

# Homework #4

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## Problem 1

Suppose that  $X$  is a positive random variable so that  $P(X > 0) = 1$ . Prove that for any  $t < 0$ ,  $M_X(t) = E[e^{tX}] < \infty$ .

*Hint: Find a relationship between  $e^{tX}$  and 1, for  $t < 0$ . Use the fact that if for all  $x$ ,  $g(x) \leq h(x)$  then  $\int g(x)dx \leq \int h(x)dx$ .*

$$\begin{aligned} M_X(t) &= E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x) dx \\ &= \int_0^{\infty} e^{tx} f_X(x) dx \end{aligned}$$

For  $t < 0$ :

$$e^{tx} f_X(x) \leq f_X(x) \forall f_X(x)$$

because  $e^{kx} \leq 1 \forall k < 0$  so it follows:

$$\begin{aligned} \int_0^{\infty} e^{tx} f_X(x) dx &\leq \int_0^{\infty} f_X(x) dx \\ \int_0^{\infty} e^{tx} f_X(x) dx &\leq P(X > 0) \\ \int_0^{\infty} e^{tx} f_X(x) dx &\leq 1 \end{aligned}$$

## Problem 2.

Let  $X$  be an exponential random variable with parameter  $\lambda > 0$ .

- a. Use Chebyshev's inequality to find an upper bound on  $Pr(|X - E[X]| \geq k)$

$$\begin{aligned} P(|X - c| \geq d) &\leq \frac{E[(X - c)^2]}{d^2} \\ P(|X - \mu| \geq k) &\leq \frac{E[(X - \mu)^2]}{k^2} \\ P(|X - \mu| \geq k) &\leq \frac{V[X]}{k^2} \\ P(|X - \mu| \geq k) &\leq \left(\frac{1}{k\lambda}\right)^2 \end{aligned}$$

- b. Calculate  $E[|X - E[X]|]$  and then use Markov's inequality to find a different upper bound on  $Pr(|X - E[X]| \geq k)$ .

*Hint: first remove the absolute value by splitting the integral into two pieces. Each piece can then be integrated using the Exponential CDF (or just integrating directly) and integration by parts.*

$$\begin{aligned} E[|X - E[X]|] &= E[|X - \mu|] = \int_{-\infty}^{\infty} |x - \mu| f_X(x) dx \\ &= \int_0^{\infty} |x - \mu| \lambda e^{-\lambda x} dx \\ &= \int_0^{1/\lambda} \left(\frac{1}{\lambda} - x\right) \lambda e^{-\lambda x} dx + \int_{1/\lambda}^{\infty} \left(x - \frac{1}{\lambda}\right) \lambda e^{-\lambda x} dx \\ &= \int_0^{1/\lambda} e^{-\lambda x} dx - \int_0^{1/\lambda} x e^{-\lambda x} dx + \int_{1/\lambda}^{\infty} x e^{-\lambda x} dx - \int_{1/\lambda}^{\infty} e^{-\lambda x} dx \\ &= -\frac{e^{-\lambda x}}{\lambda} \Big|_0^{1/\lambda} - \frac{\lambda x e^{-\lambda x}}{\lambda^2} \Big|_0^{1/\lambda} - \frac{\lambda x e^{-\lambda x}}{\lambda^2} \Big|_{1/\lambda}^{\infty} + \frac{e^{-\lambda x}}{\lambda} \Big|_{1/\lambda}^{\infty} \\ &= \left(-\frac{1}{e\lambda} + \frac{1}{\lambda}\right) - \left(\frac{1}{e\lambda^3} - 0\right) - \left(0 - \frac{1}{e\lambda^3}\right) - \left(-0 + \frac{1}{e\lambda}\right) \\ &= \frac{1}{\lambda} - \frac{1}{e\lambda} - \frac{1}{e\lambda} = \frac{1}{\lambda} - \frac{2}{e\lambda} \\ &= \frac{e-2}{e\lambda} \end{aligned}$$

Markov's inequality is then:

$$P(|X - \mu| \geq k) = \frac{e-2}{e\lambda k}$$

- (c) For which values of  $k$  is the bound in (b) lower than that given in (a)?

$$\begin{aligned} 1 &> \frac{P_b}{P_a} \\ 1 &> \frac{(e-2)\lambda^2 k^2}{e\lambda k} \\ 1 &> \frac{e-2}{e} \lambda k \\ k &< \frac{e}{\lambda(e-2)} \approx \frac{3.784}{\lambda} \end{aligned}$$

### Problem 3

If the joint probability density of the two random variables  $X$  and  $Y$  is given by:

$$f_{XY}(x, y) = \begin{cases} 2 & \text{for } 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

- a. Find  $P(Y < 0.5)$ . *Hint: Draw a graph.*

$$\begin{aligned} P(Y < 0.5) &= \int_0^1 \int_0^y f(x, y) dx dy \\ &= \int_0^1 \int_0^y 2 dx dy = 2 \frac{y^2}{2} \Big|_0^1 \\ &= 1 \end{aligned}$$

