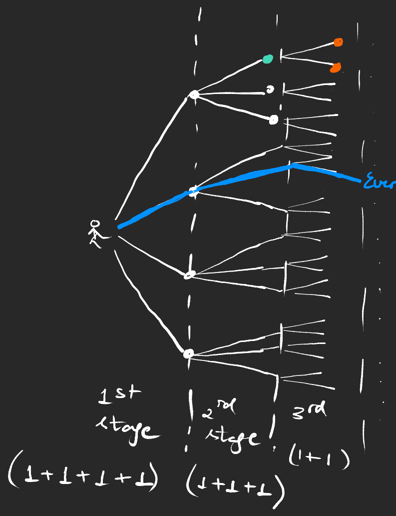
Lecture-7 (10 Aug) 22

16th c
- Pascal
- Fermat

} Gamblers

③ Combinatorics (Basics) : classical

A. Basic Counting principles



which specific Path you took ?

↓

Probability answers that

Ex: $n_1 = 4 = 1 \overset{or}{+} 1 + 1 + 1$

$n_2 = 3 = 1 + 1 + 1$

$n_3 = 2 = 1 + 1$

# of possible leaves/nodes/choices = 4 & 3 & 2 = 24 (& = ×)

Tree diagram / Sequential diag.

* r stages, n choices at stage i

$\boxed{\#\text{ of choices} = n_1 \times n_2 \times n_3 \times \cdots \times n_r}$

fundamental Principle of Counting

- Rule of Sum : "Either ... or ..." ≡ a + b = # of choices = b + a
  (both can't occur)
- Rule of product : "... this AND that ..." ≡ a × b = # of choices = b × a

Ex1 : Situation 1 - # of license plates with 3 letters & 4 digits

letters = 26
digits = 10

a) 26 · 26 · 26 · 10 · 10 · 10 · 10 (Maximal No.) No constraint

Sample Z Z R 1 1 4 9

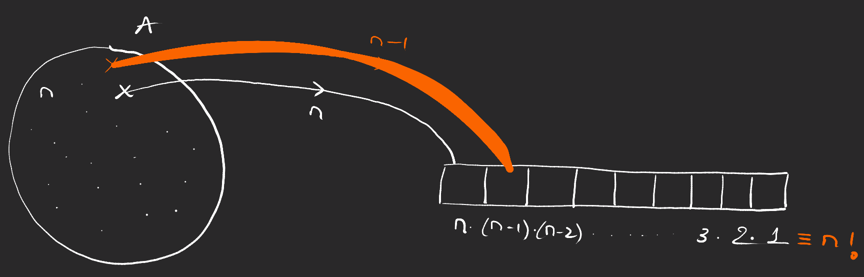
∨

b) 26 · 25 · 24 · 10 · 9 · 8 · 7 Constraint : Rep. is prohibited (tot. # reduces)

Sample Z Q R 1 3 4 9

Model 1

*



a list

# of choices / Arrangement of element / Permutation of n objects

n(A) = n

Ex2 $A = \{1,2,3,4\}$

↓

(2,3,4,1) 1 possible arrangement

4 · 3 · 2 · 1 = 4! = 24

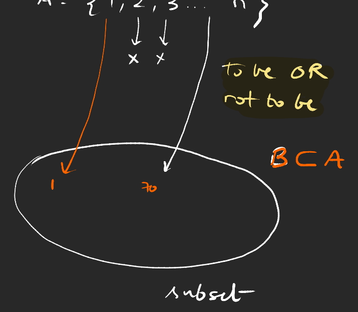
sample 3 4 1 2

∃ 24 diff. possible ways/ arrangement of 4 numbers

Ex 3 $A = \{1, 2, 3 \ldots n\}$

to be OR not to be

$B \subset A$



# of subsets $= \underbrace{2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2}_{n} = 2^n$

2 choices for 1^{st} element

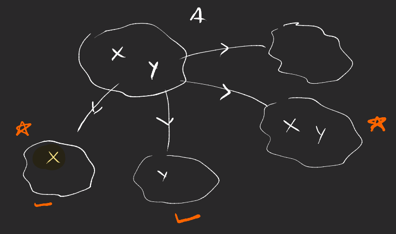
Binary $\begin{cases} \text{In the subset} \\ \text{not in the subset} \end{cases}$

Set theory

- $n = 1$ $\quad A = \{1\}$ $\quad$ subsets of $A = \{1\}, \phi \Rightarrow \#(\text{subset}) = 2$

$n = 2$ $\quad A = \{1, 2\}$ $\quad$ " " " $= \{1\}, \{2\}, \{1, 2\} = A, \phi \Rightarrow \#(\text{subset}) = 2^2$

Combinatorics:



$\underset{\text{Y OR N}}{\underline{2}} \cdot \underset{\text{Y OR N}}{\underline{2}} = 2^2$

Problems on factorial : HW

Q1. $\dfrac{1}{9!} + \dfrac{1}{10!} = \dfrac{x}{11!}$ $\quad x = ?$

2. $\dfrac{n!}{2!(n-2)!} : \dfrac{n!}{4!(n-4)!} = 2:1$ $\quad n = ?$

5. $\dfrac{n!}{(n-r)!\,r!} + \dfrac{n!}{(n-r+1)!\,(r-1)!} = \dfrac{(n+1)!}{r!\,(n-r+1)!}$

3. $\dfrac{(2n)!}{n!} = [1 \cdot 3 \cdot 5 \ldots (2n-1)]\, 2^n$

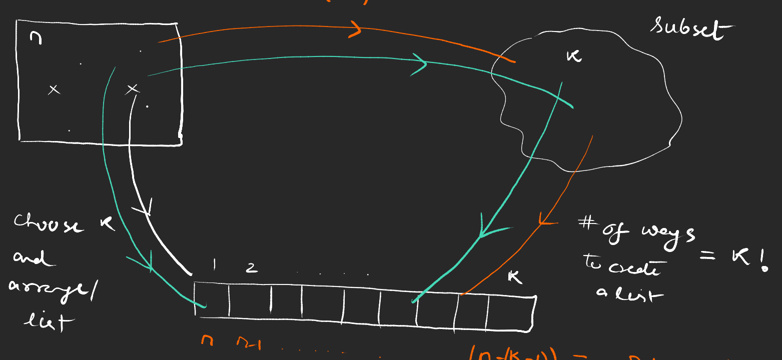
4. $\dfrac{(2n+1)!}{n!} = [1 \cdot 3 \cdot 5 \ldots (2n-1)(2n+1)]\, 2^n$

Q6. a) $n(n-1)(n-2) \ldots 4\,3\,2 \cdot 1 = n!$

b) $n(n-1)(n-2) \ldots (n-(k-1)) = \dfrac{n(n-1)(n-2) \ldots (n-k+1)(n-k)(n-k-1) \ldots 3 \cdot 2 \cdot 1}{(n-k)(n-k-1) \ldots 3 \cdot 2 \cdot 1} \equiv \dfrac{n!}{(n-k)!}$

B. n choose k / Binomial Coefficient $\boxed{\text{Model 2}}$

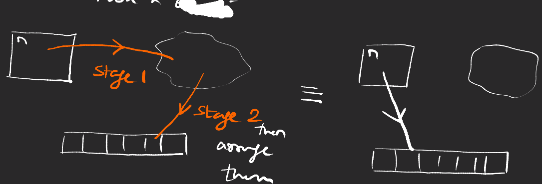
$\binom{n}{k}$ "n choose k"



$(n-(k-1)) = \dfrac{n!}{(n-k)!} \equiv {}^nP_k$

Stages

choose k



Approach 1 in stages

Approach 2 direct

logic $\quad 1^{st}$ place $\longrightarrow$ can fill n objects

2^{nd} " $\longrightarrow$ " " $(n-1)$ "

