Lecture-20 (16/dec)

1c Log Maps (Contd)

* $\boxed{N = a^x \iff x = \log_a N}$ "the most fundamental Eqⁿ" ∃ a dictionary to translate b/w powers & log
* log scale ⇒ spacing ≠ invariant ⇒ non-linearity
* $N = a^x$, $x = \log_a N \Rightarrow \boxed{N = a^{(\log_a N)}}$ (I) representation of any # in terms of log

Ex 1 find the log of $32\sqrt[5]{4}$ to the base $2\sqrt{2}$, using the defⁿ only.

* $\log_{2\sqrt{2}}(32\sqrt[5]{4}) = x \iff \underbrace{(2\sqrt{2})}^x = 32\sqrt[5]{4} \Rightarrow 2^{\frac{3}{2}x} = 2^5\, 2^{2/5} \Rightarrow \frac{3}{2}x = \frac{27}{5} \Rightarrow x = \frac{18}{5}$

Properties of Log (intro)

* $a^0 = 1 \Rightarrow \log_a 1 = 0$ II
* $a^1 = a \Rightarrow \log_a a = 1$ III
* $M = a^x$, $N = a^y \Rightarrow \underbrace{MN} = a^x \cdot a^y = a^{x+y} \iff \boxed{\log_a MN = x+y = \log_a M + \log_a N}$ IV

$\Updownarrow$ $\log_a M = x$, $\log_a N = y$

$\log(\underbrace{MNLP}) = \log M + \log N + $

ex

$\log 42 = \log(2\times3\times7) = \log 2 + \log 3 + \log 7$

* $M = a^x$, $N = a^y \Rightarrow \frac{M}{N} = \frac{a^x}{a^y} = a^{x-y} \iff \boxed{\log_a \frac{M}{N} = x-y = \log_a M - \log_a N}$ V

$\Downarrow$ $\log_a M = x$, $\log_a N = y$

ex

$\log\left(4\tfrac{2}{7}\right) = \log\left(\frac{30}{7}\right) = \log 30 - \log 7 = \log(2\times3\times5) - \log 7 = \log 2 + \log 3 + \log 5 - \log 7$

* $M = a^x \rightarrow \underbrace{M^p} = (a^x)^p = a^{xp} \iff \log_a M^p = xp \Rightarrow \boxed{\log_a M^p = p\log_a M}$ VI

$\Updownarrow$ $\log_a M = x$

"log of p^{th} power of a # is p times log(#)"

* $\log_a M^{\frac{1}{n}} = \frac{1}{n}\log_a M$ "log of n^{th} root of a # is $\frac{1}{n}$ times log(#)"

Ex 2 $\log \frac{\sqrt{a^3}}{c^5 b^2} = \frac{3}{2}\log a - 5\log c - 2\log b$

Ex 3 $a^x c^{-2x} = b^{3x+1}$, $x = ?$ $\longrightarrow x = \dfrac{\log b}{\log a - 2\log c - 3\log b}$

Common Logarithms ≡ log to the base 10 {Briggs, 1615 / Napier → e} (Later: log table)

* $10^x = N \iff \log 10^x = \log N \Rightarrow x \underbrace{\log_{10} 10}_{1} = \log_{10} N \Rightarrow x = \log_{10} N$
* $\underbrace{10^3 < 7154 < 10^4}$ $\longrightarrow 7154 > 10^3 \Rightarrow \log_{10} 7154 = 3 + \text{a fraction}$

$\downarrow$

$3 < \log 7154 < 4$

* $10^{-2} < 0.06 < 10^{-1} \rightarrow \log 0.06 = -2 + \text{a fraction}$

$\downarrow$

$-2 < \log 0.06 < -1$

Integer value ≡ characteristic of log

decimal part ≡ Mantissa of log

* N = # whose integer part contains n digits

$\boxed{N = 10^{(n-1)} + \text{residue}}$ $\Rightarrow \log_{10} N = \underbrace{(n-1)}_{\text{characteristic}} + \text{a fract}$

$329 = 10^2 + \underbrace{229}$

$\log_{10} 329 = 2 + \text{a fraction}$

* $N = b^y \Rightarrow \log_a N = \log_a b^y \Rightarrow \log_a N = y \log_a b \Rightarrow y = \dfrac{\log_a N}{\log_a b} = \log_b N \Rightarrow \boxed{\log_b N = \dfrac{\log_a N}{\log_a b}}$ VII

$\Downarrow$

$\log_b N = y$

Lecture-21 (18/dec) 2

* $a^x = N \Leftrightarrow \log_a N = x$ $\quad a > 0,\ a \neq 1,\ N > 0 \ \} \in \mathbb{R}$

* $N = a^{\log_a N}$ log representation of any #

* $\log_a (MN) = \log M + \log N$, $\log_a \dfrac{M}{N} = \log M - \log N$, $\log_a M^\alpha = \alpha \log_a M$ ✓

* $\log_b M = \dfrac{\log_a M}{\log_a b}$ Base change thm (base a ⟷ base b)

* $\log_{a^\beta} M = \dfrac{\log_a M}{\log_a a^\beta} = \dfrac{\log_a M}{\beta \cancel{\log_a a}} = \dfrac{1}{\beta} \log_a M$

Most Imp Identities

Remarks :

* $\log_a 1 = 0$, $\log_a a = 1$, $\log_{1/a} a = -1$

* $\log_b a = \dfrac{\log_a a}{\log_a b} = \dfrac{1}{\log_a b} \Rightarrow \log_b a \cdot \log_a b = 1$

$\log_b a \ \log_c b \cdot \log_a c = 1$

* $a^x = e^{\ln a^x} = e^{x \ln a}$, $x = e^{\ln x}$, $\pi = e^{\ln \pi}$, $e = e^{\ln e}$

$\log_e x \equiv \ln x$

Natural log

* $\log_{10} 2 = 0.3010$, $\log_{10} 3 = 0.4771$, $\ln 2 = 0.693$, $\ln 10 = 2.303$ (ETM)

$\boxed{\text{Practice 1}}$ Compute/Convert into exponents (Easy type)

Q1 $\dfrac{1}{\log_2 36} + \dfrac{1}{\log_3 36} = \log_{36} 2 + \log_{36} 3 = \log_{36} 6 = \log_{6^2} 6 = \frac{1}{2} \log_6 6 = \frac{1}{2}$

Q2 $\log_{ab} \dfrac{\sqrt[3]{a}}{\sqrt{b}} = ?$ if $\log_{ab} a = 4$

Trick $\underbrace{\log_{ab} ab}_{1} = \underbrace{\log_{ab} a}_{4} + \log_{ab} b$

* $\log_{ab} \dfrac{\sqrt[3]{a}}{\sqrt{b}} = \log_{ab} a^{1/3} - \log_{ab} b^{1/2} = \frac{1}{3} \overbrace{\log_{ab} a}^{4} - \frac{1}{2} \log_{ab} b = \frac{4}{3} - \frac{1}{2} \overbrace{\log_{ab} b}^{-3} = \frac{4}{3} + \frac{3}{2} = \frac{17}{6}$

Q3 $\alpha = \sqrt{\left(\frac{1}{\sqrt{27}}\right)^{2-\frac{\log_5 13}{2\log_5 9}}} = \sqrt{\left(\frac{1}{\sqrt{27}}\right)^{2-\frac{1}{2}\log_9 13}} = \sqrt{\left(\frac{1}{\sqrt{27}}\right)^{2}\cdot\left(\frac{1}{\sqrt{27}}\right)^{-\frac{1}{2}\log_9 13}} = \sqrt{(3^{-3})\cdot\left(3^{\frac{-3}{2}}\right)^{\frac{-1}{2}\log_9 13}}$

$= \sqrt{(3^{-3})\; 3^{\frac{3}{4}\log_{3^2} 13}} = \sqrt{(3^{-3})\; 3^{\frac{3}{4}\frac{1}{2}\log_3 13}} = \sqrt{(3^{-3})\; 3^{\frac{3}{8}\log_3 13}}$

$= \sqrt{(3^{-3})\; 3^{\log_3 (13)^{3/8}}} = \sqrt{(3^{-3})(13)^{3/8}} = 3^{-3/2}\cdot 13^{3/16}$ { HW Redo it

nw

Q4 $\log_2 a = s$, $\log_4 b = s^2$, $\log_{c^2} 8 = \frac{2}{s^3+1}$, $\log_2 \frac{a^2 b^5}{c^4} = ?$ s indep parameter

Lecture 22 (19/dec) 15

nw

Q4 $\log_2 a = s$, $\log_4 b = s^2$, $\log_{c^2} 8 = \frac{2}{s^3+1}$, $\log_2 \frac{a^2 b^5}{c^4} = ?$ s indep parameter

$\log_2 \frac{a^2 b^3}{c^4} = -3s^3 + 10s^2 + 2s - 3$

Q5 $\underbrace{\alpha = \log_6 15 \;,\; \beta = \log_{12} 18}_{\text{given}}$, $\log_{25} 24 = x = ?$ 8/10

log Eq^n
$x = y \iff \log_a x = \log_a y$

$\log_{a^b} M^p = \frac{p}{b}\log_a M$ ✓

$\log_a b = \frac{\log_c b}{\log_c a}$ ✓

* $x = \log_{25} 24 = \log_{5^2}(3\cdot 2^3) = \frac{1}{2}\left\{\log_5 3 + 3\log_5 2\right\}$ —— ①

