Exponential Distribution

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Chapter 2

Exponential Distribution

2.1 Introduction

In establishing a probability model for a real-world phenomenon, it is always necessary to make certain simplifying assumptions to render the mathematics tractable. On the other hand, however, we cannot make too many simplifying assumptions, for then our conclusions, obtained from the probability model, would not be applicable to the real-world phenomenon. Thus, we must make enough simplifying assumptions to enable us to handle the mathematics but not so many that the model no longer resembles the real-world phenomenon.

The reliability of instruments and systems can be measured by the survival probabilities such as $P[X > x]$. During the early 1950's, Epstein and Sobel, and Davis analyzed statistical data of the operating time of an instrument up to failure, and found that in many cases the lifetime has exponential distribution. Consequently, the exponential distribution became the underlying life distribution in research on reliability and life expectancy in the 1950's. Although further research revealed that for a number of problems in reliability theory the exponential distribution is inappropriate for modeling the life expectancy, however, it can be useful to get a first approximation (see the reference by Barlow and Proschan (1975)). Exponential distributions are also used in measuring the length of telephone calls and the time between successive impulses in the spinal cords of various mammals.

This chapter is devoted to the study of exponential distribution, its properties and characterizations, and models which lead to it and illustrate its applications.

2.2 Exponential Distribution

A continuous nonnegative random variable X ($X \geq 0$) is called to have an exponential distribution with parameter λ, $\lambda > 0$, if its probability density function is given by

$$f(x) = \lambda e^{-\lambda x}, \quad \text{for} \quad x \geq 0,$$

or equivalently, if its distribution function is given by

$$F(x) = \int_{-\infty}^{x} f(t)dt = 1 - e^{-\lambda x}, \quad \text{for} \quad x \geq 0.$$

If follows that the survival function $\bar{F}(x)$ is given by

$$\bar{F}(x) = 1 - F(x) = e^{-\lambda x}, \quad \text{for} \quad x \geq 0.$$

We list several properties of exponential distribution:

Property 1.

$$E(X) = \frac{1}{\lambda}, \quad Var(X) = \frac{1}{\lambda^2},$$

and the moment generating function is given by

$$M(t) = E[e^{tX}] = \frac{\lambda}{\lambda - t},$$

for $t < \lambda$.

In fact,

$$\begin{aligned} M(t) &= E[e^{tX}] = \int_0^\infty e^{tx}\lambda e^{-\lambda x}dx \\ &= \frac{\lambda}{\lambda - t}\int_0^\infty (\lambda - t)e^{-(\lambda - t)x}dx \\ &= \frac{\lambda}{\lambda - t}, \end{aligned}$$

since the last integral is an exponential density with parameter $\lambda - t$. The first two moments of X can be calculated as

$$E[X] = \frac{d}{dt}M(t)|_{t=0}$$

$$
\begin{aligned}
&= \frac{\lambda}{(\lambda - t)^2}|_{t=0} \\
&= \frac{1}{\lambda}, \\
E[X^2] &= \frac{d^2}{dt^2} M(t)|_{t=0} \\
&= \frac{2\lambda}{(\lambda - t)^3}|_{t=0} \\
&= \frac{2}{\lambda^2},
\end{aligned}
$$

and thus

$$
\begin{aligned}
Var(X) &= E[X^2] - (E[X])^2 \\
&= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} \\
&= \frac{1}{\lambda^2}.
\end{aligned}
$$

Property 2. *The failure rate function $r(t) = \lambda$, i.e. a constant.*

In fact,

$$
\begin{aligned}
r(t) &= \frac{f(t)}{\bar{F}(t)} \\
&= \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda.
\end{aligned}
$$

Property 3. (Lack of Memory) *The residual lifetime $(X - t)|X > t$ has the same distribution as X. That means, for all $x, t \geq 0$,*

$$P[X > x + t|X > t] = P[X > x].$$

This is clear by observing that

$$
\begin{aligned}
P[X > x + t|X > t] &= \frac{\bar{F}(x + t)}{\bar{F}(t)} \\
&= \frac{e^{-\lambda(x+t)}}{e^{-\lambda t}} \\
&= e^{-\lambda x} = P[X > x].
\end{aligned}
$$

In reliability terms, if we think X as the lifetime of some instrument, then the memoryless property implies that if the instrument is working at time t, then the distribution of the residual lifetime is the same as the original lifetime distribution.

In particular, the mean residual lifetime $\mu(t) = E[X - t|X > t]$ is the same as the original mean $\mu = E[X]$:

$$\begin{aligned}\mu(t) &= E[X - t|X > t] \\ &= E[X] = \mu = \frac{1}{\lambda}.\end{aligned}$$

Property 4.(Extreme Value) *Suppose $X_1, ..., X_n$ are independent following the same distribution as X. Then $n\min(X_1, ..., X_n)$ also has the same exponential distribution.*

In fact,

$$\begin{aligned}P[n\min(X_1, ..., X_n) > x] &= P[\min(X_1, ..., X_n) > \frac{x}{n}] \\ &= P[X_1 > \frac{x}{n}]...P[X_n > \frac{x}{n}] \\ &= (P[X > \frac{x}{n}])^n \\ &= (e^{-\lambda x/n})^n = e^{-\lambda x}.\end{aligned}$$

Example 2.1 Suppose the lifetime of certain items follows exponential distribution with parameter $\lambda = 0.1$.

(i) From Property 1, $\mu = \frac{1}{0.1} = 10$; $Var(X) = \frac{1}{0.1^2} = 100$.

