

Homework #8

Winter 2020, STATS 509

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Problem 1: Properties of Estimators and Confidence Intervals

Goldberger Qu. 11.2

Hints: For (a) use common sense / the analogy principle; for (b) read Goldberger §10.1, p.107; for part (c) use the analogy principle and p.108; for (d) recall that the standard error of T is simply an estimate of the standard deviation of T .¹

a. We can pick $T = \bar{X} - \bar{Y}$ since:

$$E[T] = E[\bar{X}] - E[\bar{Y}] = \mu_X - \mu_Y = \theta$$

b.

$$\begin{aligned} V(T) &= Cov(T, T) = Cov(\bar{X} - \bar{Y}, \bar{X} - \bar{Y}) \\ &= V(\bar{X}) + V(\bar{Y}) - 2Cov(\bar{Y}, \bar{X}) \\ &= V(\bar{X}) + V(\bar{Y}) - 2Cov\left(\frac{1}{n} \sum_i y_i, \frac{1}{n} \sum_i x_i\right) \\ &= V(\bar{X}) + V(\bar{Y}) - \frac{2}{n^2} \sum_{i,j=1}^n Cov(y_i, x_j) \\ &= V(\bar{X}) + V(\bar{Y}) - \frac{2}{n^2} \sum_{i,j=1}^n \sigma_{XY} \\ &= \frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{n} - \frac{2}{n^2} n \sigma_{XY} \\ &= \frac{\sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY}}{n} \end{aligned}$$

c. By analogy with b) we can just say:

$$V(T) = \frac{S_X^2 + S_Y^2 - 2S_{XY}}{n}$$

but following Goldberger's definitions of these values to make them unbiased:

$$V(T) = \frac{\frac{n}{n-1} S_X^2 + \frac{n}{n-1} S_Y^2 - 2 \frac{n}{n-1} S_{XY}}{n} = \frac{S_X^2 + S_Y^2 - 2S_{XY}}{n-1}$$

d. For standard error of T I would report $\sqrt{V(T)}$

¹This usage of the term 'standard error' follows Goldberger (p.123) who defines the 'standard error' of \bar{Y} to be s/\sqrt{n} which is an **estimate** of σ/\sqrt{n} , the standard deviation of \bar{Y} . (Here assuming σ is unknown.)

However, other authors use 'standard error of \bar{Y} ' to refer to σ/\sqrt{n} ; such authors will then refer to s/\sqrt{n} as an **estimated** standard error. (For Goldberger, adding the word 'estimated' to 'standard error' would be redundant.)

Problem 2

Goldberger Qu. 11.3 *Hints: see p.119. For part (a), express the estimator as $T = a_1\bar{Y}_1 + a_2\bar{Y}_2$; find a constraint on a_1 and a_2 in order for T to be unbiased for μ ; use this to solve for a_2 in terms of a_1 ; find the variance of T ; substitute for a_2 , and then differentiate the variance with respect to a_1 .*

- a. We are asked to consider $T = c_1\bar{Y}_1 + c_2\bar{Y}_2$. We want it to be unbiased so:

$$\mu = E[T] = c_1E[\bar{Y}_1] + c_2E[\bar{Y}_2] = c_1\mu + c_2\mu = (c_1 + c_2)\mu$$

we must conclude that $c_1 + c_2 = 1 \implies c_2 = 1 - c_1$ if unbiased-ness is to hold true. We are asked to find the minimum variance unbiased estimator so:

$$\begin{aligned} V(T) &= V(c_1\bar{Y}_1 + c_2\bar{Y}_2) \\ &= V(c_1\bar{Y}_1) + V(c_2\bar{Y}_2) \\ &= c_1^2V(\bar{Y}_1) + c_2^2V(\bar{Y}_2) \\ &= c_1^2V(\bar{Y}_1) + (1 - c_1)^2V(\bar{Y}_2) \end{aligned}$$