* $\alpha = \log_6 15 = \log_6(3\cdot 5) = \log_6 3 + \log_6 5 = \frac{\log_3 3}{\log_3 6} + \frac{\log_3 5}{\log_3 6} = \frac{1}{\log_3 6} + \frac{\log_3 5}{\log_3 6}$

$= \frac{1+\log_3 5}{\log_3 6} = \frac{1+\log_3 5}{\log_3 3 + \log_3 2} = \frac{1+\log_3 5}{1+\log_3 2}$

* $\beta = \log_{12} 18 = \log_{12}(2\cdot 3^2) = \log_{12} 2 + 2\log_{12} 3 = \frac{\log_3 2}{\log_3 12} + \frac{2\log_3 3}{\log_3 12} = \frac{\log_3 2 + 2}{\log_3(3\,2^2)} = \frac{2+\log_3 2}{1+2\log_3 2}$

* $\alpha = \frac{1+\log_3 5}{1+\log_3 2} = \frac{1+A}{1+B}$

$A \equiv \log_3 5$
$B \equiv \log_3 2$

$\beta = \frac{2+\log_3 2}{1+2\log_3 2} = \frac{2+B}{1+2B} \Rightarrow \beta(1+2B) = (2+B) \Rightarrow \beta + 2B\beta = 2+B$

$\frac{\beta-2}{1-2\beta} = B \Rightarrow \boxed{B = \frac{\beta-2}{1-2\beta}} = \log_3 2$ —— ②

$\alpha = \frac{1+A}{1+B} = \frac{1+A}{1+\frac{\beta-2}{1-2\beta}} = \frac{(1+A)(1-2\beta)}{1-2\beta+\beta-2} = \frac{(1+A)(1-2\beta)}{-\beta-1} \Rightarrow \frac{\alpha(-\beta-1)}{1-2\beta} = 1+A$

$A = \frac{-\alpha\beta-\alpha}{1-2\beta} - 1 = \frac{-\alpha\beta-\alpha-1+2\beta}{1-2\beta} = \frac{2\beta-\alpha\beta-\alpha-1}{1-2\beta} \equiv \log_3 5$ —— ③

* $x = \log_{25} 24 = \log_{5^2}(3 \cdot 2^3) = \frac{1}{2}\left\{ \log_5 3 + 3\log_5 2 \right\}$ —— ①

$$= \frac{1}{2}\left\{ \frac{1}{\log_3 5} + 3\frac{\log_3 2}{\log_3 5} \right\} = \frac{1}{2}\left\{ \frac{1}{A} + \frac{3B}{A} \right\} = \frac{1+3B}{2A}$$

$$= \frac{1 + 3\left(\frac{\beta-2}{1-2\beta}\right)}{2\left(\frac{2\beta-\alpha\beta-\alpha-1}{1-2\beta}\right)} = \frac{1-2\beta+3\beta-6}{4\beta-2\alpha\beta-2\alpha-2} = \frac{\beta-5}{4\beta-2\alpha\beta-2\alpha-2} = \frac{5-\beta}{2\alpha+2\alpha\beta-4\beta+2} \quad \square$$

f6 $\log 25 = a$, $\log 225 = b$; $\log\left(\left(\frac{1}{9}\right)^2\right) + \log\left(\frac{1}{2250}\right) = ?$ Base is 10

* $\log 9^{-2} + \log 2250^{-1} = -2\log 3^2 - \underbrace{\log 2250}_{\log(225 \cdot 10)} = -4\log_{10} 3 - \{\log 225 + \underbrace{\log_{10} 10}_{1}\} = -4\log 3 - \log 225 - 1$

$= -4\log 3 - b - 1$

Trick

* $\log 225 = b \Rightarrow \log(25 \cdot 9) = b \Rightarrow \underbrace{\log 25}_{a} + 2\log 3 = b \Rightarrow \log 3 = \frac{b-a}{2}$

$-4\left(\frac{b-a}{2}\right) - b - 1$

$= -3b + 2a - 1 = 2a - 3b - 1$

f7 $\log_{12} 27 = a$, $\log_6 16 = ?$

* $x = \log_6 16 = \log_6 2^4 = 4\log_6 2 = 4\frac{\log_2 2}{\log_2 6} = \frac{4}{\log_2 6} = \frac{4}{\log 2 + \log 3} = \frac{4}{1+\log_2 3}$ —— ①

* $\log_{12} 27 = \log_{12} 3^3 = 3\log_{12} 3 = \frac{3}{\log_3 12} = \frac{3}{\log_3 3 + 2\log_3 2} = \frac{3}{1+2\log_3 2} = \frac{3}{1+\frac{2}{\log_2 3}} = a$

$\frac{3\log_2 3}{2+\log_2 3} = a \Rightarrow 3\log 3 = 2a + a\log 3 \Rightarrow (3-a)\log 3 = 2a \Rightarrow \log 3 = \frac{2a}{3-a}$

* $x = \frac{4}{1+\frac{2a}{3-a}} = \frac{4(3-a)}{3+a}$ ⊠

Lecture-23 (2/Jan/2024)

Q8 $\log_2(\log_2(\log_3 x)) = \log_2(\log_3(\log_2 y)) = 0$, $x+y=?$

$$\left.\begin{matrix} x = 9 \\ y = 8 \end{matrix}\right\} \quad x+y=17$$

Q9 $$\frac{1}{\underbrace{\log_{\sqrt{bc}} abc}_{\log_{abc}\sqrt{bc}}} + \frac{1}{\log_{\sqrt{ca}} abc} + \frac{1}{\log_{\sqrt{ab}} abc} = ? = 1$$

Q Same type of questions

$$\frac{1}{\log_b abc} + \frac{1}{\log_c abc} + \frac{1}{\log_a abc} = ? = 1$$

Q10 $\log_x(\log_8(\sqrt{2}+\sqrt{8})) = \frac{1}{3} \Rightarrow x = \frac{1}{8}$

Q11 $$\frac{2^{\log_{2^{1/4}} a} - 3^{\log_{27}(a^2+1)^3} - 2a}{7^{4\log_{49} a} - a - 1} = ?$$

Simplify / Is it linear or quadratic

$$\frac{a^4 - \overbrace{a^2 - 1}^{2} - 2a}{a^2 - a - 1} = a^2 + a + 1$$

$x = a^{\log_a x}$

$\log x^n = n \log x$

$\log_{x^n} x = \frac{1}{n}\log x$

Q12 $a = \log_{12} 27$ (given) , $\log_6 16 = ?$ $\frac{a+3}{4(3-a)}$

* $a = \log_{12} 3^3 = 3\log_{12} 3 = \frac{3}{\log_3 12} = \frac{3}{\log_3 3 + 2\log_3 2} = \frac{3}{1+2\log_3 2}$ — ①

$\log_m ab = \log_m a + \log_m b$

$\log_{ab} m$

* $\log_6 16 = \log_6 2^4 = 4\log_6 2 = \frac{4}{\log_2 6} = \frac{4}{\log_2 3 + \log_2 2} = \frac{4}{1+\log_2 3}$

* $a = \frac{3}{1+\frac{2}{\log_2 3}} \Rightarrow a = \frac{3\log_2 3}{2+\log_2 3} \Rightarrow \log_2 3 = \frac{2a}{3-a}$

$\log_6 16 = \frac{4}{1+\frac{2a}{3-a}} = \frac{4(3-a)}{a+3}$

Q13 $N = \frac{\log_3 135}{\log_{15} 3} - \frac{\log_3 5}{\log_{405} 3}$

Comment on what type of # is it?

- Integer

$$N = \frac{\log_3 135}{\frac{1}{\log_3 15}} - \frac{\log_3 5}{\frac{1}{\log_3 405}} = \log_3 135 \log_3 15 - \log_3 5 \log_3 405 = \log_3 135 \log_3 5\cdot 3 - \log_3 5 \overbrace{\log_3(3\cdot 135)}^{\log 3 + \log 135}$$

$$N = \log 135(\cancel{\log 5}+\log 3) \cdot \log 5(\log 3 + \cancel{\log 135}) = \log 3 \log 135 - \log 3 \log 5 = \log 135 - \log 5 = \log \frac{135}{5} = \log_3 27 = 3$$

Lecture-24 (7/Jan/2024) '5

Q14 $\log_3 x + \log_3 y = 2 + \log_3 2$
$\log_{27}(x+y) = \frac{2}{3}$
} Sol^n set of Eq^{n}s ?

