Homework #8

Winter 2020, STATS 509
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Problem 1: Properties of Estimators and Confidence Intervals

Goldberger Qu. 11.2
Hints: For (a) use common sense / the analogy principle; for (b) read Goldberger §10.1, p.107; for part (c) use the analogy principle and p.108; for (d) recall that the standard error of T is simply an estimate of the standard deviation of T.[1]

a. We can pick $T = \bar{X} - \bar{Y}$ since:

$$E[T] = E[\bar{X}] - E[\bar{Y}] = \mu_X - \mu_Y = \theta$$

b.

$$\begin{aligned}
V(T) &= Cov(T, T) = Cov(\bar{X} - \bar{Y}, \bar{X} - \bar{Y}) \\
&= V(\bar{X}) + V(\bar{Y}) - 2Cov(\bar{Y}, \bar{X}) \\
&= V(\bar{X}) + V(\bar{Y}) - 2Cov\left(\frac{1}{n}\sum_i^n y_i, \frac{1}{n}\sum_i^n x_i\right) \\
&= V(\bar{X}) + V(\bar{Y}) - \frac{2}{n^2}\sum_{i,j=1}^n Cov(y_i, x_j) \\
&= V(\bar{X}) + V(\bar{Y}) - \frac{2}{n^2}\sum_{i,j=1}^n \sigma_{XY} \\
&= \frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{n} - \frac{2}{n^2}n\sigma_{XY} \\
&= \frac{\sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY}}{n}
\end{aligned}$$

c. By analogy with b) we can just say:

$$V(T) = \frac{S_X^2 + S_Y^2 - 2S_{XY}}{n}$$

but following Goldberger's definitions of these values to make them unbiased:

$$V(T) = \frac{\frac{n}{n-1}S_X^2 + \frac{n}{n-1}S_Y^2 - 2\frac{n}{n-1}S_{XY}}{n} = \frac{S_X^2 + S_Y^2 - 2S_{XY}}{n-1}$$

d. For standard error of T I would report $\sqrt{V(T)}$

[1] This usage of the term 'standard error' follows Goldberger (p.123) who defines the 'standard error' of $\bar{Y}$ to be $s/\sqrt{n}$ which is an **estimate** of $\sigma/\sqrt{n}$, the standard deviation of $\bar{Y}$. (Here assuming σ is unknown.)
However, other authors use 'standard error of $\bar{Y}$' to refer to $\sigma/\sqrt{n}$; such authors will then refer to $s/\sqrt{n}$ as an **estimated** standard error. (For Goldberger, adding the word 'estimated' to 'standard error' would be redundant.)

Problem 2

Goldberger Qu. 11.3 *Hints: see p.119. For part (a), express the estimator as $T = a_1\bar{Y}_1 + a_2\bar{Y}_2$; find a constraint on a_1 and a_2 in order for T to be unbiased for μ; use this to solve for a_2 in terms of a_1; find the variance of T; substitute for a_2, and then differentiate the variance with respect to a_1.*

a. We are asked to consider $T = c_1\bar{Y}_1 + c_2\bar{Y}_2$. We want it to be unbiased so:

$$\mu = E[T] = c_1 E[\bar{Y}_1] + c_2 E[\bar{Y}_2] = c_1\mu + c_2\mu = (c_1 + c_2)\mu$$

we must conclude that $c_1 + c_2 = 1 \Longrightarrow c_2 = 1 - c_1$ if unbiased-ness is to hold true. We are asked to find the minimum variance unbiased estimator so:

$$\begin{aligned} V(T) &= V(c_1\bar{Y}_1 + c_2\bar{Y}_2) \\ &= V(c_1\bar{Y}_1) + V(c_2\bar{Y}_2) \\ &= c_1^2 V(\bar{Y}_1) + c_2^2 V(\bar{Y}_2) \\ &= c_1^2 V(\bar{Y}_1) + (1 - c_1)^2 V(\bar{Y}_2) \end{aligned}$$