(ii)The density function is

$$f(x) = 0.1e^{-0.1x},$$

and the survival function is

$$\bar{F}(x) = e^{-0.1x}.$$

Therefore as stated in Property 2, the failure rate is equal to

$$r(t) = \frac{f(t)}{\bar{F}(t)} = 0.1.$$

(iii) Suppose one item has survived 10 units of time, then from Property 3, the conditional survival function given $X > 10$ is

$$P[X > x + 10|X > 10] = P[X > x] = e^{-0.1x}.$$

In particular, the mean residual life $\mu(t) = \mu = 10$.

(iv) (Serial system) Consider a serial system consists of five identical items which are working independently. The system fails as long as one item fails. Then the lifetime of the system is $X = \min(X_1, X_2, ...X_5)$, where X_i denotes the lifetime of i-th item. From Property 4, as X follows exponential distribution with parameter 0.5, then the mean system lifetime will be $\frac{\mu}{0.5} = 2$.

Example 2.2 (Location-transformed exponential distribution) Suppose the time headway X between consecutive cars in highway during a period of heavy traffic follows the location-transformed exponential density function

$$f(x) = 0.15e^{-0.15(x-0.5)}, \quad \text{for} \ \ x \geq 0.5.$$

(i) $X - 0.5$ is a regular exponential random variable with parameter $\lambda = 0.15$.

(ii) $E[X] = 0.5 + \frac{1}{0.15}$ and $Var(X) = \frac{1}{0.15^2}$.

(iii) The survival function can be calculated as

$$\bar{F}(t) = P[X > t] = P[X - 0.5 > t - 0.5] = e^{-0.15(t-0.5)}, \ \ \text{for} \ \ t \geq 0.5.$$

(iv) The memoryless property still holds with minor modification. For $t \geq 0.5$ and $x > 0$,

$$P[X > t + x|X > t] = P[X - 0.5 > t - 0.5 + x|X - 0.5 > t - 0.5] = e^{-0.15x}.$$

(v) However, the mean residual life is still constant. For $t \geq 0.5$,

$$E[X - t|X > t] = \frac{1}{0.15}.$$

2.3 Characterization of Exponential Distribution

It turns out that Properties 2 to 4 can all be used to characterize exponential distribution in the sense that if a distribution possesses one of these properties, it must be exponential.

2.3.1 Memoryless Property

Theorem 2.1 *Let X be a non-degenerate lifetime random variable and its distribution has the memoryless property, i.e.*

$$P[X > x + t | X > t] = P[X > x], \quad \text{for} \quad x, t \geq 0.$$

Then X has an exponential distribution.

Proof. The memoryless property implies that

$$\frac{\bar{F}(x+t)}{\bar{F}(t)} = \bar{F}(x), \quad \text{for} \quad x, t \geq 0,$$

equivalently,

$$\bar{F}(x+t) = \bar{F}(x)\bar{F}(t), \quad \text{for} \quad x, t \geq 0.$$

Following the lines of the proof for Theorem 1.2, this implies that for any rational number $r = m/n$,

$$\bar{F}(r) = [\bar{F}(1)]^r = e^{-\lambda r},$$

where $\lambda = -\ln \bar{F}(1)$. Since X is not degenerate, $\bar{F}(1) > 0$. (Otherwise, $\bar{F}(x) = 0$ for all $x > 0$, a contradiction.) For any irrational x, let $\{r_n\}$ be a sequence of rational numbers such that $x < r_n$ and $\lim_{n\to\infty} r_n = x$. Since $\bar{F}(x)$ is right continuous, we have

$$\begin{aligned} \bar{F}(x) &= \lim_{n\to\infty} \bar{F}(r_n) = \lim_{n\to\infty} e^{-\lambda r_n} \\ &= e^{\lambda \lim r_n} = e^{-\lambda x}. \qquad \diamond \end{aligned}$$

Remark. A more delicate analysis shows that we only have to check the memoryless property at two points t_1 and t_2 such that t_1/t_2 is irrational and

$$P[X > x + t_i | X > t_i] = P[X > x], \quad \text{for} \quad x \geq 0, \ i = 1, 2.$$

In fact, from

$$\bar{F}(x + t_i) = \bar{F}(x)\bar{F}(t_i), \quad \text{for} \quad i = 1, 2,$$

we see that

$$\bar{F}(x + mt_1 + nt_2) = \bar{F}(x)\bar{F}(mt_1 + nt_2), \quad \text{for all} \quad m, n \in \mathbf{Z}.$$

Without loss of generality, we assume $t_1 < t_2$. Then we can find a sequence $\{x_i\}$ for $i = 1, 2, \cdots$, such that

$$\begin{aligned} t_2 &= n_0 t_1 + x_1 \\ t_1 &= n_1 x_1 + x_2 \\ x_i &= n_{i+1} x_{i+1} + x_{i+2}, \quad \text{for} \quad i = 1, 2, \cdots . \end{aligned}$$

Note that each x_i is of the form $mt_1 + nt_2$ by looking at the equations backward, and more important,

$$x_1 > x_2 > \cdots \to 0.$$

Thus, for any given $\epsilon > 0$, there exists a k such that $0 < x_k < \epsilon$. This implies that for any $x > 0$, there exists a positive integer j such that

$$(j-1)x_k < x \leq jx_k.$$

Let $z_j = jx_k$ and z_j is of the form $mt_1 + nt_2$. We see that

$$0 \leq z_j - x \leq x_k < \epsilon.$$

Hence, we can find a monotone decreasing sequence

$$z_1 > z_2 > \cdots \to x.$$

By the right continuity of $\bar{F}(x)$, we have

$$\bar{F}(x+y) = \bar{F}(x)\bar{F}(y) \quad \text{for all} \quad x, y \geq 0.$$

That means, $\bar{F}(x)$ is exponential.

