Incompressible Flow Optimization in Haemodynamics

Data assimilation in an arterial bifurcation

1 Problem Formulation

We want to solve the optimal control problem given by

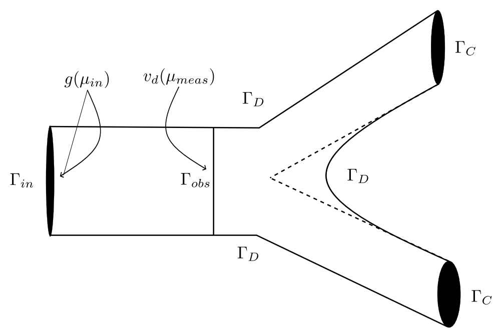
$$\min J(v,p,u) = \frac{1}{2}\int_{\Gamma_{obs}} |v - v_d|^2 ds + \frac{\alpha_1}{2}\int_{\Gamma_C} |\nabla_t u|^2 ds + \frac{\alpha_2}{2}\int_{\Gamma_C} |u|^2 ds$$

s.t.

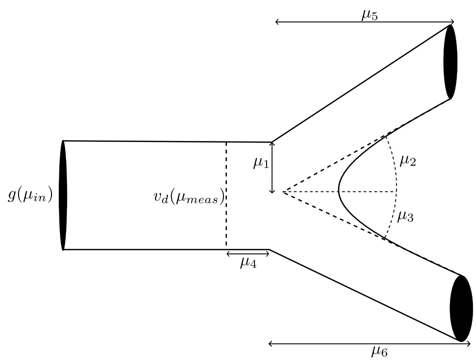
$$\begin{cases} -\nu\Delta v + \nabla p = f & \text{in } \Omega \\ \text{div} v = 0 & \text{in } \Omega \\ v = g & \text{on } \Gamma_{in} \\ v = 0 & \text{on } \Gamma_w \\ pn - \nu\partial_n v = u & \text{on } \Gamma_C \end{cases} . \tag{1}$$

where we assume $\alpha_2 = 0.1\alpha_1$ and $\alpha_1 = 10^{-3}$.

We further consider a 2D realistic representation of an arterial bifurcation, parametrized as shown in Fig. **??**. We consider an inverse problem in hemodynamics, focusing on a simplified model of an arterial bifurcation. The computational domain is parametrized, with an inflow boundary denoted as Γ_{in}, two outflow boundaries Γ_C, and the physical vessel wall represented by Γ_D. The primary variables of interest are the velocity $\vec{v}$ and pressure p, which are assumed to satisfy (1).



(a) Boundary conditions: no-slip conditions on $\Gamma_D(\mu)$, Poiseuille velocity profile $g(\mu_{in})$ on Γ_{in}, unknown Neumann flux on the outflow sections.



(b) Parametrization of the original domain

Figure 1: 2D representation of the arterial bifurcation considered for this implementation.

We assume that the velocity profile is known along a segment of the inflow boundary Γ_{in}, but no direct measurements of the Neumann flux at the outflow boundaries Γ_C are available. The control variable in this problem is the unknown Neumann flux at Γ_C. The goal is to deduce this control variable using the velocity data along Γ_{in}, and consequently, to recover the velocity and pressure fields across the entire domain.

The problem involves several parameters, including geometric parameters μ_{geom}, which describe the dimensions of the bifurcation (e.g., the length of each branch, the angle of the bifurcation), a parametrized velocity profile μ_{meas}, and a parametrized inflow velocity profile $g(\mu_{\text{in}})$, given by the following Poiseuille parabolic profile

$$g(\mu_{\text{in}}) = \begin{cases} 10\mu_8(x_2+1)(1-x_2) & \text{in } \Omega \\ 0 & \text{elsewhere} \end{cases}$$

with a parametrized peak velocity equal to $\tilde{v} = 10\mu_8\,\mathrm{cm/s}^{-1}$. The kinematic viscosity is $\nu = 0.04\,\mathrm{cm}^2\mathrm{s}^{-1}$, resulting in a Reynolds number $Re = \frac{\tilde{v}l}{\nu} \approx 500$, assuming l is the diameter of the large vessel and $\mu_8 = 1$.

These parameters together influence the flow field that we aim to reconstruct. This formulation represents a simplified version of the more complex hemodynamic flow problem, assuming two-dimensional geometry, steady-state conditions, and simplified constitutive laws. We further assume that the measured velocity profile can be approximated by a simple analytical function, parametrized for the data assimilation process as

$$v_d(\mu) = \begin{cases} \mu_8((\mu_7\eta_1(x_2)) + (1-\mu_7)\eta_2(x_2)) & \text{in } \Gamma_{obs} \\ 0 & \text{elsewhere} \end{cases}$$

where it is assumed that $\eta_1(x_3) = 10(x_3^3 - x_3^2 - x_3 + 1)$ and $\eta_2(x_3) = 10(-x_3^3 - x_3^2 + x_3 + 1)$. To prevent potential flow reversals in either branch when the vertical velocity remains unmonitored, we impose a zero vertical velocity condition despite its physical shortcomings. This constraint is particularly questionable when $\mu_7 \neq 0.5$, as such conditions would realistically produce non-zero vertical flow at Γ_{obs}. Meanwhile, our horizontal velocity profile is designed with a parameterized form that stays positive throughout, which has the advantage of automatically preserving mass balance regardless of how μ_7 and μ_8 vary.

Finally, the parameter domain is given by

$$D = \{\mu = (\mu_1, \ldots, \mu_8) \in \mathbb{R}^8 : \ \mu_i \in [\mu_{\mathrm{min},i}, \mu_{\mathrm{Max},i}] \ \ \forall i = 1, \ldots, 8\}.$$

where

$$\begin{aligned} \mu_{\mathrm{min}} &= (0.7 \quad \pi/7 \quad \pi/7 \quad 0.7 \quad 1.5 \quad 1.5 \quad 0.0 \quad 0.5), \\ \mu_{\mathrm{Max}} &= (1.3 \quad \pi/3 \quad \pi/3 \quad 1.2 \quad 2.5 \quad 2.5 \quad 1 \quad 1.5). \end{aligned}$$

Therefore, we want to solve optimal control problem stated as

$$\text{minimize } J(\cdot;\mu) \quad \text{subject to (1)}, \quad \text{given } \mu \in D.$$

utilizing a one-shot approach, i.e., we solve for the control, state and adjoint simultaneously.