3^{rd} " $\longrightarrow$ " " $(n-2)$ "

$n = 4$

1	2
3	4

$k = 2$

ex: $\underline{2} \quad \underline{4}$

choices $\quad 4 \cdot 3 \cdot$ $\quad \frac{4}{}$

$(n-(3-1))$ $4-(2-1)$

k^{th} place ... $(n-(k-1))$

$$\binom{n}{k}\cdot k! = \frac{n!}{(n-k)!} \Rightarrow \boxed{\binom{n}{k} = \frac{n!}{(n-k)!\,k!}}$$

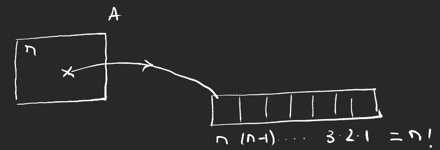
$\downarrow$ $^{n}C_{k}$

+ve Binomial Coefficient $(n \geq k)$

Lecture-8 (11/Aug) 2.22

ϕ
$\Uparrow$
$\binom{n}{0} = 1 = \binom{n}{n}$
$\Uparrow$ set itself as subset

*



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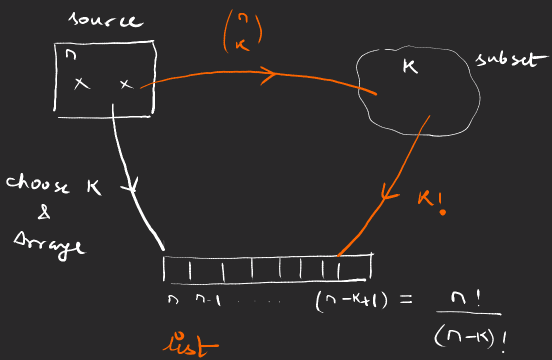


double counting proof

$$\binom{n}{k}\cdot k! = \frac{n!}{(n-k)!}$$

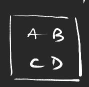
$\Downarrow$

$$\boxed{\binom{n}{k} = \frac{n!}{k!\,(n-k)!}} \quad n \geq k$$

Binomial Coeff.

ways to create k elem. subset starting from n element

$n=4$



$k=2$

D | B

choices $4 \cdot 3 \cdot \frac{2 \cdot 1}{2!} = \frac{4!}{2!}$, $\quad 3 = 4-3+1$

B1 Properties of Binomial Coeff $\binom{n}{k}$ or $^{n}C_{k}$

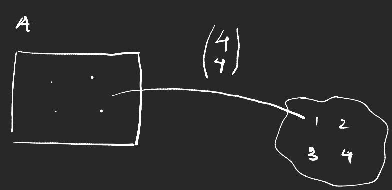
Ex: $A = \{1,2,3,4\}$ $\quad n=4$ $\quad$ # subsets $= 2\cdot 2\cdot 2\cdot 2 = 2^4 = 16$ $\quad$ set theory

subsets: $\{1\}$ $\{2\}$ $\{3\}$, $\{4\}$

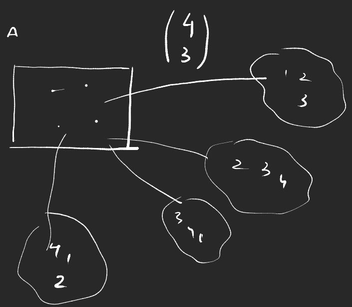
$\{1\ 2\}$ $\{2\ 3\}$ $\{3\ 4\}$ $\{2,4\}$ $\{3,1\}$, $\{1,4\}$

$\{1\ 2\ 3\}$ $\{2\ 3\ 4\}$ $\{3,4\ 1\}$ $\{1\ 2\ 4\}$

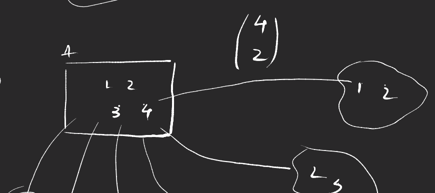
$\{1\ 2\ 3\ 4\}$ ϕ

1) 

$\binom{4}{4} = 1$

2) 

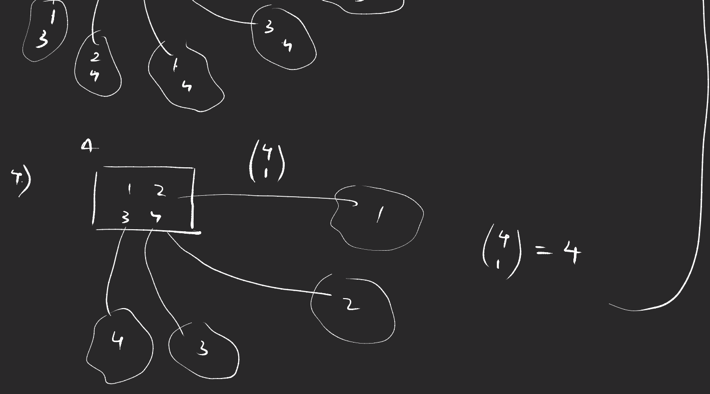
$\binom{4}{3} = 4$

3) 

$\binom{4}{2} = 6$

Combinatorics

$$\binom{4}{4} + \binom{4}{3} + \binom{4}{2} + \binom{4}{1} + \binom{4}{0} = 16 = 2^4$$



$\binom{4}{1} = 4$

Property 1

$$\boxed{\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n}$$

subsets w/ 0 elements

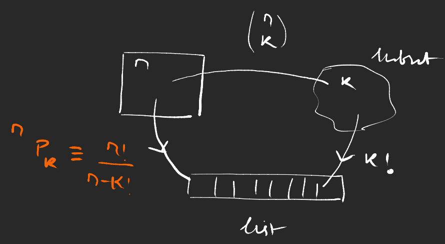
given n

Sum of Binomial coeffs

Property 2

Ex: $A = \{1, 2, 3\}$ $n = 3$

subsets : $\{1\}\ \{2\}, \{3\}$
$\{1,2\}\ \{2,3\}\ \{3,1\}$
$\{1,2,3\}\ \phi$



✓ $\binom{3}{0} = 1$ ϕ

✓ $\binom{3}{1} = 3$ $\{1\}\ \{2\}\ \{3\}$

$\boxed{\binom{3}{2} = 3}$ $\{1,2\}\ \{2,3\}\ \{3,1\}$ indistinguishable
$\{2,1\}\ \{3,2\}\ \{1,3\}$

$\binom{n}{k} \cdot k! = {}^nP_k$

$^3P_2 = \frac{3!}{1!} = 6$ ⟿ "double counting" ⟶ $\boxed{\binom{n}{k} = \frac{^nP_k}{k!}}$

"distinguishability"

Can distinguish / order matters

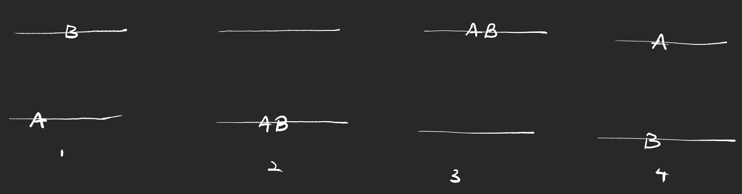
(1,2) & (2,1)

distinguishability

"indistinguish." ⟶ to eliminate distinguish. make it

- Combination is selection without duplicacy (it reduces the #)

Ex 2 labelled Balls
A B



distinguish A, B
choices = 4

2 unlabelled Balls

indistinguishable
choices = 3

1 2 3

A

B

(independence is accounted for)

