Homework #8

Winter 2020, STATS 509

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Problem 1: Properties of Estimators and Confidence Intervals

Goldberger Qu. 11.2

Hints: For (a) use common sense / the analogy principle; for (b) read Goldberger §10.1, p.107; for part (c) use the analogy principle and p.108; for (d) recall that the standard error of T is simply an estimate of the standard deviation of T.[1]

a. We can pick $T = \bar{X} - \bar{Y}$ since:

$$E[T] = E[\bar{X}] - E[\bar{Y}] = \mu_X - \mu_Y = \theta$$

b.

$$\begin{aligned}
V(T) &= Cov(T, T) = Cov(\bar{X} - \bar{Y}, \bar{X} - \bar{Y}) \\
&= V(\bar{X}) + V(\bar{Y}) - 2Cov(\bar{Y}, \bar{X}) \\
&= V(\bar{X}) + V(\bar{Y}) - 2Cov\left(\frac{1}{n}\sum_i^n y_i, \frac{1}{n}\sum_i^n x_i\right) \\
&= V(\bar{X}) + V(\bar{Y}) - \frac{2}{n^2}\sum_{i,j=1}^n Cov(y_i, x_j) \\
&= V(\bar{X}) + V(\bar{Y}) - \frac{2}{n^2}\sum_{i,j=1}^n \sigma_{XY} \\
&= \frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{n} - \frac{2}{n^2} n\sigma_{XY} \\
&= \frac{\sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY}}{n}
\end{aligned}$$

c. By analogy with b) we can just say:

$$V(T) = \frac{S_X^2 + S_Y^2 - 2S_{XY}}{n}$$

but following Goldberger's definitions of these values to make them unbiased:

$$V(T) = \frac{\frac{n}{n-1}S_X^2 + \frac{n}{n-1}S_Y^2 - 2\frac{n}{n-1}S_{XY}}{n} = \frac{S_X^2 + S_Y^2 - 2S_{XY}}{n-1}$$

d. For standard error of T I would report $\sqrt{V(T)}$

[1] This usage of the term 'standard error' follows Goldberger (p.123) who defines the 'standard error' of $\bar{Y}$ to be $s/\sqrt{n}$ which is an **estimate** of $\sigma/\sqrt{n}$, the standard deviation of $\bar{Y}$. (Here assuming σ is unknown.

However, other authors use 'standard error of $\bar{Y}$' to refer to $\sigma/\sqrt{n}$; such authors will then refer to $s/\sqrt{n}$ as an **estimated** standard error. (For Goldberger, adding the word 'estimated' to 'standard error' would be redundant.)

Problem 2

Goldberger Qu. 11.3 *Hints: see p.119. For part (a), express the estimator as $T = a_1\bar{Y}_1 + a_2\bar{Y}_2$; find a constraint on a_1 and a_2 in order for T to be unbiased for μ; use this to solve for a_2 in terms of a_1; find the variance of T; substitute for a_2, and then differentiate the variance with respect to a_1.*

a. We are asked to consider $T = c_1\bar{Y}_1 + c_2\bar{Y}_2$. We want it to be unbiased so:

$$\mu = E[T] = c_1 E[\bar{Y}_1] + c_2 E[\bar{Y}_2] = c_1\mu + c_2\mu = (c_1 + c_2)\mu$$

we must conclude that $c_1 + c_2 = 1 \Longrightarrow c_2 = 1 - c_1$ if unbiased-ness is to hold true. We are asked to find the minimum variance unbiased estimator so:

$$\begin{aligned} V(T) &= V(c_1\bar{Y}_1 + c_2\bar{Y}_2) \\ &= V(c_1\bar{Y}_1) + V(c_2\bar{Y}_2) \\ &= c_1^2 V(\bar{Y}_1) + c_2^2 V(\bar{Y}_2) \\ &= c_1^2 V(\bar{Y}_1) + (1 - c_1)^2 V(\bar{Y}_2) \end{aligned}$$

where we, in line 2, used the fact that the problem tells us that the two samples are independent so $Cov(\bar{Y}_1, \bar{Y}_2) = 0$. We minimize the variance as a function of the constants:

$$\begin{aligned} 0 &= \frac{\partial}{\partial c_1} V(T) \\ 0 &= \frac{\partial}{\partial c_1}\left(c_1^2 V(\bar{Y}_1) + (1 - c_1)^2 V(\bar{Y}_2)\right) \\ 0 &= 2c_1 V(\bar{Y}_1) - 2(1 - c_1)V(\bar{Y}_2) \\ 0 &= 2c_1(V(\bar{Y}_1) + V(\bar{Y}_2)) - 2V(\bar{Y}_2) \\ c_1 &= \frac{V(\bar{Y}_2)}{V(\bar{Y}_1) + V(\bar{Y}_2)} \\ &\to c_2 = 1 - c_1 = \frac{V(\bar{Y}_1)}{V(\bar{Y}_1) + V(\bar{Y}_2)} \end{aligned}$$

