4.7 ANTIDERIVATIVES

A physicist who knows the velocity of a particle might wish to know its position at a given time. An engineer who can measure the variable rate at which water is leaking from a tank wants to know the amount leaked over a certain time period. A biologist who knows the rate at which a bacteria population is increasing might want to deduce what the size of the population will be at some future time. In each case, the problem is to find a function F whose derivative is a known function f. If such a function F exists, it is called an *antiderivative* of f.

DEFINITION A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

F is differentiable (F' exists)

Examples:

$f(x) = x^2$, $F(x) = \frac{x^3}{3}$

$G(x) = \frac{x^3}{3} + 1$

$\frac{x^3}{3} + C$, C= constant

1 THEOREM If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

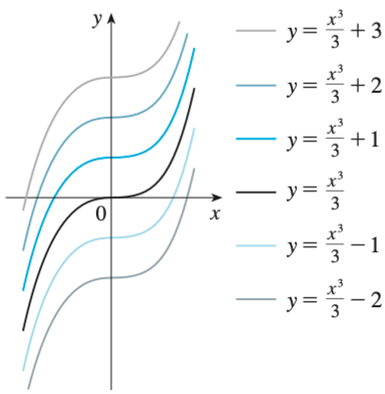
$$F(x) + C$$

where C is an arbitrary constant.

Example: (1) $f(x) = \sin x$

$$F(x) = -\cos x$$

General antiderivative: $\boxed{-\cos x + C}$



$\csc^2 x$ $-\cot x$

$\csc x \cot x$ $-\csc x$

2 TABLE OF ANTIDIFFERENTIATION FORMULAS

Function	Particular antiderivative	Function	Particular antiderivative
$cf(x)$	$cF(x)$	$\sec^2 x$	$\tan x$
$f(x) + g(x)$	$F(x) + G(x)$	$\sec x \tan x$	$\sec x$
$x^n \quad (n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$	$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1}x$ = arcsin x
$\dfrac{1}{x}$	$\ln \lvert x \rvert$	$\dfrac{1}{1+x^2}$	$\tan^{-1}x$ = arctan x
e^x	e^x	~~$\cosh x$~~	~~$\sinh x$~~
$\cos x$	$\sin x$	~~$\sinh x$~~	~~$\cosh x$~~
$\sin x$	$-\cos x$		

$\dfrac{1}{x\sqrt{x^2-1}}$ arcsec x

Notation

the variable of integration

$$\int f(x)\,dx = F(x) + C$$

the constant of integration

the antiderivative

the integral sign

the integrand

$$\frac{d}{dx}(4x^2) = 8x$$

$$\frac{d}{dt}(4x^2) = 0$$

$$\int x^2\,dx = \frac{x^3}{3} + C$$

$$\int x^2\,dt = x^2 t + C$$

$$\int 4\,dx = 4x + C$$

Examples: Find the antiderivative of:

4. $f(x) = \sqrt[3]{x^2} + x\sqrt{x}$

$$\int \left(\sqrt[3]{x^2} + x\sqrt{x}\right) dx = \int \left(x^{\frac{2}{3}} + x^{\frac{3}{2}}\right) dx$$

$$= \frac{x^{\frac{5}{3}}}{\frac{5}{3}} + \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + C = \frac{3}{5}x^{\frac{5}{3}} + \frac{2}{5}x^{\frac{5}{2}} + C$$

$$= \frac{3}{5}\sqrt[3]{x^5} + \frac{2}{5}\sqrt{x^5} + C$$

$$= \frac{3}{5}x\sqrt[3]{x^2} + \frac{2}{5}x^2\sqrt{x} + C$$

13. $f(x) = \dfrac{x^5 - x^3 + 2x}{x^4}$

$$\int \frac{x^5 - x^3 + 2x}{x^4} dx = \int \left(\frac{x^5}{x^4} - \frac{x^3}{x^4} + \frac{2x}{x^4}\right) dx$$

$$= \int \left(x - \frac{1}{x} + 2x^{-3}\right) dx = \frac{x^2}{2} - \ln|x| + \frac{\cancel{2}x^{-2}}{\cancel{-2}} + C$$

$$= \frac{x^2}{2} - \ln|x| - \frac{1}{x^2} + C$$

14. $f(x) = \dfrac{2+x^2}{1+x^2}$

$$\int \frac{2+x^2}{1+x^2}\,dx = \int \frac{1+1+x^2}{1+x^2}\,dx = \int\left(\frac{1}{1+x^2} + \cancel{\frac{1+x^2}{1+x^2}}\right)dx$$

$$= \arctan x + x + C$$

The Initial Value Problem

17–34 ▪ Find f.

