Homework #1 B

Spring 2020, CSE 446/546: Machine Learning
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B2

Use just the definitions above and let $||\cdot||$ be a norm.

a. *[3 points]* Show that $f(x) = ||x||$ is a convex function.
See A.1. problem, this follows directly from the definition of a norm's absolute scalability and triangle inequality. Start from definition of a convex function $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \forall \lambda \in [0,1]$ and for any $x, y \in \mathcal{D}$ where $\mathcal{D}$ is a non-empty domain of f and f is our norm:

$$||\lambda x + (1-\lambda)y|| \leq ||\lambda x|| + ||(1-\lambda)y||$$
$$||\lambda x + (1-\lambda)y|| \leq \lambda||x|| + (1-\lambda)||y||$$

b. *[3 points]* Show that $x \in \mathbb{R}^n : ||x|| \leq 1$ is a convex set.
This one is more logically involved than previous problem but the same approach applies. For two points x, y from the functions codomain ($[0,1]$) we must show that the expression $\lambda x + (1-\lambda)y)$ is also in the codomain. Assume that $x, y \in [0,1]$ and $||x|| < ||y||$ and apply the definition :

$$||\lambda x + (1-\lambda)y|| \leq \lambda||x|| + (1-\lambda)||y|| < \lambda||x|| + (1-\lambda)||x||$$
$$||\lambda x + (1-\lambda)y|| < \lambda||x|| + ||x|| - \lambda||x|| = ||x|| \leq 1$$
$$||\lambda x + (1-\lambda)y|| < 1$$

c. *[2 points]* Draw a picture of the set $(x1, x2) : g(x_1, x_2) \leq 4$ where $g(x1, x2) = (|x_1|^{1/2} + |x_2|^{1/2})^2$.(This is the function considered in 1b above specialized to $n = 2$.) We know g is not a norm. Is the defined set convex? Why not? This is also a partial answer to question in A.0. Regularizer "spikiness" produces sparser weights.

Context: It is a fact that a function f defined over a set $A \subseteq R^n$ is convex if and only if the set $(x, z) \in \mathbb{R}^{n+1} : z \geq f(x), x \in A$ is convex. Draw a picture of this for yourself to be sure you understand it.