Homework #4

Winter 2020, STATS 509

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Problem 1

Suppose that X is a positive random variable so that $P(X > 0) = 1$. Prove that for any $t < 0$, $M_X(t) = E[e^{tX}] < \infty$.

Hint: Find a relationship between e^{tX} and 1, for $t < 0$. Use the fact that if for all x, $g(x) \leq h(x)$ then $\int g(x)dx \leq \int h(x)dx$.

$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx$$
$$= \int_0^{\infty} e^{tx} f_X(x)dx$$

For $t < 0$:
$$e^{tx} f_X(x) \leq f_X(x) \forall f_X(x)$$

because $e^{kx} \leq 1 \forall k < 0$ so it follows:

$$\int_0^{\infty} e^{tx} f_X(x)dx \leq \int_0^{\infty} f_X(x)dx$$
$$\int_0^{\infty} e^{tx} f_X(x)dx \leq P(X > 0)$$
$$\int_0^{\infty} e^{tx} f_X(x)dx \leq 1$$

Problem 2.

Let X be an exponential random variable with parameter $\lambda > 0$.

a. Use Chebyshev's inequality to find an upper bound on $Pr(|X - E[X]| \geq k)$

$$P(|X - c|) \geq d) \leq \frac{E[(X - c)^2]}{d^2}$$

$$P(|X - \mu|) \geq k) \leq \frac{E[(X - \mu)^2]}{k^2}$$

$$P(|X - \mu|) \geq k) \leq \frac{V[X]}{k^2}$$

$$P(|X - \mu|) \geq k) \leq \left(\frac{1}{k\lambda}\right)^2$$

b. Calculate $E[|X - E[X]|]$ and then use Markov's inequality to find a different upper bound on $Pr(|X - E[X]| \geq k)$.

Hint: first remove the absolute value by splitting the integral into two pieces. Each piece can then be integrated using the Exponential CDF (or just integrating directly) and integration by parts.

$$E[|X - E[X]|] = E[|X - \mu|] = \int_{-\infty}^{\infty} |x - \mu| f_X(x) dx$$

$$= \int_0^{\infty} |x - \mu| \lambda e^{-\lambda x} dx$$

$$= \int_0^{1/\lambda} \left(\frac{1}{\lambda} - x\right) \lambda e^{-\lambda x} dx + \int_{1/\lambda}^{\infty} \left(x - \frac{1}{\lambda}\right) \lambda e^{-\lambda x} dx$$

$$= \int_0^{1/\lambda} e^{-\lambda x} dx - \int_0^{1/\lambda} \lambda x e^{-\lambda x} dx + \int_{1/\lambda}^{\infty} \lambda x e^{-\lambda x} dx - \int_{1/\lambda}^{\infty} e^{-\lambda x} dx$$

$$= -\frac{e^{-\lambda x}}{\lambda}\Big|_0^{1/\lambda} + \lambda \frac{\lambda x + 1}{\lambda^2} e^{-\lambda x}\Big|_0^{1/\lambda} - \lambda \frac{\lambda x + 1}{\lambda^2} e^{-\lambda x}\Big|_{1/\lambda}^{\infty} + \frac{e^{-\lambda x}}{\lambda}\Big|_{1/\lambda}^{\infty}$$

$$= -\frac{e^{-\lambda x}}{\lambda}\Big|_0^{1/\lambda} + \lambda \frac{\lambda x + 1}{\lambda^2} e^{-\lambda x}\Big|_0^{1/\lambda} - \lambda \left(\frac{\lambda x}{\lambda^2} e^{-\lambda x} + \frac{e^{-\lambda x}}{\lambda^2}\right)\Big|_{1/\lambda}^{\infty} + \frac{e^{-\lambda x}}{\lambda}\Big|_{1/\lambda}^{\infty}$$

$$= \left(-\frac{1}{e\lambda} + \frac{1}{\lambda}\right) + \lambda \left(\frac{2}{e\lambda^2} - \frac{1}{\lambda^2}\right) - \lambda \left(0 + 0 - \frac{2}{e\lambda^2} - \frac{1}{\lambda^2}\right) + \left(0 - \frac{1}{e\lambda}\right)$$

