$\boxed{\text{St. Lines in Non-rectang. C.S} \longrightarrow \text{Not linear functions}}$

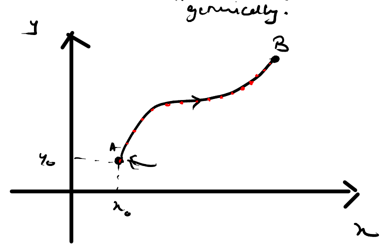
Implicit form.

* $\boxed{y=f(x)}$ f : linear $\Rightarrow$ St. line $\left\{\begin{array}{l} x : \text{indep var / i/p} \\ y : \text{dep. var / o/p} \end{array}\right.$; $y = mx + c$, $x \in \mathbb{R}$

already endowed with a set

Prescription 1

"generically"



$S(A) = \text{given} = 3$ (say)

* $\exists$ Parameter 's' : $\boxed{\begin{array}{l} x = x(s) \\ y = y(s) \end{array}}$ s : indep. / x : dep.

Parametric Form

s : indep. / y : dep. Prescription 2

s linear $\Rightarrow$ st. linear

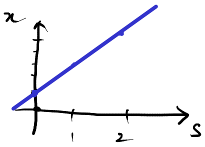
linear functions

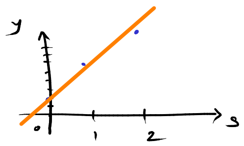
Ex: $\left.\begin{array}{l} x = 2s + 1 = x(s) \\ y = 3s + 2 = y(s) \end{array}\right\} \xrightarrow[s]{\text{Elim.}} \frac{x-1}{2} = \frac{y-2}{3} \Rightarrow y - 2 = \underset{\sim}{\frac{3}{2}}(x-1) \sim y - y_0 = m(x - x_0) \Rightarrow y = y(x)$

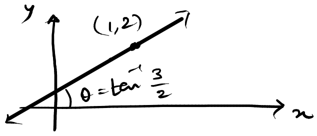
s	0	1	2
x	1	3	5

s	0	1	2
y	2	5	8

x	.	.	.
y	.	.	.







linear.

Ex: $\left.\begin{array}{l} x = x_0 + v_0 t = x(t) \\ y = y_0 + v_1 t = y(t) \end{array}\right\} \xrightarrow[t]{\text{Elim}} \frac{x - x_0}{v_0} = \frac{y - y_0}{v_1} \Rightarrow y - y_0 = \frac{v_1}{v_0}(x - x_0)$

Eqn
Condn
Equan
accln
fun
rxn
Reln
dyn
tangn

* $\left.\begin{array}{l} x = x(t) \Rightarrow \frac{dx}{dt} = v_0 = \frac{\Delta x}{\Delta t} \\ y = y(t) \Rightarrow \frac{dy}{dt} = v_1 = \frac{\Delta y}{\Delta t} \end{array}\right\}$ $\boxed{\underbrace{\frac{d^2x}{dt^2}}_{a_x} = 0 = \overbrace{\frac{d^2y}{dt^2}}^{a_y}}$ ✓

operation / 1st hand defn of St. Line

$v = \frac{dx}{dt}$ $x = x(t)$

$\frac{dv}{dt} = 0$

$\Downarrow$

St. line

$x = x(t)$

$y = y(t)$

Ex: $x\cos\alpha + y\sin\alpha = p \xrightarrow{(x,y)\to(r,\theta)}$ $\underbrace{\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}}_{\text{Transfn}}$ $r\cos(\theta - \alpha) = p$

linear in (x,y) not linear function

$\updownarrow$ $\downarrow$

Eqn of St. Line Eqn of St. Line

$\left.\begin{array}{l} r = r(s) \\ \theta = \theta(s) \end{array}\right\}$ $\frac{d^2r}{ds^2} \neq 0$????

St. Line

$f(x,y) = ax + by + c = 0$ OK!

$\downarrow$

$\frac{d^2x}{ds^2} = \frac{d^2y}{ds^2} = 0$ Good! till Rect. Cartesian syst.

$\downarrow$

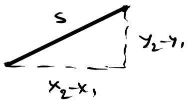
'Geodesic Eqn' $\boxed{\frac{d^2x}{ds^2} + \text{Correction term} = 0}$

Eqn of St. Line

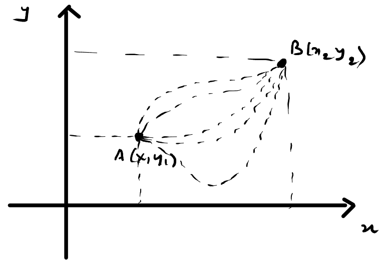
later (Calculus + geometry)

Lecture-21 (18/Aug.) 2

4. Little chat on Differentials / Arc Length / Polar Coor.

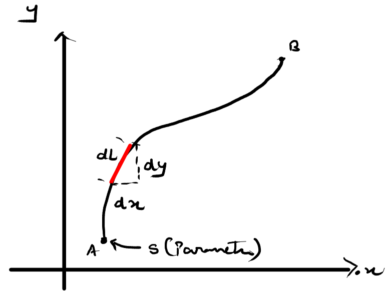


* $S^2 = \underbrace{(x_2-x_1)^2}_{\Delta x} + (y_2-y_1)^2 = (\Delta x)^2 + (\Delta y)^2$ $\Big\{$ To calc. directly the shortest route (length) b/w A & B



* $S^2 = \sum_{a=1}^{2}\sum_{b=1} g_{ab}\, x^a x^b \xrightarrow[g_{ab}=\delta_{ab}]{} s^2 = \sum_{a,b} \delta_{ab}\, x^a x^b = (x^1)^2 + (x^2)^2$ 'Distance formula'

$\exists\ \infty$ curves joining A & B



* AB is an Arc. ⟶ It has a length ($L = |AB|$)

Leibniz's prescription ≡ Arc length is a parameter

$\exists$ ds = infinitely small length element / Infinitesimal Arc length.

* $\underbrace{ds^2}_{\text{Infinitesimal quantity}} = \underbrace{dx^2 + dy^2}_{\text{Diff. Eq}^n}$

$\boxed{\text{Extremization process}}$ → The How? is a Black Box for you right now (Calculus of variation)

↳ length of the complete curve $\shortparallel$ Solution to diff^n Eq^n

↓

Minimized distance b/w 2 pts "Straight Line"

(History of) Variational Principles

Euclid / Hero
Fermat 1662
Leibniz
Maupertuis 1744
Lagrange / Jacobi 1842
Hilbert 1915
Dirac 1933
Feynman 1948

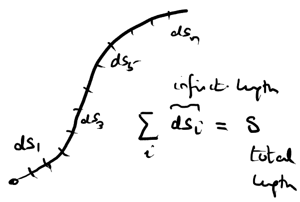
$y = y(x)$: tray.

$\exists$ s at every pt : $x = x(s)$
$y = y(s)$

Parameter form.

4.1 Infinitesimal Arc length in (x,y)

* (x,y) : Eq^n of a straight line $\boxed{x\cos\alpha + y\sin\alpha = p}$ —— ①

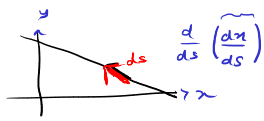


* $\boxed{ds^2 = dx^2 + dy^2}$ —② Infnt. Arc length prescription ≡ choose Arc. length to be the Parameter itself to give the length of the complete curve

Extremiz^n Process ↓ if I knew the Black Box Process

St. Line Eq^n "But I've already given the Eq^n of St. Line / No need to do EXTREMIZATION / VARIATION"

* $\underset{\text{Linear }(x,y)\text{ rel.}}{\overset{\text{Eq}^n\ (\alpha,p)\text{ const}}{x\cos\alpha + y\sin\alpha = p}} \longrightarrow \cos\alpha + \frac{dy}{dx}\sin\alpha = 0 \Rightarrow \boxed{\overbrace{\frac{dy}{dx}}^{\text{Slope}} = -\cot\alpha}$



$\frac{d}{ds}\left(\widetilde{\frac{dx}{ds}}\right)$

Notation Alert

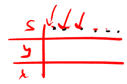
$dx^2 = (dx)^2 = (dx)(dx)$

$ds^2 = dx^2 + dy^2 \underset{\text{wrt } x}{\Rightarrow} \left(\frac{ds}{dx}\right)^2 = 1 + \underbrace{\left(\frac{dy}{dx}\right)^2} = 1 + \cot^2\alpha = \csc^2\alpha \Rightarrow \frac{ds}{dx} = \csc\alpha \Rightarrow \frac{dx}{ds} = \sin\alpha$.

$\Downarrow$

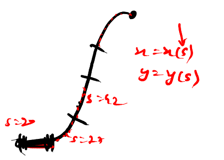
$\boxed{\frac{d}{ds}\left(\frac{dx}{ds}\right) = \frac{d^2x}{ds^2} = 0}$

$ds^2 = dx^2 + dy^2 \underset{}{\overset{\text{wrt } y}{\Rightarrow}} \left(\frac{ds}{dy}\right)^2 = \left(\frac{dx}{dy}\right)^2 + 1 = 1 + \tan^2\alpha = \sec^2\alpha \Rightarrow \frac{ds}{dy} = \sec\alpha \Rightarrow \frac{dy}{ds} = \cos\alpha \Rightarrow \boxed{\frac{d^2y}{ds^2} = 0}$



Conclusion :

* $x = x(s)$
$y = y(s)$
s = arc length ; $ds^2 = dx^2 + dy^2 \longrightarrow \underset{\frac{d}{ds}\left(\frac{dx}{ds}\right)}{\frac{d^2x}{ds^2}} = 0$, $\frac{d^2y}{ds^2} = 0$



HW : Think

* $r\cos(\theta-\alpha) = p$ SL-Line in (r,θ)

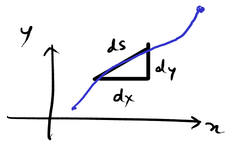
$\left.\begin{array}{l} r = r(s) \\ \theta = \theta(s) \end{array}\right\}$ arc lgth

