Proofs for Chapter 9 Differentiation

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Definition 1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a function and x in $\mathbb{R}$ and let $h > 0$ be given. Then*

1. ***Newton's difference quotient*** *at x for f is given by*

$$\frac{\Delta f(x)}{\Delta x} = \frac{f(x+h) - f(x)}{h}$$

2. *f is **differentiable** at x iff*

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

*exists and is a real number. This limit is said to be the **derivative** of f at x*

Theorem 1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. If f is differentiable at x, then f is also continuous at x.*

Suppose f is differentiable at $x_0 \in \mathbb{R}$. Then equivalent to Definition 1,

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \in \mathbb{R}$$

Now

$$\begin{aligned}
\lim_{x \to x_0} f(x) - f(x_0) &= \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} \right) (x - x_0) \\
&= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \lim_{x \to x_0} x - x_0 \\
&= f'(x_0) \cdot 0 \\
&= 0
\end{aligned}$$

so $\lim_{x \to x_0} f(x) = \lim_{\to x \to x_0} f(x_0) = f(x_0)$ as required. $\square$

Theorem 2 *If f is differentiable in (a, b), then*

1. *f is strictly increasing at $x_0 \in (a, b)$ iff $f'(x_0) > 0$*
2. *f is strictly decreasing at $x_0 \in (a, b)$ iff $f'(x_0) < 0$*

3. *f is a local maximum or minimum at $x_0 \in (a, b)$ iff $f'(x_0) = 0$*

If f is strictly increasing at x_0, then $f(x) > f(x_0)$ for $x > x_0$, so $f(x) - f(x_0) > 0$ and hence $\frac{f(x)-f(x_0)}{x-x_0} > 0$, so also $f'(x_0) > 0$. Likewise, if $f'(x_0) > 0$, then $\frac{f(x)-f(x_0)}{x-x_0} > 0$. Then either $f(x)-f(x_0) > 0$ for $x > x_0$ or $f(x_0)-f(x) > 0$ for $x_0 > x$, so in either case, f is increasing at x_0.

The same can be shown for (2), and (3) can be shown through contradiction (i.e. at x_0, if $f'(x_0) = 0$ then f cannot be strictly increasing or decreasing, so it must be constant at that point, and likewise a minimum or maximum in the neighbourhood of x_0).

Theorem 3 *If f and g are differentiable at x, then the product function $f \cdot g$ defined by $(f \cdot g)(x) = f(x) \cdot g(x)$ is also differentiable at x and*

$$(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

$$\begin{aligned}
\lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} &= \lim_{h \to 0} \frac{f(x+h)g(x+h) + f(x)g(x+h) - f(x)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h \to 0} f(x) \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} + \lim_{h \to 0} g(x+h) \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= f(x)g'(x) + g(x)f'(x)
\end{aligned}$$

Note that $\lim_{h \to 0} g(x+h) = g(x)$ is equivalent to stating that $\lim_{x' \to x} g(x') = g(x)$, which is immediate from the assumption that g is differentiable (and therefore continuous) at x. $\square$

Theorem 4 *If f and g are differentiable at $g(x)$ and x respectively, then $f \circ g$ is also differentiable at x and*

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

It is given that f is differentiable, so

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists. Now we define the error $v(h)$ in approximating the derivative of $f(x)$ as

$$v(h) = \begin{cases} \frac{f(x+h)-f(x)}{h} - f'(x) & \text{if } h \neq 0 \\ 0 & \text{if } h = 0 \end{cases}$$

Then $\lim_{h \to 0} v(h) = 0 = v(0)$, so v is continuous at $h = 0$.

For $h \neq 0$, we rearrange $v(h)$ to get

$$f(x+h) = f(x) + h(v(h) + f'(x))$$

Note that trivially, for $h = 0$, this becomes $f(x) = f(x)$ so this linear approximation holds for any h.

