Seminar

Panjerjev razred

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1 Problem

Za verjetnostno masno funkcijo $p : \mathbb{N}_0 \to [0,1]$ slučajne spremenljivke z vrednostmi v $\mathbb{N}_0$ pravimo, da je v Panjerjevem razredu, če obstajata realni števili a in b, taki, da je

$$p_k = p_{k-1}(a + \frac{b}{k}), \quad k \in \mathbb{N}$$

Dokaži, da je p v Panjerjevem razredu, če in samo če je ene izmed sledečih oblik (povsod teče n po $\mathbb{N}_0$):

1. $p_n = \delta_{\mathbb{N}_0}$ *(Diracova masa v 0).*
2. $p_n = \frac{\lambda^n}{n!}e^{-\lambda}$ *za nek $\lambda \in (0,\infty)$ (Poissonova porazdelitev).*
3. $p_n = \binom{\alpha+n-1}{n}p^n(1-p)^\alpha$ *za neka $\alpha \in (0,\infty), p \in (0,1)$(negativna binomska porazdelitev).*
4. $p_n = \binom{N}{n}p^n(1-p)^{N-n}$ *za neka $p \in (0,1), N \in \mathbb{N}$ (binomska porazdelitev).*

2 Rešitev

($\Leftarrow$)

1. $p_1 = p_0(a+b) = 0 = p_2 = p_3 = \ldots = p_k, \quad \forall k \in \mathbb{N} \to a = 0, b = 0$
2. $\frac{p_k}{p_{k-1}} = \frac{\lambda^k e^{-\lambda}(k-1)!}{\lambda^{k-1}e^{-\lambda}k!} = \frac{\lambda}{k} \to a = 0, b = \lambda$
3. Ker velja $\Gamma(n) = (n-1)!$ in $\Gamma(n+1) = n\Gamma(n)$ za $\forall n \in \mathbb{N}$, velja $\binom{n+\alpha-1}{n} = \frac{(n+\alpha-1)!}{n!(\alpha-1)!} = \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}$.
   $\frac{p_n}{p_{n-1}} = \frac{\binom{n+\alpha-1}{n}p^n(1-p)^\alpha}{\binom{n-1+\alpha-1}{n-1}p^{n-1}(1-p)^\alpha} = \ldots = \frac{\Gamma(\alpha+n)}{n\Gamma(\alpha+n-1)}p = \frac{(\alpha+n-1)\Gamma(\alpha+n-1)}{n\Gamma(\alpha+n-1)}p = p + p(\alpha-1)\frac{1}{n} \to a = p, b = p(\alpha-1)$
4. $\frac{p_n}{p_{n-1}} = \frac{\binom{N}{n}p^n(1-p)^{N-n}}{\binom{N}{n-1}p^{n-1}(1-p)^{N-n+1}} = \ldots = \frac{p}{1-p}(N+1)\frac{1}{n} - \frac{p}{1-p} \to a = -\frac{p}{1-p}, b = \frac{p}{1-p}(N+1)$

(⇒)

Ker mora biti $p_k \geq 0$ za $\forall k \in \mathbb{N} \Rightarrow a+b \geq 0$.

1. $a+b=0$:

$$p_1 = p_0(a+b) = 0 = p_2 = p_3 = \ldots = p_k$$

   Ker mora veljati

$$\sum_{k=0}^{\infty} p_k = 1 \to p_0 = 1$$

   dobimo Diracovo maso v točki 0.

2. $a+b>0$:

   - $0<a<1$: Določimo novo spremenljivko $\alpha = \frac{a+b}{a} \to b = a(\alpha-1)$

$$p_1 = p_0(a+b) = p_0 a\alpha$$

$$p_2 = p_1(a+\frac{b}{2}) = p_1 a(\frac{\alpha}{2}+\frac{1}{2}) = p_0 a^2 \alpha(\alpha+1)\frac{1}{2}$$

$$\vdots$$

$$p_k = p_0 a^k \frac{1}{k!}\frac{(\alpha+k-1)!}{(\alpha-1)!} = p_0 a^k \binom{\alpha+k-1}{k}$$

