Homework #4

Winter 2020, STATS 509
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Problem 1

Suppose that X is a positive random variable so that $P(X > 0) = 1$. Prove that for any $t < 0$, $M_X(t) = E[e^{tX}] < \infty$.
Hint: Find a relationship between e^{tX} and 1, for $t < 0$. Use the fact that if for all x, $g(x) \leq h(x)$ then $\int g(x)dx \leq \int h(x)dx$.

$$
\begin{aligned}
M_X(t) = E[e^{tX}] &= \int_{-\infty}^{\infty} e^{tx} f_X(x)dx \\
&= \int_0^{\infty} e^{tx} f_X(x)dx
\end{aligned}
$$

For $t < 0$:

$$0 < e^{tx} f_X(x) \leq f_X(x) \forall f_X(x)$$

because $e^{tx} \leq 1 \forall t < 0$ so it follows:

$$
\begin{aligned}
\int_0^{\infty} e^{tx} f_X(x)dx &\leq \int_0^{\infty} f_X(x)dx \\
\int_0^{\infty} e^{tx} f_X(x)dx &\leq P(X > 0) \\
\int_0^{\infty} e^{tx} f_X(x)dx &\leq 1
\end{aligned}
$$

Problem 2.

Let X be an exponential random variable with parameter $\lambda > 0$.

a. Use Chebyshev's inequality to find an upper bound on $Pr(|X - E[X]| \geq k)$

$$
\begin{aligned}
P(|X-c|) \geq d) &\leq \frac{E[(X-c)^2]}{d^2} \\
P(|X-\mu|) \geq k) &\leq \frac{E[(X-\mu)^2]}{k^2} \\
P(|X-\mu|) \geq k) &\leq \frac{V[X]}{k^2} \\
P(|X-\mu|) \geq k) &\leq \left(\frac{1}{k\lambda}\right)^2
\end{aligned}
$$

b. Calculate $E[|X - E[X]|]$ and then use Markov's inequality to find a different upper bound on $Pr(|X - E[X]| \geq k)$.

Hint: first remove the absolute value by splitting the integral into two pieces. Each piece can then be integrated using the Exponential CDF (or just integrating directly) and integration by parts.

$$
\begin{aligned}
E[|X-E[X]|] = E[|X-\mu|] &= \int_{-\infty}^{\infty} |x-\mu| f_X(x) dx \\
&= \int_0^{\infty} |x-\mu| \lambda e^{-\lambda x} dx \\
&= \int_0^{1/\lambda} \left(\frac{1}{\lambda} - x\right) \lambda e^{-\lambda x} dx + \int_{1/\lambda}^{\infty} \left(x - \frac{1}{\lambda}\right) \lambda e^{-\lambda x} dx \\
&= \int_0^{1/\lambda} e^{-\lambda x} dx - \int_0^{1/\lambda} \lambda x e^{-\lambda x} dx + \int_{1/\lambda}^{\infty} \lambda x e^{-\lambda x} dx - \int_{1/\lambda}^{\infty} e^{-\lambda x} dx \\
&= -\frac{e^{-\lambda x}}{\lambda}\Bigg|_0^{1/\lambda} + \frac{\lambda x + 1}{\lambda} e^{-\lambda x}\Bigg|_0^{1/\lambda} - \frac{\lambda x + 1}{\lambda} e^{-\lambda x}\Bigg|_{1/\lambda}^{\infty} + \frac{e^{-\lambda x}}{\lambda}\Bigg|_{1/\lambda}^{\infty} \\
&= \left(-\frac{1}{e\lambda} + \frac{1}{\lambda}\right) + \lambda\left(\frac{2}{e\lambda^2} - \frac{1}{\lambda^2}\right) - \lambda\left(0 + 0 - \frac{2}{e\lambda^2} - \frac{1}{\lambda^2}\right) + \left(0 - \frac{1}{e\lambda}\right) \\
&= \frac{e-1}{ea} + \frac{2-e}{ea} + \frac{2}{ea} - \frac{1}{ea} \\
&= \frac{2}{e\lambda}
\end{aligned}
$$

Markov's inequality is then:

$$
P(|X-\mu| \geq k) = \frac{2}{e\lambda k}
$$

(c) For which values of k is the bound in (b) lower than that given in (a)?

