Failure data and system reliability analysis

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Abstract: Discussions are given for the meanings of risk and safety assessment. Handling of data is important in the quest to perform probabilistic safety assessments (PSAs). Distribution functions of failure data and related statistical quantities are explained in details. Some topics regarding reliability analysis are also introduced in this lecture note. They include maintenance activity, matrix expressions of maintained systems, Boolean expression of a system reliability and analysis method for loop structures. Finally, dependent failure and common cause failure are discussed, paying much attention to the models of common cause failure (CCF).

Keyword: risk; safety assessment; failure data; probability distribution function; common cause failure

1 Introduction

In this lecture note, explanations are given for failure data, some topics of system reliability analysis and common cause failure.

"Risk" is a diametrical point of view of "safety" in a simplified aspect. This lecture note will start with the consideration of "risk", and then give the explanations of safety assessment, especially of probabilistic safety assessment (PSA).

Handling of data is extremely important for performing PSAs. Characteristics of data, distribution functions of failure data and related statistical quantities are explained in details.

Some topics regarding reliability analysis are introduced in this lecture note. They include maintenance activity, Boolean expression of a system reliability and analysis method for loop structures.

Finally, dependent failure and common cause failure are discussed. Parametric models for common cause failure are explained and some reference values are given. Dependent failure and common cause failure are important issues in reliability analysis because the system reliability is largely reduced by them in order to establish high reliability by adopting redundancy to comprise the system.

2 Risk and safety assessment

"Risk" is usually defined as the combination of probability of occurrence of harm and the severity of that harm. "Harm" is defined as "physical injury" or "damage to the health of people" or "damage to property or the environment", and harm is the consequence of a hazard occurring. "Hazard" is defined as a potential source of harm.

Here, "combination" does not simply mean the "product". There is a tendency for many people to dislike large severity of harm even if the expected occurrence frequency is subtle. It is termed as risk aversion. As such, the definition of risk does not simply use the "product" for the "combination".

Let us consider the risks of traffic accidents. Now take a person's death for the "harm", and product for the "combination". Risks are expressed by the number of persons’ deaths per year. The following values are obtained from Japanese traffic accident records.

Total traffic accidents: $7.9\text{x}10^{-5}$/y・p

If we consider a risk by "per action", air flight risk is expressed as $1\text{x}10^{-6}$/1 air flight・p.

If we consider a risk by "per benefit", the following risks are estimated for three different transportation methods, for transporting 1000 km distance.

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Automobile	8.4×10^{-6}/1000 km・p
Airplane	1.4×10^{-6}/1000 km・p
Railway	1.2×10^{-8}/1000 km・p

Sometimes, acceptable risk is decided by the cost required for further reduction of risks. In this case, we have to evaluate the values of human lives. This is another difficult question.

Safety assessment is an interdisciplinary approach that focuses on the scientific understanding of hazards as well as harm, and ultimately the risks associated with them. There are two different kinds of approach for safety assessment: one is a deterministic and the other is a probabilistic approach.

The deterministic analytical procedure attempts to ensure that various situations and particular accidents have been taken into account, and that engineered safety and safeguard systems will be capable to prevent fatal accidents.

In the analysis, it is confirmed that items of equipment are designed with adequate safety margins and constructed in such a way that under normal operating conditions the risk of accidents occurring in the plant is kept to be minimum.

Some incidents are assumed to occur through potential equipment failures and human errors. Therefore, verification that provisions are made to detect such incidents and designing safety systems will restore the plant to a normal state and maintain it under safe conditions.

The probabilistic approach is based on the idea that there is no perfect artificial system, and even multiple safety systems happen to reach simultaneous failures. Component failures, human errors, environmental conditions are considered as stochastic phenomena, and undesired system states are evaluated by their occurrence probability.

The first assessment carried out in the United States was the Reactor Safety Study (RSS: Rasmussen report) published in 1975[1]. In the RSS, the event tree (ET) method has been used for identifying possible scenarios to produce accidents (sequences). Failure probabilities of safety or safeguard systems have been evaluated by the fault tree (FT) analysis. The RSS quantitatively estimated the occurrence frequencies of accident sequences by the combination of ET and FT. The total core damage frequency and risks to surrounding people were evaluated by summing up accident scenarios.

After the RSS, many analysis methods [2] in addition to ET and FT have been proposed for more realistic and sophisticated analyses to be performed easily.

3 Data for reliability analysis

3.1 Failure data

To perform a quantitative analysis, events which are subjected to reliability analysis, such as failure or repair, must be collected as data. These events are stochastic phenomena, and therefore we have to make a statistical analysis for collected data.

3.1.1 Characteristics of data

Collected data is a sampling of parent population, and sampling method becomes important. Random and homogeneous data is required. Collected data should not contain unexpected alien data, such as error data produced in recording or measuring. It also should not contain arbitrary data, which are selected by certain or hidden standard.

The characteristic of parent population is estimated from the collected data. If this estimation process can be done effectively, we can accurately estimate the parent population from a few data. In this case, we usually assume a distribution for the parent population, for example by using probability paper, in which horizontal axis is spaced evenly and vertical axis is ruled according to a scale that allows a plotted probability curve to appear linear.

The simplest index of data is the center of the data. The first is "mean" which is defined in the following equation.

$$\bar{x} = \sum_{i=1}^{n} \frac{x_i}{n} \tag{1}$$

The second is "median". Variables x_i are arranged in a decreasing order sequence of values and numbered as *x(1),x(2),...x(n)*. The "median" is defined as:

$$x([0.5n]+1) \text{ or } \frac{x([0.5n])+x([0.5n]+1)}{2} \quad (2),$$

where, "[ ]" is a function which gives integer numbers not larger than the variable inside the parenthesis.

There is also an index "mode". Mode is the value that occurs most frequently in a set of data, and is generally different from the mean and median.

The mode of a discrete probability distribution is the value *x* that is most likely to be sampled. The mode of a continuous probability distribution is the value *x* at which its probability density function attains its maximum value. The mode is not necessarily unique, since the same maximum frequency can be attained at different values. For the uniform distributions, all values are equally likely.

Spread of collected data is expressed by many indexes. Maximum and minimum values and range of data simply show the spread. Also, 1/4, 2/4(=median), 3/4 quartiles are important indexes. Dispersion or sample variance is defined by the equation below:

$$s^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1} \quad (3)$$

In the denominator, (n-1) is used instead of n. This is known as Bessel's correction, which is required because the data is not a parent population but rather a sampling data.

The square root of variance is known as (sample) standard deviation.

$$S = \sqrt{s^2} \quad (4)$$

Normalized standard deviation is defined as follows,

$$cv = S/\bar{x}. \quad (5)$$

If there are statistical dependencies between two variables, the correlation "r" is defined as:

$$r = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2 \sum_{i=1}^{n}(y_i - \bar{y})^2}} \quad (6)$$

The value of "r" ranges from -1 to 1. The positive or negative correlation depends on the positive or negative value of "r". If r=1, the correlation is perfect. If r=0, then x_i and y_i are scattered randomly and there is no correlation between the variables x_i and y_i.