Corollary 2.1 *If X is a non-degenerate lifetime random variable such that X has constant mean residual lifetime, i.e.*

$$\begin{aligned} \mu(t) &= E[X - t | X > t] \\ &= \frac{1}{\bar{F}(t)} \int_t^\infty \bar{F}(x)dx = \mu, \quad \text{for} \quad \bar{F}(t) > 0, \end{aligned}$$

then X is exponential.

Proof. The constant mean residual lifetime implies

$$\begin{aligned}\frac{1}{\mu} &= \frac{\bar{F}(t)}{\int_t^\infty \bar{F}(x)dx} \\ &= -\frac{d}{dt}\ln\int_t^\infty \bar{F}(x)dx, \quad \text{for} \quad \bar{F}(t) > 0.\end{aligned}$$

Thus,

$$\ln\int_t^\infty \bar{F}(x)dx = C - \frac{t}{\mu}.$$

By taking $t = 0$, we get $C = \ln\mu$. Thus,

$$\int_t^\infty \bar{F}(x)dx = \mu e^{-\frac{t}{\mu}},$$

and hence

$$\bar{F}(t) = e^{-\frac{t}{\mu}}. \qquad \diamond$$

2.3.2 Constant Failure Rate Function

Theorem 2.2 *Let X be a nonnegative random variable with probability density function $f(x)$. If X has a constant failure rate function $r(t) = \lambda$, then X is exponentially distributed.*

This is simply observed from the fact that

$$\bar{F}(x) = e^{-\int_0^x r(t)dt} = e^{-\lambda t}.$$

2.3.3 Extreme Value Distribution

Let $\{X_1, ..., X_n\}$ be independent random variables distributed as X, which may represent the lifetimes of n similar and independent components. Denote by $X_{k,n}$ as the k-th order statistics, which represents the k-th failure time from the n components.

Theorem 2.3 *Let X be a non-degenerate nonnegative random variable. If $nX_{1,n}$ has the same distribution as X for every positive integer n, then X has exponential distribution.*

Proof.

$$\begin{aligned} P[nX_{1,n} > x] &= P[X_{1,n} > \frac{x}{n}] \\ &= P[\min(X_1, ..., X_n) > \frac{x}{n}] \\ &= P[X_1 > \frac{x}{n}, ..., X_n > \frac{x}{n}] \\ &= (P[X > \frac{x}{n}])^n. \end{aligned}$$

If $nX_{1,n}$ has the same distribution as X, then

$$\bar{F}(\frac{x}{n}) = (\bar{F}(x))^{1/n}, \quad \text{for} \quad n = 1, 2, ...$$

By a similar argument as in Theorem 2.1, $\bar{F}(x) = e^{-\lambda x}$ for some $\lambda > 0$. ⋄

Remark 1. Similar to the remark after Theorem 2.1, we can show that if

$$\bar{F}(x/n) = (\bar{F}(x))^{1/n}$$

holds for n_1 and n_2 such that $\frac{\ln n_1}{\ln n_2}$ is irrational, then X is exponential. That means, we only need two samples.

In fact, from

$$\bar{F}(x/n_i) = (\bar{F}(x))^{1/n_i}$$

for $i = 1, 2$, we have

$$\bar{F}(x) = [\bar{F}(n_1^j n_2^k x)]^{1/n_1^j n_2^k},$$

for all integers $j, k \in \mathbf{Z}$. This implies that

$$\bar{F}(n_1^j n_2^k) = [\bar{F}(1)]^{n_1^j n_2^k}.$$

For any $x > 0$, we have $\ln x = y \in \mathbf{R}$. Just like in the remark following Theorem 2.1, since $\frac{\ln n_1}{\ln n_2}$ is irrational, we can find a sequence of real numbers $y_m > y$ which converge to y and all y_m are of the form $j \ln n_1 + k \ln n_2$. By the right continuity of $\bar{F}(x)$, we have

$$\bar{F}(x) = [\bar{F}(1)]^x.$$

Remark 2. Under some extra condition for $F(x)$ as $x \to 0$, we only need one sample to characterize the exponential distribution. The following is a typical result.

Theorem 2.4 *Suppose for some n, $nX_{1,n}$ has the same distribution as X. If*

$$0 < \lim_{x\to 0^+} \frac{F(x)}{x} = \lambda < \infty,$$

then $F(x) = 1 - e^{-\lambda x}$, for $x \geq 0$.