   Veljati mora

$$\sum_{k=0}^{\infty} p_k = \sum_{k=0}^{\infty} \binom{\alpha+k-1}{k} p_0 a^k = p_0 \sum_{k=0}^{\infty} \binom{\alpha+k-1}{k} a^k = 1$$

   Velja tudi:

$$\binom{\alpha+k-1}{k} = \frac{(\alpha+k-1)(\alpha+k-2)\ldots(\alpha+k-(k-1))(\alpha+k-k)(\alpha-1)!}{k!(\alpha-1)!} =$$

$$= (-1)^k \frac{(-\alpha)(-\alpha-1)\ldots(-\alpha-k+1)}{k!} = (-1)^k \binom{-\alpha}{k}$$

   Torej je:

$$p_0 \sum_{k=0}^{\infty} \binom{\alpha+k-1}{k} a^k = p_0 \sum_{k=0}^{\infty} \binom{-\alpha}{k} (-1)^k a^k = p_0 \sum_{k=0}^{\infty} \binom{-\alpha}{k} (-a)^k = p_0(1+(-a))^{-\alpha}$$

$$p_0 = (1-a)^{\alpha}$$

   Od tod dobimo negativno binomsko porazdelitev.

- a = 0: Velja $p_k = p_{k-1}\frac{b}{k}$. Če razpišemo, dobimo:

$$p_1 = p_0 b$$
$$p_2 = p_0\tfrac{b^2}{2}$$
$$\dots$$
$$p_k = p_0\tfrac{b^k}{k!}$$

Ker mora veljati

$$\sum_{k=0}^{n} p_k = 1 \rightarrow \sum_{k=0}^{\infty} p_0\frac{b^k}{k!} = p_0\sum_{k=0}^{\infty}\frac{b^k}{k!} = p_0 e^b = 1 \rightarrow p_0 = e^{-b} \rightarrow p_k = e^{-b}\frac{b^k}{k!},$$

kar pa je ravno Poissonova porazdelitev s parametrom b.

- a < 0: $\lim_{k\to\infty}\frac{p_k}{p_{k-1}} = \lim_{k\to\infty}(a+\frac{b}{k}) = a < 0$, iz česar sledi, da obstaja tak N $\in \mathbb{N}$, da velja $a + \frac{b}{N+1} = 0$ (vsi členi od nekega N dalje so enaki 0, ker mora biti verjetnostna masna funkcija nenegativna). Če izrazimo b, dobimo $b = -a(N+1)$. To vstavimo v Panjerjevo zvezo $p_k = p_{k-1}(a+\frac{b}{k})$:

$$\begin{aligned} p_1 &= p_0(a - a(N+1)) = p_0 a(-1)N \\ p_2 &= p_1(a - a\tfrac{N+1}{2}) = p_0 a^2 \tfrac{1}{2}(-1)^2 N(N-1) \\ &\vdots \\ p_k &= p_0 a^k \tfrac{1}{k!} N(N-1)\cdots(N-k+1)(-1)^k \\ &= p_0 a^k \tfrac{1}{k!}\tfrac{N!}{(N-k)!}(-1)^k = p_0(-a)^k\tbinom{N}{k} \end{aligned}$$

Vemo, da mora veljati

$$\sum_{k=0}^{\infty} p_k = 1 \rightarrow \sum_{k=0}^{N}\binom{N}{k}p_0(-a)^k = p_0\sum_{k=0}^{N}\binom{N}{k}(-a)^k = p_0(1-a)^N = 1 \rightarrow p_0 = (1-a)^{-N}.$$

(Uporabili smo binomski izrek.)

Če vstavimo $a = \frac{-p}{1-p}$ dobimo:

$p_k = \binom{N}{k}(1+\frac{p}{1-p})^{-N}(\frac{p}{1-p})^k = \binom{N}{k}(\frac{1}{1-p})^{-N}\frac{p^k}{(1-p)^k} = \binom{N}{k}(1-p)^{N-k}p^k$

kar pa je ravno binomska porazdelitev.