where we, in line 2, used the fact that the problem tells us that the two samples are independent so $Cov(\bar{Y}_1, \bar{Y}_2) = 0$. We minimize the variance as a function of the constants:

$$\begin{aligned} 0 &= \frac{\partial}{\partial c_1} V(T) \\ 0 &= \frac{\partial}{\partial c_1} \left(c_1^2 V(\bar{Y}_1) + (1 - c_1)^2 V(\bar{Y}_2) \right) \\ 0 &= 2c_1 V(\bar{Y}_1) - 2(1 - c_1) V(\bar{Y}_2) \\ 0 &= 2c_1 (V(\bar{Y}_1) + V(\bar{Y}_2)) - 2V(\bar{Y}_2) \\ c_1 &= \frac{V(\bar{Y}_2)}{V(\bar{Y}_1) + V(\bar{Y}_2)} \\ &\rightarrow c_2 = 1 - c_1 = \frac{V(\bar{Y}_1)}{V(\bar{Y}_1) + V(\bar{Y}_2)} \end{aligned}$$

b. to verify that $V(T) < V(\bar{Y}_1), V(\bar{Y}_2)$ we can write:

$$\begin{aligned} V(T) &= V(c_1\bar{Y}_1 + c_2\bar{Y}_2) \\ &= \left(\frac{V(\bar{Y}_2)}{V(\bar{Y}_1) + V(\bar{Y}_2)} \right)^2 V(\bar{Y}_1) + \left(\frac{V(\bar{Y}_1)}{V(\bar{Y}_1) + V(\bar{Y}_2)} \right)^2 V(\bar{Y}_2) \\ &= \frac{V(\bar{Y}_2)^2 V(\bar{Y}_1) + V(\bar{Y}_1)^2 V(\bar{Y}_2)}{\left(V(\bar{Y}_1) + V(\bar{Y}_2) \right)^2} \\ &= \frac{V(\bar{Y}_1) V(\bar{Y}_2) \left(V(\bar{Y}_2) + V(\bar{Y}_1) \right)}{\left(V(\bar{Y}_1) + V(\bar{Y}_2) \right)^2} \\ &= \frac{V(\bar{Y}_1) V(\bar{Y}_2)}{V(\bar{Y}_1) + V(\bar{Y}_2)} \end{aligned}$$

To show the inequality we can write

$$\begin{aligned} \frac{V(T)}{V(\bar{Y}_1)} &= \frac{V(\bar{Y}_2)}{V(\bar{Y}_1) + V(\bar{Y}_2)} \leq 1 \\ \frac{V(T)}{V(\bar{Y}_2)} &= \frac{V(\bar{Y}_1)}{V(\bar{Y}_1) + V(\bar{Y}_2)} \leq 1 \end{aligned}$$

since variance is always positive or zero.

Problem 3

Goldberger Qu. 11.4. Assume that Y_1 and Y_2 are independent.
We are told that we have $N = 100$ samples where n_1 comes from $Y_1 \sim N(\mu_1, 50)$ and n_2 comes from $Y_2 \sim N(\mu_2, 100)$ so that $N = n_1 + n_2$.
We are estimating $T = \mu_1 - \mu_2$ just like in question 1 so we can use $T = \bar{Y}_1 - \bar{Y}_2$ as an unbiased estimator of θ. To get the best estimation of θ we want to minimize the variance of T. So we can write:

$$\begin{aligned} V(T) &= V(\bar{Y}_1) + V(\bar{Y}_2) \\ &= \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2} \\ &= \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{N - n_1} \end{aligned}$$

