Proofs for Chapter 6 Continuous Functions

Constantin Filip

October 28, 2024

Theorem 1 *Show that the limit of a function, if it exists, is unique.*

Assume that $\lim_{x\to x_0} = L$ and $\lim_{x\to x_0} = M$, where $L \neq M$. Then let $\epsilon = \frac{|L+M|}{2}$. Then there exists some $\delta_1 > 0$ such that

$$|f(x) - L| < \frac{|L-M|}{2}$$

where $0 < |x - x_0| < \delta_1$. There also exists some $\delta_1 > 0$ such that

$$|f(x) - M| < \frac{|L-M|}{2}$$

where $0 < |x - x_0| < \delta_2$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then both inequalities hold. Then

$$\begin{aligned}|L - M| &= |L - f(x) + f(x) - M| \\ &\leq |f(x) - L| + |f(x) - M| \\ &< |L - M|\end{aligned}$$

which is impossible, so the function cannot have two distinct limits. $\square$

Theorem 2 *Show that* $\lim_{x\to x_0} f(x) = l \in \mathbb{R} \cup \{\infty, -\infty\}$ *iff for all sequences* $(y_n)_{n\geq 1}$ *with* $\lim_{n\to\infty} y_n = x_0$ *we have* $\lim_{n\to\infty} f(y_n) = l$.

We first prove ($\Rightarrow$)

For all $\epsilon > 0$, there exists δ such that for all x such that $0 < |x - x_0| < \delta$, we have $|f(x) - l| < \epsilon$.

If $\lim_{n\to\infty} y_n = x_0$, then for all $\epsilon' > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$

$$|y_n - x_0| < \epsilon$$

Hence we can set $\epsilon' = \delta$ for any value of $\epsilon > 0$. Then there exists N_ϵ such that

$$0 < |y_n - x_0| < \epsilon$$

But then by assumption

$$|f(y_n) - l| < \epsilon$$

for all $n > N_\epsilon$.

Since N_ϵ exists for any value of $\epsilon > 0$, by the definition of sequence convergence,

$$\lim_{n\to\infty} f(y_n) = l$$

Now we prove ($\Leftarrow$)

Assume that $\lim_{n\to\infty} f(y_n) = l$ for all sequences $(y_n)_{n\geq 1}$ with $\lim_{n\to\infty} y_n = x_0$.

Now assume towards a contradiction that $\lim_{x\to x_0} f(x) \neq l$. Then we have

$$\text{not '}\forall\epsilon > 0\ \exists\delta > 0\ (x \in [a,b] \wedge 0 < |x - x_0| < \delta \Rightarrow |f(x) - l| < \epsilon)\text{'}$$

which is equivalent to

$$\exists\epsilon > 0\ \forall\delta > 0(\exists x_\delta\ (x_\delta \in [a,b] \wedge 0 < |x_\delta - x_0| < \delta \wedge |f(x_\delta) - l| \geq \epsilon))$$

We have $\lim_{n\to\infty} y_n = x_0$. Let $\delta = \frac{1}{n}$ where $n \geq 1$. Then by assumption, there exists some $\epsilon > 0$ such that for all $1/n > 0$, there exists a_n such that $|a_n - x_0| < \frac{1}{n}$ and $|f(a_n) - l| \geq \epsilon$. Since this is true for all $n \geq 1$, $f(a_n)$ always differs from l by ϵ which is a contradiction.

Hence the original assumption that $\lim_{x\to x_0} f(x) \neq l$ is false, so $f(x) \to l$ as $x \to x_0$. $\Box$

Theorem 3 *Suppose the two functions $f, g : [a,b] \to \mathbb{R}$ have limits $k \in \mathbb{R}$ and $l \in \mathbb{R}$ respectively at $x_0 \in [a,b]$. Then $f \pm g$ has limit $k \pm l$, $f \cdot g$ has limit kl and if $l \neq 0$, f/g has limit k/l at x_0.*

If $\lim_{x\to x_0} f(x) = k$, then for all sequences $(y_n)_{n\geq 1}$ with $\lim_{n\to\infty} y_n = x_0$, we have $\lim_{n\to\infty} f(y_n) = k$. Likewise, we also have $\lim_{n\to\infty} f(y_n) = l$.