3.2 Estimation of reliability

Reliability at time t ($R(t)$) is the probability that a component is normally and continuously running from initial time to a certain time t. Fraction of a component in failed state before or at time t is expressed by $F(t)$, and the following relation exists between $R(t)$ and $F(t)$:

$$R(t) = 1.0 - F(t) \quad (7)$$

If we observe a set of n components and find out r components are in failed state at time t, then the reliability $R(t)$ of these components is estimated as follows. There are three different methods:

Cumulative rank method: $R(t) = 1.0 - r/n$ (8)

Mean rank method: $R(t) = 1.0 - r/(n+1)$ (9)

Median rank method: $R(t) = 1.0 - (r-0.3)/(n+0.4)$ (10)

Cumulative rank method is a simple approximation method. Mean rank method gives good agreement for mean value between sample and parent populations. Median rank method estimates good median value for the sample population. If n becomes large value, the reliability values estimated by the three methods give almost the same value.

Hazard function ($\lambda(t)$) is defined as a failure probability during unit time interval, and is also referred to as failure rate.

$$\lambda(t) = -\frac{1}{R(t)}\frac{dR(t)}{dt} = \frac{1}{1-F(t)}\frac{dF(t)}{dt} \quad (11)$$

Summation of hazard function from time 0 to a certain time t is cumulative hazard function ($H(t)$).

$$H(t) = \int_0^t \lambda(x)dx \quad (12)$$

If we observe a group of component (number = m), and find out the time t_1, t_2,...t_n in which component failure occurs, these observed values are rearranged in a increasing order sequence; *t(1), t(2), ... t(n)*. Then, cumulative hazard function *H(t(k))* becomes,

$$H(t(k)) = \sum_{i=1}^{k} \frac{1}{m+1-i} \tag{13}$$

The relation existing between *R(t)* and *H(t)* is as follows:

$$R(t) = \exp\{-H(t)\} \tag{14}$$

The definition above is based on the cumulative hazard, and gives more correct value of the parent population. It gives lower values than those estimated by mean and median rank methods, but higher than the ones obtained by cumulative rank method.

(Question 1) Verify the relation (14).

3.3 Distribution of failure data

If we know the distribution of parent population, the characteristics of sampling data is well estimated. In the following subsections, probability distribution functions that are widely used in reliability analyses are explained.

3.3.1 Exponential distribution

Consider a case where failure rate λ is constant with time:

$$\frac{dR(t)}{dt} = -\lambda R(t) \tag{15}$$

With the initial condition *R(t=0)=1.0*, reliability is expressed as follows:

$$R(t) = \exp\{-\lambda t\} \tag{16}$$

Since this equation bears an exponential form, it is termed as an exponential distribution. Probability distribution function (pdf: *f(t)*) is defined as the differential (derivative) of *R(t)*.

$$f(t)dt = -\frac{dR(t)}{dt}dt = \lambda \exp\{-\lambda t\}dt \tag{17}$$

Cumulative distribution function becomes,

$$F(t) = \int_0^t f(x)dx = \int_0^t \lambda \exp\{-\lambda x\}dx = 1-\exp(-\lambda t) \tag{18}$$

Let us now derive the expressions for mean time to failure and variance of failure time. Generally, mean is defined as,

$$\bar{x} = \int_0^\infty x f(x)dx \tag{19}$$

where, *f(x)* is probability distribution function. Then,

$$\begin{aligned} \bar{t} &= \int_0^\infty t f(t)dt = \int_0^\infty t\lambda e^{-\lambda t}dt \\ &= \left[-\frac{t\lambda}{\lambda}e^{-\lambda t}\right]_0^\infty + \int_0^\infty e^{-\lambda t}dt = \left[-\frac{1}{\lambda}e^{-\lambda t}\right]_0^\infty = \frac{1}{\lambda} \end{aligned} \tag{20}$$

Also, variance is generally calculated as follows.

$$\begin{aligned} \sigma^2 &= \int (x-\bar{x})^2 f(x)dx = \int (x^2 - 2x\bar{x} + \bar{x}^2) f(x)dx \\ &= \int x^2 f(x)dx - \int 2x\bar{x} f(x)dx + \int \bar{x}^2 f(x)dx \\ &= \int x^2 f(x)dx - 2\bar{x}\int x f(x)dx + \bar{x}^2 \int f(x)dx \\ &= \overline{x^2} - 2\bar{x}^2 + \bar{x}^2 = \overline{x^2} - \bar{x}^2 \end{aligned} \tag{21}$$

Calculate the value of $\overline{t^2}$.

$$\begin{aligned} \overline{t^2} &= \int_0^\infty t^2 \lambda e^{-\lambda t}dt = \left[-\frac{t^2\lambda}{\lambda}e^{-\lambda t}\right]_0^\infty + \int_0^\infty 2te^{-\lambda t}dt \\ &= \left[-\frac{2t}{\lambda}e^{-\lambda t}\right]_0^\infty + \int_0^\infty \frac{2}{\lambda}e^{-\lambda t}dt = \left[-\frac{2}{\lambda^2}e^{-\lambda t}\right]_0^\infty = \frac{2}{\lambda^2}. \end{aligned} \tag{22}$$

Then, the variance in exponential distribution becomes,

$$\sigma^2 = \overline{t^2} - \bar{t}^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}. \tag{23}$$

Figure 1 shows the shape of exponential distribution with *λ=0.001/min and dt =10min.*

3.3.2 Weibull distribution

Weibull distribution is one of the most widely used lifetime distributions in reliability engineering. It is named after Waloddi Weibull, who described it in details in 1951 [3]. It was first identified by Fréchet (1927) [4], but first applied by Rosin & Rammler (1933) [5] to describe the size distribution of particles.

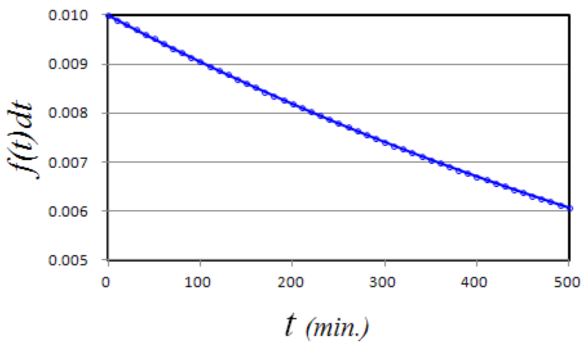

Fig. 1 Pdf of exponential distribution.

Probability distribution function is

$$f(t)dt = \frac{m}{\eta}(\frac{t-\gamma}{\eta})^{m-1}\exp\{-(\frac{t-\gamma}{\eta})^{m}\}dt \quad (24)$$

where, η, m and γ are the scale, shape and location parameters, respectively. If t is time, γ is a certain time before which there is no failure. If γ is set to time 0, η becomes the characteristic life time. Figure 2 shows the shape of probability distribution function with *(m=2, γ=1year, η=3years) and (m=4, γ=1year, η=3 years), dt=0.1 year.*

Reliability is

$$R(t) = \exp(-(\frac{t-\gamma}{\eta})^{m}). \quad (25)$$

Failure rate: $$\lambda(t) = \frac{m}{\eta^{m}}(t-\gamma)^{m-1} \quad (26)$$

Cumulative distribution function:

$$F(t) = 1-\exp(-(\frac{t-\gamma}{\eta})^{m}) \quad (27)$$

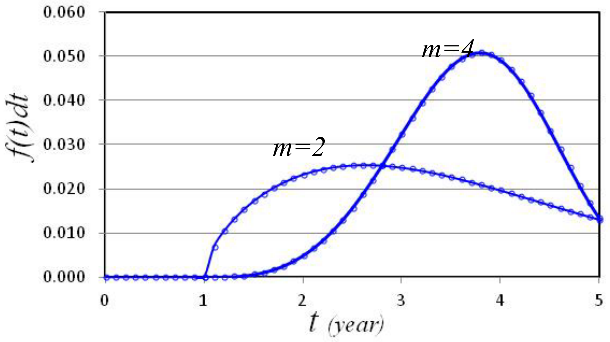

Fig. 2 Pdf of Weibull distribution.