Where we used the fact Y_1 and Y_2 are independent. Minimizing this expression with respect to number of samples n_1 will tell us how many samples of Y_1 we want to get the best estimate of θ:

$$\begin{aligned} \frac{\partial}{\partial n_1} V(T) &= \frac{\partial}{\partial n_1}\left[\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{N - n_1}\right] = 0 \\ -\frac{\sigma_1^2}{n_1^2} + \frac{\sigma_2^2}{(N - n_1)^2} &= 0 \\ \sigma_1^2 (N - n_1)^2 &= \sigma_2^2 n_1^2 \\ \sigma_2^2 n_1^2 - \sigma_1^2 (N^2 - 2Nn_1 + n_1^2) &= 0 \\ \sigma_2^2 n_1^2 - \sigma_1^2 N^2 + 2\sigma_1^2 N n_1 - \sigma_1^2 n_1^2 &= 0 \\ (\sigma_2^2 - \sigma_1^2) n_1^2 + 2\sigma_1^2 N n_1 - \sigma_1^2 N^2 &= 0 \\ \left(\frac{\sigma_2^2}{\sigma_1^2} - 1\right) n_1^2 + 2Nn_1 - N^2 &= 0 \\ \left(\frac{100}{50} - 1\right) n_1^2 + 200 n_1 - 10000 &= 0 \\ n_1^2 + 200 n_1 - 10000 &= 0 \\ \rightarrow n_{1,1} &= 41.42 \\ n_{1,2} &= -241.42 \end{aligned}$$

So we would want to draw 41 sample from Y_1 and 59 from Y_2.

Problem 4: Maximum Likelihood / ZES Estimation / GLRTs

Suppose that $X_1, \ldots, X_n$ are i.i.d. observations from the following pmf:

$$f(x \mid \theta) = \begin{cases} e^{\theta x} / \left(1 + e^{\theta}\right) & x \in \{0, 1\} \\ 0 & \text{otherwise} \end{cases}$$

where $\theta \in \mathbb{R}$.

a. Confirm that for any value of θ, this is a probability mass function.

$$\begin{aligned} \sum_x f(x|\theta) &= 1 \\ f(0|\theta) + f(1|\theta) &= 1 \\ \frac{1}{1+e^\theta} + \frac{e^\theta}{1+e^\theta} &= 1 \\ \frac{1+e^\theta}{1+e^\theta} &= 1 \ \ \forall \theta \in \mathbb{R} \end{aligned}$$

b. Write down the likelihood for one observation $f(x \mid \theta)$. Find the log-likelihood, $\ell = \log f(x \mid \theta)$.

Following the example in Goldberger 12.2, p 130:

$$\begin{aligned} L(\theta|x) &= \left(\frac{e^\theta}{1+e^\theta}\right)^x \left(\frac{1}{1+e^\theta}\right)^{1-x} \\ l(\theta|x) &= \ln\left(\frac{e^{\theta x}}{1+e^\theta}\right) = \ln e^{\theta x} - \ln\left(1+e^\theta\right) \\ &= x\theta - \ln\left(1+e^\theta\right) \end{aligned}$$

c. Find the score variable $Z = (\partial \ell / \partial \theta)$. Using the fact that $E[Z] = 0$, find $E(X)$ (see Goldberger p.128).

The score variable is:

$$Z = \frac{\partial}{\partial \theta} l(\theta|x) = x - \frac{e^\theta}{1+e^\theta}$$

and its expectation value:

$$\begin{aligned} E[Z] &= E\left[x - \frac{e^\theta}{1+e^\theta}\right] \\ 0 &= E[x] - \frac{e^\theta}{1+e^\theta} \\ E[X] &= \frac{e^\theta}{1+e^\theta} \end{aligned}$$

d. Find the maximum likelihood estimator $\hat{\theta}$ of θ based on the random sample $X_1, \ldots, X_n$.

$$
\begin{aligned}
0 &= \frac{\partial}{\partial\theta} \ln l(\theta|x_1, \ldots, x_n) \\
0 &= \frac{\partial}{\partial\theta} \ln \prod_i^n l(\theta|x_i) \\
0 &= \frac{\partial}{\partial\theta} \sum_i^n \ln l(\theta|x_i) \\
0 &= \sum_i^n \frac{\partial}{\partial\theta} \ln l(\theta|x_i) \\
0 &= \sum_i^n \frac{\partial}{\partial\theta} x_i\theta - \ln\left(1 + e^\theta\right) \\
0 &= \sum_i^n x_i - \frac{e^\theta}{1+e^\theta} \\
\frac{ne^\theta}{1+e^\theta} &= \sum_i^n x_i \\
ne^\theta &= \sum_i^n x_i + e^\theta \sum_i^n x_i \\
\left(n - \sum_i^n x_i\right) e^\theta &= \sum_i^n x_i \\
\theta_{MLE} &= \ln\left(\frac{\sum_i^n x_i}{n - \sum_i^n x_i}\right) \\
\theta_{MLE} &= \ln\left(\frac{n\bar{x}}{n - n\bar{x}}\right) \\
\theta_{MLE} &= \ln\left(\frac{\bar{x}}{1-\bar{x}}\right)
\end{aligned}
$$