where we, in line 2, used the fact that the problem tells us that the two samples are independent so $Cov(\bar{Y}_1, \bar{Y}_2) = 0$. We minimize the variance as a function of the constants:

$$\begin{aligned} 0 &= \frac{\partial}{\partial c_1} V(T) \\ 0 &= \frac{\partial}{\partial c_1} (c_1^2V(\bar{Y}_1) + (1 - c_1)^2V(\bar{Y}_2)) \\ 0 &= 2c_1V(\bar{Y}_1) - 2(1 - c_1)V(\bar{Y}_2) \\ 0 &= 2c_1(V(\bar{Y}_1) + V(\bar{Y}_2)) - 2V(\bar{Y}_2) \\ c_1 &= \frac{V(\bar{Y}_2)}{V(\bar{Y}_1) + V(\bar{Y}_2)} \\ \rightarrow c_2 &= 1 - c_1 = \frac{V(\bar{Y}_1)}{V(\bar{Y}_1) + V(\bar{Y}_2)} \end{aligned}$$

- b. to verify that $V(T) < V(\bar{Y}_1), V(\bar{Y}_2)$ we can write:

$$\begin{aligned} V(T) &= V(c_1\bar{Y}_1 + c_2\bar{Y}_2) \\ &= \left(\frac{V(\bar{Y}_2)}{V(\bar{Y}_1) + V(\bar{Y}_2)} \right)^2 V(\bar{Y}_1) + \left(\frac{V(\bar{Y}_1)}{V(\bar{Y}_1) + V(\bar{Y}_2)} \right)^2 V(\bar{Y}_2) \\ &= \frac{V(\bar{Y}_2)^2V(\bar{Y}_1) + V(\bar{Y}_1)^2V(\bar{Y}_2)}{(V(\bar{Y}_1) + V(\bar{Y}_2))^2} \\ &= \frac{V(\bar{Y}_1)V(\bar{Y}_2)(V(\bar{Y}_2) + V(\bar{Y}_1))}{(V(\bar{Y}_1) + V(\bar{Y}_2))^2} \\ &= \frac{V(\bar{Y}_1)V(\bar{Y}_2)}{V(\bar{Y}_1) + V(\bar{Y}_2)} \end{aligned}$$

To show the inequality we can write

$$\begin{aligned} \frac{V(T)}{V(\bar{Y}_1)} &= \frac{V(\bar{Y}_2)}{V(\bar{Y}_1) + V(\bar{Y}_2)} \leq 1 \\ \frac{V(T)}{V(\bar{Y}_2)} &= \frac{V(\bar{Y}_1)}{V(\bar{Y}_1) + V(\bar{Y}_2)} \leq 1 \end{aligned}$$

since variance is always positive or zero.

Problem 3

Goldberger Qu. 11.4. Assume that Y_1 and Y_2 are independent.

We are told that we have $N = 100$ samples where n_1 comes from $Y_1 \sim N(\mu_1, 50)$ and n_2 comes from $Y_2 \sim N(\mu_2, 100)$ so that $N = n_1 + n_2$.

We are estimating $T = \mu_1 - \mu_2$ just like in question 1 so we can use $T = \bar{Y}_1 - \bar{Y}_2$ as an unbiased estimator of θ . To get the best estimation of θ we want to minimize the variance of T . So we can write:

$$\begin{aligned} V(T) &= V(\bar{Y}_1) + V(\bar{Y}_2) \\ &= \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2} \\ &= \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{N - n_1} \end{aligned}$$

