UFABC

Universidade Federal do ABC

Extremal and Probabilistic Combinatorics

Diogo Eduardo Lima Alves

UNDERGRADUATE PROJECT IN COMPUTER SCIENCE PRESENTED
TO
CENTER OF MATHEMATICS, COMPUTATION AND COGNITION
OF
FEDERAL UNIVERSITY OF ABC
FOR
OBTAINING TITLE
OF
BACHELOR OF COMPUTER SCIENCE

Advisor: Prof. Dr. Guilherme Oliveira Mota

Santo André - SP

May 2019

Diogo Eduardo Lima Alves

Extremal and Probabilistic Combinatorics

Undergraduate Project presented to
Undergraduate Program in Computer Science
(Concentration area: Graph Theory),
as partial requirement to obtain the title:
Bachelor of Computer Science.

Advisor: Prof. Dr. Guilherme Oliveira Mota

Santo André - SP

May 2019

Contents

List of Figures

Chapter 1

Introduction

Computer Science was truly fundamental for the fast development of science in the last century, also being fundamental to its validation and communication. It is really hard to think about actual science without the use of computers or strong science communities connected and accessible by the internet. Computer Science is also essential for business. All multinational companies are also software development companies since the way of production, operating and delivering products are managed by software and these aspects are determinants for the success level of any company in the world. It is also common that one of the most valuable assets of a company can be connected to data, software and algorithms. In this scenario, graphs are also very interesting due its importance to Computer Science.

Graphs are one of the most flexible structures, impacting the study of Algebra, Probability and Combinatorics. They can be used for modelling many real scenarios in a very understandable graphical scheme and its properties can be explored to obtain many useful information what explains its huge importance for many knowledge areas not directly connected to Computer Science or Mathematics.

This project focus on classical results of Extremal Combinatorics and Graph Theory. It has detailed ideas and explanations about theorems and concepts developed through a large period of time which have already been intensively studied.

Extremal Combinatorics is our central focus and it studies the maximum or minimum size a collection of mathematical objects may have while satisfying certain restrictions. Since we are particularly interested on graphs, we introduce Extremal Graph Theory.

Extremal Graph Theory studies graphs' possible sizes at the same time they hold some property, which can involve bounds on the number of edges and vertices, since this property, in general, is the absence of a specific subgraph.

An important area in Extremal Combinatorics is Ramsey Theory which, roughly speaking, is the study of finding order in chaos and studies the conditions under which

order must appear.

On the other hand, the Probabilistic Method is an useful tool in Combinatorics, because it is very flexible and it can be used to prove the existence of a structure with certain properties. Simplifying the process, we define a probability space and pick randomly an element of this space then we show this element has the desired properties with positive probability.

Finally the Regularity Lemma is discussed, one of the most important tools in Combinatorics, informally it is about acquiring from any graph a good approximation with random-like subgraphs which gives us a huge flexibility to prove many statements.

Chapter 2

Preliminaries

In this chapter we introduce concepts and state the basic notation to be used in the next chapters.

2.1 Numbers and Sets

We use $\mathbb{N} = \{1, 2, \ldots\}$ to denote the set of positive integers, and $[n]$ denotes the set of all integers from 1 to n.

For a set S and an integer $k \geqslant 0$, we use $|S|$ to denote the number of elements of S. We state $\binom{S}{k}$ as the set of all subsets of S with size k and $\{x_1, \ldots, x_k\}$ refer to element in $\binom{S}{k}$. For $|S| = n$ and $0 \leqslant k \leqslant n$ the number of elements in $\binom{S}{k}$ is the binomial coefficient

$$\left| \binom{S}{k} \right| = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

2.2 Asymptotics

We write $f(n) = o(g(n))$ if and only if, for all $\varepsilon > 0$, there exist n_0 such that if $n \geqslant n_0$, then $f(n) < \varepsilon g(n)$. We also write $f(n) \ll g(n)$ for $f(n) = o(g(n))$.

We write $f(n) \approx g(n)$ if $f(n)/g(n) \to 1$ as $n \to \infty$.

2.3 Graphs

A *graph* $G = (V(G), E(G))$ or simply $G = (V, E)$ is a pair formed by the set of vertices $V(G) = [n]$, with $|V(G)| = n$ and the set of edges, $E(G) \subseteq \binom{V(G)}{2}$, is formed by pairs of vertices. We can represent an *edge* $e = \{u, v\}$ simply as uv and we use $e(G) = |E(G)|$ and $v(G) = |V(G)|$.

A *complete graph*, K_s, is a pair formed by the set of vertices $V(K_s) = [s]$ and the set of edges $E(K_s) = \binom{V(K_s)}{2}$. A *s-clique* is the set of vertices of K_s.

For a graph G an *independent set I* is a subset of vertices of G such that for every $\{u\}$ and $\{v\}$ in I the edge $\{u, v\}$ is not in $E(G)$.

The graph $K_{s,t} = (V, E)$ is a graph whose V can be partitioned in two sets $\{V_1, V_2\}$ such that V_1 and V_2 are independent sets with $|V_1| = s$, $|V_2| = t$ and $E(G) = \{uv : u \in V_1 \text{ and } v \in V_2\}$

The graph $C_\ell = (V, E)$ which, we call a *cycle* of length ℓ, is a graph with $V = \{v_1, v_2, \ldots, v_\ell\}$ and $E = \{v_1 v_2, v_2 v_3, \ldots, v_{\ell-1} v_\ell, v_\ell v_1\}$

A *vertex colouring* with r colours in a graph G is a function $c \colon V(G) \to \{1, \ldots, r\}$ and it can also be denoted as an *r-colouring* of the vertices of G. If we have a graph G such that $c(v) = i$ for every v in $V(G)$ we call it *monochromatic* or *monochromatic in colour i*.

An *edge colouring* with r colours in a graph G is a function $c \colon \binom{V(G)}{2} \to \{1, \ldots, r\}$ and it can also be denoted as a *r*-colouring of the edges of G. We denote the colour i of an edge $\{u, v\}$ as $c(\{u, v\}) = i$. If we have a graph G such that $c(\{u, v\}) = i$ for every $\{u, v\}$ in $E(G)$ we call it monochromatic or monochromatic in colour i.

A *proper k-colouring* of a graph $G = (V, E)$ is a mapping $c \colon E(G) \to \{1, 2, \ldots, k\}$ such that for every v in V and for every $e = \{vu_i\}$ in E then we have $c(vu_i) \neq c(vu_j)$.

A graph H is a *subgraph* of a graph G if every vertex in H is a vertex in G and every edge in H is an edge in G and we denote this relation by $H \subseteq G$. H is an *induced subgraph* of G if for all pair of vertices x and y of H, xy is an edge in H if and only if xy is an edge in G and we denote this relation by $G[H]$. If $H \nsubseteq G$ we call G an *H-free graph*.

Let G and H be graphs such that $V(H) \subseteq V(G)$ and $V(G) = [n]$. We define

$$\mathrm{ex}(n, H) = \max\{e(G) \colon \ G \text{ contains no copy of } H\}.$$

2.4   Neighbourhood

Given an edge $e = \{u, v\}$ in a graph G we call u and v *neighbours* or *adjacents* and we can denote $e = \{u, v\}$ as u is neighbour of v or u is adjacent to v. We also say the edge $\{u, v\}$ is incident to u and v.

The *degree* of a vertex v in a graph G, denoted by $d_G(v)$, we also write simply $d(v)$, is the quantity of neighbours of v. The set of neighbours of v is called *neighbourhood* and is denoted by $N_G(v)$ or simply $N(v)$.

The *minimum degree* of a graph G is denoted by $\delta(G)$ and is defined as

$$\delta(G) = \min\{d(v) \colon v \in V(G)\}.$$

The *maximum degree* of a graph G is denoted by $\Delta(G)$ and is defined as

$$\Delta(G) = \max\{d(v) \colon v \in V(G)\}$$

2.5 Probability

In this text, we call by *probability space* a non-empty finite set Ω and we write a function $\mathbb{P} \colon \Omega \to [0,1]$ that relates a probability $\mathbb{P}(\omega)$ to each ω in Ω. In any case the probabilities are assumed to satisfy

$$\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1.$$

Subsets $E \subseteq \Omega$ are called *events*, and we write $\mathbb{P}(E) = \sum_{\omega \in E} \mathbb{P}(\omega)$. The Inclusion-Exclusion principle shows that, for all $E_1, E_2 \subseteq \Omega$ we have

$$\mathbb{P}(E_1 \cup E_2) = \mathbb{P}(E_1) + \mathbb{P}(E_2) - \mathbb{P}(E_1 \cap E_2) \leqslant \mathbb{P}(E_1) + \mathbb{P}(E_2).$$

A mapping $X \colon \Omega \to \mathbb{R}$ is called a *random variable* and the *expected value* of X is given by

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega)\mathbb{P}(\omega).$$

The *variance* of X is $\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$. We usually write $X > t$ and $X = x$ to denote the events $\{\omega \in \Omega \colon X(\omega) > t\}$ and $\{\omega \in \Omega \colon X(\omega) = x\}$, respectively. In the text we usually omit the probability space information and focus on events and random variables.

A sequence of events $E_1, \ldots, E_m \subseteq \Omega$ is independent if for every $M \subseteq [m]$ and $M \neq \varnothing$,

$$\mathbb{P}\left(\bigwedge_{i \in M} E_i\right) = \prod_{i \in M} \mathbb{P}(E_i).$$

Chapter 3

Ramsey Theory

Ramsey's Theory is especially philosophical among the components of the Extremal Combinatorics. Ramsey's Theory is connected with the purpose of finding order in chaos, that is, it is motivated by the thought that complete chaos is impossible to achieve, then we can prove statements about any graph, independent of its particular characteristics.

3.1 Infinite Ramsey's Theorem

Consider the following question "Given a group of 6 people, are there 3 people that are mutual friends or 3 people that are mutual strangers?"

In order to answer this question think about a group of six people, call them Richard, Maria, Bete, Jack, Paul and Steven. It is possible to suppose without loss of generality that Richard has at least three friends, we assume they are Maria, Bete and Jack. If any two of these three are friends with each other, then they form a group of three mutual friends with Richard, otherwise Maria, Bete and Jack form a group of three mutual strangers to themselves.

Replacing people by vertices and a friendship by colours (blue for being friends and red otherwise) we realize that a complete graph K_6, 2-coloured, always have a monochromatic triangle independently of the edge colouring, but what about 5 vertices?

The graph K_5 can be constructed without a monochromatic triangle as show in Figure 3.1. The following definition will permit us to discuss it in terms of the Ramsey Numbers.