$\frac{d^2r}{ds^2} = ?$ $\frac{d^2\theta}{ds^2} = ?$

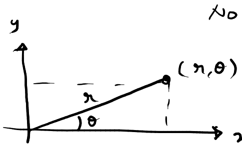
$\Delta s = s_2 - s_1$ → $\int ds = s$

4.2. Infinitesimal Arc. length in (r,θ)

No mention of Metric

* 

$\xrightarrow[\text{transfn is known}]{r,\theta}$



$x = r\cos\theta$ 'trig. (crucial) bit'
$y = r\sin\theta$

'Pythagorean thm.' (crucial bit)

$ds^2 = dx^2 + dy^2$ ⟶

Naive response
$ds^2 = dr^2 + (d\theta)^2$

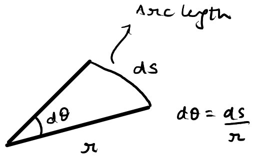
$dx = (dr)\cos\theta + r(-\sin\theta)d\theta$

$dy = dr\sin\theta + r\cos\theta\, d\theta$

$ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2$

$\boxed{ds^2 = dr^2 + r^2 d\theta^2}$ $\xrightarrow[dr=0]{r=\text{const}}$ $ds = r\,d\theta$

Infinitesimal Line element in (r,θ) circular polar coord 2D



$d\theta = \frac{ds}{r}$

Lecture-22 (19/Aug.) 2

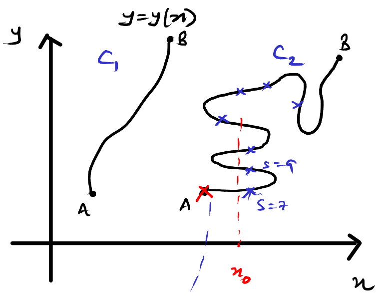
4.3. Parametrization of curves (Arc. length as a parameter)

* (x,y) ; traj./Path/Locus $y = y(x)$: C_1

* C_2 : Runs into trouble if used $y = y(x)$ ⟶ ∃ multiple y values for 1 x

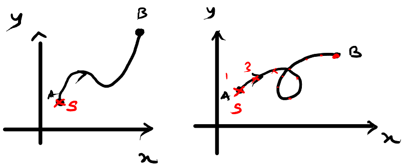
Sol^n : define a parameter (s) ; s = arc. length along curve (usually : Leibniz)

length$(AB) = \sum_{i=1}^{n} s_i = s_1 + s_2 + s_3 + s_4 \cdots$



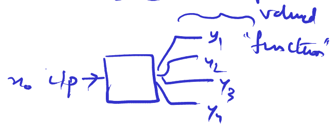
* $\boxed{\begin{array}{l} y = y(s) \\ x = x(s) \end{array}}$ Best way to talk about curves! CTM!

Parametric form of curve



$y = y(s)$
$x = x(s)$

$s = 2$ Multiple valued "function"



$y = f(x) \Rightarrow x = f^{-1}(y)$
not invertible
∄ Inverse

Parametric

* $y = y(s)$, $x = x(s)$ $\xrightarrow{\text{Eliminate } s}$ Implicit $y = y(x)$, $x(s)$

$\frac{dy}{ds}$ $\left(\frac{dx}{ds}\right)$

$\frac{dy}{dx} = \frac{dy}{ds}\cdot\frac{ds}{dx}$

$\frac{dy}{dx} = \frac{dy}{ds}\frac{ds}{dx}$
"chain rule"

$y = \sin(x^2)$ 'Intermediary' $s = x^2$

$y = \sin s$ $\frac{ds}{dx} = 2x$

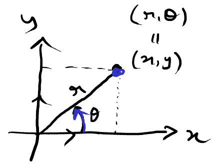
$\frac{dy}{ds} = \cos s$

$\frac{dy}{dx} = \frac{dy}{ds}\cdot\frac{ds}{dx} = (\cos x^2)2x$

s	0	2	3	. . .
x	1	7	9	
y	2	-1	7	

$\frac{\Delta x}{\Delta s} = \frac{x_2 - x_1}{s_2 - s_1} = \frac{7-1}{2-0} = \frac{6}{2} = 3$

* $x = x(s)$, $y = y(s)$ $\xrightarrow{\text{Leibniz}}$ ds = Infinitesimal Arc length "differential"
Accurate than $\Delta s = s_2 - s_1$

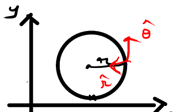


(x,y) Coordinates ⟹ $ds^2 = dx^2 + dy^2$

↓ ↓ $dr, d\theta$

(r,θ) coordinates ⟹ $ds^2 = dr^2 + r^2 d\theta^2$ $\xrightarrow{r=\text{const}}$ $ds = r\,d\theta$

Small Angle displacement



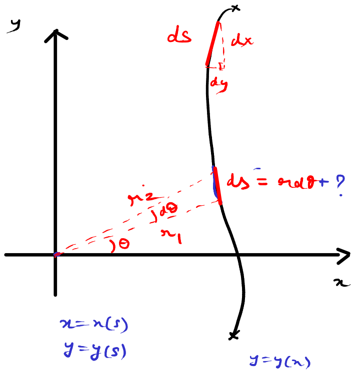
$x = r\cos\theta$
$y = r\sin\theta$

$dx = r(-\sin\theta)d\theta + \cos\theta\, dr$
$dy = r(\cos\theta)d\theta + \sin\theta\, dr$

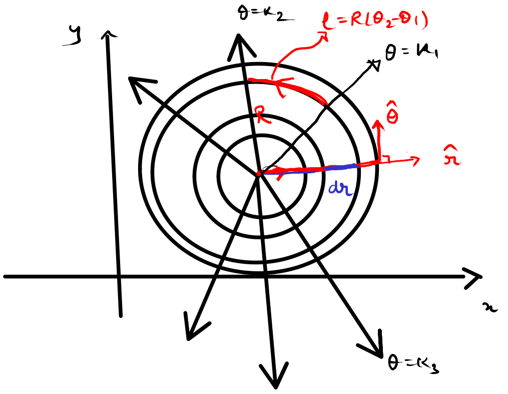
$\boxed{s = r\theta}$ ⟹ $\underbrace{\frac{d\theta}{dt}}$ $= r = \text{const}$

if s = time

$\frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} = 0$ $\omega = v_\theta = \dot\theta$ (Angular velocity)



* $ds^2 = dr^2 + r^2 d\theta^2$
  - $r = \text{const} \Rightarrow dr = 0 \Rightarrow ds = r\,d\theta$ — Small Angle displacement $(r_1 \sim r_2)$
  - $\theta = \text{const}$ Lines $\Rightarrow d\theta = 0 \Rightarrow ds = dr$



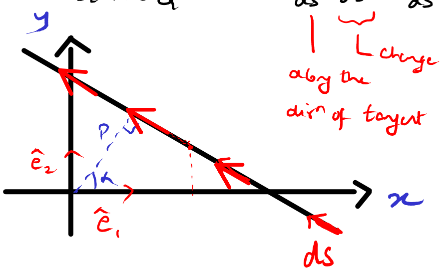
* $ds^2 = dx^2 + dy^2$ and $x\cos\alpha + y\sin\alpha = p$ St. Line eqn $\Rightarrow \frac{d}{ds}\left(\frac{dy}{ds}\right) = \frac{d^2y}{ds^2} = 0$, $\frac{d^2x}{ds^2} = 0$

tangent vect: if $s = \text{time} = t$, "velocity", "Accel^n"

$\frac{d}{ds}$: along the dir^n of tangent; $\frac{dy}{ds}$: change in tangent

↓ Black Magic (Variation principle)

Straight Line Eq^n



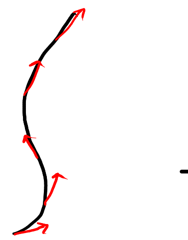
$\{\hat{e}_i\}$ basis unit vector

↓ "Straight Line"/Rectangular Coord. sys. (x, y)

$\boxed{\frac{d}{ds}\hat{e}_i = 0}$ { not true in general!

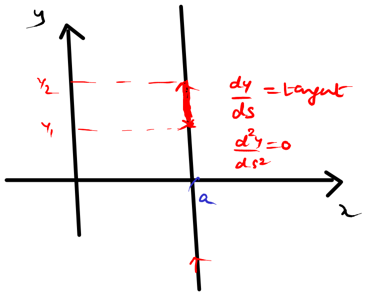
$\frac{d}{dx}\hat{x} = 0 = \frac{d}{dy}\hat{y}$

Property of basis vector



along the curve change in tangent $\neq 0$

$\boxed{\frac{d}{ds}\hat{e}_i \neq 0}$



$x = a$

$\boxed{\text{along the curve the change in tan} = 0}$

"St. Line"

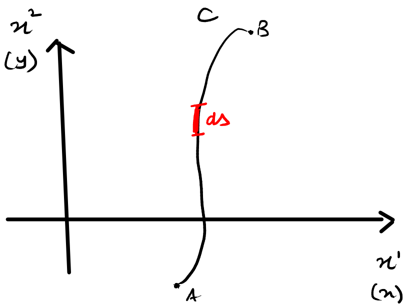
Lecture-23 (20/Aug.) 1.5

4.4. Infinitesimal Arc length Quadratic Prescription

$\left| x^a = (x^1\ x^2\ x^3) \begin{cases} \to (x\ y\ z) \\ \to (r\ \theta\ z) \\ \to (r\ \theta\ \varphi) \end{cases} \right.$ 'distance prescription'

* $s^2 = \sum_{a,b=1}^{2} g_{ab} x^a x^b \xrightarrow{g_{ab} = \delta_{ab}} s^2 = (x^1)^2 + (x^2)^2 = x^2 + y^2$

* Curve C : $\boxed{x^a = x^a(s)} \overset{2D}{\rightsquigarrow} \begin{cases} x = x(s) \\ y = y(s) \end{cases}$ Parametric form of Traj. (best way)



* s = Arc length of curve C (increasing) = Number/function/finite/finite diff./Countable/perceptible

↓ Leibniz

$\left| x^a = x^a(x^b) \rightsquigarrow y = y(x) \right.$ not unique way!

ds = Infinitesimal Arc length = Infinitely small diff. = differential / 'variation' of s / Uncountable/Imperceptible

* $\boxed{ds^2 = \sum_a \sum_b g_{ab}\, dx^a dx^b}$ Infinitesimal Arc length (quadratic) prescription { $\in$ differential geometry $\to$ More weird spaces $\left| \int ds = s \right.$

Curve C : $dx^a = dx^a(s) \overset{(x,y)}{\rightsquigarrow} \begin{cases} dx = dx(ds) \\ dy = dy(ds) \end{cases}$ Parametric form of "differential" of Traj.