Proof. For this sample size n, we have $\bar{F}(x) = (\bar{F}(x/n))^n$. Inductively, this implies that for any integer k,

$$\bar{F}(x) = (\bar{F}(x/n^k))^{n^k}, \quad \text{for} \quad x \geq 0.$$

That means

$$\begin{aligned} F(x) &= 1 - \bar{F}(x) \\ &= 1 - (\bar{F}(x/n^k))^{n^k} \\ &= 1 - (1 - F(x/n^k))^{n^k}. \end{aligned}$$

Since

$$\lim_{x\to 0^+} \frac{F(x)}{x} = \lambda > 0,$$

and

$$F(x/n^k) = \lambda x/n^k + o(1/n^k),$$

as $k \to \infty$. Consequently,

$$\begin{aligned} F(x) &= 1 - \lim_{k\to\infty} (1 - \lambda \frac{x}{n^k})^{n^k} \\ &= 1 - e^{-\lambda x}. \qquad \diamond \end{aligned}$$

Remark. The probability interpretation of the above inductive proof is that we can first take n independent and identically distributed samples of size n (total n^2 random variables), say, $\{X_1^{(i)}, ..., X_n^{(i)}\}$, as copies of $\{X_1, ..., X_n\}$ for $i = 1, 2, ..., n$. This will give a sequence of independent variables, say, $nX_{1,n}^{(i)}$ for $i = 1, 2, ..., n$ with the same distribution as $nX_{1,n}$. This process can repeatedly be carried to the total sample size n^3, n^4, ...

2.4 Order Statistics and Exponential Distribution

In life testing, when all n items are tested starting from the same time, the lifetimes $X_1, ..., X_n$ are recorded as the order statistics $X_{1,n}, ..., X_{n,n}$. We call

$$d_{r,n} = X_{r+1,n} - X_{r,n}$$

the spacing statistics for $r = 0, 1, ..., n-1$, and $D_{r,n} = (n-r)d_{r,n}$ the normalized spacing statistics. Can any of the spacing statistics characterize the exponential distribution?

2.4.1 Some Properties of Order Statistics

To find the distribution function $F_{k,n}(y)$ of $X_{k,n}$, we note that the event $\{X_{k,n} \leq y\}$ is equivalent to the event that there are at least k of X_i's $\leq y$. Note that given that there are exactly j failures at y for $j \geq k$ (thus $n-j$ survivals at y), there are total $\begin{pmatrix} n \\ j \end{pmatrix}$ combinations of j out of n. Thus

$$\begin{aligned} F_{k,n}(y) &= P[X_{k,n} \leq y] = \sum_{j=k}^{n} P[\text{Exactly j failures before y}] \\ &= \sum_{j=k}^{n} \begin{pmatrix} n \\ j \end{pmatrix} F^j(y)[1-F(y)]^{n-j} \\ &= k \begin{pmatrix} n \\ k \end{pmatrix} \int_0^{F(y)} t^{k-1}(1-t)^{n-k} dt, \end{aligned}$$

where the last equation is obtained by integrating by part consecutively for $j \geq k$.

If $F(x)$ is absolutely continuous with density function $f(x)$, then $F_{k,n}(y)$ has density

$$\begin{aligned} f_{k,n}(y) &= k \begin{pmatrix} n \\ k \end{pmatrix} F^{k-1}(y)[1-F(y)]^{n-k} f(y) \\ &= n! f(y) \frac{F^{k-1}(y)}{(k-1)!} \frac{[1-F(y)]^{n-k}}{(n-k)!}. \end{aligned}$$

This form can also be explained as follows. There are total $n!$ permutations from the n failures. One fails at y, $k-1$ fail before y, and $n-k$ are survival at

y. Thus, there are total $\frac{n!}{1!(k-1)!(n-k)!}$ combinations of observations and each occurs with the same probability $f(y)[F(y)]^{k-1}[1-F(y)]^{n-k}$.

More generally, by using the similar explanation, the joint density of

$$\{X_{r_1,n}, X_{r_2,n}, ..., X_{r_k,n}\}$$

for $1 \leq r_1 < r_2 < ... < r_k \leq n$ with $1 \leq k \leq n$ is given by

$$f_{(r_1,...,r_k)}(y_1, y_2, ..., y_k) = n! \left[\Pi_{i=1}^k f(y_i)\right] \Pi_{i=0}^k \frac{[F(y_{i+1}) - F(y_i)]^{r_{i+1}-r_i-1}}{(r_{i+1} - r_i - 1)!},$$

where

$$y_0 = 0, y_{k+1} = \infty, r_0 = 0, r_{k+1} = n+1, \text{ and } y_1 \leq y_2 \leq ... \leq y_k.$$

In particular, the density function of the spacing statistic $d_{r,n}$ is

$$f_{d_{r,n}}(x) = \frac{n!}{(r-1)!(n-r-1)!} \int_{-\infty}^{\infty} F^{r-1}(y)[1-F(x+y)]^{n-r-1} f(y) f(x+y) dy.$$

If $F(x)$ is exponential, we have the following explicit result:

Theorem 2.5 *Let $d_{r,n}$ be the r-th spacing statistic from the exponential distribution $F(x) = 1 - e^{-\lambda x}$. Then*

(1) $F_{d_{r,n}}(x) = P[d_{r,n} \leq x] = 1 - e^{-(n-r)\lambda x}$*;*

(2) $E(d_{r,n}) = \frac{1}{(n-r)\lambda}, \quad Var(d_{r,n}) = \frac{1}{(n-r)^2\lambda^2}, \quad r = 1, 2, ..., n-1$*;*

(3) $d_{1,n}, d_{2,n}, ..., d_{n-1,n}$ *are mutually independent.*

Proof. (1) The joint density function of $X_{r+1,n}$ and $X_{r,n}$ is

$$\begin{aligned} f_{r,r+1}(s,t) &= n! f(s) f(t) \frac{(F(s))^{r-1}}{(r-1)!} \frac{(1-F(t))^{n-r-1}}{(n-r-1)!} \\ &= n! \lambda e^{-\lambda s} \lambda e^{-\lambda t} \frac{(1-e^{-\lambda s})^{r-1}}{(r-1)!} \frac{e^{-(n-r-1)\lambda t}}{(n-r-1)!}, \quad 0 < s < t. \end{aligned}$$