Definition 3.1.1. *The* Ramsey Number $R(s,t)$ *is the minimum n such that any 2-colouring of $\binom{V(K_n)}{2}$ induces a complete subgraph K_s whose edges are monochromatic in colour* 1 *or a complete subgraph K_t whose edges are monochromatic in colour* 2.
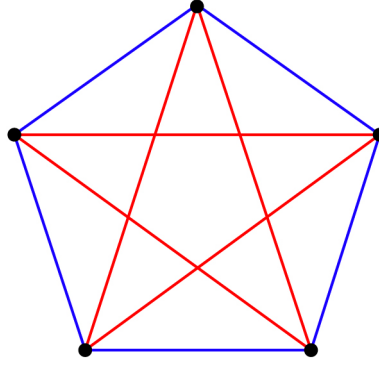
Note that $R(3,3) = 6$.

Figure 3.1: Graph K_5 without monochromatic triangle.

Theorem 3.1.2 (Ramsey's Theorem - Infinite Version [13]). *Let $r \geqslant 1$ be an integer. Every r-coloured $K_{\mathbb{N}}$ must contain an infinite monochromatic complete graph.*

Proof. Fix a r-colouring of the edges of the graph $K_{\mathbb{N}}$, with $V(K_{\mathbb{N}}) = [\mathbb{N}]$. Pick the vertex $x = \{1\}$, and since $K_{\mathbb{N}}$ is infinite there exist an infinite set $X \subset N(x)$ where all the vertices are connected to x in the same colour (if there exist more than one such infinite set we choose one arbitrarily). Consider a vertex in X, say y. Since X is infinite there exist an infinite set $Y \subset X$ such that all edges between Y and y have the same colour. Now consider a vertex in Y, say z, with $z > y$. Similarly as before, since Y is infinite, there must be an infinite number of edges with the same colour connecting z to vertices in Y, define $Z \subseteq Y$ such that for every v in Z we have $c(zv) = i$. Realize we can continue picking successive vertices indefinitely since the graph $K_{\mathbb{N}}$ is infinite, this results in a set of vertices $V = \{x, y, z, \ldots\} \subseteq V(K_{\mathbb{N}})$. We define $E = \{xy, xz, \ldots, yz, \ldots\}$. Note that the colour of any edge in E is determined by the smaller of its vertices. Then, for i, j, k in $[r]$, any edge xv for v in V must have colour i, any edge yv for v in $V \smallsetminus \{x\}$ must have colour j and any edge zv for v in $V \smallsetminus \{x, y\}$ must have colour k. We can now define a r-colouring of V, we colour any vertex in V, say v, such that $c(v) = \{c(vp) \colon p \in P \text{ and } v \notin P\}$, where P is defined in the same way that X, Y and Z were. Then V is an infinite set coloured with r colours and there must exist an infinite subset monochromatic, define graph M such that $V(M)$ is this infinite monochromatic subset of V and $E(M) = \{uv \colon uv \text{ in } E(K_{\mathbb{N}}) \text{ and } \{u\}, \{v\} \text{ in } V(M)\}$. Since every vertex in M has the same colour, so every edge in M must have the same colour. Thus the graph M is an infinite monochromatic complete graph, finishing the proof. □

3.2 Finite Ramsey's Theorem

In order to prove the next theorem we need the following definition.

Definition 3.2.1. *We define $N_i(v) = \{w : c(\{v, w\}) = i\}$, and it is denoted as the colour i neighbourhood of v.*

Theorem 3.2.2 (Erdős and Szekeres' Theorem [3,5])**.** *Considering the graph K_n we have*

$$(\sqrt{2})^k < R(k, k) < 4^k.$$

Proof. We start by proving the upper bound. First, we prove that for every $k, \ell \in \mathbb{N}$ we have

$$R(k, \ell) \leqslant R(k - 1, \ell) + R(k, \ell - 1). \tag{3.2.1}$$

Choose $n \geqslant R(k - 1, \ell) + R(k, \ell - 1)$ and pick any vertex v from $[n]$. Then, v has either at least $R(k - 1, \ell)$ red neighbours, or at least $R(k, \ell - 1)$ blue neighbours. Without loss of generality assume that v has at least $R(k - 1, \ell)$ red neighbours. Then, we have on the red neighbourhood of v a red K_{k-1} or a blue K_ℓ, if it is a red K_{k-1} we can add v to it, forming a red K_k. We just proved we always have a K_k red or a K_ℓ blue with such n and this confirms the upper bound in (3.2.1).

Now we want to prove by induction on $k + \ell$ that

$$R(k, \ell) \leqslant \binom{k + \ell - 2}{k - 1}.$$

Note that for $k = \ell = 2$ it works well. Then by induction hypothesis

$$R(k - 1, \ell) \leqslant \binom{k + \ell - 3}{k - 2},$$

and

$$R(k, \ell - 1) \leqslant \binom{k + \ell - 3}{k - 1}.$$

Since $\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$, we have

$$R(k, \ell) \leqslant \binom{k + \ell - 3}{k - 2} + \binom{k + \ell - 3}{k - 1} = \binom{k + \ell - 2}{k - 1}.$$

Confirming our induction hypothesis. Now let $k = \ell$ and using Stirling's approximation $\binom{2n}{n} \approx (2^{2n}/\sqrt{\pi n})$, assuming n is sufficiently large follows

$$R(k, k) \leqslant \binom{2k - 2}{k - 1}$$
$$\approx \frac{2^{2k-2}}{\sqrt{\pi(k - 1)}}$$
$$< 4^k.$$

Finishing the proof of the upper bound.

Now we proceed by proving the lower bound. Consider a random colouring $c\colon \binom{n}{2} \to$ {red, blue} of $E(K_n)$, with probability $p = 1/2$ of having $c(\{i,j\}) = $ red for each $\{i,j\}$ in $E(K_n)$.

We define X as the number of monochromatic k-cliques in K_n. Then, we have the expected value for X as follows

$$\mathbb{E}[X] = \binom{n}{k} 2^{1-\binom{k}{2}}$$

$$= \frac{n!}{k!(n-k)!} 2^{1-\frac{k(k-1)}{2}}$$

$$< \frac{n^k 2^{1+k/2}}{k! 2^{k^2/2}},$$

since $n!/(n-k)! < n^k$. Now we show that for $n = (\sqrt{2})^k$, we have

$$\binom{2^{k/2}}{k} 2^{1-\binom{k}{2}} < \frac{(2^{k/2})^k 2^{1+k/2}}{k! 2^{k^2/2}}$$

$$= \frac{2^{1+k/2}}{k!}$$

$$< 1,$$

since $2^{1+k/2} < k!$ for $k > 2$, then, the expected value of X is smaller than 1. Therefore, there must exist a colouring in which $X = 0$, from where we conclude that $R(k,k) > (\sqrt{2})^k$, finishing the proof. □

3.3   Schur's Theorem

Theorem 3.3.1 (Schur's Theorem [17]). *For any positive integer and any colouring* $c\colon \mathbb{N} \to [r]$*, there are* $a,b,c \in \mathbb{N}$ *such that* $a+b=c$ *and* $c(a) = c(b) = c(c)$.

Proof. For a vertex colouring given as $c\colon \mathbb{N} \to [r]$ define an edge colouring $c'\colon \binom{\mathbb{N}}{2} \to [r]$ with $c'(\{a,b\}) := c(|a-b|)$. By Theorem 3.1.2, there exists a monochromatic triangle with three vertices, say $\{x,y,z\}$, with $x < y < z$.

Using the definition of c' we have

$$c'(\{x,y\}) = i = c(y-x)$$

$$c'(\{x,z\}) = i = c(z-x)$$

$$c'(\{y,z\}) = i = c(z-y).$$

From that, $c(y-x) = c(z-x) = c(z-y)$, and $(z-y) + (y-x) = (z-x)$ this implies that there are a,b,c such that $a+b=c$ and $c(a) = c(b) = c(c)$, as required.

□

3.4 Van der Waerden's Theorem

Theorem 3.4.1 (Van der Waerden's Theorem [21])**.** *For every colouring of $\mathbb{N}$ with r colours, there is an arbitrarily long monochromatic arithmetic progression.*

Proof. To prove this theorem we need the following definition,

Definition 3.4.2. $W(r, k) = \min\{n\colon \text{ for each } c \text{ with } [n] \to \{1, \ldots, r\} \text{ there exists a monochromatic arithmetic progression of length } k\}$.

We claim that $W(r, k)$ exists and we prove it by double induction on the number of arithmetic progressions, so assume that $W(r, k - 1)$ exists. Note that if $k \leqslant 2$ the result is trivial.

First denote the arithmetic progression $\{a, a+d, a+2d, \ldots, a+(k-1)d\}$ by $AP(a, d, k)$ such that it has common difference d and length k.

We say an arithmetic progressions (AP) of length k is *focused at z* if for every i, $j \in [r]$ we have

$$a_i + (k - 1)d_i = a_j + (k - 1)d_j = z.$$

Furthermore, the AP is *colour-focused* if it is monochromatic.

Being s the number of colour-focused arithmetic progressions, if $s = 1$ it is trivial because we just need $n \geqslant r + 1$ to prove it holds.

We claim that for all $s \leqslant r$, there exists n such that whenever $[n]$ is r-coloured, there exists either a monochromatic arithmetic progression of length k, or s colour-focused arithmetic progressions of length $k - 1$.

We will proceed by induction on s. Let $s > 1$ and suppose $n = W(r, k - 1)$ is sufficient for $s - 1$ APs; we will prove that $N = 2nW(r^{2n}, k - 1)$ is sufficient for s APs.

Partition $[N]$ into blocks of length $2n$, in each block we have a monochromatic arithmetic progression of length k or $s - 1$ *colour-focused* arithmetic progressions of length $k - 1$.

Note that the r-colouring of $[N]$ induces r^{2n} different ways of colouring the blocks and from the definition of N there exists an arithmetic progression $\{B(x), B(x + y), \ldots, B(x + (k - 2)y)\}$ of blocks, such that these blocks are coloured identically.

