Proofs for Chapter 6 Continuous Functions

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December 30, 2024

Definition 1 $\epsilon - \delta$ **definition** *The function* $f : [a, b] \to \mathbb{R}$ *has a limit* $l \in \mathbb{R}$ *at* $x_0 \in [a, b]$ *iff*

$$\forall \epsilon > 0 \ \exists \delta > 0 \ \forall x \in [a, b](0 < |x - x_0| < \delta \Rightarrow |f(x) - l| < \epsilon)$$

Definition 2 **Limit as x approaches** ∞ *The function* $f : [a, \infty) \to \mathbb{R}$ *has a limit* $l \in \mathbb{R}$ *as* $x \to \infty$ *iff*

$$\forall \epsilon > 0 \ \exists c > 0 \ \forall x \in [a, \infty)(x > c \Rightarrow |f(x) - l| < \epsilon)$$

Definition 3 **Limit as x approaches** $-\infty$ *The function* $f : (-\infty, b] \to \mathbb{R}$ *has a limit* $l \in \mathbb{R}$ *as* $x \to -\infty$ *iff*

$$\forall \epsilon > 0 \ \exists c > 0 \ \forall x \in (-\infty, b](x < -c \Rightarrow |f(x) - l| < \epsilon)$$

Definition 4 **Left sided limit** *The function* $f : [a, b] \to \mathbb{R}$ *tends to the limit* l *as* $x \to b^-$ *iff*

$$\forall \epsilon > 0 \ \exists \delta > 0 \ \forall x \in [a, b] \ (b - \delta < x < b \Rightarrow |f(x) - l| < \epsilon)$$

Definition 5 **Right sided limit** *The function* $f : [a, b] \to \mathbb{R}$ *tends to the limit* l *as* $x \to a^+$ *iff*

$$\forall \epsilon > 0 \ \exists \delta > 0 \ \forall x \in [a, b] \ (a < x < a + \delta \Rightarrow |f(x) - l| < \epsilon)$$

Theorem 1 *The limit of a function, if it exists, is unique.*

Assume that $\lim_{x \to x_0} = L$ and $\lim_{x \to x_0} = M$, where $L \neq M$. Then let $\epsilon = \frac{|L-M|}{2}$. Then there exists some $\delta_1 > 0$ such that

$$|f(x) - L| < \frac{|L - M|}{2}$$

where $0 < |x - x_0| < \delta_1$. There also exists some $\delta_1 > 0$ such that

$$|f(x) - M| < \frac{|L - M|}{2}$$

where $0 < |x - x_0| < \delta_2$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then both inequalities hold. Then

$$\begin{aligned} |L - M| &= |L - f(x) + f(x) - M| \\ &\leq |f(x) - L| + |f(x) - M| \\ &< |L - M| \end{aligned}$$

which is impossible, so the function cannot have two distinct limits. $\square$

Theorem 2 $\lim_{x \to x_0} f(x) = l \in \mathbb{R} \cup \{\infty, -\infty\}$ *iff for all sequences* $(y_n)_{n \geq 1}$ *with* $\lim_{n \to \infty} y_n = x_0$*, and* $y_n \neq x_0$ *for all* $n \geq 1$*, we have* $\lim_{n \to \infty} f(y_n) = l$

We first prove $(\Rightarrow)$ for $l \in \mathbb{R}$

By assumption, for all $\epsilon > 0$, there exists $\delta > 0$ such that for all x in $[a, b]$

$$0 < |x - x_0| < \delta \Rightarrow |f(x) - l| < \epsilon \qquad (*)$$

If $\lim_{n \to \infty} y_n = x_0$, then for all $\epsilon' > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$

$$|y_n - x_0| < \epsilon'$$

Then setting $\epsilon' = \delta$, there exists N_δ such that

$$|y_n - x_0| < \delta$$

for all $n > N_\delta$. Also, $0 < |y_n - x_0|$ since $y_n \neq x_0$ so

$$0 < |y_n - x_0| < \delta$$

But then from (*)

$$|f(y_n) - l| < \epsilon$$

for all $n > N_\epsilon$.