e. Derive the ZES estimator for θ. Confirm that this leads to the same estimator for θ that you obtained in (d).

By definition:

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} z_i(y_i;\theta) &= \frac{1}{n}\sum_{i=1}^{n} \frac{\partial}{\partial\theta} \ln f(y_i;\theta = 0 \\
0 &= \frac{1}{n}\sum_{i=1}^{n}\left(x_i - \frac{e^\theta}{1+e^\theta}\right) \\
0 &= \bar{x} - \frac{e^\theta}{1+e^\theta} \\
\bar{x} &= \frac{e^\theta}{1+e^\theta} \\
e^\theta &= \bar{x}(1+e^\theta) \\
(1-\bar{x})e^\theta &= \bar{x} \\
\theta_{ZES} &= \ln\frac{\bar{x}}{1-\bar{x}}
\end{aligned}$$

f. Find the asymptotic variance of $\hat{\theta}$ (this will be a function of θ).

g. By plugging in $\hat{\theta}$ for θ in your answer to (f), find the standard error of $\hat{\theta}$. In other words, find an estimate of the standard deviation of the estimator $\hat{\theta}$.

h. Use your answer to (g) to construct an approximate 95% confidence interval for θ. *Hint: make sure that your interval is a function of $\hat{\theta}$, NOT the true value of θ, which is unknown.*

Problem 5

Suppose $Y_1, \ldots, Y_n$ are an i.i.d. sample from a population with pmf given by:

$$p(y \mid \theta) = (y!)^{-1}\theta^{y}e^{-\theta} \tag{1}$$

where $\theta > 0$, $y_i \in \{0, 1, \ldots\}$.

(a) Write down the log-likelihood for a single observation:

$$l(\theta|y) = -\ln(y!) + y\ln(\theta) - \theta$$

(b) Using your answer to (a) find the score variable for θ:
As per Goldbergers definition in chapter 12.1, p 128:

$$Z = \frac{\partial}{\partial\theta} l(\theta|x) = \frac{\partial}{\partial\theta}\left[-\ln(y!) + y\ln(\theta) - \theta\right] = \frac{y}{\theta} - 1$$

(c) Find the information variable for θ, and find its expectation:
Information variable as defined in Goldberger 12.2, p 131:

$$W = -\frac{\partial}{\partial\theta} Z = -\frac{\partial}{\partial\theta}\left(\frac{y}{\theta} - 1\right) = -\frac{y}{\theta^2}$$

Expectation of which:

$$E[W] = \frac{E[y]}{\theta^2}$$

(d) Find the maximum likelihood estimator $\hat{\theta}_{MLE}$ for θ given the sample $Y_1, \ldots, Y_n$:

$$\begin{aligned}
0 &= \frac{\partial}{\partial\theta}\ln(l(\theta|y_1, \ldots, y_n)) \\
0 &= \frac{\partial}{\partial\theta}\ln\left(\prod_{i=1}^{n} l(\theta|y_i)\right) \\
0 &= \frac{\partial}{\partial\theta}\sum_{i=1}^{n}\ln(l(\theta|y_i)) \\
0 &= \sum_{i=1}^{n}\frac{\partial}{\partial\theta}\ln(l(\theta|y_i)) \\
0 &= \sum_{i=1}^{n}\frac{y_i}{\theta} - 1 \\
0 &= \frac{1}{\theta}\sum_{i=1}^{n} y_i - n \\
n\theta &= \sum_{i=1}^{n} y_i \\
\theta_{MLE} &= \frac{1}{n}\sum_{i=1}^{n} y_i = \bar{y}
\end{aligned}$$

