

# Homework #4

Winter 2020, STATS 509

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## Problem 1

Suppose that  $X$  is a positive random variable so that  $P(X > 0) = 1$ . Prove that for any  $t < 0$ ,  $M_X(t) = E[e^{tX}] < \infty$ .

*Hint: Find a relationship between  $e^{tx}$  and 1, for  $t < 0$ . Use the fact that if for all  $x$ ,  $g(x) \leq h(x)$  then  $\int g(x)dx \leq \int h(x)dx$ .*

$$\begin{aligned} M_X(t) &= E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x) dx \\ &= \int_0^{\infty} e^{tx} f_X(x) dx \end{aligned}$$

For  $t < 0$ :

$$0 < e^{tx} f_X(x) \leq f_X(x) \forall f_X(x)$$

because  $e^{tx} \leq 1 \forall t < 0$  so it follows:

$$\begin{aligned} \int_0^{\infty} e^{tx} f_X(x) dx &\leq \int_0^{\infty} f_X(x) dx \\ \int_0^{\infty} e^{tx} f_X(x) dx &\leq P(X > 0) \\ \int_0^{\infty} e^{tx} f_X(x) dx &\leq 1 \end{aligned}$$

## Problem 2.

Let  $X$  be an exponential random variable with parameter  $\lambda > 0$ .

- a. Use Chebyshev's inequality to find an upper bound on  $Pr(|X - E[X]| \geq k)$

$$\begin{aligned} P(|X - c| \geq d) &\leq \frac{E[(X - c)^2]}{d^2} \\ P(|X - \mu| \geq k) &\leq \frac{E[(X - \mu)^2]}{k^2} \\ P(|X - \mu| \geq k) &\leq \frac{V[X]}{k^2} \\ P(|X - \mu| \geq k) &\leq \left(\frac{1}{k\lambda}\right)^2 \end{aligned}$$

- b. Calculate  $E[|X - E[X]|]$  and then use Markov's inequality to find a different upper bound on  $Pr(|X - E[X]| \geq k)$ .

*Hint: first remove the absolute value by splitting the integral into two pieces. Each piece can then be integrated using the Exponential CDF (or just integrating directly) and integration by parts.*

$$\begin{aligned} E[|X - E[X]|] &= E[|X - \mu|] = \int_{-\infty}^{\infty} |x - \mu| f_X(x) dx \\ &= \int_0^{\infty} |x - \mu| \lambda e^{-\lambda x} dx \\ &= \int_0^{1/\lambda} \left(\frac{1}{\lambda} - x\right) \lambda e^{-\lambda x} dx + \int_{1/\lambda}^{\infty} \left(x - \frac{1}{\lambda}\right) \lambda e^{-\lambda x} dx \\ &= \int_0^{1/\lambda} e^{-\lambda x} dx - \int_0^{1/\lambda} \lambda x e^{-\lambda x} dx + \int_{1/\lambda}^{\infty} \lambda x e^{-\lambda x} dx - \int_{1/\lambda}^{\infty} e^{-\lambda x} dx \\ &= -\frac{e^{-\lambda x}}{\lambda} \Big|_0^{1/\lambda} + \frac{\lambda x + 1}{\lambda} e^{-\lambda x} \Big|_0^{1/\lambda} - \frac{\lambda x + 1}{\lambda} e^{-\lambda x} \Big|_{1/\lambda}^{\infty} + \frac{e^{-\lambda x}}{\lambda} \Big|_{1/\lambda}^{\infty} \\ &= \left(-\frac{1}{e\lambda} + \frac{1}{\lambda}\right) + \lambda \left(\frac{2}{e\lambda^2} - \frac{1}{\lambda^2}\right) - \lambda \left(0 + 0 - \frac{2}{e\lambda^2} - \frac{1}{\lambda^2}\right) + \left(0 - \frac{1}{e\lambda}\right) \\ &= \frac{e-1}{e\lambda} + \frac{2-e}{e\lambda} + \frac{2}{e\lambda} - \frac{1}{e\lambda} \\ &= \frac{2}{e\lambda} \end{aligned}$$

Markov's inequality is then:

$$P(|X - \mu| \geq k) = \frac{2}{e\lambda k}$$

- (c) For which values of  $k$  is the bound in (b) lower than that given in (a)?

$$\begin{aligned} 1 &> \frac{P_b}{P_a} \\ k &< \frac{e}{2\lambda} \approx \frac{1}{\lambda} \end{aligned}$$