$$
\begin{aligned}
1 &> \frac{P_b}{P_a} \\
k &< \frac{e}{2\lambda} \approx \frac{1}{\lambda}
\end{aligned}
$$

Problem 3

If the joint probability density of the two random variables X and Y is given by:

$$f_{XY}(x,y) = \begin{cases} 2 & \text{for } 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

a. Find $P(Y < 0.5)$. *Hint: Draw a graph.*

$$\begin{aligned} P(Y < 0.5) &= \int_0^{0.5} \int_0^{y} f(x,y)dxdy \\ &= \int_0^{0.5} \int_0^{y} 2dxdy = 2\frac{y^2}{2}\Big|_0^{0.5} \\ &= \frac{1}{4} \end{aligned}$$

b. Find $P(X < 0.5)$.

$$\begin{aligned} P(X < 0.5) &= \int_0^{0.5} \int_x^{1} f(x,y)dydx \\ &= \int_0^{1} (2-2x)dx = 2x|_0^{0.5} - 2\frac{x^2}{2}\Big|_0^{0.5} \\ &= 1 - \frac{1}{4} = \frac{3}{4} \end{aligned}$$

c. Find $P(X + Y < 1)$

$$\begin{aligned} P(X + Y < 1) &= \int_0^{0.5} \int_x^{1-x} f(x,y)dydx \\ &= 2\int_0^{0.5} (1-2x)dx = 2\left(x|_0^{0.5} - 2\frac{x^2}{2}\Big|_0^{0.5} \right) \\ &= 2\frac{1}{4} = \frac{1}{2} \end{aligned}$$

d. Find the joint CDF $F(x, y)$. *(You will need to consider several cases.)*

In counter-clockwise order of appearance starting from y axis we have at least the following cases to consider:

$$\text{Case} = \begin{cases} 0 & \text{if } y > 0 \text{ and } x < 0 \\ 0 & \text{if } y < 0 \text{ and } x < 0 \\ 0 & \text{if } y < 0 \text{ and } x > 0 \\ 0 & \text{if } 0 < y < 1 \text{ and } x > y \\ \int_0^y \int_y^x & \text{if } 0 < x < y < 1 \\ 1 & \text{if } y > 1 \text{ and } x > 1 \\ \int_0^x \int_x^1 & \text{if } y > 1 \text{ and } 0 < x < 1 \end{cases}$$

Since

$$\int_0^y \int_y^x 2dxdy = 2\int_0^y (x - y)dy \qquad = 2xy - y^2$$
$$\int_0^x \int_x^1 2dxdy = 2\int_0^x (1 - x)dy \qquad = 2x - x^2$$

we can write:

$$F_{XY}(x, y) = \begin{cases} 0 & \text{if } y > 0 \text{ and } x < 0 \\ 0 & \text{if } y < 0 \text{ and } x < 0 \\ 0 & \text{if } y < 0 \text{ and } x > 0 \\ 0 & \text{if } 0 < y < 1 \text{ and } x > y \\ 2xy - y^2 & \text{if } 0 < x < y < 1 \\ 1 & \text{if } y > 1 \text{ and } x > 1 \\ 2x - x^2 & \text{if } y > 1 \text{ and } 0 < x < 1 \end{cases}$$

Problem 4

Suppose that X and Y are continuous random variables with the following joint cdf:

$$F_{XY}(x, y) = \widetilde{F}(x)\widetilde{F}(y) \tag{1}$$

where $\widetilde{F}(\cdot)$ is a (univariate) cdf.
(Aside) *It follows from* (1) *that* X *and* Y *are independent random variables with the same marginal distribution.*

(a) For a fixed value t, express $P(X \leq t, Y \leq t)$ in terms of $\widetilde{F}(\cdot)$.

Hint: Recall the definition of a joint cdf $F_{XY}(x, y)$.

By definition $F(x, y) = P(X \leq x, Y \leq y)$ so it follows that:

$$P(X \leq t, y \leq t) = F_{XY}(t, t) = \tilde{F}(t)\tilde{F}(t) = \tilde{F}^2(t)$$

(b) Let $T = \max(X, Y)$. Using your answer to (a) find the cdf $F_T(t)$ and the pdf $f_T(t)$ for T:

Hint: If $X \leq t$ **and** $Y \leq t$, *what can one say about* $\max(X, Y)$?

