Math 115 WA02 Alex Guo

I have read the Academic Integrity document on LEARN and have completed this assignment in adherence to the rules stated in that document.

Alex Guo

2. Let $\vec{x} = \begin{bmatrix}1\\-3\\2\end{bmatrix}$ and $\vec{y} = \begin{bmatrix}-1\\1\\6\end{bmatrix}$

a) Compute $\vec{x}\cdot\vec{y}$

$$\begin{aligned}\vec{x}\cdot\vec{y} &= x_1y_1 + x_2y_2 + x_3y_3\\ &= (1)(-1) + (-3)(1) + (2)(6)\\ &= -1 - 3 + 12\\ &= 8\end{aligned}$$

b) Compute $\vec{x}\times\vec{y}$

$$\vec{x}\times\vec{y} = \begin{bmatrix} \begin{vmatrix}x_2 & y_2\\ x_3 & y_3\end{vmatrix} \\ -\begin{vmatrix}x_1 & y_1\\ x_3 & y_3\end{vmatrix} \\ \begin{vmatrix}x_1 & y_1\\ x_2 & y_2\end{vmatrix}\end{bmatrix} = \begin{bmatrix}x_2y_3 - y_2x_3\\ -(x_1y_3 - y_1x_3)\\ x_1y_2 - y_1x_2\end{bmatrix} = \begin{bmatrix}(-3)(6) - (1)(2)\\ -((1)(6) - (-1)(2))\\ (1)(1) - (-1)(-3)\end{bmatrix} = \begin{bmatrix}-20\\ -8\\ -2\end{bmatrix}$$

c) Compute the cosine of the angle θ determined by $\vec{x}$ and $\vec{y}$

$$\begin{aligned}\vec{x}\cdot\vec{y} &= \|\vec{x}\|\|\vec{y}\|\cos\theta\\ 8 &= \left(\sqrt{1^2+(-3)^2+2^2}\right)\left(\sqrt{(-1)^2+1^2+6^2}\right)\cos\theta\\ 8 &= \sqrt{14}\sqrt{38}\cos\theta\\ \frac{8}{\sqrt{532}} &= \cos\theta\\ \frac{16\sqrt{133}}{532} &= \cos\theta\\ \frac{4\sqrt{133}}{133} &= \cos\theta\end{aligned}$$

d) Find the unit vector in the direction of $\vec{x}$

$\hookrightarrow \|\vec{x}\| = \sqrt{14}$

$$\therefore \text{Unit vector} = \frac{\vec{x}}{\|\vec{x}\|} = \begin{bmatrix}\frac{1}{\sqrt{14}}\\ \frac{-3}{\sqrt{14}}\\ \frac{2}{\sqrt{14}}\end{bmatrix} = \begin{bmatrix}\frac{\sqrt{14}}{14}\\ \frac{-3\sqrt{14}}{14}\\ \frac{\sqrt{14}}{7}\end{bmatrix}$$

e) Find the area of the parallelogram determined by $2\vec{x}$ and $-3\vec{y}$

$$2\vec{x} = 2\begin{bmatrix}1\\-3\\2\end{bmatrix} = \begin{bmatrix}2\\-6\\4\end{bmatrix}$$

$$-3\vec{y} = -3\begin{bmatrix}-1\\1\\6\end{bmatrix} = \begin{bmatrix}3\\-3\\-18\end{bmatrix}$$

To find the area, we can use the cross product and find the norm of the resulting vector

$$2\vec{x}\times 3\vec{y} = \begin{bmatrix} \begin{vmatrix}x_2 & y_2\\ x_3 & y_3\end{vmatrix} \\ -\begin{vmatrix}x_1 & y_1\\ x_3 & y_3\end{vmatrix} \\ \begin{vmatrix}x_1 & y_1\\ x_2 & y_2\end{vmatrix}\end{bmatrix} = \begin{bmatrix}x_2y_3 - y_2x_3\\ -(x_1y_3 - y_1x_3)\\ x_1y_2 - y_1x_2\end{bmatrix} = \begin{bmatrix}(-6)(-18) - (-3)(4)\\ -((2)(-18) - (3)(4))\\ (2)(-3) - (3)(-6)\end{bmatrix} = \begin{bmatrix}120\\ 48\\ 12\end{bmatrix}$$

$$\therefore \|2\vec{x}\times 3\vec{y}\| = \sqrt{120^2+48^2+12^2} = 129.8 \text{ units}^2$$