Let $A_j = AP(a_j, d_j, k - 1)$ for $1 \leqslant j \leqslant s - 1$ be the $s - 1$ colour-focused APs in $B(x)$, let z be their focus, and observe that the following s APs of length $k - 1$ are colour-focused at $z + 2yn(k - 1)$:

$$A'_j := AP(a_j, d_j + 2yn, k - 1) \text{ for } 1 \leqslant j \leqslant s - 1,$$

and $AP(z, 2yn, k - 1)$ that can be seen in Figure 3.2. Then the claim is proved.
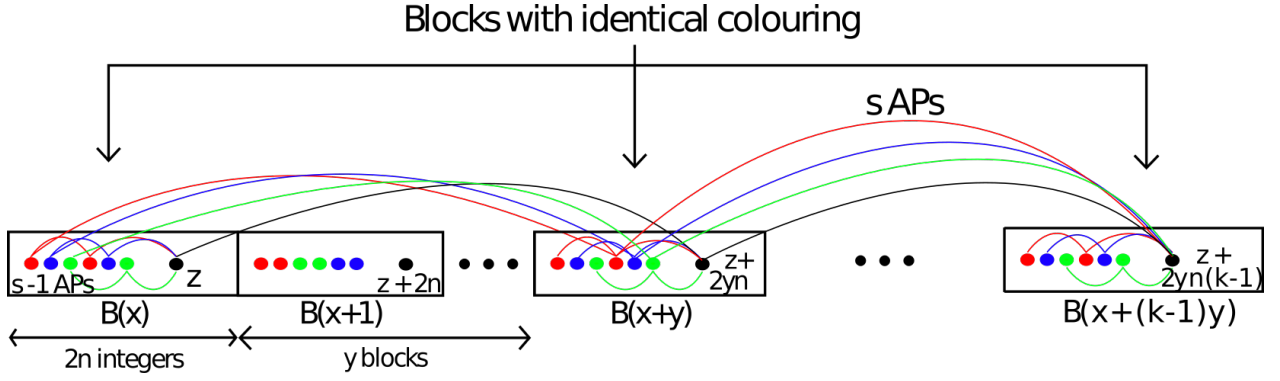
Figure 3.2: s APs in the $W(r^{2n}, k-1)$ blocks.

The theorem follows from the claim by setting $s = r$ because if we have s APs *colour-focused* at $z + 2yn(k - 1)$ and there is no other different colour to use in $z + 2yn(k-1)$, by the fact every AP has length $k-1$ independently of the colour we choose for z we will have an AP of length k. $\qquad\square$

Chapter 4

Extremal Graph Theory

In this chapter the problems and theorems are centred in extremal graphs. It embraces questions about the maximum or minimum size of a graph at the same time it satisfies certain properties.

4.1 Turán's Theorem

In this section we study forbidden subgraphs and first we shall answer the following question.

"What is the maximum number of edges in a triangle-free graph?"

Note that the complete bipartite graph with $\frac{n}{2}$ vertices in each part has $\frac{n^2}{4}$ edges and no triangles. The next theorem states that this number of edges is the maximum possible in a triangle-free graph.

Theorem 4.1.1 (Mantel's Theorem [11]). *If G is a triangle-free graph with n vertices, then*

$$e(G) \leqslant \left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor.$$

Proof. The proof follows by induction on n. Assume $n \geqslant 4$ and let G' be the graph obtained from G by removing two vertices $u, v \in G$ such that $\{u, v\} \in E(G)$. Note that if G is triangle-free, then G' is also triangle-free because it is not possible to form a triangle by removing an edge and there are at most $n - 1$ edges incident to u or v, then

$$e(G') \leqslant \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \left(\left\lceil \frac{n}{2} \right\rceil - 1 \right)$$
$$= \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil - n + 1.$$

Therefore, $e(G) \leqslant e(G') + n - 1 = \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil$. $\qquad\qquad \square$

Turán generalized Theorem 4.1.1 in 1941 and this result will require the following definitions.

Definition 4.1.2. *$T_r(n)$ is the complete r-partite graph with n vertices and $\left\lceil \frac{n}{r} \right\rceil$ or $\left\lfloor \frac{n}{r} \right\rfloor$ vertices in each part.*

Definition 4.1.3. *$t_r(n)$ is the number of edges in $T_r(n)$.*

Theorem 4.1.4 (Turán's Theorem [20])**.** *Let G be a graph with n vertices, such that G is K_{r+1}-free, then*

$$e(G) \leqslant t_r(n)$$

Proof. This proof follows by induction on n. Let G be the complete r-partite graph. Note $d(v) \approx n - (n/r)$ for each $v \in V(G)$. Now, using the fact $\sum_{v \in V(G)} d(v) = 2e(G)$ we have

$$t_r(n) \approx \frac{n}{2} \left(n - \frac{n}{r} \right) \approx \left(1 - \frac{1}{r} \right) \binom{n}{2}.$$

Construct a graph G' by removing a K_r from G as shown in Figure 4.1. Note that each vertex in G' could be neighbour of at most $r - 1$ vertices in K_r, otherwise G would have a K_{r+1}, then we have $(r - 1)(n - r)$ vertices between G' and K_r.
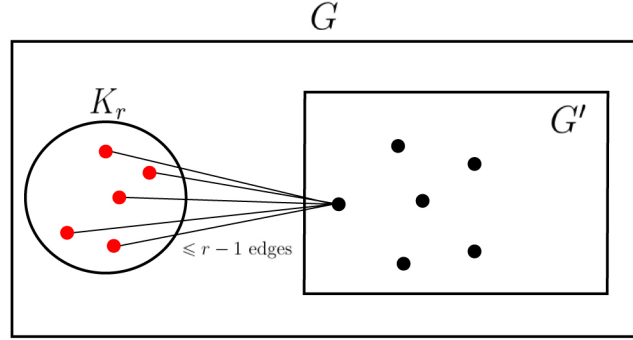

Figure 4.1: G' and K_r.

Note that inside K_r there exist $\binom{r}{2}$ edges and by the induction hypothesis we have $e(G') \leqslant t_r(n - r)$. Then it follows that

$$\begin{aligned}
e(G) &\leqslant e(G') + (r - 1)(n - r) + \binom{r}{2} \\
&\leqslant t_r(n - r) + (r - 1)(n - r) + \binom{r}{2} \\
&= t_r(n).
\end{aligned}$$

The last equality follows from the fact that $t_r(n) - t_r(n-r) = \binom{r}{2} + (r-1)(n-r)$ as we can see in Figure 4.2 removing a vertex from each part $\{1, \ldots, r\}$ we remove $\binom{r}{2}$ edges between the r vertices removed and $(r-1)(n-r)$ edges because every vertex in $T_r(n)$ is connected exactly with $r-1$ vertices of the r removed vertices.
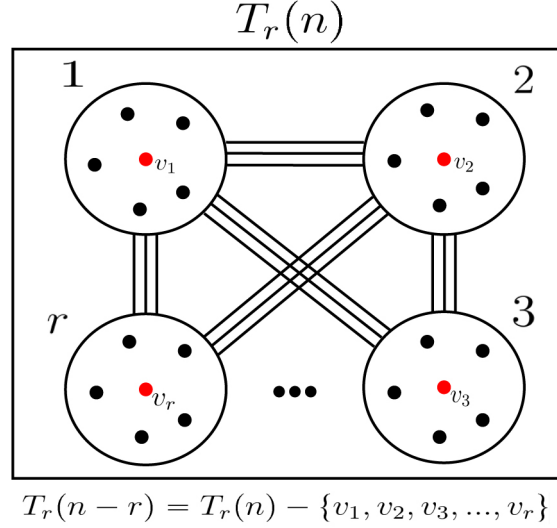

Figure 4.2: $T(n)$ and $T(n-r)$.

Then the proof is finished.   □

Theorem 4.1.5 (Erdős [2]). *Let G be a K_{r+1}-free graph with n vertices. Then, there is an r-partite graph H with $V(H) = V(G)$ such that $d_H(v) \geqslant d_G(v)$ for every $v \in V(G)$.*

Proof. The proof follows by induction on r. Let w be a vertex of maximum degree in G. For every vertex $v \in V(G) \smallsetminus N(w)$ we remove the edges incident to v and add an edge between v and each neighbour of w, this operation is called 'Zykov symmetrization'. Clearly, no vertex degree has decreased because at this point $d(v) = \Delta(G)$ for every $v \in V(G) \smallsetminus N(w)$ and the graph is still K_{r+1}-free because $N(w)$ has not changed and the $G \smallsetminus N(w)$ now is an independent set.

Note that the graph $G_1 = G[N(w)]$ must be K_r-free by our graph choice (otherwise G would not be K_{r+1}-free). By the induction hypothesis, exists an $(r-1)$-partite graph H_1 in $N(w)$ with $d_{H_1}(v) \geqslant d_{G_1}(v)$ for every v in $N(w)$. Let H be the graph obtained from G by performing Zykov symmetrization at vertex w. Note that $\{v : v \in H \smallsetminus H_1\}$ is an independent set and with H_1 we acquire the required r-partite graph, note it is valid because in any step the degrees have decreased.   □

The Theorem 4.1.5 is related to Theorem 4.1.4. In fact, Theorem 4.1.5 is stronger

than Theorem 4.1.4 and we can see this relation as follows

$$e(G) \leqslant e(H) \leqslant e(T_r(n)),$$

with G and H as stated in Theorem 4.1.5.

4.2   Forbidden Bipartite Subgraphs

In this section we study what are the consequences when small bipartite graphs are forbidden and we start with Jensen's inequality for convex functions because its importance for proving the Theorem 4.2.2.

Proposition 4.2.1 (Jensen's Inequality [9]). *If $0 \leqslant \lambda_i \leqslant 1$, with $\sum_{i=1}^{n} \lambda_i = 1$ and f is a convex function, then*

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leqslant \sum_{i=1}^{n} \lambda_i f(x_i).$$

As showed in previous subsection, then Theorem 4.1.4 implies that

$$\mathrm{ex}(n, K_{r+1}) = e(T_{r,n}) \approx \left(1 - \frac{1}{r}\right)\binom{n}{2}.$$

Erdős proved that the extremal number of C_4 is much smaller then the one of complete graphs.

Theorem 4.2.2 (Erdős [6]). *Let G be a graph with n vertices and C_4-free, then*

$$e(G) = O(n^{3/2}).$$

Proof. A C_4 is formed by two edge disjoint 'cherries' in the same pair of vertices. Counting the triples (x, y, z) of distinct vertices in G such that $xy, xz \in E(G)$ and using Proposition 4.2.1 with $\lambda_i = 1/n$ we obtain the following very useful inequality

$$\sum_{i=1}^{n} \frac{1}{n} f(x_i) \geqslant f\left(\sum_{i=1}^{n} \frac{1}{n} x_i\right).$$

Applying it we have,

$$\frac{\sum_{i=1}^{n} \binom{x_i}{2}}{n} \geqslant \binom{\frac{\sum_{i=1}^{n} x_i}{n}}{2}.$$

Replacing x_i by $d(v)$, since $\sum_{v \in V(G)} d(v) = 2e(G)$, we have

$$\sum_{v \in V(G)} \binom{d(v)}{2} \geqslant n \binom{\frac{2e(G)}{n}}{2}$$

$$= n \frac{\frac{2e(G)}{n} \left(\frac{2e(G)}{n} - 1 \right)}{2}$$

$$\geqslant \frac{n}{2} \left(\frac{2e(G)}{n} - 1 \right)^2.$$

Since the maximum number of such triples (x, y, z) in a C_4-free graph is at most $\binom{n}{2}$ because we can have only one cherry for each pair of vertices,

$$\frac{n}{2} \left(\frac{2e(G)}{n} - 1 \right)^2 \leqslant \binom{n}{2},$$

we obtain $e(G) = O(n^{3/2})$ finishing the proof. $\qquad \square$

The next result is very similar to Theorem 4.2.2. In fact, it is a generalization of Erdős' result, but instead counting cherries we count stars.

