# Proofs for Chapter 7 Integration

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**Theorem 1** *Given $f : [a, b] \to \mathbb{R}$, let $P_1$ and $P_2$ be partitions of $[a, b]$ such that $P_1 \subset P_2$. Then*

$$L(f, P_1) \leq L(f, P_2) \leq U(f, P_2) \leq U(f, P_1)$$

Suppose $P_1 := a = p_0 < p_1 < \cdots < p_n = b$ and $P_2 := a = q_0 < q_1 < \cdots < q_m = b$. Then since $P_1 \subset P_2$, for all $q_i \in P_2$, there exists $0 \leq j < m$ such that

$$p_j \leq q_i \leq p_{j+1}$$

We assume, without loss of generality, that there only exists one such $q_i$ (since we can continue to refine the partition to add additional points to each interval).

Since $[p_j, q_i] \subset [p_j, p_{j+1}]$ and $[q_i, p_{j+1}] \subset [p_j, p_{j+1}]$, we have

$$\inf_{x \in [p_j, p_j+1]} f(x) \leq \inf_{x \in [p_j, q_i]} f(x)$$

and

$$\inf_{x \in [p_j, p_j+1]} f(x) \leq \inf_{x \in [q_i, p_{j+1}]} f(x)$$

Therefore, setting $q_{j-1} := p_i$ and $q_{j+1} := p_{i+1}$:

$$\begin{aligned}
L(f, P_1) &= \sum_{i=0}^{n-1} (q_{j+1} - q_{j-1}) \inf_{x \in [q_{j-1}, q_{j+1}]} f(x) \\
&= \sum_{i=0}^{n-1} (q_{j+1} - q_j + q_j - q_{j-1}) \inf_{x \in [q_{j-1}, q_{j+1}]} f(x) \\
&\leq \sum_{i=0}^{n-1} \left[ (q_{j+1} - q_j) \inf_{x \in [q_j, q_{j+1}]} f(x) + (q_j - q_{j-1}) \inf_{x \in [q_j, q_{j+1}]} f(x) \right] \\
&\leq \sum_{j=0}^{m} (q_{j+1} - q_j) \inf_{x \in [q_j, q_{j+1}]} f(x) = L(f, P_2)
\end{aligned}$$

We can show analogously that $U(f, P_2) \leq U(f, P_1)$. Since also $L(f, P) \leq U(f, P)$ for any partition $P$ of $f$, we have

$$L(f, P_1) \leq L(f, P_2) \leq U(f, P_2) \leq U(f, P_1)$$

as required. □

**Theorem 2** *A bounded function $f : [a,b] \to \mathbb{R}$ is Riemann integrable with Riemann integral $c \in \mathbb{R}$ iff for each $\epsilon > 0$ there exists a partition $P$ of $[a,b]$ with $c - L(f,P) < \epsilon$ and $U(f,P) - c < \epsilon$*

We first prove the $\Rightarrow$ direction.

Assume that $f$ is Riemann integrable with integral $c \in \mathbb{R}$. Then

$$\sup_P L(f,P) = \inf_P U(f,P) = c$$

by definition.

Let $\epsilon > 0$ be given. $c - \epsilon$ is not a supremum of the set of lower sums of $f$, so there exists some partition $P_1$ of $[a,b]$ such that

$$c - \epsilon < L(f,P_1) \leq c \Rightarrow c - L(f,P_1) < \epsilon$$

Likewise, $c + \epsilon$ is not a supremum of the set of upper sums of $f$, so there exists some partition $P_2$ of $[a,b]$ such that

$$c \leq U(f,P_2) < c + \epsilon \Rightarrow U(f,P) - c < \epsilon$$

as required.

Now we prove the $\Leftarrow$ direction.

By assumption, for all $\epsilon > 0$, we have $c - L(f,P) < \epsilon$ and $U(f,P) - c < \epsilon$, where $c \in \mathbb{R}$.

