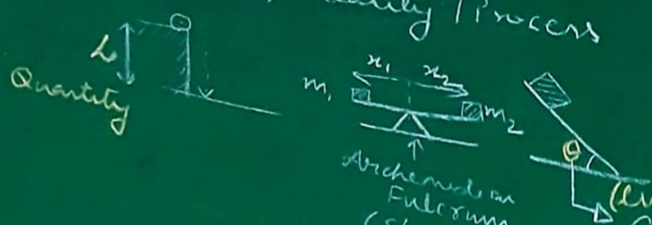
Lecture – 1 (7/5/2022) {Trigonometry Vol 1}

* Inequality => Phenomena.
* (Quality) Difference => Reality / Process

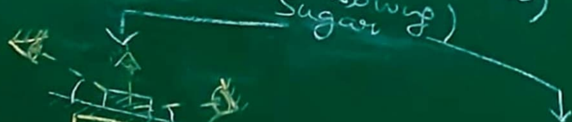


Archimedean Fulcrum.
(Statics / Equilm)
assumption Equality
⇓
No Phenomena
(Mechanical)
'laws of levers'

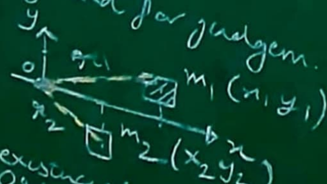
(Lucretius) Cline
||
Inclination
||
Bend / Angle

* ∃ Minimal Angle => Movement (flow)

* (dissolving Sugar)



Spatial
Relative
∃ ground for judgem.



existence of Origin => distance (x_1, x_2) b/w object
⇑
counting

Space => distance

Notion of distance / Ar. / Vol. are in Space

Difference in Degree

t-val. 10.05 10:07

∄ diff. in degree

Machine
||
Angle / Rotation / transform

* Infinitesimal change in Balance (Equalm)
⇓
judgement / Value assignment /
Motion understood in terms of rest
– Birth of physics, M. Serres

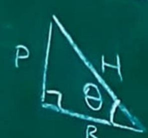
Lecture-2 (9/5/2022) {Trignomitry (Vol. 1)}

* Pythagoras' POV

Everything ⟶ $\underbrace{\text{fire}}$ + Math. order
(Heraclitus)

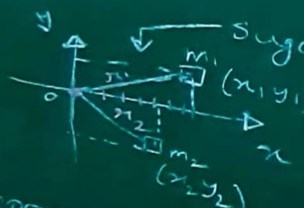
order ⇓ Represented Numerically / Geometrically

* $A^2+B^2=H^2$ ⟷ θ
Sides — Angle
⇓
Trigonometry



$\sin(\theta)$

*



sugar/ice

∃ diff. in Kind from ITSELF

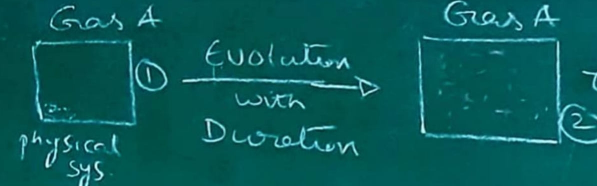
$A = A$
Self Identity
$A.1 = 1.A = A$
Math.

$A \neq A$ self-diffent
⇓
∄ Concept of 1 (Identity)

space ⇒ distance
⇓
Counting (in detail later)

Theory of Space
⇓
Number Theory

∃ difference in $\underbrace{\text{degree}}$
angle/Function/Machine/transformation

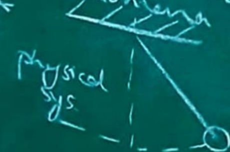
* 

* irreversible ⇒ $t \to -t$ (t val.) sys is not invariant

∄ any physical transform to take m from ② to ①

$\boxed{\text{Gas A} \neq \text{Gas A} \quad \text{Not self-identical}}$

* wave/oscillatory Phenomena



t val. — harmonic oscillator EOS

$x(t) = A\cos(\omega t)$

$x(-t) = A\cos(-\omega t) = x(t)$
⇓
sys is invariant (Reversible)

✓ $\boxed{\cos(-x) = \cos(x)}$
(later: derivation)

* $F = m\dfrac{d^2x}{dt^2}$ NLM-2

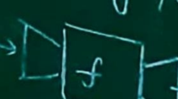
$t \to -t$ Eqⁿ remains same.
t-rev. invariant (Problem of NL)

* $f(x) = \sum_{n=0}^{n} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots a_n x^n$

$n \in \mathbb{N}, x \in \mathbb{R}$

Polynomial Machine
∃ couplings/connections

1. $\underbrace{\text{Transcendental Machines}}_{\text{Beyond Algebra}}$

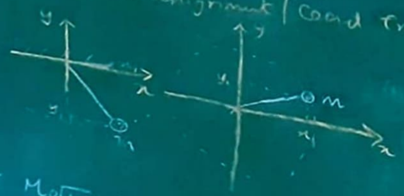


- Trigonometric Machines
- Logarithmic "
- Exponential "

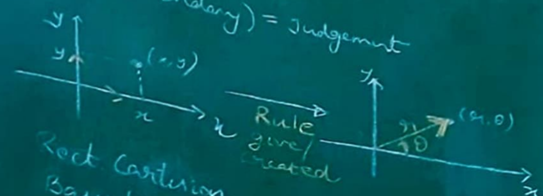
Lecture - 3 (14/5/2022) { Trigonometry (Vol. 1) }

1. Trigonometric Machines

a Pos value assignment / Coord Transf^n



* Motion (Primary)
Space (secondary) = Judgement



Rule given / created

Rect Cartesian Basis / label / unit

(x, y)

$x, y : -\infty \to \infty$

Circular Polar Basis

(r, θ)

"Cartesian-like label"

$r : 0 \to \infty$

Comment :

* $(x, y) \xrightarrow{\text{Rule}} (r, \theta)$

* Coordinate transform^n

$x^{\kappa} \longrightarrow x^{\kappa'} / x'$ Condensed Notation

$\kappa = 1, 2$

$(x^1, x^2) \equiv (x, y)$

$(x^{1'}, x^{2'}) \equiv (r, \theta)$

* $(x, y) \longrightarrow (r, \theta)$ $\quad y = f(x)$

$$\boxed{\begin{aligned} x &= x(r, \theta) = ? \\ y &= y(r, \theta) = ? \end{aligned}}$$

$x \to \boxed{f} \to y$

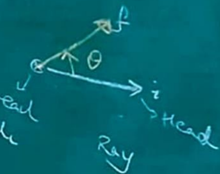
Aim x, y as a fun^c of r, θ (explicitly)

* Quality Direction → Quantity Angle (θ) Trigo / Ray → Vectors / Tensor

b. Angle

* Control parameter in circular polar C.S

* $\boxed{\text{Quantity Emerging}}$ while moving an initial ray to the final direction. The tail is called Angle



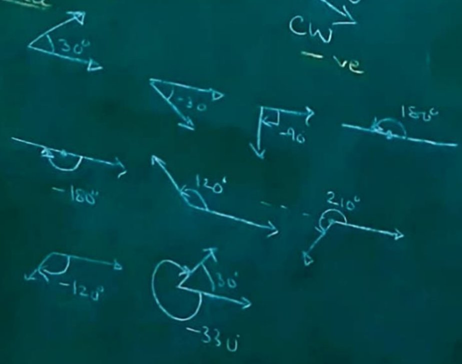
Sign Convention :

ACW +ve

CW −ve

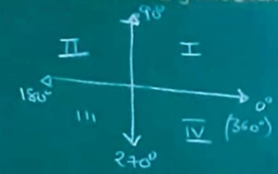
Practice

*



b.1 Quadrant formation

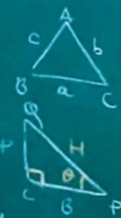
*



I. [0, 90°]
II. (90, 180°]
III. (180, 270°]
IV. (270, 360)

* $A + B + C = 180°$

* $C = 90° \Rightarrow RH\Delta$

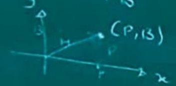


$\boxed{Q + P = 90°}$

↓

Complementary Angle

* $H^2 = P^2 + B^2$ "Pythagoras thm." (special case)



↕

Distance formula

b.2 Arc length / Radius formula

* $\pi \equiv \frac{C}{d} = \frac{C}{2r} \Rightarrow \boxed{C = 2\pi r}$ Circumference of circle

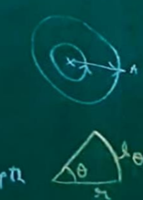
const. parameter for circle

* $(2\pi r)$ length $\longrightarrow 360°$

$\left(\frac{2\pi}{360} r\right)$ length $\longrightarrow 1°$

$\theta° \longrightarrow \underbrace{\left(\frac{\theta}{360} \cdot 2\pi r\right)}_{l_\theta = l}$ length

$l_\theta = l$ Arc length



* $l = \frac{\theta}{360} \cdot 2\pi r$

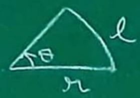
$\theta = \underbrace{\frac{360}{2\pi}}_{} \cdot \frac{l}{r}$

∃ Mathematical "Naturalness" radian (SI unit) of Angle

$$\boxed{\underset{\text{(Angle) in radians}}{\theta} = \frac{\text{arc}}{\text{radius}} : (2\pi) = 360°}$$

Lecture-4 (15/5/2022)

* $\theta = \frac{l}{r}$: $\underbrace{(2\pi)^c}_{\text{Quantity } (\#)} = \underbrace{360^\circ}_{\text{quality (direction) Angle}}$



$\Downarrow$

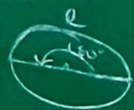
∃ a new "Measure" of Angle
Radian

(bigger unit)
Radian > Degree

* $1^c = \left(\frac{360}{2\pi}\right)^\circ = \left(\frac{180}{\pi}\right)^\circ = \left(\frac{180}{22}\cdot 7\right)^\circ$
$= \left(\frac{1260}{22}\right)^\circ = 57^\circ 16' 21''$

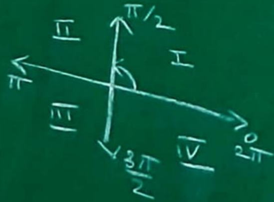
* $360^\circ = (2\pi)^c \Rightarrow \boxed{\pi^c = 180^\circ}$

$\theta = \frac{\pi r}{r} = \pi^c = 180^\circ$



* $\frac{\pi}{2} = 90^\circ$, $\frac{\pi}{4} = 45^\circ$

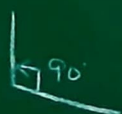
$\boxed{\left(\frac{\pi}{180}\right)^c = 1^\circ}$



* $30^\circ = \left(\frac{\pi}{6}\right)^c$, $60^\circ = \frac{\pi}{3}$
$120^\circ = \frac{2\pi}{3}$, $150^\circ = \frac{5\pi}{6}$, $210^\circ = \frac{7\pi}{6}$
$750^\circ = \frac{25\pi}{6}$, $270^\circ = \frac{3\pi}{2}$

Sexagesimal sys

* 1 R.A $= 90^\circ$
$1^\circ = 60'$ ("Minute")
$1' = 60''$ ("Seconds")



Practice

* $47^\circ 30' = \left(47\tfrac{1}{2}\right)^\circ = (47.5)^\circ$ $\quad \left(47+\tfrac{1}{2}\right)$

* $(47.5)^\circ = (?)^c = \left(\frac{19}{72}\pi\right)$

* $\left(\frac{2\pi}{15}\right)^c = 24^\circ$

* $\left(\frac{\pi}{8}\right)^c = 22^\circ 30'$

* $\left(\frac{1}{4}\right)^c = \left(\frac{315}{22}\right)^\circ = \left(14 + \frac{7}{22}\right)^\circ = 14^\circ\left(\frac{7}{22}\right)^\circ$

$\left(\frac{7}{22}\right)^\circ = \left(\frac{7}{22}\times 60\right)'$
$= \left(\frac{210}{11}\right)'$
$= \left(\frac{11\times 19}{11} + \frac{1}{11}\right)'$
$= 19'\left(\frac{1}{11}\right)'$

$\left(\frac{1}{11}\right)' = \left(\frac{1}{11}\times 60\right)'' = \left(\frac{11\times 5}{11} + \frac{5}{11}\right)''$
$\simeq 5''$

$\boxed{\left(\frac{1}{4}\right)^c = 14^\circ 19' 5''}$

* $6^c = (?)^\circ = \left(\frac{3780}{11}\right)^\circ = 343^\circ 38' 1''$

$1^\circ = 60'$
$\left(\frac{1}{2}\right)^\circ = 30'$

$a\frac{b}{c} = a + \frac{b}{c}$
$= \frac{ac+b}{c}$

22) 315 (14
22
95
88
7

$\frac{315}{22} = \frac{14\times 22 + 7}{22}$

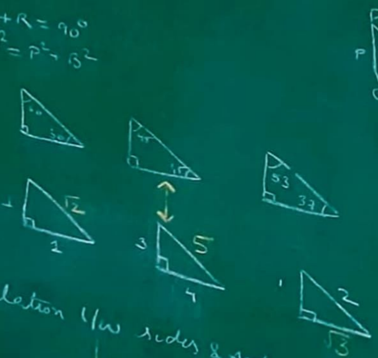
11) 210 (19
11
100
99
1

11) 60 (5
55
5

Lecture - 5 (16/5/2022)

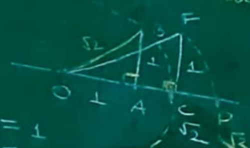
c. Circular trigonometric ratio Machine

* $P + R = 90^\circ$
* $H^2 = P^2 + B^2$

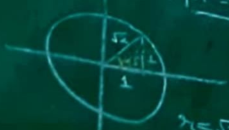


* Relation b/w sides & angles
  via
  construction

Representation of irrational # (check (Number Theory))



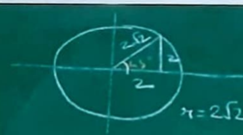
* $OA = 1$
  $AB = 1 \because AB \perp OA$
  $OB^2 = 1^2 + 1^2 \Rightarrow \boxed{OB = \sqrt{2}}$
  $CF = 1 \because CF \perp OC$
  $OF^2 = 1^2 + 2 \Rightarrow \boxed{OF = \sqrt{3}}$

* 

$r = \sqrt{2}$

$\frac{P}{H} = \frac{1}{\sqrt{2}} = \sin(45)$

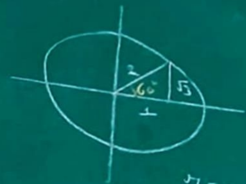
$\frac{B}{H} = \frac{1}{\sqrt{2}} = \cos(45)$

* 

$r = 2\sqrt{2}$

$\frac{P}{H} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}} = \sin 45$

$\frac{B}{H} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}} = \cos 45$

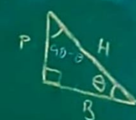


$r = 2$

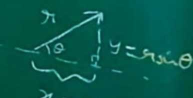
$\frac{P}{H} = \frac{\sqrt{3}}{2} = \sin 60^\circ$

$\frac{B}{H} = \frac{1}{2} = \cos 60^\circ$

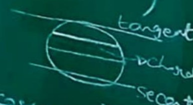
* $\sin\theta = \frac{P}{H}$ = Sinusoidal
* $\cos\theta = \frac{B}{H}$ = Cosinusoidal
* $\tan\theta = \frac{P}{B}$ = tangent



Circular Trigonometric-Ratio machine



* Geometry . Circular Polar (Circle)
  $x = x(r, \theta) = r\cos\theta$
  $y = y(r, \theta) = r\sin\theta$



* $\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{H}{P}$ = Cosecant
  $\sec\theta = \frac{1}{\cos\theta} = \frac{H}{B}$ = secant
  $\cot\theta = \frac{1}{\tan\theta} = \frac{B}{P}$ = cotangent
  } Reciprocal machines

* $\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\cot\theta = \frac{\operatorname{cosec}\theta}{\sec\theta} = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$

{Plugging 1 machine into another machine}

* Mnemonic

S (Some) $= \frac{P\text{ (People)}}{H\text{ (Have)}}$

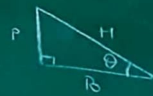
C (Curly) $= \frac{B\text{ (Black)}}{H\text{ (Hair)}}$

T (To) $= \frac{P\text{ (Paint)}}{B\text{ (Beauty)}}$

Lecture-6 (21/5/2022)

$\frac{1}{x} = x^{-1}$

* $(2\pi)^c = 360^\circ$
* $\sin\theta = \frac{P}{H}$, $\cos\theta = \frac{B}{H}$



$\tan\theta = \frac{\sin\theta}{\cos\theta}$

* $\operatorname{cosec}\theta = \underbrace{\frac{1}{\sin\theta}}_{(\sin\theta)^{-1}}$, $\sec\theta = \frac{1}{\cos\theta}$, $\cot\theta = \frac{1}{\tan\theta}$

Circular Trignometric machines

caution

* $f : \mathbb{R} \longrightarrow \mathbb{R}$

function

$y = \sin\theta$

$\mathbb{R}$ θ i/p → [Sin] → o/p $\mathbb{R}$

Number — o/p Number

Comment on Ar. under curve

* $W.D = F \cdot x$

i/p → "Functional" [$f(x)$ function] → o/p Number

* $W = F \cdot x$

function

i/p x^3 → [derivation $\frac{d}{dx}$] → o/p $3x^2$ function

$A_\Delta = \frac{1}{2} F_0 x_0 = 18$

$A_\square = F_0 x_0 = 36$

$A_\Delta = \frac{1}{2} \cdot b \cdot h$

$= \frac{1}{2}(6)(6)$

$A_\Delta = 18\,(\text{unit})^2$

* $A = \sum_i f_i x_i$ "small" $= f_0x_0 + f_1x_1 + f_2x_2 + \dots f_nx_n$

Number — function — Number

functional

* $y = \sin\theta$

↓ Mathematical Comedy

$\frac{y}{\sin} = \theta \Rightarrow \theta = \sin^{-1} y$

θ Angle → [ ] → y Number

y → [ ] → θ

$\sin^{-1}$ Inverse T-machine (vol-2 eng)

* $\frac{1}{\sin\theta} = (\sin\theta)^{-1} \neq \sin^{-1}\theta$

Notation Alert

$(x^2)^{-1} = x^{-2}$

* $(\sin\theta)^2 = \sin^2\theta$
$= (\sin\theta)(\sin\theta)$

→ [$\sin^2$] →

$\underbrace{(\sin\theta)^{-n}}_{\frac{1}{(\sin\theta)^n}} \neq \sin^{-n}\theta$

Practice 1

* $\sin\psi = \frac{3}{5}$

$\tan\psi = \frac{3}{4}$

$\underbrace{\tan\psi}_{\frac{\sin\psi}{\cos\psi}} \cos\psi = \sin\psi = \frac{3}{5}$

ψ - psi

* $\sin\alpha = \frac{1}{3}$

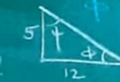
$\cos\alpha \operatorname{cosec}\alpha + \tan\alpha \sec\alpha$

$\sqrt{8} = 2\sqrt{2}$

HW

"ϕ phi"

* $\tan\phi, \sin\phi$
$\cos\psi, \sin\psi = ?$



* $\operatorname{cosec}\beta = 2.$

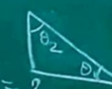
$\frac{1}{\tan\beta} + \frac{\sin\beta}{1+\cos\beta} = 2$

* $\tan\eta = \sqrt{2} - 1$ η = eta

$\sin\eta\cos\eta = ?$

* $\tan\theta = \frac{1}{\sqrt{3}}$

$\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 = ?$



Lecture - 7 (22/5/2022)

Practice 1 (equivalence)

* $\dfrac{1}{\frac{1}{\sqrt3}} + \dfrac{\frac12}{\frac{1}{1}+\frac{\sqrt3}{2}} = \sqrt3 + \dfrac{1}{2+\sqrt3}\times\dfrac{2-\sqrt3}{2-\sqrt3} = 2$

* $\sin\eta\cos\eta = \sin\eta\dfrac{\sin\eta}{\tan\eta}$

$= \dfrac{(\sin\eta)^2}{\tan\eta} = \dfrac{\frac{(\sqrt2-1)^2}{4-2\sqrt2}}{\sqrt2-1}$

$\tan\eta = \dfrac{\sin\eta}{\cos\eta}$

$= \dfrac{(\sqrt2-1)^2}{(4-2\sqrt2)(\cancel{\sqrt2-1})} = \dfrac{\sqrt2-1}{4-2\sqrt2} = \dfrac{\sqrt2}{8}$

* $\tan\lambda = 1$

$\tan\mu = \sqrt3$

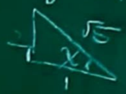
$\cos\lambda\cos\mu - \sin\lambda\sin\mu = \dfrac{1-\sqrt3}{2\sqrt2}$

* $\sec\alpha = \dfrac54$, $\dfrac{1-\tan\alpha}{1+\tan\alpha} = ?$

* $\sin\phi = \dfrac12$; $3\cos\phi - 4\cos^3\phi = ? = 0$

* $\tan\chi = 2$; $\sec\chi\sin\chi + \tan^2\chi - \operatorname{cosec}\chi = ?$

given



$\| \quad \dfrac{12-\sqrt5}{2}$

χ "chi"

$\cos^2\chi + \sin^2\chi = ? = 1$

$\cos^2\chi - \sin^2\chi = \cos^2\chi(1-\tan^2\chi)$

$= \dfrac15(-3) = -\dfrac{3}{\sqrt5}$

* $\sec\tau = \dfrac54$; $\dfrac{\tan\tau}{1+\tan^2\tau} = ? = \dfrac{12}{25}$ | τ = "tau"

$\dfrac{2\tan^2\tau}{1+\tan^2\tau} = \dfrac{18}{25}$

$\tan\tau = \dfrac34$

$\dfrac{2\tan^2\tau}{1-\tan^2\tau} = \dfrac{2\times\frac{9}{16}}{\frac11-\frac{9}{16}} = \dfrac{18}{7}$

* $16\cot\gamma = 12$, $\dfrac{\sin\gamma+\cos\gamma}{\sin\gamma-\cos\gamma} = ?$ "γ = gamma"

Simplifying

$\dfrac{\cancel{\sin\gamma}(1+\cot\gamma)}{\cancel{\sin\gamma}(1-\cot\gamma)} = \dfrac{1+\frac{12}{16}}{1-\frac{12}{16}} = 7$

* $\tan\rho = \dfrac{12}{13}$; $\dfrac{2\sin\rho\cos\rho}{\cos^2\rho-\sin^2\rho} = ?$ | ρ = "rho"

$\dfrac{2\sin\rho\cancel{\cos\rho}}{\cos^{\cancel{2}}\rho(1-\tan^2\rho)} = \dfrac{2\tan\rho}{1-\tan^2\rho} = \dfrac{312}{25}$

Level 2

* $\sin\delta = \dfrac{a^2-b^2}{a^2+b^2}$

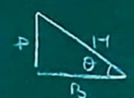
δ = "Delta"

$\tan\delta = \dfrac{a^2-b^2}{2ab}$

$\sec\delta = \dfrac{a^2+b^2}{2ab}$

Lecture - 8 (23/5/2022)

