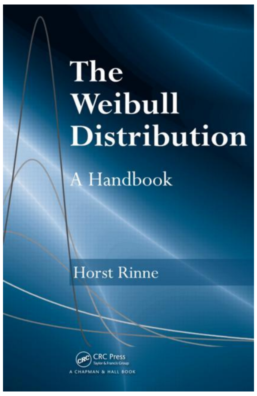

The Weibull Distribution
A Handbook

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Related distributions

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3 Related distributions

This chapter explores which other distributions the WEIBULL distribution is related to and in what manner. We first discuss (Section 3.1) how the WEIBULL distribution fits into some of the well–known systems or families of distributions. These findings will lead to some familiar distributions that either incorporate the WEIBULL distribution as a special case or are related to it in some way (Section 3.2). Then (in Sections 3.3.1 to 3.3.10) we present a great variety of distribution models that have been derived from the WEIBULL distribution in one way or the other. As in Chapter 2, the presentation in this chapter is on the theoretical or probabilistic level.

3.1 Systems of distributions and the WEIBULL distribution[1]

Distributions may be classified into families or systems such that the members of a family

- have the same special properties and/or
- have been constructed according to a common design and/or
- share the same structure.

Such families have been designed to provide approximations to as wide a variety of observed or empirical distributions as possible.

3.1.1 PEARSON system

The oldest system of distributions was developed by KARL PEARSON around 1895. Its introduction was a significant development for two reasons. Firstly, the system yielded simple mathematical representations — involving a small number of parameters — for histogram data in many applications. Secondly, it provided a theoretical framework for various families of sampling distributions discovered subsequently by PEARSON and others. PEARSON took as his starting point the skewed binomial and hypergeometric distributions, which he smoothed in an attempt to construct skewed continuous density functions. He noted that the probabilities P_r for the hypergeometric distribution satisfy the difference equation

$$P_r - P_{r-1} = \frac{(r-a)\,P_r}{b_0 + b_1\,r + b_2\,r^2}$$

for values of r inside the range. A limiting argument suggests a comparable differential equation for the probability density function

$$f'(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x} = \frac{(x-a)\,f(x)}{b_0 + b_1\,x + b_2\,x^2}\,. \tag{3.1}$$

[1] Suggested reading for this section: BARNDORFF–NIELSEN (1978), ELDERTON/JOHNSON (1969), JOHNSON/KOTZ/BALAKRISHNAN (1994, Chapter 4), ORD (1972), PEARSON/HARTLEY (1972).

The solutions $f(x)$ are the density functions of the **PEARSON system**.

The various types or families of curves within the PEARSON system correspond to distinct forms of solutions to (3.1). There are three main distributions in the system, designated types I, IV and VI by PEARSON, which are generated by the roots of the quadratic in the denominator of (3.1):

- type I with DF

$$f(x) = (1+x)^{m_1}\,(1-x)^{m_2},\ \ -1 \le x \le 1,$$

  results when the two roots are real with opposite signs (The **beta distribution of the first kind** is of type I.);

- type IV with DF

$$f(x) = (1+x^2)^{-m}\ \exp\big\{-\nu\,\tan^1(x)\big\},\ \ -\infty < x < \infty,$$

  results when the two roots are complex;

- type VI with DF

$$f(x) = x^{m_2}\,(1+x)^{-m_1},\ \ 0 \le x < \infty,$$

  results when the two roots are real with the same sign. (The ***F***– or **beta distribution of the second kind** is of type VI.)

Ten more "transition" types follow as special cases.

A key feature of the PEARSON system is that the first four moments (when they exist) may be expressed explicitly in terms of the four parameters $(a,\ b_0,\ b_1$ and $b_2)$ of (3.1). In turn, the two moments ratios,

$$\begin{aligned}\beta_1 &= \frac{\mu_3^2}{\mu_2^3}\ \text{(skewness)},\\ \beta_2 &= \frac{\mu_4}{\mu_2^2}\ \text{(kurtosis)},\end{aligned}$$

provide a complete taxonomy of the system that can be depicted in a so–called **moment–ratio diagram** with β_1 on the abscissa and β_2 on the ordinate. Fig. 3/1 shows a detail of such a diagram emphasizing that area where we find the WEIBULL distribution.

- The limit for all distributions is given by

$$\beta_2 - \beta_1 = 1,$$

  or stated otherwise: $\beta_2 \le 1 + \beta_1$.

- The line for type III (**gamma distribution**) is given by

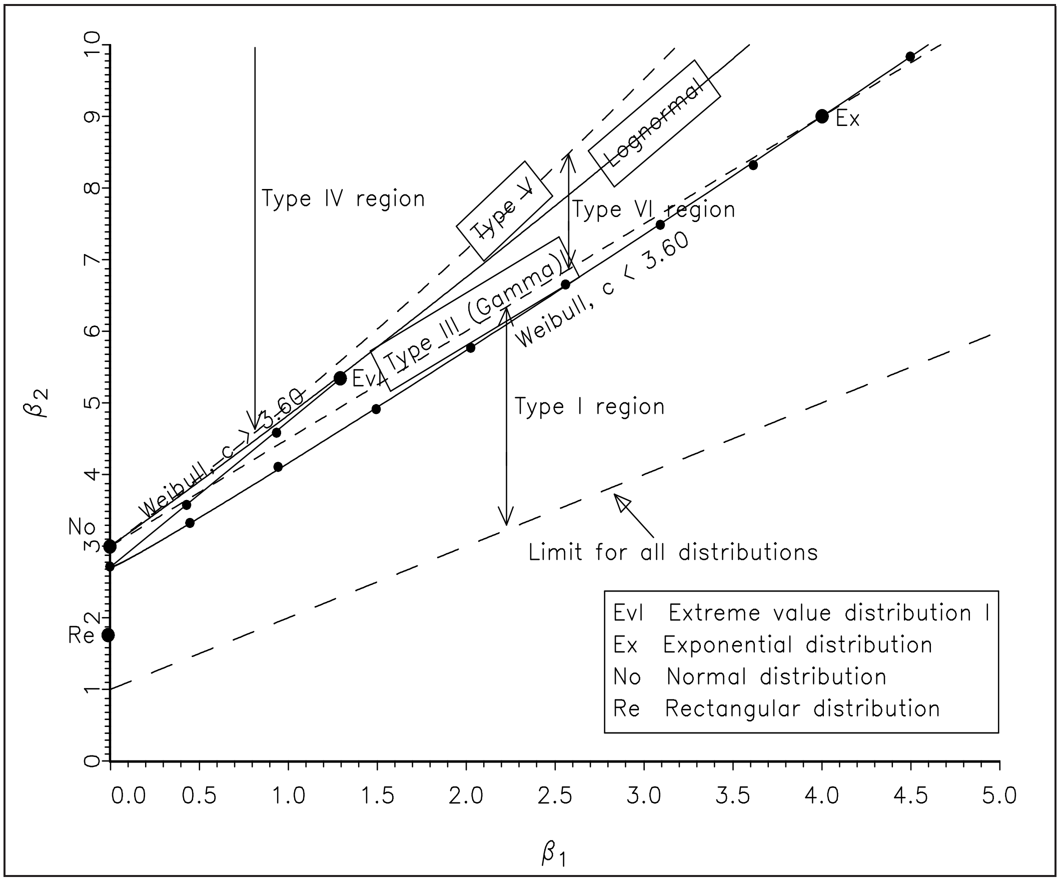
$$\beta_2 = 3 + 1.5\,\beta_1.$$

  It separates the regions of the type–I and type–VI distributions.

- The line for type V, separating the regions of type–VI and type–IV distributions, is given by

$$\beta_1\,(\beta_2+3)^2 = 4\,(4\,\beta_2 - 3\,\beta_1)\,(2\,\beta_2 - 3\,\beta_1 - 6)$$

Figure 3/1: Moment ratio diagram for the PEARSON system showing the WEIBULL distribution



or — solved for β_2 — by

$$\beta_2 = \frac{3\left(-16 - 13\,\beta_1 - 2\,\sqrt{(4+\beta_1)^3}\right)}{\beta_1 - 32}\,, \quad 0 \le \beta_1 < 32.$$

We have also marked by dots the positions of four other special distributions:

- **uniform** or **rectangular distribution** ($\beta_1 = 0;\ \beta_2 = 1.8$),
- **normal distribution** ($\beta_1 = 0;\ \beta_2 = 3$),
- **exponential distribution** ($\beta_1 = 4;\ \beta_2 = 9$) at the crossing of the gamma–line and the lower branch of the WEIBULL–distribution–line,
- **type–I extreme value distribution** ($\beta_1 \approx 1.2986;\ \beta_2 = 5.4$) at the end of the upper branch of the WEIBULL–distribution–line,

and by a line corresponding to the **lognormal distribution** which completely falls into the type–VI region.

When β_2 is plotted over β_1 for a WEIBULL distribution we get a parametric function depending on the shape parameter c. This function has a vertex at $(\beta_1, \beta_2) \approx (0, 2.72)$

corresponding to $c \approx 3.6023$ with a finite upper branch (valid for $c > 3.6023$) ending at $(\beta_1, \beta_2) \approx (1.2928, 5.4)$ and an infinite lower branch (valid for $c < 3.6023$). One easily sees that the WEIBULL distribution does not belong to only one family of the PEARSON system. For $c < 3.6023$ the WEIBULL distribution lies mainly in the type–I region and extends approximately parallel to the type–III (gamma) line until the two lines intersect at $(\beta_1, \beta_2) = (4.0, 9.0)$ corresponding to the exponential distribution (= WEIBULL distribution with $c = 1$). The WEIBULL line for $c > 3.6023$ originates in the type–I region and extends approximately parallel to the type–V line. It crosses the type–III line into the type–VI region at a point with $\beta_1 \approx 0.5$ and then moves toward the lognormal line, ending on that line at a point which marks the type–I extreme value distribution. Hence, the WEIBULL distribution with $c > 3.6023$ will closely resemble the PEARSON type–VI distribution when $\beta_1 \geq 0.5$, and for β_1 approximately greater than 1.0 it will closely resemble the lognormal distribution.

3.1.2 BURR system[2]

The **BURR system** (see BURR (1942)) fits cumulative distribution functions, rather than density functions, to frequency data, thus avoiding the problems of numerical integration which are encountered when probabilities or percentiles are evaluated from PEARSON curves. A CDF $y := F(x)$ in the BURR system has to satisfy the differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y\,(1-y)\;g(x,y), \quad y := F(x), \tag{3.2}$$

an analogue to the differential equation (3.1) that generates the PEARSON system. The function $g(x,y)$ must be positive for $0 \leq y \leq 1$ and x in the support of $F(x)$. Different choices of $g(x,y)$ generate various solutions $F(x)$. These can be classified by their functional forms, each of which gives rise to a family of CDFs within the BURR system. BURR listed twelve such families.

With respect to the WEIBULL distribution the BURR type–XII distribution is of special interest. Its CDF is

$$F(x) = 1 - \frac{1}{(1+x^c)^k}\,; \quad x, c, k > 0, \tag{3.3a}$$

with DF

$$f(x) = k\,c\,x^{c-1}\,(1+x^c)^{-k-1} \tag{3.3b}$$

and moments about the origin given by

$$\mathrm{E}\big(X^r\big) = \mu'_r = k\,\Gamma\Big(k - \frac{r}{c}\Big)\;\Gamma\Big(\frac{r}{c}+1\Big)\Big/\,\Gamma(k+1) \;\text{ for }\; c\,k > r. \tag{3.3c}$$

It is required that $c\,k > 4$ for the fourth moment, and thus β_2, to exist. This family gives rise to a useful range of value of skewness, $\alpha_3 = \pm\sqrt{\beta_1}$, and kurtosis, $\alpha_4 = \beta_2$. It may be generalized by introducing a location parameter and a scale parameter.