$$= -\frac{1}{e\lambda} + \frac{1}{\lambda} + \frac{2}{e\lambda} - \frac{1}{\lambda} + \frac{2}{e\lambda} + \frac{1}{\lambda} - \frac{1}{e\lambda}$$

$$= \frac{2}{e\lambda} + \frac{1}{\lambda}$$

Markov's inequality is then:

$$P(|X - \mu| \geq k) = \frac{1}{2e\lambda k}$$

(c) For which values of k is the bound in (b) lower than that given in (a)?

$$1 > \frac{P_b}{P_a}$$

$$k < \frac{2e}{\lambda} \approx \frac{5.44}{\lambda}$$

Problem 3

If the joint probability density of the two random variables X and Y is given by:

$$f_{XY}(x, y) = \begin{cases} 2 & \text{for } 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

a. Find $P(Y < 0.5)$. *Hint: Draw a graph.*

$$\begin{aligned} P(Y < 0.5) &= \int_0^{0.5} \int_0^y f(x, y) dx dy \\ &= \int_0^{0.5} \int_0^y 2 dx dy = 2\frac{y^2}{2} \Big|_0^{0.5} \\ &= \frac{1}{4} \end{aligned}$$

b. Find $P(X < 0.5)$.

$$\begin{aligned} P(X < 0.5) &= \int_0^{0.5} \int_x^1 f(x, y) dy dx \\ &= \int_0^1 (2 - 2x) dx = 2x|_0^{0.5} - 2\frac{x^2}{2} \Big|_0^{0.5} \\ &= 1 - \frac{1}{4} = \frac{3}{4} \end{aligned}$$

c. Find $P(X + Y < 1)$

$$\begin{aligned} P(X + Y < 1) &= \int_0^{0.5} \int_x^{1-x} f(x, y) dy dx \\ &= 2 \int_0^{0.5} (1 - 2x) dx = 2 \left(x|_0^{0.5} - 2\frac{x^2}{2} \Big|_0^{0.5} \right) \\ &= 2\frac{1}{4} = \frac{1}{2} \end{aligned}$$

d. Find the joint CDF $F(x, y)$. *(You will need to consider several cases.)*

$$F(x, y) = \begin{cases} 0 & \text{if } x < 0 \text{ or } y < 0 \\ 0 & \text{if } y > 1 \text{ or } x > y \\ 2xy & \text{if } y < 1 \text{ and } x < y \end{cases}$$

Problem 4

Suppose that X and Y are continuous random variables with the following joint cdf:

$$F_{XY}(x, y) = \widetilde{F}(x)\widetilde{F}(y) \tag{1}$$

where $\widetilde{F}(\cdot)$ is a (univariate) cdf.

(Aside) *It follows from (1) that X and Y are independent random variables with the same marginal distribution.*

(a) For a fixed value t, express $P(X \leq t, Y \leq t)$ in terms of $\widetilde{F}(\cdot)$.

   Hint: Recall the definition of a joint cdf $F_{XY}(x, y)$.

   By definition $F(x, y) = P(X \leq x, Y \leq y)$ so it follows that:

   $$P(X \leq t, y \leq t) = F_{XY}(t, t) = \tilde{F}(t)\tilde{F}(t) = \tilde{F}^2(t)$$

(b) Let $T = \max(X, Y)$. Using your answer to (a) find the cdf $F_T(t)$ and the pdf $f_T(t)$ for T:

   Hint: If $X \leq t$ and $Y \leq t$, what can one say about $\max(X, Y)$?