Mean and variance are obtained as follows.

Mean: $$\gamma + \eta\Gamma(\frac{1}{m}+1) \quad (28)$$

Variance: $$\eta^{2}\{\Gamma(\frac{2}{m}+1) - \Gamma^{2}(\frac{1}{m}+1)\} \quad (29)$$

Mean time to failure: $$\eta\Gamma(\frac{1}{m+1}) \quad (30)$$

Γ( ) is the gamma function defined as follows:

$$\Gamma(x) = \int_{0}^{\infty} e^{-t}t^{x-1}dt \quad (31)$$

The following relations are convenient when it comes to solving gamma functions:

$$\Gamma(x+1) = x\Gamma(x), \quad \Gamma(1) = 1, \quad \Gamma(x+1) = x! \quad (32)$$

(Question 2) Derive the mean and variance for Weibull distribution.

3.3.3 Gaussian distribution (Normal distribution)

Gaussian distribution (normal distribution) is considered the most prominent probability distribution in statistics. This distribution arises as the outcome of the central limit theorem. The theorem states that under mild conditions the sum of a large number of random variables is distributed approximately the normal distribution. The range of variable t is from $-\infty$ to $+\infty$. Therefore, caution should be exercised when it comes to treatment of life time. Values of mode, median, mean are the same.

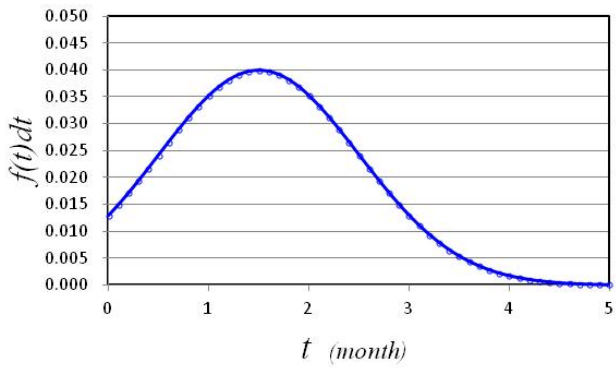

Fig. 3 Pdf of Gaussian distribution.

Figure 3 shows the shape of probability distribution function of normal distribution with *σ=1month, μ=1.5 month, dt=0.1month.*

pdf: $$f(t)dt = \frac{1}{\sqrt{2\pi}\sigma}\exp(-\frac{(t-\mu)^{2}}{2\sigma^{2}})dt \quad (33)$$

Mean: $$\mu \quad (34)$$

Variance: $$\sigma^{2} \quad (35)$$

3.3.4 Log-normal distribution

Log-normal distribution is a probability distribution of a random variable whose logarithm has normal distribution. This distribution is suitable for cases in which the values of variable change over a wide range. Since the variable t is always positive, this distribution is thus convenient for treating lifetime or some physical quantity. Values of mode, median, mean are different, and increase in this order. Figure 4 shows the shape of probability distribution function of log-normal distribution with *σ=1, μ=-3.45, ln{(t+dt)/t}=0.1151.*

pdf: $$f(t)dt = \frac{1}{\sqrt{2\pi}\sigma t}\exp(-\frac{(\ln t-\mu)^2}{2\sigma^2})dt \quad (36)$$

Mean: $$\exp(\mu+\sigma^2/2) \quad (37)$$

Variance: $$\exp(2\mu+\sigma^2)\cdot(e^{\sigma^2}-1) \quad (38)$$

(Question 3) Derive the mean and variance for log-normal distribution.

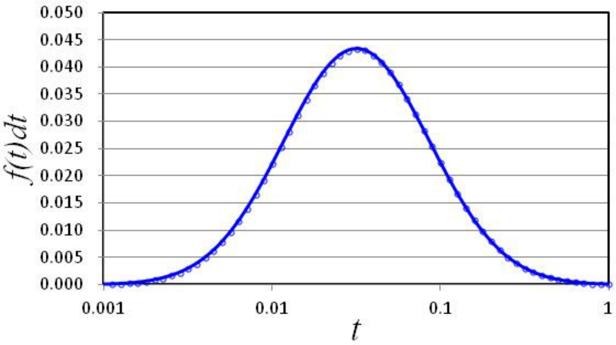

Fig. 4 Pdf of Log-normal distribution.

3.3.5 Binominal distribution

It is the discrete probability distribution of the number of successes *r* in a sequence of *n* independent yes/no experiments, each of which yields success with probability *p*. It is frequently used to model the number of successes in a sample of size *n* drawn with replacement from a population of size *N*.

If the sampling is carried out without replacement, the draws are not independent and therefore the resulting distribution is a hypergeometric distribution (not a binomial one). However, for *N* values much larger than *n*, the binomial distribution is widely used and gives a good approximation for non-replacement case. Figure 5 shows the shape of probability distribution function of binominal distribution with *p=0.25, n=15.*

pdf: $$f(r) = {}_nC_r p^r(1-p)^{n-r} = \frac{n!}{r!(n-r)!}p^r(1-p)^{n-r} \quad (39)$$

Mean: $$np \quad (40)$$

Variance: $$np(1-p) \quad (41)$$

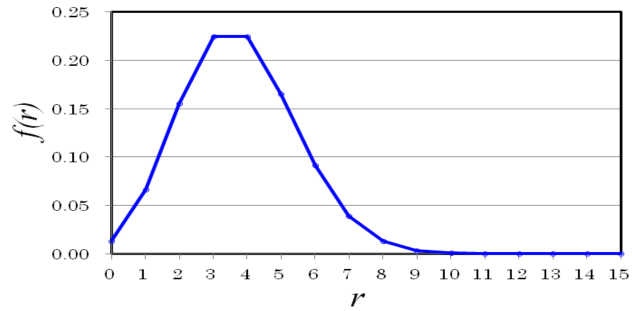

Fig. 5 Pdf of Binominal distribution.

3.3.6 Poisson distribution

The Poisson distribution (or Poisson law of small numbers) is a discrete probability distribution that expresses the probability of a given number of events occurring in a fixed interval of time if these events occur with a known average rate and independent of the time.

pdf: $$f(r) = e^{-m}\frac{m^r}{r!} \quad (42)$$

The Poisson distribution can be derived as a limiting case of the Binomial distribution, where *r* is the number of occurrences of an event, while *m* is a positive real number equivalent to the expected number of occurrences during the given interval. Figure 6 shows the shape of probability distribution function of Poisson distribution with *m=1, 4, 10.*

Mean: $$m \quad (43)$$

Variance: $$m \quad (44)$$

(Question 4) Derive the mean and variance for Poisson distribution.

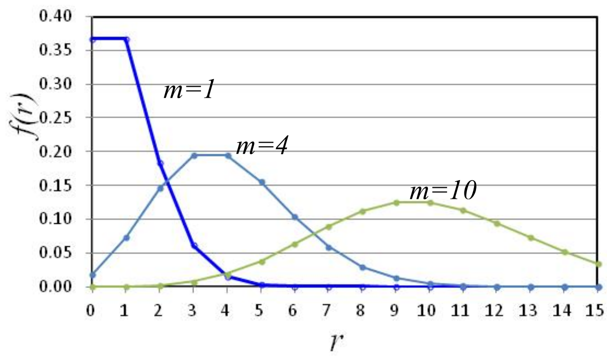

Fig. 6 Pdf of Poisson distribution.

