

Homework #4

Winter 2020, STATS 509

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Problem 1

Suppose that X is a positive random variable so that $P(X > 0) = 1$. Prove that for any $t < 0$, $M_X(t) = E[e^{tX}] < \infty$.

Hint: Find a relationship between e^{tX} and 1, for $t < 0$. Use the fact that if for all x , $g(x) \leq h(x)$ then $\int g(x)dx \leq \int h(x)dx$.

$$\begin{aligned} M_X(t) &= E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x) dx \\ &= \int_0^{\infty} e^{tx} f_X(x) dx \end{aligned}$$

For $t < 0$:

$$e^{tx} f_X(x) \leq f_X(x) \forall f_X(x)$$

because $e^{kx} \leq 1 \forall k < 0$ so it follows:

$$\begin{aligned} \int_0^{\infty} e^{tx} f_X(x) dx &\leq \int_0^{\infty} f_X(x) dx \\ \int_0^{\infty} e^{tx} f_X(x) dx &\leq P(X > 0) \\ \int_0^{\infty} e^{tx} f_X(x) dx &\leq 1 \end{aligned}$$

Problem 2.

Let X be an exponential random variable with parameter $\lambda > 0$.

- a. Use Chebyshev's inequality to find an upper bound on $Pr(|X - E[X]| \geq k)$

$$\begin{aligned} P(|X - c| \geq d) &\leq \frac{E[(X - c)^2]}{d^2} \\ P(|X - \mu| \geq k) &\leq \frac{E[(X - \mu)^2]}{k^2} \\ P(|X - \mu| \geq k) &\leq \frac{V[X]}{k^2} \\ P(|X - \mu| \geq k) &\leq \left(\frac{1}{k\lambda}\right)^2 \end{aligned}$$

- b. Calculate $E[|X - E[X]|]$ and then use Markov's inequality to find a different upper bound on $Pr(|X - E[X]| \geq k)$.

Hint: first remove the absolute value by splitting the integral into two pieces. Each piece can then be integrated using the Exponential CDF (or just integrating directly) and integration by parts.

$$\begin{aligned} E[|X - E[X]|] &= E[|X - \mu|] = \int_{-\infty}^{\infty} |x - \mu| f_X(x) dx \\ &= \int_0^{\infty} |x - \mu| \lambda e^{-\lambda x} dx \\ &= \int_0^{1/\lambda} \left(\frac{1}{\lambda} - x\right) \lambda e^{-\lambda x} dx + \int_{1/\lambda}^{\infty} \left(x - \frac{1}{\lambda}\right) \lambda e^{-\lambda x} dx \\ &= \int_0^{1/\lambda} e^{-\lambda x} dx - \int_0^{1/\lambda} \lambda x e^{-\lambda x} dx + \int_{1/\lambda}^{\infty} \lambda x e^{-\lambda x} dx - \int_{1/\lambda}^{\infty} e^{-\lambda x} dx \\ &= -\frac{e^{-\lambda x}}{\lambda} \Big|_0^{1/\lambda} + \lambda \frac{\lambda x + 1}{\lambda^2} e^{-\lambda x} \Big|_0^{1/\lambda} - \lambda \frac{\lambda x + 1}{\lambda^2} e^{-\lambda x} \Big|_{1/\lambda}^{\infty} + \frac{e^{-\lambda x}}{\lambda} \Big|_{1/\lambda}^{\infty} \\ &= -\frac{e^{-\lambda x}}{\lambda} \Big|_0^{1/\lambda} + \lambda \frac{\lambda x + 1}{\lambda^2} e^{-\lambda x} \Big|_0^{1/\lambda} - \lambda \left(\frac{\lambda x}{\lambda^2} e^{-\lambda x} + \frac{e^{-\lambda x}}{\lambda^2} \right) \Big|_{1/\lambda}^{\infty} + \frac{e^{-\lambda x}}{\lambda} \Big|_{1/\lambda}^{\infty} \\ &= \left(-\frac{1}{e\lambda} + \frac{1}{\lambda}\right) + \lambda \left(\frac{2}{e\lambda^2} - \frac{1}{\lambda^2}\right) - \lambda \left(0 + 0 - \frac{2}{e\lambda^2} - \frac{1}{\lambda^2}\right) + \left(0 - \frac{1}{e\lambda}\right) \\ &= -\frac{1}{e\lambda} + \frac{1}{\lambda} + \frac{2}{e\lambda} - \frac{1}{\lambda} + \frac{2}{e\lambda} + \frac{1}{\lambda} - \frac{1}{e\lambda} \\ &= \frac{2}{e\lambda} + \frac{1}{\lambda} \end{aligned}$$