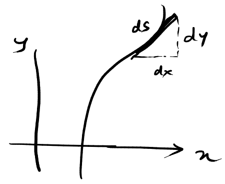
Infinitesimal Coordinates $\overset{(r,\theta)}{\rightsquigarrow}$ $dr = dr(ds)$, $d\theta = d\theta(ds)$

Rectangular Coord

* $x^a = (\underset{x^1}{x}\ \underset{x^2}{y}) \longrightarrow ds^2 = (dx)^2 + (dy)^2 = \sum_{a=1}^{2} g_{ab}\, dx^a dx^b = \underbrace{g_{11}}_{1}(dx)^2 + \overbrace{\underbrace{g_{12}}_{0} dx\,dy + \underset{0}{g_{21}}\, dy\,dx}^{0} + \underbrace{g_{22}}_{1}(dy)^2$

↓

$dx^a = (dx\ \ dy)$



$g_{ab} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \delta_{ab} = \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases}$ ↳ Comparing Coeff. of $dx\,dy$

Matrix form of metric

Matrix form

$ds^2 = \underbrace{(dx\ \ dy)_{1\times 2}}_{A^T} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times 2}}_{G} \underbrace{\begin{pmatrix} dx \\ dy \end{pmatrix}_{2\times 1}}_{A} = (dx)^2 + (dy)^2 = A^T G A$

$\left| \right.$ $\exists$ a specific transf. rule of vector: $A^i = (A^1\ A^2\ A^3)_{1\times 3}$ — List, $\underset{\sim}{A}$, $\mathbf{A}$ Matrix

* $ds^2 = \sum g_{ab} dx^a dx^b \xrightarrow[\text{Rep.}]{\text{Matrix}} A^T \cdot G \cdot A$

$A \equiv \left(dx^1 \;\; dx^2 \;\; dx^3 \cdots\right)_{1\times N}$

$A^T \equiv \begin{pmatrix} dx^1 \\ dx^2 \\ dx^3 \\ \vdots \end{pmatrix}_{N\times 1}$ Transpose of A

$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I} \underbrace{\begin{pmatrix} dx \\ dy \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} dx \\ dy \end{pmatrix}}_{X}$

Identity transf^n

Circular polar

* $x^a = (r \;\; \theta) \longrightarrow ds^2 = dr^2 + r^2 d\theta^2 = \sum_{a=1}^{2} g_{ab} dx^a dx^b = \underbrace{g_{11}}_{1}(dr)^2 + \underbrace{g_{12}}_{0} \overset{0}{\cancel{dr d\theta}} + \underbrace{g_{21}}_{0} \overset{0}{\cancel{d\theta dr}} + \underbrace{g_{22}}_{r^2}(d\theta)^2$

$\downarrow$

$dx^a = (dr \;\; d\theta)$

$$\boxed{g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \quad \text{Matrix rep. of Metric in 2D circular polar coord.}}$$

Metric is not constant

Lecture-24 (24/Aug)

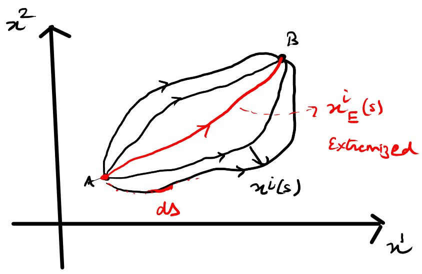
* $s \longrightarrow x^i(s) \longrightarrow dx^i(s)$

Arc length = Parameter; parametric curve; differential line element

$\downarrow$

$ds \longrightarrow$ infinitesimal Arc length

$$\boxed{ds^2 \equiv \sum_{a,b} g_{ab} dx^a dx^b \xrightarrow[\text{Pick out / choice / filter}]{\text{Extremization}} \text{"straight" Line} = \text{Extremum path}}$$

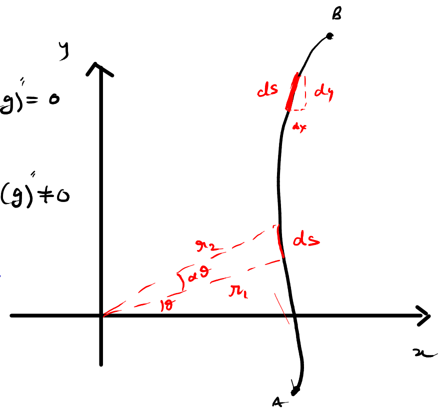


* $x^i = (x \;\; y) \longrightarrow ds^2 = \overset{\text{Euclidean}}{dx^2 + dy^2} = g_{ab} dx^a dx^b \longrightarrow g_{ab} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{Const}}$ / Metric $\longrightarrow$ "derivatives (g)" $= 0$ $\quad \frac{d}{dx}(1) = 0$

$\downarrow \qquad\qquad \downarrow$

$x^i = (r \;\; \theta) \longrightarrow ds^2 = dr^2 + r^2 d\theta^2 = g_{ab} dx^a dx^b \longrightarrow g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \longrightarrow$ "derivatives (g)" $\neq 0$

$\begin{Bmatrix} x = r\cos\theta \\ y = r\sin\theta \end{Bmatrix}$ not Const. $\quad \frac{d}{dr}(r^2) = 2r$



* $x^i = (t \;\; x) \longrightarrow ds^2 = -dt^2 + dx^2$ Not a "Pythagoras"

Car

$g_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ Minkowski

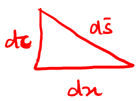
$\downarrow \; \tau = -it \qquad t = \frac{\tau}{-i} = i\tau \Rightarrow dt = i\,d\tau$

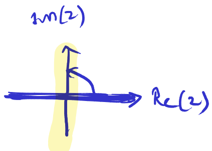
$(dt)^2 = -(d\tau)^2$

if $t \in \mathbb{R} \Rightarrow \tau \in \mathbb{C}$

$d\bar{s}^2 = (d\tau)^2 + dx^2$

"Pythagoras" (Euclidean)





$y = f(x, t, \omega)$

$\left(\frac{\partial f}{\partial x}\right)_{t,\omega} \quad \left(\frac{\partial f}{\partial t}\right)$

∂: dabba "del"

Note : Minkowski ds is not a "Pythagoras" but can be converted to "Pythagoras" by taking the problem to a Complex domain } "Gian Wick" Wick's Rotation

* $x^i = (t \;\; x) \longrightarrow ds^2 = \left(1 - \frac{a}{x}\right)dt^2 - \frac{dx^2}{\left(1 - \frac{a}{x}\right)}$ not even close to "pythagoras"

$g_{ab} = \begin{pmatrix} 1-\frac{a}{x} & 0 \\ 0 & \frac{-1}{(1-\frac{a}{x})} \end{pmatrix}$

$\downarrow \; x \to \infty$ (Asymptotically)

$ds^2 = dt^2 - dx^2 \xrightarrow{\text{wick rot}^n} d\bar{s}^2 = d\bar{x}^2 + dy^2$

Two Suggestion

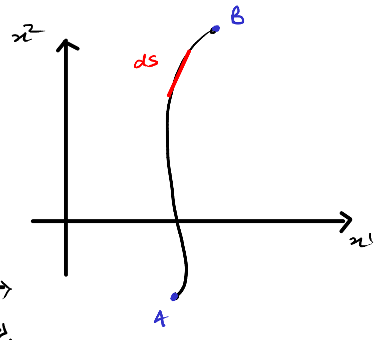
$p = \sqrt{\left(1 - \frac{a}{x}\right)}\, t$

$q = \sqrt{\left(1 - \frac{a}{x}\right)^{-1}}\, ix$

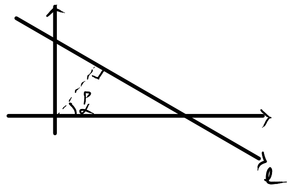
{ In general metric becomes asymptotically Euclidean/Minkowskian not globally $(\forall x)$

Lecture-25 (25/Aug.) 2

$*\quad ds^2 = \sum_{ab} g_{ab}\, dx^a dx^b$ ~ "distance/separation b/w 2 infinitesimal pts"



$*\quad g_{ab} = \delta_{ab} \Rightarrow ds^2 = dx^2 + dy^2 = dx^2\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}$



$ds = dx\sqrt{1+\left(\frac{dy}{dx}\right)^2} \Rightarrow \boxed{s = \int_A^B ds = \int_A^B dx\sqrt{1+\underbrace{\left(\frac{dy}{dx}\right)^2}_{\text{slope/tangent}}}}$

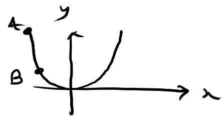
Arc length in terms of tangent/slope of a curve $y = f(x)$

$x\cos\alpha + y\sin\alpha = p$

$dx\cos\alpha + dy\sin\alpha = 0$

$\frac{dy}{dx} = -\cot\alpha$

Ex:



$y = x^2$

$\frac{dy}{dx} = 2x$

$\underbrace{\int_a^b \sqrt{1+4x^2}\, dx}$ $= \int_A^B ds$

T-BD (Calculus)

$\int\sqrt{x^2+a^2} = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log(x+\sqrt{x^2+a^2})$

$ds^2 = dx^2 + dy^2 \Rightarrow \left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \cot^2\alpha = \operatorname{cosec}^2\alpha$

$\frac{ds}{dx} = \operatorname{cosec}\alpha \Rightarrow \boxed{\frac{dx}{ds} = \sin\alpha} \Rightarrow \frac{d^2x}{ds^2} = 0$ $\quad \frac{d}{ds}\left(\frac{dx^1}{ds}\right) = 0$

dx^1/ds — tan. — const

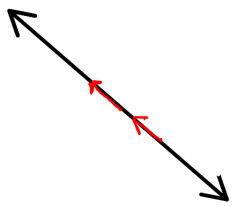
$\left(\frac{ds}{dy}\right)^2 = 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \tan^2\alpha = \sec^2\alpha \Rightarrow \boxed{\frac{dy}{ds} = \cos\alpha} \Rightarrow \frac{d^2y}{ds^2} = 0$ $\quad \frac{d}{ds}\left(\frac{dx^2}{ds}\right) = 0$

dx^2/ds — tan. — Const

$*\quad C: x = x(s),\ y = y(s)\ :\ ds^2 = dx^2 + dy^2 \Rightarrow \frac{d^2x}{ds^2} = 0 = \underbrace{\frac{d^2y}{ds^2}}$ $\equiv$ 'St. Line'