Since N_ϵ exists for any value of $\epsilon > 0$, by the definition of sequence convergence,

$$\lim_{n \to \infty} f(y_n) = l$$

If $\lim_{x \to x_0} f(x) = \infty$, then

$$\forall K > 0 \; \exists \delta > 0 \; \forall x \in [a, b] \; (0 < |x - x_0| < \delta \Rightarrow f(x) \geq K)$$

If $\lim_{n \to \infty} y_n = x_0$ as above, then given $K > 0$, for $\epsilon = \delta$, there exists N_K such that for all $n > N_K$

$$0 < |y_n - x_0| < \delta$$

. Then $f(y_n) \geq K$ so

$$\forall K > 0 \; \exists N \in \mathbb{N} \; \forall n > N \; (f(y_n) \geq K)$$

so $\lim_{n \to \infty} f(y_n) = \infty$ by definition. The proof for when $l = -\infty$ is the same.

Now we prove ($\Leftarrow$)

Assume that $\lim_{n\to\infty} f(y_n) = l$ for all sequences $(y_n)_{n\geq 1}$ with $\lim_{n\to\infty} y_n = x_0$.

Now assume towards a contradiction that $\lim_{x\to x_0} f(x) \neq l$. Then we have

$$\exists\epsilon > 0\ \forall\delta > 0\ \exists x \in [a,b]\ ((0 < |x - x_0| < \delta) \wedge (|f(x) - l| \geq \epsilon)) \qquad (*)$$

Let ϵ_0 be the ϵ for which (*) holds. Then for each $n \geq 1$, let $\delta = \frac{1}{n}$. Then by (*), we construct a sequence $(a_n)_{n\geq 1}$ for which $0 < |a_n - x_0| < \frac{1}{n}$ and $|f(a_n) - l| \geq \epsilon_0$.

We now show that $\lim_{n\to\infty} a_n = x_0$. Let $\epsilon > 0$ be given. By the Archimedean principle, there exists some smallest natural number N for which $\frac{1}{N} < \epsilon$. Then by construction, a_N satisfies

$$|a_N - x_0| < \frac{1}{N} < \epsilon$$

Then for all $n > N$

$$|a_n - x_0| < \frac{1}{n} < \frac{1}{N} < \epsilon$$

so $\lim_{n\to\infty} a_n = x_0$ by definition.

But then by assumption $\lim_{n\to\infty} f(a_n) = l$, so setting $\epsilon = \epsilon_0$, there exists $N' \in \mathbb{N}$ such that $|f(a_n) - l| < \epsilon_0$ for all $n > N'$, which is a contradiction. Hence $\lim_{x\to x_0} f(x) = l$ as required. $\square$

Theorem 3 *Suppose the two functions $f, g : [a,b] \to \mathbb{R}$ have limits $k \in \mathbb{R}$ and $l \in \mathbb{R}$ respectively at $x_0 \in [a,b]$. Then $f \pm g$ has limit $k \pm l$, $f \cdot g$ has limit kl and if $l \neq 0$, f/g has limit k/l at x_0.*

If $\lim_{x\to x_0} f(x) = k$, then for all sequences $(y_n)_{n\geq 1}$ with $\lim_{n\to\infty} y_n = x_0$, we have $\lim_{n\to\infty} f(y_n) = k$ and $\lim_{n\to\infty} g(y_n) = l$.

Now let $h(x) = f(x) + g(x)$. Then

$$\lim_{n\to\infty} h(y_n) = \lim_{n\to\infty} f(y_n) \pm g(y_n) = k \pm l$$

by the limit properties of *sequences*. Therefore, $\lim_{x\to x_0} h(x) = \lim_{x\to x_0} f(x) \pm g(x) = k \pm l$ as required.