For $T \leq t$ to hold both $X \leq t$ and $Y \leq t$ have to be satisfied. This is by definition the case for a CDF, so then the joint CDF is just

$$F_T(t) = \tilde{F}^2(t)$$

By definition the PDF:

$$\begin{aligned} f_T(t) = \frac{\partial}{\partial t} F_T(t) &= \frac{\partial}{\partial t} \tilde{F}^2(t) \\ &= 2\tilde{F}(t)\tilde{f(t)} \end{aligned}$$

Problem 5

If the joint CDF for two random variables, X and Y, is given by:

$$F_{XY}(x,y) = \begin{cases} (1-e^{-x^2})(1-e^{-y^2}) & \text{for } x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the joint pdf $f_{XY}(x,y)$.

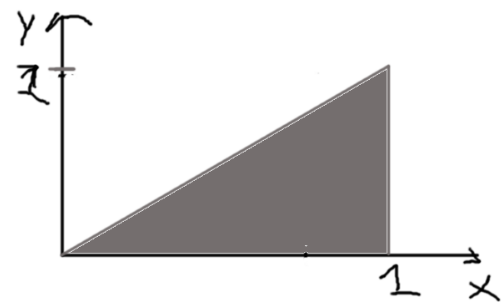
$$\begin{aligned} \frac{\partial^2}{\partial x \partial y} F(x,y) &= \frac{\partial^2}{\partial x \partial y}(1-e^{-x^2})(1-e^{-x^2}) \\ &= \frac{\partial}{\partial x}\left[(1-e^{-x^2})2ye^{-y^2}\right] \\ &= 2xe^{-x^2}2ye^{-y^2} \\ &= 4xye^{-(x^2+y^2)} \end{aligned}$$

Problem 6

Let X and Y have the joint density function (here $k > 0$ is an unknown constant):

$$f_{XY}(x,y) = \begin{cases} k(x-y), & 0 \le y \le x \le 1; \\ 0 & \text{otherwise,} \end{cases}$$

a. Sketch the support of the joint density. You will need to use this in answering the next three question parts.



b. Find the constant of proportionality k.

$$\begin{aligned} 1 &= \int_0^1 \int_0^x k(x-y)dydx \\ &= k\int_0^1 \left[xy - \frac{y^2}{2}\right]\Big|_0^x dx \\ &= k\int_0^1 \left[x^2 - \frac{x^2}{2}\right] dx \\ &= k\left[\frac{x^3}{3} - \frac{1}{2}\frac{x^3}{3}\right]\Big|_0^1 \\ &= k\left(\frac{1}{3} - \frac{1}{6}\right) = \frac{k}{6} \\ k &= 6 \end{aligned}$$

c. Find the marginal densities $f_X(x)$ and $f_Y(y)$.

$$\begin{aligned} f_X(x) &= \int_0^1 6(x-y)dy \\ &= \left[6xy - 3y^2\right]\Big|_0^1 \\ &= 6x - 3 \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \int_0^1 6(x-y)dx \\ &= \left[3x^2 - 6yx\right]\Big|_0^1 \\ &= 3 - 6y \end{aligned}$$

d. Find the conditional densities of $f_{Y|X}(y \mid x)$ and $f_{X|Y}(x \mid y)$.

Remember to carefully state for which values of the conditioning variable these are defined.

By definition : $f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$. So on $0 \leq x \leq 1$ we have:

$$f_{Y|X}(y \mid x) = \frac{6(x-y)}{6x-3} = \frac{2(x-y)}{2x-1}$$

and on $0 \leq y \leq 1$ we have:

$$f_{X|Y}(x \mid y) = \frac{6(x-y)}{3-6y} = \frac{2(x-y)}{1-3y}$$

Problem 7

If the joint probability density of the two random variables X and Y is given by:

$$f_{XY}(x,y) = \begin{cases} 4xy & \text{for } 0 < y < 1,\ 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