24. $f'(x) = 5x^4 - 3x^2 + 4$, $f(-1) = 2$

$$f(x) = \int f'(x)\,dx = \int (5x^4 - 3x^2 + 4)\,dx = \cancel{5}\frac{x^5}{\cancel{5}} - \cancel{3}\frac{x^3}{\cancel{3}} + 4x + C$$

$$f(x) = x^5 - x^3 + 4x + C$$

$$f(-1) = \cancel{-1} - \cancel{(-1)} - 4 + C = 2$$

$$C = 2 + 4 = 6 \Rightarrow f(x) = x^5 - x^3 + 4x + 6$$

31. $f''(\theta) = \sin\theta + \cos\theta, \quad f(0) = 3, \quad f'(0) = 4$

$$f'(\theta) = \int f''(\theta)\,d\theta = \int (\sin\theta + \cos\theta)\,d\theta$$

$$= -\cos\theta + \sin\theta + C$$

$$f'(0) = -\cos 0 + \sin 0 + C = -1 + C = 4$$

$$C = 5$$

$$f'(\theta) = -\cos\theta + \sin\theta + 5$$

$$f(\theta) = \int f'(\theta)\,d\theta = \int (-\cos\theta + \sin\theta + 5)\,d\theta$$

$$= -\sin\theta - \cos\theta + 5\theta + D$$

$$f(0) = -\sin 0 - \cos 0 + 0 + D = 3$$

$$D = 4$$

$$\underline{f(\theta) = -\sin\theta - \cos\theta + 5\theta + 4}$$

34. $f''(t) = 2e^t + 3\sin t, \quad f(0) = 0, \quad f(\pi) = 0$

homework

RECTILINEAR MOTION

Antidifferentiation is particularly useful in analyzing the motion of an object moving in a straight line. Recall that if the object has position function $s = f(t)$, then the velocity function is $v(t) = s'(t)$. This means that the position function is an antiderivative of the velocity function. Likewise, the acceleration function is $a(t) = v'(t)$, so the velocity function is an antiderivative of the acceleration. If the acceleration and the initial values $s(0)$ and $v(0)$ are known, then the position function can be found by antidifferentiating twice.

$s(t)$
$v(t) = s'(t)$
$a(t) = v'(t) = s''(t)$

Examples:

39–42 ▪ A particle is moving with the given data. Find the position of the particle.

42. $a(t) = t^2 - 4t + 6, \quad s(0) = 0, \quad s(1) = 20$

$v(t) = \int a(t)\,dt = \int (t^2 - 4t + 6)\,dt$

$= \frac{t^3}{3} - \cancel{4}\frac{t^2}{\cancel{2}} + 6t + C$

$s(t) = \int v(t)\,dt = \int \left(\frac{t^3}{3} - 2t^2 + 6t + C\right) dt$

$= \frac{t^4}{12} - 2\frac{t^3}{3} + 6\frac{t^2}{2} + Ct + D$

$s(t) = \frac{t^4}{12} - 2\frac{t^3}{3} + 3t^2 + Ct + D$

$s(0) = 0$

$s(0) = D \Rightarrow D = 0$

$s(1) = \frac{1}{12} - \frac{2}{3} + 3 + C = 20 \Rightarrow C = 17 - \frac{1}{12} + \frac{2}{3}$

$C = \frac{204 - 1 + 8}{12} = \frac{211}{12}$

$s(t) = \frac{t^4}{12} - \frac{2t^3}{3} + 3t^2 + \frac{211}{12}t$

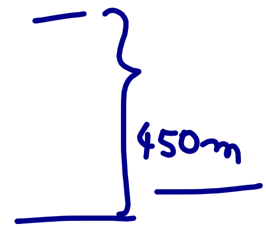
43. A stone is dropped from the upper observation deck (the Space Deck) of the CN Tower, 450 m above the ground.

(a) Find the distance of the stone above ground level at time t.

(b) How long does it take the stone to reach the ground?

(c) With what velocity does it strike the ground?

(d) If the stone is thrown downward with a speed of 5 m/s, how long does it take to reach the ground?