The other properties can be proven in an identical way by considering the functions $h(x) = f(x)g(x)$ and $h(x) = \frac{f(x)}{g(x)}$

$\square$

Definition 6 *Let $f : [a,b] \to \mathbb{R}$ be a function. We say f is continuous at $x_0 \in [a,b]$ iff $\lim_{x\to x_0} f(x) = f(x_0)$. We say that f is continuous in $[a,b]$ iff $\forall x_0 \in [a,b]\ (\lim_{x\to x_0} f(x) = f(x_0))$.*

Theorem 4 *If the two functions $f, g : [a,b] \to \mathbb{R}$ are continuous at $x_0 \in [a,b]$, then we have*

- *$f \pm g$ is continuous at x_0*

- *The product $f \cdot g$ is continuous at x_0*
- *If $g(x_0) \neq 0$ then f/g is continuous at x_0*

These results are immediate from the limit properties of functions.

Theorem 5 *Any polynomial $f(x)$ of degree n is continuous for all real x*

We first show that
$$\lim_{x \to a} x = a$$
where a is some arbitrary real number.

Let $\epsilon > 0$ be given, and let $\delta = \epsilon$. Then for all x such that $0 < |x - a| < \delta$, we have
$$|f(x) - a| = |x - a| < \delta = \epsilon \Rightarrow |f(x) - a| < \epsilon$$

so by the definition of the limit, $\lim_{x \to a} x = a$. Since this is true for an arbitrary a, we have $\lim_{x \to x_0} x = x_0$ for all $x_0 \in \mathbb{R}$.

Now we show that for any $n \in \mathbb{N}$, x^n is continuous for all real x.

For $n = 0$, $x^0 = 1$, for which it is trivial to show that $\lim_{x \to a} 1 = 1$ for all real x.

Assume that x^k is continuous over the real numbers for some $k > 1$. Then $\lim_{x \to a} x^k = a^k$ for all real a. Then for some arbitrary $c \in \mathbb{R}$
$$\lim_{x \to c} x^{k+1} = (\lim_{x \to c} x^k)(\lim_{x \to c} x) = c^{k+1}$$

Since this is true for any real c, x^{k+1} is continuous over the real numbers by definition. Since x^{k+1} is continuous under the assumption that x^k is continuous, and given the base case $n = 0$, we have x^n is continuous over the real numbers for all $n \in \mathbb{N}$.

Then also for any $\lambda > 0$, we have
$$\lim_{x \to x_0} \lambda x^n = \lim_{x \to x_0} \lambda \cdot \lim_{x \to x_0} x^n = \lambda x_0^n$$

for any real x_0, so λx_0^n is continuous over the real numbers.

Lastly, we show that any polynomial of degree n is continuous over the real numbers by induction.

From above, a polynomial of degree 0 is a constant which is continuous.

Assume that a polynomial $f_k(x)$ of degree k where $k \in \mathbb{N}$ is continuous over the real numbers. Then for $n = k + 1$
$$\lim_{x \to x_0} f_{k+1}(x) = \lim_{x \to x_0} (ax^{k+1} + f_k(x)) = \lim_{x \to x_0} ax^{k+1} + \lim_{x \to x_0} f_k(x) = ax_0^{k+1} + f_k(x_0) = f_{k+1}(x_0)$$

Hence $f_{k+1}(x)$ is continuous over all real x under the assumption that $f_k(x)$ is continuous over the real numbers. Since the base case also holds, by induction, $f_n(x)$ is continuous over the real numbers for all $n \in \mathbb{N}$. $\square$

Theorem 6 *Suppose $f : (a,b) \to \mathbb{R}$ is a function with $x_0 \in (a,b)$ and $g : (c,d) \to \mathbb{R}$ is a function with $Im(f) \subset (c,d)$. If f is continuous at x_0 and g is continuous at $f(x_0)$, then the composition $g \circ f : (a,b) \to \mathbb{R}$ with $(g \circ f)(x_0) = g(f(x_0))$ is continuous at x_0.*

If f is continuous at x_0, then $\lim_{x\to x_0} f(x) = f(x_0)$ by definition, so

$$\forall \epsilon > 0\ \exists \delta > 0\ (0 < |x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon) \tag{1}$$

Likewise, if g is continuous at $f(x_0)$, then $\lim_{x\to f(x_0)} g(x) = (g \circ f)(x_0)$, so

$$\forall \epsilon > 0\ \exists \delta > 0\ (0 < |x - f(x_0)| < \delta \implies |g(x) - (g \circ f)(x_0)| < \epsilon) \tag{2}$$

Let $\epsilon > 0$ be given. Then let $\delta_2 > 0$ be the value for which the limit inequality in (2) holds.