We verify that this is the minimum:

$$\frac{\partial^2}{\partial c_1^2} V(T) = \frac{\partial}{\partial c_1} 2c_1(V(\bar{Y}_1) + V(\bar{Y}_2)) - 2V(\bar{Y}_2) = 2(V(\bar{Y}_1) + V(\bar{Y}_2)) \geq 0$$

since variance is always positive or zero.

b. to verify that $V(T) < V(\bar{Y}_1), V(\bar{Y}_2)$ we can write:

$$\begin{aligned} V(T) &= V(c_1\bar{Y}_1 + c_2\bar{Y}_2) \\ &= \left(\frac{V(\bar{Y}_2)}{V(\bar{Y}_1) + V(\bar{Y}_2)}\right)^2 V(\bar{Y}_1) + \left(\frac{V(\bar{Y}_1)}{V(\bar{Y}_1) + V(\bar{Y}_2)}\right)^2 V(\bar{Y}_2) \\ &= \frac{V(\bar{Y}_2)^2 V(\bar{Y}_1) + V(\bar{Y}_1)^2 V(\bar{Y}_2)}{\left(V(\bar{Y}_1) + V(\bar{Y}_2)\right)^2} \\ &= \frac{V(\bar{Y}_1)V(\bar{Y}_2)\left(V(\bar{Y}_2) + V(\bar{Y}_1)\right)}{\left(V(\bar{Y}_1) + V(\bar{Y}_2)\right)^2} \\ &= \frac{V(\bar{Y}_1)V(\bar{Y}_2)}{V(\bar{Y}_1) + V(\bar{Y}_2)} \end{aligned}$$

To show the inequality we can write

$$\begin{aligned} \frac{V(T)}{V(\bar{Y}_1)} &= \frac{V(\bar{Y}_2)}{V(\bar{Y}_1) + V(\bar{Y}_2)} \leq 1 \\ \frac{V(T)}{V(\bar{Y}_2)} &= \frac{V(\bar{Y}_1)}{V(\bar{Y}_1) + V(\bar{Y}_2)} \leq 1 \end{aligned}$$

Problem 3

Goldberger Qu. 11.4. Assume that Y_1 and Y_2 are independent.
We are told that we have $N = 100$ samples where n_1 comes from $Y_1 \sim N(\mu_1, 50)$ and n_2 comes from $Y_2 \sim N(\mu_2, 100)$ so that $N = n_1 + n_2$.
We are estimating $T = \mu_1 - \mu_2$ just like in question 1 so we can use $T = \bar{Y}_1 - \bar{Y}_2$ as an unbiased estimator of θ. To get the best estimation of θ we want to minimize the variance of T. So we can write:

$$
\begin{aligned}
V(T) &= V(\bar{Y}_1) + V(\bar{Y}_2) \\
&= \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2} \\
&= \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{N - n_1}
\end{aligned}
$$

Where we used the fact Y_1 and Y_2 are independent. Minimizing this expression with respect to number of samples n_1 will tell us how many samples of Y_1 we want to get the best estimate of θ:

