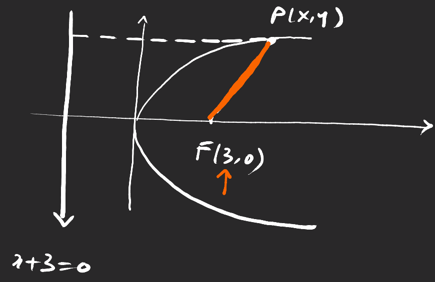
Lecture-14 (20/Jun) 1.5

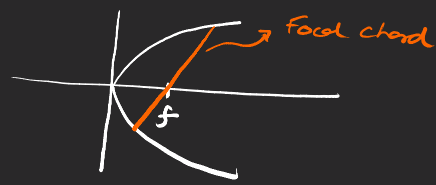


* $|PF| = \sqrt{(x-3)^2+y^2}$, $y^2=12x$

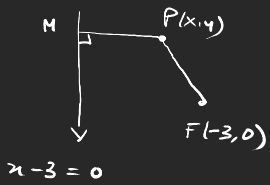
$= \sqrt{x^2+9-6x+12x} = \sqrt{x^2+6x+9}$

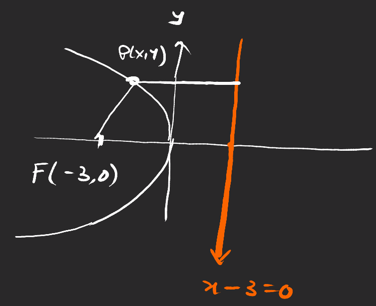
$= \sqrt{(x+3)^2} = x+\underbrace{3}_{a}$ = focal length of any pt.

$P(0,0) \Rightarrow |PF| = 3 =$ focal length



Type II : $y^2=-4ax$



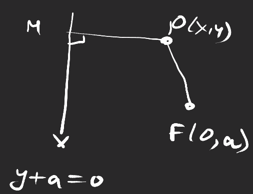


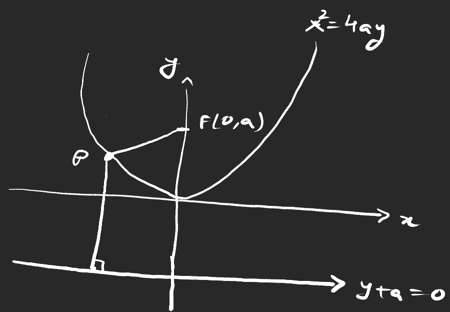
* $|QF|^2 = |PF|^2 = (x+3)^2+y^2 = |x-3|^2$

$\boxed{y^2=-12x}$

- symm. in y $\longrightarrow$ $x<0$ always
- passes thru. 0
- vertex $(0,0)$
- LR = 12
- $|PF| = \sqrt{(x+3)^2+y^2} = \sqrt{x^2+9+6x-12x} = x-3$ (expression)

Type 3 : $x^2=4ay$





* $|PM|^2=|PF|^2 \Rightarrow x^2+(y-a)^2 = |y+a|^2$

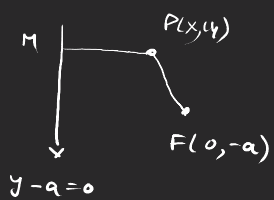
$\boxed{x^2=4ay}$

- passes thru 0
  vertex $(0,0)$
- sym in x $\rightarrow$ $y>0$

$y = \frac{1}{4a}x^2$

- $|PF| = y+a$
- LR = 4a

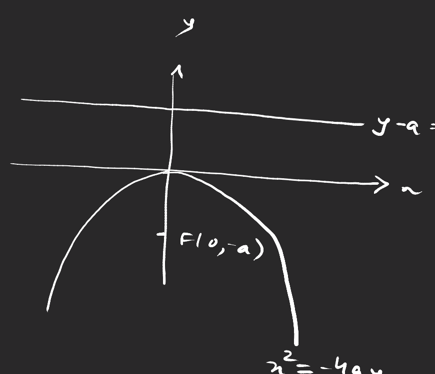
Type 4 : $x^2=-4ay$



* $|MP|^2 = |PF|^2$

$x^2+(y+a)^2 = |y-a|^2$

$x^2+2ay = -2ay \Rightarrow \boxed{x^2=-4ay}$

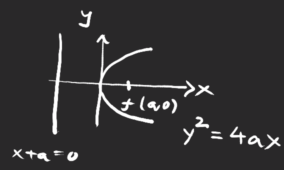


- passes thru 0
- sym. in x $\quad y<0 \Rightarrow x \in \mathbb{R}$
- LR = 4a

• If $L| = y-a$

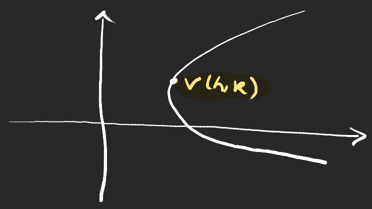
$y^2 = 4ax \qquad y^2 = -4ax$

$x^2 = 4ay \qquad x^2 = -4ay$



Type 5 : Non origin vertex

Method 1 (By inspection)



* ∵ $C \equiv y^2 = 4ax$ for $V(0,0)$

$\downarrow$ $y \to y-k$, $x \to x-h$

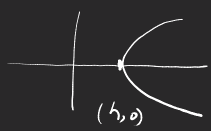
$\left\{ \begin{array}{l} C(x,y) \text{ satisfies } (0,0) \\ C(x-h, y-k) \text{ satisfies } V(h,k) \end{array} \right\}$

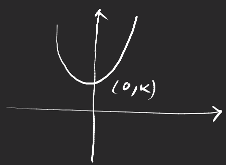
$C_1 \equiv \boxed{(y-k)^2 = 4a(x-h)}$ for $V(h,k)$

$LR = 4a$

Simpler case : $V(h,0) \Rightarrow \boxed{y^2 = 4a(x-h)}$

$V(0,k) \Rightarrow (y-k)^2 = 4ax$

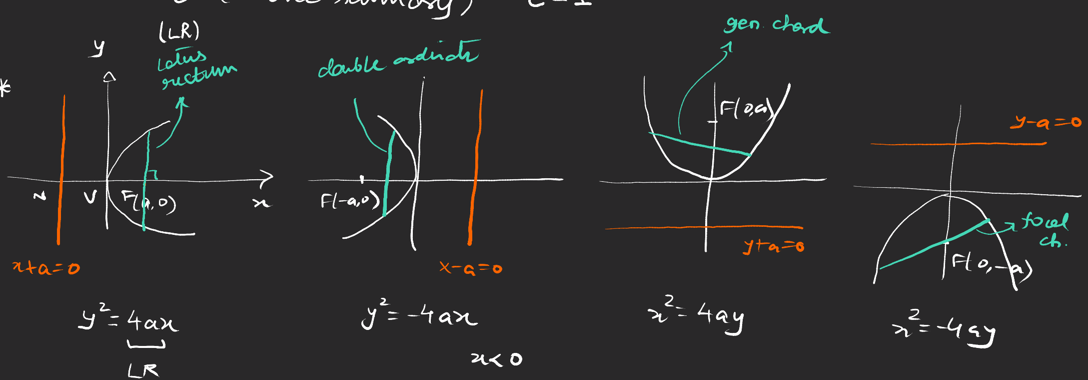


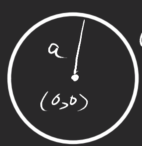


Lecture-20 (21/Jan) 2

4. Parabola (Basic summary) $e = 1$

*





$C \equiv x^2 + y^2 = a^2$

$C \equiv x^2 + y^2 + 2gx + 2fy + c = 0$

centre $(-g, -f)$

$r = \sqrt{g^2 + f^2 - c}$

"fundamental Parabolas"