Now let $h(x) = f(x) + g(x)$. Then

$$\lim_{n\to\infty} h(y_n) = \lim_{n\to\infty} f(y_n) \pm g(y_n) = k \pm l$$

by the limit properties of *sequences*. Therefore, $\lim_{x\to x_0} h(x) = \lim_{x\to x_0} f(x) \pm g(x) = k \pm l$ as required.

The other properties can be proven in an identical way by considering the functions $h(x) = f(x)g(x)$ and $h(x) = \frac{f(x)}{g(x)}$

$\Box$

Theorem 4 *If the two functions $f, g : [a,b] \to \mathbb{R}$ are continuous at $x_0 \in [a,b]$, then we have*

- *$f \pm g$ is continuous at x_0*
- *The product $f \cdot g$ is continuous at x_0*
- *If $g(x_0) \neq 0$ then f/g is continuous at x_0*

These results are immediate from the limit properties of functions.

Theorem 5 *Any polynomial $f(x)$ of degree n is continuous for all real x*

We first show that

$$\lim_{x \to a} x = a$$

where a is some arbitrary real number.

Let $\epsilon > 0$ be given, and let $\delta = \epsilon$. Then for all x such that $0 < |x - a| < \delta$, we have

$$|f(x) - a| = |x - a| < \delta \Rightarrow |f(x) - a| < \epsilon$$

so by the definition of the limit, $\lim_{x \to a} x = a$. Since this is true for an arbitrary a, we have $\lim_{x \to x_0} x = x_0$ for all $x_0 \in \mathbb{R}$.

Now we show that for any $n \in \mathbb{N}$, x^n is continuous for all real x.

For $n = 0$, $x^0 = 1$, for which it is trivial to show that $\lim_{x \to a} 1 = 1$ for all real x.

Assume that x^k is continuous over the real numbers for some $k > 1$. Then $\lim_{x \to a} x^k = a^k$ for all real a. Then for some arbitrary $c \in \mathbb{R}$

$$\lim_{x \to c} x^{k+1} = (\lim_{x \to c} x^k)(\lim_{x \to c} x) = c^{k+1}$$

Since this is true for any real c, x^{k+1} is continuous over the real numbers by definition. Since x^{k+1} is continuous under the assumption that x^k is continuous, and given the base case $n = 0$, we have x^n is continuous over the real numbers for all $n \in \mathbb{N}$.

Then also for any $\lambda > 0$, we have

$$\lim_{x \to x_0} \lambda x^n = \lim_{x \to x_0} \lambda \cdot \lim_{x \to x_0} x^n = \lambda x_0^n$$

for any real x_0, so λx_0^n is continuous over the real numbers.

Lastly, we show that any polynomial of degree n is continuous over the real numbers by induction.

From above, a polynomial of degree 0 is a constant which is continuous.

Assume that a polynomial $f_k(x)$ of degree k where $k \in \mathbb{N}$ is continuous over the real numbers. Then for $n = k + 1$

$$\lim_{x \to x_0} f_{k+1}(x) = \lim_{x \to x_0} (ax^{k+1} + f_k(x)) = \lim_{x \to x_0} ax^{k+1} + \lim_{x \to x_0} f_k(x) = ax_0^{k+1} + f_k(x_0) = f_{k+1}(x_0)$$

Hence $f_{k+1}(x)$ is continuous over all real x under the assumption that $f_k(x)$ is continuous over the real numbers. Since the base case also holds, by induction, $f_n(x)$ is continuous over the real numbers for all $n \in \mathbb{N}$. $\square$

Theorem 6 *Suppose $f : (a, b) \to \mathbb{R}$ is a function with $x_0 \in (a, b)$ and $g : (c, d) \to \mathbb{R}$ is a function with $Im(f) \subset (c, d)$. If f is continuous at x_0 and g is continuous at $f(x_0)$, then the composition $g \circ f : (a, b) \to \mathbb{R}$ with $(g \circ f)(x_0) = g(f(x_0))$ is continuous at x_0.*