* $(x, y) = (6, 3)$

Q15 $\log_{x^2} 16 + \log_{2x} 64 = 3$ How many roots ? , $x = ?$

* $\log_{x^2} 2^4 + \log_{2x} 2^6 = 3 \Rightarrow 4\log_{x^2} 2 + 6\log_{2x} 2 = 3 \Rightarrow 4\frac{\log_x 2}{\log_x x^2} + 6\frac{\log_x 2}{\log_x 2x} = 3$

$\frac{\cancel{4}^{2} \log_x 2}{\cancel{2} \log_x x} + \frac{6\log_x 2}{\log_x 2 + \log_x x} = 3 \Rightarrow 2\log_x 2 + \frac{6\log_x 2}{(1 + \log_x 2)} = 3$

$2\log_x 2 + 2(\log_x 2)^2 + 6\log_x 2 - (3 + 3\log_x 2) = 0$

$2(\log_x 2)^2 + 5\log_x 2 - 3 = 0 \Rightarrow \log_x 2 = \frac{-5 \pm \sqrt{25+24}}{4} = \frac{-5 \pm \sqrt{49}}{4}$ $\to \frac{-5+7}{4} = \frac{1}{2}$, $\to \frac{-5-7}{4} = \frac{-12}{4} = -3$

$\log_x 2 = \frac{1}{2} \Rightarrow x^{1/2} = 2 \Rightarrow x = 4$ Integer

$\log_x 2 = -3 \Rightarrow x^{-3} = 2 \Rightarrow x = 2^{-1/3}$ irrational

Q16 $x^{\{(\log_3 x)^2 - \frac{9}{2}\log_3 x + 5\}} = 3\sqrt{3}$ How many roots ? $x = ?$

$(2\alpha^2 - 7\alpha + 3)(\alpha - 1) = 0$ $\alpha = \log_3 x$ $\to 1, \to 3, \to \frac{1}{2}$ $\Rightarrow x$ $\to 3, \to 27, \to \sqrt{3}$ } Exactly 3 real roots

Q17 If $4^A + 9^B = 10^C$, $A = \log_{16} 4$, $B = \log_3 9$, $C = \log_x 83$ $x = ?$

$83 = 10^{\log_x 83} \Rightarrow x = 10$

Q18 $\log_b a \log_c a + \log_a b \underbrace{\log_c b} + \log_a c \underbrace{\log_b c} = 3$ relation b/w $a, b, c = ?$

$\frac{1}{\log_a b \log_a c} + \log_a b \frac{\log_a b}{\log_a c} + \log_a c \frac{\log_a c}{\log_a b}$

$\frac{1}{\log_a b \log_a c} + \frac{(\log_a b)^2}{\log_a c} + \frac{(\log_a c)^2}{\log_a b} = 3$

$\log_a a = \log_b b = 1$

$$\frac{1+(\log_a b)^3+(\log_a c)^3}{(\log_a b\ \log_a c)} = 3 \Rightarrow 1^3+(\log b)^3+(\log c)^3 - 3(1)(\log b)(\log c) = 0$$

$\{x^3+y^3+z^3-3xyz=0 \iff x+y+z=0$ or $x=y=z\}$

$$\log_a a + \log_a b + \log_a c = 0 \Rightarrow \log_a abc = 0 \Rightarrow a^0 = abc \Rightarrow \boxed{abc=1}$$

HW

Q19 $a=\log_{12} 18$, $b=\log_{24} 54$ $\qquad ab+5(a-b)=?$

Lecture-25 (8 Jan)

Q19. $$a=\log_{12}18 = \frac{\log_2 18}{\log_2 12} = \frac{\log_2 2\cdot 3^2}{\log_2 2^2 3} = \frac{1+2\log_2 3}{2+\log_2 3} \equiv \frac{1+2x}{2+x}$$

$$b=\log_{24}54 = \frac{\log_2 54}{\log_2 24} = \frac{\log_2 2\cdot 3^3}{\log_2 2^3 3} = \frac{1+3\log_2 3}{3+\log_2 3} \equiv \frac{1+3x}{3+x}$$

$$* \quad ab+5(a-b) = \left(\frac{1+2x}{2+x}\right)\left(\frac{1+3x}{3+x}\right) + 5\left(\frac{1+2x}{2+x} - \frac{1+3x}{3+x}\right) = \frac{(1+2x)(1+3x)+5\{\overbrace{(1+2x)(3+x)-(1+3x)(2+x)}^{3+\cancel{x}+\cancel{6x}+2x^2-2-\cancel{x}-\cancel{6x}-3x^2}\}}{(2+x)(3+x)}$$

$$= \frac{1+3x+2x+6x^2+5(1-x^2)}{(x+2)(x+3)} = \frac{6x^2+5x+1+5-5x^2}{(x+2)(x+3)} = \frac{\cancel{x^2+5x+6}}{\cancel{(x+2)(x+3)}} = 1$$

Q20 $$\frac{\log_\pi a}{b-c} = \frac{\log_\pi b}{c-a} = \frac{\log_\pi c}{a-b} \equiv \lambda \qquad a^a b^b c^c = ?$$

$*$ $\log_\pi a = \lambda(b-c) \Rightarrow a = \pi^{\lambda(b-c)}$

$\log_\pi b = \lambda(c-a) \Rightarrow b = \pi^{\lambda(c-a)}$

$\log_\pi c = \lambda(a-b) \Rightarrow c = \pi^{\lambda(a-b)}$

$$\longrightarrow a^a b^b c^c = \pi^{\lambda(\cancel{ab}-\cancel{ac}+\cancel{bc}-\cancel{ba}+\cancel{ac}-\cancel{bc})} = \pi^0 = 1$$

$\frac{x}{y} = \frac{z}{w} = \frac{l}{m} = k$

Q21 a) $\log_2 3 \ \boxed{>}\ \underbrace{\log_{1/2} 5}$ $\qquad \{>, <, =, \sim\}$

$\log_{2^{-1}} 5 = -1\log_2 5$

b) $\log_7 11 \ \boxed{>}\ \log_8 5$

$\underbrace{\log_7 11}_{x} > 1 \qquad \underbrace{\log_8 5}_{y} = \frac{1}{\log_5 8} < 1$

$x \overset{>}{\underset{<}{=}} y \iff \log_a x \overset{>}{\underset{<}{=}} \log_a y$

$3<5 \Rightarrow \log 3 < \log 5$

$3>-5 \Rightarrow \log 3 > -\log_2 5$

{Later characteristics / mantissa

Trivial type

Q22 a. $\log_{10}(x^2-12x+36)=2 \Rightarrow x=-4, 16$

b. $\log_4(\log_3(\log_2 x))=0 \Rightarrow x=8$

c. $\log_3(\log_9 x+\frac{1}{2}+9^x)=2x \Rightarrow x=1/3$

Q23 $2\log_4(4-x)=4-\log_2(-2-x)$

$2\log_{2^2}(4-x)+\log_2(-2-x)=4 \Rightarrow \cancel{2}\frac{1}{\cancel{2}}\log_2(4-x)+\log_2(-2-x)=4 \Rightarrow \log_2(4-x)(-2-x)=4$

$(4-x)(-2-x)=2^4=16 \Rightarrow \boxed{x=6,\ x=-4}$

* $4-x>0 \Rightarrow 4>x \Rightarrow x<4 \Rightarrow x\neq 6 \Rightarrow x=-4$ ✓

$-2-x>0 \Rightarrow -x>2 \Rightarrow x<-2$

Q24 $\log^2_{10} x+\log_{10} x^2=\log^2_{10} 2-1$

$(\log x)^2+2\log x+1=(\log_{10} 2)^2$

$(\log x+1)^2=(\log_{10} 2)^2 \Rightarrow \log x+1=\log_{10} 2 \Rightarrow x=1/5$

$\log(x^2)\neq(\log x)^2$

$\sin x^2\neq(\ln x)^2$

$(\log x)^2=\log^2 x$

HW

Q25 $x^{\frac{\log x+5}{3}}=10^{5+\log x}$

Q26 $(\log_5 x)^2+\log_{5x}\left(\frac{5}{x}\right)=1$

Q27 $\log_4(\log_2 x)+\log_2(\log_4 x)=2$

lecture-26 (12/Jan/24) 2

Q25 $x^{\frac{\log x+5}{3}} = 10^{5+\log x} \Rightarrow x^{\log x+5} = \underbrace{10^{3(5+\log x)}}_{1000^{5+\log x}} \Rightarrow x^{\log x+5} = 1000^{\log x+5}$

$\Downarrow$

$x = 10^3$ or $\log x + 5 = 0$

$\Downarrow$

$\log_{10} x = -5 \Rightarrow x = 10^{-5}$

$x^{1/3} = A$

$\Downarrow$

$x = A^3$

Q26 $(\log_5 x)^2 + \log_{5x}\left(\frac{5}{x}\right) = 1$, $\log_5 x \equiv \alpha$

* $\alpha^2 + \frac{\log_5\left(\frac{5}{x}\right)}{\log_5 5x} = 1 \Rightarrow \frac{\alpha^2}{1} + \frac{1-\alpha}{1+\alpha} = 1 \Rightarrow \alpha^3 + \alpha^2 + \cancel{1} - \alpha = \cancel{1} + \alpha \Rightarrow \alpha^3 + \alpha^2 - 2\alpha = 0$

$\alpha(\alpha^2 + \alpha - 1) = 0$

$\left.\begin{array}{l} \alpha = 0 \Rightarrow \log_5 x = 0 \Rightarrow x = 5^0 \\ \alpha = 1 \quad \rightarrow x = 5^1 \\ \alpha = -2 \quad \rightarrow x = 5^{-2} \end{array}\right\}$

Q27 $\log_4(\log_2 x) + \log_2(\log_4 x) = 2$

$\log_{2^2}(\log_2 x) + \log_2(\log_{2^2} x) = 2 \longrightarrow x = 2^4 = 16$

Q28 $2\log_2(\log_2 x) + \log_{1/2}(\log_2 2\sqrt{2}x) = 1$

$\log_2 x = 3$ $\quad$ $\log_2 x = -1$

$\Downarrow$ $\quad$ $\Downarrow$

$x = 2^3$ $\quad$ $x = 2^{-1}$

$\left\{ \because \log_2 x > 0 \right\}$

note

$y = \log_a x$ $\quad$ $x > 0$

(check this and all the time)