### Problem 3

If the joint probability density of the two random variables  $X$  and  $Y$  is given by:

$$f_{XY}(x, y) = \begin{cases} 2 & \text{for } 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

- a. Find  $P(Y < 0.5)$ . *Hint: Draw a graph.*

$$\begin{aligned} P(Y < 0.5) &= \int_0^{0.5} \int_0^y f(x, y) dx dy \\ &= \int_0^{0.5} \int_0^y 2 dx dy = 2 \frac{y^2}{2} \Big|_0^{0.5} \\ &= \frac{1}{4} \end{aligned}$$

- b. Find  $P(X < 0.5)$ .

$$\begin{aligned} P(X < 0.5) &= \int_0^{0.5} \int_x^1 f(x, y) dy dx \\ &= \int_0^{0.5} (2 - 2x) dx = 2x \Big|_0^{0.5} - 2 \frac{x^2}{2} \Big|_0^{0.5} \\ &= 1 - \frac{1}{4} = \frac{3}{4} \end{aligned}$$

- c. Find  $P(X + Y < 1)$

$$\begin{aligned} P(X + Y < 1) &= \int_0^{0.5} \int_x^{1-x} f(x, y) dy dx \\ &= 2 \int_0^{0.5} (1 - 2x) dx = 2 \left( x \Big|_0^{0.5} - 2 \frac{x^2}{2} \Big|_0^{0.5} \right) \\ &= 2 \frac{1}{4} = \frac{1}{2} \end{aligned}$$

d. Find the joint CDF  $F(x, y)$ . (You will need to consider several cases.)

In counter-clockwise order of appearance starting from y axis we have at least the following cases to consider:

$$\text{Case} = \begin{cases} 0 & \text{if } y > 0 \text{ and } x < 0 \\ 0 & \text{if } y < 0 \text{ and } x < 0 \\ 0 & \text{if } y < 0 \text{ and } x > 0 \\ 0 & \text{if } 0 < y < 1 \text{ and } x > y \\ \int_0^y \int_y^x & \text{if } 0 < x < y < 1 \\ 1 & \text{if } y > 1 \text{ and } x > 1 \\ \int_0^x \int_x^1 & \text{if } y > 1 \text{ and } 0 < x < 1 \end{cases}$$

Since

$$\begin{aligned} \int_0^y \int_y^x 2dx dy &= 2 \int_0^y (x - y) dy &= 2xy - y^2 \\ \int_0^x \int_x^1 2dx dy &= 2 \int_0^x (1 - x) dy &= 2x - x^2 \end{aligned}$$

we can write:

$$F_{XY}(x, y) = \begin{cases} 0 & \text{if } y > 0 \text{ and } x < 0 \\ 0 & \text{if } y < 0 \text{ and } x < 0 \\ 0 & \text{if } y < 0 \text{ and } x > 0 \\ 0 & \text{if } 0 < y < 1 \text{ and } x > y \\ 2xy - y^2 & \text{if } 0 < x < y < 1 \\ 1 & \text{if } y > 1 \text{ and } x > 1 \\ 2x - x^2 & \text{if } y > 1 \text{ and } 0 < x < 1 \end{cases}$$

## Problem 4

Suppose that  $X$  and  $Y$  are continuous random variables with the following joint cdf:

$$F_{XY}(x, y) = \tilde{F}(x)\tilde{F}(y) \quad (1)$$

where  $\tilde{F}(\cdot)$  is a (univariate) cdf.

(Aside) It follows from (1) that  $X$  and  $Y$  are independent random variables with the same marginal distribution.

- (a) For a fixed value  $t$ , express  $P(X \leq t, Y \leq t)$  in terms of  $\tilde{F}(\cdot)$ .

