Proofs for Chapter 5 Sequences

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December 24, 2024

Definition 1 *A* ***sequence*** *is a function* $f : \mathbb{N} \to \mathbb{R}$*, denoted* $(a_n)_{n\geq 1}$*, where* $a_n = f(n)$

Definition 2 *An* ***arithmetic sequence*** *is the sequence* $f : \mathbb{N}^+ \to \mathbb{R}$ *defined by*

$$f(n) = \begin{cases} a_1 & n = 1 \\ a_1 + (n-1)d & \textit{otherwise} \end{cases}$$

Theorem 1 *The sum of the first* n *terms of the arithmetic sequence* $(a_n)_{n\geq 1} = a_1 + (n-1)d$ *is* $S_n = \frac{n}{2}(a_1 + a_n)$

We can write

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_1 + (n-2)d) + (a_1 + (n-1)d)$$
$$S_n = (a_n - (n-1)d) + (a_n - (n-2)d) + \cdots + (a_n - 2d) + (a_n - d) + a_n$$

Summing both sides of the equation, the terms involving d cancel:

$$2S_n = n(a_1 + a_n)$$

Dividing both sides by 2:

$$S_n = \frac{n}{2}(a_1 + a_n) \qquad \square$$

Definition 3 *A* ***geometric sequence*** *is the sequence* $f : \mathbb{N}^+ \to \mathbb{R}$ *defined by*

$$f(n) = ar^{n-1}$$

Theorem 2 *The sum of the first* n *terms of the geometric sequence* $(a_n)_{n\geq 1} = ar^{n-1}$ *is* $S_n = \frac{a(1-r^n)}{1-r}$

$$S_n = ar^0 + ar^1 + ar^2 + \cdots + ar^{n-1}$$

Multiplying both sides by $(1-r)$:

$$\begin{aligned} (1-r)S_n &= (1-r)(ar^0 + ar^1 + ar^2 + \cdots + ar^{n-1}) \\ &= (ar^0 + ar^1 + ar^2 + \cdots + ar^{n-1}) - (ar^1 + ar^2 + ar^3 + \cdots + ar^n) \\ &= a - ar^n \end{aligned} \qquad \square$$

Definition 4 *A sequence* $(a_n)_{n\geq 1}$ *is increasing if* $a_{n+1} \geq a_n$ *for* $n \geq 1$*, and decreasing if* $a_{n+1} \leq a_n$ *for* $n \geq 1$*. The sequence is called monotonic if either increasing or decreasing.*

Definition 5 *The* ϵ*-neighbourhood of a point* $a \in \mathbb{R}$ *is defined as*

$$U_\epsilon(a) \triangleq \{x \in \mathbb{R} : |a - x| < \epsilon\} = (a - \epsilon, a + \epsilon)$$

where $\epsilon > 0$

Definition 6 *The sequence* $(a_n)_{n\geq 1}$ *converges to a limit* l *in* $\mathbb{R}$ *iff it settles in each* ϵ*-neighbourhood of* a*, that is,*

$$\forall \epsilon > 0 \; \exists N \in \mathbb{N} \; \forall n > N \; (|a_n - l| < \epsilon)$$

Definition 7 *The sequence* $(a_n)_{n\geq 1}$ *converges to* ∞ *iff*

$$\forall r \in \mathbb{R} \; \exists N \in \mathbb{N} \; \forall n > N(a_n > r)$$

Definition 8 *The sequence* $(a_n)_{n\geq 1}$ *converges to* $-\infty$ *iff*

$$\forall r \in \mathbb{R} \; \exists N \in \mathbb{N} \; \forall n > N(a_n < r)$$

Theorem 3 $(a_n)_{n\geq 1} = \left(\frac{1}{n^c}\right)_{n\geq 1} \to 0$ *as* $n \to \infty$*, where* $c > 0$*.*

We want to show that given any $\epsilon > 0$, we can find N_ϵ such that for all $n > N$

$$\left|\frac{1}{n^c}\right| < \epsilon$$

Since n is positive, $n^c > \frac{1}{\epsilon} \Leftrightarrow n > 1/\epsilon^{1/c}$. Hence we can set

$$N_\epsilon = \left\lfloor \frac{1}{\sqrt[c]{\epsilon}} \right\rfloor$$

.

