Proofs for Chapter 8 Series

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Definition 1 *A **series** is an infinite sum of real numbers*

$$\sum_{i=1}^{\infty}$$

Definition 2 *The **partial sum** $(S_n)_{n\geq 1}$ is defined as*

$$S_n = \sum_{i=1}^{n} a_i$$

Definition 3 *Let $\sum_{i=1}^{\infty} a_i$ be a series. Then the series has a limit l in $\mathbb{R}$ iff its sequences of partial sums $(S_n)_{n\geq 1}$ has limit l. The series is said to diverge if it does not converge to some l in $\mathbb{R}$.*

Theorem 1 *Suppose that the series of real numbers $\sum_{i=1}^{\infty} a_i$ converges. Then the sequence of summands has limit 0:* $\lim_{n\to\infty} a_n = 0$

Let $\epsilon > 0$ be given. By definition of series convergence, the sequence of partial sums $(S_n)_{n\geq 1}$ converges to a limit in $\mathbb{R}$, so it is a Cauchy sequence.

Then there exists $N \in \mathbb{N}$ such that for $n, m \geq N$, we have $|S_m - S_n| < \epsilon$. Set $m = n + 1$ so that $|S_{n+1} - S_n| < \epsilon$ for $n > N$. But $S_{n+1} - S_n = a_{n+1}$, so $|a_{n+1}| < \epsilon$ for $n > N$, so $|a_n| < \epsilon$ for $n > N + 1$. Then by the definition of sequence convergence, $\lim_{n\to\infty} a_n = 0$ as required. $\square$

Definition 4 *The geometric series is defined as*

$$G(x) = \sum_{n=1}^{\infty} x^n$$

Theorem 2 $\sum_{i=1}^{\infty} x^i$ *converges iff* $|x| < 1$

$$\begin{aligned} G_n(x) &= x + \sum_{i=2}^{n} x^i \\ &= x + x \cdot \sum_{i=1}^{n-1} x^i \\ &= x + x \cdot (G_n(x) - x^n) \end{aligned}$$

Making $G_n(x)$, the subject, we obtain

$$G_n(x) = \frac{x}{1-x} - \frac{1}{1-x} \cdot x^{n+1}$$

Hence $(G_n)_{n\geq 1}$ converges iff x^{n+1} converges iff $|x| < 1$. $\square$

Definition 5 *The harmonic series is defined as*

$$S = \sum_{n=1}^{\infty} \frac{1}{n}$$

Theorem 3 *The harmonic series diverges*

$$S = 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{> \frac{1}{4} + \frac{1}{4}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{> \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}} + \cdots > 1 + \frac{1}{2} + \frac{1}{2} + \cdots$$

Hence

$$S_{2^n} = \sum_{i=1}^{2^n} \frac{1}{i} > 1 + \frac{n}{2}$$

Then $(S_n)_{n\geq 1}$ is not bounded above, so $(S_n)_{n\geq 1}$ has no limit. Therefore, the harmonic series diverges by definition of series divergence. $\square$

Theorem 4 *The series of inverse squares converges*

Let $T_n = \sum_{i=1}^{n} \frac{1}{i(i+1)}$. Then using partial fractions,

$$\begin{aligned} T_n &= \sum_{i=1}^{n} \frac{1}{i(i+1)} \\ &= \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right) \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

Now for $i \geq 2$

$$\frac{1}{i(i+1)} < \frac{1}{i^2} < \frac{1}{i(i-1)}$$

Therefore

$$T_n - \frac{1}{2} < \sum_{i=2}^{\infty} \frac{1}{i^2} < T_{n-1}$$

$$\frac{1}{2} - \frac{1}{n+1} < S_n - 1 < 1 - \frac{1}{n}$$

$$\frac{3}{2} - \frac{1}{n+1} < S_n < 2 - \frac{1}{n}$$

Hence $(S_n)_{n\geq 1}$ is increasing and bounded above, so $(S_n)_{n\geq 1}$ has a limit, so the series of inverse squares converges by definition. $\square$

Theorem 5 ***Comparison test*** *For a series $\sum_{i=1}^{\infty} a_i$ with positive terms, let $\lambda > 0$ and $N \in \mathbb{N}$. Further, let $\sum_{i=1}^{\infty} c_i$ be a converging series and $\sum_{i=1}^{\infty} d_i$ a diverging series*

1. *If $a_i \leq \lambda c_i$ for all $i > N$, then $\sum_{i=1}^{\infty} a_i$ converges*
2. *If $a_i \geq \lambda d_i$ for all $i > N$, then $\sum_{i=1}^{\infty} a_i$ diverges*

If $\sum_{i=1}^{\infty} c_i$ is convergent, then its sequence $(U_m)_{m\geq 1}$ of partial sums converges and is bounded above by some M. Since $a_i \leq \lambda c_i$, the sequence of partial sums $(S_n)_{n\geq 1}$ is bounded above by λM and therefore also converges, so $\sum_{i=1}^{\infty} a_i$ converges by definition.