From the transformation

$$(X_{r,n}, X_{r+1,n}) \longrightarrow (X_{r,n}, d_{r,n}),$$

we obtain the joint density function of $X_{r,n}, d_{r,n}$, with $s = x, t = x + y$, and $|J| = 1$ (Jacobian is unit), as

$$f_{X_{r,n},d_{r,n}}(x, y) = n!\lambda e^{-\lambda x}\lambda e^{-\lambda(x+y)}\frac{(1 - e^{-\lambda x})^{r-1}}{(r-1)!}\frac{e^{-(n-r-1)\lambda(x+y)}}{(n-r-1)!},$$

for $0 < x, y < \infty$. Since the form is separable in x and y, the marginal density function for $d_{r,n}$ is

$$f_{d_{r,n}}(y) \propto e^{-(n-r)\lambda y},$$

where $\propto$ means equality except for a constant. Hence

$$F_{d_{r,n}}(x) = 1 - e^{-(n-r)\lambda x}.$$

(2) follows from (1).

For (3), we note that the joint density function of $\{X_{1,n}, ..., X_{n,n}\}$ is

$$\begin{aligned} f_{(X_{1,n},...,X_{n,n})}(x_1, ..., x_n) &= n!\,[\Pi_{i=1}^n f(x_i)]] \\ &= n!\Pi_{i=1}^n \lambda e^{-\lambda x_i}. \end{aligned}$$

Consider the transformation

$$(X_{1,n}, ..., X_{n,n}) \longrightarrow (X_{1,n}, d_{1,n}, ..., d_{n-1,n})$$

with Jacobian $|J| = 1$. The joint density function of $(X_{1,n}, d_2, ..., d_n)$ is

$$\begin{aligned} &f_{(X_{1,n},d_{1,n},...,d_{n-1,n})}(y_1, y_2, ..., y_n) \\ &= n!\lambda^n exp[-\lambda(y_1 + (y_1 + y_2) + ... + (y_1 + ... + y_n))] \\ &= n!\lambda^n exp[-\lambda(ny_1 + (n-1)y_2 + ... + y_n)] \\ &= n!\Pi_{i=1}^n \lambda e^{-(n-i+1)\lambda y_i}. \end{aligned}$$

It follows that $X_{1,n}$ and $d_{1,n}, ..., d_{n-1,n}$ are mutually independent. $\diamond$

Corollary 2.2 *For the exponential distribution, the normalized spacings $D_{r,n} = (n-r)d_{r,n}$, for $r = 1, 2, ..., n-1$, are identically and independently distributed with common distribution function $F(x) = 1 - e^{-\lambda x}$.*

Proof. The independence of $D_{r,n}$ follows from the last theorem. Since the density function of $d_{r,n}$ is $(n-r)\lambda e^{-(n-r)\lambda x}$, the density function of $D_{r,n} = (n-r)d_{r,n}$ is thus $\lambda e^{-\lambda x}$. $\diamond$

Example 2.3 Consider the life testing for n identical components which follow exponential distribution with λ.

(i) $X_{1,n}$ is exponential with mean $\frac{\mu}{n} = \frac{1}{n\lambda}$.

(ii) In general, we can write

$$\begin{aligned} X_{k,n} &= X_{1,n} + X_{2,n} - X_{1,n} + ... + X_{k,n} - X_{k-1,n} \\ &= \frac{1}{n} n X_{1,n} + \frac{1}{n-1}(n-1)(X_{2,n} - X_{1,n}) + ... \\ &\quad + \frac{1}{n-k+1}(n-k+1)(X_{k,n} - X_{k-1,n}), \end{aligned}$$

which is a weighted sum of independent exponential random variables. In particular,

$$E[X_{k,n}] = \left[\frac{1}{n} + \frac{1}{n-1} + ... + \frac{1}{n-k+1}\right]\frac{1}{\lambda},$$

and

$$Var[X_{k,n}] = \left[\frac{1}{n^2} + \frac{1}{(n-1)^2} + ... + \frac{1}{(n-k+1)^2}\right]\frac{1}{\lambda^2}.$$

(iii) The total time on test (TTT) up to the k-th failure $X_{k,n}$ can be written as

$$TTT(X_{k,n}) = X_{1,n} + X_{2,n} + ... + X_{k,n} + (n-k)X_{k,n}$$

$$= nX_{1,n} + (n-1)(X_{2,n} - X_{1,n}) + ... + (n-k+1)(X_{k,n} - X_{k-1,n}),$$

which is a sum of k independent exponential random variable with parameter λ.

(iv) In general, we denote by R the total number of failures before time t, then the total time on test up to time t can be written as

$$TTT(t) = X_{1,n} + X_{2,n} + ... + X_{R,n} + (n-R)t$$

$$= nX_{1,n} + (n-1)(X_{2,n} - X_{1,n}) + ... + (n-R+1)(X_{R,n} - X_{R-1,n}) + (n-R)(t - X_{R,n}).$$

2.4.2 Characterization based on Order Statistics

We first give a characterization of exponential distribution based on the first two spacing statistics.