Theorem 4.2.3 (Kovari-Sós-Turán [10]). *If G is a $K_{s,t}$-free graph with n vertices and $s \leqslant t$, then*

$$e(G) = O(n^{2-1/s}).$$

Proof. For this proof we do a simple double counting on the number of stars, which are graphs with one central vertex and a fixed number of neighbours as you can see at Figure 4.3.

Figure 4.3: A cherry and a star.

The graph $K_{1,s}$ is the star we need to count. We count it by taking all possible sets of s neighbours of each vertex, as follows,

$$|\{K_{1,s} \colon K_{1,s} \subseteq G\}| = \sum_{v \in V(G)} \binom{d(v)}{s}.$$

Now we use the fact $\left(\frac{n}{m} \right)^m \leqslant \binom{n}{m} \leqslant \left(\frac{en}{m} \right)^m$ and Jensen's Inequality (Proposition 4.2.1) to obtain the following inequality,

$$\sum_{v \in V(G)} \binom{d(v)}{s} \geqslant n \binom{\sum_{v \in V(G)} d(v)/n}{s}$$

$$\geqslant \frac{1}{s^s} n \left(\frac{\sum_{v \in V(G)} d(v)}{n} \right)^s$$

$$= \frac{1}{s^s} n \left(\frac{2e(G)}{n} \right)^s$$

$$= c(s) \frac{e(G)^s}{n^{s-1}},$$

where $c(s) = (2/s)^s$. We already have a lower bound for $K_{1,s}$ that depends of the number of edges, now we construct an upper bound counting the maximum number of generalized cherries. As $K_{s,t}$ is the complete bipartite graph with two partitions whose sizes are $|s|$ and $|t|$, we know the maximum number of $K_{1,s}$ is $t\binom{n}{s}$ but since we want the maximum number of edges in G without a $K_{s,t}$ we have,

$$c(s) \frac{e(G)^s}{n^{s-1}} \leqslant \max\{e(G') : G' \text{ is a } K_{s,t}\text{-free graph}\}$$

$$= (t-1)\binom{n}{s}$$

$$\Rightarrow e(G)^s \leqslant \frac{1}{c(s)} \frac{e^s (t-1)}{s^s} n^{2s-1}$$

$$\Rightarrow e(G) \leqslant c(s,t) n^{2-1/s}$$

$$\Rightarrow e(G) = O(n^{2-1/s}),$$

where $c(s,t) = \sqrt[s]{\frac{1}{c(s)} \frac{e^s (t-1)}{s^s}}$ and since $\binom{n}{m} \leqslant \left(\frac{en}{m} \right)^m$ the proof is finished. $\qquad\square$

Chapter 5

Random Graphs

5.1 The Probabilistic Method

This method has really simple applications, but not all applications of the method are so simple or easy to understand. A central fact for the simpler version of the probabilistic method is "a object with property A exists if and only if $\mathbb{P}(\text{random object has property } A) > 0$".

It is necessary to introduce the Erdős-Rényi random graph model $G(n, p)$. $G(n, p)$ is a probability distribution on graphs, more informally, it is an edge distribution on n vertices with probability p and with the existence of edges being independent events.

Definition 5.1.1. *The (Erdős-Rényi) random graph $G(n, p)$ or $G_{n,p}$ is the graph on n vertices obtained by choosing each edge independently and uniformly at random with probability p. In other words, we assume that we have a probability space Ω and independent random variables I_{vw} for each $\{v, w\} \in \binom{[n]}{2}$, such that*

$$\mathbb{P}(I_{vw} = 1) = 1 - \mathbb{P}(I_{vw} = 0),$$

and let G be the graph with vertex set $[n]$ and edge set $\{\{v, w\} \in \binom{[n]}{2} : I_{vw} = 1\}$.

Theorem 5.1.5 introduces the method and we present two different proofs. In order to prove it we will need the following definitions.

Definition 5.1.2. *$g(G)$ denotes the girth of G which is the length of the shortest cycle in G.*

Definition 5.1.3. *$\alpha(G) = \max\{|A| : A \text{ is an independent set of } G\}$.*

Definition 5.1.4. *$\chi(G)$ is the minimum number of colours used in a proper colouring of G.*

Assuming $\chi(G(n,p)) = r$ there is a $\{V_1, V_2, \ldots, V_r\}$ partition of $V(G(n,p))$ on r independent sets (see Figure 5.1).
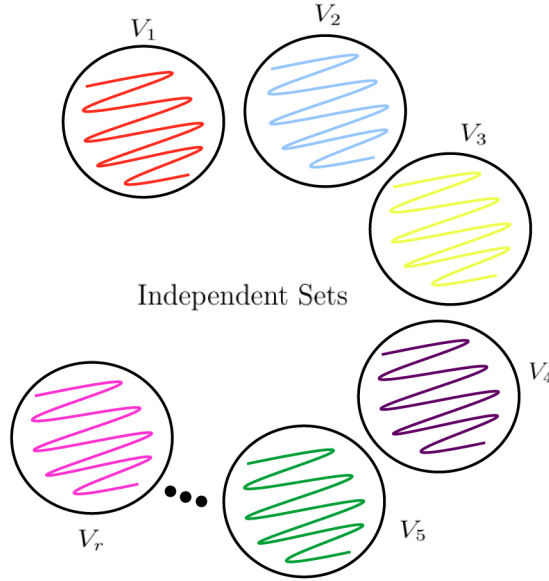
Figure 5.1: r-colouring partition.

Theorem 5.1.5. [*Erdős* [1]] *For every $k \geqslant 1$, there exist a graph G with $g(G) \geqslant k$ and $\chi(G) \geqslant k$.*

Proof. 1. We will prove for arbitrarily large k that

$$\mathbb{P}(\chi(G(n,p)) \geqslant k \text{ and } g(G(n,p)) \geqslant k) > 0,$$

for some $p = p(n) \in (0,1)$ and a sufficiently large n.

Assuming $\chi(G(n,p)) = r$ we have a partition $\{V_1, V_2, \ldots, V_r\}$ of $V(G(n,p))$ on r independent sets.

Since we have r parts and n vertices, then $\alpha(G(n,p)) \geqslant \frac{n}{r}$. Therefore,

$$\chi(G(n,p)) \geqslant \frac{n}{\alpha(G(n,p))},$$

and we want to show that

$$\chi(G(n,p)) \geqslant \frac{n}{\alpha(G(n,p))} \geqslant k.$$

But

$$\alpha(G(n,p)) \leqslant m \iff X_{m+1} = 0,$$

where $X_{m+1} = \left|\{S \in V(G(n,p)): S \text{ is an independent set with exactly } m+1 \text{ vertices in } G(n,p)\}\right|$, and let $m = \frac{n}{k}$.

$$\mathbb{E}[X_{m+1}] = \sum_{\substack{S \subseteq V(G) \\ |S| = m+1}} \mathbb{P}(S \text{ is independent})$$

$$= (1-p)^{\binom{m+1}{2}} \binom{n}{m+1}.$$

For $p \gg \ln n / n$, we have $pn \gg \ln n$. Then $e^{pm} \gg n$. Using this, the fact that $\left(\frac{n}{m}\right)^m \leqslant \binom{n}{m} \leqslant \left(\frac{en}{m}\right)^m$ and $1 - x \leqslant e^{-x}$, we obtain

$$\mathbb{E}[X_{m+1}] \leqslant \left(\frac{en}{m+1}\right)^{m+1} e^{(-p(m+1)^2)}$$

$$= \left(\frac{en}{m+1} e^{(-p(m+1))}\right)^{m+1}$$

$$= \left(\frac{en}{m+1} \frac{1}{e^{p(m+1)}}\right)^{m+1}$$

$$\ll 1.$$

By Markov's inequality,

$$\mathbb{P}(X_{m+1} \geqslant 1) \leqslant \mathbb{E}[X_{m+1}]$$

$$\leqslant \frac{1}{100}$$

$$\Rightarrow \mathbb{P}\left(\alpha(G(n,p)) \leqslant \frac{n}{k}\right) \geqslant \frac{99}{100}$$

$$\Rightarrow \mathbb{P}\left(\chi(G(n,p)) \geqslant k\right) \geqslant \frac{99}{100}.$$

Now we turn our attention to the girth of $G(n,p)$.

$$g(G(n,p)) \leqslant k \iff \sum_{\ell=3}^{k} Y_\ell \geqslant 1,$$

where $Y_\ell = |C_\ell : C_\ell \subseteq G(n,p)|$,

$$\mathbb{E}[Y_\ell] = \sum_{C_\ell \subseteq K_n} \mathbb{P}(C_\ell \subseteq G(n,p)) \leqslant n^\ell p^\ell.$$

Then $\sum_{\ell=3}^{k} Y_\ell \leqslant kn^k p^k$. Now we need to limit our short cycles by $\sqrt{n}$ in order to show with high probability we will have few short cycles so we can destroy them,

$$kn^k p^k < \sqrt{n} \text{ if } p < \frac{n^{\frac{1}{2k}}}{nk^{1/k}},$$

which implies $\sum_{\ell=3}^{k} Y_\ell < \sqrt{n}$ then with p as follows

$$\frac{k \ln k}{n} \ll p < \frac{n^{\frac{1}{2k}}}{nk^{1/k}}.$$

Now we can remove a vertex of each cycle with length $< k$, so we have $g(G(n,p)) \geqslant k$. Note that removing a cycle's vertex does not make $\alpha(G(n,p))$ bigger and consequently does not make $\chi(G(n,p))$ smaller, maintaining all our previous calculations valid. We can assume we have a graph G with $g(G) \geqslant k$ and $\chi(G) \geqslant k$ because we proved it with high probability.   □

Proof. 2. Let $G = G(n,p)$ and $\chi(G) = r$, being $\{S_1, \ldots, S_r\}$ the partition of $V(G)$ in r independent sets.

$$|V(G)| = \sum_{i=1}^{\chi(G)} |S_i| \leqslant \chi(G) \max |S_i| \leqslant \chi(G)\alpha(G).$$

Note that if $\alpha(G) \leqslant |G|/k$, then $\chi(G) \geqslant$ k. Let $\varepsilon > 0$ be sufficiently small ($\varepsilon < 1/k$) and $p = n^{\varepsilon-1}$.