for any value of $\epsilon > 0$. $\square$

Theorem 4 $(a_n)_{n\geq 1} = \left(\frac{1}{c^n}\right)_{n\geq 1} \to 0$ *as* $n \to \infty$ *for* $|c| > 1$

We want to show that given any $\epsilon > 0$, we can find $N_\epsilon \in \mathbb{N}^+$ such that for all $n > N_\epsilon$

$$\left|\frac{1}{c^n}\right| < \epsilon$$

Then $|c^n| = |c|^n > \frac{1}{\epsilon} \Leftrightarrow n \ln |c| > \ln \epsilon^{-1} \Leftrightarrow n > -\frac{\ln \epsilon}{\ln |c|}$ so we can set

$$N_\epsilon = \left\lfloor -\frac{\ln \epsilon}{\ln |c|} \right\rfloor \qquad \square$$

Theorem 5 *If* $a_n \to a$ *as* $n \to \infty$, $\lim_{n\to\infty} \lambda a_n = \lambda a$

If $\lambda = 0$, the problem is trivial. If $\lambda \neq 0$, then it is given that for any $\epsilon > 0$, there exists some $N \in \mathbb{N}^+$ such that

$$|a_n - a| < \frac{\epsilon}{|\lambda|}$$

for all $n > N$.

(For added clarity, we could instead state as normal that $|a_n - a| < \epsilon$ and then say that there exists $k = \epsilon|\lambda|$; as ϵ can take any positive value, so does k)

$$|\lambda a_n - \lambda a| = |\lambda||a_n - a| < \epsilon$$

for all $n > N$. $\square$

Lemma 1 *For any real a and b, $|a + b| \leq |a| + |b|$*

$$\begin{aligned}
|a + b|^2 = (a + b)^2 &= a^2 + 2ab + b^2 \\
&= |a|^2 + 2ab + |b|^2 \\
&\leq |a|^2 + 2|ab| + |b|^2 \\
&= |a|^2 + 2|a||b| + |b|^2 \\
&= (|a| + |b|)^2
\end{aligned}$$

Theorem 6 *If $a_n \to a$ and $b_n \to b$ as $n \to \infty$, then $\lim_{n\to\infty} a_n + b_n = a + b$*

Since $a_n \to a$ as $n \to \infty$, we can find an N_1 such that for any $n > N_1$, $|a_n - a| < \frac{1}{2}\epsilon$ (1).

Similarly, we can find an N_2 such that for any $n > N_2$, $|b_n - b| < \frac{1}{2}\epsilon$ (2).

Let $N = \max\{N_1, N_2\}$. Then if $n > N$, both (1) and (2) are true. Hence for any $n > N$

$$\begin{aligned}
|(a_n + b_n) - (a + b)| &= |(a_n - a) + (b_n - b)| \\
&\leq |a_n - a| + |b_n - b| \\
&< \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon
\end{aligned}$$

Theorem 7 *If $a_n \to a$ and $b_n \to b$ as $n \to \infty$, then $\lim_{n\to\infty} a_n - b_n = a - b$*

$\lim_{n\to\infty} a_n - b_n = \lim_{n\to\infty} a_n + (-b_n) = a + (\lim_{n\to\infty} -1 \cdot b_n) = a_n - b_n$ $\square$

Definition 9 *A subset $A \subset \mathbb{R}$ is bounded if there exists $k > 0$ such that $|a| \leq k$ for all $a \in A$*

Lemma 2 *If $(a_n)_{n\geq 1}$ converges to $a \in \mathbb{R}$, then it is bounded.*

Let $\epsilon = 1$. Then there exists $N \in \mathbb{N}$ such that $|a_n - a| < 1$ for $n > N$. Hence $|a_n| = |a_n - a + a| \leq |a| + |a_n - a| < |a| + 1$ for all $n > N$. Hence the bound $k = \max\{|a_i| : 1 \leq i \leq N\} \cup \{|a| + 1\}$

Theorem 8 *If $a_n \to a$ and $b_n \to b$ as $n \to \infty$, then $\lim_{n\to\infty} a_n b_n = ab$*

Since b_n converges, by lemma 2, there exists $K \in \mathbb{R}^+$ such that $|b_n| < K$ for all $n \in \mathbb{N}^+$.