$A = \{1,2,3\}$ set

$A' = \{1,1,1,2,3\}$ MultiSet

degen.$(1) = 3$

Ex 3 Apples, 3 oranges # obj = 6

$\underline{6}\cdot\underline{5}\cdot\underline{4}\cdot\underline{3}\cdot\underline{2}\cdot\underline{1} = 6!$
$= 720$

no constraint

But

we have counted
all apples distinctly.
all oranges distinctly
(overcounting)
" distinguishable fashion "

indistinguishable

ways to arrange $= \dfrac{6!}{3!\,3!} = \dfrac{720}{36} = 20$

repeated element

{ later in Partition Problem

Property 3 : Pascal's Triangle

$k = 0 \ldots$

$n = 0$

$n = 3$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$n \uparrow$ by 1

* $n = 3$, $0 \le k \le 3$

$\binom{3}{0} = 1$ $\binom{3}{1} = 3$ $\binom{3}{2} = 3$ $\binom{3}{3} = 1$

$\underset{1}{\binom{3}{3}} + \underset{3}{\binom{3}{2}} = \underset{4}{\binom{4}{3}} \Rightarrow$

HW

$$\boxed{\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}}$$

Recurrence rel^n (Pascal's rule)

* Obviously, $\exists$ "mirror" symmetry $\boxed{\binom{n}{k} = \binom{n}{n-k}}$

* $(x+y)^0 = 1$

$(x+y)^1 = x + y = \binom{1}{0}x + \binom{1}{1}y = \sum_{k=0}^{1} \binom{1}{k} x^{1-k} y^k$

$(x+y)^2 = x^2 + 2xy + y^2 = \binom{2}{0}x^2 + \binom{2}{1}xy + \binom{2}{2}y^2 = \sum_{k=0}^{2} \binom{2}{k} x^{2-k} y^k$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$$\boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k}$$

Binomial Coeffs

(true) Binomial Theorem $\longrightarrow$ Binomial probability distribution

$n \geq 0$

HW

- $\binom{n}{k} = \frac{n}{k} \cdot \binom{n-1}{k-1}$

- $\boxed{n \cdot \binom{n-1}{k-1} = (n-k+1)\binom{n}{k-1}}$

HW

$$\boxed{\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}}$$

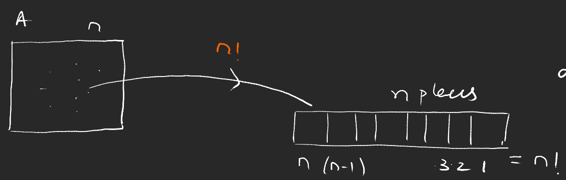
Recurrence

Lecture-9 (14/Aug) 2.20 Enumerative Combinatorics = Counting

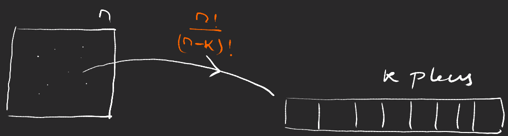
Boolean logic

ROS : $a + b$ (OR)

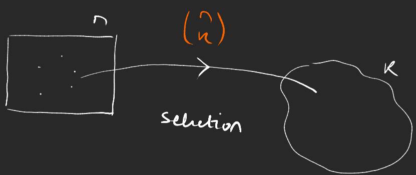
ROP : $a \times b$ (OR)

* 

ordered list = Arrangement

* 

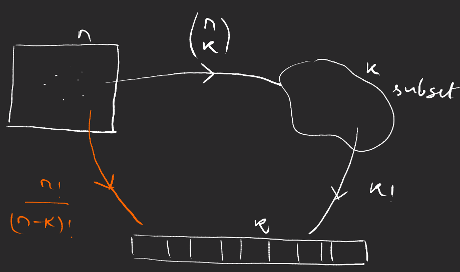
$n \; n-1 \; \cdots \; n-(k-1) = \frac{n!}{(n-k)!} \equiv {}^nP_k$ "double counted"

* 

$\binom{n}{k} = \frac{n!}{(n-k)!\,k!} = \frac{{}^nP_k}{k!}$ → destroy ordinality; $k!$ → to eliminate destroy ordinality

Binomial Coefficient

- $n \geq k$
- $n, k \geq 0$
- $n, k \in \mathbb{Z}^+$

* 

$\binom{n}{k} \cdot k! = \frac{n!}{(n-k)!}$ $\boxed{\text{Model 1}}$

Product/Sum Rule (Type 1)

Q1. R, Q, S, E

choice

- $\underline{4} \cdot \underline{4} \cdot \underline{4} \cdot \underline{4} = 4^4$ no constraint / Rep ✓
- $\underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = 4!$ Rep ✗

Q2. # 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 ; Create 3 digit Numbers

- $\underline{10} \cdot \underline{10} \cdot \underline{10} = 10^3$ rep ✓
- $\underline{10} \cdot \underline{9} \cdot \underline{8}$ Rep ✗
- $\underline{4} \cdot \underline{4} \cdot \underline{4} = 4^3$ Rep ✓
- $\underline{4} \cdot \underline{3} \cdot \underline{2}$ Rep ✗

} using 1, 7, 8, 9

Q3. How many #s are there b/w 100 & 1000   # = {0, 1, 2, 3 ... 9}   HTO

- $\underline{9} \cdot \underline{9} \cdot \underline{8}$ No rep.   H place can't have 0 ⇒ # choices = 9
  T place can have any of 9 choices
- $\underline{2} \cdot \underline{2} \cdot \underline{2}$ every digit is Either 2 or 9
- $\underline{9} \quad \underline{10} \cdot \underline{\underset{7}{1}}$ 7 is on U's place   H place can't have 0 ⇒ # choices = 9

- at least one of them is 7

  $\underline{\underset{7}{1}} \quad \underline{9} \cdot \underline{9} = 81$ OR
  $\underline{8} \cdot \underline{\underset{7}{1}} \cdot \underline{9} = 72$ OR   one - 7
  $\underline{8} \cdot \underline{9} \cdot \underline{\underset{7}{1}} = 72$

  $\underline{8} \cdot \underline{9} \cdot \underline{9}$ No - 7

  $\underline{9} \quad \underline{10} \cdot \underline{10}$ total 3 digit #s

  OR

  $\underline{\underset{7}{1}} \cdot \underline{9} \cdot \underline{\underset{7}{1}} = 9$ OR
  $\underline{\underset{7}{1}} \quad \underline{\underset{7}{1}} \cdot \underline{9} = 9$ OR   two - 7
  $\underline{8} \cdot \underline{\underset{7}{1}} \quad \underline{\underset{7}{1}} = 8$

  ≡   # (at least one 7) = # (total) − # (No 7 at all)
  = 252

  OR

  $\underline{\underset{7}{1}} \quad \underline{\underset{7}{1}} \cdot \underline{\underset{7}{1}} = 1$ three - 7

(at least 1 of them 7) = one-7 + two-7 + three-7
= (81 + 72 + 72) + (9 + 9 + 8) + 1
= 252 ✓

- $\underline{\underset{7}{1}} \quad \underline{9} \cdot \underline{9} = 81$ exactly one-7

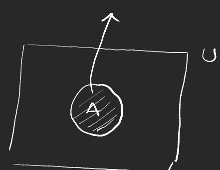
  $\underline{8} \cdot \underline{\underset{7}{1}} \cdot \underline{9} = 72$