Now we need two claims to reach the result. We claim that, with high probability, G has at most $n/2$ cycles of length $\leqslant k$ and has no independent set of size $n/2k$ then, consequently $\chi(G) \geqslant k$ and $g(G) \geqslant k$.

First claim. $V(G) > n/2$. First of all we need some variables to count the cycles,

$$X_\ell = |C_\ell \colon C_\ell \subseteq G|$$

$$X = \text{number of short cycles in } G = \sum_{\ell=3}^{k} X_\ell$$

$$
\begin{aligned}
\mathbb{E}[X] &= \sum_{\ell=3}^{k} \mathbb{E}[X_\ell] \\
&= \sum_{\ell=3}^{k} \sum_{C_\ell \subseteq K_n} \mathbb{P}(C_\ell \subseteq G) \\
&\leqslant \sum_{\ell=3}^{k} n^\ell p^\ell \\
&= \sum_{\ell=3}^{k} n^{\varepsilon\ell}, \text{ with } p = n^{\varepsilon-1} \\
&\leqslant 2n^{\varepsilon k} \\
&= o(n).
\end{aligned}
$$

Now we remove a vertex of each cycle, let G' be the graph by removing the vertices of G, note that $V(G') \geqslant n - X$. Now we show the general case for this useful inequality

for the probabilistic method. The Markov's inequality holds for any random variable which takes non-negative values and for $a > 0$,

$$\mathbb{P}(X > a) \leqslant \frac{\mathbb{E}(X)}{a}.$$

Using it and replacing $\mathbb{E}[X]$ by its upper bound we have,

$$
\begin{aligned}
\mathbb{P}(X \geqslant n/2) &\leqslant \frac{\mathbb{E}[X]}{n/2} \\
&= o(n)/n \\
&= o(1).
\end{aligned}
$$

Then, we proved that, with high probability, the graph do not have more than $n/2 - 1$ cycles with length at most k and consequently we have $|V(G')| > n/2$.

Second claim. $\alpha(G') \leqslant \alpha(G) < n/2k$. We need to count the independent sets of G. Let $Y_\ell = $ the number of independent sets with length ℓ and note that for $\ell = \frac{n}{2k}$ we have

$$
\begin{aligned}
\mathbb{E}[Y_\ell] &= \sum_{S:\, |S|=\ell} (1-p)^{\binom{\ell}{2}} \\
&= (1-p)^{\binom{\ell}{2}} \binom{n}{\ell} \\
&\leqslant \left(\frac{en}{\ell}\right)^{\ell} e^{-p\binom{\ell}{2}} \\
&= o(1).
\end{aligned}
$$

Thus,

$$\frac{\mathbb{E}[|\{C_\ell \subseteq G' \colon \ell \leqslant k\}|]}{n/2} + \mathbb{E}[|\{S \colon |S| \geqslant n/2k \text{ and } e(S) = 0\}|] \to 0$$

as $n \to \infty$. So G' has the desired properties with high probability and the proof is finished. $\square$

Realize that the key point in both proofs was the fact we could choose the most comfortable probability distribution, what gives us a huge flexibility to construct proofs.

5.2 Subgraphs of the Random Graph

In this section we will manipulate p in order to have a specific subgraph contained in G.

Definition 5.2.1. *A function $p^* = p^*(n)$ is a threshold for a monotone increasing property $\mathcal{P}$ in the random graph $G(n, p)$ if*

$$\lim_{n \to \infty} \mathbb{P}(G(n, p) \in \mathcal{P}) = \begin{cases} 0, & \textit{if } p/p^* \to 0, \\ 1, & \textit{if } p/p^* \to \infty \,. \end{cases}$$

We shall prove the threshold for the property "containing a triangle" in G. We will need the Chebyshev inequality presented below.

Theorem 5.2.2 (Chebyshev's Inequality [19])**.** *Let X be a random variable having finite mean μ and finite variance σ^2. Let a be a strictly positive real number. Then, the following inequality holds*

$$\mathbb{P}(|X - \mu| \geqslant a) \leqslant \sigma^2/a^2.$$

Using $a = \mu$, we have the following very useful inequality,

$$\mathbb{P}(X = 0) \leqslant \mathbb{P}(|X - \mu| \geqslant \mu) \leqslant \sigma^2/\mu^2,$$

which gives us an upper bound for $\mathbb{P}(X = 0)$. We shall use it to prove the following result.

Theorem 5.2.3. *Let $G = G(n, p)$ be a random graph. Then,*

$$\mathbb{P}(G \textit{ contains a triangle}) \to \begin{cases} 0, & \textit{if } p \ll 1/n, \\ 1, & \textit{if } p \gg 1/n \,. \end{cases}$$

Proof. In the first part we need a random variable X and Markov's inequality as follows,

$$X = \text{ the number of triangles in } G.$$

$$\mathbb{P}(G \text{ contains a triangle}) \leqslant \mathbb{E}(X)$$
$$\leqslant \binom{n}{3} p^3$$
$$\ll 1,$$

if $p \ll 1/n$, finishing the first part of the proof. To prove the second part we need to bound the variance of the random variable.

$$Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$
$$= \mathbb{E}\left(\sum_{(u,v)} \mathbb{1}[u]\mathbb{1}[v]\right) - \left(\sum_u \mathbb{P}(u)\right)^2$$
$$= \sum_{u,v} \left(\mathbb{P}(u \wedge v) - \mathbb{P}(u)\mathbb{P}(v)\right),$$

where v and u denote the events that a triangle occurs and the $\mathbb{1}$ denotes the indicator function, assuming value 1 if the event occurs and 0 otherwise. Note that u and v are independent and appear in both terms as follows

$$
\begin{aligned}
Var(X) &= \sum_{u,v} \left(\mathbb{P}(u \wedge v) - \mathbb{P}(u)\mathbb{P}(v) \right) \\
&\leqslant \sum_{|u \cap v|=2} \mathbb{P}(u \wedge v) + \sum_{|u \cap v|=3} \mathbb{P}(u \wedge v) \\
&\leqslant n^4 p^5 + n^3 p^3,
\end{aligned}
$$

and using Chebyshev's inequality,

$$
\mathbb{P}(G \text{ contains no triangles}) \leqslant \frac{Var(X)}{\mathbb{E}(X)^2} \leqslant \frac{n^4 p^5 + n^3 p^3}{(n^6 p^6/12^2)} \ll 1,
$$

for $p \gg 1/n$, finishing the proof. $\qquad\square$

5.3 The Janson and FKG Inequalities

In this section we focus on the occurrence of bad events, asking questions as the form "Could we count the number of bad events?" or "What is the probability that none of the bad events occurs?". The Janson and FKG inequalities give us tools to answer this type of questions.

5.3.1 The FKG Inequality

In this subsection we shall use FKG inequality to acquire an upper bound for Theorem 5.3.5, but first we introduce some concepts and present the FKG inequality.

Definition 5.3.1. *Monotone Increasing. Let $\mathcal{A}$ be a collection of graphs on n vertices, $\mathcal{A}$ is monotone increasing if for every $G \in \mathcal{A}$ and any H such that $E(G) \subseteq E(H)$ and $V(G) = E(G)$ we have $H \in \mathcal{A}$. Then if $G \in \mathcal{A}$ this means that adding edges does not make G loose the characteristics that makes it be in $\mathcal{A}$.*

Definition 5.3.2. *Monotone Decreasing. Let $\mathcal{A}$ be a collection of graphs on n vertices, $\mathcal{A}$ is monotone decreasing if for every $G \in \mathcal{A}$ and any H such that $E(H) \subseteq E(G)$ and $V(G) = E(G)$ we have $H \in \mathcal{A}$. Then if $G \in \mathcal{A}$ this means that removing edges does not make G loose the characteristics that makes it be in $\mathcal{A}$.*

Example 5.3.3. "G contains a triangle" is monotone increasing because it is impossible to destroy a triangle by adding edges.

Theorem 5.3.4 (FKG Inequality [7])**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be monotone increasing and let $\mathcal{C}$ be monotone decreasing, then*

$$\mathbb{P}((G(n,p) \in \mathcal{A}) \wedge (G(n,p) \in \mathcal{B})) \geqslant \mathbb{P}(G(n,p) \in \mathcal{A})\mathbb{P}(G(n,p) \in \mathcal{B}),$$

and

$$\mathbb{P}((G(n,p) \in \mathcal{A}) \wedge (G(n,p) \in \mathcal{C})) \leqslant \mathbb{P}(G(n,p) \in \mathcal{A})\mathbb{P}(G(n,p) \in \mathcal{C}).$$

Now we can prove the following theorem.

Theorem 5.3.5. *Let c be a constant and $p = \frac{c}{n}$. Then*

$$\limsup_{n \to \infty} \mathbb{P}(G(n,p) \text{ contains a triangle }) = 1 - e^{\frac{-c^3}{6}}.$$

Proof. We start by using FKG inequality.

Let X be a triple in $[n]$ and B_X be the event that this triple forms a triangle in $G_{n,p}$. Note that the events $\overline{B_X}$ are monotone decreasing and have probability $(1-p)^3$.

$$
\begin{aligned}
\mathbb{P}(G(n,p) \text{ has no triangles }) = \mathbb{P}&\left(\bigwedge_{X \in \binom{[n]}{3}} \overline{B_X} \right) \\
&\geqslant \prod_{X \in \binom{[n]}{3}} \mathbb{P}(\overline{B_X}) \\
&\geqslant (1-p^3)^{\binom{n}{3}} \\
&\to e^{-\mathbb{E}[X]}, \text{as } n \to \infty \\
&= e^{-c^3/6},
\end{aligned}
$$

finishing the upper bound proof. Thus we have a set of $\binom{n}{3}$ events B_X such that $B_X \subseteq E(G(n,p)) \subseteq \binom{n}{2}$. Note that

$$\mu = \sum_X \mathbb{P}(B_X) = p^3 \binom{n}{3},$$

and we define

$$\Delta = \sum_{X \sim Y} \mathbb{P}(B_X \wedge B_Y) = O(n^4 p^5) \ll 1.$$

Then using the Janson's inequality

$$\mathbb{P}\left(\bigwedge_{j=1}^{m} \overline{B_j} \right) \leqslant e^{-\mu + \Delta/2},$$

then

$$\mathbb{P}(G(n,p) \text{ has no triangles }) \leqslant e^{-\mu + \Delta/2} \to e^{-c^3/6}$$

as $n \to \infty$, finishing the proof. $\qquad \qquad \square$

5.3.2 Janson's Inequality

In this subsection we shall use Janson's inequalities to prove the bound given by the FKG inequality is sharp under certain circumstances.