If f is continuous at x_0, then $\lim_{x \to x_0} f(x) = f(x_0)$ by definition, so

$$\forall \epsilon > 0 \; \exists \delta > 0 \; (0 < |x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon) \tag{1}$$

Likewise, if g is continuous at $f(x_0)$, then $\lim_{x \to f(x_0)} g(x) = (g \circ f)(x_0)$, so

$$\forall \epsilon > 0 \; \exists \delta > 0 \; (0 < |x - f(x_0)| < \delta \implies |g(x) - (g \circ f)(x_0)| < \epsilon) \tag{2}$$

Let $\epsilon > 0$ be given. Then let $\delta_2 > 0$ be the value for which the limit inequality in (2) holds.

Now set ϵ in (1) to δ_2. Then there exists $\delta_1 > 0$ such that (1) holds. Therefore, we have

$$0 < |x - x_0| < \delta_1 \implies |f(x) - f(x_0)| < \delta_2$$

But then by (2)

$$0 < |f(x) - f(x_0)| < \delta_2 \implies |(g \circ f)(x) - (g \circ f)(x_0)| < \epsilon$$

Since δ_1 is a function of δ_2 which is itself a function of ϵ, where ϵ can be any positive real number, we have

$$\forall \epsilon > 0 \; \exists \delta > 0 \; (0 < |x - x_0| < \delta \implies |(g \circ f)(x) - (g \circ f)(x_0) < \epsilon)$$

so $\lim_{x \to x_0} (g \circ f)(x) = (g \circ f)(x_0)$ so $g \circ f$ is continuous at x_0 by definition. $\square$

Lemma 1 *If f is continuous in $[a, b]$ then f is bounded.*

Suppose that f is continuous but not bounded. Then

$$\forall k \; \exists x_k \in [a, b] \; s.t. \; |f(x_k)| \geq k$$

We can form a sequence $(a_n)_{n \geq 1}$ such that for all $N \in \mathbb{N}^+$, there exists $a_n \in [a, b]$ such that $|f(a_n)| \geq N$. Hence $|f(a_n)| \to \infty$ by definition of convergence to ∞.

Since $(a_n)_{n \geq 1}$ is bounded, as proven in Chapter 5, $(a_n)_{n \geq 1}$ must have a convergent subsequence $(a_{n_i})_{i \geq 1}$.

Now since f is continuous at l, we have $\lim_{x \to l} f(x) = f(l)$. Hence as proven earlier, we have $\lim_{n_i \to \infty} f(a_{n_i}) = f(l)$, so also $\lim_{n_i \to \infty} |f(a_{n_i})| = |f(l)|$.

But since $|f(a_n)| \to \infty$, we must also have $|f(a_{n_i})| \to \infty$ since any subsequence of a sequence converges to the limit of the sequence. This is a contradiction, so we reject the assumption that f is unbounded. $\square$

Theorem 7 *If $f : [a, b] \to \mathbb{R}$ is a continuous function with $a, b \in \mathbb{R}$ then there exists $r, s \in [a, b]$ such that $f(r) = \sup_{x \in [a,b]} f(x)$ and $f(s) = \inf_{x \in [a,b]} f(x)$ (i.e. f has a maximum and minimum in $[a, b]$).*

By the lemma above, f is bounded. So by the axiom of Dedekind-completeness, f has a supremum and infimum. Let $M = \sup_{x\in[a,b]} f(x)$ and $m = \inf_{x\in[a,b]} f(x)$.

We want to find $v_M \in [a, b]$ such that $f(v_M) = M$. For every $n > 0$, by the definition of the supremum, there exists $x_n \in [a, b]$ with $M \geq f(x_n) > M - \frac{1}{n}$. Then we can construct the sequence $(x_n)_{n\geq 1}$ with the property that $\lim_{n\to\infty} f(x_n) = M$. Since x_n is bounded, it has a convergent subsequence $(x_{n_i})_{i\geq 1}$; let v_M be $\lim_{i\to\infty} x_{n_i}$. It follows that $\lim_{i\to\infty} f(x_{n_i}) = M$. By the continuity of f at v_M we have $f(v_M) = \lim_{i\to\infty} f(n_i) = M$. There also exists $v_m \in [a, b]$ such that $f(v_m) = m$.

