Proofs for Chapter 10 Power Series

Constantin Filip

Definition 1 *A **power series** is a series of the form*

$$\sum_{i=0}^{\infty} a_i \cdot (x-c)^i$$

where x is a real variable, c is a constant in $\mathbb{R}$, and $(a_n)_{n\geq 0}$ is a sequence of reals.

Definition 2 ***Radius of convergence*** *Let c be a constant in $\mathbb{R}$ and $(a_n)_{n\geq 0}$ a sequence of reals. The power series $\sum_{i=0}^{\infty} a_i \cdot (x-c)^i$ has a radius of convergence r in $[0,\infty)\cup\{\infty\}$ such that*

1. *If $r \neq \infty$, then*
   - (a) *the power series converges for all x in $\mathbb{R}$ such that $|x-c| < r$*
   - (b) *the power series diverges for all x in $\mathbb{R}$ such that $|x-c| > r$*
2. *If $r = \infty$, then the power series converges for all x in $\mathbb{R}$*

Theorem 1 *Every power series $\sum_{n=0}^{\infty} a_n \cdot (x-c)^n$ has a radius of convergence r which is given by*

$$r^{-1} = \limsup_{n\to\infty} |a_n|^{1/n}$$

We apply the nth root test to $\sum_{n=0}^{\infty} a_i \cdot (x-c)^i$. Suppose $l = \limsup_{n\to\infty} |a_n|^{1/n}$, where l is a real number. We then have

$$\limsup_{n\to\infty}(|a_n||x-c|^n)^{1/n} = \limsup_{n\to\infty}(|a_n|)^{1/n}|x-c| = l|x-c|$$

Then the series converges absolutely if $l|x-c| < 1$ and diverges for $l|x-c| > 1$, or equivalently, the series converges for $|x-c| < 1/l$ and diverges for $|x-c| > 1/l$. If $\limsup_{n\to\infty} |a_n|^{1/n} = \infty$ then the series converges only for $|x-c| = 0$ for $x = c$.

Lemma 1 ***Ratio test for radius of convergence*** *Suppose that the sequence*

$$\left(\frac{|a_{n+1}|}{|a_n|}\right)_{n\geq 1}$$

has a limit l in $\mathbb{R}$. Then l^{-1} is the radius of convergence of any power series $\sum_{i=0}^{\infty} a_i \cdot (x-c)^i$

This is simply an application of d'Alembert's ratio test for series convergence.

Theorem 2 *Two power series with radii of convergence r_1 and r_2 respectively can be added term by term to get the sum of the two power series, absolutely convergent with the radius of convergence $r \geq \min\{r_1, r_2\}$*

Suppose we have two power series

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

$$g(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n + \cdots$$

with radii of convergence of r_1 and r_2 respectively. Then we can write

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + R_{1n}(x)$$

$$g(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n + R_{2n}(x)$$

where $R_{1n}(x) \to 0$ for $|x| < r_1$ and $R_{2n}(x) \to 0$ for $|x| < r_2$ as $n \to \infty$. Let

$$S_{1n}(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

and

$$S_{2n}(x) = b_0 + b_1 x + b_2 x^2 + \cdots b_n x^n$$

We have for $|x| < \min\{r_1, r_2\}$

$$f(x) = S_{1n}(x) + R_{1n}(x)$$

$$g(x) = S_{2n}(x) + R_{2n}(x)$$

so

$$f(x)+g(x)-R_{1n}(x)-R_{2n}(x) = S_{1n}(x)+S_{2n}(x) = (a_0+b_0)+(a_1+b_1)x+\cdots+(a_n+b_n)x^n$$

Then for $|x| < \min\{r_1, r_2\}$ as $n \to \infty$

$$f(x) + g(x) = f(x) + g(x) - 0 - 0 = a_0 + b_0 + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n + \cdots$$

as required. Note that $\min\{r_1, r_2\}$ is only a lower bound for the radius of convergence. For example, if $f(x) = \sum_{n \geq 0} a_n x^n$ has radius of convergence $r_1 < \infty$, then $g(x) = \sum_{n \geq 0} (-a_n) x^n$ also has radius of convergence $r_2 = r_1$ with $\min\{r_1, r_2\} = r_1$. However, the radius of convergence of $f(x) + g(x) = \sum_{n \geq 0} (a_n - a_n) x^n = 0$ is $\infty > r_1$. $\square$

Definition 3 *A function $f : \mathbb{R} \to \mathbb{R}$ is* ***smooth*** *at x_0 if for all $k \geq 1$ the kth derivative of f exists at x_0. These kth derivatives are defined inductively by*

$$f^{(1)} = f' \qquad f^{(k+1)} = \left(f^{(k)}\right)' \textit{for all } k \geq 1$$

Definition 4 *A function f is called a* ***real analytical function*** *if there exists a power series that has the same outputs as f within its radius of convergence.*

Theorem 3 *Not every smooth real function is analytical*

Consider the function

$$f(x) = \begin{cases} e^{-x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

It can be shown inductively that $f^{(k)}(0) = 0$ for all $k \geq 0$, so $f(x)$ is smooth. Then the Maclaurin series expansion of $f(x)$ evaluates to 0 (and therefore has radius of convergence ∞), which is clearly not equal to $f(x)$, so $f(x)$ is not analytical.

Definition 5 *The general form of the Taylor series for a function $f : \mathbb{R} \to \mathbb{R}$ that is infinitely differentiable at point c in $\mathbb{R}$ is derived as*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Theorem 4 *Suppose*

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

with radius of convergence $r > 0$. Then $f(x)$ is continuous for x with $|x - x_0| < r$ and moreover f is differentiable and integrable with

$$f'(x) = \sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1}$$

$$\int_{x_0}^{x} f(t)\mathrm{d}t = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+1}/(n+1)$$

By definition, the power series converges and equality holds for $f(x)$ when $|x - x_0| < r$. Each term of the power series is a polynomial and continuous, so the entire power series is continuous, and likewise $f(x)$. Similarly, each term of the power series is differentiable, so the power series is differentiable due to the linearity of the derivative (and likewise with the integral). The derivative and integral of $f(x)$ are obtained using the product rule and reverse chain rule respectively. □

Theorem 5 *The power series expansion of f around x_0 is unique and is given by the Taylor series expansion.*

By differentiating $f(x)$ recursively n times,

$$f^{(n)}(x_0) = n! a_n$$

so

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

Then each a_n is unique so the power series of f around x_0 is unique and given by its Taylor series expansion. □