Theorem 2.6 *Let X_1, X_2 be independently distributed random variables with common continuous distribution $F(x)$. If $X_{1,2}$ and $X_{2,2} - X_{1,2}$ are independent, then $F(x) = 1 - e^{-\lambda x}$.*

Proof. The independence of $X_{1,2}$ and $d_{1,2}$ implies

$$P[X_{2,2} - X_{1,2} > x | X_{1,2} = y] = P[X_{2,2} - X_{1,2} > x],$$

for all $y \geq 0$, which is free of y. On the other hand, the distribution of $X_{2,2}$ given $X_{1,2} = y$ is indeed equivalent to the survival distribution of X given $X > y$. Thus, for $y \geq 0$,

$$\begin{aligned} P[X_{2,2} - X_{1,2} > x | X_{1,2} = y] &= P[X_{2,2} > x + y | X_{1,2} = y] \\ &= P[X > x + y | X > y] = \frac{\bar{F}(x+y)}{\bar{F}(y)}. \end{aligned}$$

This implies

$$\frac{\bar{F}(x+y)}{\bar{F}(y)} = \bar{F}(x), \quad \text{for all} \quad x, y \geq 0.$$

That means $F(x)$ is exponential from the memoryless property. $\diamond$

Remark. If we know that the normalized spacings are all exponential with $F(x) = 1 - e^{-\lambda x}$, does it imply that the population distribution is also exponential without additional assumptions? The answer is no.

Example 2.4 Let X_1, X_2 be independent and identically distributed, with common distribution function

$$F(x) = 1 - e^{-x}[1 + 4b^{-2}(1 - cos(bx))], \quad \text{for} \quad x \geq 0, b \geq 2\sqrt{2}.$$

Then it can be shown that

$$d_{1,2} = X_{2,2} - X_{1,2} = |X_1 - X_2|$$

has exponential distribution.

We further give a result without proof.

Theorem 2.7 *Let the population distribution $F(x)$ be such that $F(0^+) = 0$ and $F(x) > 0$ for all $x > 0$. Assume that*

$$u_r(s) = \int_0^\infty e^{-sx} dF^r(x) \neq 0$$

for all s such that $Re(s) \geq 0$. If for some $r \geq 1$,

$$P[d_{r,n} \leq x] = 1 - e^{-(n-r)x}, \quad \text{for} \quad x \geq 0,$$

then $F(x) = 1 - e^{-x}$ for $x \geq 0$.

2.4.3 Record Values

Highly related to the order statistics are the following record values. Let $\{X_1, ..., X_n, ...\}$ be a sequence of nonnegative random variable which are observed one by one in time-ordered way (longitudinal observations). Let $F(x)$ be their distribution function with density $f(x)$.

Definition 2.1 *X_j is called a record value if $X_j > \max(X_1, ..., X_{j-1})$.*

Denote by $R_1 < R_2 < ...$ the successive record values. The following simple result is due to Tata (1969).

Theorem 2.8 *Let $\{X_1, X_2, ...\}$ be a sequence of identically and independently distributed nonnegative random variables. The distribution function $F(x)$ is exponential if, and only if, $R_1 = X_1$ and $R_2 - R_1$ are independent.*

Proof. Notice that given $R_1 = X_1 = x$, $R_2 - R_1 = R_2 - x$ is equivalent to the residual life $X - x | X > x$. Thus, the conditional distribution of $R_2 - R_1$ given $R_1 = x$ is just

$$P[R_2 - R_1 > y | R_1 = x] = P[X - x > y | X > x] = \frac{\bar{F}(x+y)}{\bar{F}(x)},$$

for $x, y \geq 0$. Thus, R_1 and $R_2 - R_1$ are independent if, and only if, $\bar{F}(x+y)/\bar{F}(x)$ is free of x. That means,

$$\frac{\bar{F}(x+y)}{\bar{F}(x)} = \bar{F}(y),$$

by letting $x = 0$. $\diamond$

2.5 More Applications

Example 2.5(Time of accident)

Insurance companies can collect accident records (history) of drivers. A driver is considered in "good" category if his probability of having an accident (in statistical sense) remains the same small number regardless of the passage of time. Therefore, if X denotes the random time period up to the first accident of the driver in question, then

$$P[X > x + y | X > x] = P[X > y].$$

This is exactly the Memoryless Property. In this case, the insurance company does not need to guess the risk of such a driver, but can go ahead with a well-defined model of setting the amount of premium based on the driver's record.

Example 2.6 Suppose a store serves a community of n persons. These persons visit the store independently of each other, and their actual times of entering the store have the same distribution. Therefore, the n individuals can be associated with n identically and independently distributed random variables, say $\{X_1, ..., X_n\}$. The store owner observes the order statistics $\{X_{1,n}, ..., X_{n,n}\}$ as the successive arrival times at the store. Assume the store owner observes that the spacing statistics $\{X_{1,n}, d_{1,n}, ..., d_{n-1,n}\}$ are also independent. Then from Theorem 2.9, X_j has exponential distribution. This also provides a well defined model and basis for decision on number of employees, availability of items, etc.

Example 2.7 (Geometric Sums) Consider a system with exponential lifetime with parameter λ. Upon a failure, a repair can restore the system like new with probability p and the failure is irreparable with probability $1 - p$.