[2] Suggested reading for this section: RODRIGUEZ (1977), TADIKAMALLA (1980a).

Whereas the third and fourth moment combinations of the PEARSON families do not overlap[3] we have an overlapping when looking at the BURR families. For depicting the type–XII family we use another type of moment–ratio diagram with $\alpha_3 = \pm\sqrt{\beta_1}$ as abscissa, thus showing positive as well as negative skewness, and furthermore it is upside down. Thus the upper bound in Fig. 3/2 is referred to as "lower bound" and conversely in the following text. The parametric equations for $\sqrt{\beta_1}$ and β_2 are

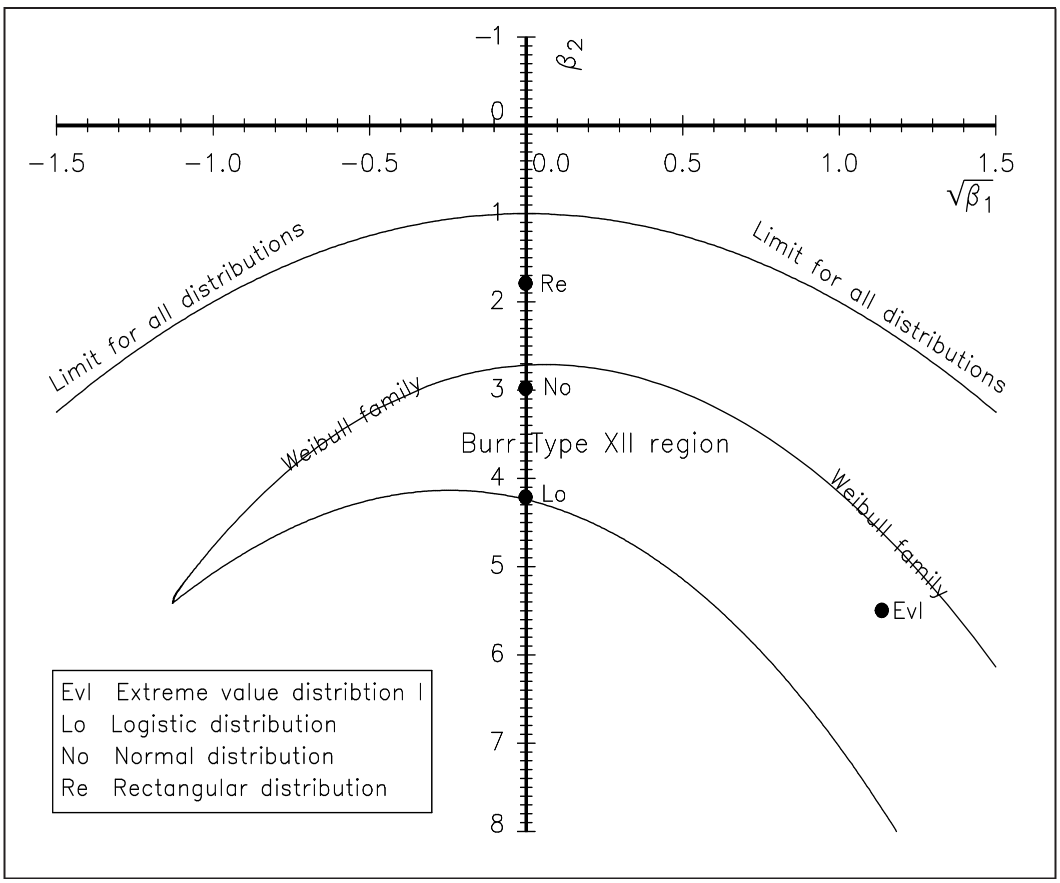
$$\sqrt{\beta_1} = \frac{\Gamma^2(k)\,\lambda_3 - 3\,\Gamma(k)\,\lambda_2\,\lambda_1 + 2\,\lambda_1^3}{\left[\Gamma(k)\,\lambda_2 - \lambda_1^2\right]^{3/2}} \tag{3.3d}$$

$$\beta_2 = \frac{\Gamma^3(k)\,\lambda_4 - 4\,\Gamma^2(k)\,\lambda_3\,\lambda_1 + 6\,\Gamma(k)\,\lambda_2\,\lambda_1^2 - 3\,\lambda_1^4}{\left[\Gamma(k)\,\lambda_2 - \lambda_1^2\right]^{3/2}} \tag{3.3e}$$

where

$$\lambda_j := \Gamma\left(\frac{j}{c} + 1\right)\,\Gamma\left(k - \frac{j}{c}\right)\;;\;\; j = 1, 2, 3, 4.$$

Figure 3/2: Moment ratio diagram for the BURR type-XII family and the WEIBULL distribution



[3] The same is true for the JOHNSON families (see Sect. 3.1.3).

RODRIGUEZ states: "The type–XII BURR distributions occupy a region shaped like the prow of an ancient Roman galley. As k varies from $4/c$ to $+\infty$ the c–constant curve moves in a counter–clockwise direction toward the tip of the 'prow'. For $c < 3.6$ the c–constant lines terminate at end–points with positive $\sqrt{\beta_1}$. For $c > 3.6$ the c–constant lines terminate at end–points with negative $\sqrt{\beta_1}$." These end–points form the **lower bound** of the type–XII region. RODRIGUEZ (1977) gives the following parametric equations of the end–points:

$$\lim_{k\to\infty} \sqrt{\beta_1} = \frac{\Gamma_3 - 3\,\Gamma_2\,\Gamma_1 + 2\,\Gamma_1^3}{\left(\Gamma_2 - \Gamma_1^2\right)^{3/2}}, \tag{3.4a}$$

$$\lim_{k\to\infty} \beta_2 = \frac{\Gamma_4 - 4\,\Gamma_3\,\Gamma_1 + 6\,\Gamma_2\,\Gamma_1 - 3\,\Gamma_1^4}{\left(\Gamma_2 - \Gamma_1^2\right)^{2}}, \tag{3.4b}$$

where

$$\Gamma_i := \Gamma\left(1 + \frac{i}{c}\right).$$

This lower bound is identical to the WEIBULL curve in the $\left(\sqrt{\beta_1}, \beta_2\right)$–plane; compare (3.4a,b) to (2.93) and (2.101). The identification of the lower bound with the WEIBULL family can also be explained as follows, starting from (3.3a):

$$\begin{aligned}
\Pr\left[X \le \left(\frac{1}{k}\right)^{1/c} y\right] &= 1 - \left(1 + \frac{y^c}{k}\right)^{-k} \\
&= 1 - \exp\left\{-k\,\ln\left(1 + \frac{y^c}{k}\right)\right\} \\
&= 1 - \exp\left\{-k\left[\frac{y^c}{k} - \frac{1}{2}\left(\frac{y^c}{k}\right)^2 + \ldots\right]\right\} \\
&\Rightarrow 1 - \exp\left(-\,y^c\right) \ \text{ for } \ k \to \infty.
\end{aligned}$$

Hence, the WEIBULL family is the limiting form $(k \to \infty)$ of the BURR type–XII family.

We finally mention the **upper bound** for the BURR type–XII region. In the negative $\sqrt{\beta_1}$ half–plane the bound is given for $c = \infty$ stretching from the point $\left(\sqrt{\beta_1}, \beta_2\right) \approx (-1.14, 5.4)$ to $\left(\sqrt{\beta_1}, \beta_2\right) = (0.4, 2)$ which is associated with the **logistic distribution**. In the positive $\sqrt{\beta_1}$ half–plain this bound corresponds to BURR type–XII distributions for which $k = 1$ and $c > 4$ has been proved by RODRIGUEZ (1977).

3.1.3 JOHNSON system

The transformation of a variate to normality is the basis of the JOHNSON system. By analogy with the PEARSON system, it would be convenient if a simple transformation could be found such that, for any possible pair of values $\sqrt{\beta_1}, \beta_2$, there will be just one member of the corresponding family of distributions. No such single simple transformation is available, but JOHNSON (1949) has found sets of three such transformations that, when combined, do provide one distribution corresponding to each pair of values $\sqrt{\beta_1}$ and β_2.

The advantage with JOHNSON's transformation is that, when inverted and applied to a normally distributed variable, it yields three families of density curves with a high degree of shape flexibility.

Let T be a standardized normal variate, i.e., having $\mathrm{E}(T) = 0$ and $\mathrm{Var}(T) = 1$, then the system is defined by

$$T = \gamma + \delta\, g(Y). \tag{3.5a}$$

Taking

$$\begin{aligned} g(Y) &= \ln\Big[Y + \sqrt{(1+Y^2)}\Big] \\ &= \sinh^{-1} Y \end{aligned} \tag{3.5b}$$

leads to the **S_U system with unbounded range**: $-\infty < Y < \infty$.

$$g(Y) = \ln Y \tag{3.5c}$$

leads to the **lognormal family S_L** with $Y > 0$. The third **system S_B with bounded range** $0 < Y < 1$ rests upon the transformation

$$g(Y) = \ln\left(\frac{Y}{1-Y}\right). \tag{3.5d}$$

The variate Y is linearly related to a variate X that we wish to approximate in distribution:

$$Y = \frac{X - \xi}{\lambda}. \tag{3.5e}$$

The density of Y in the S_U system is given by

$$f(y) = \frac{\delta}{\sqrt{2\,\pi}}\,\frac{1}{\sqrt{1+y^2}}\,\exp\left\{-\frac{\big[\gamma + \delta\,\sinh^{-1}\,y\big]^2}{2}\right\}, \quad y \in \mathbb{R}, \tag{3.6a}$$

with

$$\sqrt{\beta_1} = -\left[\frac{1}{2}\,\omega\,(\omega-1)\right]^{1/2} A^{-3/2}\left[\omega\,(\omega+2)\,\sinh(3\,B) + 3\,\sinh B\right] \tag{3.6b}$$

$$\beta_2 = \left[a_2\,\cosh(4\,B) + a_1\,\cosh(2\,B) + a_0\right]\Big/(2\,B^2), \tag{3.6c}$$

where

$$\begin{aligned} \omega &:= \exp(1/\delta^2), \quad B := \gamma/\delta, \quad A := \omega\,\cosh(2\,B) + 1, \\ a_2 &:= \omega^2\left(\omega^4 + 2\,\omega^3 + 3\,\omega^2 - 3\right), \;\; a_1 := 4\,\omega^2\,(\omega+2), \;\; a_0 := 3\,(2\,\omega+1). \end{aligned}$$

In the S_L system the density is that of a lognormal variate:

$$f(y) = \frac{\delta}{\sqrt{2\,\pi}}\,\frac{1}{y}\,\exp\left\{-\frac{[\gamma + \delta\,\ln y]^2}{2}\right\}, \quad y > 0, \tag{3.7a}$$

with

$$\sqrt{\beta_1} = (\omega - 1)\,(\omega + 2)^2 \tag{3.7b}$$

$$\beta_2 = \omega^4 + 2\,\omega^3 + 3\,\omega^2 - 3. \tag{3.7c}$$

The density in the S_B system is given by

$$f(y) = \frac{\delta}{\sqrt{2\,\pi}}\,\frac{1}{y\,(1-y)}\,\exp\left\{-\frac{\big(\gamma + \delta\,\ln[y/(1-y)]\big)^2}{2}\right\},\quad 0 < y < 1, \tag{3.8}$$

and there exist no general and explicit formulas for the moments.