If $\sum_{i=1}^{\infty} d_i$ diverges, then its sequence of partial sums $(L_m)_{m\geq 1}$diverges, so for all real k, there exists $N \in \mathbb{N}$ such that $L_m > \frac{k}{\lambda}$ for all $m > N$. Since $a_i \geq \lambda c_i$, the sequence of partial sums $(S_m)_{m\geq 1}$ of $\sum_{i=1}^{\infty} a_i$ has $S_m \geq \lambda L_m > k$ for all $m > N$ so $(S_m)_{m\geq 1}$ diverges and $\sum_{i=1}^{\infty}$ diverges by definition. $\square$

Theorem 6 ***Limit comparison text*** *Let $\sum_{i=1}^{\infty} c_i$ be a convergent series and let $\sum_{i=1}^{\infty} d_i$ be a divergent series.*

1. *If $\lim_{i\to\infty} \frac{a_i}{c_i} \in \mathbb{R}$ exists, then $\sum_{i=1}^{\infty} a_i$ converges*
2. *If $\lim_{i\to\infty} \frac{d_i}{a_i} \in \mathbb{R}$ exists, then $\sum_{i=1}^{\infty} a_i$ diverges*

Let $\epsilon > 0$ be given. Then there exists N such that for all $i > N$

$$\left| \frac{a_i}{c_i} - l \right| < \epsilon$$

Then

$$\left| \frac{a_i}{c_i} \right| \leq \left| \frac{a_i}{c_i} - l \right| + |l| < \epsilon + |l| = \lambda$$

where λ is constant. Hence $a_i \leq \lambda c_i$ (a_i and c_i are both positive) for all $i > N$ so by the comparison test $\sum_{i=1}^{\infty} a_i$ converges. The same reasoning can be used to show (2).

Theorem 7 ***D'Alembert's Ratio Test***

1. *If $\frac{a_{i+1}}{a_i} \geq 1$ for all $i \geq N$, where $N \in \mathbb{N}$, then $\sum_{i=1}^{\infty}$ diverges*
2. *If there exists some $k < 1$ in $\mathbb{R}$ such that $\frac{a_{i+1}}{a_i} \leq k$ for all $i \geq N$ where $N \in \mathbb{N}$, then $\sum_{i=1}^{\infty} a_i$ converges*

If $\frac{a_{i+1}}{a_i} \geq 1$ for all $i \geq N$, then $a_N \leq a_{N+1} \leq a_{N+2} \leq \ldots$. Since $a_N > 0$ then $a_n \geq a_N$ for $n > N$ so a_n does not tend to zero. Hence by the contrapositive of Theorem 1, the series is not convergent.

If $\frac{a_{i+1}}{a_i} \leq k < 1$ for all $i \geq N$, then $a_{N+m} \leq k^m a_N$. Setting $\lambda = \frac{a_N}{k^N}$, which is a constant, and $n = N + m$, we have $a_n \leq \lambda k^n$. Since $|k| < 1$, the geometric series $\sum_{i=1}^{\infty} k^i$ converges, so by the comparison test $\sum_{n=1}^{\infty} a_n$ converges.

Theorem 8 ***D'Alembert's Limit Ratio Test*** *For a series $\sum_{i=1}^{\infty} a_i$, suppose $\lim_{i\to\infty} \frac{a_{i+1}}{a_i}$ exists. Then*

1. *If $\lim_{i\to\infty} \frac{a_{i+1}}{a_i} > 1$ then $\sum_{i=1}^{\infty} a_i$ diverges*
2. *If $\lim_{i\to\infty} \frac{a_{i+1}}{a_i} = 1$, then the test is inconclusive*
3. *If $\lim_{i\to\infty} \frac{a_{i+1}}{a_i} < 1$ then $\sum_{i=1}^{\infty} a_i$ converges*

We first consider when $\lim_{i\to\infty} \frac{a_{i+1}}{a_i} = l > 1$. For $\epsilon = l - 1$ (ϵ is positive as $l > 1$), there exists $N \in \mathbb{N}$ such that for all $i > N$

$$\left| \frac{a_{i+1}}{a_i} - l \right| < \epsilon$$

$$l - \epsilon < \frac{a_{i+1}}{a_i} < l + \epsilon$$

$$1 < \frac{a_{i+1}}{a_i} < 2l - 1$$

Since $\frac{a_{i+1}}{a_i} > 1$, by d'Alembert's ratio test, $\sum_{i=1}^{\infty} a_i$ diverges.