Let $A_1, \ldots, A_m$ be subsets of $[N]$ and Y be a set in $[N]$ whose elements are chosen independently, with $\mathbb{P}(j \in Y) = p$ for each j and $N = \binom{n}{2}$.

$$B_j = \{A_j \colon A_j \subseteq Y\},$$

note that the events are monotone increasing and that B_i and B_j are independent if and only if $A_i \cap A_j \neq \varnothing$.

Define $\mu = \sum_j \mathbb{P}(B_j)$ denoting the expected number of bad events and let

$$\Delta = \sum_{\substack{i \sim j \\ (i,j) \in [N]^2}} \mathbb{P}(B_i \wedge B_j),$$

where the sum is over ordered pairs (i, j) such that A_i and A_j intersect. Follows the Janson inequalities.

Theorem 5.3.6 (Janson's Inequality [8]). *Let $A_1, \ldots, A_m \subseteq [n]$ and let $B_1, \ldots, B_m$ be the events defined above. Then*

$$\mathbb{P}\left(\bigwedge_{j=1}^m \overline{B_j}\right) \leqslant e^{-\mu + \Delta/2}.$$

And if $\Delta \geqslant \mu$, then

$$\mathbb{P}\left(\bigwedge_{j=1}^m \overline{B_j}\right) \leqslant e^{-\mu^2/2\Delta}.$$

Note that from FKG inequality we have,

$$\mathbb{P}\left(\bigwedge_{j=1}^m \overline{B_j}\right) \geqslant \prod_j (1 - \mathbb{P}(B_j)) \approx e^{-\mu},$$

if $\mathbb{P}(B_j) \ll 1$ for each j, so the Janson inequalities are close to being sharp. We shall prove the first inequality.

Proof. We claim that, for each $j \in [m]$,

$$\mathbb{P}\left(B_j \Big| \bigwedge_{i=1}^{j-1} \overline{B_i}\right) \geqslant \mathbb{P}(B_j) - \sum_{i \sim j, i < j} \mathbb{P}(B_i \wedge B_j).$$

In order to prove the claim we need to divide the event $\bigwedge_{i=1}^{j-1} \overline{B_i}$ into two parts, E and F. Let E denote the event $\bigwedge_{i \in I} \overline{B_i}$, where

$$I = \{i \in [j-1] : i \sim j\},$$

(i.e., the set of indices i such that $A_i \cap A_j$ is non-empty), and let F denote the event $\bigwedge_{i \in J} \overline{B_i}$, where $J = [j-1] \smallsetminus I$. Note that the event B_j is independent of the event E, and that E and F are both monotone decreasing. By the FKG inequality,

$$
\begin{aligned}
\mathbb{P}(B_j | E \cap F) &\geqslant \mathbb{P}(B_j \wedge E | F) \\
&= \mathbb{P}(B_j | F)\mathbb{P}(E | B_j \wedge F) \\
&\geqslant \mathbb{P}(B_j)\mathbb{P}(E | B_j) \\
&\geqslant \mathbb{P}(B_j) - \mathbb{P}(B_j)\mathbb{P}(\overline{E} | B_j) \\
&\geqslant \mathbb{P}(B_j) - \sum_{i \sim j, i < j} \mathbb{P}(B_j)\mathbb{P}(B_i | B_j).
\end{aligned}
$$

Since the claim is proved. Note that, from $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B | A)$ we have,

$$
\begin{aligned}
\mathbb{P}\left(\bigwedge_{j=1}^{m} \overline{B_j}\right) &= \prod_{j=1}^{m} \mathbb{P}\left(\overline{B_j} \mid \bigwedge_{i=1}^{j-1} \overline{B_i}\right) \\
&= \prod_{j=1}^{m} \left(1 - \mathbb{P}\left(B_j \mid \bigwedge_{i=1}^{j-1} \overline{B_i}\right)\right) \\
&\leqslant \exp\left(-\sum_{j=1}^{m} \mathbb{P}(B_j \mid \bigwedge_{i=1}^{j-1} \overline{B_i})\right).
\end{aligned}
$$

Using the claim we just proved,

$$
\begin{aligned}
\mathbb{P}\left(\bigwedge_{j=1}^{m} \overline{B_j}\right) &\leqslant \exp\left(-\sum_{j=1}^{m} \mathbb{P}(B_j) + \sum_{i \sim j, i < j} \mathbb{P}(B_i \wedge B_j)\right) \\
&= \exp(-\mu + \Delta/2).
\end{aligned}
$$

Then we finish the proof of the first inequality. Now we proceed with the proof of the second inequality, with $\Delta \geqslant \mu$.

$$\mathbb{P}(\bigwedge_{j=1}^{m} \overline{B_j}) \leqslant \mathbb{P}(\bigwedge_{j \in T} \overline{B_j}),$$

for all T in $\binom{n}{2}$ with T chosen randomly. We define $\mathbb{P}(j \in T) = q$ for all j independently, then we have,

$$\mathbb{P}(\bigwedge_{j \in T} \overline{B_j}) \leqslant e^{-\mu(T) + \Delta(T)/2},$$

with

$$\mu(T) = \sum_{j \in T} \mathbb{P}(B_j),$$

and

$$\mathbb{E}[\mu(T)] = \sum_{j=1}^{m} \mathbb{P}(j \in T)\mathbb{P}(B_j)$$

$$= q\mu.$$

And we have

$$\mathbb{E}[\Delta(T)] = q^2 \Delta.$$

Then,

$$\mathbb{E}[\mu(T) - \Delta(T)/2] = q\mu - q^2\Delta/2,$$

which by the fact $\mathbb{E}[X] \geqslant a \Rightarrow \mathbb{P}(X \geqslant a) > 0$,

$$\Rightarrow \mathbb{P}(\mu(T) - \Delta(T)/2 \geqslant q\mu - q^2\Delta/2) > 0$$

$$\Rightarrow \exists T \subseteq \binom{n}{2} \text{ such that } \mu(T) - \Delta(T)/2 \geqslant q\mu - q^2\Delta/2.$$

Now maximizing $q = \mu/\Delta \leqslant 1$ we have

$$\mu(T) - \Delta(T)/2 \geqslant \frac{\mu^2}{\Delta} - \frac{\mu^2}{2\Delta}$$

$$\Rightarrow -\mu(T) + \frac{\Delta(T)}{2} \leqslant -\frac{\mu^2}{2\Delta}.$$

Then we finish the proof.

$\square$

5.4   Connectedness

In this section we will answer the question "For which p is $G(n, p)$ likely to be connected?" Observe this property is monotone increasing, because it is not possible to disconnect a graph by adding edges. The following theorem gives a threshold for this property.

Theorem 5.4.1 ([14])**.** *For every $\varepsilon > 0$,*

$$\mathbb{P}(G(n,p) \text{ is connected}) \rightarrow \begin{cases} 0, & \text{if } p \leqslant \frac{(1-\varepsilon)\ln n}{n}, \\ 1, & \text{if } p \geqslant \frac{(1+\varepsilon)\ln n}{n}, \end{cases}$$

as $n \rightarrow \infty$.

Proof. We start proving what happens when $p \leqslant \frac{(1-\varepsilon)\ln n}{n}$. In order to have a connected $G \in G(n,p)$ we need to avoid isolated vertices. Let $X = X(G)$ denote the number of isolated vertices in G and let $p \leqslant ((1-\varepsilon)\ln n)/n$. Then using the fact $1 - x > e^{-x-x^2}$,

$$\mathbb{E}[X(G)] = n(1-p)^{n-1} > ne^{-pn-p^2n} \gg n^{\varepsilon/2},$$

for sufficiently large n. Now we can bound the variance of $X(G)$ as below,

$$
\begin{aligned}
Var(X) &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\
&= \sum_{x,y} \mathbb{P}(d_x = d_y = 0) - \left(\sum_x \mathbb{P}(d_x = 0) \right)^2 \\
&= 2\binom{n}{2}(1-p)^{2n-3} + n(1-p)^{n-1} - n^2(1-p)^{2n-2} \\
&= (n^2 - n)(1-p)^{2n-3} + n(1-p)^{n-1} - n^2(1-p)^{2n-2} \\
&= n^2(1-p)^{2n-3}\left(\frac{n-1}{n} - (1-p) \right) + n(1-p)^{n-1} \\
&= n^2(1-p)^{2n-3}\left(-\frac{1}{n} + p \right) + n(1-p)^{n-1} \\
&= pn^2(1-p)^{2n-3} + n(1-p)^{n-1} - n(1-p)^{2n-3} \\
&\ll n^2(1-p)^{2n-2} \\
&= \mathbb{E}[X]^2.
\end{aligned}
$$

Then we just showed $Var(X) \ll \mathbb{E}[X]^2$ and now we can use Chebyshev's inequality,

$$\mathbb{P}(X = 0) \leqslant \mathbb{P}(X - \mathbb{E}(X) \geqslant \mathbb{E}(X)) \leqslant \frac{Var(X)}{\mathbb{E}(X)^2} \to 0.$$

We conclude that $\mathbb{P}(G \text{ is connected}) \to 0$ as $n \to \infty$ finishing the prove of the first part.

To prove the second part, we define the random variable $Y_k(G)$, which counts the number of components of size k in G. Bounding the expected number of $Y_k(G)$ we have

$$
\begin{aligned}
\mathbb{E}[Y_k(G)] &\leqslant \sum_{\substack{S \subseteq V(G) \\ |S|=k}} \mathbb{P}(S \text{ is a component}) \\
&\leqslant \binom{n}{k}(1-p)^{k(n-k)} \\
&\leqslant \left(\frac{en}{k} \right)^k e^{-pk(n-k)} \\
&= \left(\frac{en}{k} e^{-p(n-k)} \right)^k.
\end{aligned}
$$

Note that if G is not connected, there must exist a component of size at most $n/2$ then $1 \leqslant k \leqslant n/2$. Since the function $\left(\frac{en}{k}e^{-p(n-k)}\right)$ is convex, the highest value of it occurs in one of its extremal points thus we have

$$\mathbb{E}[Y_k(G)] \leqslant (\max\{ene^{-p(n-1)}, 2ee^{-pn/2}\})^k.$$

Note that for $ene^{-p(n-1)}$ and $p \geqslant \frac{(1+\varepsilon)\ln n}{n}$ we have $\mathbb{E}[Y_k(G)] \to 0$ and for $2ee^{-pn/2}$ and $p \geqslant 4/n$ we have $\mathbb{E}[Y_k(G)] \to 0$.