$$
\begin{aligned}
\frac{\partial}{\partial n_1} V(T) &= \frac{\partial}{\partial n_1}\left[\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{N - n_1}\right] = 0 \\
-\frac{\sigma_1^2}{n_1^2} + \frac{\sigma_2^2}{(N - n_1)^2} &= 0 \\
\sigma_1^2 (N - n_1)^2 &= \sigma_2^2 n_1^2 \\
\sigma_2^2 n_1^2 - \sigma_1^2 (N^2 - 2Nn_1 + n_1^2) &= 0 \\
\sigma_2^2 n_1^2 - \sigma_1^2 N^2 + 2\sigma_1^2 N n_1 - \sigma_1^2 n_1^2 &= 0 \\
(\sigma_2^2 - \sigma_1^2) n_1^2 + 2\sigma_1^2 N n_1 - \sigma_1^2 N^2 &= 0 \\
\left(\frac{\sigma_2^2}{\sigma_1^2} - 1\right) n_1^2 + 2Nn_1 - N^2 &= 0 \\
\left(\frac{100}{50} - 1\right) n_1^2 + 200 n_1 - 10000 &= 0 \\
n_1^2 + 200 n_1 - 10000 &= 0 \\
&\rightarrow n_{1,1} = 41.42 \\
&\quad\; n_{1,2} = -241.42
\end{aligned}
$$

So we would want to draw 41 sample from Y_1 and 59 from Y_2. We can verify that this is a minimum:

$$
\begin{aligned}
\frac{\partial^2}{\partial n_1^2} V(T) &= \frac{\partial}{\partial n_1}\left[\left(\frac{\sigma_2^2}{\sigma_1^2} - 1\right) n_1^2 + 2Nn_1 - N^2\right] \\
&= 2\left(\frac{\sigma_2^2}{\sigma_1^2} - 1\right) n_1|_{n_1 = 41.42} + 2N \\
&= 2n_1|_{n_1 = 41.42} + 200 = 282.84 > 0
\end{aligned}
$$

Problem 4: Maximum Likelihood / ZES Estimation / GLRTs

Suppose that $X_1, \ldots, X_n$ are i.i.d. observations from the following pmf:

$$f(x \mid \theta) = \begin{cases} e^{\theta x} / \left(1 + e^{\theta}\right) & x \in \{0, 1\} \\ 0 & \text{otherwise} \end{cases}$$

where $\theta \in \mathbb{R}$.

a. Confirm that for any value of θ, this is a probability mass function.

$$\begin{aligned} \sum_x f(x|\theta) &= 1 \\ f(0|\theta) + f(1|\theta) &= 1 \\ \frac{1}{1+e^\theta} + \frac{e^\theta}{1+e^\theta} &= 1 \\ \frac{1+e^\theta}{1+e^\theta} &= 1 \ \ \forall \theta \in \mathbb{R} \end{aligned}$$

b. Write down the likelihood for one observation $f(x \mid \theta)$. Find the log-likelihood, $\ell = \log f(x \mid \theta)$.

Following the example in Goldberger 12.2, p 130:

$$\begin{aligned} L(\theta|x) &= \left(\frac{e^\theta}{1+e^\theta}\right)^x \left(\frac{1}{1+e^\theta}\right)^{1-x} \\ l(\theta|x) &= \ln\left(\frac{e^{\theta x}}{1+e^\theta}\right) = \ln e^{\theta x} - \ln\left(1+e^\theta\right) \\ &= x\theta - \ln\left(1+e^\theta\right) \end{aligned}$$

c. Find the score variable $Z = (\partial\ell/\partial\theta)$. Using the fact that $E[Z] = 0$, find $E(X)$ (see Goldberger p.128).

The score variable is:

$$Z = \frac{\partial}{\partial\theta} l(\theta|x) = x - \frac{e^\theta}{1+e^\theta}$$

and its expectation value:

$$\begin{aligned} E[Z] &= E\left[x - \frac{e^\theta}{1+e^\theta}\right] \\ 0 &= E[x] - \frac{e^\theta}{1+e^\theta} \\ E[X] &= \frac{e^\theta}{1+e^\theta} \end{aligned}$$

d. Find the maximum likelihood estimator $\hat{\theta}$ of θ based on the random sample $X_1, \ldots, X_n$.

$$\begin{aligned}
0 &= \frac{\partial}{\partial\theta} \ln l(\theta|x_1, \ldots, x_n) \\
0 &= \frac{\partial}{\partial\theta} \ln \prod_i^n l(\theta|x_i) \\
0 &= \frac{\partial}{\partial\theta} \sum_i^n \ln l(\theta|x_i) \\
0 &= \sum_i^n \frac{\partial}{\partial\theta} \ln l(\theta|x_i) \\
0 &= \sum_i^n \frac{\partial}{\partial\theta} x_i\theta - \ln(1+e^\theta) \\
0 &= \sum_i^n x_i - \frac{e^\theta}{1+e^\theta} \\
\frac{ne^\theta}{1+e^\theta} &= \sum_i^n x_i \\
ne^\theta &= \sum_i^n x_i + e^\theta \sum_i^n x_i \\
\left(n - \sum_i^n x_i\right) e^\theta &= \sum_i^n x_i \\
\theta_{MLE} &= \ln\left(\frac{\sum_i^n x_i}{n - \sum_i^n x_i}\right) \\
\theta_{MLE} &= \ln\left(\frac{n\bar{x}}{n - n\bar{x}}\right) \\
\theta_{MLE} &= \ln\left(\frac{\bar{x}}{1-\bar{x}}\right)
\end{aligned}$$