3. Let $\vec{u}, \vec{v} \in C^3$ with $\vec{u} = \begin{bmatrix} 1+j \\ 2-3j \\ 6+2j \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 2j \\ 4 \\ -2+j \end{bmatrix}$

a) $\langle \vec{u}, \vec{v} \rangle = \bar{u}_1 v_1 + \bar{u}_2 v_2 + \bar{u}_3 v_3$

$= (1-j)(2j) + (2+3j)(4) + (6-2j)(-2+j)$

$= 2j + 2 + 8 + 12j + \quad (-12+4j+6j+2)$

$= 2j + 2 + 8 + 12j - 10 + 10j$

$= 24j$

$\langle \vec{v}, \vec{u} \rangle = \bar{v}_1 u_1 + \bar{v}_2 u_2 + \bar{v}_3 u_3$

$= (-2j)(1+j) + (4)(2-3j) + (-2-j)(6+2j)$

$= -2j + 2 + 8 - 12j + (-12 - 6j - 4j + 2)$

$= -2j + 2 + 8 - 12j - 10 - 10j$

$= -24j$

b) $\|\vec{u}\| = \sqrt{\bar{u}_1 u_1 + \bar{u}_2 u_2 + \bar{u}_3 u_3}$

$= \sqrt{(1-j)(1+j) + (2+3j)(2-3j) + (6-2j)(6+2j)}$

$= \sqrt{1+1+4+9+36+4}$

$= \sqrt{55}$

$\|\vec{v}\| = \sqrt{\bar{v}_1 v_1 + \bar{v}_2 v_2 + \bar{v}_3 v_3}$

$= \sqrt{(-2j)(2j) + (4)(4) + (-2-j)(-2+j)}$

$= \sqrt{4+16+4+1}$

$= \sqrt{25} = 5$

c) $\langle \vec{u} + j\vec{v}, \vec{z} \rangle = \langle \vec{u}, \vec{z} \rangle + \langle j\vec{v}, \vec{z} \rangle$

$= \overline{\langle \vec{z}, \vec{u} \rangle} - j\,\overline{\langle \vec{z}, \vec{v} \rangle}$

$= 3 - j - 5j$

$= 3 - 6j$

4. $P(1,2,3)$, $Q(1,1,-1)$, $R(2,-1,0)$

↳ To check for a right angle, we have to dot product the vector sides of the $\triangle$

$\vec{PQ} = (\vec{OQ} - \vec{OP})$

$= \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$= \begin{bmatrix} 0 \\ -1 \\ -4 \end{bmatrix}$

$\vec{QR} = (\vec{OR} - \vec{OQ})$

$= \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$\vec{PR} = (\vec{OR} - \vec{OP})$

$= \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ -3 \\ -3 \end{bmatrix}$

$\vec{PQ} \cdot \vec{QR} = (0)(1) + (-1)(-2) + (-4)(1)$
$= -2 \Rightarrow \neq 0$

$\vec{QR} \cdot \vec{PR} = (1)(1) + (-2)(-3) + (1)(-3)$
$= 4 \Rightarrow \neq 0$

$\vec{PQ} \cdot \vec{PR} = (0)(1) + (-1)(-3) + (-4)(-3)$
$= 15 \neq$

Since none of the dot products are 0, none of the triangle's angles are 90°, and it is not a right angle triangle.