For $\lim_{i\to\infty} \frac{a_{i+1}}{a_i} = l < 1$, set $\epsilon = \frac{1-l}{2}$ which is positive as $l < 1$. Then there exists $N \in \mathbb{N}$ such that for all $i > N$,

$$\left| \frac{a_{i+1}}{a_i} - l \right| < \epsilon$$

$$l - \epsilon < \frac{a_{i+1}}{a_i} < l + \epsilon = \frac{1+l}{2}$$

Hence setting $k = \frac{1+l}{2}$

$$\frac{a_{i+1}}{a_i} < k < 1$$

so by d'Alembert's ratio test $\sum_{i=1}^{\infty} a_i$ converges.

Theorem 9 ***Integral test*** *Let $f : \mathbb{R} \to \mathbb{R}^+$ be a function that is continuous and decreasing and positive on the interval $[1, \infty)$. Let $a_n = f(n)$ for all n in $\mathbb{N}$. Then we have*

1. *If $\int_1^{\infty} f(x)\mathrm{d}x$ converges, then the series $\sum_{i=1}^{\infty} a_i$ converges*

2. If $\int_1^\infty f(x)\mathrm{d}x$ diverges, then $\sum_{i=1}^\infty a_i$ diverges

If $\int_1^\infty f(x)\mathrm{d}x$ converges, then $\lim_{r\to\infty}\int_1^r f(x)\mathrm{d}x = L \in \mathbb{R}$ by definition. Hence the sequence $(a_n)_{n\geq 1} = \int_1^n f(x)\mathrm{d}x$ converges to L.

Let the partition $P = \{i \mid 1 \leq i \leq n\}$ for some $n \in \mathbb{N}$. Then

$$L(f,P) = \sum_{i=2}^{n} f(i) = \sum_{i=1}^{n} a_i - a_1$$

Hence

$$\sum_{i=1}^{n} a_i < \int_1^n f(x)\mathrm{d}x + a_1$$

Since $\int_1^n f(x)\mathrm{d}x + a_1 \to L + a_1$ as $n \to \infty$, the sequence of partial sums of $\sum_{i=1}^\infty a_i$ converges so $\sum_{i=1}^\infty a_i$ converges by definition.

If $\int_1^\infty f(x)\mathrm{d}x$ diverges, then the sequence $(b_n)_{n\geq 1}$ where $b_n = \int_1^n f(x)\mathrm{d}x$ also diverges.

Using the same partition as before

$$U(f,P) = \sum_{i=1}^{n-1} f(i) = \sum_{i=1}^{n} a_i - a_n$$

Hence

$$\sum_{i=1}^{n} a_i > \int_1^n f(x)\mathrm{d}x + a_n$$

Since $\int_1^n f(x)\mathrm{d}x$ diverges, $\int_1^n f(x)\mathrm{d}x + a_n$ also diverges so $\sum_{i=1}^n a_i$ diverges, so the series $\sum_{i=1}^\infty$ diverges by definition.

Definition 6 *A **permutation** π over the natural numbers is a bijective function $\pi : \mathbb{N} \to \mathbb{N}$*

Definition 7 *A series $\sum_{i=1}^\infty a_i$ is **unconditionally convergent** iff it converges and the permuted series*

$$\sum_{i=1}^{\infty} a_{\pi(i)}$$

converges to the same limit for all permutations $\pi : \mathbb{N} \to \mathbb{N}$

Definition 8 *A series $\sum_{i=1}^\infty a_i$ is absolutely convergent iff the corresponding series of absolute values of summands*

$$\sum_{i=1}^{\infty} |a_i|$$

converges

Theorem 10 ***Absolute value comparison test*** *Suppose b_i is a non-negative sequence such that $\sum_{i=1}^{\infty} b_i$ converges and suppose a_i is a sequence such that $|a_i| \leq b_j$ for all $i \geq 1$. Then $\sum_{i=1}^{\infty} a_i$ converges.*

Denote the partial sums of $\sum_{i=1}^{\infty} a_i$ by S_n and that of $\sum_{i=1}^{\infty} b_i$ by S'_n.