Practice 1

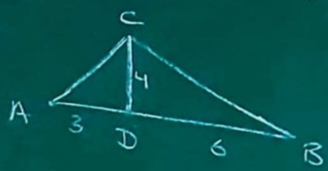


$H^2 = P^2 + B^2$

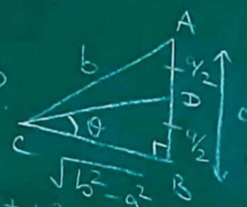
* $\tan B = \frac{2}{3}$

$\sin A = \frac{4}{5}$

$\sin^2 B + \cos^2 B = 1$



* $\sin\theta, \cos\theta, \sin^2\theta + \cos^2\theta = ?$



$CD^2 = \frac{a^2}{4} + b^2 - a^2 = -\frac{3}{4}a^2 + b^2 = \frac{4b^2 - 3a^2}{4}$

$CD = \frac{\sqrt{4b^2 - 3a^2}}{2}$

$\sin\theta = \frac{a}{\sqrt{4b^2 - 3a^2}}$; $\cos\theta = \frac{2\sqrt{b^2 - a^2}}{\sqrt{4b^2 - 3a^2}}$

$\sin^2\theta + \cos^2\theta = \frac{a^2}{(4b^2 - 3a^2)} + \frac{4(b^2 - a^2)}{4b^2 - 3a^2} = 1$

* $OQ - PQ = 1$

$\sec^2\theta - \tan^2\theta = ?$

$\frac{1}{\left(\frac{7}{OQ}\right)^2} - \frac{QP^2}{7^2} = \frac{OQ^2 - QP^2}{7^2} = \frac{OP^2}{7^2} = 1$

* $\tan\psi + \frac{1}{\tan\psi} = 2 \rightarrow \tan^2\psi + \frac{1}{\tan^2\psi} + 2 = 4$

$\tan^2\psi + \frac{1}{\tan^2\psi} = ?$

9/10

* $\sin B = \sin Q \Rightarrow \angle B = \angle Q$ P.T

$\frac{AC}{AB} = \frac{PR}{PQ} = K$

* $AB^2 = AC^2 + BC^2$

* $PQ^2 = PR^2 + \underbrace{RQ^2} \Rightarrow RQ = \sqrt{PQ^2 - PR^2}$

$BC = \sqrt{AB^2 - AC^2}$

$= \sqrt{K^2 PQ^2 - K^2 PR^2}$

$= K\sqrt{PQ^2 - PR^2}$

$\frac{AC}{AB} = \frac{PR}{PQ}$

$AC \cdot PQ = PR \cdot AB$

$\frac{AC}{PR} = \frac{AB}{PQ} = K$

* $\frac{BC}{RQ} = K$

$\frac{AC}{PR} = \frac{AB}{PQ} = \frac{BC}{RQ}$

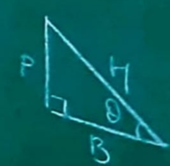
$\Delta_1 \sim \Delta_2$

$\angle B = \angle C$

d. Trigno. machines' Identities

* $H^2 = P^2 + B^2 \Rightarrow 1 = \underbrace{\frac{P^2}{H^2}}_{\sin^2\theta} + \underbrace{\frac{B^2}{H^2}}_{\cos^2\theta}$

Not true globally



* $\boxed{\sin^2\theta + \cos^2\theta = 1}$

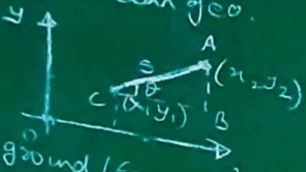
locally defined

"fundamental Identity"

Lecture - 9 (28/5/2022)

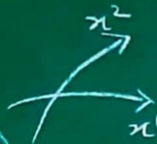
d Trigonometric Identities (Part 1)

Euclidean geo.



ground/Eye

"Non-Eucl." Differential Geometry



curvi-linear Coordinates

genuine curved

$\{x, y\} \longrightarrow \{r, \theta\} \longrightarrow \{x^1, x^2\}$

Basis "flat Coord"

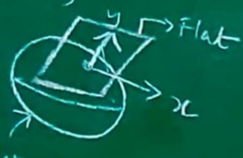
* $s^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

$\updownarrow$

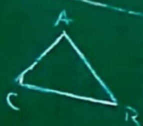
$H^2 = P^2 + B^2$ (Phyth.)

$\Uparrow$

$\boxed{A + B + C = 180^\circ}$



curved



Angle sum property "Not sacred" $\longrightarrow$

Hyperbolic $\Sigma < 180^\circ$

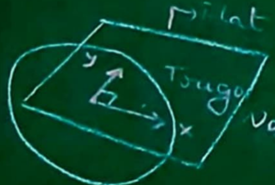
Elliptic $\Sigma > 180^\circ$

forget difference (Inequality)

$\Downarrow$

=

"distortion" of platonic shapes



$A + B + C \neq 180$

$\Downarrow$

$H^2 \neq P^2 + B^2$

$\Downarrow$

$S^2 = \sum_{a,b=1,2} g_{ab} \underbrace{x^a x^b}_{\text{basis}}$ Metric

Special Case

$g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$S^2 = 1(x^1)^2 + 1(x^2)^2$

$= 1 \cdot x^2 + 1 \cdot y^2$

* $H^2 = P^2 + B^2$

$\Downarrow$

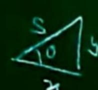
$\sin\theta = \frac{P}{H}, \cos\theta = \frac{B}{H}$

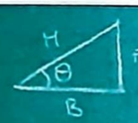
$\tan\theta = \frac{P}{B}$

* $H^2 = P^2 + B^2$

$1 = \frac{P^2}{H^2} + \frac{B^2}{H^2} \Rightarrow \boxed{\sin^2\theta + \cos^2\theta = 1}$

~ Pythagoras-trigno Identity





* $\sin^2\theta + \cos^2\theta = 1$ .... (1)

* $\frac{\sin^2\theta}{\cos^2\theta} + 1 = \frac{1}{\cos^2\theta} \Rightarrow 1 + \tan^2\theta = \sec^2\theta$ —— (2)

* $1 + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \Rightarrow 1 + \cot^2\theta = \mathrm{cosec}^2\theta$ —— (3)

* $\sec^2\theta = \tan^2\theta + 1 \Rightarrow \underbrace{\sec^2\theta - \tan^2\theta}_{(\sec\theta + \tan\theta)(\sec\theta - \tan\theta)} = 1$

* $\sec\theta + \tan\theta = \frac{1}{\sec\theta - \tan\theta}$

* $\mathrm{cosec}^2\theta = \cot^2\theta + 1 \Rightarrow \mathrm{cosec}\theta + \cot\theta = \frac{1}{\mathrm{cosec}\theta - \cot\theta}$

Practice 2 (Intermediate) decipher Comb^n

Expressions $\longrightarrow$ simplify | $1 = \sin^2\theta + \cos^2\theta$

* $\frac{1}{1 + \sin\theta} + \frac{1}{1 - \sin\theta} = 2\sec^2\theta$

P.T

* $\mathrm{cosec}^2\theta + \sec^2\theta = \mathrm{cosec}^2\theta \sec^2\theta$

* $\sqrt{\sec^2\theta + \mathrm{cosec}^2\theta} = \tan\theta + \cot\theta$

Rev. Engineer tech

$\sec^2\theta + \mathrm{cosec}^2\theta = (\tan\theta + \cot\theta)^2$

$= \tan^2\theta + \cot^2\theta + \overset{1+1}{2}\underbrace{\tan\theta\cot\theta}_{1}$

$\Uparrow = (\tan^2\theta + 1) + (\cot^2\theta + 1)$

$= \sec^2\theta + \mathrm{cosec}^2\theta$

HW

* $(\sin^4\theta - \cos^4\theta + 1)\mathrm{cosec}^2\theta = 2$

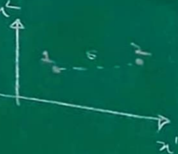
* $\sqrt{\frac{1 - \sin\theta}{1 + \sin\theta}} = \sec\theta - \tan\theta$

Lecture-10 (29/5/2022)

d': Tensors/Matrices/"Distance" formula

* $\boxed{\text{Prescription}}$ for distance"

$s^2 = \sum_{a=1,2} g_{ab} x^a x^b$



Examp 1

* $g_{ab} = \delta_{ab} \overset{\text{def.}}{\equiv} \begin{cases} 1 & a=b \\ 0 & a\neq b \end{cases}$

$a, b = 1, 2$

$\delta_{11} = 1$

$\delta_{21} = 0$

$\delta_{12} = 0$

$\delta_{22} = 1$

Euclidean 2D space

$\updownarrow$

* $\delta_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

[Quantity] list = Juxtaposition

Matrix representation of Kronecker delta.

* $s^2 = \sum \underbrace{g_{ab}}_{\delta_{ab}} x^a x^b = \sum_{a,b=1,2} \delta_{ab} x^a x^b$

$= \delta_{11}x^1x^1 + \overset{0}{\delta_{12}}x^1x^2 + \overset{0}{\delta_{21}}x^2x^1 + \delta_{22}x^2x^2$

$= 1(x^1)^2 + 1(x^2)^2 + 0(x^1)(x^2) + 0(x^2)(x^1)$

* $\boxed{s^2 = x^2 + y^2}$

Comment on g_{ab}

$\delta_{ab} = \begin{cases} 1 & a=b \\ 0 & a\neq b \end{cases} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I = \mathbb{1}$

unit Matrix

$|A|$

* $\det(\delta_{ab}) \equiv |\delta_{ab}| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$ (unit)

* Identity Matrix
$\|$
unit matrix $\Rightarrow \det|A| = 1$
A

* $\delta_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times 2}$ → off-diag

diagonal terms (symmetric part)

diagonal terms : $\delta_{11}, \delta_{22} \quad \delta_{nn}$

$\|$

* $Tr(\delta_{ab}) \equiv \underbrace{\sum_{i=1}^{n} \delta_{ii}}_{\text{Sum of diag. terms}} = \delta_{11} + \delta_{22} + \delta_{nn} = n$

$\boxed{Tr(\text{Id. Matrix}) = n}$

* $\text{Trace}(A) = Tr(A) = \sum_{i=1}^{n} A_{ii}$

$\uparrow$ matrix $(n\times n)$

* $A \equiv \begin{cases} a & i=j \\ 0 & i\neq j \end{cases} \Rightarrow$ Diagonal Matrix

$\downarrow a = 1$

$A = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases} \Rightarrow$ Identity Matrix

$\|$

I

* All diag. mat. are not Identity mat.

All Id. mat ARE diag mat.

* $\delta_{ab} = \delta_{ba}$ symmetric Matrix

* $A_{ij} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad A_{11} = -1 = A_{22}$, $A_{21} = 0 = A_{12}$

$A_{ij} = A_{ji}$ $\quad$ sym.

* $B_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad B_{11} = 1, B_{22} = -1$, $B_{21} = 0 = B_{12}$

$B_{ij} = B_{ji}$ $\quad$ sym.

* $A_{ij} = A_{ji} \Rightarrow A$: symmetric matrix / Tensor

$i \leftrightarrow j$ interchange

* $A_{ij} = -A_{ji} \Rightarrow$ Anti-symmetric matrix / tensor

$i \leftrightarrow j$ interchange

* $C_{ij} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\quad C_{11} = 0 = C_{22}$, $C_{12} = -1$, $C_{21} = 1$

$C_{ji} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad C_{11} = 0 = C_{22}$, $C_{12} = 1$, $C_{21} = -1$

$= (-1)\underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{C_{ij}} = -C_{ij}$

scaling transformation of a Matrix

* $A_{n\times n} \longrightarrow \lambda A \sim \lambda A_{ij}$

$\|$

$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \longrightarrow \lambda \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} \lambda A_{11} & \lambda A_{12} \\ \lambda A_{21} & \lambda A_{22} \end{pmatrix}$

* $A - \lambda A \neq 0$ → Later [Eigenvalue of a Matrix]

↓ Imp role

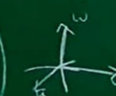
Matrix Mechanics of $\boxed{\text{QM}}$

Copenhagen Intp. (Platonism)

* λ = scalar factor
= Scaling factor
= Dilation factor
= Stretching / Squeezing Factor
= Conformal Factor (Maybe Never)
= 1×1 matrix

Exampl. 2 HW

* $g_{ij} = \begin{pmatrix} v^2+w^2 & uv & uw \\ uv & w^2+v^2 & vw \\ uw & wv & v^2+u^2 \end{pmatrix}_{3\times 3}$



basis (ground for judgment) : $x^i = (u, v, w)$

* distance $(s^2) = ?$

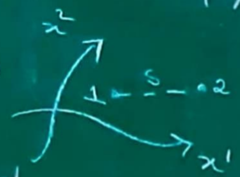
Lecture - 11 (30/5/2022)

d': Tensors / Matrices / "Distance" formula

d.1 Examples of Metric tensor (Matrix rep)

Prescription

* $s^2 = \sum_{a,b} \overset{\text{metric tensor}}{g_{ab}} x^a x^b = g_{ab} x^a x^b$ (a,b ← indices)



$= g_{11} x^1 x^1 + g_{12} x^1 x^2 + \dots + g_{1n} x^1 x^n$
$+ g_{21} x^2 x^1 + g_{22} x^2 x^2 + \dots + g_{2n} x^2 x^n$
$\vdots$
$g_{n1} x^n x^1 + g_{n2} x^n x^2 + \dots g_{nn} x^n x^n$

Sum Notation
||
sum over index one on top & one on bottom.

* $$g_{ab} = \begin{pmatrix} g_{11} & g_{12} & \dots & g_{1n} \\ g_{21} & g_{22} & \dots & g_{2n} \\ \vdots & & & \vdots \\ g_{n1} & g_{n2} & & g_{nn} \end{pmatrix}_{n\times n}$$ Matrix Representation of metric

$s^2 = \sum_{a,b} g_{ab} x^a x^b$; $x^a = (x^1, x^2, \dots x^n)$

↓ $a \leftrightarrow b$ unchanged

$s'^2 = \sum g_{ba} x^b x^a = \sum g_{ba} x^a x^b = s^2$

[later More rigorous proof]

⇓

$\boxed{g_{ab} = g_{ba}}$

Symmetric Matrix

* $s^2 = \sum g_{ab} x^a x^b$

* $g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \boxed{s^2 = x^2 + y^2}$

basis $\{x, y\}$

Eq^n of Circle

$\boxed{s^2 \geq 0}$

det $g_{ab} = 1$

$\mathbb{R}^2$

Example (2)

* $g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow s^2 = g_{11} x^1 x^1 + \overset{0}{g_{12}} x^1 x^2 + \overset{0}{g_{21}} x^2 x^1 + g_{22} x^2 x^2$

basis $\{x, y\}$

$\boxed{s^2 \gtreqless 0}$

$\boxed{s^2 = x^2 - y^2}$ Eq^n of Hyperbola

det $g_{ab} = -1$

Example (3) : Consider Platonic Heaven

* $g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow s^2 = g_{11}(x^1)^2 + g_{22}(x^2)^2$

$\boxed{s^2 = c^2t^2 - x^2}$

basis : $\{\underset{x^1}{ct}, \underset{x^2}{x}\}$

$s^2 = c^2t^2\left(1 - \frac{1}{c^2}\underbrace{\frac{x^2}{t^2}}\right)$

$v \equiv \frac{x}{t}$

$= c^2t^2\left(1 - \frac{v^2}{c^2}\right)$

$\gamma \equiv \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} > 1$

$s^2 = \frac{c^2t^2}{\gamma^2} = c^2\tau^2$

$\tau \equiv \frac{t}{\gamma} \Rightarrow \boxed{\tau = t\overbrace{\sqrt{1-\frac{v^2}{c^2}}}^{<1}}$

↓ $v = 0$: $\tau = t$; $t = 10$, $\tau = 10$

↓ $v \neq 0$: $\sqrt{1-\frac{v^2}{c^2}} = 0.5$; $t = 10$ sec, $\tau = 5$

[Time dilation]

Comment including Physics (Interpretation)

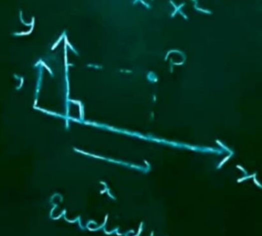
* x = pos-val
t = t-val
v = m/s
c = m/s
τ = time

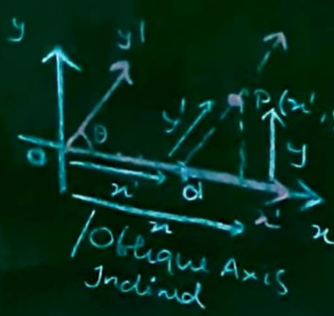
} (later) Relativity

Example (4)

* $g_{ab} = \begin{pmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{pmatrix} \Rightarrow s^2 = x'^2 + y'^2 + 2\cos\theta\, x'y'$

basis $\{\underset{x^1}{x'}, \underset{x^2}{y'}\}$





Lecture - 12 (11/6/2022)

$\sum_a \sum_b$

d' : Tensors / Matrices / "Distance" formula

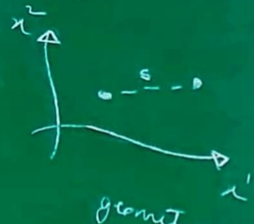
d.1 examples of metric tensor (Matrix Rep)

* $x^a = \{x^1, x^2, \ldots x^N\}$

* $s^2 = \sum g_{ab} x^a x^b = g_{ab} x^a x^b$

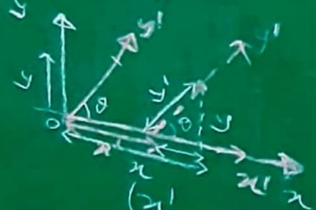
distance prescription

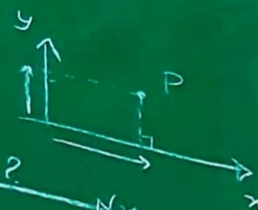
Algebraic



geometrical

Ex-4

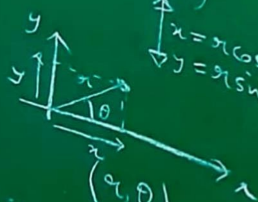




$(x'\ y') \leftarrow ? \rightarrow (x, y)$

$x = x' + y'\cos\theta$

$y = y'\sin\theta$



$x = r\cos\theta$

$y = r\sin\theta$

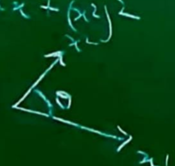
(r, θ)

* $g_{ab} = \begin{pmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{pmatrix}$; basis $\{x^1\ x^2\}$

$s^2 = (x^1)^2 + \underbrace{\cos\theta\, x^1 x^2 + \cos\theta\, x^2 x^1}_{2\cos\theta\, x^1 x^2} + (x^2)^2$ | $x^1 x^2 = x^2 x^1$

$s^2 = (x^1)^2 + 2\cos\theta\, x^1 x^2 + (x^2)^2$

$\downarrow \theta = 90°$



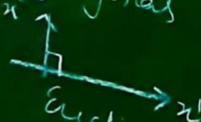
$s^2 = (x^1)^2 + (x^2)^2$

$g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Pythagoras

$\cos 0 = 1$

$\cos 90 = 0$



Euclidean geometry

Equilm / Privileged frame

* $s^2 = g_{ab} x^a x^b$

$\downarrow$ Differential Picture (dx) (locality)

Analysis (Variation Calculus)

$\downarrow$

EL Eqn $\longrightarrow$ CM / GR / QFT

---

* g_{ab} : arbitrary metric can't be Analysis
symmetrical sys. easy to handle

Ex-5

* $g_{ab} = \begin{pmatrix} 5 & -3 & 2 \\ -3 & 3 & 0 \\ 2 & 0 & 4 \end{pmatrix}$; basis $\{x^1, x^2, x^3\} = \{x, y, z\}$

commutative

* $s^2 = \sum_{a,b=1}^{3} g_{ab} x^a x^b$

$= g_{11}(x^1)^2 + g_{12} x^1 x^2 + g_{13} x^1 x^3 + \ldots$

$s^2 = 5x^2 + 3y^2 + 4z^2 - 6xy + 4xz$

$\downarrow g_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$s^2 = x^2 + y^2 + z^2$

Ex-6

* $g_{ab} = \begin{pmatrix} 1/t^2 & 0 \\ 0 & -1/t^2 \end{pmatrix}$; basis $\{x, y\}$

* $s^2 = \frac{1}{t^2}(x^2 - y^2)$

$\rightarrow s^2 > 0 \Rightarrow x^2 > y^2$

$\rightarrow s^2 = 0 \Rightarrow x = y$

$\rightarrow \boxed{s^2 < 0 \Rightarrow x^2 < y^2}$

* $\det g_{ab} = -\frac{1}{t^4}$

Ex-7

* $g_{ab} = \begin{pmatrix} f(r) & 0 \\ 0 & -\frac{1}{f(r)} \end{pmatrix}$ basis $\{t, x\}$

$s^2 = f(r)\, t^2 - \frac{1}{f(r)} x^2$; $\det g_{ab} = -1$

$\downarrow$ const fn

$f(r) = 1$

Minkowskian space

$\downarrow$ quadratic

$f(r) = \left(1 - \frac{\Lambda r^2}{3}\right)$

De Sitter space

$\updownarrow$

Universe

$f(r) = \left(1 + \frac{\Lambda r^2}{3}\right)$

$\boxed{\text{Anti-De Sitter sp.}}$

Lecture - 13. (13/6/2022)

d.2 Matrices / Euclidean - Non Euclidean

Multiplicity - Bergson

(Countable) Numerical Multiplicity

(Uncountable) Non-Numerical

Numbers/space/ spacetime

Quantity

Representation

Manifold

list/Juxtaposition

Duration

Quality

feelings

* $x^i \equiv (x^1, x^2, \ldots x^N)$ 1D list

N Entries

Countable multiple.

N-vector

* $g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $x^i = (x^1, x^2)$

2D list

2-vector

* $x_i = \sum_j g_{ij} x^j$

free index

Dummy index / Summed over

$i, j = 1, 2$

$x_i = \sum_{j=1}^{2} g_{ij} x^j = g_{i1} x^1 + g_{i2} x^2$

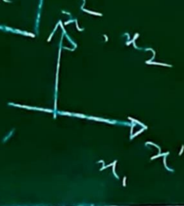
(2D list) (1D list)

$i=1$, $x_1 = g_{11} x^1 + \cancel{g_{12}} x^2 \Rightarrow \boxed{x_1 = x^1}$

$i=2$, $x_2 = \cancel{g_{21}} x^1 + g_{22} x^2 \Rightarrow \boxed{x_2 = x^2}$

In Euclidean geometry where you put indices doesn't Matter.

$x_i = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = x^i$



Ex-2

$s^2 = \sum g_{ij} x^i x^j$

Riemannian prescription

* $g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Contravariant

$x^i = (x^1, x^2)$

superguy

$s^2 = (x^1)^2 - (x^2)^2 \Rightarrow$ Non Euclidean Geometry/metric

Covariant

$x_i = \sum_j g_{ij} x^j = g_{i1} x^1 + g_{i2} x^2$

$i=1$: $x_1 = g_{11} x^1 + g_{12} x^2 \Rightarrow \boxed{x_1 = x^1}$

$i=2$: $x_2 = g_{21} x^1 + g_{22} x^2 \Rightarrow \boxed{x_2 = -x^2}$

It matters where you put indices.