Theorem 8 Intermediate Value theorem *If $f : [a, b] \to \mathbb{R}$ is continuous and $s \in \mathbb{R}$ is such that $f(a) < s < f(b)$, then there exists $c \in (a, b)$ such that $f(c) = s$.*

If $c = a$ is such that $f(c) = s$, then $f(a) = s$ which is a contradiction. Likewise, if $c = b$ then $f(b) = s$ which is also a contradiction, so c cannot equal a or b.

Consider the set $A = \{x \in [a, b] : f(x) \leq s\}$. Since A is a bounded subset of the real numbers by definition, by the axiom of Dedekind-completeness, A has a supremum; let this supremum be c. We then have three possibilities:

(1) $f(c) = s$; then we are done

(2) $f(c) < s$. Let $f(c) = k$. Since it is given that f is continuous over $[a, b]$, $\lim_{x\to c} f(x) = k$, so for all $\epsilon > 0$, there exists $\delta > 0$ such that if $|x-c| < \delta$, then $|f(x) - k| < \epsilon$.

Let $x = c + \frac{\delta}{2}$ and let $\epsilon = s - k$. Then there exists $\delta > 0$ such that $|c + \frac{\delta}{2} - c| = \frac{\delta}{2} < \delta$, so $|f(c + \delta 2) - k| < s - k$

Therefore

$$\begin{aligned} f(c + \frac{\delta}{2}) &= f(c + \frac{\delta}{2}) - k + k \\ &\leq |f(c + \frac{\delta}{2}) - k| + k \\ &< s - k + k \\ &= s \end{aligned}$$

Then also $c + \frac{\delta}{2} \in A$ by definition which is a contradiction since c then is not the supremum.

(3) $f(c) > s$

Since f is continuous at c, then for $\epsilon = k - s$, there exists $\delta > 0$ such that for all x such that $0 < |x - c| < \delta$, we have $|f(x) - k| < k - s$. Then $s - k < f(x) - k < k - s$ so $f(x) > s$.

Now since c is the supremum of A, $c - \delta$ is not an upper bound of A. Then there exists $c' \in A$ such that

$$c - \delta < c' < c$$

. Since $c' \in A$, we have $f(c') \leq s$ by definition. But since $|c - c'| < \delta$ so $f(c') > s$. This is a contradiction, so it cannot be the case that $f(c) > s$.

It is impossible for $f(c) > s$ or $f(c) < s$, so the only possibility is that $f(c) = s$. $\square$

Theorem 9 *If $f : [a,b] \to \mathbb{R}$ for $a, b \in \mathbb{R}$ is continuous, then it is uniformly continuous on $[a,b]$.*

Suppose that f is not uniformly continuous on $[a,b]$. Then there exists $\epsilon > 0$ such that for all $n \geq 1$, there exists $x_n, y_n \in [a,b]$ with $|x_n - y_n|$ with $|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \epsilon$.

As $(x_n)_{n \geq 1}$ is a bounded sequence, it has a convergent subsequence $(x_{n_i})_{i \geq 1}$ with limit l in $[a,b]$. But then

$$\begin{aligned} |l - y_{n_i}| &\leq |l - x_{n_i}| + |x_{n_i} - y_{n_i}| \\ &< |l - x_{n_i}| + \frac{1}{n_i} \end{aligned}$$

So y_{n_i} also converges to l. Since f is continuous at l, we have $\lim_{i \to \infty} f(x_{n_i}) = f(l) = \lim_{i \to \infty} f(y_{n_i}) f(y_{n_i})$. However, this contradicts $|f(x_{n_i} - f(y_{n_i}))| \geq \epsilon$ for all $i \geq 1$. $\square$