3.3.7 Beta distribution

This distribution is adequate to describe the failure which has bathtub curve failure rate.

pdf: $$f(\lambda)d\lambda = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\ \lambda^{\alpha-1}\ (1-\lambda)^{\beta-1}d\lambda \quad ;0 \le \lambda \le 1 \qquad (45)$$

Mean: $$\alpha/(\alpha+\beta) \qquad (46)$$

Variance: $$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \qquad (47)$$

Figure 7 shows the shape of probability distribution function of beta distribution with *α=0.7, β=0.6, dλ=0.02.*

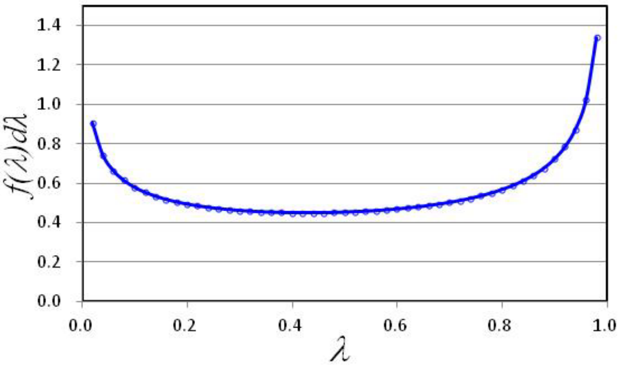

Fig. 7 Pdf of beta distribution.

3.3.8 Gamma distribution

It has a scale parameter θ and a shape parameter k.

pdf: $$f(x)dx = \frac{1}{\theta^k\Gamma(k)}\ x^{k-1}\ \mathrm{e}^{-\frac{x}{\theta}}dx\ ; x \ge 0,\ \ k,\theta > 0 \qquad (48)$$

Mean: $$k\theta \qquad (49)$$

Variance: $$k\theta^2 \qquad (50)$$

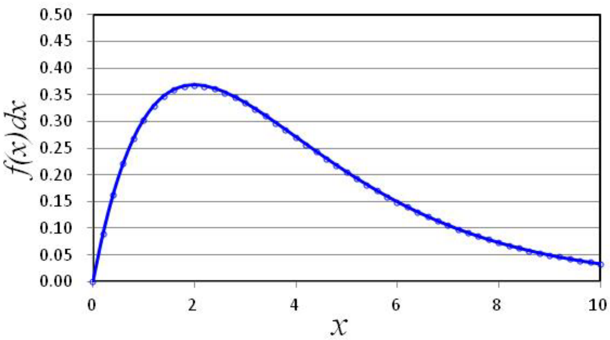

Fig. 8 Pdf of gamma distribution.

Figure 8 shows the shape of probability distribution function of gamma distribution with *k=2, θ=2.0, dx=0.2*

3.3.9 Generalized extreme value distribution

Generalized extreme value distribution (*FL(t)*) is used as an approximation to model the maxima of long (finite) sequences of random variables. A generalized extreme value distribution for minima (*FS(t)*) can be obtained, for example by substituting (-*x*) for *x* in the distribution function *FL(t)*, and subtracting from one: this yields a separate family of distributions.

$$FS(t) = 1 - FL(-t) \qquad (51)$$

There are three types of GEV; the Gumbel, Fréchet and Weibull families.

Type I – the Gumbel distribution has following expression.

pdf: $$f(t)dt = 1/\alpha \exp\left((t-u)/\alpha\right)\exp\{-\exp\left((t-u)/\alpha\right)\}dt \qquad (52)$$

Mean: $$u+\alpha\gamma \qquad (53)$$

where γ is Euler-Mascheroni constant = 0.57721.....

Variance: $$\frac{\pi^2}{6}\alpha^2 \qquad (54)$$

Figure 9 shows the shape of probability distribution function of GEV distribution with *u=3, α=4, dt=0.5.*

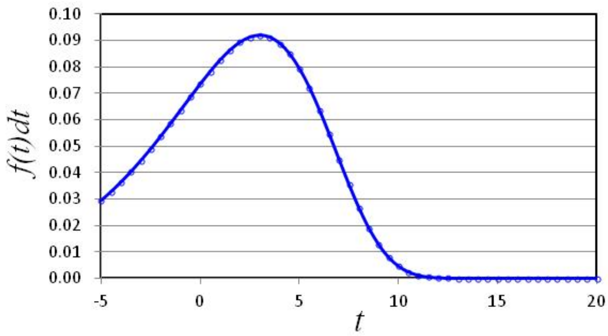

Fig. 9 Pdf of Generalized extreme value distribution.

Table 1 shows the summary of distribution functions and related statistical quantities. Appropriate distribution is selected based on the properties of data, for example, range of data spread, continuous distribution or not, corresponding phenomena.

Table 1 Summary of distribution functions and related statistical quantities

Distribution	Probability density function $f(x)dx$	Parameters	Mean $\bar{x}=\int_0^\infty xf(x)dx$	Variance $\int_0^\infty (x-\bar{x})^2 f(x)dx$	Cumulative probability distribution function $\int_0^t f(x)dx$
Exponential	$\lambda \exp\{-\lambda t\}dt$	$t>0, \lambda>0$	$1/\lambda$	$1/\lambda^2$	$1-\exp(-\lambda t)$
Weibull	$\frac{m}{\eta}(\frac{t-\gamma}{\eta})^{m-1}\exp\{-(\frac{t-\gamma}{\eta})^m\}dt$	$t>\gamma,\ m>0,\ \eta>0$	$\gamma+\eta\Gamma(\frac{1}{m}+1)$	$\eta^2\{\Gamma(\frac{2}{m}+1)-\Gamma^2(\frac{1}{m}+1)\}$	$1-\exp\{-(\frac{t-\gamma}{\eta})^m\}$
Normal	$\frac{1}{\sqrt{2\pi}\sigma}\exp(-\frac{(t-\mu)^2}{2\sigma^2})dt$	$-\infty<t<\infty,\ \sigma>0,\ \mu$	μ	σ^2	$\frac{1}{2}\left\{1+erf(\frac{(t-\mu)}{\sqrt{2\sigma^2}})\right\}$
Log normal	$\frac{1}{\sqrt{2\pi}\sigma t}\exp(-\frac{(\ln t-\mu)^2}{2\sigma^2})dt$	$t>0,\ \sigma>0,\ \mu$	$\exp(\mu+\sigma^2/2)$	$\exp(2\mu+\sigma^2)\cdot(e^{\sigma^2}-1)$	$\frac{1}{2}\left\{1-erf(-\frac{\ln t-\mu}{\sqrt{2\sigma^2}})\right\}$
Binominal	$\frac{n!}{r!(n-r)!}p^r(1-p)^{n-r}$	$n>0,\ 1\ge p\ge 0$	np	$np(1-p)$	$\sum_{r=0}^{r=c} {}_nC_r p^r(1-p)^{n-r}$
Poisson	$e^{-m}\frac{m^r}{r!}$	$r\ge 0,\ \ m>0$	m	m	$\sum_{r=0}^{r=c} e^{-m}\frac{m^r}{r!}$
Beta	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\lambda^{\alpha-1}(1-\lambda)^{\beta-1}d\lambda$	$1\ge\lambda\ge 0$	$\alpha/(\alpha+\beta)$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	$I_\lambda(\alpha,\beta)$
Gamma	$\frac{1}{\theta^k\Gamma(k)}x^{k-1}e^{-\frac{x}{\theta}}dx$	$x\ge 0,\ k,\theta>0$	$k\theta$	$k\theta^2$	$\exp(-\lambda t)\sum_{i=k}^{\infty}\frac{1}{i!}(\lambda t)^i$
Generalized extreme value	$1/\alpha \exp((t-u)/\alpha)\exp\{-\exp((t-u)/\alpha)\}dt$	$-\infty<t<\infty,\ \alpha>0$	$u+\alpha\gamma$	$\frac{\pi^2}{6}\alpha^2$	$\exp\left\{-e^{-(t-u)/\alpha}\right\}$

Gamma function: $\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt$ $\quad \Gamma(x+1)=x\Gamma(x),\quad \Gamma(1)=1,\quad \Gamma(x+1)=x!$, $\quad erf(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$, $\quad I_\lambda(\alpha,\beta)=\frac{\int_0^\lambda t^{\alpha-1}(1-t)^{\beta-1}dt}{\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt}$

γ in generalized extreme value: Euler–Mascheroni constant = 0.57721.....