$x_i = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x^1 \\ -x^2 \end{pmatrix}$ 1D list

Operations

d.2.1 Addition of Matrices

* $C_{ij} = A_{ij} + B_{ij}$

$C_{11} = A_{11} + B_{11}$

$A - \underset{\sim}{A} = 0$

Set theory

Ex 1 2 2D lists

* $A_{ij} = \begin{array}{c} A_{11}\ A_{12}\ A_{13} \\ \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}_{2\times 3} \end{array}$, $B_{ij} = \begin{array}{c} B_{11}\ B_{12}\ B_{13} \\ \begin{pmatrix} 6 & 5 & 4 \\ 3 & 2 & 1 \end{pmatrix}_{2\times 3} \end{array}$

$C_{ij} = A_{ij} + B_{ij}$

$C_{11} = A_{11} + B_{11} = 7$

$C_{12} = A_{12} + B_{12} = 7$

$C_{23} = A_{23} + A_{23} = 7$

; $g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ Euclidean

$C_{ij} = \begin{pmatrix} 7 & 7 & 7 \\ 7 & 7 & 7 \end{pmatrix}_{2\times 2}$

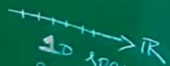
2D list

Lecture - 14. (18/6/2022)

* uncountable Multiplicity
  Movement (primary)
  $\downarrow$
  Space (secondary) $\Rightarrow$ Number/Counting = Countable Multiplicity (list/juxtaposition)
  "suddenly assumed"
  (Involuntary)

* $\phi \equiv$ Scalar 0D space
* $A_i = (A_1, A_2, \ldots A_N)_{1\times N}$ Row rep.

$$= \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{pmatrix}_{N\times 1}$$ Column Rep.



\# entries $= n(A) = N$

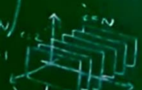
$$g_{ij} = \begin{pmatrix} g_{11} & g_{12} & \ldots & g_{1N} \\ g_{21} & \ldots & \ldots & g_{2N} \\ \vdots & & & \vdots \\ g_{N1} & \ldots & \ldots & g_{NN} \end{pmatrix}$$ Matrix Rep.

\# entries $= N^2$

* R_{ijk}

$i = 1$ $\quad R_{\underbrace{jk}} = \begin{pmatrix} R_{111} & R_{112} & \ldots & R_{11N} \\ \vdots & & & \\ R_{1N1} & R_{1N2} & \ldots & R_{1NN} \end{pmatrix}$

$i = 2$
$\vdots$
$i = N$ $\quad R_{Njk} = \begin{pmatrix} R_{N11} & R_{N12} & \ldots & R_{N1N} \\ \vdots & & & \\ R_{NN1} & R_{NN2} & \ldots & R_{NNN} \end{pmatrix}$



\# entries $= N^3$
\# slices $= N \longrightarrow$ each slice has N^2 entries.

2.2.1 Laws of Addition of Matrices

a) Comment on Same dim

* $C_{ij} = A_{ij} + B_{ij} \quad i,j \in [1, N]$

Space = Euclid like
$\downarrow$
upstairs (Contra) = downstairs (Covariant)

* $A_{ij} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}_{2\times 3}$ $\quad B_{ij} = \begin{pmatrix} 3 & 2 \\ 7 & 1 \end{pmatrix}_{2\times 2}$

$n(A) = 6$ $\quad n(B) = 4$

$A_{13} = 6$ $\quad B_{13} \neq 0 \Rightarrow C_{13} =$ unknown
$\Downarrow$
$C_{ij} = A_{ij} + B_{ij}$ Not defined

$\underbrace{N\times N'}$ $\quad \underbrace{N\times N'}$ Machine $\quad \underbrace{N\times N'}$

$C_{ij} = A_{ij} + B_{ij}$

$i: N, \; j: N'$

Same dimension Matrices Can be Added.

* $A_{ij} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$, $B_i = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$A_{ij} + B_i$
$\uparrow$
Not defined

b) Basic Arithmetic like properties

* $C_{ij} = A_{ij} + B_{ij} = B_{ij} + A_{ij}$ Pre requisite satisfied
  Commutative under +
* $(A+B)+C = A+(B+C)$
  Associative under +
* A_{ij} ; $\exists\, O_{ij}$: $A+O = O+A = A$
  Existence of Identity under +
  $O_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Existence of Additive Inverse

* $A_{ij}, O_{ij} \Rightarrow \exists\, (-A_{ij}) : A_{ij} + (-A_{ij}) = (-A_{ij}) + A_{ij} = O_{ij}$

Ground for Assignment

Lecture-15 (19/6/2022)

* $C_{ij} = A_{ij} \overset{\text{Addition Machine}}{+} B_{ij}$ "Natural" addition

$C_{ij} = B_{ij} + A_{ij}$ $\quad$ || Same dim. matrices get added

* $\exists\; 0_{ij} = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & & & \\ \vdots & & & \end{pmatrix}_{N\times N'}$ $\quad A+0=A$

* $\forall A_{ij} \; \exists (-A_{ij}) : A_{ij} + (-A_{ij}) = 0_{ij}$

Additive Inverse

* $\cancel{A} + B = \cancel{A} + C \Rightarrow \boxed{B=C}$ Cancellation law.

$(-A)+(A+B) = (-A)+(A+C)$

$\underbrace{(-A+A)}_{0} + B = \underbrace{(-A+A)}_{0} + C \Rightarrow \underbrace{0+B}_{B} = \underbrace{0+C}_{C}$

2.2.2 Multiplication by a scalar

* Scalar ϕ, ψ, λ
* A_{ij}, ϕ (Phi) $\downarrow$ No index $\longrightarrow$ $\phi A_{ij} = \begin{pmatrix} \phi A_{11} & \phi A_{12} & \cdots \\ \vdots & & \\ \phi A_{N1} & \cdots & \phi A_{NN} \end{pmatrix}$
* $\sin^2\alpha$ $\downarrow$ Scalar $\begin{pmatrix} 2 & \ln i \\ i & \sqrt{i} \end{pmatrix}_{2\times 2} = \begin{pmatrix} 2\sin^2\alpha & \sin^2\alpha \ln i \\ i\sin^2\alpha & \sqrt{i}\sin^2\alpha \end{pmatrix}$
* $\phi(A_{ij}+B_{ij}) = \phi A_{ij} + \phi B_{ij}$
* $(\phi+\psi)A_{ij} = \phi A_{ij} + \psi A_{ij}$
* $(\phi\psi)A_{ij} = (\psi\phi)A_{ij} = \psi\phi A_{ij}$

* $(-\phi)A_{ij} = -(\phi A_{ij}) = \phi(-A_{ij})$
* $\boxed{1\cdot A_{ij} = A_{ij}1 = A_{ij}}$

$A = A$ Self-Identical (Mathematical)

$A \neq A$ self-different (Reality)

2.2.3 Subtraction

* $A_{ij} = \begin{pmatrix} 2 & 5 & 0 & 7 \\ -1 & 6 & 2 & 4 \\ 3 & -4 & 8 & -2 \end{pmatrix}$ $\quad B_{ij} = \begin{pmatrix} -1 & 0 & 3 & -3 \\ 2 & 2 & -2 & 5 \\ -3 & 5 & 6 & 4 \end{pmatrix}$

$A+B = \begin{pmatrix} 1 & 5 & 3 & 4 \\ 1 & 8 & 0 & 9 \\ 0 & 1 & 14 & 2 \end{pmatrix}$

* $A_{ij} = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}$ $\quad B_{ij} = \begin{pmatrix} 1 & 4 \\ 7 & 2 \end{pmatrix}$

$3A_{ij} - 2B_{ij} = \begin{pmatrix} 4 & -11 \\ -5 & -1 \end{pmatrix}$

* $\cos\alpha \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} + \sin\alpha \begin{pmatrix} \sin\alpha & -\cos\alpha \\ \cos\alpha & \sin\alpha \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$

* $X_{ij} + Y_{ij} = \begin{pmatrix} 7 & 0 \\ 2 & 5 \end{pmatrix}$

$X_{ij} - Y_{ij} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ $\quad X_{ij}, Y_{ij} = ?$

Interesting Eq^n. Matrix Equation

$X_{ij} = \begin{pmatrix} 5 & 0 \\ 1 & 4 \end{pmatrix}, \quad Y_{ij} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$

* $\begin{pmatrix} x^2 \\ y^2 \end{pmatrix} - 3\begin{pmatrix} x \\ 2y \end{pmatrix} = \begin{pmatrix} -2 \\ 9 \end{pmatrix}$ Matrix Equation

$\begin{pmatrix} x^2 - 3x + 2 \\ y^2 - 6y - 9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$ Elements $x = 2, 1$

$y = 3 \pm 3\sqrt{2}$

Lecture-16 (20/6/2022)

d24 Kronecker Sum of Matrices

$A = \mathrm{diag}(1,2,3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

* $\underbrace{C_{ij}}_{N\times N'} = \underbrace{A_{ij}}_{N\times N'} + \underbrace{B_{ij}}_{N\times N'} = B_{ij} + A_{ij}$ Commutative = Abelian

* {Order of Matrices has to be same

$A_{ij} - N\times N$

$B_{ij} - M\times M \longrightarrow C \equiv A \oplus B \neq B \oplus A$ direct sum/Kronecker sum

Square (Container)

$(N+M)\times(N+M) = \begin{pmatrix} A_{N\times N} & O_{N\times M} \\ O_{M\times N} & B_{M\times M} \end{pmatrix}$

Non Commutative

* $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $B = \begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & \alpha \\ \beta & \gamma & \mu \end{pmatrix}$, $D = \psi_{1\times 1}$

$C = A \oplus B = \begin{pmatrix} A_{2\times 2} & O_{2\times 3} \\ O_{3\times 2} & B_{3\times 3} \end{pmatrix}_{5\times 5} = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 \\ 0 & 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 & \alpha \\ 0 & 0 & \beta & \gamma & \mu \end{pmatrix}_{5\times 5}$

* $E \equiv \overset{2\times 2}{A} \oplus \overset{3\times 3}{B} \oplus \overset{1\times 1}{D} = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 6 & 7 & 0 \\ 0 & 0 & 8 & 9 & \alpha & 0 \\ 0 & 0 & \beta & \gamma & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \psi \end{pmatrix}$

Block diagonalized form

* $M = A_1 \oplus A_2 \oplus \cdots \oplus A_k \equiv \bigoplus_{i=1}^{k} A_i$ direct sum Notation

$a_1, a_2, \ldots, a_k \longleftarrow$ order of Matrices

$\underbrace{(a_1 + a_2 + \cdots a_k)}$ order of Resultant Matrix

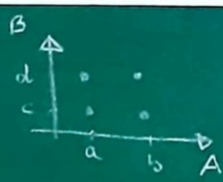
* Addition of different orders of Matrices Now Possible (Win Win!)

Sum Notation

$\sum_{i=1}^{n} A_i \equiv A_1 + A_2 + \cdots A_n$

$\prod_{i=1}^{n} A_i \equiv A_1 A_2 \cdots A_n$

Comment on Direct Sum in Set Theory



* $A = \{a, b\}$ Possibility, $B = \{c, d\}$

* $A \times B = \{(a,c), (a,d), (b,c), (b,d)\} \neq B \times A$ Algebraic rep

Cartesian Product

* $(a,b) + (c,d) = (a+c, b+d)$ Direct Sum of ordered Pair

d25 Multiplication

a) Feeling of Contraction

* $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot 4 = \begin{pmatrix} 4 & 8 \\ 12 & 16 \end{pmatrix}$

* $\underset{C_{ij}}{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}_{2\times 2}} \bullet \underset{B_{ij}}{\begin{pmatrix} \alpha & \beta \\ \gamma & \iota \end{pmatrix}_{2\times 2}}$

1) overlap/superposition

$= \begin{pmatrix} 1\alpha & 2\beta \\ 3\gamma & 4\iota \end{pmatrix}$

2) 2D list 2×2 A_{ij}

$= \begin{pmatrix} 1\alpha & 1\beta & 2\alpha & 2\beta \\ 1\gamma & 1\iota & 2\gamma & 2\iota \\ 3\alpha & 3\beta & 4\alpha & 4\beta \\ 3\gamma & 3\iota & 4\gamma & 4\iota \end{pmatrix}_{4\times 4}$ 2D list

3) Sum Element Contraction $\longrightarrow 1\alpha + 2\beta + 3\gamma + 4\iota$ = Number, 0D list

$\phi = \sum_{i,j} C_{ij} B^{ij}$

$\phi = \sum_{i,j=1,2} A_{ij}$

* $\sum$ Reduced the dimension = Contraction

$\sum_{i,j=1,2} C_{ij} B^{ij} = C_{11} B^{11} + C_{12} B^{12} + C_{21} B^{21} + C_{22} B^{22} = \phi$

Lecture-17 (23/6/2022) | FAPP — for all practical purposes

Ex1 1D list × Scalar

* $A_{1i} = \begin{pmatrix} 2 & 3 & 5 \end{pmatrix} = \underset{\sim}{A_i} \equiv (A_1\ A_2\ A_3)$ $\qquad \underset{\sim}{A} = \vec{A}$

1D list/set/Row Matrix/3-vector

$\Phi = \pi$

$\underbrace{\Phi A_i} = (2\pi\ \ 3\pi\ \ 5\pi) = B_i$

* $\boxed{B_i = \Phi A_i}$ $\qquad i = 1, 2, \ldots n$

$B_j = \Phi A_j$ $\qquad \uparrow$ n Entries

$M^a = \Phi N^a$

Ex-2 2D list × Scalar

* $A_{ij} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}_{2\times 2}, \ \Phi = \ln i$

2D list/Matrix

$\Phi A_{ij} = \ln i \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} \ln i & 2\ln i \\ 3\ln i & 4\ln i \end{pmatrix}_{2\times 2} = B_{ij}$

* $\boxed{B_{ij} = \Phi A_{ij}}$

b) Kronecker product/Direct prod/Outer prod

* A_{lm} — $L\times M$ order

B^{pq} — $P\times Q$

$\Big\} \longrightarrow C^{pq}_{lm} \equiv A_{lm} \otimes B^{pq}$

$(LP \times MQ)$ order

any dim. of Matrix can be multiplied with any other Matrix dim.

$C^{pq}_{lm} = \begin{pmatrix} A_{lm}B^{11} & A_{lm}B^{12} & \cdots & A_{lm}B^{1q} \\ A_{lm}B^{p1} & A_{lm}B^{p2} & \cdots & A_{lm}B^{pq} \end{pmatrix}$

2D representation of 4 index obj

$= \begin{pmatrix} A_{11}B^{11} & A_{12}B^{11} & A_{1m}B^{11} & \\ A_{l1}B^{11} & A_{l2}B^{11} & A_{lm}B^{11} & \cdots \end{pmatrix}_{(LP\times MQ)}$

* $M_{ab} = \begin{array}{cc} M_{11} & M_{12} \\ \end{array}$ $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}_{2\times 2}$ $\qquad N_a = \begin{array}{ccc} N_1 & N_2 & N_3 \end{array}$ $(\alpha\ \ \beta\ \ \gamma)_{1\times 3} = N_c = N^c$

* $\underbrace{M_{ab} \otimes N^c} = \begin{pmatrix} 1(\alpha\ \beta\ \gamma) & 2(\alpha\ \beta\ \gamma) \\ 3(\alpha\ \beta\ \gamma) & 4(\alpha\ \beta\ \gamma) \end{pmatrix}$

$= S^c_{ab}$

$= \begin{pmatrix} 1\alpha & 1\beta & 1\gamma & 2\alpha & 2\beta & 2\gamma \\ 3\alpha & 3\beta & 3\gamma & 4\alpha & 4\beta & 4\gamma \end{pmatrix}_{2\times 6}$

2D representation of a "3 index obj"

$\equiv A_{ij}$ $\qquad i = 1, 2$ $\quad j = 1, 2, 3, 4, 5, 6$

$c = 1, 2, 3;\ a, b = 1, 2$

* $S^1_{11} = M_{11} \otimes N^1 = 1\alpha$

$S^1_{12} = M_{12} \otimes N^1 = 2\cdot\alpha$

$S^1_{21} = M_{21} \otimes N^1 = 3\alpha$

$S^1_{22} = M_{22} \otimes N^1 = 4\alpha$

c = 1 slice

$S^1_{ab} = \begin{pmatrix} 1\alpha & 2\alpha \\ 3\alpha & 4\alpha \end{pmatrix}$

* $S^2_{11} = M_{11} \otimes N^2 = 1\cdot\beta$

$S^2_{12} = M_{12} \otimes N^2 = 2\beta$

$S^2_{21} = M_{21} \otimes N^2 = 3\cdot\beta$

$S^2_{22} = M_{22} \otimes N^2 = 4\cdot\beta$

c = 2 slice

$S^2_{ab} = \begin{pmatrix} 1\beta & 2\beta \\ 3\beta & 4\beta \end{pmatrix}$

* $S^3_{11} = M_{11} \otimes N^3 = 1\gamma$

$S^3_{12} = M_{12} \otimes N^3 = 2\gamma$

$S^3_{21} = M_{21} \otimes N^3 = 3\cdot\gamma$

$S^3_{22} = M_{22} \otimes N^3 = 4\gamma$

c = 3 slice

$S^3_{ab} = \begin{pmatrix} 1\gamma & 2\gamma \\ 3\gamma & 4\gamma \end{pmatrix}$

* $S^c_{ab} = M_{ab} \otimes N^c =$

3D representation of a "3 index obj"

"2×2×3" — row i, col. j, depth k

Not very helpful that's why go for slices (2D)

Lecture – 18 (25/6/2022)

* $M^a = \phi N^a$ (Rank 0 tensor (Scalar)) $\Longleftrightarrow \begin{pmatrix} M^1 \\ M^2 \\ \vdots \\ M^N \end{pmatrix} = \phi \begin{pmatrix} N^1 \\ N^2 \\ \vdots \\ N^N \end{pmatrix}$; $M^1 = \phi N^1$

Rank-1 "Tensor" (vector)

* $M^{ab} = \phi N^{ab} \Longleftrightarrow \begin{pmatrix} M^{11} & \cdots & M^{1B} \\ M^{A1} & & M^{AB} \end{pmatrix}_{A\times B} = \phi \begin{pmatrix} N^{11} & \cdots & N^{1B} \\ N^{A1} & \cdots & N^{AB} \end{pmatrix}_{A\times B}$

Rank-2 "Tensor"

$M^{11} = \phi N^{11}$

* $S_{ab}{}^{cd} = M_{ab} \otimes N^{cd}$ Rank-4 "Tensor"

$M_{ab} = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}_{3\times 2}$, $N^{cd} = \begin{pmatrix} r & s & t \\ x & y & z \end{pmatrix}_{2\times 3}$

$a = 1,2,3$
$b = 1,2$

$c = 1,2$
$d = 1,2,3$

"2D rep of 4 index obj."

$$S_{ab}{}^{cd} = \begin{pmatrix} ar & as & at & br & bs & bt \\ ax & ay & az & bx & by & bz \\ cr & cs & ct & dr & ds & dt \\ cx & cy & cz & dx & dy & dz \\ er & es & et & fr & fs & ft \\ ex & ey & ez & fx & fy & fz \end{pmatrix}_{6\times 6}$$

* $S^{cd}{}_{ab} = N^{cd} \otimes M_{ab} = \begin{pmatrix} ra & rb & sa & sb & ta & tb \\ rc & rd & sc & sd & tc & td \\ re & rf & se & sf & te & tf \\ \vdots & \vdots & & \vdots & & \vdots \end{pmatrix}$

* Direct product is Non commutative (Non abelian)

Special Case of Direct pdt

* $M_{ab} = \phi \longrightarrow S^{cd} = \phi \otimes N^{cd} = \phi N^{cd}$

Scalar Multiplication

* Benefit of Direct Product is to Create higher Rank/dim. object/tensor/Matrix

c) Index Contraction / Index Juggling $\to$ Inner Pdt.

Type 1: 1 index Vector $\otimes$ 1 index Vector $\xrightarrow{\text{Contract.}}$ 0 index Scalar

* $A^i = (1 \ 2 \ 3)_{1\times 3}$, $B_j = (\alpha \ \beta \ \gamma)_{1\times 3}$

Tensor

1D repres.