Using the maximum one we have $p \geqslant \frac{(1+\varepsilon)\ln n}{n}$ and by Markov's inequality

$$\mathbb{P}(Y_k \geqslant 1) \leqslant \mathbb{E}[Y_k] \to 0,$$

which implies $\mathbb{P}(\text{G is connected}) \to 1$, finishing the proof.

$\square$

Chapter 6

Regularity

In this chapter we present the Szemerédi's Regularity Lemma, which is a really powerful tool in Combinatorics. It gives us a method to "approximate" any arbitrary graph to a random graph, which means any arbitrary graph can share useful properties with random graphs.

Example 6.0.1. What is the minimum number of edges that ensure a graph G contains a triangle?

For arbitrary graphs,

$$e(G) > \frac{n^2}{4} \Rightarrow G \text{ has a triangle.}$$

For random graphs,

$$e(G) \gg n \Rightarrow G \text{ has a triangle w.h.p.}$$

As we can see random graphs give us different type of bounds based on probability and regularity lemma enables us to apply random graph techniques to any graph.

6.1 The Regularity Lemma

Informally, the lemma states that if you partition a given graph enough then the partition pairs look random-like. Before showing the formal form we need to define ε-regular pairs and to show the properties that make it looks random-like.

Definition 6.1.1. *Given a graph G and disjoint sets A and B of vertices, we say that (A, B) is ε-regular if for every $X \subseteq A$ and every $Y \subseteq B$ with $|X| \geqslant \varepsilon|A|$ and $|Y| \geqslant \varepsilon|B|$ we have*

$$\left| \frac{e(X, Y)}{|X||Y|} - \frac{e(A, B)}{|A||B|} \right| \leqslant \varepsilon.$$

A property shared between random graphs and ε-regular pairs is that most vertices have roughly the same number of neighbours.

Fact 6.1.2. *Let (A, B) be an ε-regular pair of density at least δ in a graph G. Then for at most $2\varepsilon|A|$ vertices v of A we have*

$$||N_G(v) \cap B| - \delta|B|| > \varepsilon|B|$$

Proof. Let $X = \{v \in A : |N_G(v) \cap B| < (\delta - \varepsilon)|B|\}$ and let $Y = B$. Then note that the density of the pair (X, Y) is at most $\delta - \varepsilon$, which implies $|X| < \varepsilon|A|$, otherwise it would contradict the definition of ε-regular pairs. By symmetry, the same holds for $Z = \{v \in V(A) : |N_G(v) \cap B| > (\delta + \varepsilon)|B|\}$. Then we have at most $2\varepsilon|A|$ vertices of A that do not have roughly the same number of neighbours. $\square$

Now we prove the embedding lemma that makes the regularity useful to solve problems. It states that for the purpose of finding small subgraphs in a graph G, dense ε-regular pairs can be treated as complete bipartite graphs.

Lemma 6.1.3. *(The Embedding Lemma - simple version). Let H and G be graphs and $\delta > \varepsilon > 0$, then there exists $M \in \mathbb{N}$ such that if $m \geqslant M$ and there exists a partition of G in $\{V_1, \ldots, V_{V(H)}\}$ with $|V_i| = \lceil m \rceil$ or $|V_i| = \lfloor m \rfloor$ for $1 \leqslant i \leqslant V(H)$ and all pairs (V_i, V_j) being ε-regular and δ-dense, then $H \subseteq G$.*

Proof. Here we only give the proof for $H = K_3$. Let A, B and C be ε-regular and at least δ-dense pairs of G, with $|A| = |B| = |C| = m$. Note that if we do not consider the sets of vertices $\{w \in A : d_B(w) < (\delta - \varepsilon)|B|\}$ and $\{u \in A : d_C(u) < (\delta - \varepsilon)|C|\}$ then we have at least $(1 - 2\varepsilon)|A|$ vertices in A such that $|N(v) \cap B| \geqslant \varepsilon|B|$ and $|N(v) \cap C| \geqslant \varepsilon|C|$ by Fact 6.1.2. Then let $X = N(v) \cap B$ and $Y = N(v) \cap C$ so there must be an edge between X and Y (note that any edge between X and Y forms a triangle). To be more precise we have $|X| \geqslant (\delta - \varepsilon)|B|$ and $|Y| \geqslant (\delta - \varepsilon)|C|$, as follows

$$\begin{aligned}
e(X, Y) &\geqslant (\delta - \varepsilon)|X||Y| \\
&\geqslant (\delta - \varepsilon)(\delta - \varepsilon)|B|(\delta - \varepsilon)|C| \\
&= (\delta - \varepsilon)^3 m^2.
\end{aligned}$$

Then we have at least $(\delta - \varepsilon)^3 m^2$ triangles for each v. Therefore, we have at least $(1 - 2\varepsilon)(\delta - \varepsilon)^3 m^3$ triangles in G.
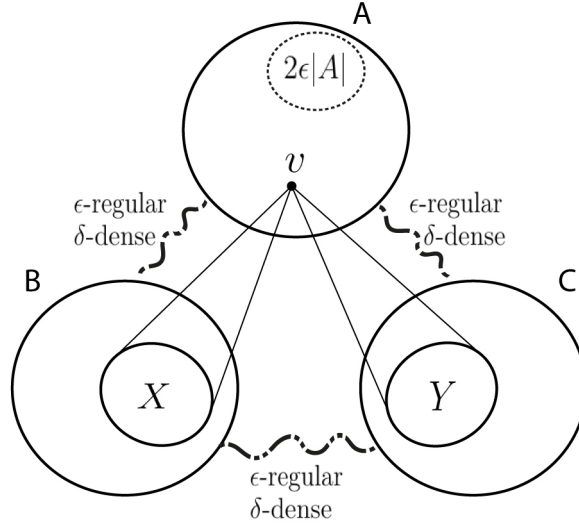
Figure 6.1: Triangles in the ε-regular pairs.

$\square$

Theorem 6.1.4 (The Szemerédi Regularity Lemma [18]). *For every $\varepsilon > 0$ there exists an integer $M = M(\varepsilon)$ such that the following holds: for any graph G, there exists a partition $V(G) = \{V_1 \cup \ldots \cup V_k\}$ of the vertex set into k classes, such that*

- $k \leqslant M$,

- $||V_i| - |V_j|| \leqslant 1$ *for all i, j,*

- $|V_i| \leqslant \lceil \varepsilon |V(G)| \rceil$ *for every i,*

- *all but εk^2 of the pairs (V_i, V_j) are ε-regular.*

Note the lemma holds for all graphs, but it is useful only for graphs with large n and with a minimum density or it becomes trivial because can be approximated by the empty graph.

6.2 Applications

Applying the Regularity Lemma, in general, makes its use easier to understand than the definitions or the proof of the lemma, then in this section we will show some classical applications. In order to apply the lemma we use the following strategy.

1. Apply the Szemerédi Regularity Lemma in order to get a partition $\{A_1, \ldots, A_k\}$ as defined in the lemma.

2. Remove a few edges (edges inside the parts, between irregular pairs and sparse pairs). There exist at most $O(\varepsilon n^2)$ such edges.

3. Define a graph R such that $V(R) = [k]$ and $\{i, j\} \in E(R)$ if the pair (A_i, A_j) is δ-dense and ε-regular.

4. Apply some result (e.g. Turán's Theorem) to R.

5. Apply the Embedding Lemma to find a copy of the desired subgraph H in G.

6.2.1   Erdős-Stone Theorem

Theorem 6.2.1 (Erdős and Stone [4]). *Let H be an arbitrary graph. Then*

$$\mathrm{ex}(n, H) = \left(1 - \frac{1}{\chi(H) - 1} + o(1)\right) \binom{n}{2}.$$

Proof. Let $\varepsilon = 1/k$ and $\delta \geqslant \varepsilon$. Let G be a H-free graph and assume $e(G) \geqslant (1 - 1/(\chi(H) - 1) + \delta)\binom{n}{2}$ and follow the steps above.

1. Apply the Regularity Lemma 6.1.4 with ε defined previously then we assume there exists M such that $k \leqslant M$ and we have a partition $\{V_1, \ldots, V_k\}$ as stated in the theorem.

2. Remove edges inside the parts, between irregular pairs and sparse pairs obtaining G' as in Figure 6.2. Note that
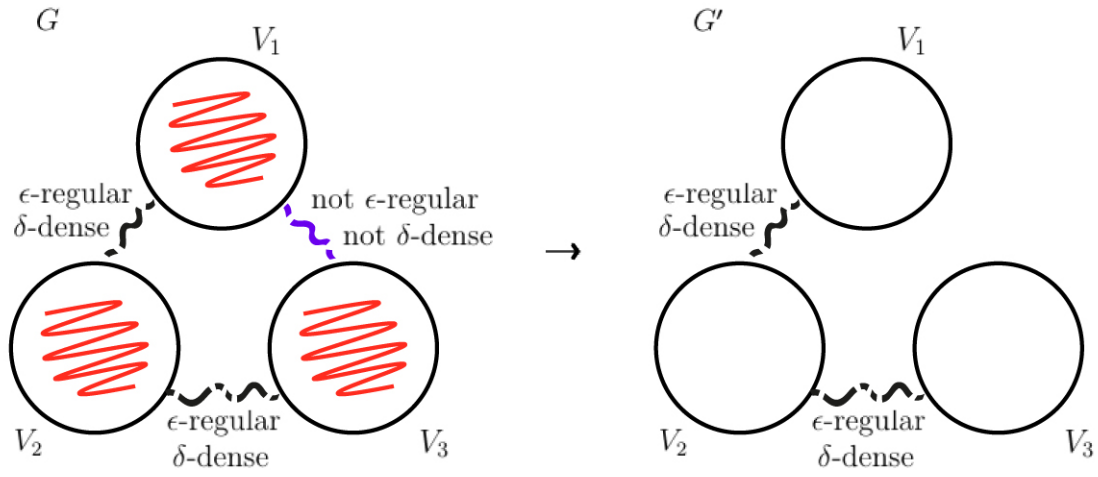
|edges inside the parts| $\leqslant k\binom{n/k}{2} \leqslant k(n/k)^2 \leqslant \varepsilon n^2$ (recall $1/\varepsilon = k$).

|edges between not ε-regular pairs| $\leqslant \varepsilon k^2 (n/k)^2 = \varepsilon n^2$.