Markov's inequality is then:

$$P(|X - \mu| \geq k) = \frac{1}{2e\lambda k}$$

- (c) For which values of k is the bound in (b) lower than that given in (a)?

$$\begin{aligned} 1 &> \frac{P_b}{P_a} \\ k &< \frac{2e}{\lambda} \approx \frac{5.44}{\lambda} \end{aligned}$$

Problem 3

If the joint probability density of the two random variables X and Y is given by:

$$f_{XY}(x, y) = \begin{cases} 2 & \text{for } 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

- a. Find $P(Y < 0.5)$. *Hint: Draw a graph.*

$$\begin{aligned} P(Y < 0.5) &= \int_0^{0.5} \int_0^y f(x, y) dx dy \\ &= \int_0^{0.5} \int_0^y 2 dx dy = 2 \frac{y^2}{2} \Big|_0^{0.5} \\ &= \frac{1}{4} \end{aligned}$$

- b. Find $P(X < 0.5)$.

$$\begin{aligned} P(X < 0.5) &= \int_0^{0.5} \int_x^1 f(x, y) dy dx \\ &= \int_0^{0.5} (2 - 2x) dx = 2x \Big|_0^{0.5} - 2 \frac{x^2}{2} \Big|_0^{0.5} \\ &= 1 - \frac{1}{4} = \frac{3}{4} \end{aligned}$$

- c. Find $P(X + Y < 1)$

$$\begin{aligned} P(X + Y < 1) &= \int_0^{0.5} \int_x^{1-x} f(x, y) dy dx \\ &= 2 \int_0^{0.5} (1 - 2x) dx = 2 \left(x \Big|_0^{0.5} - 2 \frac{x^2}{2} \Big|_0^{0.5} \right) \\ &= 2 \frac{1}{4} = \frac{1}{2} \end{aligned}$$

- d. Find the joint CDF $F(x, y)$. (*You will need to consider several cases.*)

$$F(x, y) = \begin{cases} 0 & \text{if } x < 0 \text{ or } y < 0 \\ 0 & \text{if } y > 1 \text{ or } x > y \\ 2xy & \text{if } y < 1 \text{ and } x < y \end{cases}$$

Problem 4

Suppose that X and Y are continuous random variables with the following joint cdf:

$$F_{XY}(x, y) = \tilde{F}(x)\tilde{F}(y) \quad (1)$$

where $\tilde{F}(\cdot)$ is a (univariate) cdf.

(Aside) It follows from (1) that X and Y are independent random variables with the same marginal distribution.

- (a) For a fixed value t , express $P(X \leq t, Y \leq t)$ in terms of $\tilde{F}(\cdot)$.

Hint: Recall the definition of a joint cdf $F_{XY}(x, y)$.

By definition $F(x, y) = P(X \leq x, Y \leq y)$ so it follows that:

$$P(X \leq t, Y \leq t) = F_{XY}(t, t) = \tilde{F}(t)\tilde{F}(t) = \tilde{F}^2(t)$$

- (b) Let $T = \max(X, Y)$. Using your answer to (a) find the cdf $F_T(t)$ and the pdf $f_T(t)$ for T :

*Hint: If $X \leq t$ **and** $Y \leq t$, what can one say about $\max(X, Y)$?*

For $T \leq t$ to hold both $X \leq t$ and $Y \leq t$ have to be satisfied. This is by definition the case for a CDF, so then the joint CDF is just

$$F_T(t) = \tilde{F}^2(t)$$

By definition the PDF:

$$\begin{aligned} f_T(t) &= \frac{\partial}{\partial t} F_T(t) = \frac{\partial}{\partial t} \tilde{F}^2(t) \\ &= 2\tilde{F}(t)f(\tilde{t}) \end{aligned}$$

Problem 5

If the joint CDF for two random variables, X and Y , is given by:

$$F_{XY}(x, y) = \begin{cases} (1 - e^{-x^2})(1 - e^{-y^2}) & \text{for } x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the joint pdf $f_{XY}(x, y)$.