Let $Q$ be some partition of $[a,b]$. Since $f$ is bounded, for any partition $P$ of $[a,b]$, we have

$$L(f,P) \leq L(f,P \cup Q) \leq U(f,P \cup Q) \leq U(f,Q)$$

as shown earlier. Hence the set of all lower sums of $f$ is upper bounded and has a supremum $d$ by the axiom of Dedekind-completeness. By the definition of supremum,

$$L(f,P) \leq d$$

for any partition $P$

Suppose that $d < c$. Then setting $\epsilon = c - d$, we have

$$c - L(f,P) < c - d \Rightarrow L(f,P) > d$$

which is a contradiction. Therefore, $c$ is the supremum of the set of lower sums of $f$. It can be shown similarly, that $c$ is the infimum of the set of upper sums of $f$, so we have

$$\sup_P L(f,P) = \inf_P U(f,P) = c$$

so $f$ is integrable with integral $c$ as required.$\square$

**Theorem 3 Cauchy condition for Riemann integrability** *A bounded function $f : [a,b] \to \mathbb{R}$ is Riemann integrable if and only if for every $\epsilon > 0$ there exists a partition $P$ of $[a,b]$ such that*

$$U(f,P) - L(f,P) < \epsilon$$

From the previous theorem, if the integral is $c \in \mathbb{R}$, then we can add the inequalities to get

$$U(f, P) - L(f, P) < \epsilon$$

To prove the $\Leftarrow$ direction, given some partition $P$ of $[a, b]$, we note that the set of lower sums must have a supremum $c$ since it is upper bounded by $U(f, P)$. It remains to be shown that $c$ is also the infimum of the set of upper sums of $f$.

Assume that the infimum of the set of upper sums is $d > c$. Then

$$U(f, P) \geq d$$

for any partition $P$.

Let $\epsilon = d - c$. Then there exists some partition $Q$ such that

$$U(f, Q) - c \leq U(f, Q) - L(f, Q) < d - c$$

But then

$$U(f, Q) < d$$

which is a contradiction, so $d = c$. Hence

$$\sup_P L(f, P) = \inf_P U(f, P) = c$$

so $f$ is integrable over $[a, b]$ with integral $c$ as required. $\square$

**Theorem 4** *A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable with Riemann integral $c \in \mathbb{R}$ iff for each $\epsilon > 0$, there exists a $\delta > 0$ such that for all partitions $P$ of $[a, b]$ with $||P|| < \delta$ we have $|S(f, P, (s_i)_{0 \leq i \leq n-1} - c)| < \epsilon$*

We first prove the $\Rightarrow$ direction.

Let $\epsilon > 0$ be given. Then by the Cauchy condition for Riemann integrability, there exists a partition $P_\epsilon$ such that

$$U(f, P_\epsilon) - L(f, P_\epsilon) = \sum_{i=0}^{n} (\max_{x \in [i, i+1]} f(x) - \min_{x \in [i, i+1]} f(x))(x_{i+1} - x_i) < \frac{\epsilon}{2}$$

We use the maximum and minimum for the upper and lower sums respectively since any bounded function always attains its supremum or infimum over a closed interval.

Let $N$ be the number of points in $P$. Choose $\delta = \epsilon/2KN$ where $K = \sup\{|f(x)\} + 1$. If $P$ is any partition of $[a, b]$ with $||P|| < \delta$, then we split $U(f, P) - L(f, P)$ into two parts $S_1$ and $S_2$. $S_1$ has terms with intervals which do not contain any points in $P_\epsilon$ and $S_2$ has terms with the remaining intervals (i.e. intervals that cut across the intervals made by $P_\epsilon$).

We have

$$S_1 \leq U(f, P_\epsilon) - L(f, P_\epsilon) < \frac{\epsilon}{2}$$

For $S_2$, the subintervals used contain points of $P_\epsilon$, but the length of each interval is less than $\delta$ by definition of $P$. Furthermore, the difference in the heights between any two rectangles must also be less than or equal to $K$. Hence

$$S_2 < KN\delta = \frac{\epsilon}{2}$$

Therefore,

$$U(f, P) - L(f, P) < \epsilon$$

for any $P$ with $||P|| < \delta$, so by the Cauchy condition for Riemann integrability, $f$ is integrable with integral $c$. But since

$$c - \epsilon < L(f, P) \leq S(f, P, (s_i)_{0 \leq i \leq n-1}) \leq U(f, P) < c + \epsilon$$

we have

$$|S(f, P, (s_i)_{0 \leq i \leq n-1}) - c| < \epsilon$$

Now we show the converse.