Let $\epsilon > 0$. Since S'_n converges in the real numbers, it is a Cauchy sequence, so there exists N such that for all $m \geq n > N$, we have $S'_m - S'_n < \epsilon$. Then for $m \geq n > N$ and using the general triangle inequality, we have

$$|S_m - S_n| = \left|\sum_{i=n+1}^{m} a_i\right| \leq \sum_{i=n+1}^{m} |a_i| \leq \sum_{i=n+1}^{m} b_i = S'_m - S'_n < \epsilon$$

Hence S_n is a Cauchy sequence so it converges.

Theorem 11 *If $\sum_{i=1}^{\infty} a_i$ is absolutely convergent then it is convergent.*

Set $b_n = |a_n|$ in the absolute value comparison test.

Theorem 12 *If a series is absolutely convergent, then it is unconditionally convergent*

Suppose that $\sum_{i=1}^{\infty} a_i$ is absolutely convergent. Then from Theorem 10, $\sum_{i=1}^{\infty} a_i$ is convergent to A. Hence for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$\left|\sum_{i=1}^{n} a_i - A\right| = \left|A - \sum_{i=1}^{n} a_i\right| = \left|\sum_{i=n+1}^{\infty} a_i\right| < \frac{\epsilon}{2}$$

Let $\sum_{i=1}^{\infty} b_i$ be some permutation of $\sum_{i=1}^{\infty} a_i$, that is, there exists some bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $b_i = a_{\sigma(i)}$. Let $S = \{k \mid \sigma(k) \leq i\}$. It follows that $|S| = n$ since σ is bijective. Then let $J = \max S$. For $m \geq J$, we have

$$\sum_{i=1}^{m} b_i = b_1 + \cdots + b_m = a_{\sigma(1)} + \cdots + a_{\sigma(m)}$$

Every term $a_1, \ldots, a_n$ must be include in the sum as J is the maximum of S. Then for $m \geq J$, $\sum_{i=1}^{m} b_i - \sum_{j=1}^{n} a_j = \sum_{k=n+1}^{\infty} a_k$, so by the general triangle inequality

$$\left|\sum_{i=1}^{m} b_i - \sum_{j=1}^{n} a_j\right| = \left|\sum_{k=n+1}^{\infty} a_k\right| \leq \sum_{k=n+1}^{\infty} |a_k| < \frac{\epsilon}{2}$$

Therefore

$$\begin{aligned}\left|A - \sum_{i=1}^{m} b_i\right| &\leq \left|A - \sum_{i=1}^{n} a_i\right| + \left|\sum_{i=1}^{m} b_i - \sum_{j=1}^{n} a_j\right| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon\end{aligned}$$

so $\sum_{i=1}^{\infty} b_i$ converges to the same limit and therefore $\sum_{i=1}^{\infty} a_i$ is unconditionally convergent.

Theorem 13 ***Limit absolute value ratio test*** *Consider the series* $\sum_{i=0}^{\infty} a_i$ *with* $a_i \neq 0$ *for* $i \geq 1$.

1. *If* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ *then* $\sum_{n=1}^{\infty} a_n$ *converges absolutely*
2. *If* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ *then* $\sum_{n=1}^{\infty} a_n$ *diverges*
3. *If* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ *then* $\sum_{n=1}^{\infty} a_n$ *may converge or diverge*

This is immediate from D'Alembert's limit ratio test.

Definition 9 *Let* $(a_n)_{n\geq 1}$ *be a sequence, and let the sequence* $(b_n)_{n\geq 1}$ *be the sequence such that*

$$b_n = \sup\{a_m | m \geq n\}$$

The ***limit superior*** *of* $(a_n)_{n\geq 1}$ *is defined as* $\lim_{n\to\infty} b_n$, *denoted* $\limsup_{n\to\infty} a_n$

Definition 10 *Let* $(a_n)_{n\geq 1}$ *be a sequence, and let the sequence* $(c_n)_{n\geq 1}$ *be the sequence such that*

$$c_n = \inf\{a_m | m \geq n\}$$

The ***limit inferior*** *of* $(a_n)_{n\geq 1}$ *is defined as* $\lim_{n\to\infty} c_n$, *denoted* $\liminf_{n\to\infty} a_n$

Lemma 1 *Given a sequence* $(a_n)_{n\geq 1}$, *show that there exist subsequences* $(a_{m_i})_{i\geq 1}$ *and* $(a_{p_i})_{i\geq 1}$ *such that* $\limsup_{n\to\infty} a_n = \lim_{i\to\infty} a_{m_i}$ *and* $\liminf_{n\to\infty} a_n = \lim_{i\to\infty} a_{p_i}$.