Calculus $\to$ Difference = Uncountable Multiplicity = Quality

Representation = Quantity = Countable Multiplicity

2 index obj / Matrix

* $S^i{}_j = A^i \otimes B_j = A^i B_j$

$= (1\alpha \ 1\beta \ 1\gamma \ 2\alpha \ 2\beta \ 2\gamma \ 3\alpha \ 3\beta \ 3\gamma)_{1\times 9}$ $\to$ Matrix

$i = j$ and sum over i (Prescription) : Contraction

* $\sum_{i=1}^{3} S^i{}_i = \sum_{i=1}^{3} A^i B_i = A^1 B_1 + A^2 B_2 + A^3 B_3$ ("Trace")

$= \alpha + 2\beta + 3\gamma$ = Scalar / Number = ϕ

Comment on Inner pdt

$\underset{\sim}{A} \cdot \underset{\sim}{B} = (A^1 \ A^2 \ A^3) \cdot (B_1 \ B_2 \ B_3)$

$= A^1 B_1 + A^2 B_2 + A^3 B_3 = \sum A^i B_i$ = Scalar Inner pdt

Contraction $\equiv$ Set 2 index equal ($\overset{i}{\text{up}}$ $\underset{i}{\text{down}}$) and Sum over it

Reduction of dimension / Rank / order / Index

* $\sum S^i{}_i = \sum A^i B_i = A^i B_i = \phi$

Lecture - 19 (26/6/2022)

Type 1. 1 index vector $\otimes$ 1 index vector $\xrightarrow{\text{contraction}}$ 0-index scalar

Ex 1

$L^a = (1\ \ 2\ \ 3)_{1\times3}$, $a = 1,2,3$ $\qquad M_b = (\alpha\ \ \beta\ \ \gamma)_{1\times3}$, $b = 1,2,3$

* $S^a{}_b \equiv L^a \otimes M_b = (1\alpha\ \ 1\beta\ \ 1\gamma\ \ 2\alpha\ \ 2\beta\ \ 2\gamma\ \ 3\alpha\ \ 3\beta\ \ 3\gamma)_{1\times9}$

$\downarrow$ $a=b$

$\sum_{a=1}^{3} S^a{}_a = \sum_{a=1}^{3} L^a M_a = L^1M_1 + L^2M_2 + L^3M_3 = 1\alpha + 2\beta + 3\gamma \equiv \phi$ scalar Number

* $D_b{}^a \equiv M_b \otimes L^a = (\alpha 1\ \ \alpha 2\ \ \alpha 3\ \ \beta 1\ \ \beta 2\ \ \beta 3\ \ \gamma 1\ \ \gamma 2\ \ \gamma 3)_{1\times9}$

$\downarrow$ $a=b$

$\sum D_a{}^a \equiv \sum_{a=1}^{3} M_a L^a = M_1L^1 + M_2L^2 + M_3L^3 = \alpha 1 + \beta 2 + \gamma 3 \equiv \phi'$ = Scalar Number

* if $\phi' = \phi \Rightarrow \boxed{\sum_{a=1}^{n} L^a M_a = \sum_{a=1}^{n} M_a L^a}$ commutes

* $L^a \otimes M_b \neq M_b \otimes L^a \Leftrightarrow L^a M_b \neq M_b L^a$

Non commutation

Ex 2

$L^a = (1\ \ 2\ \ 3)_{1\times3}$ $\qquad M_b = \begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}_{3\times1}$

* $S^a{}_b = L^a \otimes M_b = \begin{pmatrix} 1\alpha & 2\alpha & 3\alpha\\ 1\beta & 2\beta & 3\beta \\ 1\gamma & 2\gamma & 3\gamma\end{pmatrix}_{3\times3}$

$\downarrow$ $a=b$

$\sum S^a{}_a = \sum_{a=1}^{3} L^a M_a = 1\alpha + 2\beta + 3\gamma = \# \equiv \lambda$ $\qquad$ Tr(S)

* $D_b{}^a = M_b \otimes L^a = \begin{pmatrix}\alpha 1 & \alpha 2 & \alpha 3\\ \beta 1 & \beta 2 & \beta 2\\ \gamma 1 & \gamma 2 & \gamma 3\end{pmatrix}_{3\times3}$

$\downarrow$ $a=b$

$\sum D_a{}^a = \sum M_a L^a = \alpha 1 + \beta\cdot 2 + \gamma\cdot 3 = \# \equiv \lambda'$

* $\boxed{L^a M_b = M_b L^a}$ closed prod commutes. Weird! will come back

$\downarrow$ contraction

$L^a M_a = M_a L^a$ commute

Ex 3

$A^i = (i\ \ 2i)_{1\times2}$, $i = 1,2$ $\qquad B_j = (\ln\pi\ \ 3\ln10)_{1\times2}$, $j = 1,2$

* $S^i{}_j \equiv A^i B_j = (i\ln\pi\ \ 3i\ln10\ \ 2i\ln\pi\ \ 6i\ln10)_{1\times4}$

$\downarrow$ $i=j$

$\sum_{i=1}^{2} S^i{}_i = \sum_{i=1}^{2} A^i B_i = A^1B_1 + A^2B_2 = i\ln\pi + 6i\ln10 \equiv \phi$

* $D_j{}^i \equiv B_j A^i = (i\ln\pi\ \ 2i\ln\pi\ \ 3i\ln10\ \ 6i\ln10)_{1\times4}$

$\downarrow$ $i=j$

$\sum_{i=1}^{2} D_i{}^i = \sum_{i=1}^{2} B_i A^i = B_1A^1 + B_2A^2 = i\ln\pi + 6i\ln10 = \phi$

* $\boxed{A^i B_j \neq B_j A^i \longrightarrow A^i B_i = B_i A^i}$

No commutation $\qquad$ Commutes

Ex 4

$A^i = (7\ \ e)_{1\times2}$, $i = 1,2$ $\qquad B_j = \begin{pmatrix}\sin\alpha\\ \cos\alpha\end{pmatrix}_{2\times1}$, $j = 1,2$

* $D^i{}_j = A^i B_j = \begin{pmatrix} 7\sin\alpha & e\sin\alpha \\ 7\cos\alpha & e\cos\alpha\end{pmatrix}_{2\times2}$

$\downarrow$ $i=j$

* $D^i{}_i = A^i B_i = 7\sin\alpha + e\cos\alpha = Tr(D) = Tr(A^i B_j)$ $\qquad$ Sum of diag elements

* $L_j{}^i = B_j A^i = \begin{pmatrix}7\sin\alpha & e\sin\alpha\\ 7\cos\alpha & e\cos\alpha\end{pmatrix}_{2\times2} = A^i B_j$

$\downarrow$

$L^i{}_i = B_i A^i = 7\sin\alpha + e\cos\alpha$

$\boxed{A^i B_j = B_j A^i \longrightarrow A^i B_i = B_i A^i}$

Ex 5

$A^i = (1\ \ 2\ \ 3)_{1\times3}$, $i = 1,2,3$ $\qquad B_j = (\alpha\ \ \beta)_{1\times2}$, $j = 1,2$

* $S^i{}_j = A^i B_j = (1\alpha\ \ 1\beta\ \ 2\alpha\ \ 2\beta\ \ 3\alpha\ \ 3\beta)_{1\times6}$

$\downarrow$ $i=j$ 2 term sum

$3 \neq 2$

* $\sum_{i=1}^{3} S^i{}_i = \sum_{i=1}^{3} A^i B_i = A^1B_1 + A^2B_2 + A^3B_3 = 1\alpha + 2\beta + 3B_3 = DNE$

B_3 unknown

$\left\{ \begin{array}{l} \text{contraction DNE / impossible / not doable} \\ \because i \,\&\, j \text{ are different} \Rightarrow i=j \text{ set Not pos.} \end{array} \right\}$

* $D_j{}^i \equiv B_j A^i = (\alpha\ \ 2\alpha\ \ 3\alpha\ \ \beta\ \ 2\beta\ \ 3\beta)_{1\times6}$

condition

* $L^a = (L^1\ L^2 \ldots L^n)_{1\times n}$ $\qquad a = 1,2,\ldots n$

$M_b = (M_1\ M_2 \ldots M_n)_{1\times n}$ $\qquad b = 1,2,\ldots n$

$M^T = \begin{pmatrix}M_1\\ M_2\\ \vdots \\ M_n\end{pmatrix}_{n\times1} = M_b'$ $\qquad$ Transpose of M

* $L^a \otimes M_b \neq M_b \otimes L^a$

$L^a \otimes M_b' = M_b' \otimes L^a$ $\qquad$ a and b are same $(1 \ldots n)$

* $\sum_{a=1}^{n} L^a M_a = \sum_{a=1}^{n} M_a L^a$ $\qquad$ Contraction possible $\sum_{a=1}^{n} \ldots$

Lecture-20 (27/6/2022)

Type 2: 2 index ⊗ 1 index $\xrightarrow{\text{contraction}}$ 1 index Matrix
1 index ⊗ 2 index

Ex 1: $K_{ab} = \begin{pmatrix}1&2\\3&4\end{pmatrix}_{2\times2}$ a = 1, 2; b = 1, 2 $\qquad L^c = (\alpha\ \ \beta)_{1\times2}$ c = 1, 2

* $S_{ab}^{\ \ c} \equiv K_{ab}L^c = \begin{pmatrix}1\alpha & 1\beta & 2\alpha & 2\beta\\ 3\alpha & 3\beta & 4\alpha & 4\beta\end{pmatrix}_{2\times4}$

Contraction 1: a = c

* $\sum_{a=1} S_{ab}^{\ \ a} = \sum_{a=1}^{2} K_{ab}L^a = K_{1b}L^1 + K_{2b}L^2 \neq \text{Number} \equiv Q_b$ Matrix/1-index obj

free index

b=1: $Q_1 = K_{11}L^1 + K_{21}L^2 = 1\alpha + 3\beta$

b=2: $Q_2 = K_{12}L^1 + K_{22}L^2 = 2\alpha + 4\beta$ $\Rightarrow Q_b = \begin{pmatrix}Q_1\\Q_2\end{pmatrix} = \begin{pmatrix}1\alpha+3\beta\\2\alpha+4\beta\end{pmatrix}_{2\times1}$

Contraction 2: b = c

* $\sum_{b=1}^{2} S_{ab}^{\ \ b} = \sum_{b=1}^{2} K_{ab}L^b = K_{a1}L^1 + K_{a2}L^2 \equiv P_a$ Matrix/1-index obj.

free index

a=1: $P_1 = K_{11}L^1 + K_{12}L^2 = \alpha + 2\beta$

a=2: $P_2 = K_{21}L^1 + K_{22}L^2 = 3\alpha + 4\beta$ $\Rightarrow P_a = \begin{pmatrix}P_1\\P_2\end{pmatrix}_{2\times1} = \begin{pmatrix}\alpha+2\beta\\3\alpha+4\beta\end{pmatrix}_{2\times1}$

Flipped $K_{ab}L^c$

* $D^c_{\ ab} \equiv L^cK_{ab} = \begin{pmatrix}\alpha & 2\alpha & \beta & 2\beta\\ 3\alpha & 4\alpha & 3\beta & 4\beta\end{pmatrix}_{2\times4}$

Cont. 1

* $\sum D^a_{\ ab} = \sum_{a=1} L^aK_{ab} = L^1K_{1b} + L^2K_{2b} \equiv R_b = \begin{pmatrix}L^1K_{11}+L^2K_{21}\\L^1K_{12}+L^2K_{22}\end{pmatrix} = Q_b$

Cont. 2

* $\sum_b D^b_{\ ab} = \sum_b L^bK_{ab} = L^1K_{a1} + L^2K_{a2} \equiv \underset{\underset{P_a}{\parallel}}{U_a} = \begin{pmatrix}L^1K_{11}+L^2K_{12}\\L^1K_{21}+L^2K_{22}\end{pmatrix}$

$$\boxed{\begin{array}{c} * \ K_{ab}L^c \neq L^cK_{ab} \longrightarrow \sum_b K_{ab}L^b = \sum_b L^bK_{ab} \\ \downarrow \\ \sum_a K_{ab}L^a = \sum_a L^aK_{ab} \end{array}}$$

Comment on Minkowski Metric (Matrix)

Ex 2: $g_{ab} = \begin{pmatrix}1&0\\0&-1\end{pmatrix}_{2\times2}$ a, b = 1, 2 ; $X^c = \begin{pmatrix}t\\y\end{pmatrix}_{2\times1}$ c = 1, 2

* $M_{ab}^{\ \ c} = g_{ab}\otimes X^c = \begin{pmatrix}t&0\\y&0\\0&-t\\0&-y\end{pmatrix}_{4\times2}$

Contraction 1: a = c

* $\sum_a M_{ab}^{\ \ a} = \sum_a g_{ab}X^a = g_{1b}X^1 + g_{2b}X^2 = P_b$

$P_1 = g_{11}X^1 + g_{21}X^2 = t$

$P_2 = g_{12}X^1 + \underbrace{g_{22}}_{-1}X^2 = -y$ $\Rightarrow P_b = \begin{pmatrix}P_1\\P_2\end{pmatrix} = \underbrace{\begin{pmatrix}t\\-y\end{pmatrix}}_{\equiv X_b}$

(Non Euclidean Convention)

Contraction 2: b = c

* $\sum M_{ab}^{\ \ b} = \sum_b g_{ab}X^b = g_{a1}X^1 + g_{a2}X^2 = R_a$

$R_a = \begin{pmatrix}g_{11}X^1+g_{12}X^2\\g_{21}X^1+g_{22}X^2\end{pmatrix} = \begin{pmatrix}t\\-y\end{pmatrix}$

Flipped

* $D^c_{\ ab} = X^c\otimes g_{ab} = \begin{pmatrix}t&0\\0&-t\\y&0\\0&-y\end{pmatrix}_{4\times2}$

Contract. 1

* $\sum D^a_{\ ab} = \sum_a X^a g_{ab} \equiv Q_b = \begin{pmatrix}t\\-y\end{pmatrix}$

Contraction 2.

* $\sum D^b_{\ ab} = \sum_b X^b g_{ab} \equiv S_b = \begin{pmatrix}t\\-y\end{pmatrix}$

$$\boxed{\begin{array}{c} * \ g_{ab}X^c \neq X^c g_{ab} \\ \downarrow \\ \sum_a g_{ab}X^a = \sum_b g_{ab}X^b = \sum_a X^a g_{ab} = \sum_b X^b g_{ab} \end{array}}$$

Special Result for Minkowski Matrix

HW

Ex 3 $K_{ab} = \begin{pmatrix}1&2\\3&4\end{pmatrix}_{2\times2}$; $\psi^c = \begin{pmatrix}\psi^1\\\psi^2\end{pmatrix}_{2\times1}$ ★

Ex 4 $K_{ab} = \begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}_{3\times3}$; $M^i = (\alpha\ \ \beta\ \ \gamma)_{1\times3}$

Ex 5 $K_{ab} = \begin{pmatrix}7&8&9\\4&5&6\\1&2&3\end{pmatrix}_{3\times3}$; $L^j = \begin{pmatrix}\wedge\\\Gamma\\\Delta\end{pmatrix}_{3\times1}$ ★

Lecture-20 (27/6/2022)

Type 2: 2 index $\otimes$ 1 index, 1 index $\otimes$ 2 index $\xrightarrow{\text{contraction}}$ 1 index Matrix

Ex1: $K_{ab} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}_{2\times 2}$, $a=1,2$, $b=1,2$; $\quad L^c = (\alpha \;\; \beta)_{1\times 2}$, $c=1,2$

* $S_{ab}{}^{c} \equiv K_{ab} L^c = \begin{pmatrix} 1\alpha & 1\beta & 2\alpha & 2\beta \\ 3\alpha & 3\beta & 4\alpha & 4\beta \end{pmatrix}_{2\times 4}$

Contraction 1: $a=c$

* $\sum_{a=1}^{2} S_{ab}{}^{a} = \sum_{a=1}^{2} K_{ab} L^a = K_{1b}L^1 + K_{2b}L^2 \neq \text{Number} \equiv Q_b$ Matrix/1-index obj

free index

$b=1$: $Q_1 = K_{11}L^1 + K_{21}L^2 = 1\alpha + 3\beta$

$b=2$: $Q_2 = K_{12}L^1 + K_{22}L^2 = 2\alpha + 4\beta$

$\Rightarrow Q_b = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} 1\alpha + 3\beta \\ 2\alpha + 4\beta \end{pmatrix}_{2\times 1}$

Contraction 2: $b=c$

* $\sum_{b=1}^{2} S_{ab}{}^{b} = \sum_{b=1}^{2} K_{ab} L^b = K_{a1}L^1 + K_{a2}L^2 \equiv P_a$ Matrix/1-index obj.

free index

$a=1$: $P_1 = K_{11}L^1 + K_{12}L^2 = \alpha + 2\beta$

$a=2$: $P_2 = K_{21}L^1 + K_{22}L^2 = 3\alpha + 4\beta$

$\Rightarrow P_a = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}_{2\times 1} = \begin{pmatrix} \alpha + 2\beta \\ 3\alpha + 4\beta \end{pmatrix}_{2\times 1}$

flipped $K_{ab} L^c$

* $D^c{}_{ab} \equiv L^c K_{ab} = \begin{pmatrix} \alpha & 2\alpha & \beta & 2\beta \\ 3\alpha & 4\alpha & 3\beta & 4\beta \end{pmatrix}_{2\times 4}$

Cont 1

* $\sum_{a=1} D^a{}_{ab} = \sum_{a=1} L^a K_{ab} = L^1K_{1b} + L^2K_{2b} \equiv R_b = \begin{pmatrix} L^1K_{11} + L^2K_{21} \\ L^1K_{12} + L^2K_{22} \end{pmatrix} = Q_b$

Cont 2

* $\sum_{b} D^b{}_{ab} = \sum_{b} L^b K_{ab} = L^1K_{a1} + L^2K_{a2} \equiv U_a = \begin{pmatrix} L^1K_{11} + L^2K_{12} \\ L^1K_{21} + L^2K_{22} \end{pmatrix}$, $U_a = P_a$

---

* $$\boxed{\begin{array}{l} K_{ab}L^c \neq L^c K_{ab} \longrightarrow \sum_b K_{ab}L^b = \sum_b L^b K_{ab} \\ \downarrow \\ \sum_a K_{ab}L^a = \sum_a L^a K_{ab} \end{array}}$$

Comment on Minkowski Metric (Matrix)

Ex2: $g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{2\times 2}$, $a,b=1,2$; $\quad X^c = \begin{pmatrix} t \\ y \end{pmatrix}_{2\times 1}$, $c=1,2$

* $M_{ab}{}^{c} = g_{ab} \otimes X^c = \begin{pmatrix} t & 0 \\ y & 0 \\ 0 & -t \\ 0 & -y \end{pmatrix}_{4\times 2}$

Contraction 1: $a=c$

* $\sum_a M_{ab}{}^{a} = \sum_a g_{ab} X^a = g_{1b}X^1 + g_{2b}X^2 = P_b$

$P_1 = g_{11}X^1 + g_{21}X^2 = t$

$P_2 = g_{12}X^1 + \underbrace{g_{22}}_{-1}X^2 = -y$

$\Rightarrow P_b = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \underbrace{\begin{pmatrix} t \\ -y \end{pmatrix}}_{\equiv X_b}$ (Non Euclidean Convention)

Contraction 2: $b=c$

* $\sum_b M_{ab}{}^{b} = \sum_b g_{ab} X^b = g_{a1}X^1 + g_{a2}X^2 = R_a$

$R_a = \begin{pmatrix} g_{11}X^1 + g_{12}X^2 \\ g_{21}X^1 + g_{22}X^2 \end{pmatrix} = \begin{pmatrix} t \\ -y \end{pmatrix}$

flipped

* $D^c{}_{ab} = X^c \otimes g_{ab} = \begin{pmatrix} t & 0 \\ 0 & -t \\ y & 0 \\ 0 & -y \end{pmatrix}_{4\times 2}$

Contract. 1

* $\sum D^a{}_{ab} = \sum_a X^a g_{ab} \equiv Q_b = \begin{pmatrix} t \\ -y \end{pmatrix}$

Contraction 2.

* $\sum D^b{}_{ab} = \sum_b X^b g_{ab} \equiv S_b = \begin{pmatrix} t \\ -y \end{pmatrix}$

---

* $$\boxed{\begin{array}{l} g_{ab}X^c \neq X^c g_{ab} \\ \downarrow \\ \sum_a g_{ab}X^a = \sum_b g_{ab}X^b = \sum_a X^a g_{ab} = \sum_b X^b g_{ab} \end{array}}$$

Special Result for Minkowski Matrix

HW

Ex3 $K_{ab} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}_{2\times 2}$; $\psi^c = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}_{2\times 1}$ ★

Ex4 $K_{ab} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}_{3\times 3}$; $M^c = (\alpha \;\; \beta \;\; \gamma)_{1\times 3}$

Ex5 $K_{ab} = \begin{pmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}_{3\times 3}$; $L^j = \begin{pmatrix} \Lambda \\ \Gamma \\ \Delta \end{pmatrix}_{3\times 1}$ ★

Lecture - 21 (2/7/2022)

* $A^i = (A^1 \; A^2 \ldots A^n)_{1\times n}$; $B_j = (B_1 \ldots B_m)_{1\times m} \longrightarrow x$

$A^T_i = \begin{pmatrix} A^1 \\ \vdots \\ A^n \end{pmatrix}_{n\times 1}$; $B^T_j = \begin{pmatrix} B_1 \\ \vdots \\ B_m \end{pmatrix}_{m\times 1}$ $\uparrow y$

$C^{kl} = \begin{pmatrix} C^{11} & C^{12} & \cdots & C^{1q} \\ C^{21} & & & C^{2q} \\ C^{p1} & & \cdots & C^{pq} \end{pmatrix}_{p\times q} \xrightarrow{\text{transp.}} C^{lk} = \begin{pmatrix} C^{11} & C^{21} & \cdots & C^{q1} \\ \vdots & & & \\ C^{1p} & & & C^{qp} \end{pmatrix}_{q\times p}$

* $A^i \otimes B_j = A^i B_j \equiv \underbrace{S^i_j}_{1\times nm}$: $A^i B_j \neq \underbrace{B_j A^i}_{R^i_j \, (1\times mn)}$

* $A^i \otimes B^T_j = A^i B^T_j \equiv \underbrace{P^i_j}_{m\times n}$: $A^i B^T_j = \underbrace{B^T_j A^i}_{m\times n}$

* $\boxed{A^i B_i = B_i A^i = \phi \text{ iff } n = m \quad (i = j)}$

* $C^{kl} \otimes B_j = C^{kl} B_j \equiv \underbrace{X^{kl}_j}_{p\times qm}$: $C^{kl} B_j \neq \underbrace{B_j C^{kl}}_{Y^{kl}_j \, (p\times mq)}$

* $C^{kl} \otimes B^T_j = C^{kl} B^T_j \equiv \underbrace{Z^{kl}_j}_{(pm\times q)}$: $C^{kl} B^T_j \neq \underbrace{B^T_j C^{kl}}_{(mp\times q)}$

* $l = j$: $C^{kl} B_l = B_l C^{kl} \equiv S^l$ iff $q = m$

$(k = j)$: $C^{kl} B_k = B_k C^{kl}$ iff $p = m$

Cond for Contraction

Ex-3 $K_{ab} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}_{2\times 2}$, $a, b = 1, 2$; $\psi^c = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}_{2\times 1}$, $c = 1, 2$

* $K_{ab}\psi^c = \begin{pmatrix} \psi^1 & 2\psi^1 \\ \psi^2 & 2\psi^2 \\ 3\psi^1 & 4\psi^1 \\ 3\psi^2 & 4\psi^2 \end{pmatrix}_{4\times 2}$; $\psi^c K_{ab} = \begin{pmatrix} 1\psi^1 & 2\psi^1 \\ 3\psi^1 & 4\psi^1 \\ 1\psi^2 & 2\psi^2 \\ 3\psi^2 & 4\psi^2 \end{pmatrix}_{4\times 2} \neq K_{ab}\psi^c$

$K_{ab}\psi^a \equiv S_b = K_{1b}\psi^1 + K_{2b}\psi^2 = \begin{pmatrix} 1\psi^1 + 3\psi^2 \\ 2\psi^1 + 4\psi^2 \end{pmatrix} = \psi^a K_{ab}$

$K_{ab}\psi^b \equiv R_a = K_{a1}\psi^1 + K_{a2}\psi^2 = \begin{pmatrix} \psi^1 + 2\psi^2 \\ 3\psi^1 + 4\psi^2 \end{pmatrix} = \psi^b K_{ab}$

HW

Ex 4 / Ex 5

Ex 6 $K_{ab} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}_{2\times 2}$, $a, b = 1, 2$; $L^c = (i \;\; \pi \;\; e)_{1\times 3}$, $c = 1, 2, 3$

$K_{ab} L^a \Rightarrow$ Not possible

$(b = c)$

$2 \neq 3$

Lecture-22 (3/7/2022)

| A=A Law of Identity

* $A^i B_j = A^i B^j = A_i B_j = A_i B^j$ ⇐ Euclidean space

$g_{ab} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$, basis: {x y}

* $\underbrace{A^i}_{1\times n} \otimes \underbrace{B_j}_{1\times m} = \underbrace{A^i B_j}_{1\times nm} \neq \underbrace{B_j A^i}_{1\times mn}$

* $\underbrace{\underbrace{A^i}_{1\times n} \underbrace{B_j}_{m\times 1}}_{m\times n} = \begin{cases} = B_j A^i & n=m \\ \neq \underbrace{B_j A^i}_{m\times n} & n\neq m \end{cases}$ special matrices (2x2, 3x3)

* $A^i B_i = B_i A^i = \phi$ # of pos contract = 1 (i=j) iff n=m

* $\underbrace{\underbrace{A^{ij}}_{p\times q} \underbrace{B_m}_{1\times n}}_{p\times qn} \neq \underbrace{B_m A^{ij}}_{p\times qn}$

* $A^{ij} B_j = B_j A^{ij}$ iff q=n   # Pos contract = 2 $\begin{cases} i=m \\ j=m \end{cases}$

$\underbrace{A^{ij} B_i}_{S^j} = B_i A^{ij}$ iff p=n

Type 3: 2 index Matrix ⊗ 2 index Matrix Contractions

Ex-1 $i,j=1,2$ $A^{ij} = \begin{pmatrix}1&2\\3&4\end{pmatrix}_{2\times2}$, $k,l=1,2$ $B_{kl} = \begin{pmatrix}5&6\\7&8\end{pmatrix}_{2\times2}$

