Extremal and Probabilistic Combinatorics

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Chapter 1

Introduction

Computer Science is truly fundamental for the fast development of Science in the last century, also being fundamental for its validation and communication. It is really hard to think about actual Science without the use of computers or strong science communities connected and accessible by the internet. Computer Science is also essential for the business. All multinational company is also a software company since the way of production, operating and delivering products are managed by software and these aspects are determinants for the success level of any company in the world, it is also not uncommon that one of the most valuable assets of a company can be connected to data, software and algorithms. In this scenario, graphs are also very interesting due its importance to Computer Science.

Graphs are one of the most flexible structures, even for math theory either for efficient algorithms, impacting the study of Algebra, Probability and Combinatorics. They can be used even for modeling many real scenarios in a very easy understandable graphical scheme whose properties can be explored to obtain many useful information what explains its huge importance for many knowledge areas not directly connected to Computer Science or Math Theory.

This project, more specifically, focus on classical results of Extremal Combinatorics Theory and Graph Theory. It has detailed ideas and explanations about theorems and concepts of a large period of time which have already been intensively studied.

Extremal Combinatorics studies the maximum or minimum size a collection of objects can be at the same time it satisfies certain restrictions. Here these objects will be focus on graphs and graphs' substructures.

Ramsey Theory is the study of finding order in chaos and, in general, solves mathematical equations in the integers.

Extremal Graph Theory studies graphs' size, as the number interval of edges and vertices, since a specific substructure is forbidden inside the main graph.

Chapter 2

Ramsey's Theory

Consider the following question: "Given a group of 6 people, are there 3 people that are mutual friends or 3 people are mutual strangers?"

It is possible to suppose without loss of generality that Richard knows at least three people, Maria, Bete and Jack. If any pair of these friends know each other, then it is formed a friendly triangle, otherwise Maria, Bete and Jack form a unfriendly triangle themselves. Replacing people with vertices and a friend relationship with colours (blue for friends and red otherwise) we realize that with 6 peoples we always have a monochromatic triangle independently of the colouring, but what about 5 people?

Definition 2.0.1. *The minimum n such that any 2-colouring on K_n induces a complete subgraph K_s whose edges are monochromatic in color 1 or a complete subgraph K_t whose edges are monochromatic in color 2 is the Ramsey Number $R(s,t)$.*

The K_5 can be constructed without a monochromatic triangle as Figure 2.1, this shows that Ramsey number cannot be less than 6, what can be written as R(3,3) = 6.

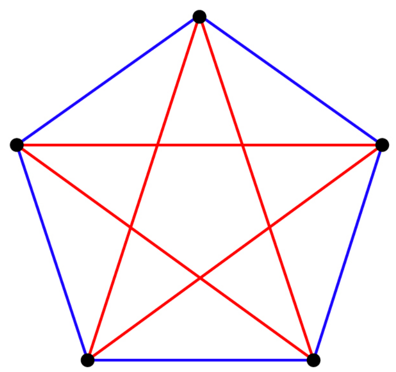

Figure 2.1: K_5 without monochromatic triangle.

Theorem 2.0.2. *(Ramsey's Theorem). Let $r \geqslant 1$ be an integer. Every colouring $c\colon \binom{\mathbb{N}}{2} \to [r]$ of the pairs in $\mathbb{N}$ contains an infinite monochromatic subset of pairs.*

In order to prove this theorem, it is needed only the pigeonhole principle as it follows:

"If a infinite number of letters lie in a finite number of pigeonholes, then some pigeonhole must contain an infinite number of letters"

Imagine we have a finite number of parts to divide $\mathbb{N}$ whose elements are infinite, let the number of parts be r. We can start giving part 1 a finite quantity of elements, doing the same for parts $\{2, 3, ..., r-1\}$. So at this point we have $\{1, 2, 3, ..., r-1\}$ being a set of parts with a finite number of elements, but $|\mathbb{N}| = \infty$ and once $|\{1, 2, 3, ..., r-1\}|$ is finite we still have infinite elements and only one part to use, so at least one of these parts must have infinite elements.

Given a colouring $c\colon \binom{\mathbb{N}}{2} \to [r]$, being i a colour and v a vertex $\in \mathbb{N}$, define $N_i(v) = \{w : c(vw) = i\}$, can be read as the colour i neighbourhood of v.