Q29 a) $9^{1+\log_3 x} - 3^{1+\log_3 x} = 210 \longrightarrow x = 5$

b) $\log_{1/3} \sqrt[4]{729 \sqrt[3]{9^{-1}\, 27^{-4/3}}} = ?$

$\log_{1/3} \sqrt[4]{729 \sqrt[3]{3^{-2}(3^3)^{-4/3}}} = \log_{1/3} \sqrt[4]{729 \sqrt[3]{3^{-6}}} = \log_{1/3} \sqrt[4]{729\; 3^{-2}} = \log_{1/3}(3^6\, 3^{-2})^{1/4}$

$= \log_{1/3} 9^{1/2} = -1$

Q30 $\ln\left(\log_5\left(\sqrt{2x-2}+3\right)\right) = 0 \longrightarrow x = 3$

Revision (Pre-test)

Q1 $\log_{100}|x+y| = \frac{1}{2}$

$\log_{10} y - \log_{10}|x| = \log_{100} 4$ $\Big\}$ $(x,y)=?$ $\quad (x,y) = \left\{ \left(\frac{10}{3}, \frac{20}{3}\right), (-10, 20) \right.$

Q2 $|x-1|^{\Delta} = (x-1)^7$ $\quad \Delta \equiv \log_3 x^2 - 2\log_x 9$ $\quad x = ?$

Q3 If $x = \log_3 4$, $y = \log_5 3$

a) $\log_3 10 = ?$

b) $\log_3(1.2) = ?$ $\Big\}$ in terms of x & y

decimal pt

a. $\frac{xy+2}{2y}$

b. $\frac{xy+2y-2}{2y}$

Q4 $\frac{2}{\log_4 (2000)^6} + \frac{3}{\log_5 (2000)^6} = ? = \frac{1}{6}$

Lecture-27 (13/Jan/2024) 2 10

Q31 $4^{\{5\log_{4\sqrt{2}}(3-\sqrt{6}) - 6\log_8(\sqrt{3}-\sqrt{2})\}} = ? = 9$

Q32 $\frac{81^{\frac{1}{\log_5 9}} + 3^{\frac{3}{\log_{\sqrt{6}} 3}}}{409}\left\{(\sqrt{7})^{\frac{2}{\log_{25} 7}} - (125)^{\log_{25} 6}\right\} = 1$

Q33. $5^{\log_{1/5}\left(\frac{1}{2}\right)} + \log_{\sqrt{2}}\frac{4}{\sqrt{7}+\sqrt{3}} + \log_{1/2}\frac{1}{10+2\sqrt{21}} = 6$

Q34. $49^{(1-\log_7 2)} + 5^{-\log_5 4} = ? = 25/2$

Q35 $4^{\log_{10}x+1} - 6^{\log_{10}x} - 2\left(3^{\log_{10}x^2+2}\right) = 0$ $\quad x = ?$ (sol^n to the log eq^n)

* $2^{2(\log_{10}x+1)} - (2\cdot 3)^{\log_{10}x} - 2\cdot 3^{\log x^2}\, 3^2 = 0$

$2^{2\log x}\, 2^2 - 2^{\log x}\cdot 3^{\log x} - 18\, 3^{\log x^2} = 0$

$\boxed{4\cdot 2^{\log x^2} - 2^{\log x}\, 3^{\log x} - 18\cdot 3^{\log x^2} = 0}$

$a \equiv 2^{\log_{10}x}$, $b \equiv 3^{\log_{10}x}$ $\quad a^2 = (2^{\log x})^2 = 2^{2\log x} = 2^{\log x^2}$, $b^2 = (3^{\log x})^2 = (3^{\log x})\, 3^{\log x} = 3^{2\log x} = 3^{\log x^2}$

* $\boxed{4a^2 - ab - 18b^2 = 0}$ Quadratic eq^n in 2 variables

$4a^2 + 8ab - 9ab - 18b^2 = 0 \Rightarrow 4a(a+2b) - 9b(a+2b) = 0 \Rightarrow (a+2b)(4a-9b) = 0$

Analyze

* $4a - 9b = 0 \Rightarrow \underbrace{4}_{2^2}\cdot 2^{\log x} = 3^2\cdot 3^{\log x} \Rightarrow 2^{\log x+2} = 3^{\log x+2} \xrightarrow[\text{iff}]{\text{Possible}} \log_{10}x + 2 = 0 \Rightarrow \boxed{x = 10^{-2} = \frac{1}{100}}$

(or)

* $a+2b=0 \Rightarrow \underbrace{2^{\log x}}_{+ve} + 2\,\underbrace{3^{\log x}}_{+ve} = 0$

sum of 2 +ve terms $(\#) \neq 0 \Rightarrow$ ignore $a+2b=0$

$2^{\log x} \to 2^{+\alpha},\ 2^{-\alpha} > 0$

$3^{\log x} > 0$

* $\boxed{x = \frac{1}{100}}$

Dark side of Log. (laughs and cries)

Key points

* $y = \log_a x$: $x>0$ remember to check for this inequality ☆☆☆
* Don't deal with composite bases, convert them into primes $\log_{24} x \longrightarrow \frac{1}{\log_x 24}$
* Convert log eq^n into an Algebraic eq^n & then solve it
* Use substitutions for common terms as much as you can, to clean up the mess in the eq^{n}s

Q1
$$\left.\begin{aligned} \log_8 x + \log_4 y^2 &= 5 \\ \log_8 y + \log_4 x^2 &= 7 \end{aligned}\right\} \quad xy = ?$$

* $\log_{\underset{2^3}{8}} x + \log_{\underset{2^2}{4}} y^2 = 5 \Rightarrow \frac{1}{3}\log_2 x + \frac{1}{2}\log y^2 = 5 \Rightarrow \underbrace{\log x^{1/3} + \log y}_{\log(x^{1/3}y)} = 5 \Rightarrow x^{1/3}y = 2^5$ — ①

* II $\Rightarrow y^{1/3}x = 2^7$ — ②

* $(xy)^{4/3} = 2^{12} \Rightarrow xy = 2^{\frac{12\times 3}{4}} \Rightarrow xy = 2^9$

Q2. $2\log(2y-3x) = \log x + \log y$, $\frac{x}{y} = ?$ $\begin{cases} 1 \\ 4/9 \end{cases}$

$\log(2y-3x)^2 = \log xy \Rightarrow (2y-3x)^2 = xy \Rightarrow 4y^2+9x^2-12xy = xy \Rightarrow 9x^2+4y^2-13xy=0$ quad in 2 var

$9\left(\frac{x}{y}\right)^2 + 4 - 13\left(\underbrace{\frac{x}{y}}_{=t}\right) = 0 \Rightarrow 9t^2-13t+4=0$ quad in 1 var

$\Downarrow$

$9t^2-9t-4t+4=0 \Rightarrow 9t(t-1)-4(t-1)=0 \Rightarrow$ $t=1 = \frac{x}{y} \not< \frac{2}{3}$ ✗ (reject) ; $t=\frac{4}{9} = \frac{x}{y} < \frac{2}{3}$ ✓

Cond^n: $x>0$, $y>0$ $2y-3x>0 \to \frac{y}{x} > \frac{3}{2} \Rightarrow \frac{x}{y} < \frac{2}{3}$

Q3 $\sqrt{2008}\,(x)^{\log_{2008} x} = x^2$ Prod of the roots of the eq^n = ?

* $\log_{2008}\left((2008)^{1/2}\, x^{\log_{2008} x}\right) = \log_{2008} x^2 \Rightarrow \frac{1}{2} + \underbrace{\log x^{\log_{2008} x}}_{(\log_{2008} x)(\log_{2008} x)} = 2\log_{2008} x$

* $2(\log_{2008} x)^2 - 4\log_{2008} x + 1 = 0$

{ quad eq^n in $\log_{2008} x$ with roots α' β'

* $\log_{2008} x \to \alpha' \to x_1 = 2008^{\alpha'}$, $\beta' \to x_2 = 2008^{\beta'}$

$\alpha'\beta' = \frac{1}{2}$ $\quad \alpha'+\beta' = -\left(\frac{-4}{2}\right) = 2$

$$\boxed{x_1 x_2 = 2008^{\alpha'+\beta'} = 2008^2}$$

Lecture-28 (14/Jan/24) 2

Q4 $7^{\log_3 5} + 3^{\log_5 7} - 5^{\log_3 7} - 7^{\log_5 3} = 0$

* $A^{\log_B C} = (C^{\log_C A})^{\log_B C} = C^{\log_C A \cdot \frac{1}{\log_C B}} = C^{\log_B A}$ $\quad A = C^{\log_C A}$

$$\boxed{A^{\log_B C} = C^{\log_B A}}$$ Another imp property

Q5 $k^{\log_2 5} = 16$, $k^{(\log_2 5)^2} = ?$

$k^{(\log_2 5)^2} = k^{(\log 5)(\log 5)} = \underbrace{(k^{\log 5})}_{=16}{}^{\log_2 5} = 16^{\log_2 5} = (2^4)^{\log_2 5} = 2^{4\log_2 5} = 2^{\log_2 5^4} = 5^4$

Q6 $\log_3 x \log_4 x \log_5 x = \log_3 x \log_4 x + \log_4 x \log_5 x + \log_5 x \log_3 x \to x = 60$

$x = 1, 1, 1$

Q7 $\underbrace{\log^2 (4-x)}_{\equiv \alpha} + \log(4-x)\cdot \underbrace{\log\left(x+\frac{1}{2}\right)}_{\equiv \beta} - 2\log^2\left(x+\frac{1}{2}\right) = 0$ $\quad x = ?$

quad in 2 var

$\alpha^2 + \underbrace{\alpha\beta}_{2\alpha\beta - \alpha\beta} - 2\beta^2 = 0 \implies (\alpha-\beta)(\alpha+2\beta) = 0$