Denote by N the total number of working periods, which is a geometric random variable (Problem 2.8) with success probability $1 - p$. That means,

$$P[N = k] = (1 - p)p^{k-1},$$

for $k = 1, 2, ...$ Denote by $X_1, ..., X_N$ the corresponding working period lengths, which are independent exponential random variables. Then the total lifetime for the system is

$$Y = \sum_{i=1}^{N} X_i.$$

(i) By conditioning on the values of N, we can calculate the mean of Y as

$$\begin{aligned} E[Y] &= \sum_{k=1}^{\infty} E[\sum_{i=1}^{k} X_i]P[N=k] \\ &= = \sum_{k=1}^{\infty} \frac{k}{\lambda} P[N=k] \\ &= \frac{1}{(1-p)\lambda}. \end{aligned}$$

(ii) Generally, we can calculate the moment generating function of Y as

$$\begin{aligned} E[E^{tY}] &= \sum_{k=1}^{\infty} E[e^{t\sum_{i=1}^{k} X_i}]P[N=k] \\ &= \sum_{k=1}^{\infty} \left(\frac{\lambda}{\lambda-t}\right)^{k} (1-p)p^{k-1} \\ &= \frac{(1-p)\lambda}{\lambda-t} \sum_{k=1}^{\infty} \left(\frac{p\lambda}{\lambda-t}\right)^{k-1} \\ &= \frac{(1-p)\lambda}{(1-p)\lambda-t}. \end{aligned}$$

(iii) By matching the moment generating function, we see that Y is also an exponential random variable with parameter $(1-p)\lambda$. In other words, geometric sum of identically and independently distributed exponential random variables is still exponential.

Example 2.8 (Valuation of defaultable Zero-Coupon Bonds)

(i)Consider a unit of bond of face value \$1.00 with maturity time T. Suppose the interest(yield) rate is constant r. Then at time t, the value of the bond is $e^{-(T-t)r}$ (discounted to time t).

(ii) Suppose the bond's default time X follows an exponential distribution with parameter λ and there is no recovery at default. Then the value of this defaultable bond at time t given it is not defaulted will be

$$e^{-(T-t)r}P[X > T|X > t] = e^{-(T-t)(r+\lambda)}.$$

(iii) Further we consider the partial recovery case by assuming that w proportion of the market value will be recovered at default. Thus, the (discounted) value at time t given no default up to time t can be evaluated as

$$
\begin{aligned}
& e^{-(r+\lambda)(T-t)} + E[e^{-r(X-t)} we^{-r(T-X)} I_{[X<T]}|X > t] \\
&= e^{-(r+\lambda)(T-t)} + we^{-r(T-t)} P[X < T|X > t] \\
&= e^{-(r+\lambda)(T-t)} + we^{-r(T-t)}(1 - e^{-\lambda(T-t)}) \\
&= (1 - w)e^{-(r+\lambda)(T-t)} + we^{-r(T-t)}.
\end{aligned}
$$

Thus, the value at time t is a mixture of values of no recovery and full recovery (no default).

Example 2.9 (Two-Unit Reliability Systems)

(i) (Serial System) Suppose the two units are connected in a serial system and are working independently. That means, as long as one unit fails, the system will fail. Suppose the lifetimes X_1 and X_2 follow the exponential distribution with parameter λ_1 and λ_2 respectively. Then the system has an exponential lifetime $\min(X_1, X_2)$ with parameters $\lambda_1 + \lambda_2$.

(ii) (Parallel System or Warm Standby System) Suppose the system works as long as at least one unit is working. Then the system's lifetime $\max(X_1, X_2)$ follows the distribution

$$F(x) = P[\max(X_1, X_2) \leq x] = (1 - e^{-\lambda_1 x})(1 - e^{-\lambda_2 x}).$$

(iii)(Cold-Redundant System) Suppose the second unit is in cold standby and it will be put in working as long as the first unit fails. Therefore the system lifetime is $X_1 + X_2$ and its distribution function is the convolution of the two exponential distribution.

$$
\begin{aligned}
P[X_1 + X_2 \leq y] &= \int_0^y P[X_2 \leq y - x] dP[X_1 \leq x] \\
&= \int_0^y \lambda_1 e^{-\lambda_1 x}(1 - e^{-\lambda_2(y-x)}) dx \\
&= 1 - e^{-\lambda_1 y} - \frac{\lambda_1}{\lambda_1 - \lambda_2} e^{-\lambda_2 y}(1 - e^{-(\lambda_1 - \lambda_2)y}).
\end{aligned}
$$

In particular, when $\lambda_1 = \lambda_2 = \lambda$,

$$P[X_1 + X_2 \leq y] = 1 - (1 + \lambda y)e^{-\lambda y}.$$

Problems

1. Suppose the lifetime of a certain model of car battery follows an exponential distribution with the mean lifetime of 5 years.
a) Write down the survival function.
b) What is the probability that the lifetime will be over 2 years?
c) What is probability that the battery will work more than 4 years given that it worked at least two years?

2. The time to repair a machine is an exponentially distributed random variable with parameter $\lambda = 1/2$. What is
a) the probability that the repair time exceeds 2 hours;
b) the conditional probability that a repair takes at least 10 hours, given that its duration exceeds 9 hours?

3. Let X and Y be independent $Exp(\lambda)$. Prove that the density function of $Z = X/Y$ is given by

$$h(z) = (1+z)^{-2}, \quad z > 0.$$

4. Let X be a nonnegative random variable such that $P[X > 0] > 0$. Then X is exponential if and only if

$$E[X|X > a] = a + E[X], \quad \text{for all} \quad a \geq 0.$$

5. Let X be an exponential random variable with rate λ.
a) Use the definition of conditional expectation to determine $E[X|X \leq c]$.
b) Now determine $E[X|X \leq c]$ by using the following identity:

$$E[X] = E[X|X \leq c]P[X \leq c] + E[X|X > c]P[X > c].$$

6. Let X_1 and X_2 be independent exponential random variables with same rate λ. Let

$$X_{(1)} = \min(X_1, X_2) \quad \text{and} \quad X_{(2)} = \max(X_1, X_2)$$

be the order statistics. Find
a) the mean and variance of $X_{(1)}$;
b) the mean and variance of $X_{(2)}$.