(a) $s(t) = ?$ $\quad S(0) = 450\,m \quad V(0) = 0$

$a(t) = -9.8\,m/s^2 = s''(t)$

$V(t) = \int -9.8\,dt = -9.8t + C$

$V(0) = C = 0 \Rightarrow V(t) = -9.8t$

$S(t) = \int -9.8t\,dt = -9.8\frac{t^2}{2} + D$

$\left.\begin{array}{l} s(t) = -4.9t^2 + D \\ S(0) = 450 = D \end{array}\right\} \Rightarrow D = 450$

$S(t) = -4.9t^2 + 450$

(b) $s(t) = 0$

$-4.9t^2 + 450 = 0$

$-4.9t^2 = -450$

$t^2 = \frac{450}{4.9}$

$t = \pm\sqrt{\frac{450}{4.9}}$, $t > 0$ $\Rightarrow t = \sqrt{\frac{450}{4.9}} \cong \underline{9.58\ s}$

(c) $v(t) = -9.8t$

$v(9.58) = -9.8(9.58) \cong -93.88\ m/s$

(d) $s(t) = ?$ $s(0) = 450\ m$, $v(0) = -5\ m/s$

$a(t) = -9.8\ m/s^2$

$\downarrow$ homework

46. Two balls are thrown upward from the edge of the cliff in Example 6. The first is thrown with a speed of 48 ft/s and the other is thrown a second later with a speed of 24 ft/s. Do the balls ever pass each other?

1. $s_1(0) = 432 \text{ ft}$

$v_1(0) = 48 \text{ ft/s}$

$a_1(t) = -32 \text{ ft/s}^2$

$v_1(t) = -32t + 48$

$s_1(t) = -32\frac{t^2}{2} + 48t + 432$

$s_1(t) = -16t^2 + 48t + 432$

2. $v_2(1) = 24 \text{ ft/s}$

$s_2(1) = 432 \text{ ft}$

$a_2(t) = -32$

$v_2(t) = -32t + C$

$v_2(1) = 24$

$-32 + C = 24$

$C = 56$

$v_2(t) = -32t + 56$

$s_2(t) = -32\frac{t^2}{2} + 56t + D$

$s_2(1) = 432 \Rightarrow$

$-16 + 56 + D = 432$

$D = 392$

$s_2(t) = -16t^2 + 56t + 392$

$s_1(t) = s_2(t)$

$\cancel{-16t^2} + 48t + 432 = \cancel{-16t^2} + 56t + 392$

$40 = 8t \Rightarrow t = 5\text{s}$

47. A stone was dropped off a cliff and hit the ground with a speed of 120 ft/s. What is the height of the cliff?

$a(t) = -32 \text{ ft/s}^2$

$v(t) = -32t + c$

$V(0) = 0$

$c = 0$

$V(t) = -32t$

$-120 = -32t$

$t = 3.75\text{s}$

$s(t) = \frac{-32t^2}{2} + D$

$s(3.75) = 0$

$-16(3.75)^2 + D = 0$

$D = 225 \text{ ft}$

50. A car is traveling at 50 mi/h when the brakes are fully applied, producing a constant deceleration of 22 ft/s^2. What is the distance traveled before the car comes to a stop?

homework

Homework - questions:

36

$$\begin{cases} f(x) = ? \\ f'(x) = x^3 \end{cases}$$

$x + y = 0 \Rightarrow$ tangent to f

$m = f'(x)$

$y = -x$

$m = -1$

$$f(x) = \int x^3 \, dx = \frac{x^4}{4} + C$$

$m = f'(x)$

$-1 = x^3 \Rightarrow x = -1$

$x + y = 0 \Rightarrow \boxed{y = -x}$

$x = -1 \Rightarrow y = 1$

$A(-1, 1) \rightarrow$ tangent

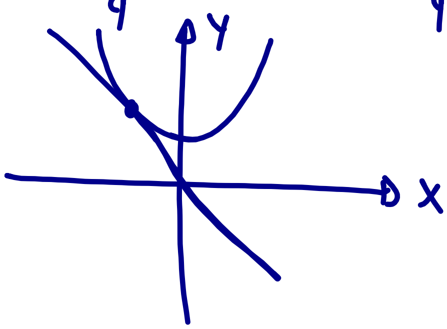
$\rightarrow$ graph of f

$\boxed{f(-1) = 1}$

$f(x) = \frac{x^4}{4} + C$

$f(-1) = \frac{1}{4} + C = 1 \Rightarrow C = \frac{3}{4}$

$\Rightarrow f(x) = \frac{x^4}{4} + \frac{3}{4}$



Homework:

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