Let $\limsup_{n\to\infty} a_n = l$. Then by definition of the limit superior, given $\frac{\epsilon}{2} > 0$, there exists N_1 such that for all $n > N_1$

$$|\sup_{m\geq n}\{a_m\} - l| < \frac{\epsilon}{2}$$

We construct a subsequence $(a_{m_i})_{i\geq 1}$ of (a_n) where for all $i \geq 1$, a_{m_i} satisfies

$$|a_{m_i} - \sup_{m\geq i}\{a_m\}| < \frac{1}{i}$$

Let N_2 be the smallest integer such that

$$|a_{m_N} - \sup_{m\geq N}\{a_m\}| < \frac{1}{N_2} < \frac{\epsilon}{2}$$

Let $N = \max\{N_1, N_2\}$. Then for all $n > N$

$$\begin{aligned}|a_{m_n} - l| &\le |a_{m_n} - \sup_{m\ge n}\{a_m\}| + |\sup_{m\ge n}\{a_m\} - l| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon\end{aligned}$$

so $(a_{m_i})_{i\ge 1} \to \limsup_{n\to\infty} a_n$ as required. The same can be shown similarly for $\liminf_{n\to\infty} a_n$.

Lemma 2 *If $(a_{q_i})_{i\ge 1}$ is any convergent subsequence of $(a_n)_{n\ge 1}$, then we have*

$$\liminf_{n\to\infty} a_n \le \lim_{i\to\infty} a_{q_i} \le \limsup_{n\to\infty} a_n$$

By definition,

$$a_{q_i} \ge \inf_{m\ge q_i}\{a_m\} \ge \inf_{m\ge n}\{a_m\}$$

for all $i \ge 1$, where $i = n$. Since a_{q_n} is greater than $\inf_{m\ge n}\{a_m\}$ for all $n \ge 1$, $\lim_{i\to\infty} a_{q_i} \ge \lim_{n\to\infty}(\inf_{m\ge n}\{a_m\}) = \liminf_{n\to\infty} a_n$.

The same reasoning can be used to show that $\lim_{i\to\infty} a_{q_i} \le \limsup_{n\to\infty} a_n$.

Lemma 3 *If a sequence $(a_n)_{n\to\infty}$ converges, then*

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \lim_{n\to\infty} a_n$$

We can construct a subsequence $(a_{m_i})_{i\ge 1}$ that converges to $\liminf_{n\to\infty} a_n$ and a second subsequence $(a_{p_i})_{i\ge 1}$ that converges to $\limsup_{n\to\infty} a_n$. But since any convergent subsequence converges to the limit of the sequence, $\liminf_{n\to\infty} a_n$ and $\limsup_{n\to\infty} a_n$ must both be equal to the limit of $(a_n)_{n\ge 1}$.

Theorem 14 ***nth root test*** *For the series $\sum_{n=1}^{\infty} a_n$, let $l = \limsup_{n\to\infty} |a_n|^{1/n}$*

1. *If $l < 1$, then the series converges absolutely*
2. *If $l > 1$, then the series diverges*
3. *If $l = 1$, then the series absolutely converges, converges or diverges*

For (1), let $l = \limsup_{n\to\infty} |a_n|^{1/n} < 1$. Then $l < (l+1)/2 < 1$. By the definition of the limit superior, there exists N such that for all $n > N$, $|a_n|^{1/n} < (l+1)/2$, that is, $|a_n| < c^n$ for $n > N$ where $c = (l+1)/2 < 1$, so the series converges absolutely by the comparison test with the geometric series $\sum_{n=1}^{\infty} c_n$.

Now suppose $l > 1$ for (2). Then $c = (l+1)/2 > 1$. Since l is the limit superior of $|a_n|^{1/n}$, there exists a subsequence $(|a_{n_i}|^{1/n_i})_{i\ge 1}$ such that $\lim_{i\to\infty} |a_{n_i}|^{1/n_i} = l$. Then there exists N such that for $i > N$, we have $|a_{n_i}|^{1/n_i} > c > 1$ or $|a_{n_i}| > c^{n_i} > 1$, so $|a_n|$ does not converge to 0 so the series diverges.

For (3), consider the series (i) $\sum_{n=1}^{\infty} 1/n^2$, (ii) $\sum_{n=1}^{\infty} (-1)^{n+1}/2$ and (iii) $\sum_{n=1}^{\infty} 1/n$. All three have limit superior equal to 1, but (i) converges (absolutely), (ii) converges conditionally and (iii) diverges.