4 Topics in system reliability analysis

4.1 Consideration of repair actions

In most engineering systems, repair actions are taken for failed component. This is called maintenance. For a repairable system, "availability" is an interesting aspect and is defined as the fraction of normal operating time duration during required time period.

The probability to return to normal state per unit time duration is defined as repair rate ($\mu(t)$). $M(t)$ is defined as a maintainability function which is the cumulative probability at time t that component has been repaired.
Then, the probability distribution function (pdf) of maintainability is defined.

$$m(t) = \frac{dM(t)}{dt} \tag{55}$$

The repair rate $\mu(t)$ is expressed by the following equation:

$$\mu(t) = \frac{m(t)}{1 - M(t)} \tag{56}$$

The above expression can be expressed as a differential equation:

$$\frac{dM(t)}{dt} = \mu(t)\left\{1.0 - M(t)\right\} \tag{57}$$

If $\mu(t)$ is constant, the equation can be easily solved with the initial condition set as $M(t=0)=0$.

$$M(t) = 1.0 - \exp(-\mu t) \tag{58}$$

The above equation is derived based on the assumption that there is no failure during repair actions.

Now, consider the case where failure rate λ and repair rate μ are constant with time. $P_0(t)$, $P_1(t)$ are defined as the probabilities that the component is in normal state and failed state, respectively. Therefore, the relation existing among λ, μ, $P_0(t)$, and $P_1(t)$ can be expressed as follows:

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t) \tag{59}$$

Substitute the relation $P_1(t) = 1 - P_0(t)$ into equation (59).

$$\frac{dP_0(t)}{dt} = \mu - (\lambda + \mu)P_0(t) \tag{60}$$

With the initial condition of $P_0(t=0)=1$, the solution becomes,

$$P_0(t) = \frac{\mu}{\mu+\lambda} + \frac{\lambda}{\mu+\lambda}\exp(-(\lambda+\mu)t). \tag{61}$$

It is an instantaneous availability, and $P_0(\infty) = \dfrac{\mu}{\mu+\lambda}$.

Mean availability is,

$$A(t) = \frac{1}{T}\int_0^T P_0(t)dt = \frac{\mu}{\mu+\lambda} + \frac{\lambda}{(\mu+\lambda)^2 T} - \frac{\lambda}{(\mu+\lambda)^2 T}\exp(-(\lambda+\mu)T). \tag{62}$$

(Question 5) Solve equation (60) with the initial condition of $P_0(t=0)=P_0$.

Next, consider a system consisting of N components connected in series. Assume that the failure and repair rates are constant, and $P_i(t)$ is used to express the probability of the state only when the i-th component is in failed state among N components. Then, $P_0(t)$ means that all the components are in sound condition. The following equations are held provided that we neglect the simultaneous failures of more than two components.

$$\begin{aligned}
\frac{dP_0(t)}{dt} &= -\sum_{i=1}^{N}\lambda_i P_0(t) + \mu_1 P_1(t) + \mu_2 P_2(t) + \mu_3 P_3(t) + \cdots + \mu_N P_N(t) \\
\frac{dP_1(t)}{dt} &= \lambda_1 P_0(t) - \mu_1 P_1(t) \\
\frac{dP_2(t)}{dt} &= \lambda_2 P_0(t) - \mu_2 P_2(t) \\
\frac{dP_N(t)}{dt} &= \lambda_N P_0(t) - \mu_N P_N(t)
\end{aligned} \tag{63}$$

From the equilibrium condition,

$$P_i(\infty) = \frac{\lambda_i}{\mu_i}P_0(\infty) = \rho_i P_0(\infty), \tag{64}$$

$$1.0 = \sum_{i=0}^{N} P_i(\infty) = P_0(\infty) + \sum_{i=1}^{N} P_i(\infty) = (1 + \sum_{i=1}^{N}\rho_i)P_0(\infty). \tag{65}$$

Mean availability is obtained as follows:

$$A(\infty) = P_0(\infty) = \frac{1}{(1 + \sum_{i=1}^{N}\rho_i)} \tag{66}$$

When one component in the system fails, mean idle period of the system becomes,

$$\frac{1}{\lambda_s}\int_0^\infty t\sum_{i=1}^{N}\lambda_i \exp(-\mu_i t)dt = \frac{1}{\lambda_s}\sum_{i=1}^{N}\lambda_i\int_0^\infty t\exp(-\mu_i t)dt$$

$$= \frac{1}{\lambda_s}\sum_{i=1}^{N}\lambda_i\left[-\frac{e^{-\mu_i t}}{\mu_i}(t+\frac{1}{\mu_i})\right]_0^\infty = \frac{1}{\lambda_s}\sum_{i=1}^{N}\lambda_i\frac{1}{\mu_i^2} = \frac{1}{\lambda_s}\sum_{i=1}^{N}\frac{\rho_i}{\mu_i}. \quad (67)$$

where, $\lambda_s = \sum_{i=1}^{N}\lambda_i$.

4.2 Matrix expressions of maintained systems

The equation (59) is transformed into a difference equation (68), where failure and repair rates are assumed as constant.

$$P_0(t+dt) = (1-\lambda dt)P_0(t) + \mu dt P_1(t) \quad (68)$$

For the failed state $P_1(t)$, a similar relation is also obtained.

$$P_1(t+dt) = \lambda dt P_0(t) + (1-\mu dt)P_1(t) \quad (69)$$

Equations (68) and (69) can be combined and expressed by one matrix expression as follows.

$$(P_0(t+dt), P_1(t+dt)) = (P_0(t), P_1(t))\begin{bmatrix} 1-\lambda dt & \lambda dt \\ \mu dt & 1-\mu dt \end{bmatrix} \quad (70)$$

Next, consider a system consisting of 2 components connected in series. Assume both components have the same failure rate λ, and repair rate μ. $P_0(t), P_1(t), P_2(t)$ are used for the expression of the probabilities that *0,1,2* components are in failed state, respectively. The following matrix is held for cases where two maintenance personnel take care of the system.

$$\left[P_0(t+dt),\ P_1(t+dt),\ P_2(t+dt)\right]$$
$$= \left[P_0(t),\ P_1(t),\ P_2(t)\right]\begin{bmatrix} 1-2\lambda dt & 2\lambda dt & 0 \\ \mu dt & 1-(\lambda+\mu)dt & \lambda dt \\ 0 & 2\mu dt & 1-2\mu dt \end{bmatrix} \quad (71)$$

Availability of the system at the equilibrium condition is obtained from the above matrix.

