

Homework #1 B

Spring 2020, CSE 446/546: Machine Learning

Dino Bektsev

B2

Use just the definitions above and let $\|\cdot\|$ be a norm.

- a. [3 points] Show that $f(x) = \|x\|$ is a convex function.

See A.1. problem, this follows directly from the definition of a norm's absolute scalability and triangle inequality. Start from definition of a convex function $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \forall \lambda \in [0, 1]$ and for any $x, y \in \mathcal{D}$ where \mathcal{D} is a non-empty domain of f and f is our norm:

$$\|\lambda x + (1 - \lambda)y\| \leq \|\lambda x\| + \|(1 - \lambda)y\|$$

$$\|\lambda x + (1 - \lambda)y\| \leq \lambda \|x\| + (1 - \lambda)\|y\|$$

- b. [3 points] Show that $x \in \mathbb{R}^n : \|x\| \leq 1$ is a convex set.

This one is more logically involved than previous problem but the same approach applies. For two points x, y from the functions codomain $([0, 1])$ we must show that the expression $\lambda x + (1 - \lambda)y$ is also in the codomain. Assume that $x, y \in [0, 1]$ and $\|x\| < \|y\|$ and apply the definition :

$$\|\lambda x + (1 - \lambda)y\| \leq \lambda \|x\| + (1 - \lambda)\|y\| < \lambda \|x\| + (1 - \lambda)\|x\|$$

$$\|\lambda x + (1 - \lambda)y\| < \lambda \|x\| + \|x\| - \lambda \|x\| = \|x\| \leq 1$$

$$\|\lambda x + (1 - \lambda)y\| < 1$$

- c. [2 points] Draw a picture of the set $(x_1, x_2) : g(x_1, x_2) \leq 4$ where $g(x_1, x_2) = (|x_1|^{1/2} + |x_2|^{1/2})^2$. (This is the function considered in 1b above specialized to $n = 2$.) We know g is not a norm. Is the defined set convex? Why not? This is also a partial answer to question in A.0. Regularizer "spikiness" produces sparser weights.

Context: It is a fact that a function f defined over a set $A \subseteq \mathbb{R}^n$ is convex if and only if the set $(x, z) \in \mathbb{R}^{n+1} : z \geq f(x), x \in A$ is convex. Draw a picture of this for yourself to be sure you understand it.