$\alpha(\alpha+2\beta) - \beta(\alpha+2\beta)$

$\alpha = \beta \Rightarrow \log(4-x) = \log\left(x+\frac{1}{2}\right) \Rightarrow 4 - x = x + \frac{1}{2} \Rightarrow \boxed{x = \frac{7}{4}}$ ✓

$\alpha = -2\beta \Rightarrow \log(4-x) + 2\log\left(x+\frac{1}{2}\right) = 0$

$\log\left((4-x)\left(x+\frac{1}{2}\right)^2\right) = 0 \Rightarrow (4-x)\left(x+\frac{1}{2}\right)^2 = 1$ ✓

$(4-x)\left(x^2+\frac{1}{4}+x\right) = 1 \Rightarrow 4x^2 + \cancel{1} + 4x - x^3 - \frac{x}{4} - x^2 - \cancel{1} = 0$

$-4x^3 + 12x^2 + 15x = 0 \Rightarrow x(4x^2 - 12x - 15) = 0$

$\boxed{x = 0}$, $x = \frac{12 \pm \sqrt{(12)^2 + 4\cdot 4\cdot 15}}{8}$

$x = \frac{3 \pm \sqrt{24}}{2}$ $\to \boxed{\frac{3+\sqrt{24}}{2}}$, $\frac{3-\sqrt{24}}{2}$ (reject)

Q8 $\log_{3/4}\left(\log_8 (x^2+7)\right) + \log_{\frac{1}{2}} \log_{1/4} (x^2+7)^{-1} = -2$ { one of the most difficult doubts 7

* $\log_{3/4}\left(\log_{2^3}(x^2+7)\right) + \log_{1/2}\left(\log_{2^{-2}}(x^2+7)^{-1}\right) = -2$

$\log_{3/4}\left(\frac{1}{3}\underbrace{\log_2(x^2+7)}_{\equiv t}\right) + \log_{\frac{1}{2}}\left(\frac{1}{2}\log_2(x^2+7)\right) = -2 \Rightarrow \log_{3/4}\left(\frac{t}{3}\right) + \log_{\frac{1}{2}}\left(\frac{t}{2}\right) = -2$ —— log eq^n

substitution

* $\frac{\log_2 t/3}{\log_2 3/4} + \frac{\log_2 t/2}{\log_2 1/2} = -2 \Rightarrow \frac{\log t - \log_2 3}{\log_2 3 - \log_2 2^2} + \frac{\log t - \log 2}{\cancel{\log 1} - \log 2} = -2$

$\frac{\log_2 t - \log_2 3}{\log_2 3 - 2} + \frac{\log_2 t - 1}{-1} = -2$ $\left\{\begin{array}{l}\log_2 t \equiv z\\ \log_2 3 \equiv \alpha\end{array}\right.$

$\frac{z-\alpha}{\alpha-2} + \frac{z-1}{-1} = -2$ Algebraic eq^n

$(z-\alpha) + (1-z)(\alpha-2) = -2(\alpha-2) \Rightarrow z - \cancel{\alpha} + \cancel{\alpha} - 2 - z\alpha + 2z = -2\alpha + 4$

$3z + 2\alpha - z\alpha = 6 \Rightarrow \cancel{(3-\alpha)}z = 6 - 2\alpha = 2\cancel{(3-\alpha)} \Rightarrow z = 2 \Rightarrow \log_2 t = 2 \Rightarrow t = 4 \Rightarrow \log_2(x^2+7) = 4$

$\Downarrow$

$x^2 + 7 = 2^4$

$\Downarrow$

$x^2 = 16 - 7 = 9$

$\boxed{x = \pm 3}$

$\log(x^2+7)$ ← $(\pm3)^2$ no effect

Condition check $\boxed{\log x,\ x > 0}$

Q9. $\log_2^2\left(1+\frac{4}{x}\right) + \log_2^2\left(1 - \frac{4}{x+4}\right) = 2\log_2^2\left(\frac{2}{x-1} - 1\right)$

$\log_2^2\left(\frac{x+4}{x}\right) + \log_2^2\left(\frac{x}{x+4}\right) = 2\log^2\left(\frac{3-x}{x-1}\right)$

$\cancel{2}\log^2\left(\frac{x+4}{x}\right) = \cancel{2}\log^2\left(\frac{3-x}{x-1}\right)$

$\log^2\left(\frac{x+4}{x}\right) - \log^2\left(\frac{3-x}{x-1}\right) = 0 \Rightarrow \underbrace{\left(\log\frac{x+4}{x} + \log\frac{3-x}{x-1}\right)}_{\log\frac{(x+4)(3-x)}{x(x-1)}}\underbrace{\left(\log\frac{x+4}{x} - \log\frac{3-x}{x-1}\right)}_{\log\left(\frac{x+4}{x}\cdot\frac{x-1}{3-x}\right)} = 0$

$\log\frac{a}{b} = \log a - \log b$

$\log\frac{b}{a} = \log b - \log a = -\log\frac{a}{b}$

$\boxed{\left(\log\frac{b}{a}\right)^2 = \left(\log\frac{a}{b}\right)^2}$

Another imp property

* $\log\left(\frac{(x+4)(3-x)}{x(x-1)}\right) = 0$ or $\log\left(\frac{x+4}{x}\cdot\frac{(x-1)}{(3-x)}\right) = 0$ ←

$\downarrow$ $\downarrow$

$3x - x^2 + 12 - 4x = x^2 - x$ $\cancel{\boxed{x = \pm\sqrt{2}}}$

$\cancel{2}x^2 = 1\cancel{2} \Rightarrow \boxed{x = \pm\sqrt{6}}$

* Constraint check $\frac{x+4}{x} > 0 \Rightarrow 1 + \frac{4}{(x)} > 0$ → $x = +\sqrt{2}$, → $x = +\sqrt{6}$ } neglect $-\sqrt{2}, -\sqrt{6}$

$1 - \frac{4}{\sqrt{2}} < 0$, $1 - \frac{4}{\sqrt{6}} < 0$

Q10 $\exists$ 3 #, $\{a, b, c\}$: given $\dfrac{\log_a N}{\log_c N} = \dfrac{\log_a N - \log_b N}{\log_b N - \log_c N}$ $N>0$, $N \neq 1$ $a,b,c>0$, $\neq 1$

Comment on $\{a,b,c\}$ / find out rel^n b/w a, b, c

$$\frac{\frac{1}{\log_N a}}{\frac{1}{\log_N c}} = \frac{\frac{1}{\log_N a} - \frac{1}{\log_N b}}{\frac{1}{\log_N b} - \frac{1}{\log_N c}} \Rightarrow \underbrace{\frac{\log_N c}{\log_N a}}_{\cancel{\log_a c}} = \frac{\log b - \log a}{\log c - \log b} \cdot \frac{\cancel{\log b}\log c}{\cancel{\log b}\log a} = \frac{\log\left(\frac{b}{a}\right)}{\log\left(\frac{c}{b}\right)} \cdot \cancel{\log_a c}$$

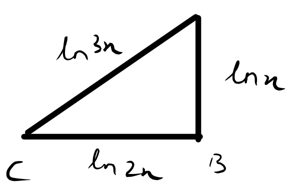
$\log \dfrac{c}{b} = \log \dfrac{b}{a} \Rightarrow \boxed{b^2 = ac}$ a, b, c are in GP

GP
a, b, c
$\dfrac{b}{a} = \dfrac{c}{b}$ = Common Ratio
$b^2 = ac$

Lecture-29 (15 Jan) 15+15

TEST → 15/25 → light!
→ 20/25 → generosity!

1



Is it a right RA Δ? , what is x if yes?

2

$$\frac{1 + \log_2(x-4)}{\log_{\sqrt{2}}\left(\sqrt{x+3} - \sqrt{x-3}\right)} = 1$$

3. $\log_5 120 + (x-3) - 2\log_5(1 - 5^{x-3}) = -\log_5(0.2 - 5^{x-4})$ Decimal pt

4 $5^{\log_{10} x} - 3^{\log_{10} x - 1} = 3^{\log_{10} x + 1} - 5^{\log x - 1}$

5 $\log 4 + \left(1 + \frac{1}{2x}\right)\log 3 = \log\left(\sqrt[x]{3} + 27\right)$

Q11 $\frac{3}{2}\log_4 (x+2)^2 + 3 = \log_4 (4-x)^3 + \log_4 (6+x)^3$

$y = \log x$ $x>0$

$y = \log x^2 = 2\log|x|$

* $\cancel{3}\log_4|x+2| + \cancel{3} = \cancel{3}\log_4(4-x) + \cancel{3}\log_4(6+x)$

$\log_4|x+2| + \underbrace{1}_{\log_4 4} = \log_4\left((4-x)(6+x)\right) \Rightarrow \log_4\left(4|x+2|\right) = \log_4\left((4-x)(6+x)\right)$

$\boxed{4|x+2| = (4-x)(6+x)}$ Algebraic eq^n

Case 1 $x+2 \geq 0$

* $4(x+2) = (4-x)(6+x) \Rightarrow \cancel{4x}+8 = 24 + \cancel{4x} - 6x - x^2 \Rightarrow x^2 + \underbrace{6x}_{8x - 2x} - 16 = 0 \Rightarrow x = 2, -8$