Looking at a moment–ratio diagram (see Fig. 3/1) one sees that the system S_B holds for the region bounded by $\beta_1 = 0$, the bottom line $\beta_2 - \beta_1 - 1 = 0$ and the lognormal curve, given by S_L. S_U holds for the corresponding region above S_L. In relation to the PEARSON system, S_B overlaps types I, II, III and part of VI; similarly S_U overlaps types IV, V, VII and part of VI. As for shapes of the density, S_U is unimodal, but S_B may be bimodal. The WEIBULL curve is completely located in the S_L region, thus telling that the WEIBULL distribution is a member of the JOHNSON S_L system.

3.1.4 Miscellaneous

There exists a great number of further families. Some of them are based on transformations like the JOHNSON system, e.g., the TUKEY's lambda distributions for a variate X and starting from the reduced uniform variable Y with DF

$$f(y) = 1 \quad \text{for} \quad 0 < y < 1$$

and defined by

$$X := \begin{cases} \dfrac{Y^\lambda - (1-Y)^\lambda}{\lambda} & \text{for} \quad \lambda \neq 0, \\[2ex] \ln\left(\dfrac{Y}{1-Y}\right) & \text{for} \quad \lambda = 0. \end{cases}$$

Other families are based on expansions e.g., the GRAM–CHARLIER series, the EDGEWORTH series and the CORNISH–FISHER expansions.

Last but not least, we have many families consisting of two members only. This is a dichotomous classification, where one class has a special property and the other class is missing that property. With respect to the WEIBULL distribution we will present and investigate

- location–scale distributions,
- stable distributions,
- ID distributions and
- the exponential family of distributions.

A variate X belongs to the **location–scale family** if its CDF $F_X(\cdot)$ may be written as

$$F_X(x \,|\, a, b) = F_Y\left(\frac{x-a}{b}\right) \tag{3.9}$$

and the CDF $F_Y(\cdot)$ does not depend on any parameter. a and b are called location parameter and scale parameter, respectively. $Y := (X - a)\big/ b$ is termed the **reduced variable** and $F_Y(y)$ is the reduced CDF with $a = 0$ and $b = 1$. The three–parameter WEIBULL distribution given by (2.8) evidently does not belong to this family, unless $c = 1$; i.e., we have an exponential distribution (see Sect. 3.2.1). Sometimes a suitable transformation $Z := g(X)$ has a location–scale distribution. This is the case when X has a two–parameter WEIBULL distribution with $a = 0$:

$$F(x) = 1 - \exp\left\{ - \left(\frac{x}{b} \right)^c \right\}. \tag{3.10a}$$

The log–transformed variate

$$Z := \ln X$$

has a type–I extreme value distribution of the minimum (see Sect. 3.2.2) with CDF

$$F(z) = 1 - \exp\left\{ - \exp\left(\frac{z - a^*}{b^*} \right) \right\}, \tag{3.10b}$$

where

$$a^* = \ln b \quad \text{and} \quad b^* = 1/c.$$

This transformation will be of importance when making inference on the parameters b and c (see Chapters 9 ff).

Let $X, X_1, X_2, \ldots$ be independent identically distributed variates. The distribution of X is **stable in the broad sense** if it is not concentrated at one point and if for each $n \in \mathbb{N}$ there exist constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that $(X_1 + X_2 + \ldots + X_n)/a_n - b_n$ has the same distribution as X. If the above holds with $b_n = 0$ for all n then the distribution is **stable in the strict sense**.[4] For every stable distribution we have $a_n = n^{1/\alpha}$ for some characteristic exponent α with $0 < \alpha \leq 2$. The family of GAUSSIAN distributions is the unique family of distributions that are stable with $\alpha = 2$. CAUCHY distributions are stable with $\alpha = 1$. The WEIBULL distribution is not stable, neither in the broad sense nor in the strict sense.

A random variable X is called **infinitely divisible distributed** ($\widehat{=}$ IDD) if for each $n \in \mathbb{N}$, independent identically distributed ($\widehat{=}$ iid) random variables $X_{n,1},\ X_{n,2},\ \ldots,\ X_{n,n}$ exist such that

$$X \stackrel{\text{d}}{=} X_{n,1} + X_{n,2} + \ldots + X_{n,n},$$

where $\stackrel{\text{d}}{=}$ denotes "equality in distribution." Equivalently, denoting the CDFs of X and $X_{n,1}$ by $F(\cdot)$ and $F_n(\cdot)$, respectively, one has

$$F(\cdot) = F_n(\cdot) * F_n(\cdot) * \ldots * F_n(\cdot) =: F_n^*(\cdot)$$

[4] Stable distributions are mainly used to model certain random economic phenomena, especially in finance, having distributions with "fat tails," thus indicating the possibility of an infinite variance.

where the operator $*$ denotes "convolution." All stable distributions are IDD. The WEIBULL distribution is not a member of the IDD family.

A concept related to IDD is **reproductivity through summation**, meaning that the CDF of a sum of n iid random variables X_i $(i = 1, \ldots, n)$ belongs to the same distribution family as each of the terms in the sum. The WEIBULL distribution is not reproductive through summation, but it is **reproductive through formation of the minimum** of n iid WEIBULL random variables.

Theorem: Let $X_i \overset{\text{iid}}{\sim} We(a, b, c)$ for $i = 1, \ldots, n$ and $Y = \min(X_1, \ldots, X_n)$, then $Y \sim We(a, b\,n^{-1/c}, c)$.

Proof: Inserting $F(x)$, given in (2.33b), into the general formula (1.3a)

$$F_Y(y) = 1 - \big[1 - F_X(y)\big]^n$$

gives

$$\begin{aligned} F_Y(y) &= 1 - \left[\exp\left\{-\left(\frac{y-a}{b}\right)^c\right\}\right]^n \\ &= 1 - \exp\left\{-n\left(\frac{y-a}{b}\right)^c\right\} \\ &= 1 - \exp\left\{-\left(\frac{y-a}{b\,n^{-1/c}}\right)^c\right\} \Rightarrow We\big(a, b\,n^{-1/c}, c\big). \end{aligned}$$

Thus the scale parameter changes from b to $b\big/\sqrt[c]{n}$ and the location and form parameters remain unchanged. ■

The **exponential family** of continuous distributions is characterized by having a density function of the form

$$f(x \,|\, \theta) = A(\theta)\, B(x)\, \exp\big\{Q(\theta) \circ T(x)\big\}, \tag{3.11a}$$

where θ is a parameter (both x and θ may, of course, be multidimensional), $Q(\theta)$ and $T(x)$ are vectors of common dimension, m, and the operator $\circ$ denotes the "inner product," i.e.,

$$Q(\theta) \circ T(x) = \sum_{j=1}^{m} Q_j(\theta)\, T_j(x). \tag{3.11b}$$

The exponential family plays an important role in estimating and testing the parameters $\theta_1, \ldots, \theta_m$ because we have the following.

Theorem: Let $X_1, \ldots, X_n$ be a random sample of size n from the density $f(x \,|\, \theta_1, \ldots, \theta_m)$ of an exponential family and let

$$S_j = \sum_{i=1}^{n} T_j(X_i).$$

Then $S_1, \ldots, S_m$ are jointly **sufficient** and **complete** for $\theta_1, \ldots, \theta_m$ when $n > m$. ■

Unfortunately the three–parameter WEIBULL distribution is not exponential; thus, the principle of sufficient statistics is not helpful in reducing WEIBULL sample data. Sometimes a distribution family is not exponential in its entirety, but certain subfamilies obtained by fixing one or more of its parameters are exponential. When the location parameter a and the form parameter c are both known and fixed then the WEIBULL distribution is exponential with respect to the scale parameter.

3.2 WEIBULL distributions and other familiar distributions

Statistical distributions exist in a great number as may be seen from the comprehensive documentation of JOHNSON/KOTZ (1992, 1994, 1995) et al. consisting of five volumes in its second edition. The relationships between the distributions are as manifold as are the family relationships between the European dynasties. One may easily overlook a relationship and so we apologize should we have forgotten any distributional relationship, involving the WEIBULL distribution.

3.2.1 WEIBULL and exponential distributions

We start with perhaps the most simple relationship between the WEIBULL distribution and any other distribution, namely the exponential distribution. Looking to the origins of the WEIBULL distribution in practice (see Section 1.1.2) we recognize that WEIBULL as well as ROSIN, RAMMLER and SPERLING (1933) did nothing but adding a third parameter to the two–parameter exponential distribution (= generalized exponential distribution) with DF

$$f(y\,|\,a,b) = \frac{1}{b}\,\exp\!\left[-\frac{y-a}{b}\right];\;\; y \geq a,\; a \in \mathbb{R},\; b > 0. \tag{3.12a}$$

Now, let

$$Y = \left(\frac{X-a}{b}\right)^{c}, \tag{3.12b}$$

and we have the reduced exponential distribution $(a = 0,\; b = 1)$ with density

$$f(y) = e^{-y},\;\; y > 0, \tag{3.12c}$$

then X has a WEIBULL distribution with DF

$$f(x\,|\,a,b,c) = \frac{c}{b}\left(\frac{x-a}{b}\right)^{c-1}\exp\!\left[-\left(\frac{x-a}{b}\right)^{c}\right];\;\; x \geq a,\; a \in \mathbb{R},\; b,\; c > 0. \tag{3.12d}$$

The transformation (3.12b) is referred to as the **power–law transformation**. Comparing (3.12a) with (3.12d) one recognizes that the exponential distribution is a special case (with $c = 1$) of the WEIBULL distribution.

3.2.2 WEIBULL and extreme value distributions

In Sect. 1.1.1 we reported on the discovery of the three types of extreme value distributions. Introducing a location parameter a $(a \in \mathbb{R})$ and a scale parameter b $(b > 0)$ into the functions of Table 1/1, we arrive at the following general formulas for extreme value distributions:

Type–I–maximum distribution:[5] $Y \sim Ev_I(a,b)$

$$f_I^M(y) = \frac{1}{b}\exp\left\{-\frac{y-a}{b} - \exp\left[-\frac{y-a}{b}\right]\right\}, \; y \in \mathbb{R}; \tag{3.13a}$$

$$F_I^M(y) = \exp\left\{-\exp\left[-\frac{y-a}{b}\right]\right\}; \tag{3.13b}$$

Type–II–maximum distribution: $Y \sim Ev_{II}(a,b,c)$

$$f_{II}^M(y) = \frac{c}{b}\left(\frac{y-a}{b}\right)^{-c-1}\exp\left\{-\left(\frac{y-a}{b}\right)^{-c}\right\}, \; y \ge a, \; c > 0; \tag{3.14a}$$

$$F_{II}^M(y) = \left\{\begin{array}{cl} 0 & \text{for} \;\; y < a \\ \exp\left\{-\left(\dfrac{y-a}{b}\right)^{-c}\right\} & \text{for} \;\; y \ge a \end{array}\right\}; \tag{3.14b}$$

Type–III–maximum distribution: $Y \sim Ev_{III}(a,b,c)$

$$f_{III}^M(y) = \frac{c}{b}\left(\frac{a-y}{b}\right)^{c-1}\exp\left\{-\left(\frac{a-y}{b}\right)^{c}\right\}, \; y < a, \; c > 0; \tag{3.15a}$$

$$F_{III}^M(y) = \left\{\begin{array}{cl} \exp\left\{-\left(\dfrac{a-y}{b}\right)^{c}\right\} & \text{for} \;\; y < a \\ 1 & \text{for} \;\; y \ge a \end{array}\right\}. \tag{3.15b}$$

The corresponding distributions of

$$X - a = -(Y - a) \tag{3.16}$$

are those of the minimum and are given by the following:

Type–I–minimum distribution: $X \sim Ev_i(a,b)$ or $X \sim Lw(a,b)$

$$f_I^m(x) = \frac{1}{b}\exp\left\{\frac{x-a}{b} - \exp\left[\frac{x-a}{b}\right]\right\}, \; x \in \mathbb{R}; \tag{3.17a}$$

$$F_I^m(x) = 1 - \exp\left\{-\exp\left[\frac{x-a}{b}\right]\right\}; \tag{3.17b}$$

[5] On account of its functional form this distribution is also sometimes called **doubly exponential distribution**.

