Line integral on rectifiable curves

Theorem 1 (Existence of $\int_C f(z)\,dz$ for continuous f on a rectifiable curve)**.** *Let $\gamma : [a,b] \to \mathbb{C}$ be continuous and rectifiable with length $L(\gamma) < \infty$, and let $f : \gamma([a,b]) \to \mathbb{C}$ be continuous. For a partition $P :\ a = t_0 < \cdots < t_n = b$ and tags $\xi_k \in [t_{k-1}, t_k]$, set*

$$S(P,\xi) := \sum_{k=1}^{n} f(\gamma(\xi_k))\big(\gamma(t_k) - \gamma(t_{k-1})\big).$$

Then $S(P,\xi)$ converges to a limit as $\|P\| \to 0$, independent of the choice of tags. This limit is denoted $\int_C f(z)\,dz$.

Proof. Since $f \circ \gamma$ is continuous on the compact interval $[a,b]$, it is uniformly continuous. Fix $\varepsilon > 0$ and choose $\delta > 0$ such that

$$|t - s| < \delta \ \Rightarrow\ |f(\gamma(t)) - f(\gamma(s))| < \varepsilon/L(\gamma).$$

Let P be a partition with $\|P\| < \delta$ and let ξ, η be two choices of tags. Then for each k we have $|\xi_k - \eta_k| \le t_k - t_{k-1} < \delta$, hence $|f(\gamma(\xi_k)) - f(\gamma(\eta_k))| < \varepsilon/L(\gamma)$. Therefore

$$|S(P,\xi) - S(P,\eta)| \le \sum_{k=1}^{n} |f(\gamma(\xi_k)) - f(\gamma(\eta_k))|\, |\gamma(t_k) - \gamma(t_{k-1})| < \frac{\varepsilon}{L(\gamma)} \sum_{k=1}^{n} |\gamma(t_k) - \gamma(t_{k-1})|.$$

By definition of length, $\sum_{k=1}^{n} |\gamma(t_k) - \gamma(t_{k-1})| \le L(\gamma)$, so $|S(P,\xi) - S(P,\eta)| < \varepsilon$.

Now let (P,ξ) and (Q,η) be tagged partitions with $\|P\|, \|Q\| < \delta$, and let $R : a = r_0 < r_1 < \cdots < r_m = b$ be a common refinement of P and Q. For each subinterval $[r_{j-1}, r_j] \subset [t_{k-1}, t_k]$ of P, choose an arbitrary tag $\tilde{\xi}_j \in [r_{j-1}, r_j]$; similarly, for each $[r_{j-1}, r_j] \subset [s_{\ell-1}, s_\ell]$ of Q, choose a tag $\tilde{\eta}_j \in [r_{j-1}, r_j]$. Then $(R, \tilde{\xi})$ and $(R, \tilde{\eta})$ are honest tagged partitions.

Since $[r_{j-1}, r_j] \subset [t_{k-1}, t_k]$, both $\tilde{\xi}_j$ and ξ_k lie in $[t_{k-1}, t_k]$. Therefore

$$|\tilde{\xi}_j - \xi_k| \le t_k - t_{k-1} < \delta.$$

Since $f \circ \gamma$ is uniformly continuous, it follows that

$$|f(\gamma(\tilde{\xi}_j)) - f(\gamma(\xi_k))| < \varepsilon/L(\gamma).$$

Writing each curve increment additively,

$$\gamma(t_k) - \gamma(t_{k-1}) = \sum_{j \in J_k} \big(\gamma(r_j) - \gamma(r_{j-1})\big),$$

where J_k indexes those refined subintervals $[r_{j-1}, r_j] \subset [t_{k-1}, t_k]$, we obtain

$$\begin{aligned}
\Big| S(P,\xi) - S(R,\tilde{\xi}) \Big| &= \left| \sum_{k=1}^{n} f(\gamma(\xi_k)) \sum_{j \in J_k} \big(\gamma(r_j) - \gamma(r_{j-1})\big) - \sum_{j=1}^{m} f(\gamma(\tilde{\xi}_j))\big(\gamma(r_j) - \gamma(r_{j-1})\big) \right| \\
&\le \sum_{j=1}^{m} \big| f(\gamma(\xi_{k(j)})) - f(\gamma(\tilde{\xi}_j)) \big|\, |\gamma(r_j) - \gamma(r_{j-1})| \\
&< \frac{\varepsilon}{L(\gamma)} \sum_{j=1}^{m} |\gamma(r_j) - \gamma(r_{j-1})| \le \varepsilon,
\end{aligned}$$

where $k(j)$ denotes the index such that $[r_{j-1}, r_j] \subset [t_{k(j)-1}, t_{k(j)}]$. Hence

$$|S(P, \xi) - S(R, \tilde{\xi})| < \varepsilon.$$

An identical argument gives

$$|S(Q, \eta) - S(R, \tilde{\eta})| < \varepsilon.$$

Finally, by the triangle inequality,

$$|S(P, \xi) - S(Q, \eta)| \leq |S(P, \xi) - S(R, \tilde{\xi})| + |S(R, \tilde{\xi}) - S(R, \tilde{\eta})| + |S(R, \tilde{\eta}) - S(Q, \eta)|.$$

Since $\|R\| < \delta$, the estimate proved earlier yields

$$|S(R, \tilde{\xi}) - S(R, \tilde{\eta})| < \varepsilon.$$

Therefore,

$$|S(P, \xi) - S(Q, \eta)| < 3\varepsilon,$$

which establishes the Cauchy criterion.

Hence $S(P, \xi)$ converges as $\|P\| \to 0$ to a limit independent of tags; define this limit to be $\int_C f(z)\, dz$. □

Note. The refinement step is written in a tag-honest way: refined tags are chosen inside each refined subinterval, and uniform continuity controls the retagging error.