We can verify this is a maximum:

$$\begin{aligned}
\frac{\partial^2}{\partial\theta^2} \ln l(\theta|x_1, \ldots, x_n) &= \frac{\partial}{\partial\theta} \sum_i^n x_i - \frac{e^\theta}{1+e^\theta} = -n\frac{\partial}{\partial\theta}\frac{e^\theta}{1+e^\theta} \\
&= -n\left(e^\theta \frac{\partial}{\partial\theta}\frac{1}{1+e^\theta} + \frac{1}{1+e^\theta}\frac{\partial}{\partial\theta}e^\theta\right) \\
&= -n\left(e^\theta \frac{-1}{(1+e^\theta)^2}e^\theta + \frac{1}{1+e^\theta}e^\theta\right) \\
&= -n\left(\frac{-e^{2\theta}}{(1+e^\theta)^2} + \frac{e^\theta}{1+e^\theta}\right) \\
&= -n\left(\frac{-e^{2\theta} + (1+e^\theta)e^\theta}{(1+e^\theta)^2}\right) \\
&= -\frac{ne^\theta}{(1+e^\theta)^2} < 0
\end{aligned}$$

No need to evaluate at θ_{MLE} this will always be negative.

e. Derive the ZES estimator for θ. Confirm that this leads to the same estimator for θ that you obtained in (d).

By definition:

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} z_i(y_i;\theta) &= \frac{1}{n}\sum_{i=1}^{n}\frac{\partial}{\partial\theta}\ln f(y_i;\theta) = 0 \\
0 &= \frac{1}{n}\sum_{i=1}^{n}\left(x_i - \frac{e^\theta}{1+e^\theta}\right) \\
0 &= \bar{x} - \frac{e^\theta}{1+e^\theta} \\
\bar{x} &= \frac{e^\theta}{1+e^\theta} \\
e^\theta &= \bar{x}(1+e^\theta) \\
(1-\bar{x})e^\theta &= \bar{x} \\
\theta_{ZES} &= \ln\frac{\bar{x}}{1-\bar{x}}
\end{aligned}$$

f. Find the asymptotic variance of $\hat{\theta}$ (this will be a function of θ).

By definition asymptotic variance is given as :

$$V_a = \frac{1}{I(\theta)}$$

where I is the Fischer information given by:

$$\begin{aligned}
I(\theta) &= -E\left[\frac{\partial^2}{\partial\theta^2}\ln f(y_i,\theta)\right] \\
&= -E\left[\frac{\partial}{\partial\theta}x - \frac{e^\theta}{1+e^\theta}\right] \\
&= -E\left[-\left(e^\theta\frac{\partial}{\partial\theta}\frac{1}{1+e^\theta} + \frac{1}{1+e^\theta}\frac{\partial}{\partial\theta}e^\theta\right)\right] \\
&= -E\left[-\left(e^\theta\frac{-1}{(1+e^\theta)^2}e^\theta + \frac{1}{1+e^\theta}e^\theta\right)\right] \\
&= -E\left[-\left(\frac{-e^{2\theta}}{(1+e^\theta)^2} + \frac{e^\theta}{1+e^\theta}\right)\right] \\
&= -E\left[-\left(\frac{-e^{2\theta}+(1+e^\theta)e^\theta}{(1+e^\theta)^2}\right)\right] \\
&= -E\left[-\frac{e^\theta}{(1+e^\theta)^2}\right] \\
&= \frac{e^\theta}{(1+e^\theta)^2}
\end{aligned}$$

so asymptotic variance is:

$$V_a = \frac{(1+e^\theta)^2}{e^\theta}$$

g. By plugging in $\hat{\theta}$ for θ in your answer to (f), find the standard error of $\hat{\theta}$. In other words, find an estimate of the standard deviation of the estimator $\hat{\theta}$.