  $\underline{8} \cdot \underline{9} \cdot \underline{\underset{7}{1}} = 72$

  # = 225

Q4. 3 prizes among 4 boys   A B C D

- 1st prize 2nd 3rd
  $\underline{4} \cdot \underline{3} \cdot \underline{2} = 24$ No boy gets more than 1   } HW do it using Altern method.
- 1st 2nd 3rd
  $\underline{4} \quad \underline{4} \quad \underline{4} = 4^3$ a boy may get any # of prizes
- ? ? ?   No boy gets all prizes



Setup:
(Any no of prizes) − # (one boy gets all) = # (No boy gets all) = 60

$A^c = U - A$

$\underbrace{\qquad\qquad}_{64}$ $\qquad$ $\underbrace{\qquad\qquad}_{4}$

$$\left.\begin{array}{lll} \underline{A}\ \underline{A}\ \underline{A} & \underline{1}\cdot\underline{1}\cdot\underline{1} = 1 & \\ \underline{B}\ \underline{B}\ \underline{B} \equiv & \underline{1}\cdot\underline{1}\ \underline{1} = 1 & \text{or} \\ \underline{C}\ \underline{C}\ \underline{C} & \underline{1}\cdot\underline{1}\cdot\underline{1} = 1 & \text{or} \\ \underline{D}\ \underline{D}\ \underline{D} & \underline{1}\ \underline{1}\cdot\underline{1} = 1 & \text{or} \end{array}\right\} = 1+1+1+1$$

Q5. 5 letters, 4 letter box

$\underline{4}\cdot\underline{4}\cdot\underline{4}\cdot\underline{4}\cdot\underline{4} = 4^5$ letters { Rep. ✓

$\underline{4}\ \underline{3}\cdot\underline{2}\cdot\underline{1}\cdot\underline{0}$ { No rep ✗ / Not logical # letters > # boxes

Q6. 4 flags of diff. colors, # of diff. signals can be generated if

signal ≡ 2 flags one below the other

choices

Signal

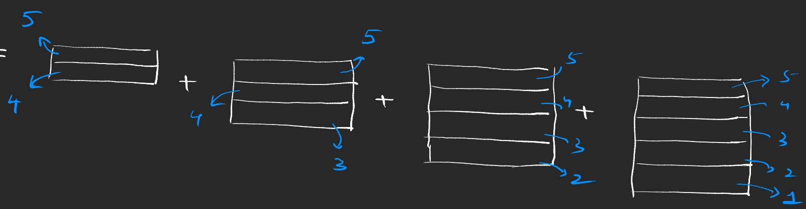
→ # choices = 4

And $\Rightarrow$ $4\times3 = 12 =$ # diff. signals

→ # choice = 3

Q7. # of diff. signals can be generated by putting at least 2 flags (one below another) if avail. flags are 5.

- # (tot signal) = # (2 flg sign.) + # (3 ,,) + # (4 ,,) + # (5 ,,) =



$= 5\cdot4 + 5\cdot4\cdot3 + 5\cdot4\cdot3\cdot2 + 5\cdot4\cdot3\cdot2\cdot1$

alt:

# of ways $= {}^5P_2 + {}^5P_3 + {}^5P_4 + {}^5P_5$

		Permutations
HW	RD	Ex – 2

Common

- Signal can be created using any # of flags

$W = {}^5P_1 + {}^5P_2 + {}^5P_3 + {}^5P_4 + {}^5P_5$

Lecture-10 (15/Aug) 2:30

# of ways to do ... = W

Q8. 3 rings, 4 fingers : atmost 1 ring in each

Method 1 ✓

$\underline{4} \cdot \underline{3} \cdot \underline{2} = 24$

Alt. A

$\frac{4 \cdot 3 \cdot 2 \cdot 1}{1} = \frac{4!}{(4-3)!} = {}^4P_3$ ← # of arrangements of 4 fingers, taken 3 at a time

Q9. 7 athletes in a race; 3 prizes

Method 1:

1st prize, 2nd, 3rd

$\underline{7} \cdot \underline{6} \cdot \underline{5} = 210$

Alt: A

$\frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4!} = {}^7P_3$ = # of arrangements of 7 athletes, taken 3 at a time

Q10. 5 Men, 4 Women seated in a row / standing in a queue

- $\underline{9} \cdot \underline{8} \cdot \underline{7} \cdot \underline{6} \cdot \underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = 9!$ {no constraint

- $\underline{\ }_5 \ \underline{4} \cdot \underline{\ }_4 \ \underline{3} \cdot \underline{\ }_3 \ \underline{2} \cdot \underline{\ }_2 \ \underline{1} \cdot \underline{\ }_1 = 4! \times 5!$ { Constraint: W on Even places

# of ways $= 4! \times 5!$

Q11. 3 Men, Wardrobe: 4 coats, 5 waist coats, 6 caps/hats

# of ways they can wear them ?

- $\underline{4} \cdot \underline{3} \cdot \underline{2}$ { Coats

and ×

$\underline{5} \cdot \underline{4} \cdot \underline{3}$ { waist coat

and ×

$\underline{6} \cdot \underline{5} \cdot \underline{4}$ { Caps

$W = (4 \cdot 3 \cdot 2) \times (5 \cdot 4 \cdot 3) \times (6 \cdot 5 \cdot 4)$

$= {}^4P_3 \times {}^5P_3 \times {}^6P_3$

Q12 chairs in a hall, 4 rows → each row has 8 chairs : 2 classes
C_1 → # students = 16
C_2 → # students = 16
} total 32

	R_1	R_2	R_3	R_4
I	C_1	C_2	C_1	C_2

Scenario - 1

OR (+)

	R_1	R_2	R_3	R_4
II	C_2	C_1	C_2	C_1

Scenario 2

# of ways stud. can be seated : In each row, all students belong to the same class & no 2 adjacent rows are allotted to the same class ?

Scenario 1

C_1 : 16 students → 8 chairs in R_1, → 8 chairs in R_3 } 16 places ⇒ $W_1 = 16!$

And ×

C_2 " → " in R_2, → " in R_4 } 16 places ⇒ $W_2 = 16!$

(+)

Scenario 2

C_1 : 16 students → " " R_2, → " " R_4 } $W_3 = 16!$

×

C_2 " " " R_1, R_3 } $W_4 = 16!$

$$W_T = (16! \times 16!) + (16! \times 16!)$$
$$= 2 \times (16!)^2$$

More kinds of Constraint (Type-2)

Q13. E Q U A T I O N

words can be formed w/ the letters of the word ?

Direct Method :

• _ _ _ _ _ _ _ _ = 8!   { No cond^n

E _ _ _ _ _ _ _
1 · 7 · 6 · 5 · 4 · 3 · 2 · 1   { word starts w/ E

E _ _ _ _ _ _ N
1 · 6 · 5 · 4 · 3 · 2 · 1 · 1   { " " w/ E, ends w/ N

sample "T" _ _ _ _ _ _ "N"
W = 3 · 6 · 5 · 4 · 3 · 2 · 1 · 2   { word begins w/ & ends w/ consonants

|||

stupid : $\boxed{QTN}$ AEIOU

A E I O U

Q T N

$$W = {}^{3}P_{2} \times {}^{6}P_{6}$$

consonant   vowels / remaining

Q14. PENCIL

- $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 6!$ {no rep^n}

- $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$ { N is always next to E
  P, EN, C, I, L

- $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$
  (+)
  $5\ 4\ 3\ 2\ 1 = 5!$