   For $T \leq t$ to hold both $X \leq t$ and $Y \leq t$ have to be satisfied. This is by definition the case for a CDF, so then the joint CDF is just

   $$F_T(t) = \tilde{F}^2(t)$$

   By definition the PDF:

   $$\begin{aligned} f_T(t) = \frac{\partial}{\partial t}F_T(t) &= \frac{\partial}{\partial t}\tilde{F}^2(t) \\ &= 2\tilde{F}(t)\tilde{f}(t) \end{aligned}$$

Problem 5

If the joint CDF for two random variables, X and Y, is given by:

$$F_{XY}(x,y) = \begin{cases} (1 - e^{-x^2})(1 - e^{-y^2}) & \text{for } x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases}$$
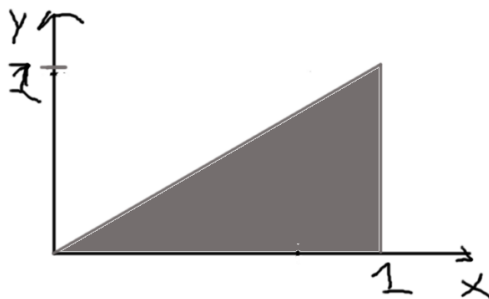
Find the joint pdf $f_{XY}(x,y)$.

$$\begin{aligned}
\frac{\partial^2}{\partial x \partial y} F(x,y) &= \frac{\partial^2}{\partial x \partial y}(1 - e^{-x^2})(1 - e^{-x^2}) \\
&= \frac{\partial}{\partial x}\left[(1 - e^{-x^2})2ye^{-y^2}\right] \\
&= 2xe^{-x^2}2ye^{-y^2} \\
&= 4xye^{-(x^2+y^2)}
\end{aligned}$$

Problem 6

Let X and Y have the joint density function (here $k > 0$ is an unknown constant):

$$f_{XY}(x,y) = \begin{cases} k(x-y), & 0 \leq y \leq x \leq 1; \\ 0 & \text{otherwise,} \end{cases}$$

a. Sketch the support of the joint density. You will need to use this in answering the next three question parts.

b. Find the constant of proportionality k.

$$1 = \int_0^1 \int_0^x k(x-y)\,dy\,dx$$

$$= k \int_0^1 \left[xy - \frac{y^2}{2} \right]\Big|_0^x dx$$

$$= k \int_0^1 \left[x^2 - \frac{x^2}{2} \right] dx$$

$$= k \left[\frac{x^3}{3} - \frac{1}{2}\frac{x^3}{3} \right]\Big|_0^1$$

$$= k \left(\frac{1}{3} - \frac{1}{6} \right) = \frac{k}{6}$$

$$k = 6$$

c. Find the marginal densities $f_X(x)$ and $f_Y(y)$.

$$f_X(x) = \int_0^1 6(x-y)\,dy$$

$$= \left[6xy - 3y^2 \right]\Big|_0^1$$

$$= 6x - 3$$

$$f_Y(y) = \int_0^1 6(x-y)\,dx$$

$$= \left[3x^2 - 6yx \right]\Big|_0^1$$

$$= 3 - 6y$$

d. Find the conditional densities of $f_{Y|X}(y \mid x)$ and $f_{X|Y}(x \mid y)$.

Remember to carefully state for which values of the conditioning variable these are defined.

By definition : $f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$. So on $0 \le x \le 1$ we have:

$$f_{Y|X}(y \mid x) = \frac{6(x-y)}{6x-3} = \frac{2(x-y)}{2x-1}$$

and on $0 \le y \le 1$ we have:

$$f_{X|Y}(x \mid y) = \frac{6(x-y)}{3-6y} = \frac{2(x-y)}{1-3y}$$

Problem 7

If the joint probability density of the two random variables X and Y is given by:

$$f_{XY}(x, y) = \begin{cases} 4xy & \text{for } 0 < y < 1,\ 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $D = (X - Y)$ and $S = (X + Y)$.