Where we used the fact Y_1 and Y_2 are independent. Minimizing this expression with respect to number of samples n_1 will tell us how many samples of Y_1 we want to get the best estimate of θ :

$$\begin{aligned} \frac{\partial}{\partial n_1} V(T) &= \frac{\partial}{\partial n_1} \left[\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{N - n_1} \right] = 0 \\ -\frac{\sigma_1^2}{n_1^2} + \frac{\sigma_2^2}{(N - n_1)^2} &= 0 \\ \sigma_1^2(N - n_1)^2 &= \sigma_2^2 n_1^2 \\ \sigma_2^2 n_1^2 - \sigma_1^2(N^2 - 2Nn_1 + n_1^2) &= 0 \\ \sigma_2^2 n_1^2 - \sigma_1^2 N^2 + 2\sigma_1^2 Nn_1 - \sigma_1^2 n_1^2 &= 0 \\ (\sigma_2^2 - \sigma_1^2)n_1^2 + 2\sigma_1^2 Nn_1 - \sigma_1^2 N^2 &= 0 \\ \left(\frac{\sigma_2^2}{\sigma_1^2} - 1 \right) n_1^2 + 2Nn_1 - N^2 &= 0 \\ \left(\frac{100}{50} - 1 \right) n_1^2 + 200n_1 - 10000 &= 0 \\ n_1^2 + 200n_1 - 10000 &= 0 \\ \rightarrow n_{1,1} &= 41.42 \\ n_{1,2} &= -241.42 \end{aligned}$$

So we would want to draw 41 sample from Y_1 and 59 from Y_2 .

Problem 4: Maximum Likelihood / ZES Estimation / GLRTs

Suppose that X_1, \dots, X_n are i.i.d. observations from the following pmf:

$$f(x | \theta) = \begin{cases} e^{\theta x} / (1 + e^{\theta}) & x \in \{0, 1\} \\ 0 & \text{otherwise} \end{cases}$$

where $\theta \in \mathbb{R}$.

- a. Confirm that for any value of θ , this is a probability mass function.

$$\begin{aligned} \sum_x f(x|\theta) &= 1 \\ f(0|\theta) + f(1|\theta) &= 1 \\ \frac{1}{1 + e^{\theta}} + \frac{e^{\theta}}{1 + e^{\theta}} &= 1 \\ \frac{1 + e^{\theta}}{1 + e^{\theta}} &= 1 \quad \forall \theta \in \mathbb{R} \end{aligned}$$

- b. Write down the likelihood for one observation $f(x | \theta)$. Find the log-likelihood, $\ell = \log f(x | \theta)$.

Following the example in Goldberger 12.2, p 130:

$$\begin{aligned} L(\theta|x) &= \left(\frac{e^{\theta}}{1 + e^{\theta}} \right)^x \left(\frac{1}{1 + e^{\theta}} \right)^{1-x} \\ l(\theta|x) &= \ln \left(\frac{e^{\theta x}}{1 + e^{\theta}} \right) = \ln e^{\theta x} - \ln(1 + e^{\theta}) \\ &= x\theta - \ln(1 + e^{\theta}) \end{aligned}$$

- c. Find the score variable $Z = (\partial \ell / \partial \theta)$. Using the fact that $E[Z] = 0$, find $E(X)$ (see Goldberger p.128).

The score variable is:

$$Z = \frac{\partial}{\partial \theta} l(\theta|x) = x - \frac{e^{\theta}}{1 + e^{\theta}}$$

and its expectation value:

$$\begin{aligned} E[Z] &= E \left[x - \frac{e^{\theta}}{1 + e^{\theta}} \right] \\ 0 &= E[x] - \frac{e^{\theta}}{1 + e^{\theta}} \\ E[X] &= \frac{e^{\theta}}{1 + e^{\theta}} \end{aligned}$$

d. Find the maximum likelihood estimator $\hat{\theta}$ of θ based on the random sample X_1, \dots, X_n .