Type–II–minimum distribution: $X \sim Ev_{ii}(a,b,c)$

$$f_{II}^m(x) = \frac{c}{b}\left(\frac{a-x}{b}\right)^{-c-1}\exp\left\{-\left(\frac{a-x}{b}\right)^{-c}\right\},\ x<a,\ c>0; \quad (3.18a)$$

$$F_{II}^m(x) = \left\{\begin{array}{cl} 1-\exp\left\{-\left(\dfrac{a-x}{b}\right)^{-c}\right\} & \text{for}\quad x<a \\ 1 & \text{for}\quad x\geq a \end{array}\right\}; \quad (3.18b)$$

Type–III–minimum distribution or WEIBULL distribution:
$X \sim Ev_{iii}(a,b,c)$ or $X \sim We(a,b,c)$

$$f_{III}^m(x) = \frac{c}{b}\left(\frac{x-a}{b}\right)^{c-1}\exp\left\{-\left(\frac{x-a}{b}\right)^{c}\right\},\ x\geq a,\ c>0; \quad (3.19a)$$

$$F_{III}^M(x) = \left\{\begin{array}{cl} 0 & \text{for}\quad x<a \\ 1-\exp\left\{-\left(\dfrac{x-a}{b}\right)^{c}\right\} & \text{for}\quad x\geq a \end{array}\right\}. \quad (3.19b)$$

The transformation given in (3.16) means a reflection about a vertical axis at $y = x = a$, so the type–III–maximum distribution (3.15a/b) is the **reflected WEIBULL distribution**, which will be analyzed in Sect. 3.3.2.

The type–II and type–III distributions can be transformed to a type–I distribution using a suitable logarithmic transformation. Starting from the WEIBULL distribution (3.19a,b), we set

$$Z = \ln(X-a),\ \ X\geq a, \quad (3.20)$$

and thus transform (3.19a,b) to

$$f(z) = c\exp\big\{c\,(z-\ln b)-\exp\big[c\,(z-\ln b)\big]\big\} \quad (3.21a)$$

$$F(z) = 1-\exp\big\{-\exp\big[c\,(z-\ln b)\big]\big\}. \quad (3.21b)$$

This is a type–I–minimum distribution which has a location parameter $a* = \ln b$ and a scale parameter $b^* = 1/c$. That is why a type–I–minimum distribution is called a **Log–WEIBULL distribution**;[6] it will be analyzed in Sect. 3.3.4.

We finally mention a third transformation of a WEIBULL variate, which leads to another member of the extreme value class. We make the following reciprocal transformation to a WEIBULL distributed variate X:

$$Z = \frac{b^2}{X-a}\,. \quad (3.22)$$

[6] We have an analogue relationship between the normal and the lognormal distributions.

Applying the well–known rules of finding the DF and CDF of a transformed variable, we arrive at

$$f(z) = \frac{c}{b}\left(\frac{z}{b}\right)^{-c-1} \exp\left\{-\left(\frac{z}{b}\right)^{-c}\right\}, \tag{3.23a}$$

$$F(z) = \exp\left\{-\left(\frac{z}{b}\right)^{-c}\right\}. \tag{3.23b}$$

Comparing (3.23a,b) with (3.14a,b) we see that Z has a type–II–maximum distribution with a zero location parameter. Thus a type–II–maximum distribution may be called an **inverse WEIBULL distribution**, it will be analyzed in Sect. 3.3.3.

3.2.3 WEIBULL and gamma distributions[7]

In the last section we have seen several relatives of the WEIBULL distributions originating in some kind of transformation of the WEIBULL variate. Here, we will present a class of distributions, the gamma family, which includes the WEIBULL distribution as a special case for distinct parameter values.

The **reduced form** of a gamma distribution with only **one parameter** has the DF:

$$f(x\,|\,d) = \frac{x^{d-1}\,\exp(-x)}{\Gamma(d)}\,;\;\; x \ge 0,\; d > 0. \tag{3.24a}$$

d is a shape parameter. $d = 1$ results in the reduced exponential distribution. Introducing a scale parameter b into (3.24a) gives the **two–parameter gamma distribution**:

$$f(x\,|\,b,d) = \frac{x^{d-1}\,\exp(-x/b)}{b^d\,\Gamma(d)}\,;\;\; x \ge 0;\; b,d > 0. \tag{3.24b}$$

The **three–parameter gamma distribution** has an additional location parameter a:

$$f(x\,|\,a,b,d) = \frac{(x-a)^{d-1}\,\exp\left(-\dfrac{x-a}{b}\right)}{b^d\,\Gamma(d)}\,;\;\; x \ge a,\; a \in \mathbb{R},\; b,d > 0. \tag{3.24c}$$

STACY (1965) introduced a second shape parameter c $(c > 0)$ into the two–parameter gamma distribution (3.24b). When we introduce this second shape parameter into the three–parameter version (3.24c), we arrive at the **four–parameter gamma distribution**, also named **generalized gamma distribution**,[8]

$$f(x\,|\,a,b,c,d) = \frac{c\,(x-a)^{c\,d-1}}{b^{c\,d}\,\Gamma(d)}\,\exp\left\{-\left(\frac{x-a}{b}\right)^c\right\};\;\; x \ge a;\; a \in \mathbb{R};\; b,\; c,\; d > 0. \tag{3.24d}$$

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[7] Suggested reading for this section: HAGER/BAIN/ANTLE (1971), PARR/WEBSTER (1965), STACY (1962), STACY/MIHRAM (1965).

[8] A further generalization, introduced by STACY/MIHRAN (1965), allows $c < 0$, whereby the factor c in $c\,(x-a)^{c\,d-1}$ has to be changed against $|c|$.

The DF (3.24d) contains a number of familiar distributions when $b,\ c$ and/or d are given special values (see Tab. 3/1).

Table 3/1: Special cases of the generalized gamma distribution

$f(x\,\vert\,a,b,c,d)$	Name of the distribution
$f(x\,\vert\,0,1,1,1)$	reduced exponential distribution
$f(x\,\vert\,a,b,1,1)$	two–parameter exponential distribution
$f(x\,\vert\,a,b,1,\nu)$	ERLANG distribution, $\nu \in \mathbb{N}$
$f(x\,\vert\,0,1,c,1)$	reduced WEIBULL distribution
$f(x\,\vert\,a,b,c,1)$	three–parameter WEIBULL distribution
$f\Big(x\,\vert\,0,2,1,\dfrac{\nu}{2}\Big)$	χ^2–distribution with ν degrees of freedom, $\nu \in \mathbb{N}$
$f\Big(x\,\vert\,0,\sqrt{2},2,\dfrac{\nu}{2}\Big)$	χ–distribution with ν degrees of freedom, $\nu \in \mathbb{N}$
$f\Big(x\,\vert\,0,\sqrt{2},2,\dfrac{1}{2}\Big)$	half–normal distribution
$f(x\,\vert\,0,\sqrt{2},2,1)$	circular normal distribution
$f(x\,\vert\,a,b,2,1)$	RAYLEIGH distribution

LIEBSCHER (1967) compares the two–parameter gamma distribution and the lognormal distribution to the WEIBULL distribution with $a = 0$ and gives conditions on the parameters of these distributions resulting in a stochastic ordering.

3.2.4 WEIBULL and normal distributions[9]

The relationships mentioned so far are exact whereas the relation of the WEIBULL distribution to the normal distribution only holds approximately. The quality of approximation depends on the criteria which will be chosen in equating these distributions. In Sect. 2.9.4 we have seen that there exist values of the shape parameter c leading to a skewness of zero and a kurtosis of three which are typical for the normal distribution. Depending on how skewness is measured we have different values of c giving a value of zero for the measure of skewness chosen:

- $c \approx 3.60235$ for $\alpha_3 = 0$,
- $c \approx 3.43954$ for $\mu = x_{0.5}$, i.e., for mean = median,
- $c \approx 3.31247$ for $\mu = x^*$, i.e., for mean = mode,
- $c \approx 3.25889$ for $x^* = x_{0.5}$, i.e., for mode = median.

[9] Suggested for this section: DUBEY (1967a), MAKINO (1984).

Regarding the kurtosis α_4, we have two values of c ($c \approx 2.25200$ and $c \approx 5.77278$) giving $\alpha_4 = 3$.

MAKINO (1984) suggests to base the approximation on the mean hazard rate $\mathrm{E}\big[(h(X)\big]$; see (2.53a). The standardized normal and WEIBULL distributions have the same mean hazard rate $\mathrm{E}\big[h(X)\big] = 0.90486$ when $c \approx 3.43927$, which is nearly the value of the shape parameter such that the mean is equal to the median.

In the sequel we will look at the closeness of the CDF of the standard normal distribution

$$\Phi(\tau) = \int\limits_{-\infty}^{\tau} \frac{1}{\sqrt{2\,\pi}}\, e^{-t^2/2}\,\mathrm{d}t \tag{3.25a}$$

to the CDF of the standardized WEIBULL variate

$$T = \frac{X-\mu}{\sigma} = \frac{X-\big(a+b\,\Gamma_1\big)}{b\,\sqrt{\Gamma_2-\Gamma_1^2}}\,,\ \ \Gamma_i := \Gamma\left(1+\frac{i}{c}\right),$$

given by

$$\begin{aligned} F_W(\tau) &= \Pr\left(\frac{X-\mu}{\sigma} \le \tau\right) = \Pr\big(X \le \mu+\sigma\,\tau\big) \\ &= 1-\exp\left[-\left(\frac{\mu+\sigma\,\tau-a}{b}\right)^c\right] \\ &= 1-\exp\left[-\left(\Gamma_1+\tau\,\sqrt{\Gamma_2-\Gamma_1^2}\right)^c\right], \end{aligned} \tag{3.25b}$$

which is only dependent on the shape parameter c. In order to achieve $F_W(\tau) \ge 0$, the expression $\Gamma_1 + \tau\,\sqrt{\Gamma_2-\Gamma_1^2}$ has to be non–negative, i.e.,

$$\tau \ge -\frac{\Gamma_1}{\sqrt{\Gamma_2-\Gamma_1^2}}\,. \tag{3.25c}$$

The right-hand side of (3.25c) is the reciprocal of the coefficient of variation (2.88c) when $a = 0$ und $b = 1$.