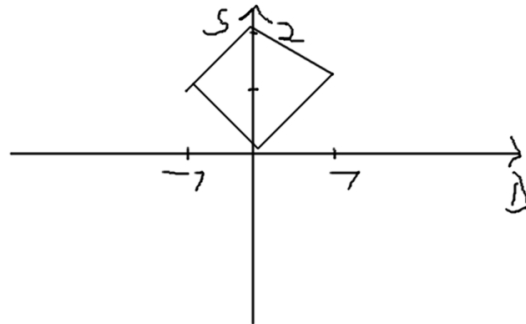 a. Solve for X and Y in terms of D and S.

$$D + S = X - Y + X + Y$$
$$D + S = 2X$$
$$X = \frac{D + S}{2}$$

$$y = S - \frac{D + S}{2}$$
$$y = \frac{S - D}{2}$$

  b. Find the support of $f_{DS}(d, s)$ and draw a sketch.

The boundary lines are given by the following equations, left to right in a clockwise direction:

$$s = -d$$
$$s = d + 2$$
$$s = -d + 2$$
$$s = d$$

where we notice that boundaries are symmetrical around $s = 0$.

c. Find the joint pdf $f_{DS}(d, s)$.

By definition:

$$f_{DV}(d, v) = f_{XY}(g_1(x, y), g_2(x, y))|J|$$

where $g_1(x, y) = 0.5(D + S)$, $g_2(x, y) = 0.5(S - D)$ and:

$$J = \begin{vmatrix} \frac{\partial x}{\partial d} & \frac{\partial y}{\partial d} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix}$$

$$= \frac{1}{4} + \frac{1}{4}$$

$$= -\frac{1}{2}$$

So putting it all back into the definition:

$$f_{DS}(u, v) = f_{XY}\left(\frac{D + S}{2}, \frac{S - D}{2}\right)\left|\frac{1}{2}\right|$$

$$= \frac{1}{2}f_{XY}\left(\frac{D + S}{2}, \frac{S - D}{2}\right)$$

$$= \begin{cases} \frac{1}{2}4\frac{s+d}{2}\frac{s-d}{2} = s^2 - d^2 & \text{when inside the diamond} \\ 0 & \text{otherwise.} \end{cases}$$

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d. Find the marginal densities $f_D(d)$ and $f_S(s)$.

$$f_D(d) = \int_{-\infty}^{\infty} s^2 - d^2 ds$$

$$= \begin{cases} \int_{-d}^{d+2} s^2 - d^2 ds & \text{for } -1 < d < 0, \\ \int_{d}^{2-d} s^2 - d^2 ds & \text{for } 0 \le d < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let's split the integrals out for simplicity:

$$\int_{-d}^{d+2} s^2 - d^2 ds = \int_{-d}^{d+2} s^2 ds - d^2 \int_{-d}^{d+2} ds$$

$$= \frac{s^3}{3}\Big|_{-d}^{d+2} - d^2 s\big|_{-d}^{d+2}$$

$$= \frac{(d+2)^3}{3} - \frac{-d^3}{3} - d^2(2d+2)$$

$$= \frac{(d+2)^3}{3} - \frac{5}{3}d^3 - 2d^2$$

and

$$\int_{d}^{2-d} s^2 - d^2 ds = \int_{d}^{2-d} s^2 ds - d^2 \int_{d}^{2-d} ds$$

$$= \frac{s^3}{3}\Big|_{d}^{2-d} - d^2 s\big|_{d}^{2-d}$$

$$= \frac{(2-d)^3}{3} - \frac{d^3}{3} - d^2(2-2d)$$

$$= \frac{(d+2)^3}{3} - \frac{7}{3}d^3 - 2d^2$$

So in the end:

$$f_D(d) = \int_{-\infty}^{\infty} s^2 - d^2 ds$$

$$= \begin{cases} \frac{(d+2)^3}{3} - \frac{5}{3}d^3 - 2d^2 & \text{for } -1 < d < 0, \\ \frac{(d+2)^3}{3} - \frac{7}{3}d^3 - 2d^2 & \text{for } 0 \le d < 1 \\ 0 & \text{otherwise.} \end{cases}$$

We repeat the same for $f_S(s)$ and get something like:

$$f_S(s) = \int_{-\infty}^{\infty} s^2 - d^2 \, dd$$

$$= \begin{cases} \int_{s-2}^{2-s} s^2 - d^2 \, dd & \text{for } 1 < s < 2, \\ \int_{-s}^{s} s^2 - d^2 \, dd & \text{for } 0 < s \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} 4s^2 - 2s^3 - \frac{2}{3}(2-s)^3 & \text{for } 1 < s < 2, \\ \frac{4}{3}s^3 & \text{for } 0 < s \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

e. Find the conditional densities of $f_{D|S}(d \mid s)$ and $f_{S|D}(s \mid d)$.