$\frac{d}{ds}\left(\frac{dy}{ds}\right)$ — change in tan. vector

along the tangent (s)

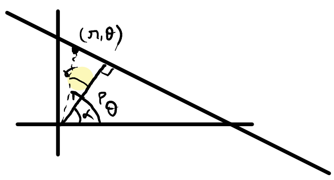


$l \equiv x\cos\alpha + y\sin\alpha = p$

$*\quad g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \Rightarrow ds^2 = dr^2 + r^2 d\theta^2 \stackrel{\text{wrt } r}{\Rightarrow} \left(\frac{ds}{dr}\right)^2 = 1 + r^2\underbrace{\left(\frac{d\theta}{dr}\right)^2}$

$x^i = (r\ \theta)$

$= 1 + \cot^2(\theta-\alpha) = \operatorname{cosec}^2(\theta-\alpha)$



$\frac{dr}{ds},\ \frac{d\theta}{ds} \stackrel{?}{\neq}$ const

$\frac{dx^1}{ds} = \underbrace{\boxed{\frac{dr}{ds} = \sin(\theta-\alpha)}}_{\text{Not const}}$ $\quad \frac{d}{ds}\left(\frac{dx^1}{ds}\right) + \underbrace{(\cdots)} = 0$

due to such Coord. sys. You chose!

$r\cos(\theta-\alpha) = p$

$r(-\sin(\theta-\alpha))\,d\theta + dr\cos(\theta-\alpha) = 0$

$r\,d\theta = \cot(\theta-\alpha)\,dr \Rightarrow \frac{d\theta}{dr} = \frac{\cot(\theta-\alpha)}{r}$

$\frac{dr}{d\theta} = r\tan(\theta-\alpha)$

$\int$ wrt θ

$\left(\frac{ds}{d\theta}\right)^2 = \underbrace{\left(\frac{dr}{d\theta}\right)^2} + r^2 = r^2\sec^2(\theta-\alpha) \Rightarrow \boxed{\underbrace{\frac{d\theta}{ds}}_{dx^2/ds} = \underbrace{\frac{\cos(\theta-\alpha)}{r}}_{\text{Not const}}}$ $\quad \frac{d}{ds}\left(\frac{dx^2}{ds}\right) + \ldots = 0$

$\boxed{\text{change in tangent is not zero even though this is a STRAIGHT LINE}}$ Problem

$\underbrace{}$ change(tan) $= 0$

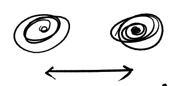
Looks like Newton's Eq? if $s =$ Time

(Generalized Coord. sys.) for arbitrarily chosen Coord. systems

$\boxed{\frac{d}{ds}\left(\frac{dx^i}{ds}\right) + \text{Correction term} = 0}$

"Straight" line

Geodesics



$1\,\text{Mpc} \sim 3\times10^{24}\,\text{cm}$

$R_{univ} \sim 6000$ Mpc

* $\frac{dr}{ds} = \sin(\theta-\alpha) \Rightarrow \frac{d}{ds}\left(\frac{dr}{ds}\right) = \frac{d}{d\theta}\left(\frac{dr}{ds}\right)\frac{d\theta}{ds} = \cos(\theta-\alpha)\cdot\frac{\cos(\theta-\alpha)}{r} = \frac{\cos^2(\theta-\alpha)}{r} \Rightarrow \underbrace{\frac{d^2r}{ds^2}}_{\text{Change (tangent)}} = \frac{\cos^2(\theta-\alpha)}{\underbrace{r}_{\text{Correction}}} \neq 0$

$\frac{d\theta}{ds} = \frac{\cos(\theta-\alpha)}{r}$

Lecture-26 (21/Aug) 2

4.5 Metric Examples

* $ds^2 = g_{ab}\,dx^a dx^b \xrightarrow[g_{ab} = \begin{pmatrix} (1-\frac{a}{r}) & 0 \\ 0 & \frac{-1}{(1-a/r)}\end{pmatrix}]{x^a = (t\ \ r)} ds^2 = \left(1-\frac{a}{r}\right)dt^2 - \frac{dr^2}{(1-a/r)} \xrightarrow{?}$ simplify to $ds^2 = dT^2 - dX^2$

* $x^a = (x\ \ y)$, $g_{ab} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \Rightarrow ds^2 = dx^2 + dy^2$

* $g_{ab} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \Rightarrow ds^2 = dt^2 - dx^2$, $x^a = (t\ \ x)$

* $g_{ab} = \begin{pmatrix}1 & 0\\ 0 & r^2\end{pmatrix} \Rightarrow ds^2 = dr^2 + r^2 d\theta^2$, $x^a = (r\ \ \theta)$

* $ds^2 = \left(1-\frac{a}{r}\right)dt^2 - \frac{dr^2}{\left(1-\frac{a}{r}\right)}$

$= y\,dt^2 - \frac{1}{y}\left(\frac{-a}{r^2}dy\right)^2$ trouble not simple / More Complicated ! ✓

Coord. transformations

$1-\frac{a}{r} = y$ trouble!

$-\frac{a}{r^2}dr = dy$.

$dr = -\frac{a}{r^2}dy$

* $ds^2 = dr^2 + r^2 d\theta^2 \longrightarrow ds^2 = dx^2 + dy^2$

{nothing crazy at $r=0$}

$g_{ab} = \begin{pmatrix}1 & 0\\ 0 & r^2\end{pmatrix}$

$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\}$

(x,y) (r,θ)

$dx = dr\cos\theta + r(-\sin\theta)d\theta$

$dy = dr\sin\theta + r(\cos\theta)d\theta$

$\Downarrow$

$(r\ \theta)$ (x,y)

$\left.\begin{aligned} r^2 d\theta &= x\,dy - y\,dx \\ r\,dr &= x\,dx + y\,dy \end{aligned}\right\}$

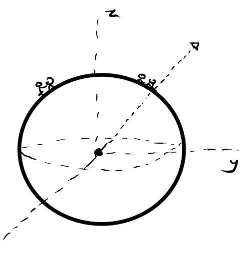
$(x,y) \xrightarrow{\text{transf}} (r,\theta)$

Inverse

* $g_{ab} = \begin{pmatrix}1 & 0\\ 0 & \sin^2\theta\end{pmatrix}$ $\quad ds^2 = g_{ab}\,dx^a dx^b = d\theta^2 + \sin^2\theta\,d\varphi^2$ ✓ (θ,φ) $\quad R = \text{const}$; $\boxed{\sin\theta = \mu}$

$x^i = (\theta\ \ \varphi)$ $\quad = \frac{d\mu^2}{1-\mu^2} + \mu^2 d\varphi^2$

$\cos\theta\,d\theta = d\mu \Rightarrow d\theta = \frac{d\mu}{\cos\theta} = \frac{d\mu}{\sqrt{1-\mu^2}}$



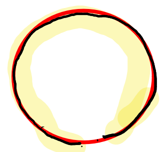
$ds^2 = \frac{d\mu^2}{1-\mu^2} + \mu^2 d\varphi^2$ $\quad \boxed{\mu = \pm 1 \quad ds^2 \to \infty}$

$(\mu\ \ \varphi)$

topological fact

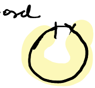
Can't cover full sphere with 1 Coordinate Chart

1 chart



$\exists$ atleast 2 Coordinates

2nd coord



Analysis of Special Metric (Schwarzschild-type) : Conversion to a simpler form

* $ds^2 = \underbrace{\left(1-\frac{a}{r}\right)}_{f(r)}dt^2 - \frac{dr^2}{\left(1-\frac{a}{r}\right)} \xrightarrow[\text{Is it doable?}]{??} ds^2 = dT^2 - dX^2$ $\quad (T, X)$

$f(r) = 1 - \frac{a}{r}$ $\quad$ $\left.\begin{aligned} f(r=a) &= 0 \Rightarrow ds^2 \to \infty \\ f(r=0) &\to \infty \Rightarrow ds^2 \to \infty \end{aligned}\right\}$ Singularity $r = 0, a$

$f'(r) = \frac{a}{r^2}$ $\quad f'(r=a) = \frac{1}{a} \equiv 2K$

Taylor exp. about $x=a$

$$\frac{df}{dx}(x)=\frac{f(x+h)-f(x)}{h}$$

$$f(x+h)=f(x)+h\frac{df}{dx}$$

* $f(x)=\cancel{f(x=a)}_{0}+(x-a)f'(x=a)=\frac{x-a}{a}=\frac{l}{a}=2\kappa l$

$x-a\equiv l$

$dx=dl$

$$ds^2\simeq 2\kappa l\,dt^2-\frac{dl^2}{2\kappa l} \qquad (t\ l)$$

$$\simeq \kappa^2\bar{x}^2dt^2-(d\bar{x})^2 \qquad (t\ \bar{x})$$

✓ $2\kappa l\equiv \kappa^2\bar{x}^2$

$l=\frac{1}{2}\kappa\bar{x}^2$

$dl=\underbrace{\kappa\bar{x}}\,d\bar{x}$

$=\sqrt{2\kappa l}\,d\bar{x}$

$\frac{dl}{\sqrt{2\kappa l}}=d\bar{x}$

not gonna work: circular Trig. fun^n

$X\equiv \bar{x}\cos\kappa t \Rightarrow dX=d\bar{x}\cos\kappa t-\kappa\bar{x}\sin\kappa t\,dt$

$Y\equiv \bar{x}\sin\kappa t\Rightarrow dY=d\bar{x}\sin\kappa t+\kappa\bar{x}\cos\kappa t\,dt$

$$\boxed{dX^2+dY^2=d\bar{x}^2+\kappa^2\bar{x}^2dt^2}$$

HW $X\equiv ?$

$Y\equiv ?$

Lecture-27 (27/Aug) 2

* $ds^2=\underbrace{\left(1-\frac{a}{x}\right)}dt^2-\frac{dx^2}{\left(1-\frac{a}{x}\right)} \qquad (t,x)$

$\simeq \underbrace{2\kappa l}\,dt^2-\frac{dl^2}{2\kappa l} \qquad (t\ l)$

$\kappa^2\bar{x}^2dt^2-(d\bar{x})^2 \qquad (t\ \bar{x})$

$x-a\equiv l$, $\frac{1}{a}\equiv 2\kappa$, $f(x)\equiv 1-\frac{a}{x}=\cancel{f(a)}^{0}+\underset{\leftarrow l\rightarrow}{(x-a)}f'(a)=2\kappa l$ $\quad f'(x)=\frac{a}{x^2}\Big|_a=\frac{1}{a}$

$2\kappa l\equiv\kappa^2\bar{x}^2\Rightarrow l=\frac{1}{2}\kappa\bar{x}^2\Rightarrow \frac{dl}{d\bar{x}}=\kappa\bar{x}\Rightarrow dl=\kappa\bar{x}\,d\bar{x}$

$\sqrt{\frac{2l}{\kappa}}=\bar{x}$

Doesn't work

$X\equiv\bar{x}\cos\kappa t$

$Y\equiv\bar{x}\sin\kappa t$

$dX^2+dY^2=d\bar{x}^2+\kappa^2\bar{x}^2dt^2$

$dX^2-dY^2\sim \underbrace{\cos^2-\sin^2}_{\cos 2\kappa t}$

quick detour to hyperbolic trig. functions (= trig. + C.N.)

