Homework #4

Winter 2020, STATS 509
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Problem 1

Suppose that X is a positive random variable so that $P(X > 0) = 1$. Prove that for any $t < 0$, $M_X(t) = E[e^{tX}] < \infty$.
Hint: Find a relationship between e^{tX} and 1, for $t < 0$. Use the fact that if for all x, $g(x) \leq h(x)$ then $\int g(x)dx \leq \int h(x)dx$.

$$
\begin{aligned}
M_X(t) = E[e^{tX}] &= \int_{-\infty}^{\infty} e^{tx} f_X(x)dx \\
&= \int_0^{\infty} e^{tx} f_X(x)dx
\end{aligned}
$$

For $t < 0$:

$$e^{tx} f_X(x) \leq f_X(x) \forall f_X(x)$$

because $e^{kx} \leq 1 \forall k < 0$ so it follows:

$$
\begin{aligned}
&\int_0^{\infty} e^{tx} f_X(x)dx \leq \int_0^{\infty} f_X(x)dx \\
&\int_0^{\infty} e^{tx} f_X(x)dx \leq P(X > 0) \\
&\int_0^{\infty} e^{tx} f_X(x)dx \leq 1
\end{aligned}
$$

Problem 2.

Let X be an exponential random variable with parameter $\lambda > 0$.

a. Use Chebyshev's inequality to find an upper bound on $Pr(|X - E[X]| \geq k)$

$$
\begin{aligned}
P(|X - c|) \geq d) &\leq \frac{E[(X-c)^2]}{d^2} \\
P(|X - \mu|) \geq k) &\leq \frac{E[(X-\mu)^2]}{k^2} \\
P(|X - \mu|) \geq k) &\leq \frac{V[X]}{k^2} \\
P(|X - \mu|) \geq k) &\leq \left(\frac{1}{k\lambda}\right)^2
\end{aligned}
$$

b. Calculate $E[|X - E[X]|]$ and then use Markov's inequality to find a different upper bound on $Pr(|X - E[X]| \geq k)$.

Hint: first remove the absolute value by splitting the integral into two pieces. Each piece can then be integrated using the Exponential CDF (or just integrating directly) and integration by parts.

$$
\begin{aligned}
E[|X - E[X]|] = E[|X-\mu|] &= \int_{-\infty}^{\infty} |x-\mu| f_X(x)dx \\
&= \int_0^{\infty} |x-\mu|\lambda e^{-\lambda x} dx \\
&= \int_0^{1/\lambda} \left(\frac{1}{\lambda} - x\right) \lambda e^{-\lambda x} dx + \int_{1/\lambda}^{\infty} \left(x - \frac{1}{\lambda}\right) \lambda e^{-\lambda x} dx \\
&= \int_0^{1/\lambda} e^{-\lambda x} dx - \int_0^{1/\lambda} \lambda x e^{-\lambda x} dx + \int_{1/\lambda}^{\infty} \lambda x e^{-\lambda x} dx - \int_{1/\lambda}^{\infty} e^{-\lambda x} dx \\
&= -\frac{e^{-\lambda x}}{\lambda}\Bigg|_0^{1/\lambda} + \lambda \frac{\lambda x + 1}{\lambda^2} e^{-\lambda x}\Bigg|_0^{1/\lambda} - \lambda \frac{\lambda x + 1}{\lambda^2} e^{-\lambda x}\Bigg|_{1/\lambda}^{\infty} + \frac{e^{-\lambda x}}{\lambda}\Bigg|_{1/\lambda}^{\infty} \\
&= -\frac{e^{-\lambda x}}{\lambda}\Bigg|_0^{1/\lambda} + \lambda \frac{\lambda x + 1}{\lambda^2} e^{-\lambda x}\Bigg|_0^{1/\lambda} - \lambda \left(\frac{\lambda x}{\lambda^2} e^{-\lambda x} + \frac{e^{-\lambda x}}{\lambda^2}\right)\Bigg|_{1/\lambda}^{\infty} + \frac{e^{-\lambda x}}{\lambda}\Bigg|_{1/\lambda}^{\infty} \\
&= \left(-\frac{1}{e\lambda} + \frac{1}{\lambda}\right) + \lambda\left(\frac{2}{e\lambda^2} - \frac{1}{\lambda^2}\right) - \lambda\left(0 + 0 - \frac{2}{e\lambda^2} - \frac{1}{\lambda^2}\right) + \left(0 - \frac{1}{e\lambda}\right) \\
&= -\frac{1}{e\lambda} + \frac{1}{\lambda} + \frac{2}{e\lambda} - \frac{1}{\lambda} + \frac{2}{e\lambda} + \frac{1}{\lambda} - \frac{1}{e\lambda} \\
&= \frac{2}{e\lambda} + \frac{1}{\lambda} \\
&= \frac{4 - 2\lambda}{e\lambda^2}
\end{aligned}
$$

Markov's inequality is then:

$$P(|X - \mu| \geq k) = \frac{e-2}{e\lambda k}$$

(c) For which values of k is the bound in (b) lower than that given in (a)?

$$
\begin{aligned}
1 &> \frac{P_b}{P_a} \\
1 &> \frac{(e-2)\lambda^2 k^2}{e\lambda k} \\
1 &> \frac{e-2}{e}\lambda k \\
k &< \frac{e}{\lambda(e-2)} \approx \frac{3.784}{\lambda}
\end{aligned}
$$