* $A^{ij} \otimes B_{kl} \equiv S^{ij}{}_{kl} = \begin{pmatrix} 1\cdot5 & 1\cdot6 & 2\cdot5 & 2\cdot6 \\ 1\cdot7 & 1\cdot8 & 2\cdot7 & 2\cdot8 \\ 3\cdot5 & 3\cdot6 & 4\cdot5 & 4\cdot6 \\ 3\cdot7 & 3\cdot8 & 4\cdot7 & 4\cdot8 \end{pmatrix}_{4\times4} = \Lambda^a{}_b$ $a,b=1,2,3,4$

2D rep of 4 index obj

Contractions

C1: $i=k$ ↓ $\sum A^{ij} B_{il} \equiv S^j{}_l$

C2: $i=l$ ↓ $\sum A^{ij} B_{ki} \equiv R^j{}_k$

C3: $j=k$ ↓ $\sum A^{ij} B_{jl} \equiv L^i{}_l$

C4: $j=l$ ↓ $\sum A^{ij} B_{kj} \equiv M^i{}_k$

C5: $i=k$, $j=l$ ↓ $\sum_{i,j} A^{ij} B_{ij} \equiv \phi$

C6: $i=l$, $j=k$ ↓ $\sum A^{ij} B_{ji} \equiv \lambda$

* C1: $S^j{}_l = \sum_{i=1,2} A^{ij} B_{il} = A^{1j} B_{1l} + A^{2j} B_{2l}$

$j=1$: $S^1{}_l = A^{11} B_{1l} + A^{21} B_{2l}$

$l=1$: $S^1{}_1 = A^{11} B_{11} + A^{21} B_{21} = 5+21 = 26$

$l=2$: $S^1{}_2 = A^{11} B_{12} + A^{21} B_{22} = 6+24 = 30$

$j=2$: $S^2{}_l = A^{12} B_{1l} + A^{22} B_{2l}$

$l=1$: $S^2{}_1 = A^{12} B_{11} + A^{22} B_{21} = 10+28 = 38$

$l=2$: $S^2{}_2 = A^{12} B_{12} + A^{22} B_{22} = 12+32 = 44$

$S^j{}_l = \begin{pmatrix}26&30\\38&44\end{pmatrix}_{2\times2}$

* C2: $R^j{}_k = \sum_{i=1,2} A^{ij} B_{ki} = A^{1j} B_{k1} + A^{2j} B_{k2}$

$j=1$: $R^1{}_k = A^{11} B_{k1} + A^{21} B_{k2}$ | $R^j{}_k = \begin{pmatrix}23&31\\34&46\end{pmatrix}_{2\times2}$

$k=1$: $R^1{}_1 = A^{11} B_{11} + A^{21} B_{12} = 5+18 = 23$

$k=2$: $R^1{}_2 = A^{11} B_{21} + A^{21} B_{22} = 7+24 = 31$

$j=2$: $R^2{}_k = A^{12} B_{k1} + A^{22} B_{k2}$

$k=1$: $R^2{}_1 = A^{12} B_{11} + A^{22} B_{12} = 10+24 = 34$

$k=2$: $R^2{}_2 = A^{12} B_{21} + A^{22} B_{22} = 14+32 = 46$

* C3: $\boxed{L^i{}_l = \sum_{j=1,2} A^{ij} B_{jl} = A^{i1} B_{1l} + A^{i2} B_{2l}}$ Inner product

$i=1$: $L^1{}_l = A^{11} B_{1l} + A^{12} B_{2l}$ | $L^i{}_l = \begin{pmatrix}19&21\\43&50\end{pmatrix}_{2\times2}$

$l=1$: $L^1{}_1 = A^{11} B_{11} + A^{12} B_{21} = 5+14 = 19$

$l=2$: $L^1{}_2 = A^{11} B_{12} + A^{12} B_{22} = 6+16 = 21$

$i=2$: $L^2{}_l = A^{21} B_{1l} + A^{22} B_{2l}$

$l=1$: $L^2{}_1 = A^{21} B_{11} + A^{22} B_{21} = 15+28 = 43$

$l=2$: $L^2{}_2 = A^{21} B_{12} + A^{22} B_{22} = 18+32 = 50$

* C4: $M^i{}_k \equiv \sum_{j=1,2} A^{ij} B_{kj} = A^{i1} B_{k1} + A^{i2} B_{k2}$

$i=1$: $M^1{}_k = A^{11} B_{k1} + A^{12} B_{k2}$

$k=1$: $M^1{}_1 = A^{11} B_{11} + A^{12} B_{12} = 5+12 = 17$

$k=2$: $M^1{}_2 = A^{11} B_{21} + A^{12} B_{22} = 7+16 = 23$

$i=2$: $M^2{}_k = A^{21} B_{k1} + A^{22} B_{k2}$

$k=1$: $M^2{}_1 = A^{21} B_{11} + A^{22} B_{12} = 15+24 = 39$

$k=2$: $M^2{}_2 = A^{21} B_{21} + A^{22} B_{22} = 21+32 = 53$

$M^i{}_k = \begin{pmatrix}17&23\\39&53\end{pmatrix}$

Special #

* C5: $\phi = \sum_{i,j=1,2} A^{ij} B_{ij} = \sum_{i=1} \sum_{j=1} A^{ij} B_{ij}$

* $i=1$: $\phi = \sum_{j=1,2} A^{1j} B_{1j} = A^{11} B_{11} + A^{12} B_{12}$

$i=2$: $\phi = \sum_{j=1,2} A^{2j} B_{2j} = A^{21} B_{21} + A^{22} B_{22}$

* $\phi = \sum A^{ij} B_{ij} = A^{11} B_{11} + A^{12} B_{12} + A^{21} B_{21} + A^{22} B_{22} = 5+12+21+32 = 70$

C6: $\lambda \equiv \sum A^{ij} B_{ji} = A^{11} B_{11} + A^{12} B_{21} + A^{21} B_{12} + A^{22} B_{22} = 5+14+18+32 = 69$

Traces (More contractions)

* $\sum_{i=1,2} A^{ii} = A^{11} + A^{22} = 5$

$\sum_{k=1,2} B^{kk} = B^{11} + B^{22} = 13$

$\sum_{a=1,2,3,4} \Lambda^a{}_a = \Lambda^1{}_1 + \Lambda^2{}_2 + \Lambda^3{}_3 + \Lambda^4{}_4 = 65$

* $\sum_{j=1,2} S^j{}_j = S^1{}_1 + S^2{}_2 = 70 = \sum A^{ij} B_{ij}$: $j=l$

$\sum_{j=1,2} R^j{}_j = R^1{}_1 + R^2{}_2 = 69 = \sum A^{ij} B_{ji}$: $j=k$

$\sum_{i=1,2} L^i{}_i = L^1{}_1 + L^2{}_2 = 69 = \sum A^{ij} B_{ji}$: $j=k$

$\sum_{i=1,2} M^i{}_i = M^1{}_1 + M^2{}_2 = 70 = \sum A^{ij} B_{ij}$: $i=k$

Lecture-23 (4/7/2022)

* $A^{ij} = \begin{pmatrix}1&2\\3&4\end{pmatrix}_{2\times2}$ ($i,j=1,2$), $B_{kl} = \begin{pmatrix}5&6\\7&8\end{pmatrix}_{2\times2}$ ($k,l=1,2$)

* $A^{ij}B_{kl} = \begin{pmatrix}5&6&10&12\\7&8&14&16\\15&18&20&24\\21&24&28&32\end{pmatrix}_{4\times4} = \Lambda_{ab} \neq B_{kl}A^{ij}$ ($a,b=1,2,3,4$)

"Kronecker Direct product"

* $S^j{}_l = \sum_i A^{ij}B_{il} = \begin{pmatrix}26&30\\38&44\end{pmatrix} \neq \sum B_{il}A^{ij}$

* $R^j{}_k = \sum_i A^{ij}B_{ki} = \begin{pmatrix}23&31\\34&46\end{pmatrix} \neq \sum B_{ki}A^{ij}$

* $L^i{}_l = \sum_j A^{ij}B_{jl} = \begin{pmatrix}19&22\\43&50\end{pmatrix} \neq \sum B_{jl}A^{ij}$ "Inner pdt" (usual multiplication)

* $M^i{}_k = \sum_j A^{ij}B_{kj} = \begin{pmatrix}17&23\\39&53\end{pmatrix} \neq \sum B_{kj}A^{ij}$

* $\phi = \sum_{ij} A^{ij}B_{ij} = 70 = \sum B_{ij}A^{ij}$

$\lambda = \sum A^{ij}B_{ji} = 69 = \sum B_{ji}A^{ij}$

Frobenius Inner pdt (Matrix → #)

Traces

* $\sum_i A^{ii} = A^{11}+A^{22} = 5$ $i=j$

$\sum_k B^{kk} = B^{11}+B^{22} = 13$ $k=l$

$\sum_{a=1}^{4} \Lambda_{aa} = \Lambda_{11}+\Lambda_{22}+\Lambda_{33}+\Lambda_{44} = 69$ $a=b$

* $\sum_j S^j{}_j = \sum A^{ij}B_{ij} = 70 = \sum B_{ij}A^{ij}$ $j=l$

$\sum_j R^j{}_j = \sum A^{ij}B_{ji} = 69 = \sum B_{ji}A^{ij}$ $j=k$

$\sum_i L^i{}_i = \sum A^{ij}B_{ji} = 69 = \sum B_{ji}A^{ij}$ $i=l$

$\sum_i M^i{}_i = \sum A^{ij}B_{ij} = 70 = \sum B_{ij}A^{ij}$ $i=k$

Flipped (transformation)

* $B_{kl}A^{ij} = \begin{pmatrix}5&10&6&12\\15&20&18&24\\7&14&8&16\\21&28&24&32\end{pmatrix} \neq A^{ij}B_{kl}$

* $W^j{}_l = \sum_i B_{il}A^{ij} = \begin{pmatrix}26&38\\30&44\end{pmatrix}$

* $X^i{}_l = \sum_{j=1,2} B_{jl}A^{ij} = \sum_{a=1,2} B_{al}A^{ia} = \begin{pmatrix}19&43\\22&50\end{pmatrix}$

* $Y^j{}_k = \sum_i B_{ki}A^{ij} = \begin{pmatrix}23&34\\31&46\end{pmatrix}$

* $Z^i{}_k = \sum_j B_{kj}A^{ij} = \begin{pmatrix}17&39\\23&53\end{pmatrix}$

* $\alpha = \sum B_{ij}A^{ij} = 70$

$\beta = \sum B_{ij}A^{ji} = 69$

Traces

* $\sum_j W^j{}_j = 70 = \sum B_{ij}A^{ij}$

$\sum_i X^i{}_i = 69 = \sum B_{ji}A^{ij}$

$\sum_j Y^j{}_j = 69 = \sum B_{ji}A^{ij}$

$\sum Z^i{}_i = 70 = \sum B_{ij}A^{ij}$

Lecture - 24 (7/7/2022)

c. Usual / Inner prod / Multiplication

* $\underbrace{A_{ij}}_{p\times q} \overset{n\times m}{B^{kl}} \xrightarrow[\;j=k\;]{} M_i^{\ l} \equiv \sum_j A_{ij} B^{jl}$ → def Inner product

$\boxed{q=n}$. Col. of 1st Matrix = Row of 2nd Matrix

* $\underbrace{M_a^{\ b}}_{p\times q} = \sum_{i=1}^{n} \underbrace{L_{ai}}_{p\times n} \underbrace{N^{ib}}_{n\times q} = L_{a1}N^{1b} + L_{a2}N^{2b} + \dots + L_{an}N^{nb}$

$a=1,\ b=1$

$M_1^{\ 1} = L_{11}N^{11} + L_{12}N^{21} + \dots + L_{1n}N^{n1}$

$a=1,\ b=2$

$M_1^{\ 2} = L_{11}N^{12} + L_{12}N^{22} + \dots + L_{1n}N^{n2}$

$a=p,\ b=q$

$M_p^{\ q} = L_{p1}N^{1q} + L_{p2}N^{2q} \dots L_{pn}N^{nq}$

$$\boxed{\underset{p\times q}{M}\ \underset{q\times r}{N} = \sum M_{ai} N^{ib} \equiv \underset{p\times q}{P_a^{\ b}}}$$

Matrix Multiplication

Ex1

$A_{ij} = A = \begin{pmatrix} 2 & 1 & 3 \\ 3 & -2 & 1 \\ -1 & 0 & 1 \end{pmatrix}_{3\times3}$, $i,j=1,2,3$; $B^{kl} = B = \begin{pmatrix} 1 & -2 \\ 2 & 1 \\ 4 & -3 \end{pmatrix}_{3\times2}$, $k=1,2,3,\ l=1,2$

* $\underbrace{AB}_{C} \equiv \left(A_{ij} B^{kl} \xrightarrow{j=k} \sum_j A_{ij} B^{jl} \right) = ?$

* $\underbrace{C_i^{\ l}}_{3\times2} = A_{i1}B^{1l} + A_{i2}B^{2l} + A_{i3}B^{3l}$

$C = \begin{pmatrix} 16 & -12 \\ 3 & -11 \\ 3 & -1 \end{pmatrix}$

Fixed 2nd row of A
fixed 1st col. of B

* $C_2^{\ 1} = A_{21}B^{11} + A_{22}B^{21} + A_{23}B^{31} = 3\cdot 1 + (-2)2 + 1\cdot 4 = 3$

* $M_{ab} = \begin{pmatrix} -2 & 3 & 5 \\ 0 & -1 & 6 \\ 8 & -6 & 1 \\ 5 & 4 & -3 \end{pmatrix}_{4\times3}$, $N^{cd} = \begin{pmatrix} 2 & 3 \\ 5 & -7 \\ -1 & 3 \end{pmatrix}_{3\times2}$

$M\cdot N = ?$

finger trick

1. $\underbrace{M.N}_{4\times2} \equiv \sum_{i=1}^{3} M_{ai} N^{id}$: $b=c$

$a = 1,2,3,4$
$b = 1,2,3$ } i
$c = 1,2,3$
$d = 1,2$

2. Horizontal of the 1st Mat. × Vertical of the 2nd

* $M\cdot N = \begin{pmatrix} -4+15-5 & -6-21+15 \\ 0-5-6 & 0+7+18 \\ 16-30-1 & 24+42+3 \\ 10+20+3 & 15-28-9 \end{pmatrix}_{4\times2} = \begin{pmatrix} 6 & -12 \\ -11 & 25 \\ -15 & 69 \\ 33 & -22 \end{pmatrix}_{4\times2}$

HW

* $A = \begin{pmatrix} 3 & 7 & -9 \\ 1 & -2 & 3 \\ 5 & 0 & -6 \end{pmatrix}$, $B = \begin{pmatrix} 6 & 9 & -3 \\ 2 & 0 & 5 \\ -8 & 1 & 7 \end{pmatrix}$

$AB = ?$, $BA = ?$

* $A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 5 & 6 \end{pmatrix}$

$AB = ?$ $BA = ?$

* $A = \begin{pmatrix} 1 & 0 \\ -1 & 7 \end{pmatrix}$, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$: $A^2 = 8A + kI$, $k = ?$

* $f(x) = x^2 - 5x + 6$, $A = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix}$. $f(A) = ?$

* $A = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$, $A^2 - 5A + 7I_2 = ?$

Lecture-25 (9/7/2022)

2.2.6 Pauli Matrices

* $i^2 = -1 \Rightarrow i^3 = \underbrace{i^2}_{-1} i = -i$
  $i^4 = i^2 i^2 = 1$

  Just as a Fact Right now!

* $\sigma = \{\sigma_x, \sigma_y, \sigma_z\} = \{\sigma_1, \sigma_2, \sigma_3\}$

  $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{2\times 2}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_{2\times 2}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{2\times 2}$

  $\sigma_x = \sigma_1$, $\sigma_y = \sigma_2$, $\sigma_z = \sigma_3$

* $\sigma_1 \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

* $\sigma_2 \sigma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$

* $[A, B] \equiv AB - BA$ → $= 0$ Entities Commute; → $\neq 0$ Entities Do not Commute

  "Commutator Brackets"

  $[\sigma_1, \sigma_2] = \sigma_1\sigma_2 - \sigma_2\sigma_1 = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} = 2i\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\sigma_3}$

  $\boxed{[\sigma_1, \sigma_2] = 2i\sigma_3}$ ✓

* $\sigma_2 \sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$; $\sigma_3 \sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = 2i\sigma_1$

  $\boxed{[\sigma_2, \sigma_3] = 2i\sigma_1}$ ✓

* $\sigma_1 \sigma_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$; $\sigma_3 \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

  $\underbrace{[\sigma_1, \sigma_3]}_{\neq [\sigma_3, \sigma_1]} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2i^2 \\ -2i^2 & 0 \end{pmatrix} = -2i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -2i\sigma_2$

* $\boxed{[\sigma_3, \sigma_1] = 2i\sigma_2}$ ✓

---

* $[A, B] = AB - BA = -(BA - AB) = -[B, A]$

* $[\sigma_1, \sigma_2] = 2i\sigma_3$
  $[\sigma_2, \sigma_3] = 2i\sigma_1$
  $[\sigma_3, \sigma_1] = 2i\sigma_2$

  $1 \to 2 \to 3 \to 1$
  cyclic permutation

Naive generalization

* $[\sigma_\alpha, \sigma_\beta] = 2i\,\sigma_\gamma$

  index $\alpha, \beta = 1, 2, 3$, γ

"Pro" generalization

* $\delta_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}$

  Kronecker delta (2 index entity)

  $\epsilon_{\alpha\beta\gamma} \equiv \begin{cases} 1 & \text{Even permutation} \\ -1 & \text{odd permutation} \\ 0 & \text{any 2 indices same} \end{cases}$

  Levi-Civita symbol (3-index Entity)

* $\epsilon_{123} \equiv +1$

* $(1\ 2) \longrightarrow (2\ 1)$

  # arrangements/juxtapositions

  $(1\ 2\ 3) \longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 1 \\ 3 & 1 & 2 \end{pmatrix}$

  $(1\ 2\ 3) \xrightarrow{\text{odd}} (1\ 3\ 2)$

Lecture - 26 (10/7/2022)

Convention
(greek) $\alpha, \beta, \mu, \nu .. = 1, 2, 3.$
(latin) $a, b, c \ldots = 0, 1, 2, 3$

2.2.6 Pauli Matrices / Levi Civita symbol

$\rightarrow$ Determinant

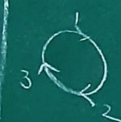
* $\sigma_\alpha = \{\sigma_1, \sigma_2, \sigma_3\}$ index $\alpha = 1, 2, 3$

$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

* $[\sigma_1, \sigma_2] = 2i\sigma_3 \, ; [\sigma_2, \sigma_3] = 2i\sigma_1 ; [\sigma_3, \sigma_1] = 2i\sigma_2$

* $\delta_{\alpha\beta} = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}$ Kronecker delta

$= \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & & & \\ \vdots & & & & \\ 0 & \ldots & 1 & & \\ & & & & 1 \end{pmatrix}_{N \times N}$ Identity Matrix (Square)



$1 \leftrightarrow 2$
$3 \leftrightarrow 1$
$1 \leftrightarrow 1$
$25 \leftrightarrow 1$
$\vdots$
$\rightarrow \boxed{\delta_{\alpha\beta} = \delta_{\beta\alpha}}$ symmetric Matrix

By inspection

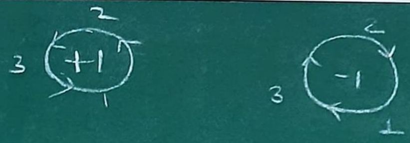
* Epsilon

$\epsilon_{\alpha\beta\mu} \equiv \begin{cases} +1 & \alpha\beta\mu \text{ Even Permutation} \\ -1 & \alpha\beta\mu \text{ odd perm.} \\ 0 & \alpha = \beta \text{ or } \beta = \gamma \text{ or } \gamma = \alpha \end{cases}$

* $\epsilon_{111} = 0 \qquad \epsilon_{121} = 0 \qquad \epsilon_{211} = 0$
$\epsilon_{112} = 0 \qquad \epsilon_{131} = 0 \qquad \epsilon_{311} = 0$
$\epsilon_{113} = 0 \qquad \epsilon_{141} = 0 \qquad \epsilon_{411} = 0$
$\vdots$
$\epsilon_{11K} = 0 \qquad \epsilon_{1Q1} = 0 \qquad \epsilon_{P11} = 0$

* $\epsilon_{223} = 0$, $\epsilon_{333} = 0 = \epsilon_{331}$

* $\epsilon_{123} = 1 \qquad \epsilon_{213} = -1$
$\epsilon_{231} = 1 \qquad \epsilon_{321} = -1$
$\epsilon_{312} = 1 \qquad \epsilon_{132} = -1$

* 

cyclic Analogy.

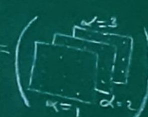
* $\boxed{\epsilon_{\alpha\beta\mu} = -\epsilon_{\beta\alpha\mu}}$ $\quad 123 \rightarrow 213$

$\alpha, \beta, \mu = 1, 2, 3$ Antisymmetric

* fix $\alpha = 1$ slice : $\begin{pmatrix} \epsilon_{111} & \epsilon_{112} & \epsilon_{113} \\ \epsilon_{121} & \epsilon_{122} & \epsilon_{123} \\ \epsilon_{131} & \epsilon_{132} & \epsilon_{133} \end{pmatrix}_{3\times3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

* $\alpha = 2$ fix ; $\begin{pmatrix} \epsilon_{211} & \epsilon_{212} & \epsilon_{213} \\ \epsilon_{221} & \epsilon_{222} & \epsilon_{223} \\ \epsilon_{231} & \epsilon_{232} & \epsilon_{233} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

* $\alpha = 3$: $\begin{pmatrix} \epsilon_{311} & \epsilon_{312} & \epsilon_{313} \\ \epsilon_{321} & \epsilon_{322} & \epsilon_{323} \\ \epsilon_{331} & \epsilon_{332} & \epsilon_{333} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\epsilon_{\alpha\beta\mu} =$ 

* $\overset{+}{(1\ 2\ 3)} \xrightarrow[1 \leftrightarrow 2]{1^{st}} \overset{-}{(2\ 1\ 3)} \xrightarrow[3 \leftrightarrow 1]{2^{nd}} \underset{+}{(2\ 3\ 1)} \xrightarrow[3^{rd}]{2 \leftrightarrow 3} \underset{-}{(3\ 2\ 1)} \xrightarrow[2 \leftrightarrow 1]{4^{th}} \underset{+}{(3\ 1\ 2)}$

* # of transpose $= \begin{cases} \text{odd} \rightarrow \text{odd permut} \\ \text{Even} \rightarrow \text{Even permut.} \end{cases}$

Levi civita sym. = Alternating symbol
= Antisym. symbol

Lecture - 27 (11/7/2022)

L2.6 Pauli Matrices / Levi Civita symbol ($\to$ Determinant)

* $\underset{p\times q}{A}\ \underset{q\times r}{B} = \sum_j A_{ij} B^{jk} = \underset{p\times r}{M}$

* $\underset{n\times m}{A_{ij}} \pm \underset{n\times m}{B_{ij}} = \underset{n\times m}{C_{ij}}$

* $[A, B] \equiv \underset{p\times q}{AB} - \underset{p\times q}{BA}$

Commutator 'Bracket' " operator/function/Machine/Transformation

* $\{A, B\} \equiv \underset{p\times q}{AB} + \underset{p\times q}{BA}$

Anti-Commutator 'Bracket'

* $\epsilon_{\alpha\beta\mu} \equiv \begin{cases} 1 & (1,2,3),(2,3,1),(3,1,2) \\ -1 & (3\,2\,1)\ (1\,3\,2)\ (2\,3\,1) \\ 0 & \alpha=\beta \text{ or } \beta=\mu \text{ or } \mu=\alpha \end{cases}$

$\epsilon_{\alpha\beta\mu} = -\epsilon_{\beta\alpha\mu} = -\epsilon_{\alpha\mu\beta} = -\epsilon_{\mu\beta\alpha}$

* $[\sigma_1, \sigma_2] = 2i\sigma_3$
$[\sigma_2, \sigma_3] = 2i\sigma_1$
$[\sigma_3, \sigma_1] = 2i\sigma_2$

$$\boxed{[\sigma_\alpha, \sigma_\beta] = 2i\epsilon_{\alpha\beta\mu}\sigma^\mu} = -2i\epsilon_{\beta\alpha\mu}\sigma^\mu$$

$\sigma_1 = \begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\sigma_2 = \begin{pmatrix}0&-i\\i&0\end{pmatrix}$, $\sigma_3 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$

* $\epsilon_{\alpha\beta\mu}\sigma^\mu \xrightarrow{\alpha=1,\ \beta=2} \epsilon_{123}\sigma^3$

$\epsilon_{\alpha\beta\mu}\sigma^\mu \xrightarrow{\alpha=3,\ \beta=1} \underbrace{\epsilon_{312}}_{+1}\sigma^2$

* $[\sigma_\alpha, \sigma_\beta] = 2i\epsilon_{\alpha\beta\mu}\sigma^\mu = \underbrace{\sum_{\mu=1,2,3}}_{\text{Sum over Notation}} 2i\epsilon_{\alpha\beta\mu}\sigma^\mu$

Why summation (contraction)?