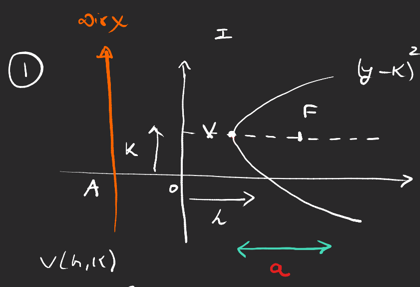
Comment on LR :

* Length (LR) = 4a

* Vertex = Mid pt of focus & (pt of Intersection of directrix & Axis) $\Rightarrow |NV| = |VF|$

* $\perp$ distance from focus on directrix = Half of LR

* 2 parabolas are "equal" if they have same LR $\rightarrow$ $\boxed{LR_1 = LR_1 \Rightarrow C_1 = C_2}$



$(y-k)^2 = 4a(x-h)$

$V(h,k)$

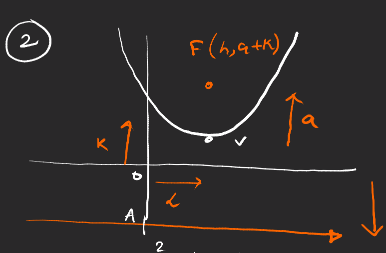
$(y-k)^2 = 4a(x-h)$

$F(a+h, k)$ focus

Coord transfⁿ

$x \to X = x-h$

$y \to Y = y-k$



$(x-h)^2 = 4a(y-k)$

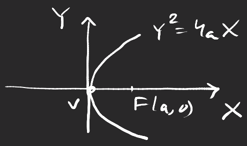
$F(h, a+k)$ focus

$C \equiv S(x,y)$

* $|AV| = |VF| \Rightarrow AO + VO = VF$

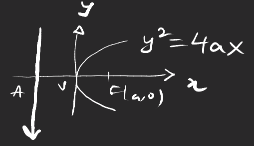
$AO = VF - VO$

$\boxed{AO = a - h}$



$V' = (0,0)$

Direx : $\boxed{y + a - k = 0}$



we know

$\boxed{|AV| = |VF|}$

$x + AV = 0$

$x + a = 0$

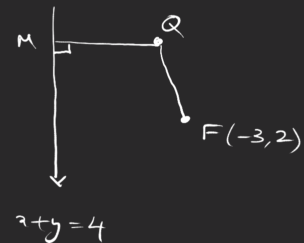
* $x + AO = 0$

$\boxed{x + a - h = 0}$ Directrix

Practice 1

eqn of curve ie Parabola

Q1. $F(-3,2)$, directrix $\equiv x + y = 4$ $\quad C \equiv ?$



* $x^2 + y^2 - 2xy + 20x + 10 = 0$

Q2. vertex, Axis, focus, directrix, latus-rectum

* $y^2 - 8y - x + 19 = 0$

* $y^2 - 8y = x - 19 \Rightarrow y^2 - 8y + (-4)^2 - (-4)^2 = x - 19$

$(y-4)^2 = x - 19 + 16 \Rightarrow \boxed{(y-4)^2 = (x-3)}$

$= x - 3$

$\leftarrow Y \rightarrow \quad \leftarrow X \rightarrow$

trick : $\boxed{\begin{aligned} y &\to Y \equiv y - 4 \\ x &\to X \equiv x - 3 \end{aligned}}$

$y = Y + 4$

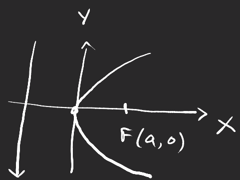
$x = X + 3$

coord. tr.

* $\boxed{Y^2 = X}$ ∽ "$y^2 = 4ax$" : $4a = 1 \Rightarrow a = 1/4$

New coord.

Parabola : $Y^2 = X$



X, Y

$V(0,0)$ is satisfied

Axis : $Y = 0$

LR $= 4a = 1 \Rightarrow a = 1/4$

$F(a, 0)$

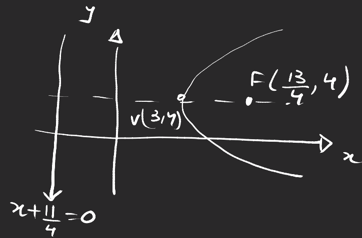
Directrix : $\boxed{X + a = 0}$

$X + \frac{1}{4} = 0$

$X = -1/4$

old coord.

$(y-4)^2 = x - 3$



vertex : $\begin{aligned} X &= 0 \\ Y &= 0 \end{aligned} \Rightarrow \left.\begin{aligned} x &= 3 \\ y &= 4 \end{aligned}\right\} V(3,4)$

Axis $Y = 0 \Rightarrow y = 4$

LR $= 1$

focus $\begin{aligned} X &= a \\ Y &= 0 \end{aligned} \Rightarrow \left.\begin{aligned} x &= a + 3 \\ y &= 4 \end{aligned}\right\} \Rightarrow F(\overset{1/4}{a} + 3, 4) = F(\frac{13}{4}, 4)$

Direct $X = -1/4 \Rightarrow x = -\frac{1}{4} + 3 = \frac{11}{4}$

$\boxed{x - \frac{11}{4} = 0}$

HW

- $y = x^2 - 2x + 3$
- $x^2 + 2y - 3x + 5 = 0$
- $9y^2 - 16x - 12y - 57 = 0$

Lecture-21 (22/Jun)

TEST (st. lines) 2 hrs

Q1. there are 3 pts. $P(m^2, 2m)$ $Q(mn, m+n)$ $R(n^2, 2n)$

there is a line $l \equiv x\cos^2\theta + y\sin\theta\cos\theta + \sin^2\theta = 0$

P_1, P_2, P_3 = length of $\perp$ from P, Q, R to l resp. $\rightarrow$ rel^n b/w P_1, P_2, P_3 = ?

Q2. $A(4,7)$, $B(\cos\theta, \sin\theta)$ $0 < \theta < \pi$

if A, B are 2 pts such that they lie on the same side of line $x+y-1=0$

then what is the cond^n on θ ? i.e. $\theta \in (\ ?\ )$

Q3. $l \equiv ax+by+c=0$, $P(x_1, y_1)$

if $L(h,k)$ is a pt at the foot of $\perp$ from P to l line

& $Q(\alpha, \beta)$ is the image of P

Prove & Compute

a) $\dfrac{h-x_1}{a} = \dfrac{k-y_1}{b} = ?$

b) $\dfrac{\alpha-x_1}{a} = \dfrac{\beta-y_1}{b} = ?$

Q4 let l be a line & $A(6,5)$, $B(2,-1)$ be 2 pts on l

if $P(4,1)$ is a pt. from which a $\perp$ is drawn to $\overline{AB}$

find the ratio in which it divides AB ?