Theorem 2.0.3. *(Schur's Theorem, 1916). Considering any colouring $c\colon \mathbb{N} \to [r]$, it implies a monochromatic x, y, z with the property $x + y = z$.*

Proof. For a vertex colouring given as $c\colon \mathbb{N} \to [r]$ define a edge colouring $c'\colon \binom{\mathbb{N}}{2} \to [r]$ as $c'(\{a, b\}) := c(|a-b|)$. By Theorem 2.0.2, there exists a monochromatic triangle with three vertices, say $\{x, y, z\}$, with $x < y < z$.
Using the definition of c' we have

$$c'(x, y) = i = c(|y - x|)$$

$$c'(x, z) = i = c(|z - x|)$$

$$c'(y, z) = i = c(|z - y|)$$

From that, $c(|y-x|) = c(|z-x|) = c(|z-y|)$, and $(z-y) + (y-x) = (z-x)$ forms a monochromatic x, y, z such that $x + y = z$, as required.

□

Theorem 2.0.4. *(Erdős and Szekeres, 1935; Erdős, 1947).*

$$(\sqrt{2})^k \ll R(k) \ll 4^k.$$

Proof. We start by proving the upper bound. Considering every $k, \ell \in \mathbb{N}$ we assume:

$$R(k, \ell) \leqslant R(k-1, \ell) + R(k, \ell-1).$$

Choose $n \geqslant R(k-1, \ell) + R(k, \ell-1)$ and pick any vertex v from $[n]$. Using Pigeonhole

principle we realize v has either at least $R(k-1,\ell)$ red neighbours, or at least $R(k,\ell-1)$ blue neighbours, for simplicity assume that v has at least $R(k-1,\ell)$ red neighbours. By definition we have on the red neighbors of v a red K_{k-1} or a blue K_ℓ, if it is a K_ℓ we are done and if it is a K_{k-1} we can add v, forming a K_k. We just proved we always have a K_k red or a K_ℓ blue with such n and it is a upper bound for equation $R(k,\ell) \leqslant R(k-1,\ell) + R(k,\ell-1)$ because it doesn't prove this n is the minimum one, only shows that for this specific n it is true.
By induction hypothesis:

$$R(k,\ell) \leqslant \binom{k+\ell}{k}.$$

So,

$$R(k-1,\ell) \leqslant \binom{k-1+\ell}{k-1}.$$

and

$$R(k,\ell-1) \leqslant \binom{k+\ell-1}{\ell-1}.$$

That implies on:

$$R(k,\ell) \leqslant \binom{k-1+\ell}{k-1} + \binom{k+\ell-1}{\ell-1} = \binom{k+\ell}{k}.$$

It is hard to find colourings whose subgraphs are not big and not monochromatic, this is counter-intuitive but is really hard to construct this type of graph. However, in 1947 Erdős made a important contribution for combinatorics showing a simple proof of an exponential lower bound on $R(k)$.

Now we proceed by proving the lower bound. Given a random coloring $c\colon \binom{n}{2} \to \{0,1\}$ let 1/2 be the probability of a red $c(i)(j)$ for any edge $ij\ ine(K_n)$.
Define X as the number of monochromatics cliques in K_n. The expected value for X is $\binom{n}{k}$ times the probability of a given clique be monochromatic, so:

$$\binom{n}{k}\frac{1}{2}^{\binom{k}{2}-1} \leqslant 2\left(\frac{en}{k}\left(\frac{1}{\sqrt{2}}\right)^{k-1}\right)^{k} \ll 1.$$

it follows if $n < (1/e\sqrt{2})k2^{k/2}$. Once the expected value for X is less than 1, then there must exist a colouring in which $X = 0$. $\square$

Theorem 2.0.5. *(Van der Waerden, 1927). Every two-coloring of* $\mathbb{N}$ *contains arbitrarily long monochromatic arithmetic progressions.*

Chapter 3

Extremal Graph Theory

In this chapter, the problems and theorems extend previous chapter. Instead of working only in questions based on sets' cardinality it embraces questions about the maximum size of graphs and graphs' substructures.

3.1 Turán's Theorem

In this section we study prohibited subgraphs to answer questions of the following form:

"What is the maximum number of edges in a tringle-free graph?"