We want to exploit (3.25b) in comparison with (3.25a) for the six values of c given in Tab. 3/2.

Table 3/2: Values of $c,\ \Gamma_1,\ \sqrt{\Gamma_2-\Gamma_1^2}$ and $\Gamma_1\big/\sqrt{\Gamma_2-\Gamma_1^2}$

c	Γ_1	$\sqrt{\Gamma_2-\Gamma_1^2}$	$\Gamma_1\big/\sqrt{\Gamma_2-\Gamma_1^2}$	Remark
2.25200	0.88574	0.41619	2.12819	$\alpha_4 \approx 0$
3.25889	0.89645	0.30249	2.96356	$x^* \approx x_{0.5}$
3.31247	0.89719	0.29834	3.00730	$\mu \approx x^*$
3.43954	0.89892	0.28897	3.11081	$\mu \approx x_{0.5}$
3.60235	0.90114	0.27787	3.24306	$\alpha_3 \approx 0$
5.77278	0.92573	0.18587	4.98046	$\alpha_4 \approx 0$

Using Tab. 3/2 the six WEIBULL CDFs, which will be compared with $\Phi(\tau)$ in Tab. 3/3, are given by

$$\begin{aligned}
F_W^{(1)}(\tau) &= 1-\exp\left[-(0.88574+0.41619\,\tau)^{2.25200}\right],\\
F_W^{(2)}(\tau) &= 1-\exp\left[-(0.89645+0.30249\,\tau)^{3.25889}\right],\\
F_W^{(3)}(\tau) &= 1-\exp\left[-(0.89719+0.29834\,\tau)^{3.31247}\right],\\
F_W^{(4)}(\tau) &= 1-\exp\left[-(0.89892+0.28897\,\tau)^{3.43954}\right],\\
F_W^{(5)}(\tau) &= 1-\exp\left[-(0.90114+0.27787\,\tau)^{3.60235}\right],\\
F_W^{(6)}(\tau) &= 1-\exp\left[-(0.92573+0.18587\,\tau)^{5.77278}\right].
\end{aligned}$$

Tab. 3/3 also gives the differences $\Delta^{(i)}(\tau) = F_W^{(i)}(\tau) - \Phi(\tau)$ for $i = 1,\ 2,\ \dots,\ 6$ and $\tau = -3.0\ (0.1)\ 3.0$. A look at Tab. 3/3 reveals the following facts:

- It is not advisable to base a WEIBULL approximation of the normal distribution on the equivalence of the kurtosis because this leads to the maximum errors of all six cases: $\left|\Delta^{(1)}(\tau)\right| = 0.0355$ for $c \approx 2.52200$ and $\left|\Delta^{(6)}(\tau)\right| = 0.0274$ for $c \approx 5.77278$.

- With respect to corresponding skewness (= zero) of both distributions, the errors are much smaller.

- The four cases of corresponding skewness have different performances.

  - We find that $c = 3.60235$ (attached to $\alpha_3 = 0$) yields the smallest maximum absolute difference $\left(\left|\Delta^{(5)}(\tau)\right| = 0.0079\right)$ followed by $\max\limits_{\tau}\left|\Delta^{(4)}(\tau)\right| = 0.0088$ for
    $c = 3.43945$, $\max\limits_{\tau}\left|\Delta^{(3)}(\tau)\right| = 0.0099$ for $c = 3.31247$ and $\max\limits_{\tau}\left|\Delta^{(2)}(\tau)\right| = 0.0105$ for $c = 3.25889$.
  - None of the four values of c with zero–skewness is uniformly better than any other. $c \approx 3.60235$ leads to the smallest absolute error for $-3.0 \le \tau \le -1.8$, $-1.0 \le \tau \le -0.1$ and $1.0 \le \tau \le 1.6$; $c \approx 3.25889$ is best for $-1.5 \le \tau \le -1.1$, $0.2 \le \tau \le 0.9$ and $2.1 \le \tau \le 3.0$; $c \approx 3.31247$ is best for $\tau = -1, 6$, $\tau = 0.1$ and $\tau = 1.9$; and, finally, $c \approx 3.43954$ gives the smallest absolute error for $\tau = -1.7$, $\tau = 0.0$ and $1.7 \le \tau \le 1.8$.
  - Generally, we may say, probabilities associated with the lower (upper) tail of a normal distribution can be approximated satisfactorily by using a WEIBULL distribution with the shape parameter $c \approx 3.60235$ ($c \approx 3.25889$).

Another relationship between the WEIBULL and normal distributions, based upon the χ^2–distribution, will be discussed in connection with the FAUCHON et al. (1976) extensions in Sections 3.3.7.1 and 3.3.7.2.

3.2.5 WEIBULL and further distributions

In the context of the generalized gamma distribution we mentioned its special cases. One of them is the **χ–distribution** with ν degrees of freedom which has the DF:

$$f(x \,|\, \nu) = \frac{2}{\sqrt{2}\,\Gamma\Big(\dfrac{\nu}{2}\Big)} \left(\frac{x}{\sqrt{2}}\right)^{\nu-1} \exp\left\{-\left(\frac{x}{\sqrt{2}}\right)^{2}\right\},\ \ x \geq 0,\ \nu \in \mathbb{N}. \tag{3.26a}$$

(3.26a) gives a WEIBULL distribution with $a = 0,\ b = \sqrt{2}$ and $c = 2$ for $\nu = 2$. Substituting the special scale factor $\sqrt{2}$ by a general scale parameter $b > 0$ leads to the following WEIBULL density:

$$f(x \,|\, 0, b, 2) = \frac{2}{b}\left(\frac{x}{b}\right) \exp\left\{-\left(\frac{x}{b}\right)^{2}\right\}, \tag{3.26b}$$

which is easily recognized as the density of a **RAYLEIGH distribution**.

Table 3/3: Values of $\Phi(\tau),\ F_W^{(i)}(\tau)$ and $\Delta^{(i)}(\tau)$ for $i = 1,\ 2,\ \ldots,\ 6$

τ	$\Phi(\tau)$	$F_W^{(1)}(\tau)$	$\Delta^{(1)}(\tau)$	$F_W^{(2)}(\tau)$	$\Delta^{(2)}(\tau)$	$F_W^{(3)}(\tau)$	$\Delta^{(3)}(\tau)$	$F_W^{(4)}(\tau)$	$\Delta^{(4)}(\tau)$	$F_W^{(5)}(\tau)$	$\Delta^{(5)}(\tau)$	$F_W^{(6)}(\tau)$	$\Delta^{(6)}(\tau)$
-3.0	.0013	–	–	–	–	.0000	$-.0013$	.0000	$-.0013$	.0001	$-.0013$	.0031	.0018
-2.9	.0019	–	–	.0000	$-.0019$	.0000	$-.0019$	.0001	$-.0018$	.0002	$-.0017$	.0041	.0023
-2.8	.0026	–	–	.0001	$-.0025$	.0001	$-.0025$	.0003	$-.0023$	.0005	$-.0020$	.0054	.0029
-2.7	.0035	–	–	.0003	$-.0032$	.0004	$-.0031$	.0007	$-.0028$	.0011	$-.0024$	.0070	.0036
-2.6	.0047	–	–	.0008	$-.0039$	.0009	$-.0037$	.0014	$-.0033$	.0020	$-.0026$	.0090	.0043
-2.5	.0062	–	–	.0017	$-.0046$	.0019	$-.0043$	.0026	$-.0036$	.0034	$-.0028$	.0114	.0052
-2.4	.0082	–	–	.0031	$-.0051$	.0035	$-.0047$	.0043	$-.0039$	.0053	$-.0028$	.0143	.0061
-2.3	.0107	–	–	.0053	$-.0054$	.0058	$-.0050$	.0068	$-.0040$	.0080	$-.0027$	.0178	.0070
-2.2	.0139	–	–	.0084	$-.0055$	.0089	$-.0050$	.0101	$-.0038$	.0115	$-.0024$	.0219	.0080
-2.1	.0179	.0000	$-.0178$	.0125	$-.0054$	.0131	$-.0048$	.0144	$-.0035$	.0159	$-.0019$	.0268	.0089
-2.0	.0228	.0014	$-.0214$	.0178	$-.0049$	.0185	$-.0043$	.0199	$-.0029$	.0215	$-.0013$	.0325	.0098
-1.9	.0287	.0050	$-.0237$	.0245	$-.0042$	.0252	$-.0035$	.0266	$-.0021$	.0283	$-.0004$	.0392	.0105
-1.8	.0359	.0112	$-.0247$	.0327	$-.0032$	.0334	$-.0025$	.0348	$-.0011$	.0365	.0006	.0470	.0110
-1.7	.0446	.0204	$-.0242$	.0426	$-.0020$	.0432	$-.0013$	.0446	.0001	.0462	.0017	.0559	.0113
-1.6	.0548	.0325	$-.0223$	.0543	$-.0005$	.0549	.0001	.0562	.0014	.0576	.0028	.0661	.0113
-1.5	.0668	.0476	$-.0192$	.0679	.0010	.0684	.0016	.0695	.0027	.0708	.0040	.0777	.0109
-1.4	.0808	.0657	$-.0150$	.0835	.0027	.0839	.0031	.0848	.0041	.0858	.0051	.0909	.0101

Table 3/3: Values of $\Phi(\tau)$, $F_W^{(i)}(\tau)$ and $\Delta^{(i)}(\tau)$ for $i = 1,\ 2,\ \ldots,\ 6$ (Continuation)