Let $\epsilon = \frac{\epsilon}{2K}$. Then there exists $N_1 \in \mathbb{N}^+$ such that

$$|a_n - a| < \frac{\epsilon}{2k}$$

There also exists $N_2 \in \mathbb{N}^+$ such that

$$|b_n - b| < \frac{\epsilon}{2|a|}$$

Then

$$\begin{aligned}
|a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \\
&\leq |b_n||a_n - a| + |a||b_n - b| \\
&< K\left(\frac{\epsilon}{2K}\right) + |a|\left(\frac{\epsilon}{2|a|}\right) \\
&= \epsilon
\end{aligned}$$

□

Theorem 9 *If $a_n \to a$ and $b_n \to b$ as $n \to \infty$, then $\lim_{n\to\infty} = \frac{a}{b}$ given that $b \neq 0$*

We first show that

$$\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{b}$$

Let $\epsilon = \frac{|b|}{2}$. Then there exists $N_1 \in \mathbb{N}$ such that

$$|b_n - b| < \frac{|b|}{2}$$

for all $n > N_1$.

Therefore

$$\begin{aligned}
|b| &= |b - b_n + b_n| \\
&\leq |b_n - b| + |b_n|
\end{aligned}$$

$$\begin{aligned}
\Rightarrow |b_n| &\geq |b| - |b_n - b| \\
&> |b| - \frac{|b|}{2} \\
&= \frac{|b|}{2}
\end{aligned}$$

for $n > N_1$

For any $\epsilon > 0$, there exists $N_2 \in \mathbb{N}$ such that

$$|b_n - b| < \frac{\epsilon |b|^2}{2}$$

for any $n > N_2$.

Therefore, for any $\epsilon > 0$

$$\begin{aligned}
\left| \frac{b_n - b}{b_n b} \right| &= \frac{|b_n - b|}{|b_n||b|} \\
&< \frac{\epsilon |b|^2/2}{\frac{|b|}{2}|b|} \\
&= \epsilon
\end{aligned}$$

for all $n > \max\{N_1, N_2\}$ $\square$

Theorem 10 *A sequence $(a_n)_{n\geq 1}$ can only have one limit.*

Suppose that $(a_n)_{n\geq 1}$ has two limits, L and M such that $L \neq M$. Let $\epsilon = \frac{|L-M|}{2}$. Then there exists N_1 such that for all $\epsilon > 0$, $|a_n - L| < \epsilon$ for all $n > N_1$. There also exists N_2 such that for all $\epsilon > 0$, $|a_n - M| < \epsilon$ for all $n > N_2$. Let $N = \max\{N_1, N_2\}$. Then both inequalities hold simultaneously. Then

$$\begin{aligned}
|L - M| &= |L - a_n + a_n - M| \\
&\leq |L - a_n| + |a_n - M| \\
&= |a_n - L| + |a_n - M| \\
&< |L - M|
\end{aligned}$$

which is impossible. Hence the assumption that $L \neq M$ is false so $L = M$

Definition 10 *A sequence $(a_n)_{n\geq 1}$ is a* **Cauchy sequence** *iff*

$$\forall \epsilon > 0 \; \exists N \in \mathbb{N} \; \forall n, m > N (|a_n - a_m| < \epsilon)$$

Theorem 11 *Every sequence that converges to a real number is a Cauchy sequence*

For every $\epsilon > 0$, there exists $N \in \mathbb{N}^+$ such that for any $n, m > N$

$$|a_n - l| < \tfrac{\epsilon}{2} \; |a_m - l| < \tfrac{\epsilon}{2}$$

Then $-\frac{\epsilon}{2} < a_n - l < \frac{\epsilon}{2}$ and $-\frac{\epsilon}{2} < a_m - l < \frac{\epsilon}{2}$, so

$$\begin{aligned}
|a_n - a_m| &= |a_n - l + l - a_m| \\
&\leq |a_n - l| + |a_m - l| \\
&< \epsilon
\end{aligned}$$

$\square$

Definition 11 *A subset $A \subset \mathbb{R}$ is said to be **complete** if any Cauchy sequence in A converges to a limit in A*

Theorem 12 Sandwich theorem *Let $(l_n)_{n\geq 1}$ and $(u_n)_{n\geq 1}$ be sequences, and l a real number where both $\lim_{n\to\infty} l_n = l$ and $\lim_{n\to\infty} u_n = l$. If for a third sequence $(a_n)_{n\geq 1}$, there is some $N \in \mathbb{N}$ such that $l_n \leq a_n \leq u_n$ for all $n \geq N$, then $\lim_{n\to\infty} a_n = l$.*

For any $\epsilon > 0$, there exists N_1 such that $|l_n - l| < \epsilon$ for all $n > N_1$. There also exists N_2 such that $|u_n - l| < \epsilon$ for all $n > N_2$.