$\because$ $x+2 \geq 0 \Rightarrow \boxed{x=2}$ ✓ neglect -8

Can 11 $\quad x+2<0$

* $-4(x+2) = (4-x)(6+x) \Rightarrow -4x-8 = 24+4x-6x-x^2 \Rightarrow x^2-2x-32=0 \Rightarrow x = \frac{2\pm\sqrt{4+4\cdot 32}}{2}$

$$x = \frac{2\pm\sqrt{132}}{2} = \frac{2\pm 2\sqrt{33}}{2} = 1\pm\sqrt{33}$$

$\therefore \; x+2<0 \Rightarrow \boxed{x = 1+\sqrt{33}}$ ✓ my bad $1-\sqrt{33}$

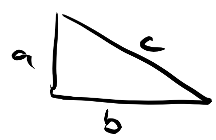
Q12 $\{a,b,c\}$: $\log_{c+b} a + \log_{c-b} a = 2\log_{c+b} a \log_{c-b} a$

$a,b,c>0$

$\begin{cases} c-b \neq 1 \\ c+b \neq 1 \end{cases}$

Comment on a,b,c / Relate a,b,c

* $$\frac{\log_{c+b} a + \log_{c-b} a}{\log_{c+b} a \; \log_{c-b} a} = 2 \Rightarrow \frac{1}{\log_{c-b} a} + \frac{1}{\log_{c+b} a} = \underbrace{\log_a (c-b) + \log_a (c+b)}_{\log_a(c^2-b^2)=2} = 2$$



$a^2 = c^2-b^2 \Rightarrow a^2+b^2=c^2$ Pythagorean triplet

Q13 $\quad x + \log_{10}(1+2^x) = x\log_{10}5 + \log_{10}6$

$x\log_{10}10$

$\log 10^x$ $\qquad \log 5^x$ $\qquad (5^x>0)$

$\underbrace{\log 10^x + \log_{10}(1+2^x)}_{\log(10^x(1+2^x))}$ $\qquad \underbrace{\log 5^x + \log_{10}6}_{\log(5^x\cdot 6)}$

Most common mistake

$x^2-2x=0$

$\cancel{x} = 2\cancel{x}$

$\Downarrow$

$x=2$

* $5^x 2^x(1+2^x) = 5^x 6 \Rightarrow 5^x\left(6-2^x-(2^x)^2\right)=0$

$5^x=0$ (DNE) $\quad$ or $\quad \underbrace{(2^x)}_{t}^2 + 2^x - 6 = 0 \longrightarrow t^2+t-6=0$

$t^2+3t-2t-6=0$

$t(t+3)-2(t+3)=0 \Rightarrow t=2 \longrightarrow 2^x=2 \longrightarrow \boxed{x=1}$

$t=-3 \longrightarrow 2^x=-3 \longrightarrow x\notin R$ (DNE)

Q14 $\quad x^{(\log_2 x+4)} = \underbrace{32}_{2^5}$

* $x^{\log_2 x}\, x^4 = 2^5 \longrightarrow \log_2\left(x^{\log_2 x}\, x^4\right) = \log_2(2^5) \Rightarrow \log_2 x\cdot\log_2 x + 4\log_2 x - 5 = 0$

$(\log_2 x)^2 + 4\log_2 x - 5 = 0$

* $t^2+5t-t-5=0$

$t(t+5)-1(t+5)=0 \Rightarrow t=1 \longrightarrow \log_2 x = 1 \Rightarrow x = 2^1 > 0$

$t=-5 \qquad \log_2 x = -5 \Rightarrow x = 2^{-5} > 0$ $\Big\}$ valid answers

Q15. $\log_4 (x-1) = \log_2 (x-3)$ # of sol^n ; $x = ?$

$\because x > 3$

$\log (x-1)^{1/2} = \log (x-3) \Rightarrow (x-1) = (x-3)^2 \longrightarrow x = 2, 5 \longrightarrow x = 5 \quad (\because x > 3)$

Q16. $\log_5 x + \log_7 x = \log_{25} x \quad \longleftrightarrow \quad 0 + 0 = 0$

$\log 1 = 0$

$\log_5 x + \log_7 x - \log_{25} x = 0 \Rightarrow \frac{\log x}{\log 5} + \frac{\log x}{\log 7} - \frac{\log x}{\log 25} = 0$

$\log x \underbrace{\left(\frac{1}{\log 5} + \frac{1}{\log 7} - \frac{1}{\log 25} \right)}_{\alpha \text{ (indep of } x)} = 0 \Rightarrow \log_n x = 0 \Rightarrow \boxed{x = 1}$

Q17. hw

$\log_{10} (\overbrace{2000xy}^{2\times 10^3 \times x \times y}) - \log x \log y = 4$

$\log 2yz - \log y \log z = 1$

$\log zx - \log z \log x = 0$

3 simult. log Eq^n; $x, y, z = ?$

* $\log_{10} 2 + 3 + \log x + \log y - \log x \log y = 4 \Rightarrow \overbrace{\log x}^{\alpha} + \overbrace{\log y}^{\beta} - \log x \log y = 1 - \log_{10} 2$ — ①

$\log 2 + \log y + \log z - \log y \log z = 1 \Rightarrow \log y + \overbrace{\log z}^{\gamma} - \log y \log z = 1 - \log_{10} 2$ — ②

$\log z + \log x - \log z \log x = 0 \Rightarrow \log x + \log z - \log z \log x = 0$ — ③

factorization

* $\alpha + \beta - \alpha\beta = \log_{10} 5 \rightarrow (\alpha - 1)(\beta - 1) = \log_{10} 2$ — ①'

$\beta + \gamma - \beta\gamma = \log_{10} 5 \rightarrow (\beta - 1)(\gamma - 1) = \log_{10} 2$ — ②'

$\alpha + \gamma - \alpha\gamma = 0 \rightarrow (\alpha - 1) - \gamma(\alpha - 1) = -1 \Rightarrow (\alpha-1)(-\gamma+1) = -1 \Rightarrow (\alpha - 1)(\gamma - 1) = 1$ — ③'

$\gamma(1-\alpha)$

$-\gamma(\alpha - 1)$

$\log_{10} 10$

$1 - \log_{10} 2 = \log_{10} 5$

$\left((\alpha-1)(\beta-1)(\gamma-1) \right)^2 = (\log_{10} 2)^2$

* $\boxed{(\alpha-1)(\beta-1)(\gamma-1) = \pm \log_{10} 2}$

use ①' $\rightarrow \gamma - 1 = \pm 1 \rightarrow \gamma = 2$; $\gamma = 0$

②' $\rightarrow \alpha - 1 = \pm 1 \rightarrow \alpha = 2$; $\alpha = 0$

③' $\rightarrow \beta - 1 = \pm \log 2 \rightarrow \log 2 + 1$; $-\log 2 + 1$; $= 1 \pm \log_{10} 2$

* $\log x = \alpha = 2 \rightarrow x = 10^2$; $0 \rightarrow x = 10^0$

$\log_{10} y = \beta = 1 + \log 2 = \log_{10} 20 \rightarrow y = 20$; $1 - \log 2 = \log_{10} 5 \rightarrow y = 5$

$\log z = \gamma = 2 \rightarrow z = 10^2$; $0 \rightarrow z = 10^0$

(Trick) : Protocol Genius Method

$xy = a$ — ①

$yz = b$ — ②

$zx = c$ — ③

$(xyz)^2 = abc$

$xyz = \sqrt{abc}$

$xy = a$: $z = \frac{\sqrt{abc}}{a}$

$yz = b$: $x = \frac{\sqrt{abc}}{b}$

$zx = c$: $y = \frac{\sqrt{abc}}{c}$

Lecture-30 (19/Jan) (Darkest of the dark!)

Q18 $\log_3\left(\sqrt{x}+|\sqrt{x}-1|\right)=\log_9\left(4\sqrt{x}-3+4|\sqrt{x}-1|\right)$

hint

$\log(a+b) \rightsquigarrow$

$\sqrt{a}$, $|a-b|$

$\log_3(a+|b|)=\log_9(4a-3+4|b|)$

$\log_3(a+|b|)=\frac{1}{2}\log_3(4a-3+4|b|) \Rightarrow a+|b|=\sqrt{4a-3+4|b|}$

* $(a+|b|)^2=4a-3+4|b|$ algebraic eq^n ; $a\equiv\sqrt{x}$, $b\equiv|\sqrt{x}-1|$

$a^2+|b|^2+2a|b|=4a-3+4|b|$

$(b)^2=(\sqrt{x}-1)^2 = x+1-2\sqrt{x}$

Plug back a & b

* $x+x+1-2\sqrt{x}+2\sqrt{x}|\sqrt{x}-1|=4\sqrt{x}-3+4|\sqrt{x}-1|$

$2x-6\sqrt{x}+2\sqrt{x}|\sqrt{x}-1|=4|\sqrt{x}-1|-3-1$

$x-3\sqrt{x}+\sqrt{x}|\sqrt{x}-1|=2|\sqrt{x}-1|-2$

$x-3\sqrt{x}+2=(2-\sqrt{x})|\sqrt{x}-1|$

$\sqrt{x}=\alpha \Rightarrow \alpha^2=x$

$x-3\sqrt{x}+2$

$\alpha^2-3\alpha+2$

$\alpha^2-\alpha-2\alpha+2$

$\alpha(\alpha-1)-2(\alpha-1)$

$(\alpha-1)(\alpha-2)$

$(\sqrt{x}-1)(\sqrt{x}-2)+(\sqrt{x}-2)|\sqrt{x}-1|=0 \Rightarrow \boxed{(\sqrt{x}-2)\left(\sqrt{x}-1+|\sqrt{x}-1|\right)=0}$

algebraic eq^n with mod

Take cases of $|x|$

Case I

* $(\sqrt{x}-2)\left(\underbrace{\sqrt{x}-1+\sqrt{x}-1}\right)=0$ if $\sqrt{x}-1\geq 0 \Rightarrow \sqrt{x}\geq 1 \Rightarrow x\geq 1$