Now set ϵ in (1) to δ_2. Then there exists $\delta_1 > 0$ such that (1) holds. Therefore, we have

$$0 < |x - x_0| < \delta_1 \implies |f(x) - f(x_0)| < \delta_2$$

If $|f(x) - f(x_0)| > 0$, then by (2)

$$0 < |f(x) - f(x_0)| < \delta_2 \implies |(g \circ f)(x) - (g \circ f)(x_0)| < \epsilon$$

Since δ_1 is a function of δ_2 which is itself a function of ϵ, where ϵ can be any positive real number, we have

$$\forall \epsilon > 0\ \exists \delta > 0\ (0 < |x - x_0| < \delta \implies |(g \circ f)(x) - (g \circ f)(x_0) < \epsilon)$$

If $0 < |x - x_0| < \delta$ but $|f(x) - f(x_0)| = 0$, then $f(x) = f(x_0)$ so also $|(g \circ f)(x) - (g \circ f)(x_0)| = |(g \circ f)(x_0) - (g \circ f)(x)| = 0 < \epsilon$.

Hence $\lim_{x\to x_0}(g\circ f)(x) = (g\circ f)(x_0)$ so $g\circ f$ is continuous at x_0 by definition. □

Lemma 1 *If f is continuous in $[a,b]$ then f is bounded.*

Suppose that f is continuous but not bounded. Then

$$\forall k\ \exists x_k \in [a,b]\ (|f(x_k)| \geq k)$$

We construct a sequence $(a_n)_{n\geq 1}$ where $|f(a_n)| \geq n$ and $a_n \in [a,b]$ for all $n \geq 1$, where $n \in \mathbb{N}$. Then given any $K > 0$, let N be the smallest integer greater than or equal to K. Then for all $n > N$

$$|f(a_n)| \geq n > N \geq K$$

so $|f(a_n)| \to \infty$ as $n \to \infty$ by definition of sequence convergence to ∞.

Since $(a_n)_{n\geq 1}$ is bounded by $|b|$, it must have a monotonically increasing subsequence $(a_{n_i})_{i\geq 1}$ which converges to a limit l in $[a,b]$.

Now since f is continuous at l, we have $\lim_{x\to l} f(x) = f(l)$. Since $\lim_{i\to\infty} a_{n_i} = l$, $\lim_{i\to\infty} f(a_{n_i}) = f(l)$, so also $\lim_{i\to\infty} |f(a_{n_i})| = |f(l)|$.

But since $|f(a_n)| \to \infty$ as $n \to \infty$, we must also have $|f(a_{n_i})| \to \infty$ since any convergent subsequence of a sequence converges to the limit of the sequence. This is a contradiction, so f must be bounded. □

Theorem 7 *If $f : [a,b] \to \mathbb{R}$ is a continuous function with $a, b \in \mathbb{R}$ then there exists $r, s \in [a,b]$ such that $f(r) = \sup_{x\in[a,b]} f(x)$ and $f(s) = \inf_{x\in[a,b]} f(x)$ (i.e. f has a maximum and minimum in $[a,b]$).*

By the lemma above, f is bounded. So by the axiom of Dedekind-completeness, f has a supremum and infimum. Let $M = \sup_{x\in[a,b]} f(x)$ and $m = \inf_{x\in[a,b]} f(x)$.

We want to find $v_M \in [a,b]$ such that $f(v_M) = M$. For every $n > 0$, by the definition of the supremum, there exists $x_n \in [a,b]$ with $M \geq f(x_n) > M - \frac{1}{n}$. Then we can construct the sequence $(x_n)_{n\geq 1}$ with the property that $\lim_{n\to\infty} f(x_n) = M$. Since x_n is bounded, it has a convergent subsequence $(x_{n_i})_{i\geq 1}$; let v_M be $\lim_{i\to\infty} x_{n_i}$. It follows that $\lim_{i\to\infty} f(x_{n_i}) = M$ so $\lim_{x\to v_M} f(x) = M$. By the continuity of f at v_M we have $f(v_M) = M$. By the same reasoning, it can be shown that there exists $v_m \in [a,b]$ such that $f(v_m) = m$.