*Hint: Recall the definition of a joint cdf  $F_{XY}(x, y)$ .*

By definition  $F(x, y) = P(X \leq x, Y \leq y)$  so it follows that:

$$P(X \leq t, Y \leq t) = F_{XY}(t, t) = \tilde{F}(t)\tilde{F}(t) = \tilde{F}^2(t)$$

- (b) Let  $T = \max(X, Y)$ . Using your answer to (a) find the cdf  $F_T(t)$  and the pdf  $f_T(t)$  for  $T$ :

*Hint: If  $X \leq t$  **and**  $Y \leq t$ , what can one say about  $\max(X, Y)$ ?*

For  $T \leq t$  to hold both  $X \leq t$  and  $Y \leq t$  have to be satisfied. This is by definition the case for a CDF, so then the joint CDF is just

$$F_T(t) = \tilde{F}^2(t)$$

By definition the PDF:

$$\begin{aligned} f_T(t) &= \frac{\partial}{\partial t} F_T(t) = \frac{\partial}{\partial t} \tilde{F}^2(t) \\ &= 2\tilde{F}(t)f(\tilde{t}) \end{aligned}$$

## Problem 5

If the joint CDF for two random variables,  $X$  and  $Y$ , is given by:

$$F_{XY}(x, y) = \begin{cases} (1 - e^{-x^2})(1 - e^{-y^2}) & \text{for } x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the joint pdf  $f_{XY}(x, y)$ .

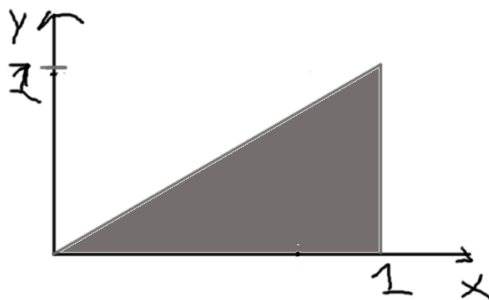
$$\begin{aligned} \frac{\partial^2}{\partial x \partial y} F(x, y) &= \frac{\partial^2}{\partial x \partial y} (1 - e^{-x^2})(1 - e^{-y^2}) \\ &= \frac{\partial}{\partial x} \left[ (1 - e^{-x^2}) 2y e^{-y^2} \right] \\ &= 2x e^{-x^2} 2y e^{-y^2} \\ &= 4xy e^{-(x^2 + y^2)} \end{aligned}$$

## Problem 6

Let  $X$  and  $Y$  have the joint density function (here  $k > 0$  is an unknown constant):

$$f_{XY}(x, y) = \begin{cases} k(x - y), & 0 \leq y \leq x \leq 1; \\ 0 & \text{otherwise,} \end{cases}$$

- a. Sketch the support of the joint density. You will need to use this in answering the next three question parts.



- b. Find the constant of proportionality  $k$ .

$$\begin{aligned} 1 &= \int_0^1 \int_0^x k(x - y) dy dx \\ &= k \int_0^1 \left[ xy - \frac{y^2}{2} \right] \Big|_0^x dx \\ &= k \int_0^1 \left[ x^2 - \frac{x^2}{2} \right] dx \\ &= k \left[ \frac{x^3}{3} - \frac{1}{2} \frac{x^3}{3} \right] \Big|_0^1 \\ &= k \left( \frac{1}{3} - \frac{1}{6} \right) = \frac{k}{6} \\ k &= 6 \end{aligned}$$

- c. Find the marginal densities  $f_X(x)$  and  $f_Y(y)$ .

$$\begin{aligned} f_X(x) &= \int_0^x 6(x - y) dy \\ &= [6xy - 3y^2] \Big|_0^x \\ &= 6x^2 - 3x^2 = 3x^2 \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \int_y^1 6(x - y) dx \\ &= [3x^2 - 6yx] \Big|_y^1 \\ &= 3 - 6y - 3y^2 + 6y^2 = 3(y^2 - 2y + 1) \\ &= 3(y - 1)^2 \end{aligned}$$

- d. Find the conditional densities of  $f_{Y|X}(y | x)$  and  $f_{X|Y}(x | y)$ .

Remember to carefully state for which values of the conditioning variable these are defined.

By definition :  $f_{Y|X}(y | x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$ . So on  $0 \leq y < x \leq 1$  we have:

$$f_{Y|X}(y | x) = \frac{6(x-y)}{3x^2} = \frac{2(x-y)}{x^2}$$

and on  $0 \leq y < x \leq 1$  we have:

$$f_{X|Y}(x | y) = \frac{6(x-y)}{3(y-1)^2} = \frac{2(x-y)}{(y-1)^2}$$



## Problem 7

If the joint probability density of the two random variables  $X$  and  $Y$  is given by:

$$f_{XY}(x, y) = \begin{cases} 4xy & \text{for } 0 < y < 1, 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let  $D = (X - Y)$  and  $S = (X + Y)$ .