Q5. Find the eq^n of line upon which perp. length (from origin) is 5 units

& the slope of the $\perp$ is $\frac{3}{4}$?

Q6. let line l pass through pt. $P(3,4)$ & make $\frac{\pi}{6}$ with $\hat{x}$ (axis)

l meets line $l' \equiv 12x+5y+10=0$ at Q

Compute the distance $|PQ|$

Q7. let line l be such that it passes thru $A(2,0)$, $B(3,1)$

if through a transformation R, line l is rotated by 15° in AntiClKwise dir^n (A being the pivot)

a) Eq^n of the rotated line, $l' \equiv$?

b) R is a rotation matrix that transforms every point on l uniquely to a point on line l'.

To what point B transforms ?

Note: Knowledge of Matrices is not required.

Q8. Prove an important fundamental thm. of geometry

Thm: translation of Axes leaves the slope of line invariant

Q9. let l be a line such that its segment b/w the lines $l_1 \equiv 5x-y+4=0$ & $l_2 \equiv 3x+4y-4=0$ is bisected at the point $(1,5)$

Eq^n of the line $l \equiv$?

Q10. Derive Bretschneider's formula for area of a non cyclic quadrilateral starting from fundamental Laws of triangles (in the context of trigonometry) and results from trigonometry (such as $\cos(A\pm B)$, $\sin(A\pm B)$ ...), compute the Area of 2 triangles formed by the 'NON CYCLIC' quadrilateral.

Also Reduce Bretschneider's to Bramagupta's and also Heron's for a cyclic quadrilateral and a triangle.

OR

let l be a line passing through a pt. $A(-5,-4)$ and meeting 3 lines $l_1 \equiv x+3y+2=0$, $l_2 \equiv 2x+y+4=0$, $l_3 \equiv x-y-5=0$ at B, C, D respectively. in such a way that $\left(\frac{15}{AB}\right)^2+\left(\frac{10}{AC}\right)^2=\left(\frac{6}{AD}\right)^2$ holds

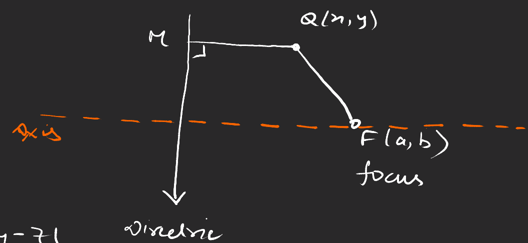
find the eq^n of line l

Lecture-22 (23/Jun) 1.5

Q3. $F(1,-1)$ focus, $V(2,1)$ vertex $\rightarrow$ Eq^n of parabola ?

$\rightarrow$ Axis ?

$\rightarrow$ LR ?

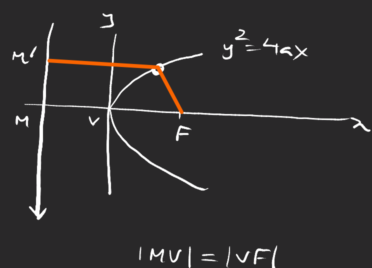


fundamental eq^n

* $e=1 \Rightarrow \underbrace{|QM|}_{?}=|QF| \Rightarrow$ Eq^n of parabola $\equiv 4x^2+y^2-4xy+8x+46y-71$

$\uparrow$

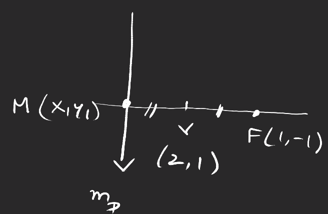
Directrix ?



Calculation

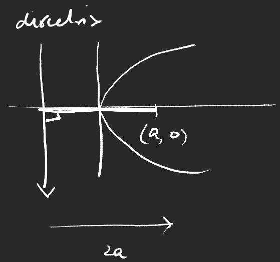
* Directrix $\equiv (y-y_1)=m_D(x-x_1)$

$V(2,1)=\left(\frac{x_1+1}{2},\frac{y_1-1}{2}\right) \Rightarrow x_1=3, \; y_1=3$



$m_{Axis}=\frac{1+1}{2-1}=2$

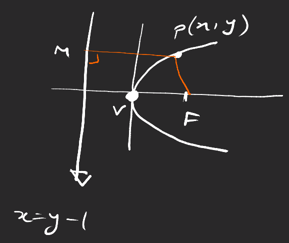
$m_D=-\frac{1}{2} \longrightarrow (y-3)=-\frac{1}{2}(x-3)$



* Axis $\equiv y+1=2(x-1)$

* $LR=4a=4\sqrt{5}$

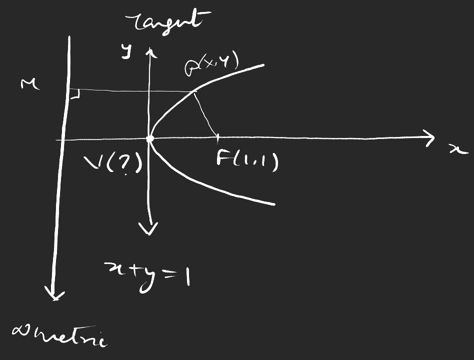
Q4. $V(2,1)$, Directrix $\equiv x = y-1$ Eq^n of parabola ?



* $e=1 \Rightarrow |MP| = \underbrace{|PF|}_{\Uparrow \text{ Focus (?)}}$

* $x^2+y^2-14x+2y+2xy+17=0$

Q5. $F(1,1)$, tangent at vertex $\equiv x+y=1$
Eq^n of parabola = ?

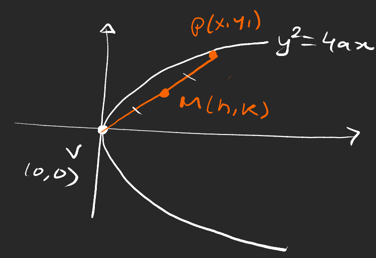


* $e=1 \Rightarrow \underbrace{|PM|}_{\Uparrow \text{ Directrix = ?}} = |PF|$

* $x^2+y^2-2xy-4x-4y+4=0$

Q6. $y^2=4ax$, find the locus of mid pts. of all chords of parabola drawn through the vertex.

let $M(h,K)$



* let the pt on parabola $P(x_1,y_1)$

Mid pt M $\Rightarrow M(h,K) = \left(\frac{x_1}{2}, \frac{y_1}{2}\right) \Rightarrow \begin{matrix} x_1 = 2h \\ y_1 = 2K \end{matrix}$

P lies on parabola $\Rightarrow y_1^2 = 4ax_1 \Rightarrow \cancel{4}K^2 = \cancel{4}a\cdot 2h \Rightarrow K^2 = 2ah$ Constraint eq^n for (h,K)

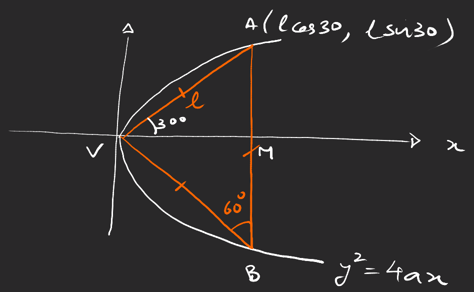
$\downarrow$

$\boxed{y^2 = 2ax}$ locus is a parabola

Lecture-23 (24 Jun) 1.5

Q7. $C \equiv y^2 = 4ax$, Equilateral Δ is inscribed in the parabola C where vertex is at the vertex of parabola } side of Δ = ?