{ P, EN, C, I, L
OR(+)
P, NE, C, I, L

N & E are always together

$W = 5! + 5!$

$W = {}^5P_5 \times {}^2P_2$

Q15. TRIANGLE

- T _ _ _ _ _ _ E
  $1\ 6\ 5\ 4\ 3\ 2\ 1\ 1 = 6!$ { 1^{st} is T, last is E

|TE| |RIANGL|

$^2P_2 \times {}^6P_6$

Q16. ORDINATE

$\underline{4}\ \_\ \underline{3}\ \_\ \underline{2}\ \_\ \underline{1}\ \_ = 4!$
$\times\ 4\ \times\ 3\ \times\ 2\ \times\ 1 = 4!$

{ vowel occupy odd places
↓
OIAE

remaining : RDNT

$W = 4! \times 4!$

Q17 DELHI

$\_\ \underline{2}\ \_\ \underline{1}\ \_ = 2! \times 3!$
$3 \times 2 \times 1$

{ vowels even places

$W = 2! \times 3!$

Q18. DAUGHTER

- $8\ 7\ 6\ 5\ 4\ 3\ 2\ 1 = 8!$ {No rep^n}

- $6 \cdot 5 \cdot 4\ 3\ 2\ 1 = 6!$
  undercounting

{ vowels go together

|AEU| D, G, H, T, R # objs = 6
unit
↓ within unit : # obj = 3

$3\ 2\ 1 = 3!$

$\boxed{W = 6! \times 3!}$

$\log_{10} 10^{90} = 90$
$\log n! << n!$
Stirling's formula

- $W = \#(tot) - \#(\text{vowel togeth.})$ { vowels never go together

$= 8! - 6! \times 3! = 36000$

Q19. 9 exam papers {1, 2, 3, 4, 5 6 7 8 9}

- $\underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{\ }$ { No rup^n

$9 \cdot 8\ 7\ 6\ 5\ 4\ 3\ 2\ 1 = 9!$

{ Best & worst go together

- $\underline{\ }\ \underline{\boxed{BW}}\ \underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{\ }$

$8\ 7\ 6\ 5\ 4\ 3\ 2\ 1 = 8!$

$\boxed{B, W}$, 2, 3, 7, 5, 6, 7, 8 # obj = 8

unit

within unit : # objs = 2

$\boxed{\ \ |\ \ }$

$2\ 1 = 2!$

$W = 8! \times 2!$

- $W = 9! - (8! \times 2!)$ { Best & worst do not go together

Q20 Code word $\equiv$ 2 distinct alphabets followed by 2 distinct numbers (1 - 9)

FG79

- $\underline{F}\ \underline{X}\ \underline{1}\ \underline{9}$ { No constraint but Code word

$26 \cdot 25 \cdot 9 \cdot 8$

$^{9}P_{2}$ 9 Numbers

$^{26}P_{2}$ 26 letters

letters = 26

numbers = 9

- $\underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{\text{Even}}$ { Number ends in Even

$26 \cdot 25 \cdot 8 \cdot 4$

2 1
~~4~~ 3
6 5
8 7
9

- $\underline{\ }\ \underline{E}\ \underline{\ }\ \underline{\ }$ { Eng alph. ends ends in vowel

$25 \cdot 5\ \ 9 \cdot 8$

A E I O U $\to$ 21 Consl.

- $\underline{\ }\ \underline{\overset{\text{vowel}}{E}}\ \underline{\ }\ \underline{\overset{\text{Even}}{4}}$ { Numb ends even, letters " vowel

$25 \cdot 5\ \ 8 \cdot 4$

Q1. 10 students in a class : 7 boys , 3 G — X, Y, Z

• _ _ _ _ _ _ _ _ _ _
10 9 8 . . . . 3 2 1 = 10!   { No constr.

7 boys

• × × × × × × ×   X Y Z   { All 3 Gs are allowed to sit together

$\underbrace{\times\ \times\ \times\ \times\ \times\ \times\ \times}_{7!}$ $\underbrace{X\ Y\ Z}_{\text{allowed}}$

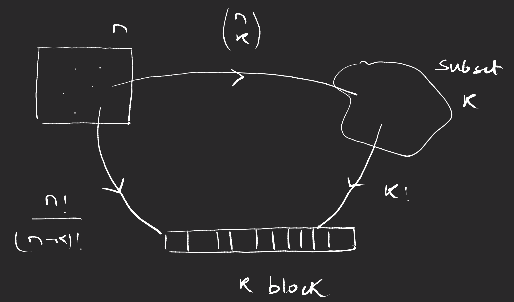
• Sample: × _ _ G 8 _ G 7 _ G 6 × × × → boys 7!   { No 2 Gs together

$W = 7! \times (8 \times 7 \times 6)$

Lecture-11 (16/Aug) 2

c. multinomial coeff / Partition Problem $\boxed{\text{Model-3}}$

*



$\binom{n}{k} \cdot k! = \frac{n!}{(n-k)!}$ Model 2

Derivation 1

* $n = n_1 + n_2 + n_3 + \dots + n_r$

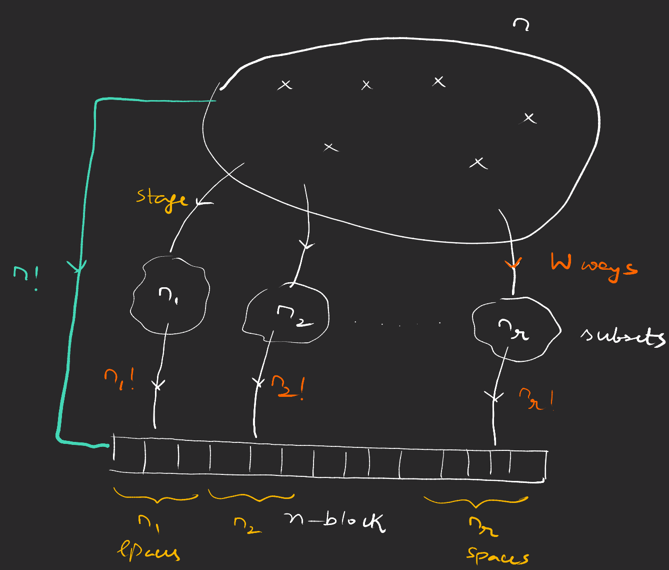
$W \cdot n_1! \, n_2! \dots n_r! = n!$

⇓

* $\boxed{W = \#\text{ of partitions} = \frac{n!}{n_1! \, n_2! \dots n_r!}}$ Multinomial Coeff

" # of ways in which n items can be partitioned into subsets "

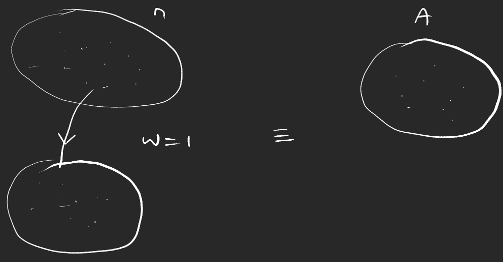
Model 3



* Another intuition : n distinct items
r persons
give n_i items to i^{th} person

Special Cases

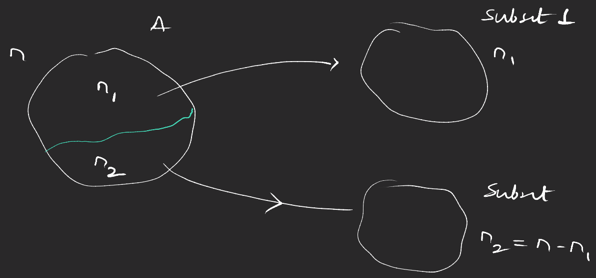
* $r=1$, $n=n_1$ $W = \frac{n!}{n_1!} = \frac{n!}{n!} = 1$