$\alpha=1$, $\beta=2$

* $[\sigma_1, \sigma_2] = \sum_{\mu=1,2,3} 2i\epsilon_{12\mu}\sigma^\mu$
$= 2i\left[\underbrace{\cancel{\epsilon_{121}}}_{0}\sigma^1 + \underbrace{\cancel{\epsilon_{122}}}_{0}\sigma^2 + \underbrace{\epsilon_{123}}_{1}\sigma^3\right]$

* $[\sigma_\alpha, \sigma_\beta] = -\sum 2i\epsilon_{\beta\alpha\mu}\sigma^\mu$

* $\{\sigma_\alpha, \sigma_\beta\} = \sigma_\alpha\sigma_\beta + \sigma_\beta\sigma_\alpha = \begin{cases} 2I_2 & \alpha=\beta \\ 0 & \alpha\neq\beta \end{cases}$

$\alpha=1$, $\beta=1$
$\{\sigma_1, \sigma_1\} = 2\sigma_1^2 = 2I_2$; $I_2 \equiv \begin{pmatrix}1&0\\0&1\end{pmatrix}$

$\alpha=2$, $\beta=2$
$\{\sigma_2, \sigma_2\} = 2\sigma_2^2 = 2I_2$

$\alpha=3$, $\beta=3$
$\{\sigma_3, \sigma_3\} = 2\sigma_3^2 = 2I_2$

$\alpha=1$, $\beta=2$
$\{\sigma_1, \sigma_2\} = \sigma_1\sigma_2 + \sigma_2\sigma_1 = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} + \begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}$
$= \begin{pmatrix}i&0\\0&-i\end{pmatrix} + \begin{pmatrix}-i&0\\0&i\end{pmatrix} = 0$

HW

* $\{\sigma_1, \sigma_3\}$
$\{\sigma_2, \sigma_1\}$
$\{\sigma_2, \sigma_3\}$
$\{\sigma_1, \sigma_3\}$
$\{\sigma_3, \sigma_1\}$

Lecture-28 (14/7/2022)

2.2.6 Pauli Matrices / Levi Civita symbol (↳ Determinant)

* $\sigma_\alpha = \sigma^\alpha = (\sigma_1, \sigma_2, \sigma_3)$ index $\alpha = 1,2,3$

$\sigma_1 = \begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\sigma_2 = \begin{pmatrix}0&-i\\i&0\end{pmatrix}$ $\sigma_3 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$

2D 'matrix' Representation of ?? (later) transformation: group/set "Rules", faceless

* $[\sigma_\alpha, \sigma_\beta] = 2i \sum_\mu \epsilon_{\alpha\beta\mu}\sigma^\mu$

* $\{\sigma_\alpha,\sigma_\beta\} = \begin{cases} 2I_2 & \alpha=\beta \\ 0 & \alpha\neq\beta\end{cases} = 2I_2\delta_{\alpha\beta}$

* $\sigma_1^2=\sigma_2^2=\sigma_3^2 = I_2 = \mathbb{1}$

↓

$\sum_{\alpha=1}^{3} \sigma_\alpha\sigma^\alpha = \sigma_1^2+\sigma_2^2+\sigma_3^2 = 3I_2$

* $\sigma_1\sigma_2 = -\sigma_2\sigma_1$ Antisym

$\epsilon_{\alpha\beta\mu} = \begin{cases}1 & \text{even perm}\\ -1 & \text{odd per.}\\ 0 & \text{any 2 same}\end{cases}$

* $\delta_{\alpha\beta} = \begin{cases}1 & \alpha=\beta \\ 0 & \alpha\ne\beta\end{cases}$

* $[\sigma_\alpha \sigma_\beta] = 2i\epsilon_{\alpha\beta\mu}\sigma^\mu$

+ $\{\sigma_\alpha \sigma_\beta\} = 2\delta_{\alpha\beta}$

$\cancel{2}\sigma_\alpha\sigma_\beta = \cancel{2}(\delta_{\alpha\beta} + i\epsilon_{\alpha\beta\mu}\sigma^\mu)$

* $\boxed{\sigma_\alpha\sigma_\beta = \delta_{\alpha\beta} + i\epsilon_{\alpha\beta\mu}\sigma^\mu}$

* $\sigma_\alpha\sigma_\beta = \delta_{\alpha\beta} + i\epsilon_{\alpha\beta\mu}\sigma^\mu$ "Index Juggling/manipulation"

$\downarrow \alpha\leftrightarrow\beta$

$\sigma_\beta\sigma_\alpha = \delta_{\beta\alpha} + i\epsilon_{\beta\alpha\mu}\sigma^\mu$

−

$[\sigma_\alpha,\sigma_\beta] = \cancel{\delta_{\alpha\beta}} - \underbrace{\delta_{\beta\alpha}}_{\cancel{\delta_{\alpha\beta}}} + i\underbrace{(\epsilon_{\alpha\beta\mu} - \underbrace{\epsilon_{\beta\alpha\mu}}_{-\epsilon_{\alpha\beta\mu}})}_{2\epsilon_{\alpha\beta\mu}}\sigma^\mu$

* $[\sigma_\alpha,\sigma_\beta] = 2i\epsilon_{\alpha\beta\mu}\sigma^\mu$ □

* trace$(\sigma_\alpha) = Tr(\sigma_\alpha) = 0$

$\alpha=1$: $Tr(\sigma_1) = 0$

$\alpha=2$: $Tr(\sigma_2) = 0$

$\alpha=3$: $Tr(\sigma_3) = 0$

traceless Pauli matrices

* $\underset{\sim}{\det}\,\sigma = -1$
determinant ⟷ Special (S) $SU(2)$ group (U: unitary)

NR 'spin' of e^-
↑
Rotation in 2D $\mathbb{C}$

* FAPP = for all practical purposes
John Bell
" Utilitarianism (Nietzsche)

2.2.7. Dirac Matrices/gamma matrices

α. Feel of Clifford Matrices (William Clifford, 1878)

* $\alpha^\mu = \begin{pmatrix}0&\sigma^\mu\\ \sigma^\mu&0\end{pmatrix}$ $\mu=1,2,3$

$\alpha^1 = \begin{pmatrix}0&\sigma^1\\\sigma^1&0\end{pmatrix} = \begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}_{4\times4} = \begin{pmatrix}0&1\\1&0\end{pmatrix}_{2\times2}\otimes\begin{pmatrix}0&1\\1&0\end{pmatrix}_{2\times2}$

$= \sigma^1\otimes\sigma^1$

* $\alpha^2 = \begin{pmatrix}0&\sigma^2\\\sigma^2&0\end{pmatrix} = \begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix}$

$\ne \begin{pmatrix}0&-i\\i&0\end{pmatrix}\otimes\begin{pmatrix}0&-i\\i&0\end{pmatrix}$

* $\alpha^3 = \begin{pmatrix}0&\sigma^3\\\sigma^3&0\end{pmatrix} = \begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix}$

$\ne \begin{pmatrix}1&0\\0&-1\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

Lecture - 29 (18/7/2022)

* symmetry $\longleftrightarrow$ Group $\longrightarrow$ Representation
  (symmetry = Transformation)
  Representation $\to$ Algebraic, geometric
  Representation $\to$ Matrix ✓, Tensor ✓

SU(2) group $\longrightarrow$ $\sigma_\alpha = \{\sigma_1, \sigma_2, \sigma_3\}$ "Generators" of SU(2) group

(dimension)

$\alpha, \beta, \mu \ldots = 1, 2, 3$

* $[\underset{\text{generator}}{\sigma_\alpha}, \sigma_\beta] = \underbrace{2i\epsilon_{\alpha\beta\mu}}_{\text{structure const (*)}}\sigma^\mu \longleftrightarrow$ group Algebra

* $\{\sigma_\alpha \sigma_\beta\} = 2I\delta_{\alpha\beta}$

* $\underbrace{SU(2)}$ $\longleftrightarrow$ "Spin" of electron
  Non-relativistic
  $v \lll c$

$\updownarrow$

Rotation in $\mathbb{C}$ (Complex plane)
Abstract

* $$\alpha^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix}, \quad \mu = 1, 2, 3$$

* $\{\alpha^\mu, \alpha^\nu\} = ?$
Check.

* $$\{\alpha^1, \alpha^2\} \equiv \alpha^1\alpha^2 + \alpha^2\alpha^1$$
$$= \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}$$

(tedious) NOT GONNA HAPPEN!

* $$\{\alpha^\mu, \alpha^\nu\} \equiv \alpha^\mu\alpha^\nu + \alpha^\nu\alpha^\mu$$
$$= \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\nu \\ \sigma^\nu & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^\nu \\ \sigma^\nu & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix}$$
$$= \begin{pmatrix} \sigma^\mu\sigma^\nu & 0 \\ 0 & \sigma^\mu\sigma^\nu \end{pmatrix} + \begin{pmatrix} \sigma^\nu\sigma^\mu & 0 \\ 0 & \sigma^\nu\sigma^\mu \end{pmatrix}$$
$$= \begin{pmatrix} \sigma^\mu\sigma^\nu + \sigma^\nu\sigma^\mu & 0 \\ 0 & \sigma^\mu\sigma^\nu + \sigma^\nu\sigma^\mu \end{pmatrix}_{2\times2}$$

* $$\{\alpha^\mu, \alpha^\nu\} = \begin{pmatrix} \sigma^\mu\sigma^\nu + \sigma^\nu\sigma^\mu & 0 \\ 0 & \sigma^\mu\sigma^\nu + \sigma^\nu\sigma^\mu \end{pmatrix}_{2\times2}$$
$$= \underbrace{(\sigma^\mu\sigma^\nu + \sigma^\nu\sigma^\mu)}_{\{\sigma^\mu\sigma^\nu\} = 2I_2\delta^{\mu\nu}}\begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix}_{2\times2}$$
$$= 2I_2\delta^{\mu\nu}\begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix} = 2\delta^{\mu\nu}I_4$$

* $$\boxed{\{\alpha^\mu, \alpha^\nu\} = 2I_4\delta^{\mu\nu}}$$

$$I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{4\times4}$$

* $$\beta \equiv \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}_{2\times2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{4\times4}$$

* $$\beta^2 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} = I_4$$

$$\boxed{\beta^2 = I_4 = (\alpha^\mu)^2}$$

Check

* $\{\alpha^\mu, \alpha^\nu\} = 2I_4\widehat{\delta^{\mu\nu}}$, $\quad \delta^{\mu\nu} = \begin{cases} 1 & \mu = \nu \\ 0 & \mu \neq \nu \end{cases}$

$\alpha^\mu\alpha^\mu + \alpha^\mu\alpha^\mu = 2I_4(1)$

$\cancel{2}\alpha^\mu\alpha^\mu = \cancel{2}I_4 \Rightarrow \underbrace{\alpha^\mu\alpha^\mu}_{(\alpha^\mu)^2} = I_4 \Rightarrow \boxed{(\alpha^\mu)^2 = I_4}$

$\alpha^\mu\alpha^\nu + \alpha^\nu\alpha^\mu = 0 \quad ; \mu \neq \nu$

* $$\{\alpha^\mu, \beta\} = \alpha^\mu\beta + \beta\alpha^\mu$$
$$= \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix}_{4\times4}\begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}_{4\times4} + \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -\sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix}_{4\times4} + \begin{pmatrix} 0 & \sigma^\mu \\ -\sigma^\mu & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_{4\times4}$$

'Clifford Algebra'

Lecture - 30 (17/7/2022)

$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\sigma^\mu = (\sigma^1 \, \sigma^2 \, \sigma^3)$

$\mu = 1,2,3$

B. Dirac Representation of Gamma Matrices

* $\alpha^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix}_{4\times4}$ (2×2), $\beta \equiv \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}_{4\times4} \equiv \gamma^0$ diagonal Matrix

def 1 of γ^μ:

* $\gamma^\mu \equiv \beta\alpha^\mu = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix}_{4\times4} = \begin{pmatrix} 0 & \sigma^\mu \\ -\sigma^\mu & 0 \end{pmatrix}_{4\times4}$

$\mu = 1$: $\gamma^1 = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}_{4\times4}$

$\mu = 2$ $\gamma^2 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}_{4\times4}$

$\mu = 3$. $\gamma^3 = \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}_{4\times4}$

* $\gamma^a \equiv (\gamma^0 \quad \gamma^\mu)$

$= (\gamma^0 \quad \gamma^1 \, \gamma^2 \, \gamma^3)$ $a,b = 0,1,2,3$

4 Entries in gamma Matrix

$[\sigma^\mu, \sigma^\nu] = 2i\epsilon^{\mu\nu\rho}\sigma_\rho$

$\{\sigma^\mu, \sigma^\nu\} = 2I_2\delta^{\mu\nu}$

* $(\gamma^0)^2 = (\beta)^2 = I_4 = \mathbb{1}$

$(\gamma^\mu)^2 = \begin{pmatrix} 0 & \sigma^\mu \\ -\sigma^\mu & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu \\ -\sigma^\mu & 0 \end{pmatrix} = \begin{pmatrix} -\overbrace{(\sigma^\mu)^2}^{I_2} & 0 \\ 0 & -\underbrace{(\sigma^\mu)^2}_{I_2} \end{pmatrix}$

$= -I_4$

* $\{\gamma^a, \gamma^b\} = ?$

check Case by case $a = 0, \overbrace{1,2,3}^{\mu}$ $\to \{\gamma^0\gamma^0\}$ (1) $\to \{\gamma^0, \gamma^\mu\}$ (2) $\to \{\gamma^\mu \gamma^\nu\}$ (3)

* $\{\gamma^0, \gamma^0\} = \gamma^0\gamma^0 + \gamma^0\gamma^0$

$= (\gamma^0)^2 + (\gamma^0)^2 = 2(\gamma^0)^2 = 2\beta^2 = 2I_4$

* $\{\gamma^0, \gamma^\mu\} = \gamma^0\gamma^\mu + \gamma^\mu\gamma^0$

$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu \\ -\sigma^\mu & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^\mu \\ -\sigma^\mu & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

* $\{\gamma^0, \gamma^\mu\} = \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\sigma^\mu \\ -\sigma^\mu & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_{4\times4}$

* $\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu$

$= \begin{pmatrix} 0 & \sigma^\mu \\ -\sigma^\mu & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\nu \\ -\sigma^\nu & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^\nu \\ -\sigma^\nu & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu \\ -\sigma^\mu & 0 \end{pmatrix}$

$= \begin{pmatrix} -\sigma^\mu\sigma^\nu & 0 \\ 0 & -\sigma^\mu\sigma^\nu \end{pmatrix} + \begin{pmatrix} -\sigma^\nu\sigma^\mu & 0 \\ 0 & -\sigma^\nu\sigma^\mu \end{pmatrix}$

$= \begin{pmatrix} -(\sigma^\mu\sigma^\nu + \sigma^\nu\sigma^\mu) & 0 \\ 0 & -(\sigma^\mu\sigma^\nu + \sigma^\nu\sigma^\mu) \end{pmatrix}$

$= -\underbrace{\{\sigma^\mu, \sigma^\nu\}}_{-2I_2\delta^{\mu\nu}} \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix} = -2I_4\delta^{\mu\nu}$

* $\{\gamma^0 \gamma^0\} = 2I_4$, $\{\gamma^0 \gamma^\mu\} = 0$, $\{\gamma^\mu \gamma^\nu\} = -2\delta^{\mu\nu}$

$\boxed{\{\gamma^a, \gamma^b\} = 2g^{ab}}$

$a,b = 0,1,2,3$

* $\delta^{\mu\nu} = \begin{cases} 1 & \mu = \nu \\ 0 & \mu \neq \nu \end{cases} = \overbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}^{I_3}$ $\mu = 1,2,3$

Kronecker delta

Euclidean Metric

* $g^{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{4\times4} = \begin{cases} 1 & a = b = 0 \\ 0 & a \neq b \\ -1 & a = b \neq 0 \end{cases}$

Minkowski Metric

$\boxed{g^{00} = 1}$ $g^{\mu\mu} = -1$ $\boxed{g^{\mu\nu} = -\delta^{\mu\nu}}$

* $[\sigma^\mu, \sigma^\nu] = 2i\epsilon^{\mu\nu\rho}\sigma_\rho$, $\{\sigma^\mu, \sigma^\rho\} = 2\delta^{\mu\rho}$

"Rotation" of NR e^- $v \ll c$ Euclidean space (3D)

* $\{\gamma^a, \gamma^b\} = 2g^{ab}$

$a,b = 0,1,2,3$ Minkowski "spacetime" (4D) = 4D - Space

"Rotation" of Relativistic e^- $v \sim c$

Lecture-31 (18/7/2022)

B. Dirac Representation of Gamma Matrices

* $\sigma^\mu_{2\times 2}$: $[\sigma^\mu, \sigma^\nu] = 2i\,\epsilon^{\mu\nu\beta}\sigma^\beta$, $\sigma^\alpha = \sigma_\alpha$ $\quad a = 0,1,2,3$; $\alpha = 1,2,3$

$\{\sigma^\mu, \sigma^\nu\} = 2\delta^{\mu\nu} I_2$ ↳ SU(2) group Algebra

* $\alpha^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix}_{4\times 4}$: $\{\alpha^\mu, \alpha^\nu\} = 2\delta^{\mu\nu} I_4$

$\beta \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{4\times 4}$ $\quad \{\alpha^\mu, \beta\} = 0$, $(\alpha^\mu)^2 = \beta^2 = 1$

* $\gamma^a = (\gamma^0 \;\; \gamma^\mu)$, $\gamma^0 \equiv \beta$, $\gamma^\mu \equiv \beta\alpha^\mu$ $\quad$ Clifford Algebra

$\{\gamma^a, \gamma^b\} = 2g^{ab}$; $g^{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \text{diag}(+---) = (+---)$

def. 2 of γ^μ : (via index juggling)

* $\gamma^\mu \equiv \alpha^\mu\beta = \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -\sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix}$

$\boxed{\text{definitions of } \gamma^\mu \text{ changed}}$

* Check: $\{\gamma^a, \gamma^b\} = ?$

* $\{\gamma^0, \gamma^0\} = \{\beta, \beta\} = \beta\beta + \beta\beta = 2\beta^2 = 2 I_4$

* $\{\gamma^0, \gamma^\mu\} = \{\beta, \alpha^\mu\beta\} = \beta\underbrace{\alpha^\mu\beta}_{-\beta\alpha^\mu} + \alpha^\mu\underbrace{\beta\beta}_{\beta^2 = 1}$

$= -\beta^2\alpha^\mu + \alpha^\mu 1$

$= -\alpha^\mu + \alpha^\mu = 0$

$\Big| \; \underbrace{\{\alpha^\mu, \beta\}}_{\alpha^\mu\beta + \beta\alpha^\mu} = 0$

* $\{\gamma^\mu, \gamma^\nu\} = \{\alpha^\mu\beta, \alpha^\nu\beta\}$

$= \alpha^\mu\underbrace{\beta\alpha^\nu}_{-\beta\alpha^\nu}\beta + \alpha^\nu\underbrace{\beta\alpha^\mu}_{-\beta\alpha^\mu}\beta = -\beta^2\underbrace{(\alpha^\mu\alpha^\nu + \alpha^\nu\alpha^\mu)}_{\{\alpha^\mu\alpha^\nu\} = 2\delta^{\mu\nu}} = -2\delta^{\mu\nu}$

* $\boxed{\{\gamma^a, \gamma^b\} = 2g^{ab}}$ No change in the Master Equation

---

* $\gamma^a = (\beta \;\; \underbrace{\beta\alpha^\mu}_{\gamma^\mu})$ $\quad$ 4 matrices

$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ -\sigma^\mu & 0 \end{pmatrix}$

$\boxed{\text{Tr}(\gamma^a) = 0}$ $\quad$ Traceless gamma matrices

gamma-five / "chirality" Matrix

* $\gamma^5 \equiv -i\gamma^0\gamma^1\underbrace{\gamma^2\gamma^3}$ $\quad \Big| \; \sigma^\mu\sigma^\nu = \delta^{\mu\nu} + i\epsilon^{\mu\nu\beta}\sigma_\beta$

$\gamma^3 = \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} \longrightarrow \gamma^2\gamma^3 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}$

$= \begin{pmatrix} -\sigma^2\sigma^3 & 0 \\ 0 & -\sigma^2\sigma^3 \end{pmatrix}$

$\gamma^2\gamma^3 = -\underbrace{\sigma^2\sigma^3}_{i\underbrace{\epsilon^{231}}_{1}\sigma_1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -i\sigma_1 I$

* $\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}(-i\sigma_1)} = -i\begin{pmatrix} 0 & (\sigma_1)^2 \\ -(\sigma_1)^2 & 0 \end{pmatrix}$

$= -i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

* $\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}(-i) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(-i)$

* $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \underbrace{(-i)(-i)}_{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}_{4\times 4}$

* $(\gamma^5)^2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_4$

* $\text{Tr}(\gamma^5) = 0$

* $\{\gamma^5, \gamma^a\} = ?$

$a = 0$: $\{\gamma^5, \gamma^0\} = \gamma^5\gamma^0 + \gamma^0\gamma^5 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 0$

$\Big| \; \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ -\sigma^\mu & 0 \end{pmatrix}$

---

* $a = \mu$: $\{\gamma^5, \gamma^\mu\} = \gamma^5\gamma^\mu + \gamma^\mu\gamma^5$

$= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu \\ -\sigma^\mu & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^\mu \\ -\sigma^\mu & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = 0$

* $\{\gamma^5, \gamma^a\} = 0$

Lecture - 32 (21/7/2022)

γ. Weyl Representation of gamma Matrices

* $\gamma^a = \begin{pmatrix} \gamma^0 \\ \gamma^\mu \end{pmatrix}$: $\{\gamma^a \gamma^b\} = 2g^{ab}$, $g^{ab} = (+ - - -)$

Dirac's choice

$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ -\sigma^\mu & 0 \end{pmatrix}$, $\gamma^5 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ ($-i\gamma^0\gamma^1\gamma^2\gamma^3$)

Weyl's 1st choice (diag)

* $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ -\sigma^\mu & 0 \end{pmatrix}$, $\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ diag

check 1

* $\gamma^5 = -i\underbrace{\gamma^0\gamma^1\gamma^2\gamma^3}_{-i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}$

$= -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

check 2

* $\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{A} = \cancel{+i}\,\gamma^0(-i)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \underbrace{\gamma^0}_{B}\underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{C}$

$A = BC$ → Num over division $\frac{A}{C} = \frac{B\cancel{C}}{\cancel{C}} = B$

→ ∃ Identity, ∃ Inverse, ∃ C^{-1} : $CC^{-1} = C^{-1}C = I$

Aristotle: $C = C \cdot 1$, $C = C$, FAPP, law of self identity

$AC^{-1} = B\underbrace{CC^{-1}}_{1} = B \Rightarrow \boxed{B = AC^{-1}}$ → Inv. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

"division" of Matrices {Expn, determinant} Matrices Vol 2 later

Weyl's 2nd choice

* $\gamma^0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ -\sigma^\mu & 0 \end{pmatrix}$ $\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ diag

check

* $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \gamma^0\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow \gamma^0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

δ. Chiral Projection operators/Matrices

* (dirac rep) $\gamma^5 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, $(\gamma^5)^2 = 1$, $\mathrm{Tr}\,\gamma^5 = 0$, $\{\gamma^5, \gamma^a\} = 0$

* $P_L \equiv \frac{1}{2}(I_2 - \gamma^5) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ left handed proj. op.