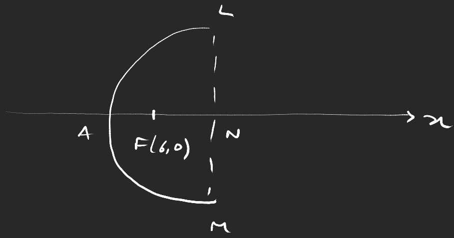
C(A) satisfies



* $\left(l\frac{1}{2}\right)^2 = 4a\left(\frac{\sqrt{3}\,l}{2}\right) = \frac{l^2}{4} = 2a\sqrt{3}\,l \Rightarrow l = 8a\sqrt{3}$

HW

Q. Focus of a parabolic mirror is at a dist. of 6cm from the vertex.
If the mirror is 20cm deep, find dist. LM = ?

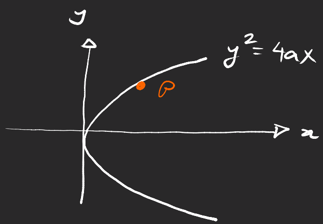


Q The towers of a bridge, hung in form of a parabola, have their tops 30m above the road & are 200m apart.
If the cable is 5m above the road at the center of the bridge, find the lgth of vertical supporting cable 30m from the center.

Q A beam is supported at its ends by supports which are 12m apart. Since the load is connected at the center, there is a deflection of 3cm at the center & the deflected beam is in the shape of a parabola.
How far from the center is the deflection 1cm ?

5. Parabola (The Real deal)

A. Parametric Representation



* $P(x,y)$ satisfies curve => there are 2 parameters / # to define a pt on a parabola

* $\boxed{\begin{matrix} x = at^2 \\ y = 2at \end{matrix}}$ Parametric eqn

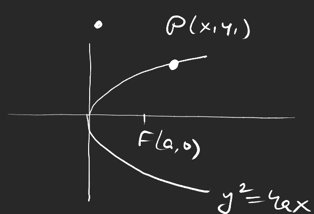
t: parameter

Param. eqn of circle
$x = \cos t$
$y = \sin t$
$\downarrow$
$x^2+y^2=1$

$t = \dfrac{y}{2a} \xrightarrow{x=at^2} x = \dfrac{a\,y^2}{4a^2} \Rightarrow \boxed{y^2 = 4ax}$ std. parabola eqn is recovered.

* $\boxed{P(x,y) \longrightarrow P(at^2, 2at)}$ pt on a parabola

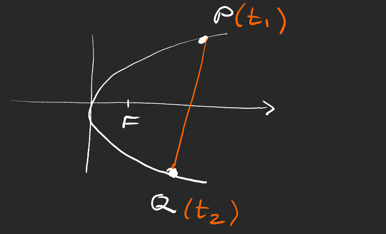
B. Relative position of a pt on a parabola



* $y^2 = 4ax$, $P(x_1, y_1)$

$y_1^2 - 4ax_1 = 0$	P on parabola
> 0	P outside "
< 0	P inside "

C. Chord joining two points



* chord PQ : $y - y_1 = \left(\dfrac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$

$\left.\begin{array}{l} x_1 = at_1^2 , \ x_2 = at_2^2 \\ y_1 = 2at_1 , \ y_2 = 2at_2 \end{array}\right\}$ Trick : go to parametric form

$(x, y) \to (at^2, 2at)$

* $y - 2at_1 = \dfrac{2\cancel{a}(t_2 - t_1)}{\cancel{a}(t_2^2 - t_1^2)}(x - at_1^2) = \dfrac{2}{(t_2 + t_1)}(x - at_1^2)$

$y = \dfrac{2x}{t_2 + t_1} \underbrace{- \dfrac{2at_1^2}{t_2 + t_1} + 2at_1}_{\frac{-2\cancel{a}t_1^2 + 2at_1t_2 + 2\cancel{a}t_1^2}{t_2 + t_1}} \Rightarrow y = \dfrac{2}{(t_2 + t_1)}x + \dfrac{2at_1t_2}{(t_2 + t_1)}$

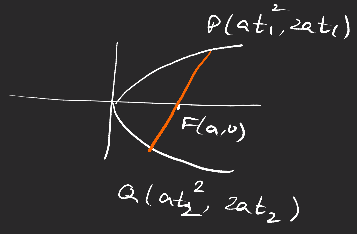
$\boxed{\text{chord } PQ \equiv y(t_2 + t_1) = 2x + 2at_1t_2}$ Eqⁿ of chord

P(t₁) Q(t₂) needed

Remark :

1) PQ : focal chord ⇒ PQ satisfies $F(a, 0)$ (x y)

$0(t_2 + t_1) = 2a + 2at_1t_2 \Rightarrow \boxed{t_1t_2 = -1}$ Condⁿ

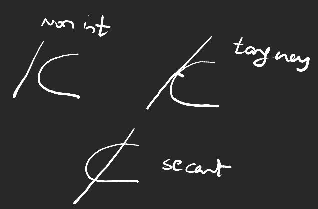


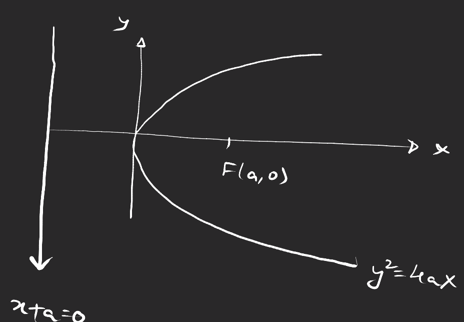
2) focal chord : $\left.\begin{array}{l} P(at_1^2, 2at_1) \\ Q\left(\dfrac{a}{t_1^2}, -\dfrac{2a}{t_1}\right) \end{array}\right\}$ Extremities of focal chord

D. St. line Interacting w/ parabola

* $y^2 = 4ax$; $y = mx + c$
Parab. line

Scenarios :





* $(mx + c)^2 = 4ax \Rightarrow m^2x^2 + c^2 + 2mcx = 4ax$

$\boxed{m^2x^2 + (2mc - 4a)x + c^2 = 0}$

* $x_{1,2} = \dfrac{-(2mc - 4a) \pm \sqrt{(2mc - 4a)^2 - 4m^2c^2}}{2m^2}$

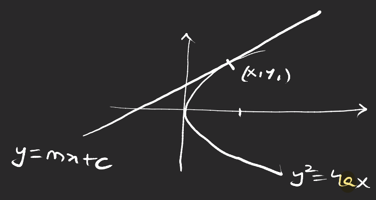
→ D = 0 ⇒ 2 same roots → tangency
→ D > 0 ⇒ 2 distinct " → secant
→ D < 0 ⇒ 2 imag. " → non intⁿ