It is given there exists $N_3 \in \mathbb{N}$ such that $l_n \leq a_n \leq u_n$ for all $n > N_3$. Let $N = \max\{N_1, N_2, N_3\}$; then all three inequalities hold simultaneously.

Hence $-\epsilon < l_n - l < \epsilon$ and $-\epsilon < u_n - l < \epsilon$ for all $n > N$. But since also $l_n \leq a_n \leq u_n$ for $n > N$, we have

$$l - \epsilon < l_n \leq a_n \leq u_n < l + \epsilon$$

$-\epsilon < a_n - l < \epsilon$ so $|a_n - l| < \epsilon$ as required. □

Theorem 13 Ratio test *Let $c \in \mathbb{R}$ be such that $0 \leq c < 1$. Suppose that there is some N in $\mathbb{N}$ such that for all $n \geq N$ we have $|\frac{a_{n+1}}{a_n}| \leq c$. Then $\lim_{n\to\infty} a_n = 0$.*

Multiplying both sides of the inequality by $|a_n|$, we get $|a_{n+1}| \leq c|a_n|$ for $n \geq N$. Similarly, $|a_n| \leq c|a_{n-1}|$ and so forth until $n = N$:

$$|a_n| \leq c|a_{n-1}| \leq c^2|a_{n-2}| \leq \cdots \leq c^{n-N}|a_N|$$

for all $n \geq N$. This means that

$$|a_n| \leq c^{n-N}|a_N| \leq kc^n$$

where $k = \frac{|a_N|}{c^N}$ is constant. Therefore, for all $n \geq N$

$$-kc^n \leq a_n \leq kc^n$$

Since $0 < c < 1$, $-kc^n \to 0$ and $kc^n \to 0$ as $n \to \infty$. Hence by the sandwich theorem, a_n also tends to 0. □.

Theorem 14 Limit ratio test *If the limit $r = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$ exists and $r < 1$, then the sequence $(a_n)_{n\geq 1}$ converges to 0.*

Let $(b_n)_{n\geq 1} = |\frac{a_{n+1}}{a_n}|$. Then for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$ we have $|b_n - r| < \epsilon$. Therefore

$$r - \epsilon < b_n < r + \epsilon$$

Assume $r < 1$. Since $r < 1$, we know that ϵ defined as $\frac{1-r}{2}$ is positive. Therefore, there exists some $N(\epsilon)$ such that for all $n > N$, the inequalities above are true, and they now read as:

$$r - \frac{1-r}{2} < b_n < r + \frac{1-r}{2}$$

which we can simplify to

$$\frac{3r-2}{2} < b_n < \frac{r+1}{2}.$$

Taking the right-hand side, we see that $b_n < \frac{r+1}{2} < 1$. Therefore, since $b_n < 1$ for $n > N(\epsilon)$, we can apply the ratio test to conclude that $(a_n)_{n\geq 1}$ converges to 0. $\square$

Definition 12 *If $f : \mathbb{N} \to \mathbb{R}$ is a sequence and $M \subset N$ an infinite subset, then $f : M \to \mathbb{R}$ is called a subsequence of f, usually written $(a_{n_i})_{i\geq 1}$ where n_i are positive integers.*

Theorem 15 *Any subsequence of a convergent sequence converges to the limit of the sequence.*

Let $l = \lim_{n\to\infty}$ and let (a_{n_i}) be any subsequence. Let $\epsilon > 0$. Then there exists $N > 0$ such that for $n > N$ we have $|a_n - l| < \epsilon$. By definition, $n_i \geq i$, so for $i > N$, $|a_{n_i} < \epsilon|$. $\square$

Theorem 16 *Any sequence of real numbers has a monotonic subsequence*

Consider a sequence $(a_n)_{n\geq 1}$. For any $m \geq 1$, we call a_m a peak of $(a_n)_{n\geq 1}$ if $a_m \geq a_n$ for all $n \geq m$.

Suppose in the first case that $(a_n)_{n\geq 1}$ has infinitely many peaks denoted $a_{n_1}, a_{n_2}, \ldots$. Then the subsequence $(a_{n_i})_{i\geq 1}$ is monotonically decreasing.