$$P_0(\infty) = \left(\frac{\mu}{\lambda+\mu}\right)^2 \quad (72)$$

If there is only one maintenance person for this system, the matrix becomes, as shown in equation. (73). In the third row there appear "μdt" and "*1-* μdt", since two failed components cannot be repaired at the same time.

$$\left[P_0(t+dt),\ P_1(t+dt),\ P_2(t+dt)\right]$$
$$= \left[P_0(t),\ P_1(t),\ P_2(t)\right]\begin{bmatrix} 1-2\lambda dt & 2\lambda dt & 0 \\ \mu dt & 1-(\lambda+\mu)dt & \lambda dt \\ 0 & \mu dt & 1-\mu dt \end{bmatrix}. \quad (73)$$

Therefore, availability becomes,

$$P_0(\infty) = \left(\frac{\mu^2}{\mu^2+2\lambda\mu+2\lambda^2}\right) \quad (74)$$

If two components are placed in parallel, the matrix is same to the series system but availability becomes $P_0(\infty)+P_1(\infty)$ and easy to calculate.

$$A(\infty) = P_0(\infty)+P_1(\infty) = \left(\frac{\mu^2+2\lambda\mu}{\mu^2+2\lambda\mu+2\lambda^2}\right) \quad (75)$$

Next try to examine a stand-by and redundant system. One component is running and the other is in stand-by state. Stand-by component can immediately start its operation if required. It also assumes that there is only one maintenance person. The matrix becomes as follows:

$$\left[P_0(t+dt),\ P_1(t+dt),\ P_2(t+dt)\right]$$
$$= \left[P_0(t),\ P_1(t),\ P_2(t)\right]\begin{bmatrix} 1-\lambda dt & \lambda dt & 0 \\ \mu dt & 1-(\lambda+\mu)dt & \lambda dt \\ 0 & \mu dt & 1-\mu dt \end{bmatrix} \quad (76)$$

and availability is expressed as follows:

$$A(\infty) = P_0(\infty)+P_1(\infty) = \left(\frac{\mu^2+\lambda\mu}{\mu^2+\lambda\mu+\lambda^2}\right) \quad (77)$$

4.3 Boolean expression of system reliability

Consider a sample system as shown in Fig. 10. The reliability of this system could be expressed in Boolean algebraic expression.

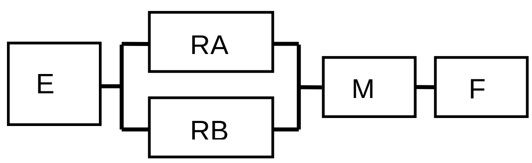

Fig. 10 A sample system.

Start with a sound state of a component and observe it for certain time duration *t*. The component will be in success or failed state at time *t*. Repeat this observation many times and collect the events the component is in success state. Let us denote a set of these events, for instance, as *E(t)* for component E (electric power source).

A fraction of the number of success events over total observation number gives the success probability of a component at time *t*. It is expressed by the notation of Pr (*E(t)*). Concrete expression of Pr (*E(t)*) is obtained, for example, based on an assigned failure model as shown in Chapter 3.

The system is expressed as follows by *E(t)*, *etc*.

$$E(t)\bullet\{RA(t)+RB(t)\}\bullet M(t)\bullet F(t) \qquad (78)$$

The symbols "$\bullet$ " and "＋" mean the "product" and "union" between sets, respectively. Equation (78) expresses a set of success events in which the system shown in Fig. 10 is functioning at time *t*. These events are supposedly collected in many observations or in many hypothetical observations.

Any block diagram or logical structure can be expressed by this kind of Boolean equation. The logic of FT and ET can also be expressed by this method.

System reliability at time *t* is obtained as,

$$\Pr(\ E(t)\bullet\{RA(t)+RB(t)\}\bullet M(t)\bullet F(t)\ ). \qquad (79)$$

For the calculation of the total system reliability, the following relations should be taken into consideration:

$$\Pr(\ \prod_i A_i(t)\ ) = \prod_i Pr(A_i(t)) \qquad (80)$$

$$\Pr(\ \sum_i A_i(t)\ ) = \sum_i Pr(A_i(t)) - \sum_{i<j}^{N} Pr(A_i(t)\cdot A_j(t))$$

$$+\sum_{i<j<k}^{N} Pr(A_i(t)\cdot A_j(t)\cdot A_k(t)) \ - \ \bullet\bullet\bullet \qquad (81)$$

where, $A_i(t)$,$A_j(t)$,$A_k(t)$ are sets of events.

(Question 6). Answer the following questions about the fault tree shown in Fig. 11.

a. Obtain the Boolean expression of the top event.
b. Obtain the top event probability if the occurrence probabilities of basic events are given as follows.

Pr(X1)=0.003
Pr(X2)=0.03
Pr(X3)=0.01
Pr(X4)=0.03
Pr(X5)=0.01

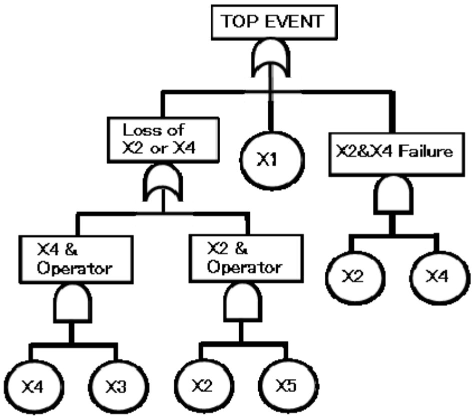

Fig. 11 An example fault tree.

4.4 Analysis of loop structure

Many kinds of components in engineering systems require supports of other components for their operation. The relations between supporting and supported components in a large system produce a closed circuit to comprise a logical loop structure. For example, a nuclear power plant uses self-generated electricity for its operation.

Logical loop was not generally solved in terms of the arithmetic operators of Boolean algebra. Many attempts have been proposed. One simple method is just to ignore the intermediate event recursively appear in a fault tree.
Now consider a system which has a loop structure as shown in Fig. 12.

The event X, which is output of component C, is expressed by the next Boolean relation.

$$X = S1\bullet A\bullet B\bullet C + S2\bullet B\bullet C + A\bullet B\bullet C\bullet X \qquad (82)$$

The output *per se* is used as an input in this equation. The equation (82) does not change its form after repeating the substitution of *X* into the right hand side. This is a situation wherein infinite circulation of unknown element appears in the process of solving a Boolean equation with unknown element(s).

However, the equation can directly be solved with the aid of Boolean fundamental theorems. The solution of the equation (82) becomes,

$$X = S1 \cdot A \cdot B \cdot C + S2 \cdot B \cdot C + mA \cdot B \cdot C. \quad (83)$$

The last term represents the loop operating state. The procedures to determine indefinite element "*m*" is comprehensively described in references [6] [7].

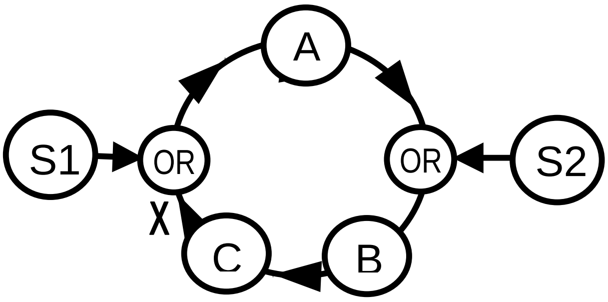

Fig. 12 A system with loop structure.