$$\begin{aligned}
 0 &= \frac{\partial}{\partial \theta} \ln l(\theta | x_1, \dots, x_n) \\
 0 &= \frac{\partial}{\partial \theta} \ln \prod_i^n l(\theta | x_i) \\
 0 &= \frac{\partial}{\partial \theta} \sum_i^n \ln l(\theta | x_i) \\
 0 &= \sum_i^n \frac{\partial}{\partial \theta} \ln l(\theta | x_i) \\
 0 &= \sum_i^n \frac{\partial}{\partial \theta} x_i \theta - \ln(1 + e^\theta) \\
 0 &= \sum_i^n x_i - \frac{e^\theta}{1 + e^\theta} \\
 \frac{ne^\theta}{1 + e^\theta} &= \sum_i^n x_i \\
 ne^\theta &= \sum_i^n x_i + e^\theta \sum_i^n x_i \\
 \left(n - \sum_i^n x_i\right) e^\theta &= \sum_i^n x_i \\
 \theta_{MLE} &= \ln \left(\frac{\sum_i^n x_i}{n - \sum_i^n x_i} \right) \\
 \theta_{MLE} &= \ln \left(\frac{n\bar{x}}{n - n\bar{x}} \right) \\
 \theta_{MLE} &= \ln \left(\frac{\bar{x}}{1 - \bar{x}} \right)
 \end{aligned}$$

- e. Derive the ZES estimator for θ . Confirm that this leads to the same estimator for θ that you obtained in (d).
- f. Find the asymptotic variance of $\hat{\theta}$ (this will be a function of θ).
- g. By plugging in $\hat{\theta}$ for θ in your answer to (f), find the standard error of $\hat{\theta}$. In other words, find an estimate of the standard deviation of the estimator $\hat{\theta}$.
- h. Use your answer to (g) to construct an approximate 95% confidence interval for θ . *Hint: make sure that your interval is a function of $\hat{\theta}$, NOT the true value of θ , which is unknown.*

Problem 5

Suppose Y_1, \dots, Y_n are an i.i.d. sample from a population with pmf given by:

$$p(y | \theta) = (y!)^{-1} \theta^y e^{-\theta} \quad (1)$$

where $\theta > 0$, $y_i \in \{0, 1, \dots\}$.

- (a) Write down the log-likelihood for a single observation:

$$l(\theta | y) = -\ln(y!) + y \ln(\theta) - \theta$$

- (b) Using your answer to (a) find the score variable for θ :

As per Goldbergers definition in chapter 12.1, p 128:

$$Z = \frac{\partial}{\partial \theta} l(\theta | x) = \frac{\partial}{\partial \theta} [-\ln(y!) + y \ln(\theta) - \theta] = \frac{y}{\theta} - 1$$

- (c) Find the information variable for θ , and find its expectation:

Information variable as defined in Goldberger 12.2, p 131:

$$W = -\frac{\partial}{\partial \theta} Z = -\frac{\partial}{\partial \theta} \left(\frac{y}{\theta} - 1 \right) = -\frac{y}{\theta^2}$$

Expectation of which:

$$E[W] = \frac{E[y]}{\theta^2}$$

- (d) Find the maximum likelihood estimator $\hat{\theta}_{MLE}$ for θ given the sample Y_1, \dots, Y_n :

$$0 = \frac{\partial}{\partial \theta} \ln(l(\theta | y_1, \dots, y_n))$$

$$0 = \frac{\partial}{\partial \theta} \ln \left(\prod_{i=1}^n l(\theta | y_i) \right)$$

$$0 = \frac{\partial}{\partial \theta} \sum_{i=1}^n \ln(l(\theta | y_i))$$

$$0 = \sum_{i=1}^n \frac{\partial}{\partial \theta} \ln(l(\theta | y_i))$$

$$0 = \sum_{i=1}^n \frac{y_i}{\theta} - n$$

$$0 = \frac{1}{\theta} \sum_{i=1}^n y_i - n$$

$$n\theta = \sum_{i=1}^n y_i$$

$$\theta_{MLE} = \frac{1}{n} \sum_{i=1}^n y_i = \bar{y}$$

- (e) Using your answers to (c) and (d) give an approximate 90% confidence interval for θ : *Hint: your answer should be a function of $\hat{\theta}_{MLE}$ and n .*

$$P \left(\bar{Y} - 1.96 \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{Y} + 1.96 \frac{\sigma}{\sqrt{n}} \right) = 0.90$$

$$P \left(\theta_{MLE} - 1.96 \frac{\sigma}{\sqrt{n}} \leq \mu \leq \theta_{MLE} + 1.96 \frac{\sigma}{\sqrt{n}} \right) = 0.90$$