* $\underset{\mathrm{ch}\theta}{\cosh\theta}\equiv\frac{e^{\theta}+e^{-\theta}}{2}$, $\sinh\theta\equiv\frac{e^{\theta}-e^{-\theta}}{2}\equiv \mathrm{sh}\theta$; $e\equiv$ Napier's const — Napier Log$_e$ (1600) — J. Bernoulli $\lim_{n\to\infty}(\ )$ C.I — Leibniz/Huygens — Euler $e^{i\theta}$ (1700)

John

$(\sin\theta)'=\cos\theta$

$(\cos\theta)'=-\sin\theta$

$$\frac{d}{d\theta}\cosh\theta=\frac{1}{2}(e^{\theta}-e^{-\theta})=\sinh\theta$$

$$\frac{d}{d\theta}\sinh\theta=\frac{1}{2}(e^{\theta}+e^{-\theta})=\cosh\theta$$

$$\cosh^2\theta-\sinh^2\theta=\frac{1}{4}\left\{e^{2\theta}+e^{-2\theta}+2-\left(e^{2\theta}+e^{-2\theta}-2\right)\right\}=1$$

* $X\equiv\bar{x}\cosh\kappa t \qquad T\equiv\bar{x}\sinh\kappa t$

$dX=d\bar{x}\cosh\kappa t+\bar{x}\kappa\sinh\kappa t\,dt$

$dT=d\bar{x}\sinh\kappa t+\bar{x}\kappa\cosh\kappa t\,dt$

$$dX^2-dT^2=(d\bar{x})^2\overbrace{(\cosh^2\kappa t-\sinh^2\kappa t)}^{1}+(dt)^2\bar{x}^2\kappa^2\overbrace{(\sinh^2\kappa t-\cosh^2\kappa t)}^{-1}$$

$$+2\bar{x}\kappa\cosh\kappa t\,\cancel{\sinh\kappa t\,d\bar{x}dt}-\cancel{2\bar{x}\kappa\sinh\kappa t}\cosh\kappa t\,d\bar{x}dt$$

$$=(d\bar{x})^2-\bar{x}^2\kappa^2dt^2$$

$$\boxed{dT^2-dX^2=\kappa^2\bar{x}^2dt^2-(d\bar{x})^2}$$

$T=\bar{x}\sinh\kappa t=\sqrt{4a(x-a)}\,\sinh\frac{t}{2a}$ $\quad\Big|\ 2\kappa=\frac{1}{a}$

$X=\bar{x}\cosh\kappa t=\sqrt{4a(x-a)}\,\cosh\frac{t}{2a}$

* $$\boxed{\underset{\text{sphere}}{ds^2=\left(1-\frac{a}{x}\right)dt^2-\frac{dx^2}{\left(1-\frac{a}{x}\right)}}\ \xrightarrow[\text{Synasthes + Taylor exp.}]{\text{Coordinate transf}^n}\ \underset{\text{hyperbola}}{ds^2\simeq dT^2-dX^2}\quad g_{ab}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}}$$

Comment on Euler's Identity :

$*$ $f(x) = \sum_{n=0}^{\infty} a_n x^n$ $\quad n \in \mathbb{Z}^+$ Power series exp. of $f(x)$

$\big\downarrow$ Taylor exp

? (To be proven later)

$*$ $f(x) = e^x \xrightarrow[\text{Taylor}]{}$ $\boxed{e^x \overset{?}{=} 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$

$*$ $\sin x \overset{?}{=} x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots$

$\cos x \overset{?}{=} 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots$

$*$ $\boxed{x = iz}$ substitution $\Rightarrow$ $e^{iz} = 1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \frac{(iz)^5}{5!} \cdots$

$$= 1 + iz - \frac{z^2}{2!} - \frac{iz^3}{3!} + \frac{z^4}{4!} + \frac{iz^5}{5!} \cdots$$

$$= \underbrace{\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} \cdots\right)}_{\cos z} + i \underbrace{\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} \cdots\right)}_{\sin z}$$

$\boxed{e^{iz} = \cos z + i \sin z}$

Leonhard Euler's Identity (e, i, θ)

Number th. C.N Trig

$*$ $\mathrm{Re}\{e^{iz}\} = \cos z$

$\mathrm{Im}\{e^{iz}\} = \sin z$

$\overbrace{f(t)}^{x} = c + vt + \frac{1}{2!}at^2 + \cdots$

$\boxed{\frac{d^2 f(t)}{dt^2} = \text{const}}$

$x(t) = c + vt + \frac{1}{2}at^2 + \cdots$

Syllabus

- geometry
- P.C
- Prob.
- Series / $\mathbb{C}$
- Pre-degree

$i^1 = i$

$i^2 = -1$

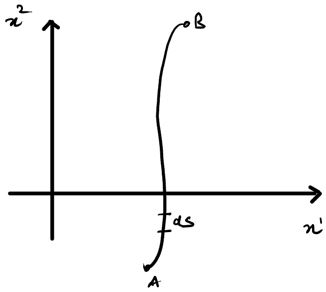
$i^3 = -i$

$i^4 = -i \cdot i = 1$

$i^5 = i^1$

Lecture-28 (31/Aug.) 2

* $ds^2 = \sum g_{ab}\, dx^a dx^b$; `s : physical' (Arc length) ; x^a = Coordinate label



$x^a = (x\ y) \Rightarrow \boxed{\frac{d^2x}{ds^2} = 0 = \frac{d^2y}{ds^2}} \Rightarrow x(s) = a_1 s + b_1$

$y(s) = a_2 s + b_2 = \frac{a_2}{a_1}(x - b_1) + b_2 = \underbrace{\frac{a_2}{a_1}}_{\equiv a} x \underbrace{- \frac{a_2 b_1}{a_1} + b_2}$

$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \frac{1}{a_1} \equiv b$

$y(x) = ax + b$ Linear $\Rightarrow$ St. Line

Physics 101

$m = 1$

$\underset{\sim}{a} = \frac{d^2x}{dt^2} = \begin{cases} = 2t \uparrow \\ = -g \\ = 0 \ \boxed{\text{St. Line}} \\ = -x \downarrow \ (\omega = 1) \end{cases}$

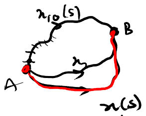
$x^a = (r\ \theta) \Rightarrow \boxed{\frac{d^2r}{ds^2} \neq 0 \neq \frac{d^2\theta}{ds^2}} \Rightarrow r(s) = \text{non-linear}$, $\theta(s) = \text{non linear}$

Still can be a straight Line (previous lecture)

$a_r = \frac{d^2r}{dt^2} = r\omega^2$

$a_T = \frac{d^2\theta}{dt^2} = \alpha = 0$

* $\boxed{ds^2 = g_{ab}\, dx^a dx^b \xrightarrow{\text{variation}} \text{variation}\left(\int ds^2\right) = 0 \quad : \quad \text{variation}(dx)\Big|_{\text{Boundary}} = 0}$



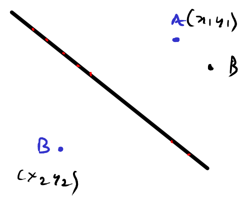
Minimization

Picks out a Min. Path ("Straight Line")

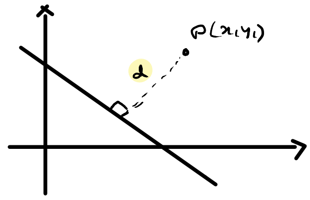
Variational Principle

$x^a = (x\ y)$ Linear Eq^n $\Rightarrow$ st. Lines

*



$f(x, y) = ax + by + c = 0$



$f(x, y) = ax + by + c = 0$

$f(x, y) = x + y + 2 = \text{const}$ ⌈ Plane

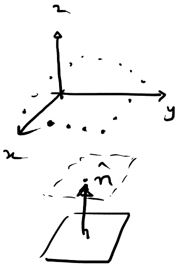
$f(1, 2) = 5$

$f(0, 0) = 2$

$f(3, 1) = 6$

$f(x_0 y_0) = 0$

$f(\alpha, \beta) = 0$



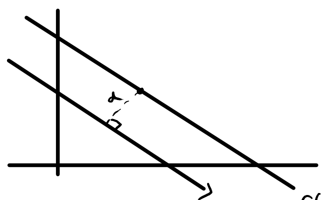
$f(A) > 0,\ < 0$; $f(B) > 0,\ < 0$ } $f(A), f(B)$ same side

$d = \frac{|f(P)|}{\sqrt{a^2 + b^2}}$ → Normalization of distance

$f(A) > 0 \quad f(B) < 0$ opp. sides.