- a. Solve for  $X$  and  $Y$  in terms of  $D$  and  $S$ .

$$D + S = X - Y + X + Y$$

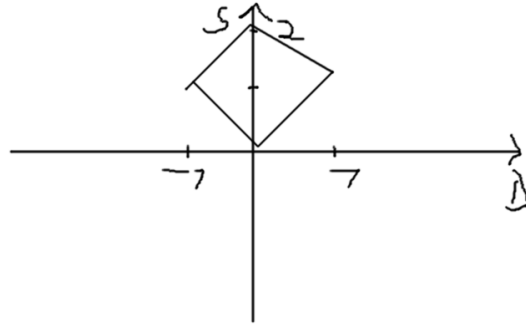
$$D + S = 2X$$

$$X = \frac{D + S}{2}$$

$$y = S - \frac{D + S}{2}$$

$$y = \frac{S - D}{2}$$

- b. Find the support of  $f_{DS}(d, s)$  and draw a sketch.



The boundary lines are given by the following equations, left to right in a clockwise direction:

$$s = -d$$

$$s = d + 2$$

$$s = -d + 2$$

$$s = d$$

where we notice that boundaries are symmetrical around  $s = 0$ .

c. Find the joint pdf  $f_{DS}(d, s)$ .

By definition:

$$f_{DV}(d, v) = f_{XY}(g_1(x, y), g_2(x, y))|J|$$

where  $g_1(x, y) = 0.5(D + S)$ ,  $g_2(x, y) = 0.5(S - D)$  and:

$$\begin{aligned} J &= \begin{vmatrix} \frac{\partial x}{\partial d} & \frac{\partial y}{\partial d} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} \\ &= \frac{1}{4} - \frac{1}{4} \\ &= -\frac{1}{2} \end{aligned}$$

So putting it all back into the definition:

$$\begin{aligned} f_{DS}(u, v) &= f_{XY}\left(\frac{D+S}{2}, \frac{S-D}{2}\right) \left|\frac{1}{2}\right| \\ &= \frac{1}{2} f_{XY}\left(\frac{D+S}{2}, \frac{S-D}{2}\right) \\ &= \begin{cases} \frac{1}{2} 4 \frac{s+d}{2} \frac{s-d}{2} = s^2 - d^2 & \text{when inside the diamond} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

d. Find the marginal densities  $f_D(d)$  and  $f_S(s)$ .

$$f_D(d) = \int_{-\infty}^{\infty} s^2 - d^2 ds = \begin{cases} \int_{-d}^{d+2} s^2 - d^2 ds & \text{for } -1 < d < 0, \\ \int_d^{2-d} s^2 - d^2 ds & \text{for } 0 \leq d < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let's split the integrals out for simplicity:

$$\begin{aligned} \int_{-d}^{d+2} s^2 - d^2 ds &= \int_{-d}^{d+2} s^2 ds - d^2 \int_{-d}^{d+2} ds \\ &= \left. \frac{s^3}{3} \right|_{-d}^{d+2} - d^2 s \Big|_{-d}^{d+2} \\ &= \frac{(d+2)^3}{3} - \frac{-d^3}{3} - d^2(2d+2) \\ &= \frac{(d+2)^3}{3} - \frac{5}{3}d^3 - 2d^2 \end{aligned}$$

and

$$\begin{aligned} \int_d^{2-d} s^2 - d^2 ds &= \int_d^{2-d} s^2 ds - d^2 \int_d^{2-d} ds \\ &= \left. \frac{s^3}{3} \right|_d^{2-d} - d^2 s \Big|_d^{2-d} \\ &= \frac{(2-d)^3}{3} - \frac{d^3}{3} - d^2(2-2d) \\ &= \frac{(d+2)^3}{3} - \frac{7}{3}d^3 - 2d^2 \end{aligned}$$

So in the end:

$$f_D(d) = \int_{-\infty}^{\infty} s^2 - d^2 ds = \begin{cases} \frac{(d+2)^3}{3} - \frac{5}{3}d^3 - 2d^2 & \text{for } -1 < d < 0, \\ \frac{(d+2)^3}{3} - \frac{7}{3}d^3 - 2d^2 & \text{for } 0 \leq d < 1 \\ 0 & \text{otherwise.} \end{cases}$$