Theorem 8 Intermediate Value theorem *If $f : [a,b] \to \mathbb{R}$ is continuous and $s \in \mathbb{R}$ is such that $f(a) < s < f(b)$, then there exists $c \in (a,b)$ such that $f(c) = s$.*

If $c = a$ is such that $f(c) = s$, then $f(a) = s$ which is a contradiction. Likewise, if $c = b$ then $f(b) = s$ which is also a contradiction, so c cannot equal a or b.

Consider the set $A = \{x \in [a,b] : f(x) \leq s\}$. Since A is a bounded subset of the real numbers by definition, by the axiom of Dedekind-completeness, A has a supremum c. We then have three possibilities:

(1) $f(c) = s$; then we are done

(2) $f(c) < s$. Let $f(c) = k$. Since it is given that f is continuous over $[a,b]$, $\lim_{x\to c} f(x) = k$, so for all $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - k| < \epsilon$.

Let $x = c + \frac{\delta}{2}$ and let $\epsilon = s - k$. Then there exists $\delta > 0$ such that $|c + \frac{\delta}{2} - c| = \frac{\delta}{2} < \delta$, so $|f(c + \delta/2) - k| < s - k$

Therefore

$$\begin{aligned} f\left(c + \frac{\delta}{2}\right) &= f\left(c + \frac{\delta}{2}\right) - k + k \\ &\leq \left| f\left(c + \frac{\delta}{2}\right) - k \right| + k \\ &< s - k + k \\ &= s \end{aligned}$$

Then also $c + \frac{\delta}{2} \in A$ by definition which is a contradiction since c then is not the supremum of A.

(3) $f(c) > s$

Let $f(c) = k$. Since f is continuous at c, then for $\epsilon = k - s$, there exists $\delta > 0$ such that for all x such that $0 < |x - c| < \delta$, we have $|f(x) - k| < k - s$. Then $s - k < f(x) - k < k - s$ so $f(x) > s$.

Now since c is the supremum of A, $c - \delta$ is not an upper bound of A. Then there exists $c' \in A$ such that

$$c - \delta < c' < c$$

. Since $c' \in A$, we have $f(c') \leq s$ by definition. But since $|c - c'| < \delta$, $f(c') > s$, which is a contradiction, so it cannot be the case that $f(c) > s$.

Therefore, it is impossible for $f(c) > s$ or $f(c) < s$, so the only possibility is that $f(c) = s$. $\square$

Definition 7 *A function $f : [a, b] \to \mathbb{R}$ is uniformly continuous on A iff*

$$\forall \epsilon > 0 \ \exists \delta > 0 \ \forall x, x_0 \in A \ (|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon)$$

Theorem 9 *If $f : [a, b] \to \mathbb{R}$ for $a, b \in \mathbb{R}$ is continuous, then it is uniformly continuous on $[a, b]$.*

Suppose that f is not uniformly continuous on $[a, b]$. Then there exists $\epsilon > 0$ such that for all $n \geq 1$, there exists $x_n, y_n \in [a, b]$ with $|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \epsilon$.

As $(x_n)_{n \geq 1}$ is a bounded sequence, it has a convergent subsequence $(x_{n_i})_{i \geq 1}$ with limit l in $[a, b]$. But then

$$\begin{aligned} |l - y_{n_i}| &\leq |l - x_{n_i}| + |x_{n_i} - y_{n_i}| \\ &< |l - x_{n_i}| + \frac{1}{n_i} \end{aligned}$$

So y_{n_i} also converges to l. Since f is continuous at l, we have $\lim_{i \to \infty} f(x_{n_i}) = f(l) = \lim_{i \to \infty} f(y_{n_i})$. However, this contradicts $|f(x_{n_i}) - f(y_{n_i})| \geq \epsilon$ for all $i \geq 1$. $\square$