Lecture-24 (27|Jan) 2

* $D = 0 \Rightarrow (2mc - 4a)^2 - \underbrace{4m^2c^2}_{(2mc)^2} = 0 \Rightarrow (4mc - 4a)4a = 0 \Rightarrow \boxed{m = \dfrac{a}{c}}$ tangency Condⁿ

* $a = mc$ ⇒ line cuts at 1 pt (tan.)
$a > mc$ ⇒ " " " 2 pts (secant)
$a < mc$ ⇒ " doesn't cut

E. Tangent to Parabola



1. Slope form

* $m = \dfrac{a}{c}$, $D = 0$ $\Rightarrow$ $x_{1,2} = \dfrac{-(2mc - 4a)}{2m^2} = \dfrac{a}{m^2}$

$y_{1,2} = mx_{1,2} + c \Rightarrow y_1 = m\left(\dfrac{a}{m^2}\right) + \left(\dfrac{a}{m}\right) = \dfrac{2a}{m}$

* $\boxed{y = mx + \dfrac{a}{m}}$ { slope form of Eqⁿ of tan

(m, a) needed

$\left(\underset{x_1}{\dfrac{a}{m^2}}, \underset{y_1}{\dfrac{2a}{m}}\right)$ = Point of Contact

2. Pt form

* $y - y_1 = m(x - x_1)$; $y_1 = \dfrac{2a}{m} \Rightarrow m = \dfrac{2a}{y_1}$, $y^2 = 4ax \Rightarrow y_1^2 = 4ax_1$

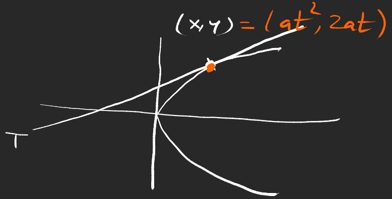
$y - y_1 = \dfrac{2a}{y_1}(x - x_1) \Rightarrow yy_1 - y_1^2 = 2ax - 2ax_1$

$\Downarrow$

$\boxed{yy_1 = 2a(x + x_1)}$ Point form of EOT

(x_1, y_1), a needed

3. Parametric form



* $(x, y) = (at^2, 2at)$ Point of Contact $= \left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right) \Rightarrow \boxed{\dfrac{1}{m} = t}$

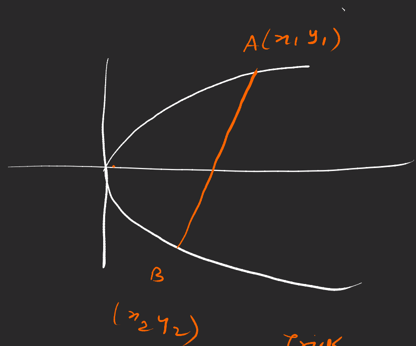
* $T \equiv y = mx + \dfrac{a}{m}$

$\Downarrow$

$\boxed{yt = x + at^2}$ Parametric eqⁿ of tangent

t : needed

F. Length of the chord



* $|AB|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

* $x_{1,2} = \dfrac{-(2mc - 4a) \pm \sqrt{(2mc - 4a)^2 - 4m^2c^2}}{2m^2}$

$x_2 - x_1 = \dfrac{\cancel{2}\sqrt{(2mc - 4a)^2 - 4m^2c^2}}{\cancel{2}m^2}$

$y_2 - y_1 = \dfrac{\cancel{2}m\sqrt{(2mc - 4a)^2 - 4m^2c^2}}{\cancel{2}m^2}$

Trick

$y_1 = mx_1 + c$

$y_2 = mx_2 + c$

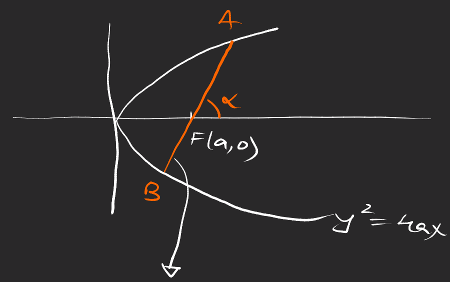
$y_2 - y_1 = m(x_2 - x_1)$

* $|AB|^2 = \dfrac{\left((2mc - 4a)^2 - 4m^2c^2\right) + m^2\left((2mc - 4a)^2 - 4m^2c^2\right)}{m^4}$

$$= \frac{(4mc-4a)(-4a) + m^2(4mc-4a)(-4a)}{m^4} = \frac{(1+m^2)16a(a-mc)}{m^4}$$

$$\boxed{|AB| = \frac{4}{m^2}\sqrt{a(1+m^2)(a-mc)}}$$ 'length of the chord'

Comment : focal chord



$y = mx + c$ satisfies $F(a,0)$

$0 = ma + c \Rightarrow \boxed{c = -ma}$

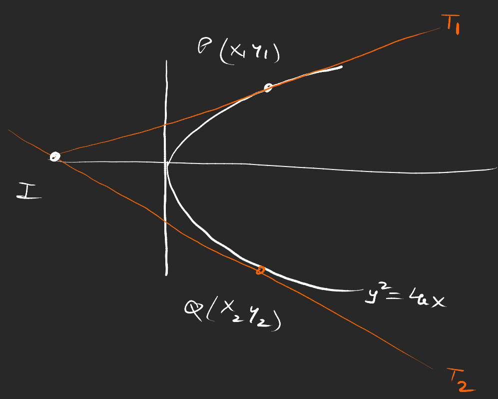
* $m_{FC} = \tan\alpha$

$$|AB| = \frac{4}{m^2}\sqrt{a(1+m^2)(a-mc)}$$

$$= \frac{4}{m^2}\sqrt{a(1+m^2)(1+m^2)a} = \frac{4}{m^2}a(1+m^2) = \frac{4}{\tan^2\alpha}a(\overbrace{1+\tan^2\alpha}^{\sec^2})$$

$$\boxed{|AB| = 4a\operatorname{cosec}^2\alpha}$$ 'length of focal chord'

G. Tangent to Parabola (Cont.) – Pt of Intersection of 2 tangents



* $P(x_1,y_1) \rightarrow P(t_1) = (at_1^2, 2at_1)$

$Q(x_2 y_2) \rightarrow Q(t_2) = (at_2^2, 2at_2)$

* $\left.\begin{aligned} T_1 &\equiv yt_1 = x + at_1^2 \\ T_2 &\equiv yt_2 = x + at_2^2 \end{aligned}\right\}$ 2 linear eqn, solve them.