By definition the standard error is:

$$\begin{aligned} SE(\theta) &= \frac{1}{\sqrt{nI(\theta)}} \\ &= \sqrt{\frac{(1+e^{\ln\left(\frac{\bar{x}}{1-\bar{x}}\right)})^2}{ne^{\ln\left(\frac{\bar{x}}{1-\bar{x}}\right)}}} \\ &= \sqrt{\frac{(1+\frac{\bar{x}}{1-\bar{x}})^2}{n\frac{\bar{x}}{1-\bar{x}}}} \end{aligned}$$

h. Use your answer to (g) to construct an approximate 95% confidence interval for θ. *Hint: make sure that your interval is a function of $\hat{\theta}$, NOT the true value of θ, which is unknown.*

Following the notes from quiz section we have:

$$P\left(\theta_{MLE} - z_{1-\frac{\alpha}{2}} SE(\theta_{MLE}) \leq \theta \leq \theta_{MLE} + z_{1-\frac{\alpha}{2}} SE(\theta_{MLE})\right) = 0.95$$

$$P\left(\ln\left(\frac{\bar{x}}{1-\bar{x}}\right) - 1.96\sqrt{\frac{(1+\frac{\bar{x}}{1-\bar{x}})^2}{n\frac{\bar{x}}{1-\bar{x}}}} \leq \theta \leq \ln\left(\frac{\bar{x}}{1-\bar{x}}\right) + 1.96\sqrt{\frac{(1+\frac{\bar{x}}{1-\bar{x}})^2}{n\frac{\bar{x}}{1-\bar{x}}}}\right) = 0.95$$

Problem 5

Suppose $Y_1, \ldots, Y_n$ are an i.i.d. sample from a population with pmf given by:

$$p(y \mid \theta) = (y!)^{-1}\theta^y e^{-\theta} \tag{1}$$

where $\theta > 0$, $y_i \in \{0, 1, \ldots\}$.

(a) Write down the log-likelihood for a single observation:

$$l(\theta|y) = -\ln(y!) + y\ln(\theta) - \theta$$

(b) Using your answer to (a) find the score variable for θ:

As per Goldbergers definition in chapter 12.1, p 128:

$$Z = \frac{\partial}{\partial\theta} l(\theta|x) = \frac{\partial}{\partial\theta}\left[-\ln(y!) + y\ln(\theta) - \theta\right] = \frac{y}{\theta} - 1$$

(c) Find the information variable for θ, and find its expectation:

Information variable as defined in Goldberger 12.2, p 131:

$$W = -\frac{\partial}{\partial\theta} Z = -\frac{\partial}{\partial\theta}\left(\frac{y}{\theta} - 1\right) = -\frac{y}{\theta^2}$$

Expectation of which:

$$\begin{aligned}
E[W] &= \frac{E[Y]}{\theta^2} \\
E[Y] &= y_1 p(y_1|\theta) + y_2 p(y_2|\theta) + \ldots = e^{\theta} \sum_{i=0}^{\infty} \frac{y_i \theta^{y_i}}{y_i!} \\
&= e^{-\theta} \sum_{i=0}^{\infty} \frac{\theta^{y_i}}{(y_i - 1)!} \\
&= e^{-\theta}\theta \sum_{i=1}^{\infty} \frac{\theta^{y_i - 1}}{(y_i - 1)!} \\
&= e^{-\theta}\theta e^{\theta} \\
&= \theta \\
\rightarrow E[W] &= \frac{\theta}{\theta^2} = \frac{1}{\theta}
\end{aligned}$$

Where we noticed that the sum can be written in the form of Taylor series expansion of the exponential function.

(d) Find the maximum likelihood estimator $\hat{\theta}_{MLE}$ for θ given the sample $Y_1, \ldots, Y_n$:

$$\begin{aligned}
0 &= \frac{\partial}{\partial\theta} \ln(l(\theta|y_1, \ldots, y_n)) \\
0 &= \frac{\partial}{\partial\theta} \ln\left(\prod_{i=1}^{n} l(\theta|y_i)\right) \\
0 &= \frac{\partial}{\partial\theta} \sum_{i=1}^{n} \ln(l(\theta|y_i)) \\
0 &= \sum_{i=1}^{n} \frac{\partial}{\partial\theta} \ln(l(\theta|y_i)) \\
0 &= \sum_{i=1}^{n} \frac{y_i}{\theta} - 1 \\
0 &= \frac{1}{\theta} \sum_{i=1}^{n} y_i - n \\
n\theta &= \sum_{i=1}^{n} y_i \\
\theta_{MLE} &= \frac{1}{n} \sum_{i=1}^{n} y_i = \bar{y}
\end{aligned}$$