As we see in the next theorem the complete bipartite graph has $\frac{n^2}{4}$ edges and no triangles.

Theorem 3.1.1. *(Mantel, 1907). If G is a triangle-free graph on n vertices, then*

$$e(G) \leqslant \left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor.$$

Proof. This is a proof by induction on n. Let G' be the graph obtained from G by removing two vertices $\{u\}, \{v\} \in G$ such that $\{uv\} \in e(G)$, note that if G is triangle-free so G' is also triangle-free because it is not possible to form a triangle removing a edge and there are at most $n-1$ edges incident with u or v, then:

$$\begin{aligned} e(G - \{\{u\},\{v\}\}) &\leqslant (\left\lfloor \frac{n}{2} \right\rfloor - 1)(\left\lceil \frac{n}{2} \right\rceil - 1) \\ &= \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil - n + 1. \end{aligned}$$

When we add the $n-1$ edges removed from G we confirm the induction hypothesis. □

Turán generalized its result in 1941. It uses $T_r(n)$ which is a complete r-partite graph with n vertices and $\lceil \frac{n}{r} \rceil$ or $\lfloor \frac{n}{r} \rfloor$ vertices on each part.

Theorem 3.1.2. *(Turán, 1941).* $ex(n, k_{r+1}) = t_r(n)$.

Proof. This is a proof by induction on n. Let G be maximum without a K_{r+1}. Construct a graph G' removing a K_r from G, so:

$$
\begin{aligned}
e(G) &\leqslant e(G') + (r-1)(n-r) + \binom{r}{2} \\
&\leqslant e(T_r(n-r)) + (r-1)(n-r) + \binom{r}{2} \\
&= e(T_r(n)).
\end{aligned}
$$

□

Theorem 3.1.3. *(Erdős, 1970). There is an r-partite graph H with $V(H) = V(G)$, with $d_H(v) \geqslant d_G(v)$ for every $v \in V(G)$.*

Proof. Let G be a $K_{r+1}-$free graph on n vertices and let w be a vertex of maximum degree in G. For every vertex $v \in V(G) \setminus N(w)$ is removed the edges incident with v and added an edge between v and each neighbor of w, this operation is called 'Zykov symmetrization'. Clearly, no vertex degree has decreased because at this point $d(v) = \Delta(G)$ for every $v \in V(G) \setminus N(w)$ and the graph is still $K_{r+1}-$free because $N(w)$ has not changed and the $G \setminus N(w)$ is a independent set now.

Being $w \in V(G)$ a vertex of maximal degree in G, note that $H = G[N(w)]$ must be K_r-free by our graph choice (otherwise G would not be $K_{r+1}-$free). So by induction hypothesis, exists a $(r-1)-$partite graph H_1 on $N(w)$ with $d_{H_1}(v) \geqslant d_H(v)$ for every $v \in N(w)$. Let G_1 be the graph obtained from G by performing Zykov symmetrization at vertex w. Now replace H by H_1 and we acquire G_1 which is the required $r-$partite graph, note it is valid because in any step the degrees have decreased. □

This theorem is related to Theorem 3.1.2. In fact Theorem 3.1.3 is stronger than Theorem 3.1.2 and we can see this relation as follows:

$$e(G) \leqslant e(H) \leqslant e(T_r(n)).$$

3.2 Forbidden Bipartite Subgraphs

In this section we study what the consequences when small bipartite graphs are forbidden and starts with Jensen's inequality for convex functions because its importance for a next Erdős' result in Theorem 3.2.2.

Proposition 3.2.1. *(Jensen's Inequality). If* $0 \leqslant \lambda_i \leqslant 1, \sum_{i=1}^{n} \lambda_i = 1$ *and* f *is convex, then:*

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leqslant \sum_{i=1}^{n} \lambda_i f(x_i).$$

As showed in previous chapter, thus Theorem 3.1.2:

$$ex(n, K_{r+1}) = e(T_{r,n}) \approx \left(1 - \frac{1}{r}\right)\binom{n}{2}.$$

Erdős proved that the extremal number is much smaller for the 4-cycle graph.