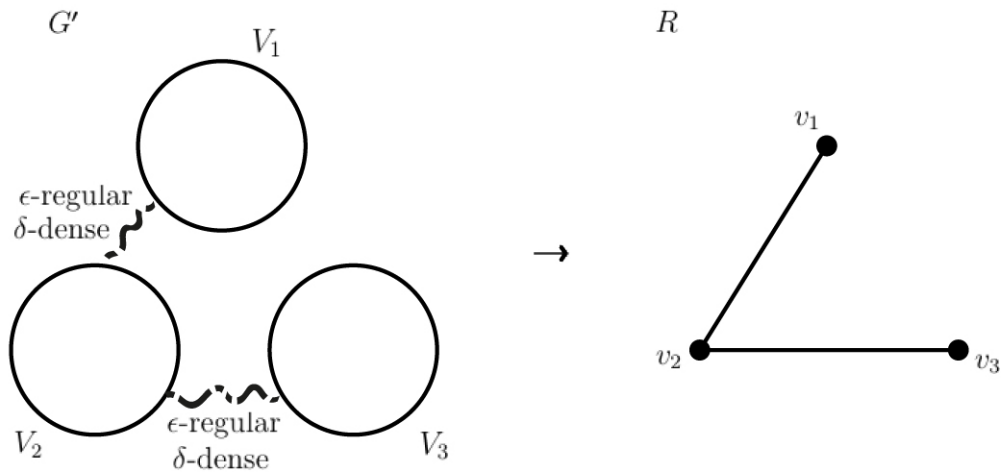|edges between not δ-dense pairs| $\leqslant \binom{k}{2}(\delta/4)(n/k)^2 = (\delta/4)n^2$ (sparse pairs with density at most $\delta/4$).

Thus we removed d edges such that $d \leqslant (2\varepsilon + \delta/4)n^2$ and let c be a constant such that $\delta\binom{n}{2} - d \geqslant \delta/c\binom{n}{2}$, therefore

$$e(G') \geqslant \left(1 - \frac{1}{\chi(H) - 1} + \frac{\delta}{c}\right) \binom{n}{2}.$$

Figure 6.2: Graphs G and G'.

3. Define R with $V(R) = [k]$ and $\{i, j\} \in E(R)$ if the pair (V_i, V_j) is δ-dense and ε-regular as in Figure 6.3

Figure 6.3: Graphs G' and R.

Note that

$$e(R) \geqslant \frac{e(G')}{\left(\frac{n}{k}\right)^2}.$$

4. Since we have

$$e(R) \geqslant \frac{\left(1 - \frac{1}{\chi(H)-1} + \frac{\delta}{c}\right)\binom{n}{2}}{\left(\frac{n}{k}\right)^2} > \left(1 - \frac{1}{\chi(H)-1} + \frac{\delta}{c}\right)\binom{k}{2} > t_r(k),$$

we can apply the Turán's Theorem 4.1.4 that implies $K_{\chi(H)} \subseteq R$.

5. Now applying the Embedding Lemma 6.1.3 with n large enough follows that $H \subseteq G$ since $K_{\chi(H)} \subseteq R$, as required.

$\square$

6.2.2   The Triangle Removal Lemma

Theorem 6.2.2. [*Triangle Removal Lemma* [16]] *For all $\alpha > 0$ exists $\beta > 0$ such that if G is a graph with at most βn^3 triangles, then it is possible to remove all triangles by removing at most αn^2 edges.*

Proof. The first, second and third steps of the method are, in general, the same for classical problems and we need $\varepsilon = 1/k$ and $\delta > \varepsilon$.

1. Apply SzRL, then we have the partition $\{V_1, \ldots, V_k\}$.

2. Remove edges inside the parts, between irregular pairs and sparse pairs obtaining G' with at most αn^2 edges removed.

3. Define R with $V(R) = [k]$ and $\{i, j\} \in E(R)$ if the pair (A_i, A_j) is δ-dense and ε-regular.

Now we have two cases, in the first one we have a triangle in R. Note if $\beta n^3 < 1$ the result is trivial, then we assume $\beta \geqslant 1/n^3$ and the parts that forms the triangle (assume $\{V_1, V_2, V_3\}$ for simplicity) has size $n/k \geqslant 1/(\beta^{1/3}k) \geqslant m$.

Applying the Embedding Lemma 6.1.3 we have that the quantity of triangles in G' is at least $\delta^3/2(n/k)^3 > \beta n^3$ if we choose β such that $\delta^3/(2k^3) > \beta$.

We conclude that if αn^2 edges are removed and the graph still has triangles the triangles quantity is more than βn^3.

In the second case there is no triangle in R and this implies there is no triangle in G' because the edges inside the pairs $\{V_1, \ldots, V_k\}$ were removed then the only possible triangles are formed between the pairs, finishing the proof. $\square$

6.2.3   Roth's Theorem

Theorem 6.2.3 (Roth's Theorem [15])**.** *For some $\delta > 0$, there exists some n_0 such that with $n > n_0$ and if $A \subseteq [n]$, with $|A| \geqslant \delta n$, then A contains an arithmetic progression*

of length three.

Proof. Let $|A \cap [n]| \geqslant \delta n$, with n sufficiently large. Let G be a graph such that $V(G) = X \cup Y \cup Z$, where $X, Y, Z = [3n]$ and

$$\begin{aligned} E(G) &= \{\{x, y\} \colon x \in X, y \in Y, \text{ and } y = x + a, a \in A\} \\ &\cup \{\{y, z\} \colon y \in Y, z \in Z, \text{ and } z = y + b, b \in A\} \\ &\cup \{\{x, z\} \colon x \in X, z \in Z, \text{ and } z = x + 2c, c \in A\}, \end{aligned}$$

then for a triangle $\{x, y, z\}$ in G we have $2c = a + b$ in A and if $a \neq b \neq c$ this triangle corresponds to a 3-AP in A. Note that if A has no 3-AP, then all the triangles in G correspond to triples $\{a, a, a\}$ and there exists $n|A| \geqslant \delta n^2$ such triangles. Suppose we have no 3-AP in A.

Then we can apply the Triangle Removal Lemma 6.2.2 to G for n sufficiently large $\delta n^2 \leqslant \beta n^3$, which means we can destroy all triangles in G by removing at most αn^2 edges. Let $\delta > \alpha$, since triangles $\{a, a, a\}$ are edge-disjoint and we have more than αn^2 such triangles which is a contradiction, finishing the proof. $\qquad \square$

Chapter 7

Conclusion

Essential classical results and methods in Extremal Combinatorics were presented in this project, although it is not a complete cover of literature it gives us a good background of this fruitful area of study and makes easier to expand knowledge and results throughout the whole area of Extremal Combinatorics. We focus our efforts in the methods and results we consider most important and useful for who is interested in continuing studying the area in postgraduate courses and contributing for the area expansion in the future.

We use as base text and initial point of study the Robert Morris and Roberto Imbuziero book [12] which cover more topics of study and is much more simplified and summarized. Completing the gaps and trying to make a self-contained text with detailed explanations for initial students of the area was a great effort by itself.

Bibliography

[1] P. Erdős, *Graph theory and probability*, Canad. J. Math. **11** (1959), 34–38. MR0102081 ↑5.1.5

[2] P. Erdős, *On the graph theorem of Turán*, Mat. Lapok **21** (1970), 249–251 (1971). MR0307975 ↑4.1.5

[3] P. Erdös, *Some remarks on the theory of graphs*, Bull. Amer. Math. Soc. **53** (1947), 292–294. MR0019911 (8,479d) ↑3.2.2

[4] P. Erdös and A. H. Stone, *On the structure of linear graphs*, Bull. Amer. Math. Soc. **52** (1946), 1087–1091. MR0018807 (8,333b) ↑6.2.1

[5] P. Erdös and G. Szekeres, *A combinatorial problem in geometry*, Compositio Math. **2** (1935), 463–470. MR1556929 ↑3.2.2

[6] P. Erdös, *On the Density of Some Sequences of Numbers: III*, J. London Math. Soc. **13** (1938), no. 2, 119–127. MR1574138 ↑4.2.2

[7] C. M. Fortuin, P. W. Kasteleyn, and J. Ginibre, *Correlation inequalities on some partially ordered sets*, Comm. Math. Phys. **22** (1971), 89–103. MR0309498 ↑5.3.4

[8] S. Janson, *Poisson convergence and Poisson processes with applications to random graphs*, Stochastic Process. Appl. **26** (1987), no. 1, 1–30. MR917244 ↑5.3.6

[9] J. L. W. V. Jensen, *Sur les fonctions convexes et les inégalités entre les valeurs moyennes*, Acta Math. **30** (1906), no. 1, 175–193. MR1555027 ↑4.2.1

[10] T. Kövari, V. T. Sós, and P. Turán, *On a problem of K. Zarankiewicz*, Colloquium Math. **3** (1954), 50–57. MR0065617 ↑4.2.3

[11] W. Mantel, *Problem 28*, Wiskundige Opgaven **10** (1907), 60–61. ↑4.1.1

[12] R. Morris and R. I. Oliveira, *Extremal and probabilistic combinatorics*, IMPA, 2011. ↑7

[13] F. P. Ramsey, *On a Problem of Formal Logic*, Proc. London Math. Soc. (2) **30** (1929), no. 4, 264–286. MR1576401 ↑3.1.2

[14] P E.-A Rényi, *On the evolution of random graphs*, Publ. Math. Inst. Hung. Acad. Sci. A **5** (1960), 17–61. ↑5.4.1

[15] K. F. Roth, *On certain sets of integers*, J. London Math. Soc. **28** (1953), 104–109. MR0051853 ↑6.2.3

[16] I. Z. Ruzsa and E. Szemerédi, *Triple systems with no six points carrying three triangles*, Coll. Math. Soc. J. Bolyai 18 (1976), 939–945. ↑6.2.2

[17] I. Schur, *Über die Kongruenz* $x^m + y^m \equiv z^m \mod \varrho$, Jber. Deutsch. Math.-Verein **25** (1916), 114–117. ↑3.3.1

[18] E. Szemerédi, *On sets of integers containing no k elements in arithmetic progression*, Acta Arith. **27** (1975), 199–245. Collection of articles in memory of Juriĭ Vladimirovič Linnik. MR0369312 ↑6.1.4

[19] P. Tchebichef, *Des valeurs moyennes*, Mathématiques Pures et Appliquées **12** (1867), 177–184. ↑5.2.2

[20] P. Turán, *On an extremal problem in graph theory*, Matematikai és Fizikai Lapok **48** (1941), 436–452. ↑4.1.4

[21] B. L. van der Waerden, *Der Multiplizitätsbegriff der algebraischen Geometrie*, Math. Ann. **97** (1927), no. 1, 756–774. MR1512387 ↑3.4.1