We can verify that this is a maximum:

$$\frac{\partial^2}{\partial\theta^2} \ln(l(\theta|y_1, \ldots, y_n)) = \frac{\partial}{\partial\theta} \frac{1}{\theta} \sum_{i=1}^{n} y_i - n = -\frac{1}{\theta^2} \sum_{i=1}^{n} y_i < 0$$

since $y_i \in \{0, 1, \ldots\}$ by definition the sum will be positive and so will θ^2 so the expression is always negative even without evaluating it at θ_{MLE}

(e) Using your answers to (c) and (d) give an approximate 90% confidence interval for θ: *Hint: your answer should be a function of* $\hat{\theta}_{MLE}$ *and* n.

In problem 4g we had the definition of SE given over via Fischer information I which is defined analogous to Golbergers W so we can determine SE:

$$SE(\theta) = \frac{1}{\sqrt{nI(\theta)}} = \frac{1}{\sqrt{nW}} = \sqrt{\frac{\theta}{n}}$$

Which gives us 90% CIs as:

$$\begin{aligned}
P\left(\bar{Y} - z_{1-\frac{\alpha}{2}} SE(\theta_{MLE}) \leq \theta \leq \bar{Y} + z_{1-\frac{\alpha}{2}} SE(\theta_{MLE})\right) &= 0.90 \\
P\left(\theta_{MLE} - 1.645\sqrt{\frac{\theta_{MLE}}{n}} \leq \theta \leq \theta_{MLE} + 1.645\sqrt{\frac{\theta_{MLE}}{n}}\right) &= 0.90
\end{aligned}$$

Problem 6

Let $X_1, \ldots, X_n$ be i.i.d. observations from a $\mathrm{N}(\mu, 1)$ population so that $f(x \mid \mu) = (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}(x-\mu)^2}$.
Hint: See quiz section notes from 12/4/20

(a) Find the MLE $\hat{\mu}_{MLE}$ for μ.

Log-likelihood for a single observation can be written as:

$$l(\theta|x) = \ln{(2\pi)^{-\frac{1}{2}}} - \ln e^{-\frac{1}{2}(x-\mu)^2}) = -\frac{1}{2}\ln 2\pi - \frac{1}{2}(x-\mu)^2$$

or for multiple samples:

$$\begin{aligned}
l(\theta|x_1 \ldots h_n) &= \ln\left(\prod_i^n f(\theta|x_i)\right) = \sum_i^n \ln f(\theta|x_i) = \\
&= \sum_i^n \left(-\frac{1}{2}\ln 2\pi - \frac{1}{2}(x_i-\mu)^2\right) \\
&= -\frac{n}{2}\ln 2\pi - \frac{1}{2}\sum_i^n (x_i-\mu)^2
\end{aligned}$$

Maximizing yields:

$$\begin{aligned}
0 &= \frac{\partial}{\partial\mu} l(\theta|x) \\
0 &= \frac{\partial}{\partial\mu}\left[-\frac{n}{2}\ln 2\pi - \frac{1}{2}\sum_i^n (x_i-\mu)^2\right] \\
0 &= -\frac{1}{2}\sum_i^n \frac{\partial}{\partial\mu}(x_i-\mu)^2 \\
0 &= \sum_i^n (x_i-\mu) \\
0 &= \sum_i^n x_i - n\mu \\
\rightarrow \mu_{MLE} &= \frac{1}{n}\sum_i^n x_i = \bar{x}
\end{aligned}$$

We can verify this is a maximum:

$$\frac{\partial^2}{\partial\mu^2} l(\theta|x) = \frac{\partial}{\partial\mu}\sum_i^n x_i - n\mu = -n < 0$$

Suppose that we wish to perform a likelihood ratio test of the hypothesis $H_0 : \mu = 0$ against $H_1 : \mu \neq 0$.