We repeat the same for  $f_S(s)$  and get something like:

$$\begin{aligned}
 f_S(s) &= \int_{-\infty}^{\infty} s^2 - d^2 dd \\
 &= \begin{cases} \int_{s-2}^{2-s} s^2 - d^2 dd & \text{for } 1 < s < 2, \\ \int_{-s}^s s^2 - d^2 dd & \text{for } 0 < s \leq 1 \\ 0 & \text{otherwise.} \end{cases} \\
 &= \begin{cases} 4s^2 - 2s^3 - \frac{2}{3}(2-s)^3 & \text{for } 1 < s < 2, \\ \frac{4}{3}s^3 & \text{for } 0 < s \leq 1 \\ 0 & \text{otherwise.} \end{cases}
 \end{aligned}$$

- e. Find the conditional densities of  $f_{D|S}(d | s)$  and  $f_{S|D}(s | d)$ .

Remember to carefully state for which values of the conditioning variable these are defined.

We follow the definition:

$$f_{Y|X}(y | x) = \frac{f_{X,Y}(x, y)}{f_X(x)}$$

but note that life is better lived afk.

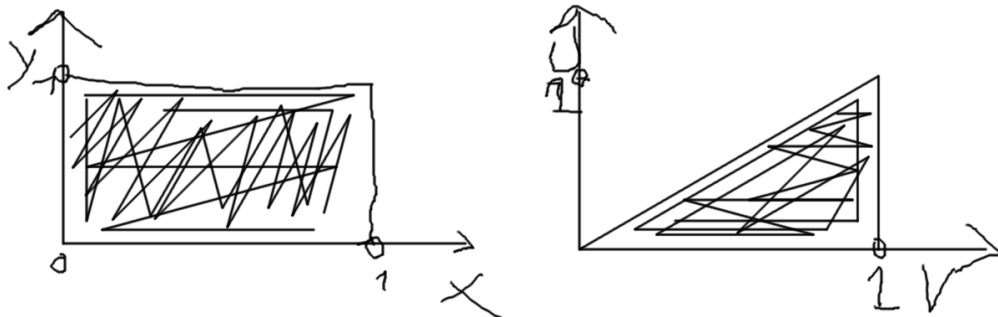
## Problem 8

The random variables  $(X, Y)$  have the following joint pdf:

$$f_{XY}(x, y) = \begin{cases} 2y & \text{for } 0 < x < 1 \text{ and } 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the transformation:  $U = XY$  and  $V = X$ .

- a. Draw the support of the joint density  $f_{XY}(x, y)$  for  $(X, Y)$ .



- b. Solve for  $X$  and  $Y$  in terms of  $U$  and  $V$ .

$$\begin{aligned} V = X &\rightarrow X = V \\ U = XY &\rightarrow Y = \frac{U}{V} \end{aligned}$$

- c. Carefully find the support for the joint density for  $(U, V)$ . Then find the joint density  $f_{UV}(u, v)$ , remember to include the support.

By definition:

$$f_{UV}(u, v) = f_{XY}(g_1(x, y), g_2(x, y))|J|$$

where  $g_1(x, y) = v$ ,  $g_2(x, y) = u/v$  and:

$$\begin{aligned} J &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & \frac{1}{v} \\ 1 & -\frac{u}{v^2} \end{vmatrix} \\ &= 0 \cdot -\frac{u}{v^2} - \frac{1}{v} \cdot 1 \\ &= -\frac{1}{v} \end{aligned}$$

So putting it all back into the definition:

$$\begin{aligned} f_{UV}(u, v) &= f_{XY}\left(v, \frac{u}{v}\right) \left| -\frac{1}{v} \right| \\ &= \frac{1}{v} f_{XY}\left(v, \frac{u}{v}\right) \\ &= \begin{cases} \frac{1}{v} 2 \frac{u}{v} = 2 \frac{u}{v^2} & \text{for } 0 < u < v < 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

- d. Find the marginal densities  $f_U(u)$  for  $U$  and  $f_V(v)$  for  $V$ .

$$\begin{aligned}f_U(u) &= \int_u^1 \frac{2u}{v^2} dv \\&= 2u \left( -\frac{1}{v} \Big|_u^1 \right) \\&= 2u \left( \frac{1}{u} - 1 \right) = 2 - 2u \\&= 2(1 - u)\end{aligned}$$

$$\begin{aligned}f_V(v) &= \int_0^v \frac{2u}{v^2} du \\&= \frac{2}{v^2} \frac{u^2}{2} \Big|_0^v \\&= \frac{2}{v^2} \frac{v^2}{2} = 1\end{aligned}$$