- b. Find  $P(X < 0.5)$ .

$$\begin{aligned} P(X < 0.5) &= \int_0^{0.5} \int_x^1 f(x, y) dy dx \\ &= \int_0^{0.5} (2 - x) dx = 2x \Big|_0^{0.5} - \frac{x^2}{2} \Big|_0^{0.5} \\ &= 1 - \frac{1}{8} = \frac{7}{8} \end{aligned}$$

- c. Find  $P(X + Y < 1)$

$$\begin{aligned} P(X + Y < 1) &= \int_0^{0.5} \int_x^{1-x} f(x, y) dy dx \\ &= 2 \int_0^{0.5} (1 - 2x) dx = 2 \left( x \Big|_0^{0.5} - 2 \frac{x^2}{2} \Big|_0^{0.5} \right) \\ &= 2 \frac{1}{4} = \frac{1}{2} \end{aligned}$$

- d. Find the joint CDF  $F(x, y)$ . (*You will need to consider several cases.*)

$$F(x, y) = \begin{cases} 0 & \text{if } x < 0 \text{ or } y < 0 \\ 0 & \text{if } y > 1 \text{ or } x > y \\ 2xy & \text{if } y < 1 \text{ and } x < y \end{cases}$$

## Problem 4

Suppose that  $X$  and  $Y$  are continuous random variables with the following joint cdf:

$$F_{XY}(x, y) = \tilde{F}(x)\tilde{F}(y) \quad (1)$$

where  $\tilde{F}(\cdot)$  is a (univariate) cdf.

(Aside) *It follows from (1) that  $X$  and  $Y$  are independent random variables with the same marginal distribution.*

- (a) For a fixed value  $t$ , express  $P(X \leq t, Y \leq t)$  in terms of  $\tilde{F}(\cdot)$ .

*Hint: Recall the definition of a joint cdf  $F_{XY}(x, y)$ .*

- (b) Let  $T = \max(X, Y)$ . Using your answer to (a) find the cdf  $F_T(t)$  and the pdf  $f_T(t)$  for  $T$ :

*Hint: If  $X \leq t$  **and**  $Y \leq t$ , what can one say about  $\max(X, Y)$ ?*

## Problem 5

If the joint CDF for two random variables,  $X$  and  $Y$ , is given by:

$$F_{XY}(x, y) = \begin{cases} (1 - e^{-x})(1 - e^{-y^2}) & \text{for } x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the joint pdf  $f_{XY}(x, y)$ .

## Problem 6

Let  $X$  and  $Y$  have the joint density function (here  $k > 0$  is an unknown constant):

$$f_{XY}(x, y) = \begin{cases} k(x - y), & 0 \leq y \leq x \leq 1; \\ 0 & \text{otherwise,} \end{cases}$$

- (a) Sketch the support of the joint density. You will need to use this in answering the next three question parts.
- (b) Find the constant of proportionality  $k$ .
- (c) Find the marginal densities  $f_X(x)$  and  $f_Y(y)$ .
- (d) Find the conditional densities of  $f_{Y|X}(y | x)$  and  $f_{X|Y}(x | y)$ .

Remember to carefully state for which values of the conditioning variable these are defined.

## Problem 7

If the joint probability density of the two random variables  $X$  and  $Y$  is given by:

$$f_{XY}(x, y) = \begin{cases} 4xy & \text{for } 0 < y < 1, 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let  $D = (X - Y)$  and  $S = (X + Y)$ .

- (a) Solve for  $X$  and  $Y$  in terms of  $D$  and  $S$ .
- (b) Find the support of  $f_{DS}(d, s)$  and draw a sketch.
- (c) Find the joint pdf  $f_{DS}(d, s)$ .
- (d) Find the marginal densities  $f_D(d)$  and  $f_S(s)$ .
- (e) Find the conditional densities of  $f_{D|S}(d | s)$  and  $f_{S|D}(s | d)$ .

Remember to carefully state for which values of the conditioning variable these are defined.

## Problem 8

The random variables  $(X, Y)$  have the following joint pdf:

$$f_{XY}(x, y) = \begin{cases} 2y & \text{for } 0 < x < 1 \text{ and } 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the transformation:  $U = XY$  and  $V = X$ .

- (a) Draw the support of the joint density  $f_{XY}(x, y)$  for  $(X, Y)$ .
- (b) Solve for  $X$  and  $Y$  in terms of  $U$  and  $V$ .
- (c) Carefully find the support for the joint density for  $(U, V)$ . Then find the joint density  $f_{UV}(u, v)$ , remember to include the support.
- (d) Find the marginal densities  $f_U(u)$  for  $U$  and  $f_V(v)$  for  $V$ .