$(\sqrt{x}-2)(\sqrt{x}-1)=0 \Rightarrow \boxed{x=1,\ 4}$

Case II

* $(\sqrt{x}-2)\left(\overbrace{\cancel{\sqrt{x}-1}-\cancel{(\sqrt{x}-1)}}^{0}\right)=0$ if $\sqrt{x}-1<0 \Rightarrow \overbrace{\sqrt{x}<1}^{\text{mod. cond}^n}$ ✓

$x<1$

$\Downarrow$

$0=0$ always true $\Rightarrow$ It is an identity for $x<1$

* $\sqrt{x}\in\mathbb{R}$, $\underbrace{x\geq 0 \quad \& \quad x<1}$

$0\leq x<1 \Rightarrow x\in[0,1)$

* find solutions $x=1$, $x=4$, $x\in[0,1) \Rightarrow \boxed{x\in[0,1],\ \{4\}}$

Last Doubts

(5) $\log 4 + \left(1+\frac{1}{2x}\right)\log 3 = \log\left(\sqrt[x]{3}+27\right)$ — $x=?$

* $\log 4 + \log 3 + \frac{1}{2x}\log 3 = \log\left(3^{1/x}+3^3\right)$

* $\log_{10}\left(12\cdot 3^{1/2x}\right) = \log_{10}\left(3^{1/x}+3^3\right) \Rightarrow 12\,\underbrace{3^{1/2x}}_{\alpha} = \underbrace{3^{1/x}}_{\alpha^2}+27$

$3^{1/2x}=\alpha$
$3^{1/x}=\alpha^2$

$\alpha^2-12\alpha+27=0$

$\alpha^2-9\alpha-3\alpha+27=0$

$\alpha(\alpha-9)-3(\alpha-9)=0 \longrightarrow \alpha = 9 \longrightarrow 3^{1/2x}=3^2 \Rightarrow x=1/4 \in Q$

$\alpha = 3 \longrightarrow 3^{1/2x}=3^1 \Rightarrow x=1/2 \in Q$

∵ $\sqrt[x]{3}$ $\quad x\in\mathbb{N} \Rightarrow$ solution $\boxed{x\in\phi}$ a null set

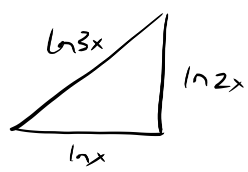
$\sqrt[2]{3}=3^{1/2}$ $\quad \sqrt[5]{4}=4^{1/5}$

$\sqrt[3]{3}=3^{1/3}$

$\sqrt[1/2]{3}$ = no sense

Lecture-31 (20/Jan) 1 + 1 20

(1) $(\ln x)^2+(\ln 2x)^2=(\ln 3x)^2$



$\underbrace{(\ln x)^2}_{t} + \underbrace{(2\ln 2-2\ln 3)}_{2\ln\frac{2}{3}}\ln x + \underbrace{(\ln 2)^2-(\ln 3)^2}=0$

$(\ln 2+\ln 3)(\ln 2-\ln 3)=\ln 6\ln\frac{2}{3}$

$t = \frac{-2\ln\frac{2}{3}\pm\sqrt{4\left(\ln\frac{2}{3}\right)^2-4\ln 6\ln\frac{2}{3}}}{2} = \ln\frac{3}{2}\pm\sqrt{\underbrace{\left(\ln\frac{2}{3}\right)^2-\ln 6\ln\frac{2}{3}}}$

$\ln\frac{2}{3}\left(\ln\frac{2}{3}-\ln 6\right)=\ln\frac{2}{3}\ln\frac{1}{9}$

$=\ln\frac{3}{2}\pm\sqrt{\ln\frac{2}{3}\ln\frac{1}{9}} = \ln\frac{3}{2}\pm\sqrt{\ln\frac{3}{2}\ln 9}$

$\ln_e x = t \Rightarrow x = e^{\ln\frac{3}{2}\pm\sqrt{\ln\frac{3}{2}\ln 9}} = \underbrace{e^{\ln_e\frac{3}{2}}}_{\frac{3}{2}}\cdot e^{\pm\sqrt{\ln\frac{3}{2}\ln 9}} = \frac{3}{2}e^{\pm\sqrt{\ln\frac{3}{2}\ln 9}}$

smooth / continuous $\Rightarrow$ Δ is a right Δ

(nothing bad is happening)

$\sqrt{-ve}$ $\quad \ln(-ve)$
"Bad"!

1 C 1 Numerical / Approx Computations of $\log_{10}$

a) $\log(\# > 1)$

* $N = 15 \longrightarrow 10^1 < \underbrace{15}_{} < 10^2$

10^{2-1} # digits = 2

general? $\longrightarrow$ $\boxed{10^{n-1} < \text{\# whose integer part has } n \text{ digits} < 10^n}$

* $N = 10^{(n-1) + \text{a fraction}}$ → $15 = 10^{1 + \text{a fraction}}$, $15 \neq 10^2$

$\downarrow$

$\log_{10} N = \underbrace{(n-1)}_{\text{Characteristic}} + \underbrace{\text{a fraction}}_{\text{Mantissa}}$

b) $\log(\# < 1)$

* $N = 0.0324 \longrightarrow 10^{-2} < 0.0324 < 10^{-1}$

* $10^{-2} <$ # with 1 zeroes after decimal $< 10^{-1}$
$10^{-3} <$ " " 2 " " " $< 10^{-2}$

$10^0 = 1$
$10^{-1} = 0.1$
$10^{-2} = 0.01$
$10^{-3} = 0.001$
$10^{-4} = 0.0001$

$\boxed{10^{-(n+1)} < \text{\# with } n \text{ zeroes after decimal} < 10^{-n}}$

* $N = 10^{-(n+1) + \text{a fraction}}$ → $0.0324 = 10^{-2 + \text{fraction}}$

$\downarrow$

$\log N = -(n+1) + \text{a fraction}$

characteristic is → -ve
→ greater by unity than # of zeroes after decimal

Problem: This method will take you so far
won't be able to compute mantissa

Q: $x = (0.15)^{20}$ $\log(0.15)^{20} = ?$: $\log_{10} 2 = 0.301$, $\log 3 = 0.477$

* $\log x = 20 \log\left(\frac{15}{100}\right) = 20\left(\log \underbrace{15}_{3 \times 5} - \log 100\right) = 20\left(\log 3 + \underbrace{\log_{10} 5}_{} - 2\right) = 20\left(\log 3 - \log 2 - 1\right)$

$\log\left(\frac{10}{2}\right) = 1 - \log 2$

$= 20\left(0.477 - 0.301 - 1\right) = -16.48$

$\log_{10}(0.15)^{20} = -16.48 = \underbrace{-17}_{\text{Characteristic}} + \underbrace{(0.52)}_{\text{Mantissa}}$

-17   -16

$\underbrace{-17}_{} < \log x < -16$

* $N = 10^{-(n+1) + \text{fraction}} \longrightarrow \log N = \underbrace{-(n+1)}_{17} + \underbrace{\text{fraction}}_{0.52}$

$(0.15)^{20} = 10^{-17+(0.52)} = 10^{-16.48}$

Q2 # of the integers are there : $\underbrace{\text{characteristic} = 3}_{3 \le \log x < 4}$ & base = 7 1457

$\overset{\parallel}{?}$

✓ $\log_7 N = x = 3 + \text{a fraction} \longrightarrow \boxed{3 \le x < 4}$

$\updownarrow$

$N = 7^x$

$\underbrace{7^3}_{343} \le N < \underbrace{7^4}_{2401} \longrightarrow$ # of integers $= 2401 - 343$

$= 2058$

Q3 Compute characteristics

1. $\log_{10}(37.203) \simeq 1.\overbrace{5705}^{\text{mantissa}}$ (using log table) Later

$10^1 < 37.203 < 10^2 \Rightarrow 37.203 = 10^{1+\text{a fraction}}$

$\log(37.203) = 1 + \text{a fraction}$

2. $\log(3.203) = 0.5705$

$10^0 < 3.203 < 10^1 \Rightarrow 3.203 = 10^{0+\text{fraction}}$

$\log(3.203) = 0 + \text{fraction}$

3. $\log(\overbrace{372030000}^{\#}) = 8.5705$

$10^8 < \# < 10^9 \Rightarrow \# = 10^{8+\text{fraction}}$

$\log \# = 8 + \text{fraction}$

4. $\log(\overset{\#}{0.0000372 03}) = -5.5705$

$10^{-5} < \# < 10^{-4} \Rightarrow \# = 10^{-5+\text{fraction}}$

$\log \# = -5 + \text{fraction}$

0.000037203

scientific notation

3.7203×10^{-5}

5. $\log(0.37203) = -1.5705$

$10^{-1} < \# < 10^0 \Rightarrow \# = 10^{-1+\text{fraction}}$

$\log \# -1 + \text{fraction}$

6. $\log(208)$

$10^2 < \# < 10^3 \Rightarrow \# = 10^{2+\text{fraction}}$

$\log \# = 2 + \text{fraction}$

7. $\log(56.7) \rightsquigarrow \# = 10^{1+\text{fraction}}$

8. $\log(6) \rightsquigarrow \# = 10^{0+\text{fraction}}$

$\boxed{\log_{10} x}$

LHS

main column

RHS

differences

c) log Table

* to compute "exactly" the log value
$\underbrace{\text{to compute "exactly" the log value}}_{\text{Mantissa of log}}$