Likewise, let $w(h)$ be the error in g for some h; then also

$$g(x+h) = g(x) + h(w(h) + g'(x))$$

Newton's difference quotient for $f \circ g(x)$ is given by

$$\frac{\Delta f \circ g(x)}{\Delta x} = \frac{f(g(x+h)) - f(g(x))}{h}$$

Now

$$\begin{aligned} f[g(x+h)] - f[g(x)] &= f\{g(x) + h[w(h) + g'(x)]\} - f[g(x)] \\ &= f[g(x)] + (h[w(h) + g'(x)])(v\{h[w(h) + g'(x)]\} + f'(g(x))) - f[g(x)] \\ &= h(k)(v(hk) + f'(g(x))) \end{aligned}$$

where $k = w(h) + g'(x)$

So the difference quotient becomes

$$\frac{\Delta f \circ g(x)}{\Delta x} = k(v(hk) + f'(g(x)))$$

Now $w(h) \to 0$ as $h \to 0$, so $k \to g'(x)$ since g is differentiable. We have shown v is continuous, so $\lim_{h \to 0} v(hk) = v(0) = 0$ from before. Hence

$$(f \circ g)'(x) = g'(x) f'(g(x))$$

as required. $\square$

Theorem 5 *If f and g are differentiable at x, then $af(x) + bg(x)$ is differentiable at x with*

$$(a \cdot f + b \cdot g)'(x) = af'(x) + bg'(x)$$

This is immediate from the limit properties of continuous functions.

Theorem 6 ***Rolle's theorem*** *If $f : [a, b] \to \mathbb{R}$ is continuous and f is differentiable in (a, b), and $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$*

Any continuous function attains its maximum and minimum over a closed interval. If the maximum or minimum occurs in (a, b), then from Theorem 2, if $x = c$ at the minimum or maximum, then $f'(c) = 0$. Otherwise, the maximum or minimum occurs at a or b. But since $f(a) = f(b)$, f is constant over $[a, b]$ and has derivative 0. $\square$

Theorem 7 ***Mean value theorem*** *If $f : [a, b] \to \mathbb{R}$ is continuous and $f : (a, b) \to \mathbb{R}$ is differentiable, then there exists $c \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Let $s = \frac{f(b)-f(a)}{b-a}$ and let $g(x) = f(x) - sx$
Then

$$\begin{aligned}
g(a) &= f(a) - a\frac{f(b)-f(a)}{b-a} \\
&= \frac{bf(a) - af(a) - af(b) + af(a)}{b-a} \\
&= \frac{bf(a) - af(b) + bf(b) - bf(b)}{b-a} \\
&= \frac{f(b)(b-a) - bf(b) + bf(a)}{b-a} \\
&= f(b) - b\frac{f(b)-f(a)}{b-a} \\
&= g(b)
\end{aligned}$$

Therefore by Rolle's theorem, there exists $c \in (a,b)$ such that $g'(c) = 0$. But since $g'(x) = f'(x) - s$, we have

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

as required. $\square$

Theorem 8 ***Taylor's theorem*** *If f is n times differentiable in (a,b) with $x_0 \in (a,b)$, then for any $x \in (a,b)$ we have*

$$f(x) = \sum_{i=0}^{n-1} \frac{(x-x_0)^i}{i!} f^{(i)}(x_0) + E_n$$

where

$$E_n = \frac{1}{n!}(x-x_0)^n f^{(n)}(x^*)$$

is the ***Lagrange error term*** *for some x^* between x and x_0*

We define $F(y)$ as the error in approximating $f(y)$

$$F(y) = f(x) - \sum_{i=0}^{n-1} \frac{(x-y)^i}{i!} f^{(i)}(y)$$

Note that $F(x) = 0$. Differentiating with respect to y, we get

$$
\begin{aligned}
F'(y) &= f'(y) - \sum_{i=1}^{n-1} \left(-\frac{(x-y)^{i-1}}{(i-1)!} f^{(i)}(y) + \frac{(x-y)^i}{i!} f^{(i+1)}(y) \right) \\
&= -f'(y) \\
&- \left(-f^{(1)}(y) + (x-y) f^{(2)}(y) \right) \\
&- \left(-(x-y) f^{(2)}(y) + \frac{(x-y)^2}{2} f^{(3)}(y) \right) \\
&- \dots \\
&- \left(-\frac{(x-y)^{n-2}}{(n-2)!} f^{(n-1)}(y) + \frac{(x-y)^n}{n!} f^{(n+1)}(y) \right) \\
&= -\frac{1}{(n-1)!} (x-y)^{n-1} f^{(n)}(y)
\end{aligned}
$$