Problem 6

Let X_1, \dots, X_n be i.i.d. observations from a $N(\mu, 1)$ population so that $f(x|\mu) = (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}(x-\mu)^2}$.
Hint: See quiz section notes from 12/4/20

- (a) Find the MLE $\hat{\mu}_{MLE}$ for μ .

Log-likelihood for a single observation can be written as:

$$l(\theta|x) = \ln(2\pi)^{-\frac{1}{2}} - \ln e^{-\frac{1}{2}(x-\mu)^2} = -\frac{1}{2} \ln 2\pi - \frac{1}{2}(x-\mu)^2$$

or for multiple samples:

$$\begin{aligned} l(\theta|x_1 \dots x_n) &= \ln \left(\prod_i^n f(\theta|x_i) \right) = \sum_i^n \ln f(\theta|x_i) = \\ &= \sum_i^n \left(-\frac{1}{2} \ln 2\pi - \frac{1}{2}(x_i - \mu)^2 \right) \\ &= -\frac{n}{2} \ln 2\pi - \frac{1}{2} \sum_i^n (x_i - \mu)^2 \end{aligned}$$

Minimizing yields:

$$\begin{aligned} 0 &= \frac{\partial}{\partial \mu} l(\theta|x) \\ 0 &= \frac{\partial}{\partial \mu} \left[-\frac{n}{2} \ln 2\pi - \frac{1}{2} \sum_i^n (x_i - \mu)^2 \right] \\ 0 &= -\frac{1}{2} \sum_i^n \frac{\partial}{\partial \mu} (x_i - \mu)^2 \\ 0 &= \sum_i^n (x_i - \mu) \\ 0 &= \sum_i^n x_i - n\mu \\ \rightarrow \mu_{MLE} &= \frac{1}{n} \sum_i^n x_i = \bar{x} \end{aligned}$$

Suppose that we wish to perform a likelihood ratio test of the hypothesis $H_0 : \mu = 0$ against $H_1 : \mu \neq 0$.

(b) Using your answer to (a) write down the generalized likelihood ratio test statistic (LRT).

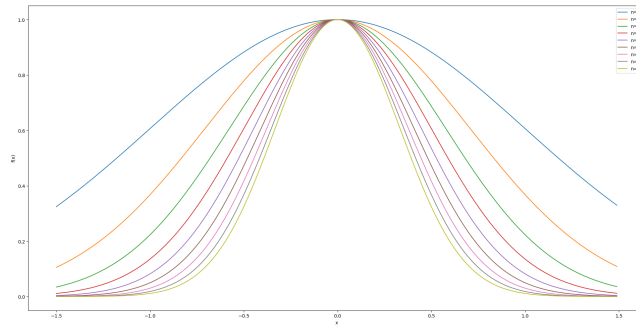
$$\begin{aligned}\Lambda &= \frac{f(x_1, \dots, x_n | \mu = 0)}{\sup_{\mu \in \mathbb{R}} f(x_1, \dots, x_n | \mu)} \\ &= \frac{\prod_i^n (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2} x_i^2}}{\prod_i^n (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2} (x_i - \bar{x})^2}} \\ &= \frac{\prod_i^n e^{-\frac{1}{2} x_i^2}}{\prod_i^n e^{-\frac{1}{2} (x_i - \bar{x})^2}} \\ &= \prod_i^n e^{\frac{1}{2} (x_i - \bar{x})^2 - \frac{1}{2} x_i^2} \\ &= e^{\sum_i^n \frac{1}{2} (x_i - \bar{x})^2 - \sum_i^n \frac{1}{2} x_i^2}\end{aligned}$$

(c) Re-express your answer to (b) as a function of \bar{X} , and draw the LRT as a function of \bar{X} .