Remember to carefully state for which values of the conditioning variable these are defined.

We follow the definition:

$$f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

Note that life is worth living so we leave writing out the expressions explicitly to the reader.
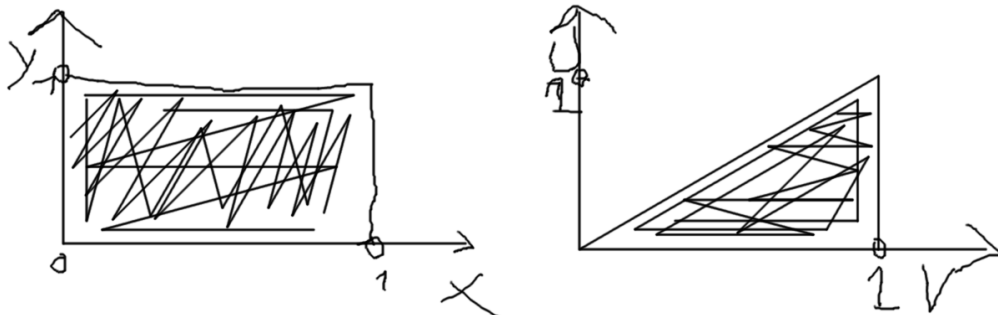
Problem 8

The random variables (X, Y) have the following joint pdf:

$$f_{XY}(x, y) = \begin{cases} 2y & \text{for } 0 < x < 1 \text{ and } 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the transformation: $U = XY$ and $V = X$.

a. Draw the support of the joint density $f_{XY}(x, y)$ for (X, Y).

b. Solve for X and Y in terms of U and V.

$$V = X \rightarrow X = V$$
$$U = XY \rightarrow Y = \frac{U}{V}$$

c. Carefully find the support for the joint density for (U, V). Then find the joint density $f_{UV}(u, v)$, remember to include the support.

By definition:

$$f_{UV}(u, v) = f_{XY}(g_1(x, y), g_2(x, y))|J|$$

where $g_1(x, y) = v$, $g_2(x, y) = u/v$ and:

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & \frac{1}{v} \\ 1 & -\frac{u}{v^2} \end{vmatrix}$$
$$= 0 \cdot -\frac{u}{v^2} - \frac{1}{v} \cdot 1$$
$$= -\frac{1}{v}$$

So putting it all back into the definition:

$$f_{UV}(u, v) = f_{XY}\left(v, \frac{u}{v}\right)\left|-\frac{1}{v}\right|$$
$$= \frac{1}{v}f_{XY}\left(v, \frac{u}{v}\right)$$
$$= \begin{cases} \frac{1}{v}2\frac{u}{v} = 2\frac{u}{v^2} & \text{for } 0 < u < v < 1 \\ 0 & \text{otherwise.} \end{cases}$$

d. Find the marginal densities $f_U(u)$ for U and $f_V(v)$ for V.

$$f_U(u) = \int_u^1 \frac{2u}{v^2} dv$$
$$= 2u \left(-\frac{1}{v} \Big|_u^1 \right)$$
$$= 2u(\frac{1}{u} - 1) = 2 - 2u$$
$$= 2(1 - u)$$

$$f_V(v) = \int_0^v \frac{2u}{v^2} du$$
$$= \frac{2}{v^2} \frac{u^2}{2} \Big|_0^v$$
$$= \frac{2}{v^2} \frac{v^2}{2} = 1$$