											Mean Difference								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	19
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13

* $\log(23.78) = ? = 1.$

Step 1 write down the # in Scientific notation & the power represents the Characteristic (C)

$23.78 = 2.378 \times 10^{1}$ $C = 1$
$4572 = 4572 \times 10^{0}$ $C = 0$ } > 0
$0.172 = 1.72 \times 10^{-1}$ $C = -1$
$0.0172 = 1.72 \times 10^{-2}$ $C = -2$ } < 0

$\log(200) \rightarrow 1^{st}\ \# = 2$

Step 2 Mantissa Calculation

always +ve

found by log table

Protocol

1. find 1^{st} non zero # = 1 $\quad 0.001724$
2. Starting from that # take Next 4 numbs. = 1724 (R, C, Mean diff)
3. 1^{st} 2 Numbs = Rows = 17
   3^{rd} Numb = Columns = 2
   Intersection No. = 2355
4. last # is the Mean diff = 10
5. Mantissa = Intersection # + Mean diff
   = 2355 + 10 = 2365
6. Representation of Mantissa = 0.2365
   Put a decimal pt just before the #

Note
this is Mantissa if there is No 4th digit

* $\log(0.001724)$

$1.724 \times 10^{-3} \Rightarrow C = -3$

$\log(0.001724) = -3 + 0.2365$
$= -2.7635$

* $\log(23.78) = 1.3762$ (R, C)

* $\log(200) = ?$

HW

q4 # of digits in $2^{75} = ?$

q5. # of integers in $6^{15} = ?$

q6 $x = 3^{-100}$ # of zeros after decimal = ?

HW

$$\begin{cases} \log 2 = 0.3010 \\ \log 3 = 0.4771 \end{cases}$$

Lecture-32 (21/Jan/24)

Q4 $N = 2^{75}$ # of digits

$\log N = \log 2^{75} = 75 \log_{10} 2 = 75\,(0.3010) = 22.5750$

$\log_{10} 2^{75} = 22 + 0.5750 \Rightarrow 2^{75} = 10^{22+0.5750} = \underbrace{10^{\overbrace{0.5750}^{0 < \text{Power} < 1}}}_{1 \text{ digit } \#} 10^{22} = \alpha\, 10^{22}$

$\boxed{\# \text{ of digits} = 23}$

* $10^0 = 1$
  $10^1 = 10$

* $5732 = 5.7 \times 10^{3}$
  # of digits = 4

Q5. $N = 6^{15}$ # of integers = ?

$\log N = \log 6^{15} = 15\{\log 2 + \log 3\} = 15\{0.3010 + 0.4771\}$

$= 11.6715 = 11 + 0.6715$

$\log 6^{15} = 11 + 0.6715 \Rightarrow 6^{15} = 10^{11}\,(\underbrace{10^{\overbrace{0.6715}^{0 < \text{Power} < 1}}}_{1 \text{ digit}}) = \alpha\, 10^{11}$

$\boxed{\# \text{ of integers} = 12}$

$10^2 = 100$
Power = 2, # of digits = 3

$\alpha = 5$
$\alpha\, 10^2 = 500$
digits = 2

$100 \to 3$
$10^{11} \to 12$
$10^{n} \to (n+1)$ digits

Q6 $x = 3^{-100}$ # of 0s after decimal = ?

$\log x = -100 \log_{10} 3 = -100\,(0.4771) = -47.71$

$= -47 + 0.71$

$3^{-100} = (10^{0.71}) \cdot 10^{-48}$

0 after decimal $= 48 - 1 = 47$

$\left\{ \begin{array}{ll} 3^{-100} & 2^{-100} \\ 5^{-100} & \vdots \\ 7^{-100} & \end{array} \right.$

$10^{-1} = \frac{1}{10} = 0.1$

$10^{-2} = 0.01$

$10^{-n} = (n-1)$ 0's

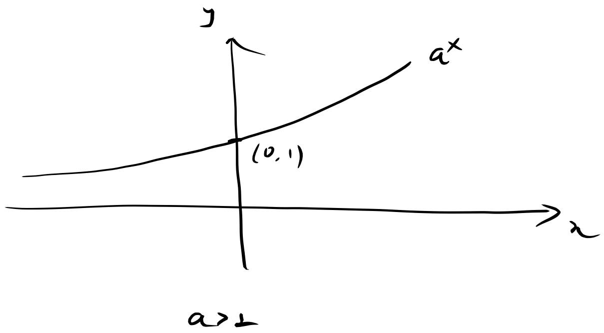
1.c.2 Graph of log(x)

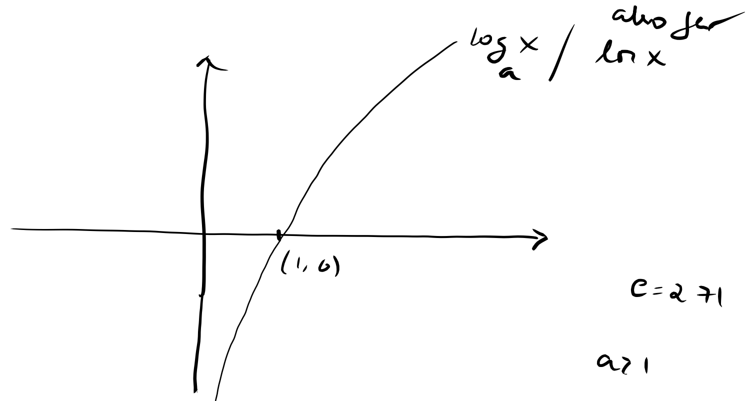
* $y = f(x) \Rightarrow x = f^{-1}(y)$ x & y are inverses of each other } not all fun^n are invertible

* $\boxed{y = a^{x} \Rightarrow x = \log_a y}$

exp^n, log — inverses of each

*

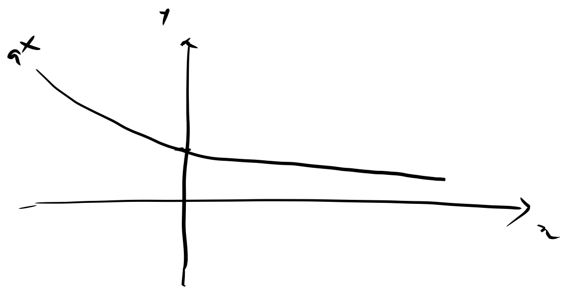


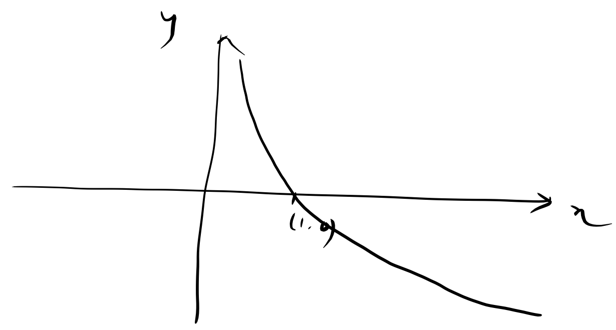


$$f(x) = a^x = \begin{cases} >1 & x>0 \\ =1 & x=0 \\ <1 & x<0 \end{cases} \quad \overset{\text{Inverse}}{\rightsquigarrow} \quad y = \log_a x = \begin{cases} <0 & 0<x<1 \\ 0 & x=1 \\ >0 & x>1 \end{cases}$$

$2^7 \qquad 2^{-7}$

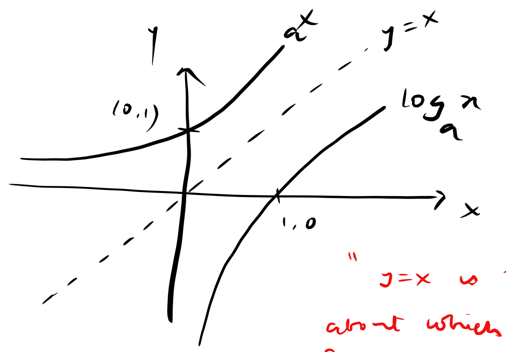
$\log 3 > 0 \qquad \log \frac{1}{3} = -\log 3 < 0$

* 



$0<a<1 \qquad\qquad 0<a<1$

$$f(x) = a^x = \begin{cases} <1 & x>0 \\ =1 & x=0 \\ >1 & x<0 \end{cases} \quad \rightsquigarrow \quad y = \log_a x = \begin{cases} >0 & 0<x<1 \\ =0 & x=1 \\ <0 & x>1 \end{cases}$$

* 

" $y=x$ is the Mirror about which a^x is reflected & you get $\log x$ "

{ Reflection symmetry (Mirror)

$\log_{1/2}(3) = -\log_2 3 < 0$

$\log_{1/2}\left(\frac{1}{2}\right) = -(\log_2 2^{-1}) > 0$

Note

$\text{Dom}(\log x) = (0, \infty)$ ⇝ rather later in $\mathbb{C}$ & make it Bigger

$\text{Range}(\log x) = \mathbb{R}$

$\exists$ change in relationships

To Be Contd