τ	$\Phi(\tau)$	$F_W^{(1)}(\tau)$	$\Delta^{(1)}(\tau)$	$F_W^{(2)}(\tau)$	$\Delta^{(2)}(\tau)$	$F_W^{(3)}(\tau)$	$\Delta^{(3)}(\tau)$	$F_W^{(4)}(\tau)$	$\Delta^{(4)}(\tau)$	$F_W^{(5)}(\tau)$	$\Delta^{(5)}(\tau)$	$F_W^{(6)}(\tau)$	$\Delta^{(6)}(\tau)$
−1.3	.0968	.0868	−.0100	.1012	.0044	.1015	.0047	.1022	.0054	.1029	.0061	.1057	.0089
−1.2	.1151	.1108	−.0043	.1210	.0060	.1212	.0062	.1216	.0065	.1220	.0069	.1223	.0072
−1.1	.1357	.1374	.0018	.1431	.0074	.1431	.0075	.1432	.0075	.1432	.0075	.1407	.0050
−1.0	.1587	.1666	.0079	.1673	.0087	.1672	.0085	.1669	.0082	.1665	.0078	.1611	.0024
−0.9	.1841	.1980	.0139	.1937	.0096	.1934	.0093	.1927	.0087	.1919	.0079	.1835	−.0006
−0.8	.2119	.2314	.0195	.2221	.0103	.2217	.0098	.2207	.0088	.2194	.0076	.2079	−.0039
−0.7	.2420	.2665	.0245	.2525	.0105	.2519	.0099	.2506	.0086	.2489	.0070	.2345	−.0075
−0.6	.2743	.3030	.0287	.2847	.0104	.2840	.0097	.2823	.0080	.2803	.0060	.2631	−.0112
−0.5	.3085	.3405	.0320	.3185	.0100	.3177	.0091	.3157	.0072	.3134	.0049	.2937	−.0148
−0.4	.3446	.3788	.0342	.3538	.0092	.3528	.0082	.3506	.0060	.3480	.0035	.3263	−.0182
−0.3	.3821	.4175	.0354	.3902	.0081	.3891	.0071	.3868	.0047	.3840	.0019	.3608	−.0213
−0.2	.4207	.4563	.0355	.4275	.0067	.4264	.0057	.4239	.0032	.4210	.0003	.3968	−.0239
−0.1	.4602	.4948	.0346	.4654	.0052	.4643	.0041	.4618	.0016	.4588	−.0014	.4343	−.0258
0.0	.5000	.5328	.0328	.5036	.0036	.5025	.0025	.5000	−.0000	.4971	−.0029	.4730	−.0270
0.1	.5398	.5699	.0301	.5417	.0019	.5407	.0009	.5383	−.0015	.5355	−.0043	.5125	−.0274
0.2	.5793	.6060	.0268	.5795	.0003	.5786	−.0007	.5763	−.0029	.5737	−.0055	.5524	−.0268
0.3	.6179	.6409	.0229	.6167	−.0012	.6158	−.0021	.6138	−.0041	.6114	−.0065	.5925	−.0254
0.4	.6554	.6742	.0188	.6528	−.0026	.6521	−.0033	.6503	−.0051	.6483	−.0071	.6323	−.0231
0.5	.6915	.7059	.0144	.6878	−.0037	.6871	−.0043	.6857	−.0058	.6840	−.0074	.6713	−.0201
0.6	.7257	.7358	.0101	.7212	−.0046	.7207	−.0051	.7196	−.0062	.7183	−.0075	.7092	−.0166
0.7	.7580	.7639	.0059	.7528	−.0052	.7525	−.0056	.7517	−.0063	.7508	−.0072	.7455	−.0125
0.8	.7881	.7901	.0020	.7826	−.0056	.7824	−.0058	.7819	−.0062	.7815	−.0067	.7799	−.0083
0.9	.8159	.8143	−.0016	.8102	−.0057	.8102	−.0057	.8101	−.0059	.8100	−.0059	.8119	−.0040
1.0	.8413	.8366	−.0047	.8357	−.0056	.8358	−.0055	.8360	−.0053	.8363	−.0050	.8415	.0001
1.1	.8643	.8570	−.0074	.8590	−.0053	.8592	−.0051	.8597	−.0047	.8603	−.0040	.8683	.0039
1.2	.8849	.8754	−.0095	.8800	−.0049	.8803	−.0046	.8810	−.0039	.8819	−.0030	.8921	.0072
1.3	.9032	.8921	−.0111	.8989	−.0043	.8992	−.0040	.9001	−.0031	.9012	−.0020	.9131	.0099

Table 3/3: Values of $\Phi(\tau)$, $F_W^{(i)}(\tau)$ and $\Delta^{(i)}(\tau)$ for $i = 1,\ 2,\ \dots,\ 6$ (Continuation)

τ	$\Phi(\tau)$	$F_W^{(1)}(\tau)$	$\Delta^{(1)}(\tau)$	$F_W^{(2)}(\tau)$	$\Delta^{(2)}(\tau)$	$F_W^{(3)}(\tau)$	$\Delta^{(3)}(\tau)$	$F_W^{(4)}(\tau)$	$\Delta^{(4)}(\tau)$	$F_W^{(5)}(\tau)$	$\Delta^{(5)}(\tau)$	$F_W^{(6)}(\tau)$	$\Delta^{(6)}(\tau)$
1.4	.9192	.9070	−.0122	.9155	−.0037	.9159	−.0033	.9169	−.0023	.9182	−.0010	.9312	.0120
1.5	.9332	.9203	−.0129	.9301	−.0031	.9306	−.0026	.9317	−.0015	.9330	−.0002	.9465	.0133
1.6	.9452	.9321	−.0131	.9427	−.0025	.9432	−.0020	.9444	−.0008	.9458	.0006	.9592	.0140
1.7	.9554	.9424	−.0130	.9535	−.0019	.9540	−.0014	.9552	−.0003	.9566	.0012	.9695	.0140
1.8	.9641	.9515	−.0126	.9627	−.0014	.9632	−.0009	.9643	.0002	.9657	.0016	.9777	.0136
1.9	.9713	.9593	−.0120	.9704	−.0009	.9708	−.0004	.9719	.0006	.9732	.0019	.9840	.0127
2.0	.9772	.9661	−.0112	.9767	−.0005	.9772	−.0001	.9781	.0009	.9793	.0021	.9889	.0116
2.1	.9821	.9719	−.0103	.9819	−.0002	.9823	.0002	.9832	.0011	.9843	.0021	.9924	.0103
2.2	.9861	.9768	−.0093	.9861	.0000	.9865	.0004	.9873	.0012	.9882	.0021	.9950	.0089
2.3	.9893	.9810	−.0083	.9895	.0002	.9898	.0005	.9905	.0012	.9913	.0020	.9968	.0075
2.4	.9918	.9845	−.0073	.9921	.0003	.9924	.0006	.9929	.0011	.9936	.0018	.9980	.0062
2.5	.9938	.9874	−.0064	.9941	.0004	.9944	.0006	.9949	.0011	.9954	.0016	.9988	.0050
2.6	.9953	.9899	−.0055	.9957	.0004	.9959	.0006	.9963	.0010	.9968	.0014	.9993	.0039
2.7	.9965	.9919	−.0046	.9969	.0004	.9971	.0005	.9974	.0008	.9977	.0012	.9996	.0031
2.8	.9974	.9935	−.0039	.9978	.0004	.9979	.0005	.9982	.0007	.9985	.0010	.9998	.0023
2.9	.9981	.9949	−.0033	.9985	.0003	.9985	.0004	.9987	.0006	.9990	.0008	.9999	.0017
3.0	.9987	.9960	−.0027	.9989	.0003	.9990	.0003	.9991	.0005	.9993	.0007	.9999	.0013

Excursus: RAYLEIGH distribution

This distribution was introduced by J.W. STRUTT (Lord RAYLEIGH) (1842 – 1919) in a problem of acoustics. Let $X_1,\ X_2,\ \dots,\ X_n$ be an iid sample of size n from a normal distribution with $\mathrm{E}(X_i) = 0\ \forall\ i$ and $\mathrm{Var}(X_i) = \sigma^2\ \forall\ i$. The density function of

$$Y = \sqrt{\sum_{i=1}^{n} X_i^2},$$

that is, the distance from the origin to a point $(X_1,\ \dots,\ X_n)$ in the n–dimensional EUCLIDEAN space, is

$$f(y) = \frac{2}{(2\,\sigma^2)^{n/2}\,\Gamma(n/2)}\,y^{n-1}\,\exp\left\{-\frac{y^2}{2\,\sigma^2}\right\};\ \ y > 0,\ \sigma > 0. \tag{3.27}$$

With $n = \nu$ and $\sigma = 1$, equation (3.27) is the χ–distribution (3.26a), and with $n = 2$ and $\sigma = b$, we have (3.26b).

The hazard rate belonging to the special WEIBULL distribution (3.26b) is

$$h(x \mid 0, b, 2) = \frac{2}{b}\,\frac{x}{b} = \frac{2}{b^2}\,x. \tag{3.28}$$

This is a linear hazard rate. Thus, the WEIBULL distribution with $c = 2$ is a member of the class of **polynomial hazard rate distributions**, the polynomial being of degree one.

Physical processes that involve nucleation and growth or relaxation phenomena have been analyzed by GITTUS (1967) to arrive at the class of distribution functions that should characterize the terminal state of replicates. He starts from the following differential equation for the CDF:

$$\frac{\mathrm{d}F(x)}{\mathrm{d}x} = K\,x^m\,\big[1 - F(x)\big]^n;\;\; K > 0,\; m > -1,\; n \geq 1. \tag{3.29a}$$

For $n = 1$ the solution is

$$F(x)_{m,1} = 1 - \exp\left\{-K\,\frac{x^{m+1}}{m+1}\right\}. \tag{3.29b}$$

(3.29b) is a WEIBULL distribution with $a = 0,\; b = \big[(m+1)/K\big]^{1/(m+1)}$ and $c = m + 1$. For $n > 1$ the solution is

$$F(x)_{m,n} = 1 - \left\{\frac{(n-1)\,K}{m+1}\,x^{m+1} + 1\right\}^{1/(1-n)}. \tag{3.29c}$$

The effects of $m,\; n$ and K on the form of the CDF, given by (3.29c), are as follows:

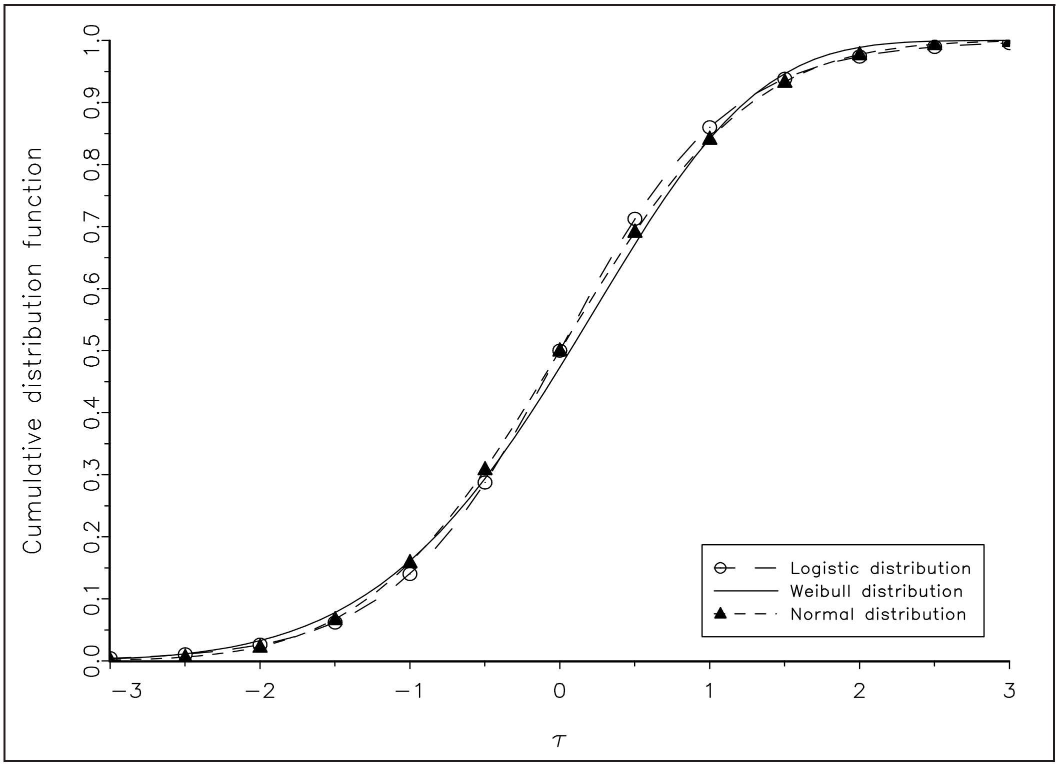
- Increasing the value of m increases the steepness of the central part of the curve.
- Increasing the value of K makes the curve more upright.
- Increasing the value of n reduces its height.