Theorem 3.2.2. *(Erdős, 1938).*

$$ex(n, C_4) = O(n^{3/2}).$$

Proof. A C_4 is formed by two 'cherries' in the same pair of vertices. Counting these triples (x,{y,z}) of distinct vertices in G such that $xy, xz \in E(G)$ and using proposition 3.2.1 with $\lambda_i = 1/n$ we obtain a very useful inequality,

$$\sum_{i=1}^{n} \frac{1}{n} f(x_i) \geqslant f\left(\sum_{i=1}^{n} \frac{1}{n} x_i\right),$$

using our convex function,

$$\frac{\sum_{i=1}^{n} \binom{x_i}{2}}{n} \geqslant \binom{\frac{\sum_{i=1}^{n} x_i}{n}}{2},$$

replacing x_i and assuming $\sum_{v \in V(G)} d(v) = 2e(G)$,

$$\begin{aligned}\sum_{v \in V(G)} \binom{d(v)}{2} &\geqslant n\binom{\frac{2e(G)}{n}}{2} \\ &= n\frac{\frac{2e(G)}{n}\left(\frac{2e(G)}{n} - 1\right)}{2} \\ &\geqslant \frac{n}{2}\left(\frac{2e(G)}{n} - 1\right)^2.\end{aligned}$$

And since the maximum number of such triples in a C_4-free graph is at most $\binom{n}{2}$ because we can have only one cherry for each pair of vertices,

$$\frac{n}{2}\left(\frac{2e(G)}{n} - 1\right)^2 \leqslant \binom{n}{2},$$

we obtain $e(G) = O(n^{3/2})$ finishing the proof. □

Let $A = \{a_1, ..., a_t\} \subset [n]$ be such that $a_i a_j \neq a_k a_l$ unless $\{i, j\} = \{k, l\}$. How big can be A with these properties? To show a easy lower bound it is necessary only a example that fits the problem, for this problem we can use the number of primes in $[n]$ which is $\pi(n)$. But is it close to the maximum possible size?

Erdos used theorem 3.2.2 to answer this question.

Corollary 3.2.3. *(Erdős, 1938). Let $A \subseteq [n]$ be a multiplicative Sidon set. Then,*

$$|A| \leqslant \pi(n) + O(n^{3/4}).$$

The next result is very similar to theorem 3.2.2. In fact, it is a generalization of Erdős' result, but instead counting cherries we count generalized cherries.

Theorem 3.2.4. *(Kovari-Sós-Turán, 1954). If $s \leqslant t$, then*

$$ex(K(s,t)) = O(n^{2-1/s}).$$

Proof. At this proof we do a double counting on generalized cherries, a generalized cherry is a vertex with a fixed number of neighbors as you can see at figure 3.1

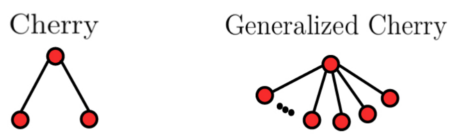

Figure 3.1: Cherry and Generalized Cherry.

$K_{1,s}$ is the generalized cherry we need to count. Note we can count it by taking all possible sets of s neighbors of each vertex, as follows,

$$\#K_{1,s} = \sum_{v \in V(G)} \binom{d(v)}{s}.$$

Now we use the fact $\left(\frac{n}{m}\right)^m \leqslant \binom{n}{m} \leqslant \left(\frac{en}{m}\right)^m$ and Jensen's Inequality 3.2.1 to acquire the following inequality,

$$\begin{aligned}
\sum_{v \in V(G)} \binom{d(v)}{s} &\geqslant \sum_{v \in V(G)} \frac{d(v)^s}{s^s} \\
&\geqslant \frac{1}{s^s} n \left(\frac{\sum_{v \in V(G)} d(v)}{n} \right)^s \\
&= \frac{1}{s^s} n \left(\frac{2e(G)}{n} \right)^s \\
&\geqslant c(s) \frac{e(G)^s}{n^{s-1}},
\end{aligned}$$

where $c(s) = \left(\frac{2}{s}\right)^s$. We already have a lower bound for $K_{1,s}$ that depends of the number of edges, now let's construct a upper bound counting the maximum number of generalized cherries. As $K(s,t)$ is the complete bipartite graph with two partitions whose sizes are $|s|$ and $|t|$ we know the maximum number of $K_{1,s}$ is $t\binom{n}{s}$ but since we want the extremal number of edges without a $K_{s,t}$ we have,

$$
\begin{aligned}
c(s)\frac{e(G)^s}{n^{s-1}} \leqslant \#K_{1,s} &\leqslant (t-1)\binom{n}{s} \\
&\Rightarrow e(G)^s \leqslant \frac{1}{c(s)}\frac{e(t-1)}{s^s}n^{2s-1} \\
&\Rightarrow e(G) \leqslant c(s,t)n^{2-1/s} \\
&\Rightarrow e(G) = O(n^{2-1/s}),
\end{aligned}
$$