* $P_R \equiv \frac{1}{2}(1 + \gamma^5) = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ right handed proj. op.

Algebraic rep — Matrix rep

* $P_L + P_R = 1$

* $(P_L)^2 = \left(\frac{1}{2}(1 - \gamma^5)\right)^2 = \frac{1}{4}\left[1 + \underbrace{(\gamma^5)^2}_{1} - 2\gamma^5\right]$

$= \frac{1}{2}(1 - \gamma^5) = P_L \Rightarrow \boxed{(P_L)^2 = P_L}$

* $(P_R)^2 = (1 - P_L)^2 = 1 + \underbrace{P_L^2}_{P_L} - 2P_L = 1 - P_L = P_R$

* $P_L P_R = (1 - P_R)P_R = P_R - P_R^2 = 0$

$P_R P_L = 0$

$\boxed{[P_L, P_R] = 0}$

Comment on Dirac Spinor

* ψ : $P_L\psi = \frac{1}{2}(1 - \gamma^5)\psi \equiv \psi_L$ (left handed spinor)

Dirac spinor/wavefunction

$P_R\psi = \frac{1}{2}(1 + \gamma^5)\psi \equiv \psi_R$ (right handed spinor)

* $\gamma^5\psi_L = \frac{1}{2}(\gamma^5 - 1)\psi = -\psi_L$

$\gamma^5\psi_R = \frac{1}{2}(1 + \gamma^5)\psi = \psi_R$

$\psi = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}_{4\times1}$ 4 Entities Matrix

$\gamma^5\psi_L = -1\psi_L$, $\gamma^5\psi_R = +1\psi_R$

operator/Matrix — Scalar $\lambda = -1$ — op./Matrix — $\lambda = 1$ E. value

$\boxed{AB = \lambda B}$ $\boxed{AC = \lambda C}$

* $A B = \lambda B$ Eigen-value Equation

A: pre-given Matrix; λ: Eigenvalue of A; B: Eigenvector of A

(Matrix vol 2 later)

"Scaling transformation" (1 imprint)

Lecture-33 (23/7/2022)

$\{\gamma^a, \gamma^b\} = 2g^{ab}$

$\{\gamma^5, \gamma^a\} = 0$

* Dirac Rep^n

$\gamma^5 = \begin{pmatrix} 0 & 1\!\!1 \\ 1\!\!1 & 0 \end{pmatrix}_{4\times4}$

Matrix/operator Affect

psi

$\psi = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}_{4\times1}$ → $\psi_L = \frac{1}{2}(1-\gamma^5)\psi$, $\psi_R = \frac{1}{2}(1+\gamma^5)\psi$

$\psi_L = P_L\psi = \frac{1}{2}\begin{pmatrix} 1\!\!1 & -1\!\!1 \\ -1\!\!1 & 1\!\!1 \end{pmatrix}\psi = \frac{1}{2}\begin{pmatrix} 1&0&-1&0 \\ 0&1&0&-1 \\ -1&0&1&0 \\ 0&-1&0&1 \end{pmatrix}_{4\times4}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}_{4\times1}$

$= \frac{1}{2}\begin{pmatrix} a-c \\ b-d \\ c-a \\ d-b \end{pmatrix}$

$\psi_R = \frac{1}{2}\begin{pmatrix} a+c \\ b+d \\ a+c \\ b+d \end{pmatrix}$

Dirac Spinor

$\psi_L + \psi_R = \psi$

Non-arbitrary splitting of ψ

* Weyl Rep^n

$\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ diagonal (Advantage!)

$\psi_L = \underbrace{\frac{1}{2}(1-\gamma^5)}_{P_L}\psi = \begin{pmatrix} 0 \\ 0 \\ c \\ d \end{pmatrix}_{4\times1}$

$\psi_R = \frac{1}{2}(1+\gamma^5)\psi = \begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}_{4\times1}$

$\psi_R + \psi_L = \psi = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}_{4\times1} = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$ 2 components of ψ

Weyl spinor

* Dirac → Weyl : $\gamma^0 \leftrightarrow \gamma^5$ Trick

* γ^5 → $\gamma^5\psi_L = -\psi_L$, → $\gamma^5\psi_R = \psi_R$

Eigenvalues of $\gamma^5 = +1, -1$

$\gamma^5 = \begin{pmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{pmatrix}$ operator

d 2 8 Eigenvalue problem of Matrix

* $A\psi = \lambda\psi$ → E vector (that which got scaled)

A: Matrix operator $p\times q$

λ: Scaling factor, E value (deviation from $\lambda = 1$)

Eigenvalue Equation, Matrix Equation

A: Known/given

ψ, λ: unknown

$\psi = 0$ Trivial Sol^n

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1q} \\ A_{21} & & & \\ \vdots & & & \vdots \\ A_{p1} & \cdots & & A_{pq} \end{pmatrix}_{p\times q}\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_q \end{pmatrix}_{q\times1} = \lambda\begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_q \end{pmatrix}$$

* $A_{11}\psi^1 + A_{12}\psi^2 + \cdots A_{1q}\psi^q = \lambda\psi^1$

$\vdots$

$A_{p1}\psi^1 + A_{p2}\psi^2 + \cdots A_{pq}\psi^q = \lambda\psi^q$

$\Big\}\ \sum_j A_{ij}\psi^j =$

* $(A_{11}-\lambda)\psi^1 + A_{12}\psi^2 + \cdots A_{1q}\psi^q = 0$

$A_{21}\psi^1 + (A_{22}-\lambda)\psi^2 \cdots A_{2q}\psi^q = 0$

$\vdots$

$A_{p1}\psi^1 + A_{p2}\psi^2 + \cdots + (A_{pq}-\lambda)\psi^q = 0$

$\Big\} \Rightarrow (A-\lambda I)\psi = 0$

* A is a square matrix $\Rightarrow p = q$

* $\boxed{\#\text{ of linear Equations} = q}$

* $\boxed{(A-\lambda I)\psi = 0}$

Trivial Sol^n: $ax = 0$, $ax^2 + bx = 0$ → $x = 0$, $x = -\frac{b}{a}$

* $A\psi = \lambda\psi \Rightarrow A\psi - \lambda\psi = 0 \Rightarrow (\underset{\text{Matrix}}{A} - \underset{\text{scalar}}{\lambda})\psi = 0$ wrong!

$\underbrace{(A-\lambda I)}_{\mathcal{L}}\psi = 0 \Rightarrow \boxed{\mathcal{L}\psi = 0}$

* $\psi = 0$ Trivial Sol^n $\Rightarrow$ Nontrivial Sol^n $\boxed{\mathcal{L} \equiv A-\lambda I = 0}$

* $\mathcal{L} \equiv A - \lambda I = 0 \not\Rightarrow \mathcal{L} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ & & & \end{pmatrix}$

$\Downarrow$

$|A-\lambda I| = \det(A-\lambda I) = 0$

Characteristic Eq^n

* Eigen = proper = characteristic = own = its own

* $\lambda = 1 \Rightarrow A\psi = \psi \Rightarrow A = I$ Identity Matrix

Ex

* $F = Hx \longleftrightarrow \begin{pmatrix} F_x \\ F_y \end{pmatrix}_{2\times1} = \underbrace{\begin{pmatrix} -2a & -b \\ -b & -2c \end{pmatrix}}\begin{pmatrix} x \\ y \end{pmatrix}_{2\times2}$

$H = \begin{pmatrix} -2a & -b \\ -b & -2c \end{pmatrix}$

* characteristic Eq^n $\Rightarrow |H - \lambda I| = 0$

$\left|\begin{pmatrix} -2a & -b \\ -b & -2c \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right| = 0$

$\begin{vmatrix} -2a-\lambda & -b \\ -b & -2c-\lambda \end{vmatrix} = 0$

$(2a+\lambda)(2c+\lambda) - b^2 = 0$

$4ac + 2a\lambda + 2c\lambda + \lambda^2 - b^2 = 0$

$\lambda^2 + (2a+2c)\lambda + (4ac - b^2) = 0$

Quadratic Eq^n in λ

λ has 2 roots : $\boxed{\begin{matrix}\lambda_1 = \alpha \\ \lambda_2 = \beta\end{matrix}}$ Evalue of H

HW

* $a = 1$, $b = -\sqrt{5}$, $c = 3$

$\alpha = ?$

$\beta = ?$

Lecture-34 (24/7/2022)

* $A\psi = \psi \Rightarrow A = I$ (Equil^m) Identity Matrix

* $A\psi = \lambda\psi \Rightarrow \overbrace{(A-\lambda I)}\psi = 0 \Rightarrow \mathcal{L}\psi = 0$

deviation from $\boxed{\lambda = 1}$ Equil^m Static value

Sq Matrix

E.val. Equation

| A : given (Known)
| λ, ψ : unknown.

$\mathcal{L}\psi = 0 \rightarrow \psi = 0$ Trivial Sol^n

$\rightarrow \psi \neq 0 \Rightarrow \mathcal{L} \equiv A - \lambda I$

Non trivial Sol^n

$$\boxed{\det(A-\lambda I) = 0 \text{ characteristic Eq}^n}$$

$\Downarrow$

* $\mathcal{L}\psi = (A-\lambda I)\psi = 0$

$\lambda_i = \{\lambda_1, \lambda_2 \dots\}$ E.val of A

Ex. $H = \begin{pmatrix} -2 & \sqrt{5} \\ \sqrt{5} & -6 \end{pmatrix}$ E.val of H = ? E.vector of H = ?

Prescription

* $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times 2} \rightarrow \lambda I = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

$H - \lambda I = \begin{pmatrix} -2-\lambda & \sqrt{5} \\ \sqrt{5} & -6-\lambda \end{pmatrix}$

* $(H-\lambda I)\psi = 0$ E.val Eq^n $\Big\}$ 2 unknown

$\boxed{\det(H-\lambda I) = 0}$

$\rightarrow \lambda$

* $|H-\lambda I| = 0 \Rightarrow \lambda_1 = -7, \lambda_2 = -1$ E.values of H

$\lambda_1 \rightarrow \psi_1$ E.vector, $\lambda_2 \rightarrow \psi_2$

* $\underbrace{(H-\lambda I)}\psi = 0$ $\rightarrow \lambda_1 = -7 \longrightarrow \psi_1$, $\rightarrow \lambda_2 = -1 \longrightarrow \psi_2$

* $\begin{pmatrix} -2-\lambda & \sqrt{5} \\ \sqrt{5} & -6-\lambda \end{pmatrix}\psi = 0$ | $\psi = \begin{pmatrix} a \\ b \end{pmatrix}$

$\lambda = \lambda_1 = -7$: $\begin{pmatrix} 5 & \sqrt{5} \\ \sqrt{5} & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

E.vector $\psi_1 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$, $a_1 = ?$, $b_1 = ?$

$5a_1 + \sqrt{5}b_1 = 0$

$\sqrt{5}a_1 + b_1 = 0$

linear Eq^n in 2 var.

| $\frac{5}{\sqrt{5}} = \frac{\sqrt{5}}{1} \neq \frac{0}{0}$
| $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

* $b_1 = -\sqrt{5}a_1$

* $\psi_1 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} a_1 \\ -\sqrt{5}a_1 \end{pmatrix}$

$$\boxed{\psi_1 = a_1\begin{pmatrix} 1 \\ -\sqrt{5} \end{pmatrix}}$$

$\lambda_1 = -7$

(Non) Un-normalized Eigenvector

HW: check if ψ_2, λ_2 is a Sol^n of $H\psi_2 = \lambda_2\psi_2$

$H\psi_1 = \lambda_1\psi_1$

* $\lambda_2 = -1$ $\quad \psi_2 = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = b_2\begin{pmatrix} \sqrt{5} \\ 1 \end{pmatrix}$

* $H\psi_1 - \lambda_1\psi_1 = 0 \Rightarrow H\psi_1 = \lambda_1\psi_1$

$a_1\begin{pmatrix} -2 & \sqrt{5} \\ \sqrt{5} & -6 \end{pmatrix}_{2\times 2}\begin{pmatrix} 1 \\ -\sqrt{5} \end{pmatrix}_{2\times 1} = a_1\begin{pmatrix} -7 \\ 7\sqrt{5} \end{pmatrix} = \text{LHS}$

$\lambda_1\psi_1 = -7a_1\begin{pmatrix} 1 \\ -\sqrt{5} \end{pmatrix} = a_1\begin{pmatrix} -7 \\ 7\sqrt{5} \end{pmatrix} = \text{RHS}$

||

Lecture - 35 (25/7/2022)

HW due : Sunday (31/7)

* $A = \begin{pmatrix} 1 & 2 \\ -3 & 6 \end{pmatrix}$
* $B = \begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix}$
* $Z = \begin{pmatrix} 1 & 2 \\ -3 & 6 \end{pmatrix}$
* $Y = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$
* $W = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

Evalues.
Evectors of Matrices?

$$\boxed{\begin{aligned} \det(A - \lambda I) &= 0 \\ (A - \lambda I)\psi &= 0 \end{aligned}}$$

check
Matrices Vol. 2
(Later!)

Practice-3 (Lec-9 Trigonometric Id)

* $\sin^2\theta + \cos^2\theta = 1 \Leftarrow H^2 = P^2 + B^2$

$1 + \cot^2\theta = \operatorname{cosec}^2\theta$

$1 + \tan^2\theta = \sec^2\theta$

$H^2 = P^2 + B^2$

$s^2 = g_{ab} x^a x^b$

$g_{ab} = \delta_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

* $(\underbrace{\sin^4\theta - \cos^4\theta}_{} + 1) \operatorname{cosec}^2\theta = ? = 2$

$(\sin^2\theta)^2 - (\cos^2\theta)^2 = (\sin^2\theta + \cos^2\theta)(\sin^2\theta - \cos^2\theta)$

* $\dfrac{\sin\alpha}{1-\cos\alpha} = \dfrac{\sin\alpha}{1-\cos\alpha} \cdot \dfrac{(1+\cos\alpha)}{(1+\cos\alpha)} = \dfrac{1+\cos\alpha}{\sin\alpha}$
$= \operatorname{cosec}\alpha + \cot\alpha$

* $\cot\theta - \tan\theta = \dfrac{\cos\theta}{\sin\theta} - \dfrac{\sin\theta}{\cos\theta} = \dfrac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta}$
$= \dfrac{1 - 2\sin^2\theta}{\sin\theta\cos\theta} = \dfrac{2\cos^2\theta - 1}{\sin\theta\cos\theta}.$

* $\dfrac{\tan\psi + \sin\psi}{\tan\psi - \sin\psi} = \dfrac{\frac{\sin\psi}{\cos\psi} + \frac{\sin\psi}{1}}{\frac{\sin\psi}{\cos\psi} - \frac{\sin\psi}{1}} = \dfrac{1+\cos\psi}{1-\cos\psi} = \dfrac{\sec\psi + 1}{\sec\psi - 1}$

* $\sqrt{\dfrac{1-\sin\beta}{1+\sin\beta}} = \dfrac{\cos\beta}{1+\sin\beta} = \dfrac{1-\sin\beta}{\cos\beta} = \sec\beta - \tan\beta$

* $\dfrac{1-\sin\gamma}{1+\sin\gamma} = \left(\dfrac{\cos\gamma}{1+\sin\gamma}\right)^2 = (\sec\beta - \tan\beta)^2$

* $\dfrac{\sin\lambda + \cos\lambda}{\sin\lambda - \cos\lambda} + \dfrac{\sin\lambda - \cos\lambda}{\sin\lambda + \cos\lambda} = \dfrac{2}{\sin^2\lambda - \cos^2\lambda} = \dfrac{2}{1 - 2\cos^2\lambda}$
$= \dfrac{2}{2\sin^2\lambda - 1}$

* $\boxed{\cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1}$!

HW (due Thur.)

* $(\operatorname{cosec}\theta - \sin\theta)(\sec\theta - \cos\theta)(\tan\theta + \cot\theta) = ?$
* $\dfrac{2\sin\theta - 2\sin^3\theta}{2\cos^3\theta - \cos\theta} = ?$
* $(\sin\theta + \operatorname{cosec}\theta)^2 + (\cos\theta + \sec\theta)^2 = ?$
* $\sec^4\theta - \sec^2\theta = ?$

$\sqrt{\sec^2\theta + \operatorname{cosec}^2\theta} = ?$

Lecture - 36 (28/7/2022)

Practice-4

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$1 + \cot^2\theta = \operatorname{cosec}^2\theta$

$H^2 = P^2 + B^2$

$1 = \left(\frac{P}{H}\right)^2 + \left(\frac{B}{H}\right)^2$

* $(\sin\theta + \operatorname{cosec}\theta)^2 + (\cos\theta + \sec\theta)^2$

$\sin^2\theta + \operatorname{cosec}^2\theta + 2 + \cos^2\theta + \sec^2\theta + 2$

$5 + \operatorname{cosec}^2\theta + \sec^2\theta = 7 + \tan^2\theta + \cot^2\theta$

* $\sec^4\theta - \sec^2\theta = \sec^2\theta\,\underbrace{(\sec^2\theta - 1)}_{\tan^2\theta} = \sec^2\theta \tan^2\theta$

$= (1+\tan^2\theta)\tan^2\theta = \tan^2\theta + \tan^4\theta$

* $\sqrt{\sec^2\theta + \operatorname{cosec}^2\theta} = \sqrt{2 + \tan^2\theta + \cot^2\theta} = \tan\theta + \cot\theta$

* $2\sec^2\alpha - \sec^4\alpha - 2\operatorname{cosec}^2\alpha + \operatorname{cosec}^4\alpha$

Alter

$(\operatorname{cosec}^4\alpha - 2\operatorname{cosec}^2\alpha) - (\sec^4\alpha - 2\sec^2\alpha)$

$x^4 - x^2 = x^2(x^2-1)$

$x^4 - 2x^2 \neq \underbrace{(x^2-1)(x^2+1)}_{x^4-1}$

* $x^4 - 2x^2 = (x^2-1)^2 - 1$

* $(\operatorname{cosec}^2\alpha - 1)^2 - (\sec^2\alpha - 1)^2 = \cot^4\alpha - \tan^4\alpha$

* $\dfrac{\sin\theta}{1-\cos\theta} + \dfrac{\tan\theta}{1+\cos\theta} = \dfrac{\sin\theta + \sin\theta\cos\theta + \tan\theta - \tan\theta\cos\theta}{\sin^2\theta}$

$\dfrac{\cancel{1} + \cos\theta + \frac{1}{\cos\theta} - \cancel{1}}{\sin\theta} = \cot\theta - \operatorname{cosec}\theta\sec\theta$

* $\dfrac{1}{\sec\beta - \tan\beta} = \sec\beta + \tan\beta$

* $\dfrac{\sec\gamma - \tan\gamma}{\sec\gamma + \tan\gamma} = (\sec\gamma - \tan\gamma)^2$ ∃ Moreness.

$\parallel$

$\dfrac{\sec^2\gamma + \tan^2\gamma - 2\sec\gamma\tan\gamma}{1+\tan^2\gamma}$

$= 1 - 2\sec\gamma\tan\gamma + 2\tan^2\gamma$

HW P.T.