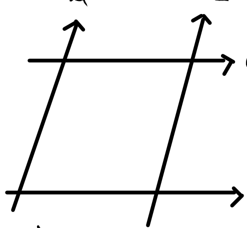
$\underset{\sim}{A} = 2\hat{i} + 3\hat{j}$

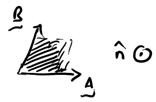
$\hat{\underset{\sim}{A}} = \frac{2\hat{i} + 3\hat{j}}{(\sqrt{2^2 + 3^2})}$



$g(x, y) \equiv ax + by + c_2 = 0$

$f \equiv ax + by + c_1 = 0$





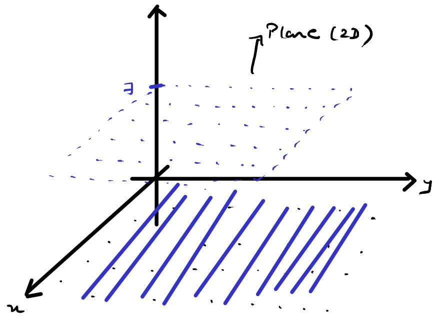
$\underset{\sim}{A} \times \underset{\sim}{B} = |A||B| \sin\theta\, \hat{n}$

$d = \frac{\overbrace{|c_1 - c_2|}^{f - g}}{\sqrt{a^2 + b^2}}$ ← Length

$\text{Area} \parallel^{gm} = \frac{|\overbrace{(c_1 - c_2)}^{l} \overbrace{(d_1 - d_2)}^{l}|}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$ ← (Length)^2

Lecture-29 (1/Sept) 2

Practice-4



Q1. $A(4,7)$, $B(\cos\theta, \sin\theta)$ $\quad 0<\theta<\pi$

A, B : lie on same side of line $x+y-1=0$ $\quad \theta \in I^{st}$ or II^{nd} ?

* $L(x,y) \equiv x+y-1=0$; $L(A) = 12$, $L(B) = \sin\theta + \cos\theta - 1$

* $\sin\theta + \cos\theta - 1 > 0$ $\quad$ T-inequality

$f(\theta) = 1\sin\theta + 1\cos\theta$

$f^2(\theta) = \underbrace{1+2\sin\theta\cos\theta}_{\sin 2\theta} > 1 \Rightarrow \sin 2\theta > 0 \Rightarrow 2\theta \in (0,\pi) \Rightarrow \theta \in (0, \frac{\pi}{2})$

Recall:

$f(\theta) \equiv a\sin\theta + b\cos\theta = r\{\cos\alpha \sin\theta + \sin\alpha\cos\theta\}$

$= r\sin(\theta+\alpha) = \sqrt{a^2+b^2}\cdot \sin(\theta+\alpha)$

$a = r\cos\alpha$

$b = r\sin\alpha$

$r^2 = a^2+b^2$

$\tan\alpha = \frac{b}{a}$

$f(\theta) = \sin\theta + \cos\theta = \sqrt{2}\sin(\theta + \frac{\pi}{4}) \overset{\text{given}}{>} 1$

$\boxed{\sin(\theta+\frac{\pi}{4}) > \frac{1}{\sqrt{2}}}$ $\quad$ HW $\quad$ $\theta \in (0, \frac{\pi}{2})$

$x, y \in \mathbb{R}$

$f(x,y) = x+y+2 = 7 = \text{const}$ $\quad$ Line Eqn

$f(1,1) = 4$

$f(1,2) = 5$

$f(0,1) = 3$

$f(0,-1) = 1$

$\vdots$

$S = \{(x,y) : x,y \in \mathbb{R}\} \subset \mathbb{R}\times\mathbb{R} = \mathbb{R}^2$

$f(x,y,z) = ax+by+cz+d = 0$ $\quad$ Plane Eqn

$\downarrow z=0$

$f(x,y) = ax+by+d = 0$ $\quad$ Line

* $-1 \le \sin\theta \le 1$; $\theta \xrightarrow{T} -(\theta-\pi)$: $\sin(\pi-\theta) = \sin\theta$ (Periodicity of function)

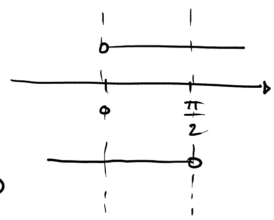
$\sin(\theta+\frac{\pi}{4}) > \frac{1}{\sqrt{2}}$ $\quad$ and $\quad$ $\sin(\pi - (\theta+\frac{\pi}{4})) > \frac{1}{\sqrt{2}}$

$\Downarrow$ $\qquad\qquad$ $\Downarrow$

$\theta + \frac{\pi}{4} > \frac{\pi}{4}$ $\qquad$ $\pi - (\theta+\frac{\pi}{4}) > \frac{\pi}{4} \Rightarrow -(\theta+\frac{\pi}{4}) > -\frac{3\pi}{4} \Rightarrow \theta+\frac{\pi}{4} < \frac{3\pi}{4}$ — (2)

$\theta > 0$ $\qquad\qquad$ $\theta < \frac{\pi}{2}$

$\boxed{\theta \in (0, \frac{\pi}{2})}$



HW

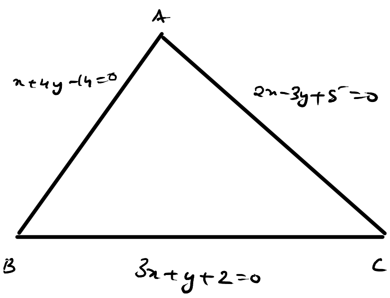
Q2. $l_1 \equiv 3x+y+2=0$

$l_2 \equiv 2x-3y+5=0$

$l_3 \equiv x+4y-14=0$

$P(0,\beta)$ $\quad \beta = ?$: P lies on or inside the Δ

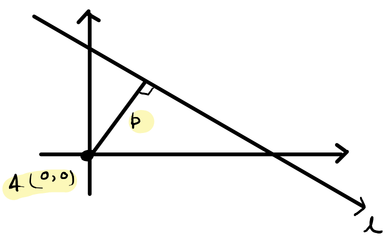
* $\{l_1, l_2, l_3\} \xrightarrow{\Delta}$ vertices A, B, C



Q3. $l \equiv \frac{x}{a} + \frac{y}{b} = 1$, $\quad p$ = length of the $\perp$ from o to line ; constraint Eqn = ?

$bx+ay-ab=0$ $\qquad\qquad$ rel^n b/w a, b, p

* $p = \frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}} = \frac{|b(0)+a(0)-ab|}{\sqrt{a^2+b^2}} = \frac{ab}{\sqrt{a^2+b^2}} \Rightarrow \frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$ ✓



Q4. $l_1 \equiv x\sec\theta + y\,\mathrm{cosec}\,\theta = a \longrightarrow p$: $\perp$ dist from o

$l_2 \equiv x\cos\theta - y\sin\theta = a\cos 2\theta \longrightarrow p'$: "

Constraint eq^n = ? rel b/w a, p, p', θ

* $(2p)^2 + p'^2 = a^2$

Q5. $\ell \equiv 4x+3y=12$; $p = \perp$ dist of line from a pt on x axis $= 4$, pt/pts = ?

$(a,0)$

* $\left|\frac{4a-12}{5}\right| = 4 \Rightarrow |a-3| = 5 \Rightarrow a-3 = \pm 5 \Rightarrow (8,0), (-2,0)$ ✓

Lecture-30 (2/sept) 2

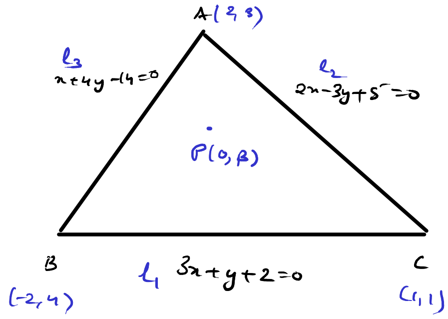
Sol^n

Q2 $\ell_1 \equiv 3x+y+2=0$

$\ell_2 \equiv 2x-3y+5=0$

$\ell_3 \equiv x+4y-14=0$

$P(0,\beta)$ $\beta = ?$: P lies on or inside the Δ



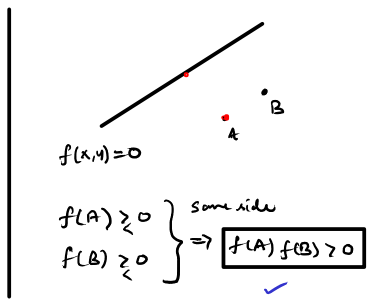
* $\{\ell_1, \ell_2, \ell_3\} \xrightarrow{\Delta}$ vertices A, B, C

* P lies ~~on or~~ inside $\Rightarrow$ A, P lies on same side of BC (ℓ_1) — ①

Imp! Cond^n (not included)

B P ,, ,, ,, ,, ,, AC (ℓ_2) — ②

C. P ,, ,, ,, AB (ℓ_3) — ③



Coord. of A, B, C :

* A(2,3), B(-2,4), C(-1,1)

* ① $\Rightarrow$ $\ell_1(A)\,\ell_1(P) > 0$

② $\Rightarrow$ $\ell_2(B)\,\ell_2(P) > 0$

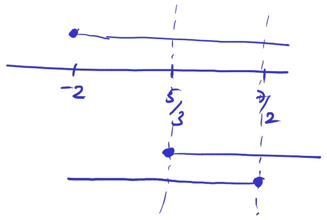
③ $\Rightarrow$ $\ell_3(C)\,\ell_3(P) > 0$

* $\boxed{\text{P lies on the line } \ell \Rightarrow \ell(P) = 0}$

* P lies on or inside $\Rightarrow$ $\ell_1(A)\,\ell_1(P) \geq 0 \Rightarrow \left(\overset{11}{3(2)+3+2}\right)\left(3(0)+\beta+2\right) \geq 0 \Rightarrow 11(\beta+2) \geq 0 \Rightarrow \boxed{\beta \geq -2}$

$\ell_2(B)\,\ell_2(P) \geq 0 \Rightarrow (2(-2)-3(4)+5)\,(2(0)-3\beta+5) \geq 0 \Rightarrow -11(-3\beta+5) \geq 0 \Rightarrow 3\beta - 5 \geq 0 \Rightarrow \boxed{\beta \geq \frac{5}{3}}$

$\ell_3(C)\,\ell_3(P) \geq 0 \Rightarrow (-1+4-14)\,(0+4\beta-14) \geq 0 \Rightarrow -11(4\beta-14) \geq 0 \Rightarrow \boxed{\beta \leq \frac{7}{2}}$