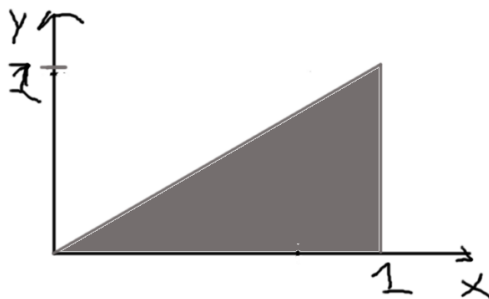
$$\begin{aligned} \frac{\partial^2}{\partial x \partial y} F(x, y) &= \frac{\partial^2}{\partial x \partial y} (1 - e^{-x^2})(1 - e^{-y^2}) \\ &= \frac{\partial}{\partial x} \left[(1 - e^{-x^2}) 2ye^{-y^2} \right] \\ &= 2xe^{-x^2} 2ye^{-y^2} \\ &= 4xye^{-(x^2+y^2)} \end{aligned}$$

Problem 6

Let X and Y have the joint density function (here $k > 0$ is an unknown constant):

$$f_{XY}(x, y) = \begin{cases} k(x - y), & 0 \leq y \leq x \leq 1; \\ 0 & \text{otherwise,} \end{cases}$$

- a. Sketch the support of the joint density. You will need to use this in answering the next three question parts.



- b. Find the constant of proportionality k .

$$\begin{aligned} 1 &= \int_0^1 \int_0^x k(x - y) dy dx \\ &= k \int_0^1 \left[xy - \frac{y^2}{2} \right] \Big|_0^x dx \\ &= k \int_0^1 \left[x^2 - \frac{x^2}{2} \right] dx \\ &= k \left[\frac{x^3}{3} - \frac{1}{2} \frac{x^3}{3} \right] \Big|_0^1 \\ &= k \left(\frac{1}{3} - \frac{1}{6} \right) = \frac{k}{6} \\ k &= 6 \end{aligned}$$

- c. Find the marginal densities $f_X(x)$ and $f_Y(y)$.

$$\begin{aligned} f_X(x) &= \int_0^1 6(x - y) dy \\ &= [6xy - 3y^2] \Big|_0^1 \\ &= 6x - 3 \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \int_0^1 6(x - y) dx \\ &= [3x^2 - 6yx] \Big|_0^1 \\ &= 3 - 6y \end{aligned}$$

- d. Find the conditional densities of $f_{Y|X}(y | x)$ and $f_{X|Y}(x | y)$.

Remember to carefully state for which values of the conditioning variable these are defined.

By definition : $f_{Y|X}(y | x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$. So on $0 \leq x \leq 1$ we have:

$$f_{Y|X}(y | x) = \frac{6(x-y)}{6x-3} = \frac{2(x-y)}{2x-1}$$

and on $0 \leq y \leq 1$ we have:

$$f_{X|Y}(x | y) = \frac{6(x-y)}{3-6y} = \frac{2(x-y)}{1-3y}$$

Problem 7

If the joint probability density of the two random variables X and Y is given by:

$$f_{XY}(x, y) = \begin{cases} 4xy & \text{for } 0 < y < 1, 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $D = (X - Y)$ and $S = (X + Y)$.

- a. Solve for X and Y in terms of D and S .

$$D + S = X - Y + X + Y$$

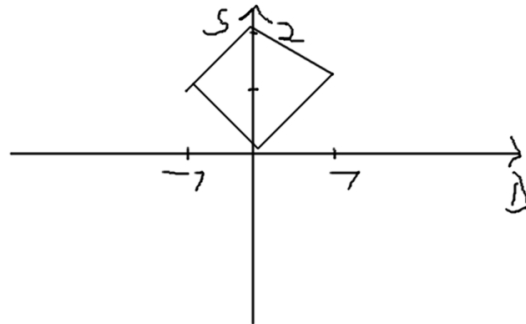
$$D + S = 2X$$

$$X = \frac{D + S}{2}$$

$$y = S - \frac{D + S}{2}$$

$$y = \frac{S - D}{2}$$

- b. Find the support of $f_{DS}(d, s)$ and draw a sketch.



The boundary lines are given by the following equations, left to right in a clockwise direction:

$$s = -d$$

$$s = d + 2$$

$$s = -d + 2$$

$$s = d$$

where we notice that boundaries are symmetrical around $s = 0$.

c. Find the joint pdf $f_{DS}(d, s)$.