Where $c(s,t) = \sqrt[s]{\frac{1}{c(s)}\frac{e(t-1)}{s^s}}$. Using again the fact $\binom{n}{m} \leqslant \left(\frac{en}{m}\right)^m$ we have what we want to show and the proof is finished. □

3.3 The Erdős-Stone Theorem

This theorem is extremely important for graph theory and can be called as the fundamental theorem of graph theory such its importance. To prove it we need the following definition.

Definition 3.3.1. *The Chromatic Number:*

$$\chi(G) = \ min\{r : \exists c : V(G) \to [r] \text{ such that } c(u) \neq c(v) \text{ for every } uv \in E(G)\}$$

Note $\chi(G)$ is related to r-partite graphs, a graph G is r-partite if and only if $\chi(G) \leqslant r$.

Theorem 3.3.2. *(Erdős and Stone, 1946). Let H be an arbitrary graph. Then*

$$ex(n,H) = \left(1 - \frac{1}{\chi(H)-1} + o(1)\right)\binom{n}{2}.$$

Chapter 4

Random Graphs

4.1 The Probabilistic Method

This method is very powerful and has incredible simple applications, when this method begun to being used it created many shocking proofs, but not all applications of the method are so simple or easy to understand. A central fact for the probabilistic method is "a object with property A exists $\iff \mathbb{P}(\text{object has property } A) > 0$"

It is necessary to introduce the Erdős-Rényi random graph $G(n,p)$ which although the name is not a graph. $G(n,p)$ is a probability distribution on graphs, more informally, it is a edge distribution on n vertices with probability p and with the existence of edges being independent events.

Definition 4.1.1. *The (Erős-Rényi) random graph, $G(n,p)$, is the graph on n vertices obtained by choosing each edge independently at random with probability p. In other words, we assume that we have a probability space Ω and independent random variables I_{vw} for each $vw \in \binom{[n]}{2}$, such that*

$$\mathbb{P}(I_{vw} = 1) = 1 - \mathbb{P}(I_{vw} = 0),$$

and let G be the graph with vertex set $[n]$ and edge set $\{vw \in \binom{[n]}{2} : I_{vw} = 1\}$.

Question: What is the probability of $G(n,p) = H$, with H being a fixed graph? Intuitively we can guess there exists a really small chance that this event occurs and the intuition in this case is true, let's see this probability below,

$$\mathbb{P}(G(n,p) = H) = p^{e(H)}(1-p)^{\binom{n}{2} - e(H)}.$$

Theorem 4.1.2. *(Erdős, 1959) There exist graphs whose girth and chromatic number are both arbitrarily large.*

For this theorem will be presented two different proofs, the second one is a little smaller.

Proof. First proof. In other words, we need to prove:

$$\mathbb{P}\begin{pmatrix} \chi(G(n,p)) \geqslant k \text{ and} \\ g(G(n,p)) \geqslant k \end{pmatrix} > 0,$$

for some $p = p(n) \in (0,1)$ and a sufficiently large n.

First of all, we need some definitions,

Definition 4.1.3. *$g(G)$ is the girth of G which is the length of the shortest cycle in G.*

Definition 4.1.4. *$\alpha(G) = max\{|A| : A$ is a independent set$\}$.*

Definition 4.1.5. *$\chi(G)$ is the minimum number of colors used in a proper coloring of G.*

Assuming $\chi(G(n,p)) = r$ we have a $\{V_1, V_2, ..., V_r\}$ partition of $V(G(n,p))$ on r independent sets illustrated at figure 4.1