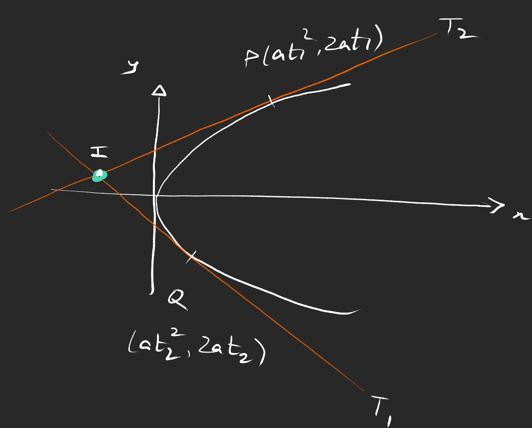
$-$

$y(t_1 - t_2) = a(t_1^2 - t_2^2) \Rightarrow \boxed{y = a(t_1 + t_2)}$

$yt_1 = x + at_1^2 \Rightarrow \cancel{at_1^2} + at_1t_2 = x + \cancel{at_1^2} \Rightarrow \boxed{x = at_1t_2}$

$$\boxed{I(at_1t_2,\ a(t_1+t_2))}$$ Point of Intersection of tangent

H. Director circle



* Pt of Intn $I = (at_1t_2,\ a(t_1+t_2))$

* if $T_1 \perp T_2 \Rightarrow$ locus of $I \equiv$ director circle

$$m_{T_1} = \frac{2at_1 - (at_1 + at_2)}{at_1^2 - at_1t_2} = \frac{(t_1 - t_2)\cancel{a}}{\cancel{a}t_1(t_1 - t_2)} = \frac{1}{t_1}$$

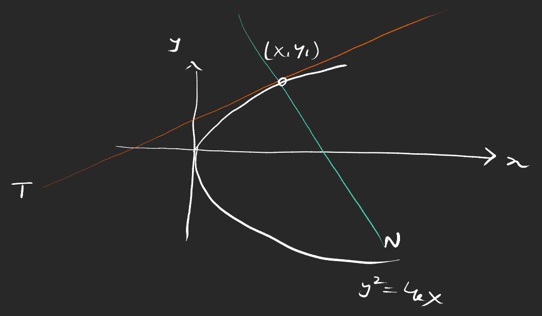
$$m_{T_2} = \frac{2at_2 - (at_1 + at_2)}{at_2^2 - at_1t_2} = \frac{a(t_2 - t_1)}{a(t_2 - t_1)t_2} = \frac{1}{t_2}$$

* $\perp \Rightarrow m_{T_1} \cdot m_{T_2} = -1 \Rightarrow \boxed{t_1 t_2 = -1}$

* pt of int^n $I = (a t_1 t_2,\ a(t_1+t_2)) \longrightarrow$ x coord. $= -a \Rightarrow \boxed{x+a=0}$ Directrix of Parabola

$\boxed{x+a=0}$ Eq^n of director circle

I. Normal to (tangent) parabola



1. ⊥ pt form

* $y-y_1 = m_N(x-x_1)$; $(x_1, y_1) = \left(\frac{a}{m_T^2}, \frac{2a}{m_T}\right)$

$y_1 = \frac{2a}{m_T} \Rightarrow m_T = \frac{2a}{y_1} \xrightarrow{m_N \cdot m_T = -1} \boxed{m_N = -\frac{y_1}{2a}}$

$\boxed{y-y_1 = -\frac{y_1}{2a}(x-x_1)}$

Eq^n of Normal
(x_1, y_1), Parabola

2. Slope form

* $(x_1, y_1) = \left(\frac{a}{m_T^2}, \frac{2a}{m_T}\right) \xrightarrow{m_N \cdot m_T = -1} (a m_N^2, -2a m_N)$

Point of Contact
in terms of m_N

* $y - y_1 = m_N(x - x_1) \Rightarrow y + 2am_N = m_N(x - a m_N^2) \Rightarrow \boxed{y = m_N x - 2am_N - am_N^3}$

EON
m_N needed

note -
• cubic in m

$\boxed{\text{Viete's formula}}$
extension of

$\alpha+\beta = \frac{-b}{a}$ | $\alpha+\beta+\gamma = ?$

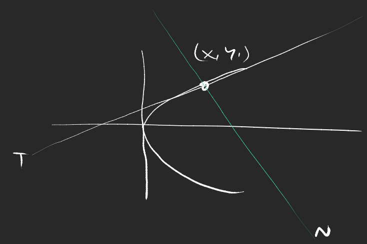
$\alpha\beta = \frac{c}{a}$ | $\alpha\beta + \beta\gamma + \gamma\alpha = ?$

$\alpha\beta\gamma = ?$

3. Parametric form

* $P(x_1, y_1) \to P(t_1)$

* generic pt $= P(at^2, 2at) \xrightarrow[\text{Point of Contact}]{\text{for a Normal}} (am_N^2, -2am_N)$



$\Downarrow$

$\boxed{m_N = -t}$

* $\because$ $N \equiv y = m_N x - 2am_N - am_N^3 = -tx + 2at + at^3$

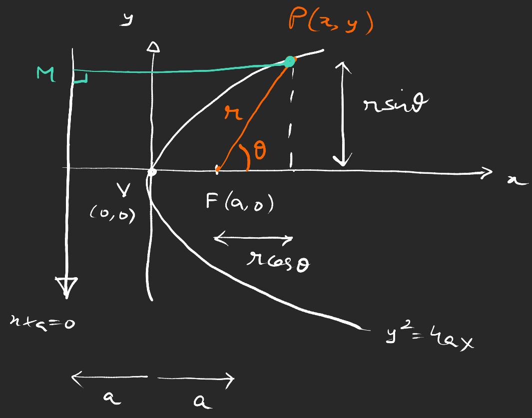
$\boxed{y + tx = 2at + at^3}$ EON

t: needed

J. Polar form of Parabola

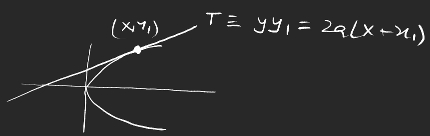
* $y^2 = 4ax$ Parabola eqⁿ in Cartesian Coordⁿ

* Parabola $e = 1 \Rightarrow \underbrace{|MP|}_{2a + r\cos\theta} = \underbrace{|PF|}_{r}$



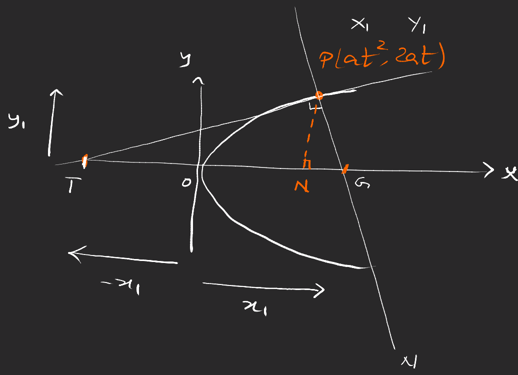
$$\boxed{r = \frac{2a}{1 - \cos\theta}}$$

Polar form of Parabola (r, θ)



Lecture-25 (6/Jul) 2.5

K. length of subtangent & subnormal



* $P(x_1, y_1) = P(at_1^2, 2at_1)$, $y^2 = 4ax$
* length of Subtangent $\equiv TN$
* length of Subnormal $\equiv NG = ?$