Let $\epsilon > 0$ be given. Then let $n$ be the smallest positive integer such that $\frac{b-a}{n} < \delta$. Then by assumption there exists some partition $P$ with $||P|| = \frac{b-a}{n}$ such that

$$|S(f, P, (s_i)_{0 \leq i \leq n-1}) - c| < \epsilon$$

Then

$$c - \epsilon < S(f, P, (s_i)_{0 \leq i \leq n-1}) < c + \epsilon$$

If the sequence $(s_i)_{0 \leq i \leq n-1}$ is taken as the infimum (equivalently minimum) of $f$ in each interval, then we have

$$c - L(f, P) < \epsilon$$

Likewise, if $(s_i)_{0 \leq i \leq n-1}$ is taken as the supremum (equivalently maximum) of $f$ in each interval, then

$$U(f, P) - c < \epsilon$$

So $f$ is Riemann integrable with integral $c$, as required. □

**Theorem 5** *Let $f : [a, b] \to \mathbb{R}$ be a function that is continuous on the interval $[a, b]$. Then the Riemann integral*

$$\int_a^b f(x)\mathrm{d}x$$

*exists.*

Let $\epsilon > 0$ be given. By uniform continuity of $f$ on $[a, b]$, and using $\epsilon/2(b-a)$ in the definition of uniform continuity, there exists some $\delta > 0$ such that $|f(x) - f(y)| < \epsilon/(b-a)$ for all $x, y \in [a, b]$ with $|x - y| < \delta$. Hence, if $||P|| < \delta$ then in each given subinterval, we have $f(x_M) - f(x_m) < \epsilon/2(b-a)$, where $x_M$ and $x_m$

are the points of the subinterval where $f$ attains its maximum and minimum respectively. Then

$$\begin{aligned}
U(f,P) - L(f,P) &= \sum_{i=0}^{n-1}(r_{i+1}-r_i)(\sup_{x\in[r_i,r_{i+1}]} - \inf_{x\in[r_i,r_{i+1}]} f(x)) \\
&< \sum_{i=0}^{n-1}(r_{i+1}-r_i)\left(\frac{\epsilon}{2(b-a)}\right) \\
&= (b-a)\left(\frac{\epsilon}{2(b-a)}\right) \\
&= \frac{\epsilon}{2}
\end{aligned}$$

so by the Cauchy condition, $f$ is Riemann integrable. □

**Theorem 6** *Suppose that $f : [a,b] \to \mathbb{R}$ is bounded and integrable on $[a,r]$ for every $a < r < b$. Then $f$ is integrable on $[a,b]$ and*

$$\int_a^b f = \lim_{r\to b^-} \int_a^r f$$

Since $f$ is bounded, $|f| \le M$ for some $M > 0$ over $[a,b]$. Given $\epsilon > 0$, let

$$r = b - \frac{\epsilon}{4M}$$

where it is assumed that $\epsilon$ is sufficiently small so that $r > a$. Since $f$ is integrable on $[a,r]$, there is a partition $P$ of $[a,r]$ such that

$$U(f,P) - L(f,P) < \frac{\epsilon}{2}$$

Then $P' = P \cup \{b\}$ is a partition of $[a,b]$ whose last interval is $[r,b]$. The boundedness of $f$ implies that

$$\sup_{[r,b]} f - \inf_{[r,b]} f \le 2M$$

Therefore

$$\begin{aligned}
U(f,P') - L(f,P') &= U(f,P) - L(f,P) + \left(\sup_{[r,b]} - \inf_{[r,b]} f\right) \cdot (b-r) \\
&< \frac{\epsilon}{2} + 2M \cdot (b-r) = \epsilon
\end{aligned}$$

so by the Cauchy condition, $f$ is integrable on $[a,b]$. Using the additivity of the integral, we also get

$$\left|\int_a^b f - \int_a^r f\right| = \left|\int_r^b f\right| \le M\cdot(b-r) \to 0 \qquad \text{as } r \to b^-$$

**Theorem 7** *If $f : [a, b] \to \mathbb{R}$ is a bounded function with finitely many discontinuities, then $f$ is Riemann integrable.*

Let the set of discontinuities be $\{c_1, c_2, \ldots, c_n\}$, where $c_i < c_{i+1}$ for all $1 \leq i < n$. $f$ is then continuous over $[a, k]$, for all $a < k < c_1$, so by the theorem above, $f$ is integrable over $[a, c_1]$. It is sufficient to show that $f$ is Riemann integrable over $[a, c_1]$, since then the integrals of $f$ over every interval $[c_i, c_{i+1}]$, where $1 \leq i < n$. $\square$