Problem 3

If the joint probability density of the two random variables X and Y is given by:

$$f_{XY}(x,y) = \left\{ \begin{array}{ll} 2 & \text{for } 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{array} \right.$$

a. Find $P(Y < 0.5)$. *Hint: Draw a graph.*

$$\begin{aligned} P(Y < 0.5) &= \int_0^1 \int_0^y f(x,y)dxdy \\ &= \int_0^1 \int_0^y 2dxdy = 2\frac{y^2}{2}\Big|_0^1 \\ &= 1 \end{aligned}$$

b. Find $P(X < 0.5)$.

$$\begin{aligned} P(X < 0.5) &= \int_0^{0.5} \int_x^1 f(x,y)dydx \\ &= \int_0^1 (2-x)dx = 2x|_0^{0.5} - \frac{x^2}{2}\Big|_0^{0.5} \\ &= 1 - \frac{1}{8} = \frac{7}{8} \end{aligned}$$

c. Find $P(X + Y < 1)$

$$\begin{aligned} P(X+Y < 1) &= \int_0^{0.5} \int_x^{1-x} f(x,y)dydx \\ &= 2\int_0^{0.5} (1-2x)dx = 2\left(x|_0^{0.5} - 2\frac{x^2}{2}\Big|_0^{0.5} \right) \\ &= 2\frac{1}{4} = \frac{1}{2} \end{aligned}$$

d. Find the joint CDF $F(x,y)$. *(You will need to consider several cases.)*

$$F(x,y) = \begin{cases} 0 & \text{if } x < 0 \text{ or } y < 0 \\ 0 & \text{if } y > 1 \text{ or } x > y \\ 2xy & \text{if } y < 1 \text{ and } x < y \end{cases}$$

Problem 4

Suppose that X and Y are continuous random variables with the following joint cdf:

$$F_{XY}(x,y) = \widetilde{F}(x)\widetilde{F}(y) \tag{1}$$

where $\widetilde{F}(\cdot)$ is a (univariate) cdf.
(Aside) *It follows from* (1) *that X and Y are independent random variables with the same marginal distribution.*

(a) For a fixed value t, express $P(X \leq t, Y \leq t)$ in terms of $\widetilde{F}(\cdot)$.

Hint: Recall the definition of a joint cdf $F_{XY}(x,y)$.

By definition $F(x,y) = P(X \leq x, Y \leq y)$ so it follows that:

$$P(X \leq t, y \leq t) = F_{XY}(t,t) = \tilde{F}(t)\tilde{F}(t) = \tilde{F}^2(t)$$

(b) Let $T = \max(X,Y)$. Using your answer to (a) find the cdf $F_T(t)$ and the pdf $f_T(t)$ for T:

Hint: If $X \leq t$ **and** *$Y \leq t$, what can one say about* $\max(X,Y)$?

Because $T = \max(X,Y)$, for $T \leq t$ to hold both $X \leq t$ and $Y \leq t$ have to be satisfied:

By definition:

$$\begin{aligned} f_T(t) = \frac{\partial}{\partial t}F_T(t) &= \frac{\partial}{\partial t}\tilde{F}^2(t) \\ &= 2\tilde{F}(t)\tilde{f(t)} \end{aligned}$$

Problem 5

If the joint CDF for two random variables, X and Y, is given by:

$$F_{XY}(x,y) = \begin{cases} (1-e^{-x^2})(1-e^{-y^2}) & \text{for } x>0, y>0 \\ 0 & \text{otherwise} \end{cases}$$

Find the joint pdf $f_{XY}(x,y)$.

Problem 6

Let X and Y have the joint density function (here $k > 0$ is an unknown constant):

$$f_{XY}(x, y) = \begin{cases} k(x - y), & 0 \le y \le x \le 1; \\ 0 & \text{otherwise,} \end{cases}$$

(a) Sketch the support of the joint density. You will need to use this in answering the next three question parts.

(b) Find the constant of proportionality k.

(c) Find the marginal densities $f_X(x)$ and $f_Y(y)$.

(d) Find the conditional densities of $f_{Y|X}(y \mid x)$ and $f_{X|Y}(x \mid y)$.

Remember to carefully state for which values of the conditioning variable these are defined.

Problem 7

If the joint probability density of the two random variables X and Y is given by:

$$f_{XY}(x,y) = \begin{cases} 4xy & \text{for } 0 < y < 1,\ 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $D = (X - Y)$ and $S = (X + Y)$.

(a) Solve for X and Y in terms of D and S.

(b) Find the support of $f_{DS}(d,s)$ and draw a sketch.

(c) Find the joint pdf $f_{DS}(d,s)$.

(d) Find the marginal densities $f_D(d)$ and $f_S(s)$.

(e) Find the conditional densities of $f_{D|S}(d \mid s)$ and $f_{S|D}(s \mid d)$.
Remember to carefully state for which values of the conditioning variable these are defined.

Problem 8

The random variables (X, Y) have the following joint pdf:

$$f_{XY}(x,y) = \begin{cases} 2y & \text{for } 0 < x < 1 \text{ and } 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the transformation: $U = XY$ and $V = X$.

(a) Draw the support of the joint density $f_{XY}(x,y)$ for (X, Y).

(b) Solve for X and Y in terms of U and V.

(c) Carefully find the support for the joint density for (U, V). Then find the joint density $f_{UV}(u,v)$, remember to include the support.

(d) Find the marginal densities $f_U(u)$ for U and $f_V(v)$ for V.