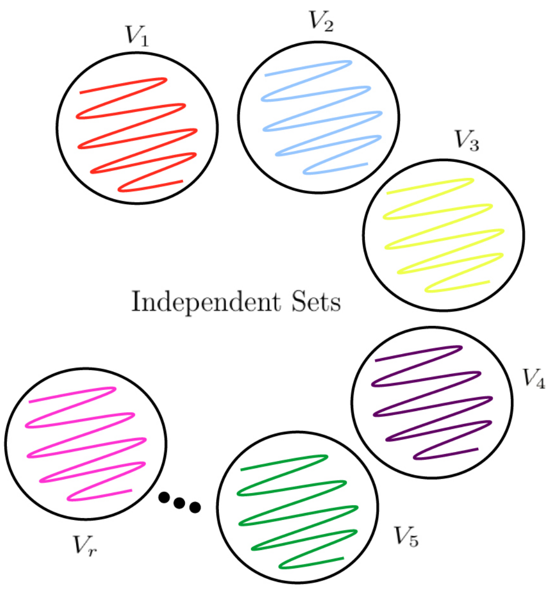

Figure 4.1: r-coloring partition.

By the pigeonhole principle we have $\alpha(G) \geqslant \frac{n}{r}$ and since $\chi(G) = r$,

$$\chi(G(n,p)) \geqslant \frac{n}{\alpha(G(n,p))},$$

and we want to show:

$$\chi(G(n,p)) \geqslant \frac{n}{\alpha(G(n,p))} \geqslant k,$$

Then,

$$\alpha(G(n,p)) < m \iff X_m = 0,$$

with $X_m = \left|\{S \in V(G(n,p))\colon S \text{ is a independent set with exactly } m \text{ vertices in } G(n,p)\}\right|$, and follows $m = \frac{n}{k}$.

$$\begin{aligned}
\mathbb{E}[X_m] &= \sum_{\substack{S \in V(G) \\ |S| = m}} \mathbb{P}(S \text{ is independent }) \\
&= (1-p)^{\binom{m}{2}} \binom{n}{m}
\end{aligned}$$

since we know that $\left(\frac{n}{m}\right)^m \leqslant \binom{n}{m} \leqslant \left(\frac{en}{m}\right)^m$ and $1 - x \leqslant e^{-x}$ replacing in the equation above,

$$\begin{aligned}
\mathbb{E}[X_m] &\leqslant \left(\frac{en}{m}\right)^m e^{(-pm^2)} \\
&= \left(\frac{en}{m} e^{(-pm)}\right)^m \\
&\leqslant \left(\frac{en}{m} e^{(-pm)}\right)^m \\
&= \left(\frac{en}{m} \frac{1}{e^{(pm)}}\right)^m \\
&\ll 1 \text{ if } e^{pm} \gg n
\end{aligned}$$

remember $m = \frac{n}{k}$ and using a logarithm function on both sides of equation to isolate p,

$$pn/k \gg \ln n \Rightarrow p \gg \frac{k \ln n}{n},$$

By Markov's inequality,

$$\mathbb{P}(X_m \geqslant 1) \leqslant \mathbb{E}[X_m],$$

and if $p \gg \frac{k \ln n}{n}$ we have,

$$\begin{aligned}
\mathbb{P}(X_m \geqslant 1) &\leqslant \mathbb{E}[X_m] \\
&\leqslant \frac{1}{100} \\
&\Rightarrow \mathbb{P}(\alpha(G(n,p)) < \frac{n}{k}) \geqslant \frac{99}{100} \\
&\Rightarrow \mathbb{P}(\chi(G(n,p)) > k) \geqslant \frac{99}{100}.
\end{aligned}$$

Since the proof for $\chi(G) \geqslant k$ is finished we will prove $g(G(n,p)) \geqslant k$,

$$g(G(n,p)) \geqslant k \iff X_k \geqslant 1,$$

onde $X_k = |B| :$ todo C_k in $K_n \in B$,

$$\mathbb{E}[X_k] = \sum_{C_k \in K_n} \mathbb{P}(C_k) \leqslant n^k p^k,$$

now we need to limit our short cycles by $\sqrt{n}$ in order to show with high probability we will have few short cycles so we can destroy then without changing anything,

$$n^k p^k \leqslant \sqrt{n} \text{ if } p \leqslant n^{-1+\frac{1}{2k}},$$

then getting together with our previous result,

$$\frac{k \ln k}{n} \ll p \leqslant n^{-1+\frac{1}{2k}}.$$

And we can remove a edge of each cycle with length $< k$, so we have $g(G) \geqslant k$. Note removing a cycle's edge doesn't make $\alpha(G)$ bigger and consequently doesn't make $\chi(G)$ smaller, maintaining all our previous calculations valid. □