* $r=2$, $n=n_1+n_2$

$$W = \frac{n!}{n_1!\,n_2!} = \frac{n!}{n_1!\,(n-n_1)!} = \binom{n}{n_1}$$

Binomial coefficient

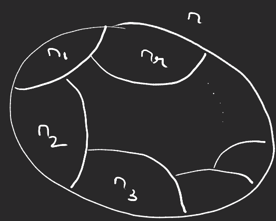


* Another derivation : • $n = n_1+n_2+\dots+n_r$

• $W = \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\cdots\binom{n-n_1-n_2\cdots(n_{r-1})}{n_r}$

$$= \frac{n!}{\cancel{(n-n_1)!}\,n_1!}\cdot\frac{\cancel{(n-n_1)!}}{\cancel{(n-n_1-n_2)!}\,n_2!}\cdot\frac{\cancel{(n-n_1-n_2)!}}{\cancel{(n-n_1-n_2-n_3)!}\,n_3!}\cdots\frac{\cancel{(n-n_1-n_2\cdots-n_{r-1})!}}{(n-n_1-n_2\cdots-n_{r-1}-n_r)!\,n_r!}$$

$$= \frac{n!}{\underbrace{(n-n_1-n_2-\cdots n_r)!}_{0}\,n_1!\,n_2!\,n_3!\cdots n_r!} = \frac{n!}{n_1!\,n_2!\cdots n_r!}$$



Practice (Type 1)

Q1. Cards # = 52

4 categories

- Red Category
  - Diamond ◇
    - 1 = A
    - 2
    - 3
    - 4
    - 5
    - 6
    - 7
    - 8
    - 9
    - 10
    - 11 = J
    - 12 = Q
    - 13 = K
    - $\underbrace{\qquad}_{13}$
  - Heart ♡
    - # = 13
- Black
  - Spade
    - 13
  - Club
    - 13

• dealt to 4 Players

Partitioning of 52 into 4 people

Given : 4 partitions of 13 cards

# of partitions of 52 cards = ?

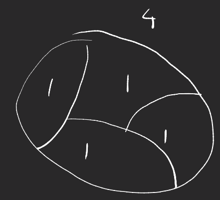
Method 1

- $W = \binom{52}{13}\binom{39}{13}\cdot\binom{26}{13}\cdot\binom{13}{13} = \frac{52!}{\cancel{39!}\,13!}\cdot\frac{\cancel{39!}}{\cancel{26!}\,13!}\cdot\frac{\cancel{26!}}{\cancel{13!}\,13!}\cdot\frac{\cancel{13!}}{0!\,13!} = \frac{52!}{13!\,13!\,13!\,13!}$

give 13 cards out of 52 to 1st person

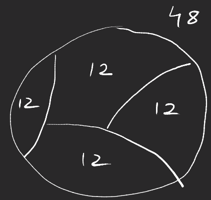
give 13 cards out of remaining 39

. . . . . .



- # ways to distribute 4 Aces = $\underline{4}\ \underline{3}\cdot\underline{2}\cdot\underline{1} = \frac{4!}{1!\,1!\,1!\,1!}$

- # ways " " 48 cards remaining = $\frac{48!}{12!\,12!\,12!\,12!}$



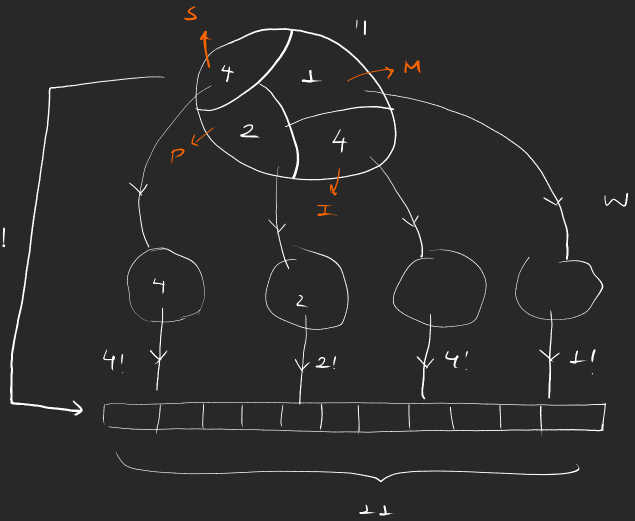
Q2. MISSISSIPPI # words that can be formed w/ letters of this word = ?

* $W = \frac{11!}{4!\,2!\,4!\,1!}$

4 S
2 P
4 I
1 M

Repetition

11!



Analysis:

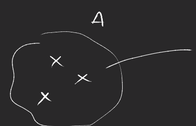
Note:

$\binom{n}{k}k! = {}^nP_k$

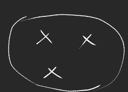
$A = \{1, 2, 3\}$ subset : $\{1,2\}, \{2,3\}, \{3,1\}$

subsets $= 2^3 = 8$

- $\binom{3}{2} = 3$ (1,2) (2,3) (3,1)



- ${}^3P_2 = 6$ (1,2) (2,3) (3,1)
  (2,1) (3,2) (1,3)



3 · 2 = 6

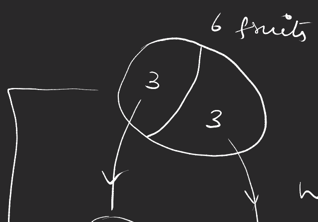
Permut^n : distinguishes
double counting

$\binom{3}{2} = \frac{{}^3P_2}{2!}$ indistinguishable

- 3 Apples & 3 oranges

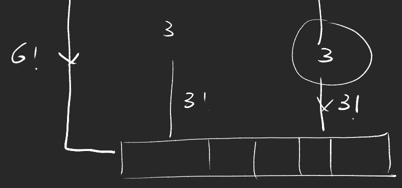
ways to arrange $= \frac{6!}{3!\,3!}$

if distinguishable = to make indistinguishable



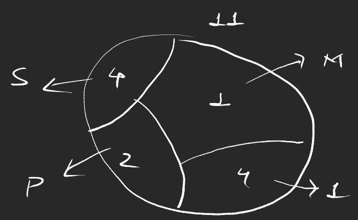
repeated element

$W = \binom{6}{3} = \dfrac{6!}{3!\,3!}$ Binomial Coeff.



$W = \binom{6}{3} \cdot \binom{3}{3} = \dfrac{6!}{3!\,3!}$

Q3. MISSISSIPPI

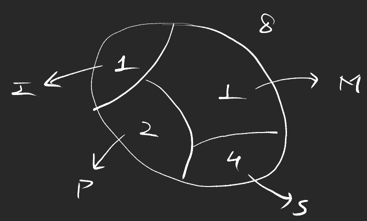


- # of partitions = # words w/ (Repetⁿ) $= \dfrac{11!}{4!\,2!\,4!\,1!}$

- # of ways $= \dfrac{8!}{2!\,4!}$ Is go together

- # (I's don't go together) = #(tot) − #(Is go togethr)

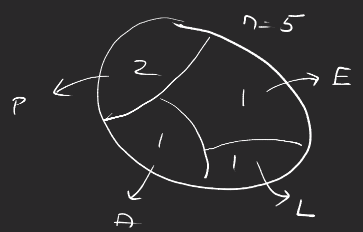
$= \dfrac{11!}{4!\,4!\,2!} - \dfrac{8!}{2!\,4!}$



Q4. APPLE

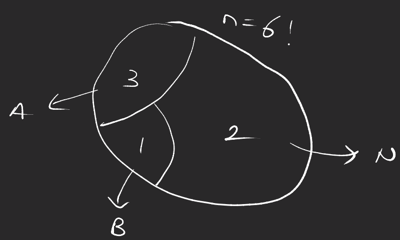
# of ways $= \dfrac{5!}{2!\,1!\,1!\,1!} = \dfrac{5!}{2!}$

Analyze !