* $\dfrac{1}{\operatorname{cosec}\alpha - \cot\alpha} - \dfrac{1}{\sin\alpha} = \dfrac{1}{\sin\alpha} - \dfrac{1}{\operatorname{cosec}\alpha + \cot\alpha}$

* $\dfrac{\cos A}{1-\tan A} + \dfrac{\sin A}{1-\cot A} = \cos A + \sin A$

* $\sin^6\alpha + \cos^6\alpha = 1 - 3\sin^2\alpha\cos^2\alpha$

* $\cot^4\theta - 1 = \operatorname{cosec}^4\theta - 2\operatorname{cosec}^2\theta$

* $\dfrac{\cos^2\theta}{1-\tan^2\theta} + \dfrac{\sin^3\theta}{\sin\theta - \cos\theta} = 1 + \sin\theta\cos\theta$

* $\dfrac{\sin^3\theta + \cos^3\theta}{\sin\theta + \cos\theta} + \cos\theta\sin\theta = 1$

Lecture - 37 (30/7/2022)

Practice - 5

* $\frac{(1+\sin\theta)^2+(1-\sin\theta)^2}{\cos^2\theta} = \frac{2+2\sin^2\theta}{\cos^2\theta} = \frac{2(1+\sin^2\theta)}{\cos^2\theta} = \frac{2(1+\sin^2\theta)}{(1-\sin^2\theta)}$

* $\frac{\sin A-\sin B}{\cos A+\cos B} + \frac{\cos A-\cos B}{\sin A+\sin B} = \frac{\sin^2A-\sin^2B+\cos^2A-\cos^2B}{\text{Denominator}}$

$= \frac{1-1}{\text{Den.}} = 0$

* if $\tan\theta+\sin\theta = m$, $\tan\theta-\sin\theta = n$ } P.T. $m^2-n^2 = 4\sqrt{mn}$

* $m^2-n^2 = 2\tan\theta \, 2\sin\theta = 4\tan\theta\sin\theta = 4\sqrt{mn}$

check

$mn = \tan^2\theta-\sin^2\theta = \sin^2\theta\left(\frac{1}{\cos^2\theta}-1\right) = \sin^2\theta\frac{(1-\cos^2\theta)}{\cos^2\theta} = \frac{\sin^2\theta}{\cos^2\theta}\sin^2\theta$

$= \tan\theta\sin\theta$

* if $\cos\theta+\sin\theta = \sqrt{2}\cos\theta$

$\cos\theta-\sin\theta = ?$

* $\cos^2\theta+\sin^2\theta+2\cos\theta\sin\theta = 2\cos^2\theta \Rightarrow \underbrace{\cos^2\theta-\sin^2\theta}_{(\cos\theta+\sin\theta)(\cos\theta-\sin\theta)} = 2\sin\theta\cos\theta$

$\cos\theta-\sin\theta = \frac{2\cancel{\cos\theta}\sin\theta}{\sqrt{2}\cancel{\cos\theta}} = \sqrt{2}\sin\theta$ $\square$

HW

* if $x = a\sin\theta$, $y = b\tan\theta$ P.T. $\frac{a^2}{x^2}-\frac{b^2}{y^2} = 1$

* if $x = r\sin A\cos C$, $y = r\sin A\sin C$, $z = r\cos A$

P.T. $r^2 = x^2+y^2+z^2$ (Spherical Polar C.S)

* if $a\cos\theta+b\sin\theta = m$, $a\sin\theta-b\cos\theta = n$ } P.T. $a^2+b^2 = m^2+n^2$

Lecture - 38 (31/7/2022)

Practice - 5

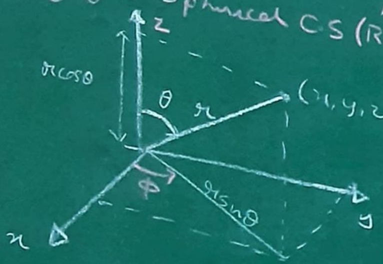
$x^3+y^3=(x+y)(x^2+y^2-xy)$

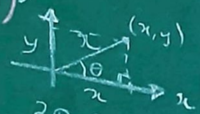
$x^3+(-y)^3=(x-y)(x^2+y^2+xy)$

* $\dfrac{\cos^3\theta}{1-\tan\theta}+\dfrac{\sin^3\theta}{\sin\theta-\cos\theta}=1+\sin\theta\cos\theta$

d.3 * Trig for Spherical C.S (Ref L-25)

Alg Vol 1





2D Euclidean

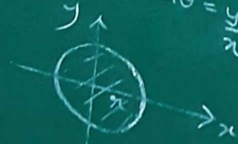
$(x,y)\xrightarrow{?}(r,\theta)$

Basis 1 Basis 2

$x=r\cos\theta$

$y=r\sin\theta$

$x^2+y^2=r^2,\ \tan\theta=\dfrac{y}{x}$



circular polar (r,θ)

Spherical C.S.

$(x,y,z)\longrightarrow(r,\theta,\phi)$

$x=(r\sin\theta)\cos\phi$

$y=(r\sin\theta)\sin\phi$

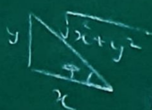
$z=r\cos\theta$

$x^2+y^2+z^2=r^2$ distance

$\boxed{\dfrac{y}{x}=\tan\phi}\Rightarrow\phi=\tan^{-1}\dfrac{y}{x}$

* $\dfrac{y}{z}=\tan\theta\ \underbrace{\sin\phi}_{?}$

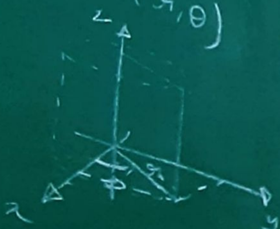
$\tan\phi=\dfrac{y}{x}$



$\sin\phi=\dfrac{y}{\sqrt{x^2-y^2}}$

$\dfrac{y}{z}=\tan\theta\ \dfrac{\cancel{y}}{\sqrt{x^2-y^2}}$

* $\boxed{\tan\theta=\dfrac{\sqrt{x^2+y^2}}{z}}$



cylindrical CS

$(x,y,z)\to(r,\phi,z)$

$x=r\cos\phi$

$y=r\sin\phi$

$z=z$

* HW if $a\cos\theta-b\sin\theta=c$

$a\sin\theta+b\cos\theta=?$

* $\dfrac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1}=\dfrac{1+\sin\theta}{\cos\theta}$

* HW : Short - Trig RD Vol. 1 (Ch-5/6 4 exercises +2)

Lecture - 39 (7/8/2022)

$A - B = \frac{n\pi}{2}$ $n = 0, 1, 2$

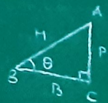
Q4 T-ratios of Allied Angles

1. $0°, 90°$

* $\sin\theta = \frac{P}{H}$

$\cos\theta = \frac{B}{H}$

$\tan\theta = \frac{P}{B}$



* $\sin 0 = \frac{0}{H} = 0$

$\theta \to 0 \Rightarrow P \to 0$

* $\theta \to 0 \Rightarrow P \to 0 \Rightarrow H \to B$

$\underbrace{\lim_{\theta \to 0}}$ $\cos\theta = \lim_{\theta \to 0} \left(\frac{B}{H}\right) = \lim_{H \to B} \left(\frac{B}{H}\right) = \frac{B}{B} = 1$

limits (later)

* $\boxed{\lim_{x \to a} [f(x)] = f(x = a)}$ Direct substitution of the limit

Machine

▷ Machine

∃ couplings & connection

* $\lim_{x \to b} g(x) = g(b)$

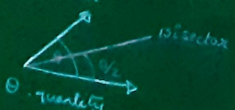
$\boxed{\lim_{x \to c} g(b) = g(b)}$

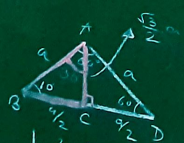
* $\tan 0 = \frac{\sin 0}{\cos 0} = \frac{0}{1} = 0$

2. $30°$ & $60°$

* Equil Δ sides a

* ⊥ bisector

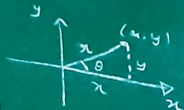




$a^2 - \frac{a^2}{4} = P^2 \Rightarrow P = \frac{\sqrt{3}}{2} a$

⟶ tends to approaches

Tendency



$g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \delta_{ab}$

$s^2 = g_{ab} x^a x^b$

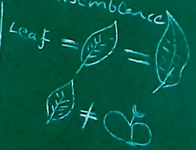
$x^a = (x \ \ y)$

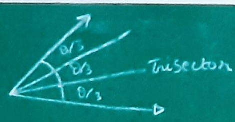
$a = 1, 2$

$s^2 = x^2 + y^2 = r^2$

Pythagoras

Resemblance





1 angle — θ

1 part — $\frac{\theta}{3}$

\# parts = 3

* $\sin 60° = \frac{P}{H} = \frac{\frac{\sqrt{3}}{2} a}{a} = \frac{\sqrt{3}}{2}$

$\cos 60° = \frac{1}{2}$

$\tan 60° = \sqrt{3}$

* $\sin 30 = \frac{a/2}{a} = \frac{1}{2}$

3. $45°$

* $A + B + C = 180°$

$A + C = 90°$

* $A = 45 \Rightarrow C = 45 \Rightarrow$ Isosceles $\Delta \Rightarrow AB = CB = a$

* $\sin 45 = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$

$\cos 45° = \frac{1}{\sqrt{2}}$

$\tan 45 = 1$

4. Ratio table for 1st quad

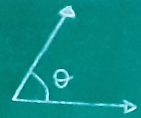
T ratio	0	30	45	60	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tanθ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	N.D.
cosec	N.D	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	ND
cot	ND	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

5. Complementary / Allied Angles

* $\sin\theta = \frac{P}{H}$

$\cos\theta = \frac{B}{H}$

$\tan\theta = \frac{P}{B}$



1 angle — θ

1 part — $\frac{\theta}{n}$

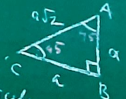
\# parts = n

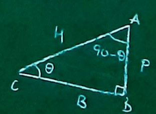
$\downarrow \lim_{n \to \infty}$

$\underbrace{\text{Lie}}$ Groups

Sophus Lie (1888)

" all physics current theories are written in this language "





* $\operatorname{cosec}\theta = \frac{H}{P}$

$\sec\theta = \frac{H}{B}$

$\cot\theta = \frac{B}{P}$

* $\sin(90 - \theta) = \frac{B}{H} = \cos\theta$

$\sin(90 - 90 + \theta) = \cos(90 - \theta)$

* $\tan(90 - \theta) = \cot\theta$

$\tan(\theta) = \cot(90 - \theta)$

* $\operatorname{cosec}(90 - \theta) = \sec\theta$

$\sec(\theta) = \operatorname{cosec}(90 - \theta)$

Examples

* $\sin 39 - \cos 51 = 0$

* $\tan 1 \tan 2 \tan 3 \ldots \tan 89 = 1$

Sheet Trigo.

Ex 5.1 / 5.2 / 5.3

6.1

Lecture - 40 (8/8/2022)

$A - B = \frac{n\pi}{2}$ $n = 0, 1, 2$

Q4 T-ratios of Allied Angles

* $2\sin 2\alpha = \sqrt{3}$ $\Rightarrow$ Particular sol^n $\alpha = 30^\circ = \frac{\pi}{6}$

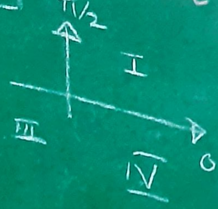
$\boxed{\alpha \in [0, \frac{\pi}{2}]}$

Linear Eq^n

$2x^2 + 3 = 5$

$x = 1$ ↑ variable

Trigonometric Equation ⇓ (Vol. 2)



$\sin 2\alpha = \frac{\sqrt{3}}{2} \Rightarrow 2\alpha = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ ✓

* $2\cos\theta + 1 = 0 \Rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \theta = \cos^{-1}\left(-\frac{1}{2}\right) = ?$

Practice 1

* $\tan\theta + \cot\theta = 2$, $\theta \in [0, \frac{\pi}{2}]$ Acute angle

$\theta = 45$

$\tan^7\theta + \cot^7\theta = (\tan 45^\circ)^7 + (\cot 45)^7 = 2$

$\tan 45^\circ = 1$, $\cot 45 = 1$

* $\frac{(\cos\theta - \sin\theta)}{(\cos\theta + \sin\theta)} = \frac{(1-\sqrt{3})}{(1+\sqrt{3})} \Rightarrow \theta = ?$

$\frac{\cancel{2}\cos\theta}{\cancel{2}\sin\theta} = \frac{\cancel{2}}{\cancel{2}\sqrt{3}} \Rightarrow \tan\theta = \sqrt{3}$

⇓

$\boxed{\theta = 60^\circ}$

Ref. A1 vol. 2

$\frac{a}{b} = \frac{c}{d}$

$\frac{a+b}{b} = \frac{c+d}{d}$

$\frac{a-b}{b} = \frac{c-d}{d}$

$\frac{a+b}{a-b} = \frac{c+d}{c-d}$

(C+D)

Lecture - 41 (9/8/2022)

Practice 2

* $\alpha, \beta, \gamma \in [0, \frac{\pi}{2}]$: $\sin(\alpha+\beta-\gamma) = \frac{1}{2}$

$\alpha, \beta, \gamma = ?$ $\quad \cos(\beta+\gamma-\alpha) = \frac{1}{2}$

* $\quad \tan(\gamma+\alpha-\beta) = 1$

$\alpha = \frac{75}{2}, \beta = 45, \gamma = \frac{105}{2}$

check

* Add ①, ② $\Rightarrow 2\beta = 90°$

* $\frac{\sin^2 20 + \sin^2 70}{\cos^2 20 + \cos^2 70} + \frac{\sin(90-\theta)\sin\theta}{\tan\theta} + \frac{\cos(90-\theta)\cos\theta}{\cot\theta} = 2$

* $p = \sin\alpha + \cos\alpha$ ✓
$q = \operatorname{cosec}\alpha + \sec\alpha$ } Eliminate α / Equation of $p(q) = ?$

$\sin^2\theta + \cos^2\theta = 1$

$1 + \tan^2 = \sec^2$

$\operatorname{cosec}\alpha + \sec\alpha = q$ ✓

* $\frac{\cos\alpha + \sin\alpha}{\cos\alpha\sin\alpha} = q \Rightarrow \frac{p}{\sin\alpha\cos\alpha} = q$

$\sin\alpha\cos\alpha = \frac{p}{q}$

$\sin\alpha + \cos\alpha = p$

$\sin^2\alpha + \cos^2\alpha + 2\sin\alpha\cos\alpha = p^2$

$\sin\alpha\cos\alpha = \frac{p^2-1}{2}$

$\frac{p^2-1}{2}$

$\boxed{q(p^2-1) = 2p}$

*

$\frac{\cos\alpha}{\cos\beta} = m, \quad \frac{\cos\alpha}{\sin\beta} = n$ } Eliminate α / write Eqn for β, n, m

*

$\sin^2\beta + \cos^2\beta = 1$

$\frac{\cos^2\alpha}{n^2} + \frac{\cos^2\alpha}{m^2} = 1 \Rightarrow (m^2+n^2)\cos^2\alpha = n^2 m^2$

*

$\boxed{(m^2+n^2)\cos^2\beta = n^2}$

* $\operatorname{cosec}\theta - \sin\theta = l$
$\sec\theta - \cos\theta = m$ } $l^2m^2(l^2+m^2+3) = 1$

$lm = \left(\frac{1}{\sin\theta} - \sin\theta\right)\left(\frac{1}{\cos\theta} - \cos\theta\right) = \frac{\left(\cancel{1} - \cos^2\theta - \sin^2\theta + \sin^2\theta\cos^2\theta\right)}{\sin\theta\cos\theta}$

$= \sin\theta\cos\theta$

$l^2 + m^2 = \operatorname{cosec}^2\theta + \sin^2\theta - 2\sin\theta\operatorname{cosec}\theta + \sec^2\theta + \cos^2\theta - 2\sec\theta\cos\theta$

$l^2 + m^2 = 1 - 4 + \operatorname{cosec}^2\theta + \sec^2\theta$

$= -3 + \frac{1}{\sin^2\theta\cos^2\theta} \Rightarrow l^2 + m^2 + 3 = \frac{1}{\sin^2\theta\cos^2\theta} = \frac{1}{l^2m^2}$

*

$l^2m^2(l^2+m^2+3) = 1$

HW

* $\tan\gamma = n\tan\beta$
$\sin\gamma = m\sin\beta$ } Eliminate β

Lecture-42 (14/8/2022)

Practice Δ (Trig + Linear Equation Elimination)

* $\tan\gamma = n\tan\beta \Rightarrow \tan\beta = \frac{1}{n}\tan\gamma$

$\sin\gamma = m\sin\beta \Rightarrow \sin\beta = \frac{1}{m}\sin\gamma$

$\sin^2\theta+\cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$1+\cot^2\theta = \csc^2\theta$

Eliminate β

$\cot\beta = \frac{n}{\tan\gamma}, \quad \csc\beta = \frac{m}{\sin\gamma}$

* $\csc^2\beta - \cot^2\beta = 1 \Rightarrow \frac{m^2}{\sin^2\gamma} - \frac{n^2}{\tan^2\gamma} = 1$

$m^2 - n^2\cos^2\gamma = \sin^2\gamma = 1-\cos^2\gamma \Rightarrow m^2-1 = (n^2-1)\cos^2\gamma$

$\Downarrow$

$\boxed{\cos^2\gamma = \frac{m^2-1}{n^2-1}}$

* $\csc\theta - \sin\theta = m$

$\sec\theta - \cos\theta = n$ } Eliminate θ

$\Downarrow$

$\frac{1}{\sin\theta} - \sin\theta = m \Rightarrow \frac{1-\sin^2\theta}{\sin\theta} = m \Rightarrow \frac{\cos^2\theta}{\sin\theta} = m$

$\frac{1}{\cos\theta} - \cos\theta = n \Rightarrow \frac{1-\cos^2\theta}{\cos\theta} = n \Rightarrow \frac{\sin^2\theta}{\cos\theta} = n$

* $\sin^2\theta + \cos^2\theta = 1$

$\boxed{\cos^3\theta = m^2n}$

$\Downarrow$

$\cos\theta = (m^2n)^{1/3}$

$\sin^3\theta = mn^2$

$\Downarrow$

$\sin\theta = (mn^2)^{1/3}$

$\boxed{(m^2n)^{2/3} + (mn^2)^{2/3} = 1}$

$mn = \sin\theta\cos\theta$

$m^2n = \frac{\cos^4\theta}{\cancel{\sin^2\theta}}\cdot\frac{\cancel{\sin^2\theta}}{\cos\theta} = \cos^3\theta$

$mn^2 = \frac{\cancel{\cos^2\theta}}{\sin\theta}\cdot\frac{\sin^4\theta}{\cancel{\cos^2\theta}} = \sin^3\theta$

* $x\sin^3\theta + y\cos^3\theta = \sin\theta\cos\theta$

$x\sin\theta = y\cos\theta$ } Eliminate θ.

* $x\underbrace{\sin^3\theta}_{\sin\theta\sin^2\theta} + y\underbrace{\cos^3\theta}_{\cos\theta\cos^2\theta} = \sin\theta\cos\theta$

$(x\sin\theta)(\sin^2\theta) + (x\sin\theta)\cos^2\theta = \cos\theta\sin\theta$

$x\cancel{\sin\theta} = \cos\theta\cancel{\sin\theta} \Rightarrow \boxed{x=\cos\theta}$

* $x\sin\theta = y\cos\theta \Rightarrow \cancel{\cos\theta}\sin\theta = y\cancel{\cos\theta}$

$\boxed{y = \sin\theta}$

* $\boxed{x^2+y^2=1}$

* $a\overbrace{\sec\theta}^{x} + b\overbrace{\tan\theta}^{y} + c = 0$

$p\underbrace{\sec\theta}_{x} + q\underbrace{\tan\theta}_{y} + r = 0$ } Eliminate θ ☆

$\frac{\sec\theta}{br-qc} = \frac{-\tan\theta}{ar-pc} = \frac{1}{aq-bp}$

$\sec\theta = \frac{br-qc}{aq-bp}, \quad \tan\theta = \frac{pc-ar}{aq-bp}$

$\sec^2\theta - \tan^2\theta = 1$

$\Downarrow$

$\boxed{(br-qc)^2 + (pc-ar)^2 = (aq-bp)^2}$

* $\tan^2\theta = 1-a^2$

$\sec\theta + \tan^3\theta\csc\theta = ?$

$\sec\theta + \frac{\sin^3\theta}{\cos^3\theta}\cdot\frac{1}{\sin\theta} = \frac{1}{\cos\theta} + \frac{\tan^2\theta}{\cos\theta}$

$= \frac{1+\tan^2\theta}{\cos\theta} = (1+\tan^2\theta)\underbrace{\sec\theta}_{\sqrt{1+\tan^2\theta}} = (1+\underbrace{\tan^2\theta}_{1-a^2})^{3/2}$

$= (2-a^2)^{3/2}$

* $\sin\theta + \cos\theta = \sqrt{3}, \quad \tan\theta + \cot\theta = ?$

* $1 + 2\sin\theta\cos\theta = 3 \Rightarrow \sin\theta\cos\theta = 1$

$\tan\theta + \cot\theta = 1$

HW

* $a\cos^3\theta + 3a\cos\theta\sin^2\theta = m$

$a\sin^3\theta + 3a\cos^2\theta\sin\theta = n$ } Eliminate θ

$(m+n)^{2/3} + (m-n)^{2/3} = ?$

* $2\sec^2\alpha - \sec^4\alpha - 2\csc^2\alpha + \csc^4\alpha = ?$

(in terms of $\tan\alpha$)

* $3(\sin\beta - \cos\beta)^4 + 6(\sin\beta+\cos\beta)^2 + 4(\sin^6\beta + \cos^6\beta) = ?$

Lecture - 43 (15/8/2022)

$\sin^2 + \cos^2 = 1$
$\tan^2 + 1 = \sec^2$
$1 + \cot^2 = \csc^2$

Practice Δ (Trig + Linear Equation Elimination)

* $2\sec^2\alpha - \sec^4\alpha - 2\csc^2\alpha + \csc^4\alpha$

$2(\cancel{1} + \tan^2\alpha) - (1+\tan^2\alpha)^2 - 2(\cancel{1} + \cot^2\alpha) + (1+\cot^2\alpha)^2$

$= \frac{1}{\tan^4\alpha} - \tan^4\alpha$

* $3(\underbrace{\sin\beta - \cos\beta)^4}) + 6(\sin\beta + \cos\beta)^2 + 4(\sin^6\beta + \cos^6\beta)$

$\left[\underbrace{(\sin\beta - \cos\beta)^2}_{1 - 2\sin\beta\cos\beta}\right]^2 = 1 + 4\sin^2\beta\cos^2\beta - 4\sin\beta\cos\beta$

$3 + 12\sin^2\beta\cos^2\beta - 12\cancel{\sin\beta}\cos\beta + 6 + 12\cancel{\sin\beta}\cos\beta + 4(\sin^6\beta + \cos^6\beta)$

$9 + 12\sin^2\beta\cos^2\beta + 4(\underbrace{\sin^6\beta + \cos^6\beta}_{(\sin^2\beta)^3 + (\cos^2\beta)^3})$

$\boxed{a^3 + b^3 = (a+b)(a^2 + b^2 - ab)}$

* $9 + 12\sin^2\beta\cos^2\beta + 4\left\{(\underbrace{\sin^2\beta + \cos^2\beta}_{1})(\sin^4\beta + \cos^4\beta - \sin^2\beta\cos^2\beta)\right\}$

$9 + \underbrace{8\sin^2\beta\cos^2\beta + 4(\sin^4\beta + \cos^4\beta)}_{4(\underbrace{\sin^2\beta + \cos^2\beta}_{1})^2} = 13$

* $a\cos\gamma - b\sin\gamma = c$

$a\sin\gamma + b\cos\gamma = x = ?$

$a^2\cos^2\gamma + b^2\sin^2\gamma - 2ab\cos\gamma\sin\gamma = c^2$

* $a^2\sin^2\gamma + b^2\cos^2\gamma + \underbrace{2ab\sin\gamma\cos\gamma}_{a^2\cos^2\gamma + b^2\sin^2\gamma - c^2} = x^2$

$\boxed{x = \pm\sqrt{a^2 + b^2 - c^2} = a\sin\gamma + b\cos\gamma}$

* $\sin\theta + \cos\theta = \sqrt{2}\underbrace{\sin(90 - \theta)}_{\cos\theta}$; $\cot\theta = ?$

$\sin\theta + (1 - \sqrt{2})\cos\theta = 0$

$\boxed{\frac{1}{\sqrt{2}-1} = \cot\theta}$

$\parallel$

$\sqrt{2} + 1$

* $\sin^8\theta - \cos^8\theta \stackrel{?}{=} (\sin^2\theta - \cos^2\theta)(1 - 2\sin^2\theta\cos^2\theta)$

HW 9/10

* $\frac{ax}{\cos\theta} + \frac{by}{\sin\theta} = a^2 - b^2$

$\frac{ax\sin\theta}{\cos^2\theta} - \frac{by\cos\theta}{\sin^2\theta} = 0$

Eliminate θ

$(ax)^{2/3} + (by)^{2/3} = ?$

$\Downarrow$

$ax\sin^3\theta = by\cos^3\theta \Rightarrow \frac{ax}{\cos^3\theta} = \frac{by}{\sin^3\theta}$

$\frac{(ax)^{1/3}}{\cos\theta} = \frac{(by)^{1/3}}{\sin\theta} \Rightarrow \frac{(ax)^{2/3}}{\cos^2\theta} = \frac{(by)^{2/3}}{\sin^2\theta}$