The normal and the logistic distributions are very similar, both being symmetric, but the logistic distribution has more kurtosis ($\alpha_4 = 4.2$). It seems obvious to approximate the **logistic distribution** by a symmetric version of the WEIBULL distribution. Comparing the CDF of the reduced logistic distribution

$$F(\tau) = \frac{1}{1 - \exp\big(-\pi\,\tau/\sqrt{3}\,\big)}$$

with the CDF of a reduced WEIBULL distribution in the symmetric cases $F_W^{(2)}(\tau)$ to $F_W^{(5)}(\tau)$ in Tab. 3/3 gives worse results than the approximation to the normal distribution. Fig. 3/3 shows the best fitting WEIBULL CDF ($c \approx 3.60235$ giving $\alpha_3 = 0$) in comparison with the logistic and normal CDFs.

Figure 3/3: CDFs of the standardized logistic, normal and WEIBULL distributions



3.3 Modifications of the WEIBULL distribution[10]

A great number of distributions have been developed that take the classical WEIBULL distribution (2.8) as their point of departure. Some of these modifications have been made in response to questions from practice while other generalizations have their origin in pure science. We hope that the grouping of the WEIBULL offshoots in the following ten subsections is neither overlapping nor incomplete.

3.3.1 Discrete WEIBULL distribution[11]

A main area of application for the WEIBULL distribution is lifetime research and reliability theory. In these fields we often encounter failure data measured as discrete variables such as number of work loads, blows, runs, cycles, shocks or revolutions. Sometimes a device is inspected once an hour, a week or a month whether it is still working or has failed in the meantime, thus leading to a lifetime counted in natural units of time. In this context, the **geometric** and the **negative binomial distributions** are known to be discrete alternatives for the exponential and gamma distributions, respectively. We are interested, from the

[10] Suggested reading for this section: MURTHY/XIE/JIANG (2004).

[11] Parameter estimation for discrete WEIBULL distributions is treated by ALI KHAN/KHALIQUE/ABOUAMMOH (1989).

viewpoints of both theory and practice, what discrete distribution might correspond to the WEIBULL distribution. The answer to this question is not unique, depending on what characteristic of the continuous WEIBULL distribution is to be preserved.

The **type–I discrete WEIBULL distribution**, introduced by NAKAGAWA/OSAKI (1975), retains the form of the continuous CDF. The **type–II discrete WEIBULL distribution**, suggested by STEIN/DATTERO (1984), retains the form of the continuous hazard rate. It is impossible to find a discrete WEIBULL distribution that mimics both the CDF and the HR of the continuous version in the sense that its CDF and its HR agree with those of the continuous WEIBULL distribution for integral values of the variate.

We will first present these two types and compare them to each other and to the continuous version. Finally, we mention another approach, that of PADGETT/SPURRIER (1985) and SALVIA (1996). This approach does not start from the continuous WEIBULL distribution but tries to generalize the notions of hazard rate and mean residual life to the discrete case. But first of all we have to comment on the functions that generally describe a discrete lifetime variable.

Excursus: Functions for a discrete lifetime variable

We define and redefine the following concepts:

- probability of failure within the k–th unit of time (**probability mass function**)

$$P_k := \Pr(X = k);\ \ k = 0, 1, 2, \ldots; \tag{3.30a}$$

- failure distribution CDF

$$F(k) := \Pr(X \le k) = \sum_{i=0}^{k} P_i;\ \ k = 0, 1, 2, \ldots \tag{3.30b}$$

with

$$F(\infty) = 1,\ \ F(-1) := 0 \ \text{ and } \ P_k = F(k) - F(k-1); \tag{3.30c}$$

- survival function CCDF

$$\begin{aligned} R(k) &:= \Pr(X > k) = 1 - F(k) \\ &= \sum_{i=k+1}^{\infty} P_i;\ \ k = 0, 1, 2, \ldots \end{aligned} \tag{3.30d}$$

with

$$R(\infty) = 0,\ \ R(-1) := 1 \ \text{ and } \ P_k = R(k-1) - R(k). \tag{3.30e}$$

The most frequently used and wide–spread definition of the hazard rate in the discrete case, see footnote 1 in Chapter 2, is

$$\left.\begin{aligned} h_k &:= \Pr(X = k \,|\, X \ge k);\ \ k = 0, 1, 2, \ldots, \\ &= \frac{\Pr(X = k)}{\Pr(X \ge k)} \\ &= \frac{F(k) - F(k-1)}{1 - F(k-1)} \\ &= \frac{R(k-1) - R(k)}{R(k-1)} \end{aligned}\right\} \tag{3.31a}$$

with corresponding cumulative hazard rate

$$H(k) := \sum_{i=0}^{k} h_i. \tag{3.31b}$$

We notice the following properties of (3.31a):[12]

- h_k is a conditional probability, thus

$$0 \le h_k \le 1. \tag{3.32a}$$

- These conditional probabilities and the unconditional probabilities P_k are linked as follows:

$$\left.\begin{array}{rcl} P_0 & = & h_0 \\ P_k & = & h_k\,(1-h_{k-1})\cdot\ldots\cdot(1-h_0);\;\; k \ge 1. \end{array}\right\} \tag{3.32b}$$

- The survival function $R(k)$ and the hazard rate h_k are linked as

$$R(k) = (1-h_0)\,(1-h_1)\cdot\ldots\cdot(1-h_k);\;\; h = 0,1,2,\ldots \tag{3.32c}$$

- The mean $\mathrm{E}(X)$, if it exists, is given by

$$\mathrm{E}(X) = \sum_{i=1}^{k} R(k) = \sum_{k=1}^{\infty}\prod_{j=0}^{k}(1-h_j). \tag{3.32d}$$

The relationships between $h(x),\ H(x)$ on the one side and $F(x),\ R(x)$ on the other side in the continuous case, which are to be found in Tab. 2/1, do not hold with h_k and $H(k)$ defined above, especially

$$R(k) \ne \exp\big[-H(k)\big] = \exp\Bigg[-\sum_{i=0}^{k} h_i\Bigg];$$

instead we have (3.32c). For this reason ROY/GUPTA (1992) have proposed an alternative discrete hazard rate function:

$$\lambda_k := \ln\left(\frac{R(k-1)}{R(k)}\right);\;\; k = 0,1,2,\ldots \tag{3.33a}$$

With the corresponding cumulative function

$$\left.\begin{array}{rcl} \Lambda(k) & := & \sum\limits_{i=0}^{k}\lambda_i \\ & = & \ln R(-1) - \ln R(k) \\ & = & -\ln R(k), \;\; \text{see (3.30e)} \end{array}\right\}, \tag{3.33b}$$

we arrive at

$$R(k) = \exp\big[-\Lambda(k)\big]. \tag{3.33c}$$

We will term λ_k the **pseudo–hazard function** so as to differentiate it from the hazard rate h_k.

[12] For discrete hazard functions, see SALVIA/BOLLINGER (1982).

The **type–I discrete WEIBULL distribution** introduced by NAKAGAWA/OSAKI (1975) mimics the CDF of the continuous WEIBULL distribution. They consider a probability mass function $P_k^I \ (k = 0, 1, 2, \ldots)$ indirectly defined by

$$\Pr(X \geq k) = \sum_{j=k}^{\infty} P_j^I = q^{k^\beta}; \ \ k = 0, 1, 2, \ldots; \ 0 < q < 1 \ \text{ and } \ \beta > 0. \tag{3.34a}$$

The probability mass function follows as

$$P_k^I = q^{k^\beta} - q^{(k+1)^\beta}; \ \ k = 0, 1, 2, \ldots; \tag{3.34b}$$

and the hazard rate according to (3.31a) as

$$h_k^I = 1 - q^{(k+1)^\beta - k^\beta}; \ \ k = 0, 1, 2, \ldots, \tag{3.34c}$$

The hazard rate

- has the constant value $1 - q$ for $\beta = 1$,
- is decreasing for $0 < \beta < 1$ and
- is increasing for $\beta > 1$.

So, the parameter β plays the same role as c in the continuous case. The CDF is given by

$$F^I(k) = 1 - q^{(k+1)^\beta}; \ \ k = 0, 1, 2, \ldots; \tag{3.34d}$$

and the CCDF by

$$R^I(k) = q^{(k+1)^\beta}; \ \ k = 0, 1, 2, \ldots \tag{3.34e}$$

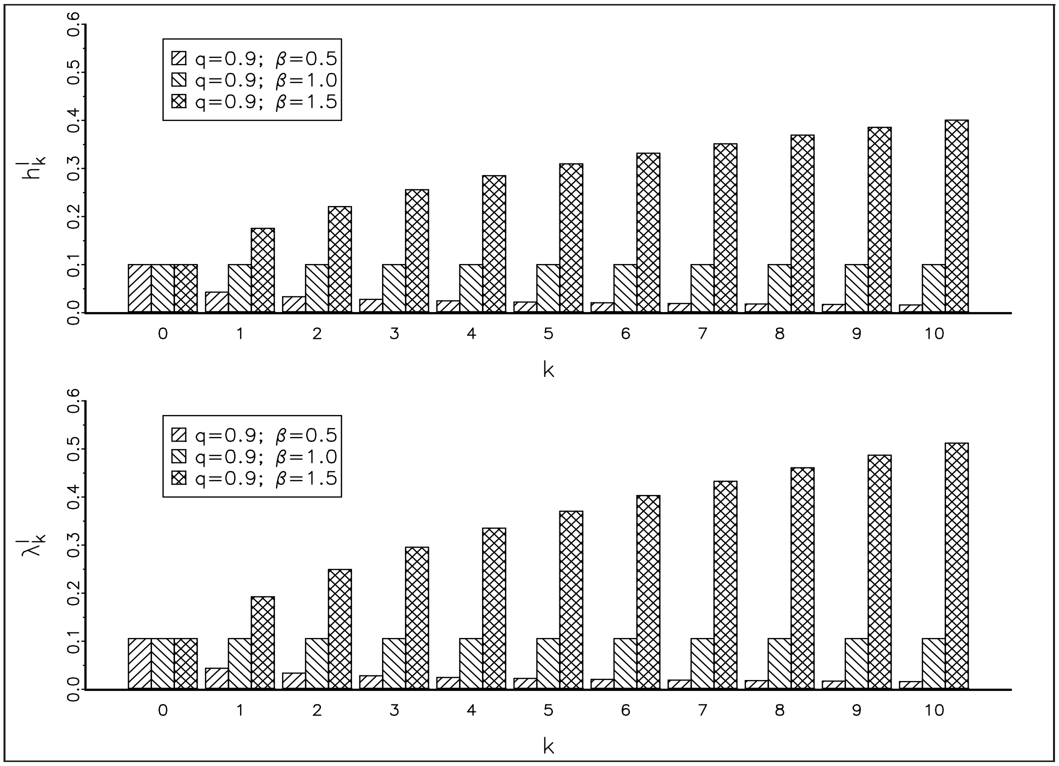
Thus, the pseudo–hazard function follows as

$$\lambda_k^I = \Big[k^\beta - (k+1)^\beta\Big] \ln q; \ \ k = 0, 1, 2, \ldots \tag{3.34f}$$

and its behavior in response to β is the same as that of h_k^I.

Fig. 3/4 shows the hazard rate (3.34c) in the upper part and the pseudo–hazard function (3.34f) in the lower part for $q = 0.9$ and $\beta = 0.5, \ 1.0, \ 1.5$. h_k^I and λ_k^I are not equal to each other but $\lambda_k^I(q, \beta) > h_k^I(q, \beta)$, the difference being the greater the smaller q and/or the greater β. λ_k^I increases linearly for $\beta = 2$, which is similar to the continuous case, whereas h_k^I is increasing but is concave for $\beta = 2$.