Proof. Second Proof. Using the same definitions of the first proof for girth 4.1.3, independence number 4.1.4 and chromatic number 4.1.5 we have,

$$|G| = \sum_{i=1}^{\chi(G)} |S_i| \leqslant \chi(G) max|S_i| \leqslant \chi(G)\alpha(G).$$

Note that if $\alpha(G) \leqslant |G|/k$, then $\chi(G)$ ⩾k.
Let $\epsilon > 0$ be sufficiently small ($\epsilon < 1/k$) and $p = n^{\epsilon-1}$. As we have already been introduced to $G(n,p)$ in definition 4.1.1 let $G = G(n,p) - \{$one vertex of each cycle C_l with $l \leqslant k\}$.

Now we need two claims to reach the result. We claim that, with high probability, G has at most $n/2$ cycles of length $\leqslant k$ and has no independent set of size $n/2k$ then, consequently $\chi(G) \geqslant k$ and $girth(G) \geqslant k$.

First claim. $V(G) > n/2$. First of all we need some variables to count the cycles,

$$X_l = \#C_l \subset G(n,p)$$

$$X = \# \text{ short cycles } = \sum_{l=3}^{k} X_l$$

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{l=3}^{k} \mathbb{E}[X_l] \\
&= \sum_{l=3}^{k} \sum_{C_l \in K_n} \mathbb{P}(C_l \subset G(n,p)) \\
&\leqslant \sum_{l=3}^{k} n^l p^l \\
&= \sum_{l=3}^{k} n^{\epsilon l}, p = n^{\epsilon - 1} \\
&\leqslant 2n^{\epsilon k} \\
&= o(n).
\end{aligned}$$

Note that since we are removing a vertex of each cycle $V(G) \geqslant n - X$. Now we show the general case for this important inequality for the probabilistic method. The Markov's inequality holds for any random variable which takes non-negative valuers and for $a > 0$,

$$\mathbb{P}(X > a) \leqslant \frac{\mathbb{E}(X)}{a}.$$

Using it and replacing $\mathbb{E}[X]$ by its upper bound we have,

$$\begin{aligned}
\mathbb{P}(X \geqslant n/2) &\leqslant \mathbb{E}[X]/(n/2) \\
&= o(n)/n \\
&= o(1)
\end{aligned}$$

Then, we proved that, with high probability, we don't have more than $n/2 - 1$ cycles with length at most k and consequently we have $V(G) > n/2$.

Second claim. $\alpha(G) \leqslant \alpha(G(n,p)) < n/2k$.
We need to count the independent sets,

$$Y_l = \# \text{ independents sets with length } l.$$

$$\begin{aligned}
\mathbb{E}[Y_l] &= \sum_{\substack{|S|=l \\ e(S)=0}} (1-p)^{\binom{l}{2}} \\
&= (1-p)^{\binom{l}{2}} \binom{n}{l} \\
&\leqslant \left(\frac{en}{l}\right)^l e^{-p\binom{l}{2}} \\
&\to 0 \text{ if } l = n/2k.
\end{aligned}$$

Thus both claims,

$$\frac{\mathbb{E}[|\{C_l \subset G : l \leqslant k\}|]}{n/2} + \mathbb{E}[|\{S : |S| \geqslant n/2k, e(S) = 0\}|] \to 0$$

as $n \to \infty$. So G has the desired properties with high probability and the prove is finished. $\square$

Realize that the key point in both proofs was the fact we could choose the most comfortable probability distribution, what gives us a huge flexibility to construct proofs.

4.2 Subgraphs of the Random Graph

Chapter 5

Schedule

Task	Month	Year
Introduction, Methodology, Justification	August	2018
Chapter 1	September	2018
Chapter 2	October	2018
Chapter 3	November	2018
	December	2018
Chapter 4	January	2019
Chapter 5	February	2019
Conclusion	March	2019
Defense	April	2019
	May	2019

Table 5.1: Schedule.