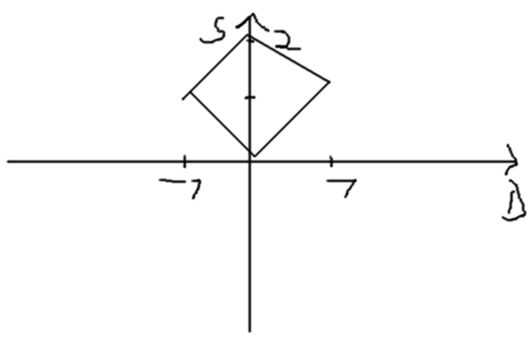
Let $D = (X - Y)$ and $S = (X + Y)$.

a. Solve for X and Y in terms of D and S.

$$\begin{aligned} D + S &= X - Y + X + Y \\ D + S &= 2X \\ X &= \frac{D+S}{2} \end{aligned}$$

$$\begin{aligned} y &= S - \frac{D+S}{2} \\ y &= \frac{S-D}{2} \end{aligned}$$

b. Find the support of $f_{DS}(d,s)$ and draw a sketch.



The boundary lines are given by the following equations, left to right in a clockwise direction:

$$\begin{aligned} s &= -d \\ s &= d + 2 \\ s &= -d + 2 \\ s &= d \end{aligned}$$

where we notice that boundaries are symmetrical around $s = 0$.

c. Find the joint pdf $f_{DS}(d,s)$.

By definition:

$$f_{DV}(d,v) = f_{XY}(g_1(x,y), g_2(x,y))|J|$$

where $g_1(x,y) = 0.5(D+S)$, $g_2(x,y) = 0.5(S-D)$ and:

$$\begin{aligned} J &= \begin{vmatrix} \frac{\partial x}{\partial d} & \frac{\partial y}{\partial d} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} \\ &= \frac{1}{4} + \frac{1}{4} \\ &= -\frac{1}{2} \end{aligned}$$

So putting it all back into the definition:

$$\begin{aligned} f_{DS}(u,v) &= f_{XY}\left(\frac{D+S}{2}, \frac{S-D}{2}\right)\left|\frac{1}{2}\right| \\ &= \frac{1}{2} f_{XY}\left(\frac{D+S}{2}, \frac{S-D}{2}\right) \\ &= \begin{cases} \frac{1}{2} 4 \frac{s+d}{2}\frac{s-d}{2} = s^2 - d^2 & \text{when inside the diamond} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

d. Find the marginal densities $f_D(d)$ and $f_S(s)$.

$$
\begin{aligned}
f_D(d) &= \int_{-\infty}^{\infty} s^2 - d^2 ds \\
&= \begin{cases} \int_{-d}^{d+2} s^2 - d^2 ds & \text{for } -1 < d < 0, \\ \int_{d}^{2-d} s^2 - d^2 ds & \text{for } 0 \leq d < 1 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

Let's split the integrals out for simplicity:

$$
\begin{aligned}
\int_{-d}^{d+2} s^2 - d^2 ds &= \int_{-d}^{d+2} s^2 ds - d^2 \int_{-d}^{d+2} ds \\
&= \left.\frac{s^3}{3}\right|_{-d}^{d+2} - d^2 s|_{-d}^{d+2} \\
&= \frac{(d+2)^3}{3} - \frac{-d^3}{3} - d^2(2d+2) \\
&= \frac{(d+2)^3}{3} - \frac{5}{3}d^3 - 2d^2
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{d}^{2-d} s^2 - d^2 ds &= \int_{d}^{2-d} s^2 ds - d^2 \int_{d}^{2-d} ds \\
&= \left.\frac{s^3}{3}\right|_{d}^{2-d} - d^2 s|_{d}^{2-d} \\
&= \frac{(2-d)^3}{3} - \frac{d^3}{3} - d^2(2-2d) \\
&= \frac{(d+2)^3}{3} - \frac{7}{3}d^3 - 2d^2
\end{aligned}
$$

So in the end:

$$
\begin{aligned}
f_D(d) &= \int_{-\infty}^{\infty} s^2 - d^2 ds \\
&= \begin{cases} \frac{(d+2)^3}{3} - \frac{5}{3}d^3 - 2d^2 & \text{for } -1 < d < 0, \\ \frac{(d+2)^3}{3} - \frac{7}{3}d^3 - 2d^2 & \text{for } 0 \leq d < 1 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