7. Let $X_1, \cdots, X_n$ be n identically and independently distributed exponential variables with rate λ, and $X_{1,n}, \cdots, X_{n,n}$ be the order statistics.
a) What are the mean and variance of $X_{1,n}$?
b) Using the property of spacing statistics to find $E[X_{k,n}]$ and $Var(X_{k,n})$.

8. Suppose that independent trials, each having a success probability p, $(0 < p < 1)$, are performed until a success occurs. Denote by X the number of trials needed. Then

$$P[X = k] = (1-p)^{k-1}p, \quad k = 1, 2, ...,$$

is called the *Geometric probability distribution*.
(a) Calculate the mean and variance of X.
(b) Calculate $P[X > k]$.
(c) Show that X has the similar memoryless property in the sense that

$$P[X > k + m | X > k] = P[X > m], \quad k, m \geq 1.$$

(d) Calculate the mean and variance of X.

9. Suppose a certain kind of insurance claim follows exponential distribution with rate $\lambda = 0.02$. Assume a deductible $D = 400$. Refer to Problem 11 of Chapter 1.
(a) Find the expected payment $E[X_D]$ from the insurer.
(b) Find the expected payment $E[X_{DF}]$ under the franchise deductible.

10. By conditioning on the value of the first record value R_1, find the distribution function of the second record value R_2.

11. If X is an exponential random variable with parameter $\lambda = 1$, compute the probability density function of the random variable Y defined by $Y = \log X$.

12. If X is uniformly distributed over $(0, 1)$, find the density function of $Y = e^X$.

13. Magnitude M of an earthquake, as measured on the Richter scale, is a random variable. Suppose the excess $M - 3.25$ for large magnitudes bigger than 3.25 is roughly exponential with mean 0.59 (or $\lambda = 1/0.59$).

(i) Find the probability that an earthquake has scale larger than 5?

(ii) Given an earthquake has scale larger than 5, what is the conditional probability that its scale is larger than 7?

14. The duration of pauses (and the duration of vocalizations) that occur in a monologue follows an exponential distribution with mean 0.70 seconds.

(i) What is the variance of the duration of pauses?

(ii) Given that the duration of a pause is longer than 1.0 seconds, what will be the expected total duration time?

15. Suppose the time X between two successive arrivals at the drive-up window of a local bank is an exponential random variable with $\lambda = 0.2$.

(i) What is the expected time between two successive arrivals?

(ii) Find the probability $P[X > 4]$.

(iii) Find the probability $P[2 < X < 6]$.

16. Consider the two similar unit parallel systems. That means the two units follow the same exponential distribution with parameter λ.

(i) What is the distribution function of system's lifetime?

(ii) What is the density function?

(iii) Calculate the failure rate.

17. Consider the two similar unit cold standby systems. That means, the two units have the same exponential distribution with parameter λ.

(i) What is the distribution function of system's lifetime?

(ii) What is the density function of system's lifetime?

(iii) Calculate the failure rate and verify that it is monotone increasing.

18. Suppose X and Y are two independent exponential random variables with parameters λ and δ respectively.

(i) Show that $\min(X, Y)$ is exponential with parameter $\lambda + \delta$.

(ii) Show that the probability $P[X > Y] = \frac{\delta}{\lambda+\delta}$.

(iii)Show by the memoryless property that given $X > Y$, Y and $X - Y$ are independent. Thus,

$$E[Y|X > Y] = E[\min(X, Y)].$$

(iv) By extending (iii), show that $\min(X, Y)$ and $|X - Y|$ are independent.

19. Suppose a bank branch has two tellers and their service times for each customer are exponential with parameters 0.2 and 0.25 respectively. When a customer arrives at the branch, he finds that both tellers are serving and no customers are waiting.

(i) What is the distribution of his waiting time?

(ii) What is the probability that he will be served by the first teller?

20. In deciding upon the appropriate premium to charge, insurance companies sometimes use the exponential principle, defined as follows. If X denotes the amount of claims which the company has to pay. Then the premium charged by the insurance company will be

$$P = \frac{1}{a} \ln E[e^{aX}],$$

where a is some specified positive constant. Suppose X is exponential with parameter λ and let $a = \alpha\lambda$ for some $0 < \alpha < 1$. Calculate the value P.

21. Refer to Problem 1.16. Suppose $F(x)$ is exponential with parameter λ.

(i) Find the Risk at Value at level 99%;

(ii) Find the corresponding expected total loss

$$T_l = E[L|L \geq l].$$

22. Let Y be an exponential random variable and X_1 and X_2 be two independent positive random variables. Show by the memoryless property of Y

$$P[Y > X_1 + X_2] = P[Y > X_1]P[Y > X_2].$$

23. Let $X_1, \cdots, X_n, \cdots,$ be a sequence of nonnegative identically and independently distributed random variables. Define N as the first time the sequence stops decreasing, i.e.

$$N = \inf\{n \geq 2 : X_1 \geq X_2 \geq \cdots \geq X_{n-1} \leq X_n\}.$$

(i) Show that for $n \geq 2$,

$$P[N \geq n] = P[X_1 \geq X_2 \geq \cdots \geq X_{n-1}] = \frac{1}{(n-1)!}.$$

(ii) Find $E[N]$.