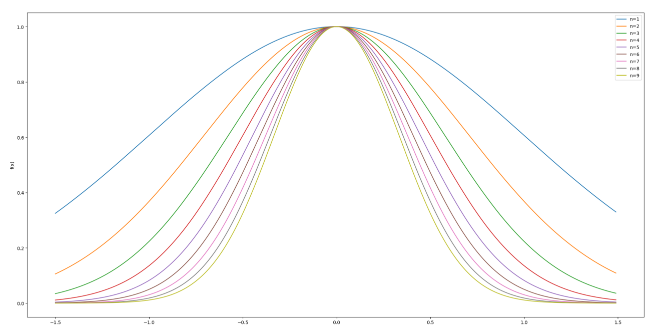
(b) Using your answer to (a) write down the generalized likelihood ratio test statistic (LRT).

$$
\begin{aligned}
\Lambda &= \frac{f(x_1, \ldots, x_n | \mu = 0)}{\sup_{\mu \in \mathbb{R}} f(x_1, \ldots, x_n | \mu)} \\
&= \frac{\prod_i^n (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2} x_i^2}}{\prod_i^n (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}(x_i - \bar{x})^2}} \\
&= \frac{\prod_i^n e^{-\frac{1}{2} x_i^2}}{\prod_i^n e^{-\frac{1}{2}(x_i - \bar{x})^2}} \\
&= \prod_i^n e^{\frac{1}{2}(x_i - \bar{x})^2 - \frac{1}{2} x_i^2} \\
&= e^{\sum_i^n \frac{1}{2}(x_i - \bar{x})^2 - \sum_i^n \frac{1}{2} x_i^2}
\end{aligned}
$$

(c) Re-express your answer to (b) as a function of $\bar{X}$, and draw the LRT as a function of $\bar{X}$.
Hint: $\sum_{i=1}^n (X_i - \bar{X})^2 = \left(\sum_{i=1}^n X_i^2\right) - n(\bar{X})^2$.

$$
\begin{aligned}
\Lambda &= e^{\sum_i^n \frac{1}{2}(x_i - \bar{x})^2 - \sum_i^n \frac{1}{2} x_i^2} \\
&= e^{\left(\sum_i^n \frac{1}{2} x_i^2\right) - \frac{n}{2}\bar{x}^2 - \sum_i^n \frac{1}{2} x_i^2} \\
&= e^{-\frac{n}{2}\bar{x}^2}
\end{aligned}
$$

Where, in line 2, we used the given hint.



```python
import numpy as np
import matplotlib.pyplot as plt


def problem6c():
    x = np.arange(-1.5, 1.5, 0.01)
    f = lambda n: np.exp(-n/2.0 * x**2)
    for i in range(1, 10):
        plt.plot(x, f(i), label=f"n={i}")
    plt.xlabel("x")
    plt.ylabel("f(x)")
    plt.legend()
    plt.show()
```

(d) If we wish to perform a hypothesis test with significance level $\alpha = 0.05$, use your answer to (c) to find the values of $\bar{X}$ for which we reject H_0. *Hint: your answer should be a function of n.*

$$
\begin{aligned}
-2\ln\Lambda &= \chi^2(1) \\
-2\ln\left(e^{-\frac{n}{2}\bar{x}^2}\right) &= \chi^2(x \le 1-\alpha, 1) \\
n\bar{x}^2 &= \chi^2(x \le 1-\alpha, 1) \\
\bar{x} &= \sqrt{\frac{\chi^2(x \le 0.95, 1)}{n}} \\
\bar{x} &= \frac{\sqrt{3.841458820694125958361}}{\sqrt{n}} \\
\bar{x} &= \frac{1.95996}{\sqrt{n}}
\end{aligned}
$$

We will reject when $\bar{x}$ is larger than the right hand side of the last expression.

Suppose that $n = 100$ and $\bar{x} = 0.16$.

(e) Using your answer to (d), would we reject H_0 in favor of H_1 using significance level $\alpha = 0.05$?

Since $\bar{x} = 0.16 < \frac{1.95996}{\sqrt{100}} = 0.195996$ we would not reject.