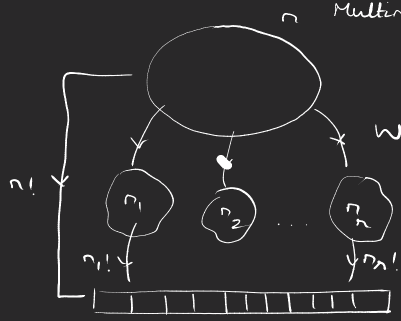
Q5 BANANA

# of ways $= \dfrac{6!}{3!\,2!}$

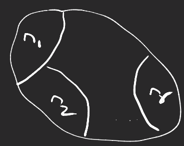


Lecture-12 (17/Aug) 2:30

Multinomial template



$W = \frac{n!}{n_1! n_2! \cdots n_r!}$ = # partition

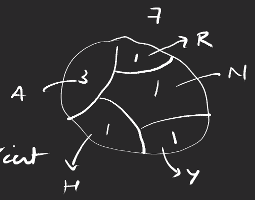


n_1 like items
n_2 like "

$W = \frac{n!}{n_1! n_2! \cdots n_r!}$

Q6. HARYANA

- # of Words = $\frac{7!}{3!}$ {No Constraint



- $\underline{H}$ _ _ _ _ _ $\underline{N}$ {Starts w/ H, Ends w/ N

$1 \cdot \frac{5!}{3!} \cdot 1$

$W = 1 \cdot \frac{5!}{3!} \cdot 1$

- $\boxed{HN}$ _ _ _ _ _ {H, N go together

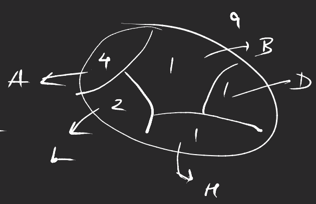
$2 \cdot 1 = 2!$

$\frac{6!}{3!}$

$W = \frac{6!}{3!} \cdot 2!$

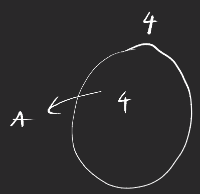
Q7. ALLAHABAD

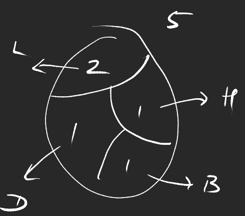
- $W = \frac{9!}{4!\,2!}$ {No constraint



vowel : A

- _ $\underset{\times}{A}$ _ $\underset{\times}{A}$ _ $\underset{\times}{A}$ _ $\underset{\times}{A}$ _ { Vowels occupying Even pos.



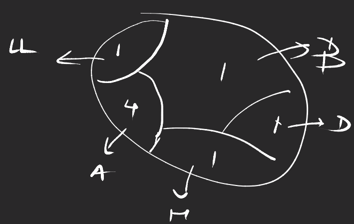


$W = \frac{4!}{4!} \times \frac{5!}{2!}$

- $\boxed{L\ L}$ _ _ _ _ _ _ _ { Ls go together

unit

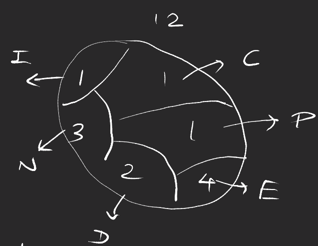
$\frac{8!}{4!}$



- $W = \frac{9!}{4!\,2!} - \frac{8!}{4!}$ {Ls don't go together

Q8. INDEPENDENCE

- $W = \frac{12!}{3!\,2!\,4!}$ {No constraint



- $\underline{P}$ _ _ _ _ _ _ _ _ _ _ _ { starts w/ P

$1 \cdot$

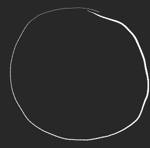
$= 1 \cdot \frac{11!}{3!\,2!\,4!}$

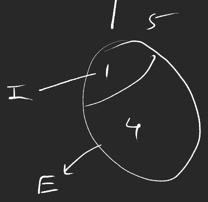
- $\boxed{\text{IEEEE}}$ _ _ _ _ _ _ _

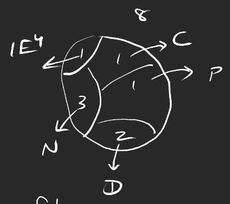
unit

{all vowels go together

vowel : I E E E E







$$\frac{5!}{4!} \times \frac{8!}{3!\,2!} = W$$

Constraint remaining

- $W = \frac{12!}{3!\,2!\,4!} - \left(\frac{5!}{4!} \times \frac{8!}{3!\,2!}\right)$

{vowell don't go together

- I _ _ _ _ _ _ _ _ _ _ _ P

1 1

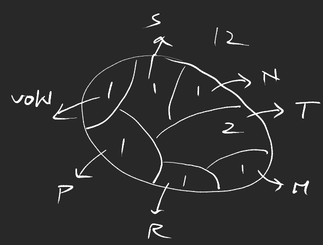
{starts I Ends P

q9. PERMUTATIONS

$\boxed{\text{AEIOU}}$ _ _ _ _ _ _ _

{all vowels together

P R M T T N S



$\frac{8!}{2!}$

5!

constraint

$$W = 5! \times \frac{8!}{2!}$$

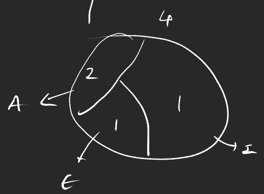
q10 MATHEMATICS

M T H M T C S

- $\boxed{\text{AEAI}}$ _ _ _ _ _ _ _

{all vowels together

$\frac{8!}{2!\,2!}$



$\frac{4!}{2!}$

$$W = \frac{4!}{2!} \times \frac{8!}{2!\,2!}$$

- C _ _ _ _ _ _ _ _ _ _

1 .

$\frac{10!}{2!\,2!\,2!}$

- T _ _ _ _ _ _ _ _ _ _

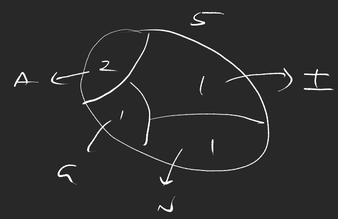
1 .

$\frac{10!}{2!\,2!}$

Q11. AGAIN — if all letters are arranged as in dictionary

AAIGN

50^{th} word = ?



A _ _ _ _

$1 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 4! = 24$

G _ _ _ _

$\underbrace{\quad\quad}_{\frac{4!}{2!} = 12}$

I _ _ _ _

$1 \cdot \underbrace{\quad\quad}_{\frac{4!}{2!} = 12}$

} 48

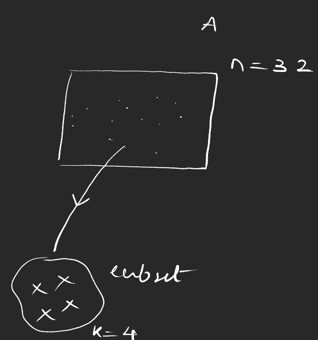
N A A G I — 49^{th} word

$\boxed{\text{N A A I G}}$ — 50^{th} word

Practice (Type-2)

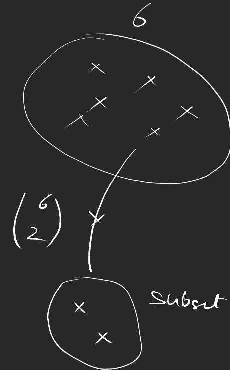
Q12. A class w/ 32 students, 4 are to be choosen for a play.