By definition:

$$f_{DV}(d, v) = f_{XY}(g_1(x, y), g_2(x, y))|J|$$

where $g_1(x, y) = 0.5(D + S)$, $g_2(x, y) = 0.5(S - D)$ and:

$$\begin{aligned} J &= \begin{vmatrix} \frac{\partial x}{\partial d} & \frac{\partial y}{\partial d} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} \\ &= \frac{1}{4} - \frac{1}{4} \\ &= -\frac{1}{2} \end{aligned}$$

So putting it all back into the definition:

$$\begin{aligned} f_{DS}(u, v) &= f_{XY}\left(\frac{D+S}{2}, \frac{S-D}{2}\right) \left|\frac{1}{2}\right| \\ &= \frac{1}{2} f_{XY}\left(\frac{D+S}{2}, \frac{S-D}{2}\right) \\ &= \begin{cases} \frac{1}{2} 4^{\frac{S+D}{2} \frac{S-D}{2}} = S^2 - D^2 & \text{for } ummm.... \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

d. Find the marginal densities $f_D(d)$ and $f_S(s)$.

$$f_D(d) = \int_{-\infty}^{\infty} s^2 - d^2 ds$$

$$= \begin{cases} \int_{-d}^{d+2} s^2 - d^2 ds & \text{for } -1 < d < 0, \\ \int_d^{2-d} s^2 - d^2 ds & \text{for } 0 \leq d < 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$f_S(s) = \int_{-\infty}^{\infty} s^2 - d^2 dd$$

$$= \begin{cases} \int_{s-2}^{2-s} s^2 - d^2 dd & \text{for } 1 < s < 2, \\ \int_d^{2-d} s^2 - d^2 dd & \text{for } 0 < s \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

e. Find the conditional densities of $f_{D|S}(d | s)$ and $f_{S|D}(s | d)$.

Remember to carefully state for which values of the conditioning variable these are defined.

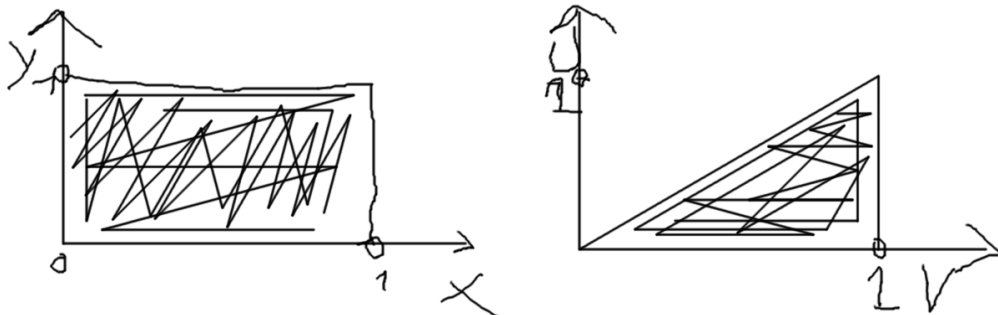
Problem 8

The random variables (X, Y) have the following joint pdf:

$$f_{XY}(x, y) = \begin{cases} 2y & \text{for } 0 < x < 1 \text{ and } 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the transformation: $U = XY$ and $V = X$.

- a. Draw the support of the joint density $f_{XY}(x, y)$ for (X, Y) .



- b. Solve for X and Y in terms of U and V .

$$\begin{aligned} V = X &\rightarrow X = V \\ U = XY &\rightarrow Y = \frac{U}{V} \end{aligned}$$

- c. Carefully find the support for the joint density for (U, V) . Then find the joint density $f_{UV}(u, v)$, remember to include the support.

By definition:

$$f_{UV}(u, v) = f_{XY}(g_1(x, y), g_2(x, y))|J|$$

where $g_1(x, y) = v$, $g_2(x, y) = u/v$ and:

$$\begin{aligned} J &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & \frac{1}{v} \\ 1 & -\frac{u}{v^2} \end{vmatrix} \\ &= 0 \cdot -\frac{u}{v^2} - \frac{1}{v} \cdot 1 \\ &= -\frac{1}{v} \end{aligned}$$

So putting it all back into the definition:

$$\begin{aligned} f_{UV}(u, v) &= f_{XY}\left(v, \frac{u}{v}\right) \left| -\frac{1}{v} \right| \\ &= \frac{1}{v} f_{XY}\left(v, \frac{u}{v}\right) \\ &= \begin{cases} \frac{1}{v} 2 \frac{u}{v} & \text{for } 0 < u < v < 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

- d. Find the marginal densities $f_U(u)$ for U and $f_V(v)$ for V .

$$\begin{aligned} f_U(u) &= \int_0^1 \frac{2u}{v^2} dv \\ &= 2u \left(-\frac{1}{v} \Big|_0^1 \right) \end{aligned}$$

$$\begin{aligned} f_V(v) &= \int_0^v \frac{2u}{v^2} du \\ &= \left(-\frac{2u^2}{2v^2} \Big|_0^v \right) \\ &= \frac{1}{v^2} \end{aligned}$$