4.5 Dependent failure and common cause failure

Dependent Failure is defined as a set of failure events, in which the probability of this phenomenon cannot be expressed as the simple product of unconditional failure probabilities of the individual events.

Common Cause Failure (CCF) is a specific type of dependent failure where simultaneous (or near simultaneous) multiple failures result from a single shared cause. In this case, a single shared cause can be clearly identified. There is a terminology "Common Mode Failure (CMF)", and it is often confused with common cause failure. CMF is one of the CCF, in which multiple equipment items fail in the same mode, probably from a single shared cause.

Dependent failure is an important issue in reliability analysis because system reliability is largely reduced by it if high reliability is established by redundancy.

The important point of the analysis of dependent failure and common cause failure is to properly investigate these kinds of failure phenomena.

4.5.1 Classification of dependent failure

Many kinds of dependent failures are investigated in the execution of PSAs. In the following sub-sections, various kinds of dependent failures are listed up. The list is useful when the readers perform a system reliability analysis and / or PSA.

(1) Physical dependencies: These are easy to understand and identify. Exemplars are common suction valve for two pumps, AC power supply, DC power supply, interlocks for pumps, circuit breakers, automatic start / alignment signals, and restart signals.

(2)Functional dependencies: They occur when components are functionally inter-related. Examples are makeup and steam relief required for secondary heat removal, operator actions to start / align equipment, timing of failures and recovery actions, coordinated outages and planned maintenance, and cooling water.

(3)Location / environmental dependencies: They occur when components have no relation with each other physically or functionally, but however locate at near places and suffer same cause. These dependencies are common cause failure. Examples are structural failures, seismic events, fires, flooding, turbine missiles, and water spray.

(4)Data-based dependencies: Examples are multiple component maintenance, coordinated outages and planned maintenance, failure of multiple similar components, plant configuration related dependencies.

(5)Human dependencies: They are caused by human activities. Examples are time window for operator response, similar functions, multiple options/priorities, procedures / training, personnel / staffing, and location (preceding system successes / failures, preceding operator successes/failures).

4.5.2 Common cause failure

Consider a system comprising *m* similar components. Each component has independent failure probability Q_1. Probability Q_k is defined as the probability that k components fail simultaneously. The total failure probability of one component then becomes:

$$Q_t = \sum_{k=1}^{m} {}_{m-1}C_{k-1} \cdot Q_k \quad (84)$$

There are a plethora of models for the prediction of the ratio Q_k and Q_t. Some representative parametric models are explained in the following section 4.5.3 to 4.5.6.

4.5.3 β-factor model

This is a simple model and numerically conservative model. Many reasonable data have been collected from plant operating experiences.

The idea of this model is that common cause failures occur in all the components of the system. As such, the model gives the following relations:

$$Q_k = 0; \, for k \neq m,$$
$$Q_m = \beta Q_t,$$
$$Q_1 = (1-\beta) Q_t$$
$$\beta = Q_m \big/ (Q_1 + Q_m) \tag{85}$$

The values of β are obtained as follows in nuclear field [8].

Breaker: 0.19, Diesel generator: 0.05, Motor operated valve: 0.08, Relief valve(PWR): 0.07, Relief valve(BWR): 0.22, Check valve: 0.06, Pumps: 0.03-0.17, Cooling system: 0.11, Ventilation: 0.13, overall components: 0.10.

4.5.4 Multiple Greek letter model

This model is logically more correct than β-factor model, and expansion of β-factor model. It is deemed to be a numerically more realistic model. The parameters are defined as follows.

$$Q_k = \frac{1}{{}_{m-1}C_{k-1}} \prod_{i=1}^{k} \rho_i (1-\rho_{k+1}) Q_t$$
$$\rho_1 = 1, \rho_2 = \beta, \rho_3 = \gamma, \rho_4 = \delta, \ldots.. \tag{86}$$

From the above definition,

$$Q_1 = (1-\beta) Q_t,$$
$$Q_2 = \frac{1}{m-1} \beta (1-\gamma) Q_t,$$
$$Q_3 = \frac{1}{m-1} \frac{2}{m-2} \beta \gamma (1-\delta) Q_t, \ldots. \tag{87}$$

The values of parameters are recommended in case of no operating data [9].

m=2: β=0.10,
m=3: β=0.10, γ=0.27,
m=4, β=0.11, γ=0.42, δ=0.40.

4.5.5 α-factor model

This is also a simple model that is based on the data of system failures. Thus, the data is easy to collect from field experiences. In this model consider the probability that k components fail simultaneously.

$$\alpha_k = \frac{{}_mC_k \cdot Q_k}{\sum_{k=1}^{m} {}_mC_k \cdot Q_k}$$
$$\alpha_1 + \alpha_2 + \cdots + \alpha_m = 1$$
$$Q_k = \frac{k \cdot \alpha_k}{{}_{m-1}C_{k-1} \cdot \alpha_t} Q_t, \quad \alpha_t = \sum_{k=1}^{m} k \cdot \alpha_k \tag{88}$$

Also, recommended values are given [9].

m=2: α_1=0.95, α_2=0.05,
m=3: α_1=0.95, α_2=0.04, α_3=0.01,
m=4, α_1=0.95, α_2=0.035, α_3=0.01, α_4=0.005,

4.5.6 Binominal failure rate mode

This model is based on the idea that there are two different kinds of failures owing to some cause, one being "Lethal"(ω) and the other being "nonlethal"(μ). Assume lethal failure occurs with probability 1.0 and nonlethal failure independently occurs with the probability P. Then following relations are obtained.

$$Q_m = \mu p^m + \omega$$
$$Q_1 = Q_i + \mu p (1-p)^{m-1}$$
$$Q_k = \mu p^k (1-p)^{m-k} \tag{89}$$

where, Q_i is an independent, random failure probability, which is not produced by some common cause.

5 Summary

In this lecture note, explanations are given for risk and safety assessment. Handling of data is crucially important for performing PSAs. Characteristics of data, distribution functions of failure data and related statistical quantities are explained in detail.

Boolean expression of a system reliability and analysis method for loop structures are introduced, since they are fundamental and important topics of system reliability analysis, maintenance activity, and matrix expressions of a maintained system. Finally, dependent failure and common cause failure are discussed. Parametric models for common cause failure are explained and some reference values are given. Dependent failure is indeed an important issue in reliability analysis.

It is the authors hope that the paper can serve as a reference for the reader's future research activities.

6 Answer of the questions

(Question 1) From equation (11),

$$\frac{dR(t)}{dt} = -\lambda(t)R(t). \quad \text{Then} \quad \frac{dR(t)}{R(t)} = -\lambda(t)dt.$$

Integrate the above equation with the initial condition $R(t=0)=1.0$.

$$\int_0^T \frac{dR(t)}{R(t)} = -\int_0^T \lambda(t)dt = -H(T)$$

$$\therefore \ \ln R(T) = -H(T).$$

Therefore, $\quad R(t) = \exp\{-H(t)\}$

(Question 2)

$$E(t^k) = \int_\gamma^\infty t^k f(t)dt = -\int_\gamma^\infty t^k \frac{dR(t)}{dt}dt = \left[-t^k R(t)\right]_\gamma^\infty$$

$$+k\int_\gamma^\infty t^{k-1}R(t)dt = 0 + \gamma^k + k\int_\gamma^\infty t^{k-1}R(t)dt$$