Hint: $\sum_{i=1}^n (X_i - \bar{X})^2 = (\sum_{i=1}^n X_i^2) - n(\bar{X})^2$.

$$\begin{aligned}\Lambda &= e^{\sum_i^n \frac{1}{2} (x_i - \bar{x})^2 - \sum_i^n \frac{1}{2} x_i^2} \\ &= e^{(\sum_i^n \frac{1}{2} x_i^2) - \frac{n}{2} \bar{x}^2 - \sum_i^n \frac{1}{2} x_i^2} \\ &= e^{-\frac{n}{2} \bar{x}^2}\end{aligned}$$

Where, in line 2, we used the given hint.



```
import numpy as np
import matplotlib.pyplot as plt

def problem6c():
    x = np.arange(-1.5, 1.5, 0.01)
    f = lambda n: np.exp(-n/2.0 * x**2)
    for i in range(1, 10):
        plt.plot(x, f(i), label=f"n={i}")
    plt.xlabel("x")
    plt.ylabel("f(x)")
    plt.legend()
    plt.show()

if __name__ == "__main__":
    problem6c()
```


- (d) If we wish to perform a hypothesis test with significance level $\alpha = 0.05$, use your answer to (c) to find the values of \bar{X} for which we reject H_0 . *Hint: your answer should be a function of n .*

$$\begin{aligned}
 -2 \ln \Lambda &= \chi^2(1) \\
 -2 \ln \left(e^{-\frac{n}{2} \bar{x}^2} \right) &= \chi^2(x \leq 1 - \alpha, 1) \\
 n \bar{x}^2 &= \chi^2(x \leq 1 - \alpha, 1) \\
 \bar{x} &= \sqrt{\frac{\chi^2(x \leq 0.95, 1)}{n}} \\
 \bar{x} &= \frac{\sqrt{3.841458820694125958361}}{\sqrt{n}} \\
 \bar{x} &= \frac{1.95996}{\sqrt{n}}
 \end{aligned}$$

We will reject when \bar{x} is larger than the right hand side of the last expression.

Suppose that $n = 100$ and $\bar{x} = 0.16$.

- (e) Using your answer to (d), would we reject H_0 in favor of H_1 using significance level $\alpha = 0.05$?

Since $\bar{x} = 0.16 < \frac{1.95996}{\sqrt{100}} = 0.195996$ we would not reject.

- (f) Find the p-value for this hypothesis test.

$$\begin{aligned}
 p_{val} &= P(\bar{X} > \bar{x} | H_0) \\
 &=
 \end{aligned}$$

Problem 7

A set of times T_1, \dots, T_n are sampled independently from a population with the following density:

$$f(t \mid \theta) = \begin{cases} e^{-(t-\theta)} & t \geq \theta \\ 0 & \text{otherwise} \end{cases}$$

where $\theta > 0$.

- (a) Find the maximum likelihood estimate for θ .

*Hint: do some plots, examining the values of θ for which $f(t_1, \dots, t_n \mid \theta) > 0$. It may help you first to think about the cases where $n = 1$ and $n = 2$. Do **not** rush into differentiating anything!*

- (b) Is there a ZES estimator for θ ? Briefly explain your answer.

[**Motivation:** (not necessary to answer the problem, but may help with intuition). For example, the observations T_1, \dots, T_n might be the observed times taken for n messages to be transmitted across a network. In this case, θ represents the (non-random) minimum time for a message to be transmitted across the network if there were no delays; the additional random component of the time $(T - \theta)$ is due to bottlenecks and queues encountered by the message.]