Let

$$G(y) = F(y) - \left(\frac{x-y}{x-x_0} \right)^n F(x_0)$$

which satisfies $G(x_0) = G(x) = 0$. By Rolle's Theorem, there exists x^* between x and x_0 such that

$$0 = G'(x^*) = F'(x^*) + \frac{n(x-x^*)^{n-1}}{(x-x_0)^n} F(x_0) = -\frac{(x-x^*)^{n-1}}{(n-1)!} f^{(n)}(x) + \frac{n(x-x^*)^{n-1}}{(x-x_0)^n} F(x_0)$$

Note that $F(x_0)$ is a constant.

Then

$$F(x_0) = \frac{1}{n!} (x-x_0)^n f^{(n)}(x^*) = E_n$$

as required. □

Theorem 9 *Suppose $f, g : (a,b) \to \mathbb{R}$ have derivatives $f', g' : (a,b) \to \mathbb{R}$ that are continuous in (a,b). If $f(c) = g(c) = 0$ for some $c \in (a,b)$ and $g'(c) \neq 0$ then*

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f(x)-f(c)}{g(x)-g(c)} = \lim_{x \to c} \frac{\left(\frac{f(x)-f(c)}{x-c}\right)}{\left(\frac{g(x)-g(c)}{x-c}\right)} = \frac{\lim_{x \to c} \left(\frac{f(x)-f(c)}{x-c}\right)}{\lim_{x \to c} \left(\frac{g(x)-g(c)}{x-c}\right)} = \frac{f'(c)}{g'(c)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

Theorem 10 ***Fundamental theorem of calculus*** *If $f : [a,b] \to \mathbb{R}$ is continuous and the function $F : [a,b] \to \mathbb{R}$ is defined by $F(y) = \int_a^y f(x) \mathrm{d}x$ then F is uniformly continuous on $[a,b]$ and $F'(x) = f(x)$ for $x \in (a,b)$*

Since f is continuous on $[a, b]$, it is bounded. Then, by definition, there exists $K > 0$ such that $|f(x)| < K$ for $x \in [a, b]$. For proving the uniform continuity of F, let $\epsilon > 0$ be given. For any h with $0 < h < \epsilon/K$, we have

$$F(y+h) - F(y) = \int_y^{y+h} f(x)\mathrm{d}x$$

Then

$$|F(y+h) - F(y)| = \left|\int_y^{y+h} f(x)\mathrm{d}x\right| \leq \int_y^{y+h} |f(x)|\mathrm{d}x \leq \int_y^{y+h} K\mathrm{d}x = Kh < \epsilon$$

so F is uniformly continuous.

Note that

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\mathrm{d}t$$

Since it is given that f is continuous at x, there exists δ such that $|y - x| < \delta$ implies $|f(y) - f(x)| < \epsilon$. Then if $|h| < \delta$

$$\begin{aligned}
\left|\frac{1}{h}\int_x^{x+h} f(t)\mathrm{d}t - f(x)\right| &= \left|\frac{1}{h}\int_x^{x+h} (f(t) - f(x))\mathrm{d}t\right| \\
&\leq \frac{1}{|h|}\left|\int_x^{x+h} |f(t) - f(x)|\mathrm{d}t\right| \\
&\leq \frac{\epsilon|h|}{|h|} \\
&= \epsilon
\end{aligned}$$

The substitution in the third step is because $x < t < x+h$ for the differential $\mathrm{d}t$ so $|t - x| < |h| < \delta$.

Taking the limits of both sides, we get

$$\left|\lim_{h\to 0} \frac{F(x+h) - F(x)}{h} - f(x)\right| < \epsilon \Rightarrow |F'(x) - f(x)| < \epsilon$$

so $F'(x) = f(x)$ as required. $\square$