(e) Using your answers to (c) and (d) give an approximate 90% confidence interval for θ: *Hint: your answer should be a function of $\hat{\theta}_{MLE}$ and n.*

$$\begin{aligned}
P\left(\bar{Y} - 1.96\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{Y} + 1.96\frac{\sigma}{\sqrt{n}}\right) &= 0.90 \\
P\left(\theta_{MLE} - 1.96\frac{\sigma}{\sqrt{n}} \le \mu \le \theta_{MLE} + 1.96\frac{\sigma}{\sqrt{n}}\right) &= 0.90
\end{aligned}$$

Problem 6

Let $X_1, \dots, X_n$ be i.i.d. observations from a $\mathrm{N}(\mu, 1)$ population so that $f(x \mid \mu) = (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}(x-\mu)^2}$.
Hint: See quiz section notes from 12/4/20

(a) Find the MLE $\hat{\mu}_{MLE}$ for μ.

Log-likelihood for a single observation can be written as:

$$l(\theta|x) = \ln{(2\pi)^{-\frac{1}{2}}} - \ln{e^{-\frac{1}{2}(x-\mu)^2}}) = -\frac{1}{2}\ln 2\pi - \frac{1}{2}(x-\mu)^2$$

or for multiple samples:

$$\begin{aligned} l(\theta|x_1 \dots h_n) &= \ln\left(\prod_i^n f(\theta|x_i)\right) = \sum_i^n \ln f(\theta|x_i) = \\ &= \sum_i^n \left(-\frac{1}{2}\ln 2\pi - \frac{1}{2}(x_i-\mu)^2\right) \\ &= -\frac{n}{2}\ln 2\pi - \frac{1}{2}\sum_i^n (x_i-\mu)^2 \end{aligned}$$

Minimizing yields:

$$\begin{aligned} 0 &= \frac{\partial}{\partial\mu} l(\theta|x) \\ 0 &= \frac{\partial}{\partial\mu}\left[-\frac{n}{2}\ln 2\pi - \frac{1}{2}\sum_i^n (x_i-\mu)^2\right] \\ 0 &= -\frac{1}{2}\sum_i^n \frac{\partial}{\partial\mu}(x_i-\mu)^2 \\ 0 &= \sum_i^n (x_i-\mu) \\ 0 &= \sum_i^n x_i - n\mu \\ \rightarrow \mu_{MLE} &= \frac{1}{n}\sum_i^n x_i = \bar{x} \end{aligned}$$

Suppose that we wish to perform a likelihood ratio test of the hypothesis $H_0 : \mu = 0$ against $H_1 : \mu \neq 0$.

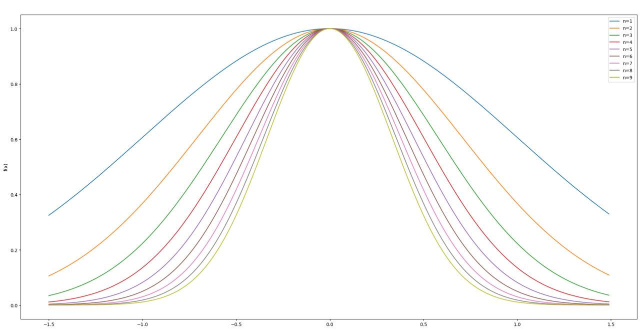
(b) Using your answer to (a) write down the generalized likelihood ratio test statistic (LRT).

$$\begin{aligned}
\Lambda &= \frac{f(x_1, \dots, x_n | \mu = 0)}{\sup_{\mu \in \mathbb{R}} f(x_1, \dots, x_n | \mu)} \\
&= \frac{\prod_i^n (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}x_i^2}}{\prod_i^n (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}(x_i - \bar{x})^2}} \\
&= \frac{\prod_i^n e^{-\frac{1}{2}x_i^2}}{\prod_i^n e^{-\frac{1}{2}(x_i - \bar{x})^2}} \\
&= \prod_i^n e^{\frac{1}{2}(x_i - \bar{x})^2 - \frac{1}{2}x_i^2} \\
&= e^{\sum_i^n \frac{1}{2}(x_i - \bar{x})^2 - \sum_i^n \frac{1}{2}x_i^2}
\end{aligned}$$