$\beta \in \left[\frac{5}{3}, \frac{7}{2}\right]$

HW

* Δ : $\ell_1 \equiv 2x+3y-1=0$

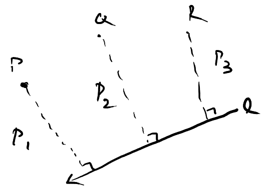
$\ell_2 \equiv x+2y-3=0$

$\ell_3 \equiv 5x-6y-1=0$

$P(x,y) \rightarrow P(\alpha,\alpha^2) = ?$: it lies inside Δ

related

Q6. $P(m^2, 2m)$, $Q(mn, m+n)$, $R(n^2, 2n)$, $\ell \equiv x\cos^2\theta + y\sin\theta\cos\theta + \sin^2\theta = 0$



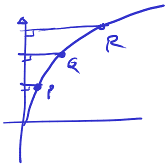
P_1, P_2, P_3 = length of $\perp$ from P, Q, R to ℓ resp. $\longrightarrow$ rel^n b/w $P_1, P_2, P_3 = ?$

$P_1 = \dfrac{|\ell(P)|}{\sqrt{(\text{Coeff } x)^2 + (\text{Coeff } y)^2}} = \dfrac{|m\cos^2\theta + 2m\sin\theta\cos\theta + \sin^2\theta|}{\sqrt{\cos^4\theta + \sin^2\theta\cos^2\theta}} = \dfrac{|(m\cos\theta + \sin\theta)^2|}{\cos\theta}$

$D = \sqrt{\cos^2\theta\,\underbrace{(\cos^2\theta + \sin^2\theta)}_{1}} = \cos\theta$

$P_2 = \dfrac{|mn\cos^2\theta + (m+n)\sin\theta\cos\theta + \sin^2\theta|}{\cos\theta}$

$$P_3 = \frac{|n^2\cos^2\theta + 2n\sin\theta\cos\theta + \sin^2\theta|}{\cos\theta} = \frac{|(n\cos\theta+\sin\theta)^2|}{\cos\theta}$$



$\{2, 4, 8\}$

$\frac{4}{2} = \frac{8}{4} = 2$ Common Ratio

P_1, P_2, P_3

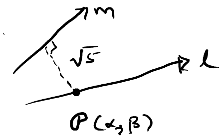
$\frac{P_2}{P_1} = \frac{P_3}{P_2} = C.R$

$P_2^2 = P_1P_3$

$P_2 = \sqrt{P_1P_3}$ | AP $c = \frac{a+b}{2}$

* $\boxed{P_2^2 = P_1P_3 \iff \{P_1, P_2, P_3\} \text{ are in Geometric progression}}$

Q7. $l \equiv x+y+3=0$, $m \equiv x+2y+2=0$, $P(\alpha,\beta)$ on the l : dist b/w P & $m = \sqrt{5}$?



* $(\alpha,\beta) = (-9,6), (1,-4)$

Lecture-31 (3/Sept) 1.50

HW

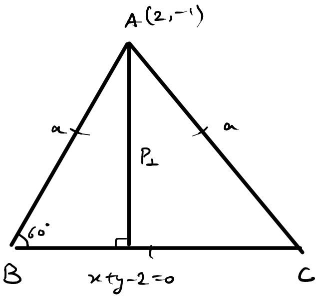
Q Δ : $l_1 \equiv 2x+3y-1=0$
$l_2 \equiv x+2y-3=0$
$l_3 \equiv 5x-6y-1=0$

$P(x,y) \longrightarrow P(\alpha,\alpha^2) = ?$: it lies inside Δ

related

Q8. base of Δ $l \equiv x+y-2=0$, opp. vertex of Δ $(2,-1)$; Area $\Delta = ?$

Equil tri.

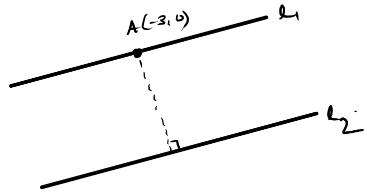


$p_\perp = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$; $a\sin 60 = P_\perp \Rightarrow a = \frac{P_\perp}{\sin 60}$; $Ar = \frac{1}{2}\cdot a \cdot P_\perp = \frac{P_\perp^2}{\sin 60} = \frac{1}{2\sqrt{3}}$ ✓

Q9. $\left.\begin{array}{l} l_1 \equiv 3x-4y+9=0 \\ l_2 \equiv 6x-8y-15=0 \end{array}\right\}$ $l_1 \parallel l_2$ dist $(l_1, l_2) = ?$

$l_2 = 3x-4y-\frac{15}{2}=0$

different method :

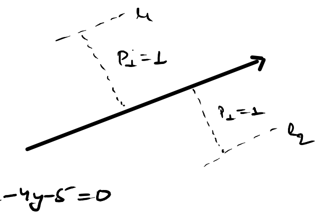


* $l_1 = 0$, $y=0 \Rightarrow x=-3$ $A(-3,0)$ lies on l_1

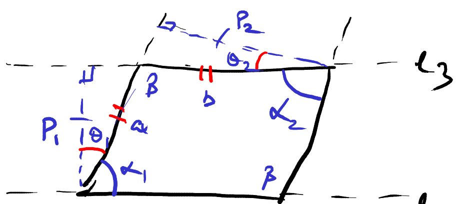
$$P_\perp = \frac{|l_2(-3,0)|}{\sqrt{6^2+8^2}} = \frac{33}{10}$$

Q10. $l \equiv 3x-4y-5=0$ Eqn of Lines $\parallel$ to l : $P_\perp = 1$?

$l_1 \equiv 3x-4y-10=0$
$l_2 \equiv 3x-4y=0$



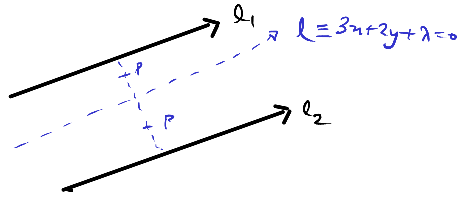
Q11. $\parallel^{gm}$: $l_1 \equiv \frac{x}{a}+\frac{y}{b}=1$, $l_3 \equiv \frac{x}{a}+\frac{y}{b}=2$
$l_2 \equiv \frac{x}{b}+\frac{y}{a}=1$, $l_4 \equiv \frac{x}{b}+\frac{y}{a}=2$ } what kind of $\parallel^{gm}$ = ?



$$P_1(l_1 \& l_3) = \frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}} = P_2(l_2 \& l_4)$$

* $\because$ $\underbrace{\parallel^{gm}}_{a\neq b} \Rightarrow \alpha_1 = \alpha_2 \Rightarrow 90-\alpha_1 = 90-\alpha_2 \Rightarrow \theta_1 = \theta_2 \Rightarrow \cos\theta_1 = \cos\theta_2 \Rightarrow \frac{P_1}{a} = \frac{P_2}{b} \Rightarrow a=b \Rightarrow$ Rhombus

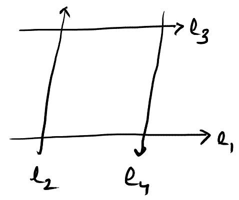
Q12. $l_1 \equiv 9x+6y-7=0$
$l_2 \equiv 3x+2y+6=0$ } Eqⁿ of line mid way b/w l_1 & l_2 $(l_2 \parallel l_1) = ?$



$l \equiv$ Mid way $(l_1$ & $l_2) = 3x+2y+\frac{11}{6}=0$

$l_1 = 3x+2y-\frac{7}{3}=0$; $P(l_1$ & $l) = P(l_2$ & $l) \Rightarrow \lambda = \frac{11}{6}$

Q13. H.W : $l_1 \equiv \frac{x}{a}+\frac{y}{b}=1$, $l_3 \equiv \frac{x}{a}+\frac{y}{b}=2$
$l_2 \equiv \frac{x}{b}+\frac{y}{a}=1$, $l_4 \equiv \frac{x}{b}+\frac{y}{a}=2$ } Area of Rhombus = ?

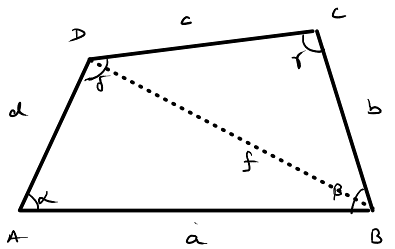


$Ar = \frac{a^2b^2}{|a^2-b^2|}$

Lecture-32 (7/Sept) 2

5. Miscellaneous stuff

5.1 Bretschneider's Ar. formula for quad.



* Area of a generic quad. = Ar ($\triangle ABD$) + Ar($\triangle DBC$) = ?

$$\Delta = \frac{1}{2}\,ad\sin\alpha + \frac{1}{2}\,bc\sin\gamma \quad — ①$$

Elimination journey of $\sin\alpha / \sin\gamma$:

* $f^2 = a^2+d^2-2ad\cos\alpha = b^2+c^2-2bc\cos\gamma$

$$\frac{a^2+d^2-b^2-c^2}{2} = ad\cos\alpha - bc\cos\gamma$$

* $$\frac{(a^2+d^2-b^2-c^2)^2}{4} = (ad)^2\cos^2\alpha + (bc)^2\cos^2\gamma - 2abcd\cos\alpha\cos\gamma \quad — ②$$

* $$①^2 \Rightarrow 4\Delta^2 = (ad)^2\sin^2\alpha + (bc)^2\sin^2\gamma + 2abcd\sin\alpha\sin\gamma \quad — ③$$

* $$② + ③ \Rightarrow 4\Delta^2 + \frac{(a^2+d^2-b^2-c^2)^2}{4} = \underbrace{(ad)^2+(bc)^2}_{(ad+bc)^2 - 2abcd} - 2abcd\cos(\alpha+\gamma)$$

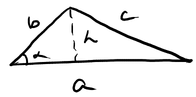
* $$4\Delta^2 + \frac{(a^2+d^2-b^2-c^2)^2}{4} = (ad+bc)^2 - 2abcd\underbrace{\left(1+\cos(\alpha+\gamma)\right)}_{2\cos^2\left(\frac{\alpha+\gamma}{2}\right)}$$