If there are only a finite number of peaks $a_{n_1}, a_{n_2}, \ldots, a_{n_k}$, then choose $t_1 = n_k + 1$. Since t_1 is not a peak, there exists $t_2 > t_1$ such that $a_{t_1} < a_{t_2}$. We can then construct t_i for all $i \geq 1$ such that $(a_{t_i})_{i\geq 1}$ is monotonically increasing sequence. If there are no peaks, then the sequence is strictly increasing (and likewise any subsequence).

Theorem 17 **reverse triangle inequality** $|x - y| \geq ||x| - |y||$

From the triangle inequality

$$|y| = |x + y - x| \leq |x| + |y - x| \;\; |x| = |y + x - y| \leq |y| + |x - y|$$

Therefore $|y - x| \geq |y| - |x|$ and $|x - y| \geq |x| - |y|$. Since $|y - x| = |x - y|$, we have $|x - y| \geq -(|x| - |y|)$ and $|x - y| \geq |x| - |y|$. Therefore

$$|x - y| \geq ||x| - |y||$$

as required.

Theorem 18 *The least upper bound of a set X is unique*

Let $L, L' \in \mathbb{R}$ be such that both are least upper bounds of X.Since L is an upper bound of X and L' is a *least* upper bound, $L' \leq L$. Since L' is an upper bound of X and L is a *least* upper bound of X, $L \leq L'$. Therefore, $L = L'$.

Definition 13 ***Axiom of Dedekind-completeness for real numbers*** *Every nonempty subset X of the real numbers $\mathbb{R}$ that is bounded above has a least upper bound.*

Theorem 19 ***Fundamental theorem of analysis*** *Let $(a_n)_{n\geq 1}$ be a sequence of real numbers that is increasing and bounded above. Then $s = \sup\{a_n\}$ and is the limit of $(a_n)_{n\geq 1}$.*

By the axiom of Dedekind-completeness, we know that the supremum s stated exists.

Let $\epsilon > 0$ be given. We assume that there does not exist $N \in \mathbb{N}$ such that $|a_N - s| < \epsilon$. Then $|a_n - s| \geq \epsilon$ for all $n \geq 1$. This implies $s - a_n \geq \epsilon$, so $s - \epsilon \geq a_n$ for all $n \geq 1$. But then $s - \epsilon$ is an upper bound of a_n. Since s is the supremum, we have $s \leq s - \epsilon$, which is a contradiction since ϵ is positive. Then there exists $N \in \mathbb{N}$ such that $|a_N - s| < \epsilon$.

Since the sequence is monotonically increasing, $n > N$ implies that $a_n \geq a_N$. Therefore $-\epsilon < 0 \leq s - a_n \leq s - a_N < \epsilon$, so $|s - a_n| < \epsilon$ for all $n > N$ as required. □

Theorem 20 *The set of real numbers is complete.*

Let $(a_n)_{n\geq 1}$ be a Cauchy sequence. Putting $\epsilon = 1$, there exists $N \in \mathbb{N}$ such that for $n, m > N$, $|a_n - a_m| < 1$. Let $K = 1 + \max\{|a_1|, |a_2|, \ldots, |a_N|, |a_{N+1}|\}$. Then for $m > N$ we have $|a_m| \leq |a_{N+1}| + |a_m - a_{N+1}| < |a_{N+1}| + 1$. Therefore $|a_n| < K$ for all $n \geq 1$, so the sequence is bounded.

Since any sequence has a monotonic subsequence, let $(a_{n_i})_{i\geq 1}$ be a monotonic subsequence of $(a_n)_{n\geq 1}$. Then by the fundamental theorem of analysis, the subsequence is convergent with limit l.

Let $\epsilon > 0$ be given. Since $(a_n)_{n\geq 1}$ is Cauchy, there exists $N_1 \in \mathbb{N}$ such that for $n, m > N_1$, we have $|a_n - a_m| < \epsilon/2$. Since $\lim_{i\to\infty} a_{n_i} = l$, there exists $N_2 > 0$ such that for $i > N_2$, we have $|a_{n_i} - l| < \epsilon/2$.

Let $N = \max\{N_1, N_2\}$. Then

$$|a_n - l| \leq |a_n - a_{n_i}| + |a_{n_i} - l| < \epsilon/2 + \epsilon/2 = \epsilon$$

for any $n, i > N$. Hence $(a_n)_{n\geq 1}$ is convergent to a real number. □