(f) Find the p-value for this hypothesis test.

$$
\begin{aligned}
p_{val} &= P(\Lambda > n\bar{x}^2) = 1 - \chi^2(x > n\bar{x}^2, 1) \\
&= 1 - \chi^2(x \le 100 * 0.0256, 1) \\
&= 1 - \chi^2(x \le 2.56, 1) \\
&= 0.1096
\end{aligned}
$$

Problem 7

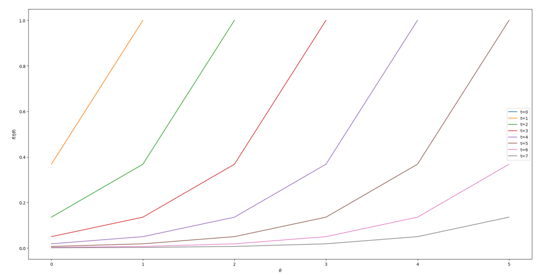
A set of times $T_1, \ldots, T_n$ are sampled independently from a population with the following density:

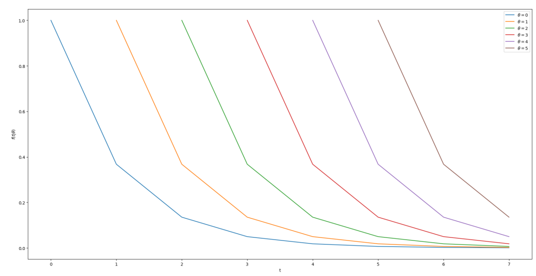
$$f(t \mid \theta) = \begin{cases} e^{-(t-\theta)} & t \geq \theta \\ 0 & \text{otherwise} \end{cases}$$

where $\theta > 0$.

(a) Find the maximum likelihood estimate for θ.

Hint: do some plots, examining the values of θ for which $f(t_1, \ldots, t_n|\theta) > 0$. It may help you first to think about the cases where $n = 1$ and $n = 2$. Do **not** *rush into differentiating anything!*





```python
import numpy as np
import matplotlib.pyplot as plt
def problem7a():
    theta = np.arange(0, 6, 1)
    t = np.arange(0, 8, 1)

    # f vs theta plot for different t
    for i in t:
        constT = np.exp(-(i -theta))
        mask = i >= theta
        plt.plot(theta[mask], constT[mask], label=f"t={i}")
    plt.xlabel(r"$\theta$")
    plt.ylabel(r"$f(t|\theta)$")
    plt.legend()
    plt.tight_layout()
    plt.show()

    # f v t plot for different theta
    for i in theta:
        constTheta = np.exp(-(t - i))
        mask = t >= i
        plt.plot(t[mask], constTheta[mask], label=fr"$\theta={i}$")
    plt.xlabel(r"t")
    plt.ylabel(r"$f(t|\theta)$")
    plt.legend()
    plt.tight_layout()
    plt.show()
```

In context of the provided motivation and right hand plot we can say that the MLE estimator for this problem will be the function:

$$\min x_1, \ldots, x_n$$

The motivation tells us that there is a minimum latency time of the system on top of which a random additional system delay time is added. The question is asking us to use this data to determine the minimum system latency. In a best case scenario, i.e. Sunday evening in a holiday season, when the system bottlenecks and queues are minimal it is conceivable that a message transmission time is equal to the system latency. But in principle we will always run into some additional time added to the system latency so in principle the closest we can get to the pure system latency is the minimal time we observe in our sample.

(b) Is there a ZES estimator for θ ? Briefly explain your answer.

By definition:

$$
\begin{aligned}
Z &= \frac{\partial}{\partial\theta} l(\theta|t) \\
&= \frac{\partial}{\partial\theta}\left(\ln \prod_{i=1}^{n} e^{-(t_i-\theta)}\right) \\
&= \frac{\partial}{\partial\theta} \ln e^{-\sum_{i=1}^{n}(t_i-\theta)} \\
&= \frac{\partial}{\partial\theta} \sum_{i=1}^{n}(\theta - t_i) \\
&= \frac{\partial}{\partial\theta}\left(n\theta - \sum_{i=1}^{n}(\theta - t_i)\right) \\
&= n \\
&\rightarrow E[Z] = E[n] = n \neq 0
\end{aligned}
$$

So it would be impossible to perform zero-expected score estimation because the estimator we want is not at a stationary point of the distribution.

[**Motivation:** (not necessary to answer the problem, but may help with intuition). For example, the observations $T_1, \ldots, T_n$ might be the observed times taken for n messages to be transmitted across a network. In this case, θ represents the (non-random) minimum time for a message to be transmitted across the network if there were no delays; the additional random component of the time $(T - \theta)$ is due to bottlenecks and queues encountered by the message.]