Variable t is changed to u by the following relation:

$$\left(\frac{t-\gamma}{\eta}\right)^m = u$$

Then,

$$E(t^k) = \gamma^k + k\int_\gamma^\infty (\eta u^{\frac{1}{m}} + \gamma)^{k-1} e^{-u} \frac{\eta}{m} u^{\frac{1}{m}-1} du$$

$$\therefore \ E(t) = \bar{t} = \gamma + \int_\gamma^{\infty} e^{-u} \frac{\eta}{m} u^{\frac{1}{m}-1} du$$

$$= \gamma + \frac{\eta}{m}\int_\gamma^\infty u^{\frac{1}{m}-1} e^{-u} du = \gamma + \frac{\eta}{m}\Gamma(\frac{1}{m})$$

$$\therefore \ E(t^2) = \gamma^2 + 2\int_\gamma^\infty (\eta u^{\frac{1}{m}} + \gamma)\, e^{-u} \frac{\eta}{m} u^{\frac{1}{m}-1} du$$

$$= \gamma^2 + 2\frac{\eta^2}{m}\int_\gamma^\infty e^{-u} u^{\frac{2}{m}-1} du + 2\frac{\eta\gamma}{m}\int_\gamma^\infty e^{-u} u^{\frac{1}{m}-1} du$$

$$= \gamma^2 + 2\frac{\eta^2}{m}\Gamma(\frac{2}{m}) + 2\frac{\eta\gamma}{m}\Gamma(\frac{1}{m})$$

$$= \gamma^2 + \eta^2\Gamma(\frac{2}{m}+1) + 2\eta\gamma\Gamma(\frac{1}{m}+1)$$

$$\sigma^2 = E(t^2) - E(t)^2 = \eta^2\left\{\Gamma(\frac{2}{m}+1) - \Gamma^2(\frac{1}{m}+1)\right\}$$

(Question 3)

$$E(t^k) = \int_0^\infty t^k \frac{1}{\sqrt{2\pi}\sigma t}\exp\left\{-\frac{(\ln t - \mu)^2}{2\sigma^2}\right\}dt$$

Variable t is changed to x by the following relation:

$$\ln\frac{t}{e^m} = x \ , \quad \frac{t}{e^m} = e^x \ , \quad dt = e^{x+m}dx$$

Then,

$$E(t^k) = \frac{e^{\mu k}}{\sqrt{2\pi}\sigma}\int_{-\infty}^\infty e^{kx}\exp(-\frac{x^2}{2\sigma^2})dx$$

$$= \frac{e^{\mu k}}{\sqrt{2\pi}\sigma}e^{\frac{k^2\sigma^2}{2}}\int_{-\infty}^\infty \exp(-\frac{(x-k\sigma)^2}{2\sigma^2})dx = e^{\frac{k^2\sigma^2}{2}+\mu k}$$

$$\therefore \ E(t) = \bar{t} = e^{\mu + \frac{\sigma^2}{2}}$$

$$\sigma^2 = E(t^2) - E(t)^2 = e^{2\mu}e^{2\sigma^2} - e^{2\mu}e^{\sigma^2} = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$$

(Question 4)

$$\bar{r} = \sum_{r=1}^\infty rf(r) = \sum_{r=1}^\infty re^{-m}\frac{m^r}{r!} = \sum_{r=1}^\infty e^{-m}\frac{m^r}{(r-1)!}$$

$$= m\sum_{r=1}^\infty e^{-m}\frac{m^{r-1}}{(r-1)!} = me^{-m}\sum_{R=0}^\infty \frac{m^R}{R!} = me^{-m}e^m = m$$

$$\sigma^2 = E(r^2) - E(r)^2 = E(r(r-1)+r) - E(r)^2$$

$$= E(r(r-1)) + E(r) - E(r)^2$$

$$E(r(r-1)) = \sum_{r=0}^\infty r(r-1)e^{-m}\frac{m^r}{r!} = m^2e^{-m}\sum_{r=2}^\infty \frac{m^{r-2}}{(r-2)!} = m^2$$

$$\therefore \sigma^2 = m^2 + m - m^2 = m$$

(Question 5)

From the equation (59), $\quad \dfrac{dP_0(t)}{\mu - (\lambda+\mu)P_0(t)} = dt$.

Then integrate, $\quad \displaystyle\int \frac{dP_0(t)}{\mu - (\lambda+\mu)P_0(t)} = \int dt$,

$$t = \int \frac{dP_0(t)}{\mu - (\lambda+\mu)P_0(t)} + C_1 = -\frac{1}{\lambda+\mu}\ln(\mu - (\lambda+\mu)P_0(t)) + C_1$$

$$-(\lambda+\mu)t = \ln(\mu - (\lambda+\mu)P_0(t)) + C_1'$$

With the initial condition $P_0(t=0)=P_0$,

$$C_1' = -\log(\mu - (\lambda+\mu)P_0)$$

$$\therefore \ -(\lambda+\mu)t = \log(\frac{\mu - (\lambda+\mu)P_0(t)}{\mu - (\lambda+\mu)P_0})$$

$$e^{-(\lambda+\mu)t} = \frac{\mu - (\lambda+\mu)P_0(t)}{\mu - (\lambda+\mu)P_0}$$

$$\mu - (\lambda+\mu)P_0(t) = (\mu - (\lambda+\mu)P_0)e^{-(\lambda+\mu)t}$$

$$P_0(t) = \frac{\mu}{\lambda+\mu} - \frac{\mu-(\lambda+\mu)P_0}{\lambda+\mu} e^{-(\lambda+\mu)t}$$

(Question 6)

a. $TOP = X1 + X2 \bullet X4 + X4 \bullet X3 + X2 \bullet X5.$

b.

$$\begin{aligned}
&\Pr(TOP) = \Pr(X1 + X2\bullet X4 + X4\bullet X3 + X2\bullet X5)\\
&= \Pr(X1) + \Pr(X2\bullet X4) + \Pr(X4\bullet X3) + \Pr(X2\bullet X5)\\
&- \Pr(X1\bullet X2\bullet X4) - \Pr(X1\bullet X4\bullet X3) - \Pr(X1\bullet X2\bullet X5)\\
&- \Pr(X2\bullet X4\bullet X3) - \Pr(X2\bullet X4\bullet X5) - \Pr(X4\bullet X3\bullet X2\bullet X5)\\
&+ \Pr(X1\bullet X2\bullet X4\bullet X3) + \Pr(X1\bullet X2\bullet X4\bullet X5)\\
&+ \Pr(X1\bullet X4\bullet X3\bullet X2\bullet X5) + \Pr(X2\bullet X4\bullet X3\bullet X5)\\
&- \Pr(X1\bullet X2\bullet X4\bullet X3\bullet X5)
\end{aligned}$$

$$\begin{aligned}
&= 0.003 + 0.03\times 0.03 + 0.03\times 0.01 + 0.03\times 0.01\\
&-0.003\times 0.03\times 0.03 - 0.003\times 0.03\times 0.01 - 0.003\times 0.03\times 0.01\\
&-0.003\times 0.03\times 0.01 - 0.003\times 0.03\times 0.01 - 0.003\times 0.01\times 0.03\times 0.01\\
&+0.003\times 0.03\times 0.03\times 0.01 + 0.003\times 0.03\times 0.03\times 0.01\\
&+0.003\times 0.03\times 0.01\times 0.03\times 0.01 + 0.03\times 0.01\times 0.03\times 0.01\\
&-0.003\times 0.03\times 0.03\times 0.01\times 0.01\\
&= 0.0045 - 0.0000045 - 0.000001809\\
&\quad + 0.00000014427 - 0.00000000027\\
&= 0.004493835 \doteq 0.004494
\end{aligned}$$

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