5. Let $\vec{v}, \vec{w} \in \mathbb{R}^n$. Evaluate $\|3\vec{v} - 2\vec{w}\|$ given that $\|\vec{v}\| = 4$, $\|\vec{w}\| = 5$ and $\vec{v} \cdot \vec{w} = 3$

$$\|3\vec{v} - 2w\|^2 = \|3\vec{v}\|^2 - 2(3v \cdot 2w) + \|2w\|^2$$

$$= 9\|\vec{v}\|^2 - 12(v \cdot w) + 4\|w\|^2$$

$$= 9(4)^2 - 12(3) + 4(5)^2$$

$$= 144 - 36 + 100$$

$$= \underline{\underline{208}}$$

$$\therefore \|3\vec{v} - 2w\| = \sqrt{208}$$

$$= 4\sqrt{13}$$

a)

6. $L_1: \; x_1 = 2+t$, $x_2 = 10+6t$, $x_3 = 5+8t$

$L_2: \; x_1 = 5+2s$, $x_2 = 2-s$, $x_3 = 4+3s$

They intersect at

$$\begin{bmatrix} 2 \\ 10 \\ 5 \end{bmatrix} + t\begin{bmatrix} 1 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix} + s\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$\therefore \; 2+t = 5+2s$$
$$10+6t = 2-s$$
$$5+8t = 4+3s$$

↳ simplify:
$$t-2s = 3 \quad \text{—} (1)$$
$$6t+s = -8 \quad \text{—} (2)$$
$$8t-3s = -1 \quad \text{—} (3)$$

$(2)\times 2: \; 12t+2s = -16$ — (4)

$(1)+(4) = $
$$\begin{array}{r} t-2s = 3 \\ +\; 12t+2s = -16 \\ \hline 13t = -13 \\ t = -1 \end{array} \quad \text{—} (5)$$

sub (5) into (1) $\Rightarrow -1-2s = 3 \Rightarrow s = -2$

$\therefore t = -1, s = -2$ is true for (1) and (2)

For (3): $8(-1) - 3(-2) = -1$

$-2 = -1$

Since $-2 \neq -1$, there is no point that lies on both lines and they do not intersect

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b) We need the cross product of L_1 and L_2 to find the line orthogonal to both lines

↳ We can use the direction vectors of L_1 and L_2 to cross

$$\therefore \begin{bmatrix} 1 \\ 6 \\ 8 \end{bmatrix} \times \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} \begin{vmatrix} 6 & -1 \\ 8 & 3 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 \\ 8 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 6 & -1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 6(3)-(-1)(8) \\ -((1)(3)-(2)(8)) \\ (1)(-1)-(2)(6) \end{bmatrix} = \begin{bmatrix} 26 \\ 13 \\ -13 \end{bmatrix} = 13\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

If $u \in \mathbb{R}$, we can use that as the magnitude of the dir. vector

We know $\begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}$ is on the line as well.

Thus, the equation is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix} + u\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \quad u \in \mathbb{R}$$

7. Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ be such that $\vec{v} = k\vec{u}$ for some $k \in \mathbb{R}$ with $k \geq 0$

↳ Show that $\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$

$$\|\vec{u} + \vec{v}\|^2 = (\vec{u} + \vec{v})(\vec{u} + \vec{v})$$
$$= \vec{u} \cdot \vec{u} + 2(\vec{u} \cdot \vec{v}) + \vec{v} \cdot \vec{v}$$
$$= \|\vec{u}\|^2 + 2(\vec{u} \cdot \vec{v}) + \|\vec{v}\|^2$$

Sub in $\vec{v} = k\vec{u}$ into the $2(\vec{u} \cdot \vec{v})$

$$\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + 2(\vec{u} \cdot k\vec{u}) + \|\vec{v}\|^2$$
$$= \|\vec{u}\|^2 + 2k(\vec{u} \cdot \vec{u}) + \|\vec{v}\|^2$$
$$= \|\vec{u}\|^2 + 2k\|u\|^2 + \|\vec{v}\|^2$$

We can see that if $\vec{v} = k\vec{u}$, $\|\vec{v}\| = \|k\vec{u}\| = k\|\vec{u}\|$

Sub in $\|\vec{v}\|$ for the $k\|u\|$ that appears in $2k\|u\|^2$

$$\therefore \|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + 2\|u\|\|\vec{v}\| + \|\vec{v}\|^2$$

↳ Takes the form $a^2 + 2ab + b^2$
↳ $(a+b)^2$

$$\|\vec{u} + \vec{v}\|^2 = (\|\vec{u}\| + \|\vec{v}\|)^2$$
$$\sqrt{\|\vec{u} + \vec{v}\|^2} = \sqrt{(\|\vec{u}\| + \|\vec{v}\|)^2}$$
$$\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$$