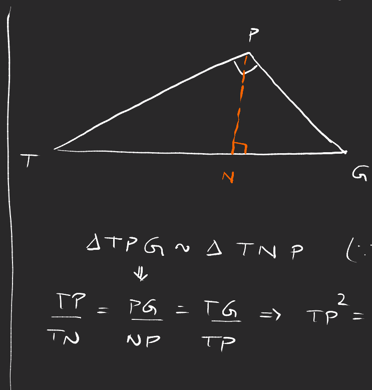
Part 1

* Eqⁿ of tangent $= T \equiv yy_1 = 2a(x + x_1)$

$\downarrow$ if $y = 0$

$\boxed{x = -x_1} \Rightarrow \boxed{|TN| = 2x_1} = 2 \times$ length of abscissa of pt. P

Basic △ property



Part 2

* $TP^2 = TG \cdot TN$

$= (TN + NG) \cdot TN = TN^2 + NG \cdot TN$

$TG = TN + NG$

$y_1^2 = 4ax_1$

$\underbrace{TP^2 - TN^2}_{PN^2} = NG \cdot TN \Rightarrow NG = \frac{PN^2}{TN} = \frac{PN^2}{2x_1} = \frac{y_1^2}{2x_1} = 2a$

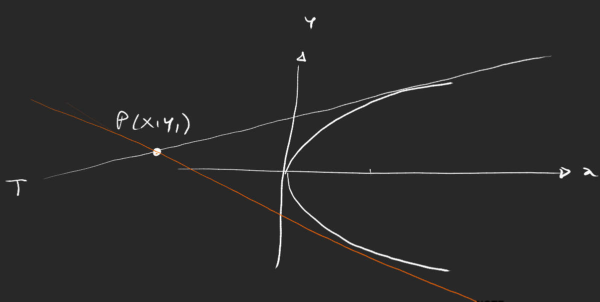
$\triangle TPG \sim \triangle TNP$ ($\because$ RHS sim.)

$\Downarrow$

$\frac{TP}{TN} = \frac{PG}{NP} = \frac{TG}{TP} \Rightarrow TP^2 = TG \cdot TN$

$\boxed{|NG| = 2a = \text{Semi latus rectum}}$

L. "2" tangents drawn from a points



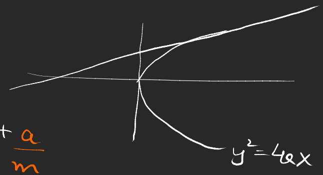
* $P(x_1, y_1)$, $C \equiv y^2 = 4ax \to \boxed{y^2 - 4ax = 0}$

$T \equiv$ Eqⁿ of tangent $\equiv y = mx + \frac{a}{m}$

* T satisfies $P(x_1, y_1) \Rightarrow y_1 = mx_1 + \frac{a}{m}$

$\boxed{m^2 x_1 - m y_1 + a = 0}$ quad. eqⁿ in m (slope of tangent)

$T \equiv y = mx + \frac{a}{m}$

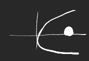


$$m = \frac{y_1 \pm \sqrt{y_1^2 - 4ax_1}}{2x_1}$$

→ $y_1^2 - 4ax_1 > 0 \Rightarrow$ 2 real/distinct roots

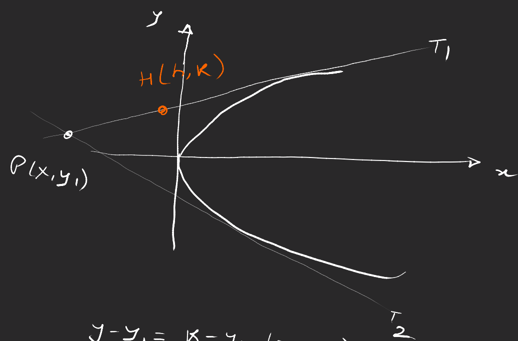
→ $y_1^2 - 4ax_1 = 0 \Rightarrow$ 2 same root
P lies on C

→ $y_1^2 - 4ax_1 < 0 \Rightarrow$ no real roots
P lies inside C



HW

M. Eqⁿ of Pair of tangents



* $C \equiv y^2 = 4ax$, $P(x_1, y_1)$

* $\frac{k - y_1}{h - x_1} = m$, $\frac{hy_1 - kx_1}{h - x_1} = \frac{a}{m} \Rightarrow a = m \frac{hy_1 - kx_1}{h - x_1} = \frac{k - y_1}{h - x_1} \cdot \frac{hy_1 - kx_1}{h - x_1}$

$a(h - x_1)^2 = (k - y_1)(hy_1 - kx_1)$

$\downarrow (h, k) \to (x, y)$

$a(x - x_1)^2 = (y - y_1)(xy_1 - yx_1) = xyy_1 - y^2x_1 - y_1^2 x + x_1 y_1 y$

$y - y_1 = \frac{k - y_1}{h - x_1}(x - x_1)$

$\Downarrow$

$y = \frac{k - y_1}{h - x_1} x - \frac{k - y_1}{h - x_1} x_1 + \frac{y_1}{1}$

$= \frac{k - y_1}{h - x_1} x - \frac{kx_1 + \cancel{x_1 y_1} + hy_1 - \cancel{x_1 y_1}}{h - x_1}$

$= \frac{k - y_1}{h - x_1} x + \frac{hy_1 - kx_1}{h - x_1} = mx + \frac{a}{m}$

$\boxed{CC_1 = T^2}$

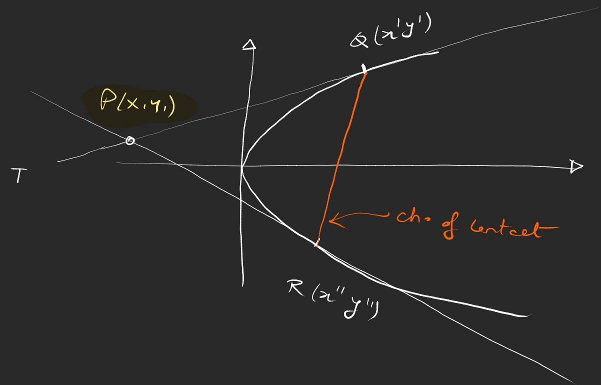
Eqⁿ of Pair of tan.

$C \equiv y^2 - 4ax$

$C_1 \equiv y_1^2 - 4ax_1$

$T \equiv yy_1 - 2a(x + x_1)$

N. Chord of Contact



* $Q(x'y')$ pt. of Contact $\Rightarrow T \equiv yy' = 2a(x + x')$

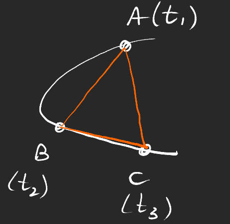
$R(x''y'')$ " " " $\Rightarrow T \equiv yy'' = 2a(x + x'')$

* $P(x_1, y_1)$ is satisfied by T $\Rightarrow \begin{cases} y_1 y' = 2a(x_1 + x') \\ y_1 y'' = 2a(x_1 + x'') \end{cases}$

$\downarrow$

$\boxed{\text{ch of Cont.} \equiv QR \equiv yy_1 = 2a(x + x_1)}$ $\longleftrightarrow$ $\boxed{T = 0}$