$\cos 2\theta = 2\cos^2\theta - 1$

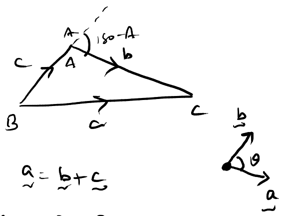
$\frac{1+\cos\theta}{2} = \cos^2\theta/2$

* $$4\Delta^2 = (ad+bc)^2 - \frac{(a^2+d^2-b^2-c^2)^2}{4} - 4abcd\cos^2\left(\frac{\alpha+\gamma}{2}\right)$$

$$16\Delta^2 = \underbrace{4(ad+bc)^2 - (a^2+d^2-b^2-c^2)^2}_{\text{Similar step as Bhramgupta form.} \equiv B} - 16abcd\cos^2\left(\frac{\alpha+\gamma}{2}\right)$$



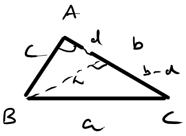
$Ar\,\Delta = \frac{1}{2}ab\sin\alpha$



$\underset{\sim}{a} = \underset{\sim}{b} + \underset{\sim}{c}$

$a^2 = b^2+c^2+2\underbrace{\underset{\sim}{b}\cdot\underset{\sim}{c}}_{bc\cos(180-A)}$

$a^2 = b^2+c^2-2bc\cos A$



$c^2 = d^2 + h^2$

$a^2 = (b-d)^2 + h^2$

$c^2 - a^2 = d^2 - (b^2 + d^2 - 2bd)$

$a^2 = b^2 + c^2 - 2b\underbrace{d}_{c\cos A}$

$a^2 = b^2+c^2-2bc\cos A$

* $$B = \left(2(ad+bc) + (a^2+d^2-b^2-c^2)\right)\left(2(ad+bc) - (a^2+d^2-b^2-c^2)\right)$$

$$= \left(\underbrace{a^2+d^2+2ad}_{(a+d)^2} - (\underbrace{b^2+c^2-2bc}_{(b-c)^2})\right)\left(\underbrace{b^2+c^2+2bc}_{(b+c)^2} - (\underbrace{a^2+d^2-2ad}_{(a-d)^2})\right)$$

$$= \underbrace{(a+d+b-c)}_{2(s-c)}\underbrace{(a+d-b+c)}_{2(s-b)}\underbrace{(b+c+a-d)}_{2(s-d)}\underbrace{(b+c-a+d)}_{2(s-a)} \rightsquigarrow \boxed{\text{Diophantine Eqn}}!$$

Euler!

$s \equiv \frac{a+b+c+d}{2}$

$s-a = \frac{b+c+d-a}{2}$

* $$\boxed{\text{Ar. of non cyclic quad} = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\underbrace{\cos^2\left(\frac{\alpha+\gamma}{2}\right)}_{\frac{1}{2}(1+\cos(\alpha+\gamma))}}}$$

1840 Bretschneider's Formula

$\cos 2\theta = 2\cos^2\theta - 1$

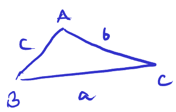
$\alpha+\gamma = 180$ | cyclic quad

$$\Delta = \sqrt{(s-a)(s-b)(s-c)(s-d)} \xrightarrow{a=0} \Delta = \sqrt{s(s-b)(s-c)(s-d)}$$

Bhramgupta's

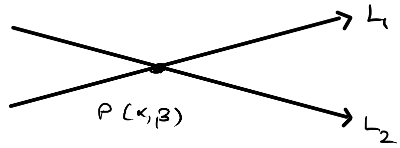
Heron's

Heronian Δ



5.2

* $L_1 \equiv a_1x + b_1y + c_1 = 0$
  $L_2 \equiv a_2x + b_2y + c_2 = 0$ } L_1 intersects L_2 ⟹ P satisfies L_1 & L_2



* $a_1\alpha + b_1\beta + c_1 = 0$, $a_2\alpha + b_2\beta + c_2 = 0$ ⟹ $(a_1\alpha + b_1\beta + c_1) + \lambda(a_2\alpha + b_2\beta + c_2) = 0$ ∵ it is a linear comb^n using L_1 & L_2

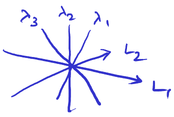
$(a_1 + \lambda a_2)\alpha + (b_1 + \lambda b_2)\beta + (c_1 + \lambda c_2) = 0$

(α,β) satisfies line formed by linear comb^n of L_1 L_2

* $\boxed{L_1 + \lambda L_2 \equiv L_3}$

⇓

$L_3 = a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = (a_1 + \lambda a_2)x + (b_1 + \lambda b_2)y + c_1 + \lambda c_2$



$\boxed{L_1 + \lambda L_2 = 0 \implies \text{a family of st. lines for diff. } \lambda\text{-values (passing thru a fixed pt)}}$

$\underbrace{\text{fixed pt}}_{\text{intersection pts.}}$

* $L_1 + \lambda L_2 = 0 \implies L_1 = 0,\ L_2 = 0$

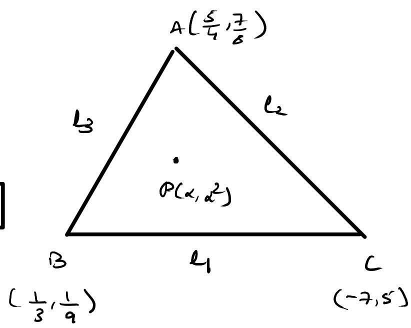
Lecture-33 (8/sept) 2

Sol^n

Q Δ : $l_1 \equiv 2x + 3y - 1 = 0$
$l_2 \equiv x + 2y - 3 = 0$
$l_3 \equiv 5x - 6y - 1 = 0$

$P(x,y) \longrightarrow P(\alpha,\alpha^2) = ?$: it lies inside Δ

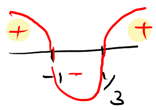
related



* A, P lies on same side of l_1 ⟹ $\underbrace{\left(2\left(\frac{5}{4}\right) + 3\left(\frac{7}{8}\right) - 1\right)}_{+}\underbrace{\left(2\alpha + 3\alpha^2 - 1\right)}_{(\alpha+1)(3\alpha-1)} > 0 \implies \boxed{(\alpha+1)(3\alpha-1) > 0}$

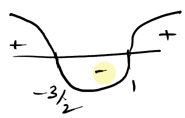
$f(\alpha)$

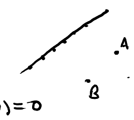
$\alpha \in (-\infty,-1) \cup (1/3,\infty)$ — ①



* B, P " " l_2 ⟹ $\underbrace{\left(\frac{1}{3} + 2\left(\frac{1}{9}\right) - 3\right)}_{-}\underbrace{\left(\alpha + 2\alpha^2 - 3\right)}_{(\alpha-1)(2\alpha+3)} > 0 \implies \boxed{(\alpha-1)(2\alpha+3) < 0}$

$\alpha \in \left(-\frac{3}{2}, 1\right)$ — ②





$f(x,y) = 0$

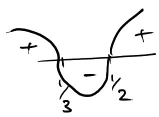
$f(A) \gtrless 0$
$f(B) \gtrless 0$

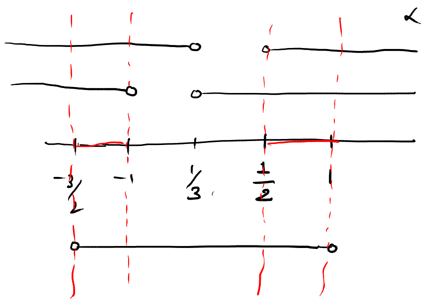
$\underbrace{\boxed{f(A)\,f(B) > 0}}_{\text{trick}}$

* C, P " " l_3 ⟹ $\underbrace{(5(-7) - 6(5) - 1)}_{\cancel{+}}\underbrace{(5\alpha - 6\alpha^2 - 1)}_{} > 0 \implies \boxed{(3\alpha - 1)(2\alpha - 1) > 0}$

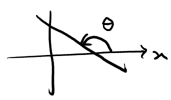
$\cancel{+}$ $(6\alpha^2 - 5\alpha + 1) = (2\alpha - 1)(3\alpha - 1)$

$\alpha \in \left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{2}, \infty\right)$ — ③

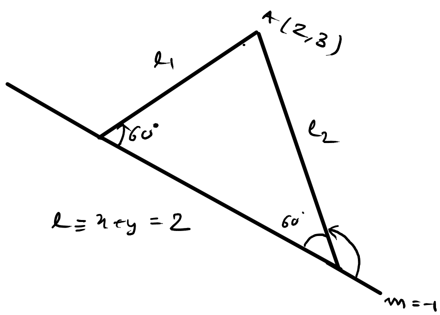




$\alpha \in \left(-\frac{3}{2}, -1\right) \cup \left(\frac{1}{2}, 1\right)$

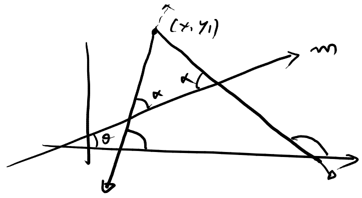


Q Δ Equil. vertex $V(2,3)$ & opp. side $\rightarrow$ $\underbrace{l \equiv x + y = 2}_{y = -x + 2 \quad m_l = -1}$; Eq^n of other sides = ?



* $(1+\sqrt{3})(y-3) = (\sqrt{3}-1)(x-2) \longrightarrow y-3 = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}(x-2)$

$(1-\sqrt{3})(y-3) = (-\sqrt{3}-1)(x-2) \longrightarrow y-3 = \dfrac{-(1+\sqrt{3})}{1-\sqrt{3}}(x-2)$

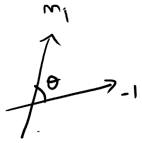


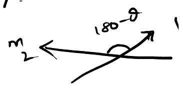
$y-y_1 = \dfrac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x-x_1)$ ✓

$\underline{1^{st}\ \text{principles}}$

* $y-3 = (2 \pm \sqrt{3})(x-2)$ ✓

$m_1 = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$, $m_2 = \dfrac{-(1+\sqrt{3})}{1-\sqrt{3}}$





* $\tan\theta = \dfrac{m_1 - m_2}{1 + m_1 m_2}$