$W = \binom{32}{4}$ {no constraint



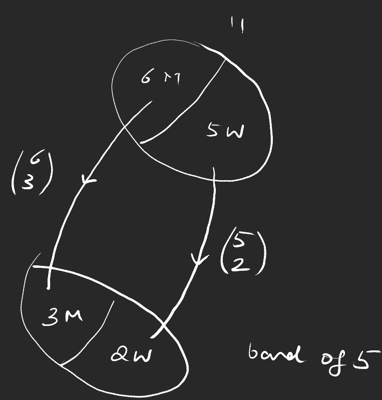
Q13. 3M, 3W ; 2 vacancies

$W = \#$ ways to fill in 2 Pos. $= \binom{6}{2} = 15$



Q14. Committe (5 members) — 3M, 2W — to be formed out of 6M, 5W

$W = \binom{6}{3} \times \binom{5}{2}$



Q15. a) out of 15 players : Team of 11 ⟶ $\# = \binom{15}{11}$

b) 15 players

$\# = \binom{14}{10}$

A 14

is always chosen

c) $\underbrace{15 \text{ players}}$

A 14

is never chosen

$\# = \binom{14}{11}$

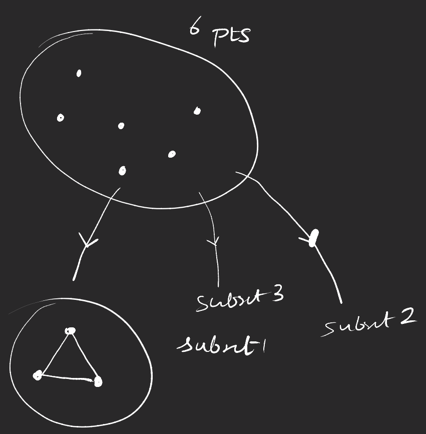
Q16. 5 posts in a company : Tot. 23 applicants
- 7 Category 1
- 16 Category 2

Company wants 2 people from Category #1 (must)

$$\# = \underset{C1}{\binom{7}{2}} \times \underset{C2}{\binom{16}{3}}$$

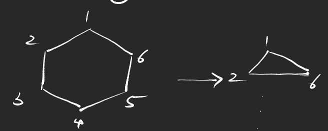
Q15.

- 6 pts in a space
- Hexagon given



$\#$ of triangle $= \binom{6}{3} = \frac{6!}{3!3!} = 20 = \#$ of subsets

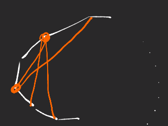
$\#$ of triangles formed by joing vertices $= \binom{6}{3} = 20$



Lecture-13 (18/Aug) 1:45

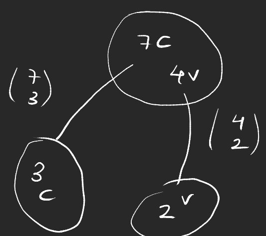
HW

Q16. Polygon w/ n sides $\#$ of diagonals are there = ?



Type 3 Combinatorics

Q17. 7 consonents, 4 vowels $\longrightarrow$ 3 conson., 2 vowels : $\#$ words that can be formed = ?
(given / initial bag)

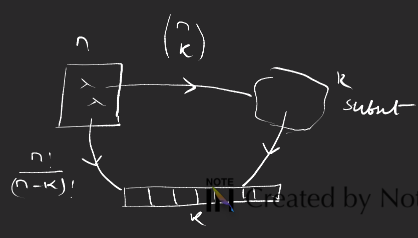


Now we have 5 distinct letters to be arranged

G T I R A

$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$

$$W = \binom{7}{3} \times \binom{4}{2} \times 5!$$



spaces

Q18. FAILURE # of 4 letter words = ?

doubt

- $\underline{L}\ \underline{F}\ \underline{E}\ \underline{U}$
  4 3 2 1

  { F is included in each word

  ~~F~~ A
  L I
  R U
  E

  $\binom{6}{3} \times 4! \longrightarrow$ 4 letters into 4 spaces

  ∵ F is already the part, only 3 options/letters are left to be chosen out of 6

- _ _ _ _

  { F is never included

  L A
  R I
  U
  E

  $W = \binom{6}{4} \times 4!$

  out of 6
  pick 4

- _ F _

  { F is incl

  ~~F~~ A
  L I
  R U
  E

  $\binom{6}{2} \times 3!$

Q19. EQUATION # of words

E
U Q
A T
I N
O

- $\boxed{AEIOU}$ $\boxed{QTN}$
  unit unit

  { V & consonants go together

  5! _ _ _ _ _ _ _ _ 3!

  $W = 2! \times 5! \times 3!$
  V & C together, V, C

- # of 5 letter words w/ 3 V & 2 C : 2 C go together = ?

  E
  U Q
  A T
  I N
  O
  5 V 3 C

  $\binom{5}{3}$ 3 V

  $\binom{3}{2}$ 2C

  example $\boxed{Q\ T}$ _ _ _
  unit

  $\binom{3}{2} \times \binom{5}{3} \times 4! \times 2!$
  Cons vowels QT as unit; _ _ 2 . 1 for QT

Q20. DAUGHTER

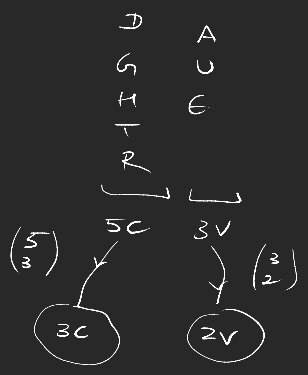
try it!

# of words w/ 2V & 3C

⇓

5 letter word

_ _ _ _ _



$W = \binom{5}{3} \times \binom{3}{2} \times 5!$

C & V &

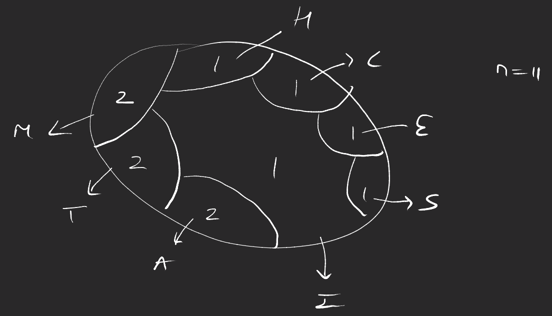
1st stage, 2nd, 3rd

Lecture-14 (19/Aug) 1.5 + 2.5 [Contd. in # theory vol-4]

Q21. MATHEMATICS

# of 4 letter words built out of this word = ?

_ _ _ _



① All distinct

OR +

② 2 distinct, 2 alike

OR +

③ 2 alike of 1 kind, " " 2 kind

H C T S

S C $\overbrace{TT}$

T T A A

n=8

M H C E S T A I

$\binom{8}{4}$

M 4 ATS

$W_1 = \binom{8}{4} \times 4!$

MATS
ATMS
MSAT

MM ~~TT~~ AA rep. less here

$\binom{3}{1}$

TT

diff. ltrs

HCESI
M, A

$W_2 = \binom{3}{1} \times \binom{7}{2} \times \frac{4!}{2!}$

alike remaining (different)

SCT_1T_2
TTSC
CSTT

rep.

TT AA MM

singular units

TT AA

separate entities

$W_3 = \binom{3}{2} \times \frac{4!}{2!\,2!}$

TTAA
AATT
ATAT
...

$\boxed{W = W_1 + W_2 + W_3}$

HW

Q22. INEFFECTIVE # of 4-letter words = ?

Q23. PERMUTATIONS # of words : there are always 4 letters b/w P & S ?

Q24. INVOLUTE # words formed : there are 3 vowels, 2 consonants ?

Syllabus

1. Functional eq^n
2. Intro. Graph theory
3. Diophantine eq^n