We repeat the same for $f_S(s)$ and get something like:

$$\begin{aligned}
f_S(s) &= \int_{-\infty}^{\infty} s^2 - d^2 dd \\
&= \begin{cases} \int_{s-2}^{2-s} s^2 - d^2 dd & \text{for } 1 < s < 2, \\ \int_{-s}^{s} s^2 - d^2 dd & \text{for } 0 < s \leq 1 \\ 0 & \text{otherwise.} \end{cases} \\
&= \begin{cases} 4s^2 - 2s^3 - \frac{2}{3}(2-s)^3 & \text{for } 1 < s < 2, \\ \frac{4}{3}s^3 & \text{for } 0 < s \leq 1 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

e. Find the conditional densities of $f_{D|S}(d \mid s)$ and $f_{S|D}(s \mid d)$.

Remember to carefully state for which values of the conditioning variable these are defined.

We follow the definition:

$$f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

but note that life is better lived afk.

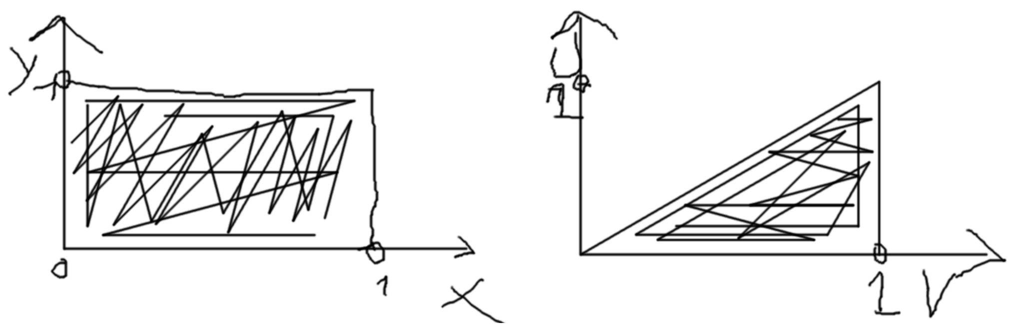
Problem 8

The random variables (X, Y) have the following joint pdf:

$$f_{XY}(x, y) = \begin{cases} 2y & \text{for } 0 < x < 1 \text{ and } 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the transformation: $U = XY$ and $V = X$.

a. Draw the support of the joint density $f_{XY}(x, y)$ for (X, Y).



b. Solve for X and Y in terms of U and V.

$$V = X \to X = V$$
$$U = XY \to Y = \frac{U}{V}$$

c. Carefully find the support for the joint density for (U, V). Then find the joint density $f_{UV}(u, v)$, remember to include the support.

By definition:

$$f_{UV}(u, v) = f_{XY}(g_1(x, y), g_2(x, y))|J|$$

where $g_1(x, y) = v$, $g_2(x, y) = u/v$ and:

$$\begin{aligned} J &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & \frac{1}{v} \\ 1 & -\frac{u}{v^2} \end{vmatrix} \\ &= 0 \cdot -\frac{u}{v^2} - \frac{1}{v} \cdot 1 \\ &= -\frac{1}{v} \end{aligned}$$

So putting it all back into the definition:

$$\begin{aligned} f_{UV}(u, v) &= f_{XY}\left(v, \frac{u}{v}\right)\left| -\frac{1}{v} \right| \\ &= \frac{1}{v} f_{XY}\left(v, \frac{u}{v}\right) \\ &= \begin{cases} \frac{1}{v} 2\frac{u}{v} = 2\frac{u}{v^2} & \text{for } 0 < u < v < 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

d. Find the marginal densities $f_U(u)$ for U and $f_V(v)$ for V.

$$\begin{aligned}
f_U(u) &= \int_u^1 \frac{2u}{v^2} dv \\
&= 2u \left(-\frac{1}{v}\bigg|_u^1 \right) \\
&= 2u(\frac{1}{u} - 1) = 2 - 2u \\
&= 2(1-u)
\end{aligned}$$

$$\begin{aligned}
f_V(v) &= \int_0^v \frac{2u}{v^2} du \\
&= \frac{2}{v^2} \frac{u^2}{2}\bigg|_0^v \\
&= \frac{2}{v^2} \frac{v^2}{2} = 1
\end{aligned}$$