(c) Re-express your answer to (b) as a function of $\bar{X}$, and draw the LRT as a function of $\bar{X}$.
Hint: $\sum_{i=1}^n (X_i - \bar{X})^2 = \left(\sum_{i=1}^n X_i^2\right) - n(\bar{X})^2$.

$$\begin{aligned}
\Lambda &= e^{\sum_i^n \frac{1}{2}(x_i - \bar{x})^2 - \sum_i^n \frac{1}{2}x_i^2} \\
&= e^{\left(\sum_i^n \frac{1}{2}x_i^2\right) - \frac{n}{2}\bar{x}^2 - \sum_i^n \frac{1}{2}x_i^2} \\
&= e^{-\frac{n}{2}\bar{x}^2}
\end{aligned}$$

Where, in line 2, we used the given hint.



```python
import numpy as np
import matplotlib.pyplot as plt


def problem6c():
    x = np.arange(-1.5, 1.5, 0.01)
    f = lambda n: np.exp(-n/2.0 * x**2)
    for i in range(1, 10):
        plt.plot(x, f(i), label=f"n={i}")
    plt.xlabel("x")
    plt.ylabel("f(x)")
    plt.legend()
    plt.show()


if __name__ == "__main__":
    problem6c()
```

(d) If we wish to perform a hypothesis test with significance level $\alpha = 0.05$, use your answer to (c) to find the values of $\bar{X}$ for which we reject H_0. *Hint: your answer should be a function of* n.

$$\begin{aligned}
-2\ln\Lambda &= \chi^2(1) \\
-2\ln\left(e^{-\frac{n}{2}\bar{x}^2}\right) &= \chi^2(x \leq 1-\alpha, 1) \\
n\bar{x}^2 &= \chi^2(x \leq 1-\alpha, 1) \\
\bar{x} &= \sqrt{\frac{\chi^2(x \leq 0.95, 1)}{n}} \\
\bar{x} &= \frac{\sqrt{3.841458820694125958361}}{\sqrt{n}} \\
\bar{x} &= \frac{1.95996}{\sqrt{n}}
\end{aligned}$$

We will reject when $\bar{x}$ is larger than the right hand side of the last expression.

Suppose that $n = 100$ and $\bar{x} = 0.16$.

(e) Using your answer to (d), would we reject H_0 in favor of H_1 using significance level $\alpha = 0.05$?

Since $\bar{x} = 0.16 < \frac{1.95996}{\sqrt{100}} = 0.195996$ we would not reject.

(f) Find the p-value for this hypothesis test.

$$\begin{aligned}
p_{val} &= P(\bar{X} > \bar{x} | H_0) \\
&=
\end{aligned}$$

Problem 7

A set of times $T_1, \ldots, T_n$ are sampled independently from a population with the following density:

$$f(t \mid \theta) = \begin{cases} e^{-(t-\theta)} & t \geq \theta \\ 0 & \text{otherwise} \end{cases}$$

where $\theta > 0$.

(a) Find the maximum likelihood estimate for θ.

Hint: do some plots, examining the values of θ for which $f(t_1, \ldots, t_n | \theta) > 0$. It may help you first to think about the cases where $n = 1$ and $n = 2$. Do **not** *rush into differentiating anything!*

(b) Is there a ZES estimator for θ ? Briefly explain your answer.

[**Motivation:** (not necessary to answer the problem, but may help with intuition). For example, the observations $T_1, \ldots, T_n$ might be the observed times taken for n messages to be transmitted across a network. In this case, θ represents the (non-random) minimum time for a message to be transmitted across the network if there were no delays; the additional random component of the time $(T - \theta)$ is due to bottlenecks and queues encountered by the message.]