O. Ar. of Δ formed by 3 pts & tangents on those pts



* $y^2 = 4ax$, $A(at_1^2, 2at_1)$, $B(at_2^2, 2at_2)$, $C(at_3^2, 2at_3)$

* $$\text{Ar}(\Delta ABC) = \frac{1}{2}\begin{vmatrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ at_3^2 & 2at_3 & 1 \end{vmatrix}$$

$$= \frac{1}{\cancel{2}}\left\{ at_1^2\,\cancel{2}a(t_2 - t_3) - at_2^2\,\cancel{2}a(t_1 - t_3) + at_3^2\,\cancel{2}a(t_1 - t_2)\right\}$$

$$= a^2\left\{ t_1^2 t_2 - t_1^2 t_3 - t_2^2 t_1 + t_2^2 t_3 + t_3^2 t_1 - t_3^2 t_2 + t_1 t_2 t_3 - t_1 t_2 t_3 \right\}$$ Algebra Trick

$$= a^2\left\{ t_1(t_1 t_2 - t_1 t_3 + t_3^2 - t_2 t_3) + t_2\underbrace{(-t_2 t_1 + t_2 t_3 - t_3^2 + t_1 t_3)}_{-t_2(t_1 t_2 - t_1 t_3 + t_3^2 - t_2 t_3)} \right\}$$

$$= a^2\left\{ (t_1 - t_2)\underbrace{(t_1 t_2 - t_1 t_3 + t_3^2 - t_2 t_3)}_{t_1(t_2 - t_3) + t_3(t_3 - t_2)} \right\} = a^2\left\{ (t_1 - t_2)(t_1 - t_3)(t_2 - t_3) \right\} = -a^2\left\{ (t_1 - t_2)(t_2 - t_3)(t_3 - t_1) \right\}$$

* T_1 at A $\equiv$ $yt_1 = x + at_1^2$

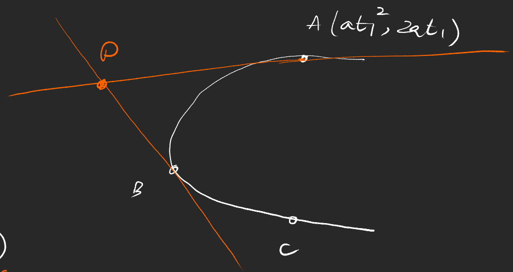
T_2 at B $\equiv$ $yt_2 = x + at_2^2$

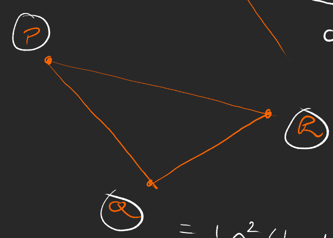
T_3 at C $\equiv$ $yt_3 = x + at_3^2$

* T_1 & T_2 pt of int $P(at_1 t_2, a(t_1 + t_2))$

T_2 & T_3 ,, ,, $Q(at_2 t_3, a(t_2 + t_3))$

T_3 & T_1 ,, ,, $R(at_3 t_1, a(t_3 + t_1))$





* $$\text{Ar}(\Delta PQR) = \frac{1}{2}\begin{vmatrix} at_1 t_2 & a(t_1 + t_2) & 1 \\ at_2 t_3 & a(t_2 + t_3) & 1 \\ at_1 t_3 & a(t_3 + t_1) & 1 \end{vmatrix}$$

$$= \frac{1}{2}a^2(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)$$

$$= \frac{1}{2}\left\{ at_1 t_2\, a(t_2 - t_1) - at_2 t_3\, a(t_1 - t_3) + at_1 t_3\, a(t_1 - t_3) \right\}$$

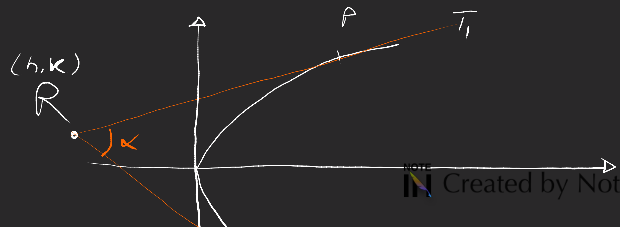
$$= \frac{1}{2}a^2\left\{ t_1 t_2 (t_2 - t_1) - t_2 t_3 (t_2 - t_3) + t_1 t_3 (t_1 - t_3) \right\} = \frac{1}{2}a^2\left\{ t_1 t_2^2 - t_1^2 t_2 - t_2^2 t_3 + t_2 t_3^2 + t_1^2 t_3 - t_1 t_3^2 + t_1 t_2 t_3 - t_1 t_2 t_3 \right\}$$

$$= \frac{1}{2}a^2\left\{ t_1(-t_1 t_2 + t_1 t_3 - t_3^2 + t_2 t_3) + t_2\underbrace{(+t_1 t_2 - t_2 t_3 + t_3^2 - t_1 t_3)}_{-t_2(-t_1 t_2 + t_2 t_3 - t_3^2 + t_1 t_3)} \right\} = \frac{1}{2}a^2(t_1 - t_2)\left\{ \underbrace{-t_1 t_2 + t_1 t_3 - t_3^2 + t_2 t_3}_{t_1(t_3 - t_2) - t_3(t_3 - t_2)} \right\}$$

$$= \frac{1}{2}a^2(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)$$

* $$\boxed{\frac{\text{Ar}(\Delta ABC)}{\text{Ar}(\Delta PQR)} = 2}$$ $\longleftrightarrow$ " Ar of Δ formed by 3 pts on parabola is twice the Ar. of Δ formed by the tangents at them points

P. Locus of a pt of intn of tangents with fix angle b/w



* $y^2 = 4ax$

* $\boxed{T_1 \text{ intersects } T_2 : \exists \text{ a given angle } \alpha \text{ between them}}$

$*\quad T \equiv y = mx + \dfrac{a}{m} \xrightarrow{R \text{ satisfied}} k = mh + \dfrac{a}{m}$

$\Downarrow$

$\boxed{hm^2 - km + a = 0}$ quad. in m

(m_1, m_2)

$\downarrow$

$\boxed{m_1 + m_2 = \dfrac{k}{h}, \quad m_1 m_2 = \dfrac{a}{h}}$

$T_1 \equiv y = m_1 x + \dfrac{a}{m_1}$

$T_2 \equiv y = m_2 x + \dfrac{a}{m_2}$

$$\tan\alpha = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{\sqrt{(m_1+m_2)^2 - 4m_1m_2}}{1 + m_1 m_2}$$

$$= \frac{\sqrt{\dfrac{k^2}{h^2} - \dfrac{4a}{h}}}{1 + \dfrac{a}{h}} \Rightarrow \tan^2\alpha \left(\frac{a+h}{\cancel{h}}\right)^2 = \frac{k^2 - 4ah}{\cancel{h^2}}$$

$(m_1+m_2)^2 = m_1^2 + m_2^2 + 2m_1 m_2 - 2m_1m_2 + 2m_1m_2$

$= (m_1 - m_2)^2 + 4m_1 m_2$

$(a+h)^2 \tan^2\alpha = k^2 - 4ah$ Constraint eqⁿ of (h, k)

$\downarrow \quad (h,k) \to (x,y)$

$$\boxed{y^2 - 4ax = (a+x)^2 \tan^2\alpha}$$