Figure 3/4: Hazard rate and pseudo–hazard function for the type-I discrete WEIBULL distribution



Compared with the CDF of the two–parameter continuous WEIBULL distribution

$$F(x\,|\,0,b,c) = 1 - \exp\Big\{-\Big(\frac{x}{b}\Big)^c\Big\}, \quad x \geq 0, \tag{3.35}$$

we see that (3.34d) and (3.35) have the same double exponential form and they coincide for all $x = k+1 \ \ (k = 0,1,2,\ldots)$ if

$$\beta = c \ \text{ and } \ q = \exp(-1/b^c) = \exp(-b_1),$$

where $b_1 = 1/b^c$ is the combined scale–shape factor of (2.26b).

Remark: Suppose that a discrete variate Y has a geometric distribution, i.e.,

$$\Pr(Y = k) = p\,q^{k-1}; \ \ k = 1,2,\ldots; \ p+q = 1 \ \text{ and } \ 0 < p < 1;$$

and

$$\Pr(Y \geq k) = q^k.$$

Then, the transformed variate $X = Y^{1/\beta}$, $\beta > 0$, will have

$$\Pr(X \geq k) = \Pr(Y \geq k^\beta) = q^{k^\beta},$$

and hence X has the discrete WEIBULL distribution introduced above. When $\beta = 1$, the discrete WEIBULL distribution reduces to the geometric distribution. This transformation is the counterpart to the power–law relationship linking the exponential and the continuous WEIBULL distributions.

The moments of the type–I discrete WEIBULL distribution

$$\mathrm{E}(X^r) = \sum_{k=1}^{\infty} k^r \left(q^{k^\beta} - q^{(k+1)^\beta}\right)$$

have no closed–form analytical expressions; they have to be evaluated numerically. ALI KHAN et al. (1989) give the following inequality for the means μ_d and μ_c of the discrete and continuous distributions, respectively, when $q = \exp(-1/b^c)$:

$$\mu_d - 1 < \mu_c < \mu_d.$$

These authors and KULASEKERA (1994) show how to estimate the two parameters β and q of (3.34b).

The **type–II discrete WEIBULL distribution** introduced by STEIN/DATTERO (1984) mimics the HR

$$h(x\,|\,0,b,c) = \frac{c}{b}\left(\frac{x}{b}\right)^{c-1} \tag{3.36}$$

of the continuous WEIBULL distribution by

$$h_k^{II} = \left\{\begin{array}{ccc} \alpha\,k^{\beta-1} & \text{for} & k = 1,2,\ldots,m \\ 0 & \text{for} & k = 0 \text{ or } k > m \end{array}\right\},\ \alpha > 0,\ \beta > 0. \tag{3.37a}$$

m is a truncation value, given by

$$m = \left\{\begin{array}{cc} \mathrm{int}\left[\alpha^{-1/(\beta-1)}\right] & \text{if} \quad \beta > 1 \\ \infty & \text{if} \quad \beta \le 1 \end{array}\right\}, \tag{3.37b}$$

which is necessary to ensure $h_k^{II} \le 1$; see (3.32a).

(3.36) and (3.37a) coincide at $x = k$ for

$$\beta = c \text{ and } \alpha = c/b^c,$$

i.e., α is a combined scale–shape factor. The probability mass function generated by (3.37a) is

$$\begin{aligned} P_k^{II} &= \begin{cases} h_1^{II} \\ h_k^{II}\,(1-h_{k-1}^{II})\ldots(1-h_1^{II}) & \text{for} \quad k \ge 2, \end{cases} \\ &= \alpha\,k^{\beta-1}\prod_{j=1}^{k-1}(1-\alpha\,j^{\beta-1});\ \ k = 1,2,\ldots,m. \end{aligned} \tag{3.37c}$$

The corresponding survival or reliability function is

$$R^{II}(k) = \prod_{j=1}^{k} \left(1 - h_k^{II}\right) = \prod_{j=1}^{k} \left(1 - \alpha\, j^{\beta-1}\right); \quad k = 1, 2, \ldots, m. \tag{3.37d}$$

(3.37d) in combination with (3.33b,c) gives the pseudo–hazard function

$$\lambda_k^{II} = -\ln\left(1 - \alpha\, k^{\beta-1}\right). \tag{3.37e}$$

For $\beta = 1$ the type–II discrete WEIBULL distribution reduces to a geometric distribution as does the type–I distribution.

The discrete distribution suggested by PADGETT/SPURRIER (1985) and SALVIA (1996) is not similar in functional form to any of the functions describing a continuous WEIBULL distribution. The only item connecting this distribution to the continuous WEIBULL model is the fact that its hazard rate may be constant, increasing or decreasing depending on only one parameter. The discrete hazard rate of this model is

$$h_k^{III} = 1 - \exp\left[-d\,(k+1)^{\beta}\right]; \; k = 0, 1, 2, \ldots; \; d > 0, \; \beta \in \mathbb{R}, \tag{3.38a}$$

where h_k^{III} is

- constant with $1 - \exp(-d)$ for $\beta = 0$,
- increasing for $\beta > 0$,
- decreasing for $\beta < 0$.

The probability mass function is

$$P_k^{III} = \exp\left[-d\,(k+1)^{\beta}\right] \prod_{j=1}^{k} \exp\left(-d\, j^{\beta}\right); \quad k = 0, 1, 2, \ldots; \tag{3.38b}$$

and the survival function is

$$R^{III}(k) = \exp\left[-d \sum_{j=1}^{k+1} j^{\beta}\right]; \quad k = 0, 1, 2, \ldots \tag{3.38c}$$

For suitable values of $d > 0$ the parameter β in the range $-1 \leq \beta \leq 1$ is sufficient to describe many hazard rates of discrete distributions. The pseudo–hazard function corresponding to (3.38a) is

$$\lambda_k^{III} = d\,(k+1)^{\beta}; \quad k = 0, 1, 2, \ldots; \tag{3.38d}$$

which is similar to (3.37a).

3.3.2 Reflected and double WEIBULL distributions

The modifications of this and next two sections consist of some kind of transformation of a continuous WEIBULL variate. The **reflected WEIBULL distribution**, introduced by COHEN (1973), originates in the following linear transformation of a classical WEIBULL variate X:

$$Y - a = -(X - a) = a - X.$$

This leads to a reflection of the classical WEIBULL DF about a vertical axis at $x = a$ resulting in

$$f_R(y\,|\,a,b,c) = \frac{c}{b}\left(\frac{a-y}{b}\right)^{c-1}\exp\left\{-\left(\frac{a-y}{b}\right)^{c}\right\};\;\; y<a;\; b,\; c>0; \tag{3.39a}$$

$$F_R(y\,|\,a,b,c) = \left\{\begin{array}{ll} \exp\left\{-\left(\frac{a-y}{b}\right)^{c}\right\} & \text{for}\quad y<a, \\ 1 & \text{for}\quad y\geq a. \end{array}\right\} \tag{3.39b}$$

This distribution has been recognized as the type–III maximum distribution in Sect. 3.2.2. The corresponding hazard rate is

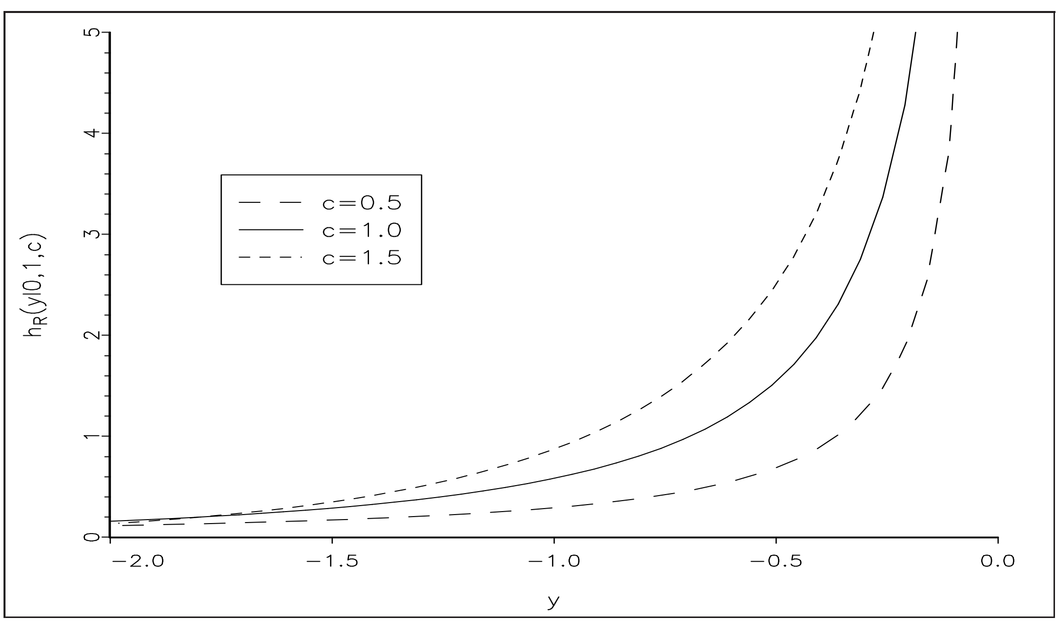
$$h_R(y\,|\,a,b,c) = \frac{c}{b}\left(\frac{a-y}{b}\right)^{c-1}\frac{\exp\left\{-\left(\frac{a-y}{b}\right)^{c}\right\}}{1-\exp\left\{-\left(\frac{a-y}{b}\right)^{c}\right\}}, \tag{3.39c}$$

which — independent of c — is increasing, and goes to ∞ with y to a^-.

Some parameters of the reflected WEIBULL distribution are

$$\begin{aligned} \mathrm{E}(Y) &= a - b\,\Gamma_1, && (3.39d)\\ \mathrm{Var}(Y) &= \mathrm{Var}(X) = b^2\,(\Gamma_2 - \Gamma_1^2), && (3.39e)\\ \alpha_3(Y) &= -\alpha_3(X), && (3.39f)\\ \alpha_4(Y) &= \alpha_4(X), && (3.39g)\\ y_{0.5} &= a - b\,(\ln 2)^{1/c}, && (3.39h)\\ y^* &= a - b\,(1-1/c)^{1/c} && (3.39i) \end{aligned}$$

Figure 3/5: Hazard rate of the reflected WEIBULL distribution ($a = 0,\; b = 1$)



By changing the sign of the data sampled from a reflected WEIBULL distribution, it can be viewed as data from the classical WEIBULL model. Thus the parameters may be estimated by the methods discussed in Chapters 9 ff.

Combining the classical and the reflected WEIBULL models into one distribution results in the **double WEIBULL distribution** with DF

$$f_D(y\,|\,a,b,c) = \frac{c}{2\,b}\left|\frac{a-y}{b}\right|^{c-1}\exp\left\{-\left|\frac{a-y}{b}\right|^{c}\right\};\;\; y,\ a\in\mathbb{R};\ b,\ c>0; \tag{3.40a}$$

and CDF

$$F_D(y\,|\,a,b,c) = \left\{\begin{array}{cl} 0.5\,\exp\left\{-\left(\dfrac{a-y}{b}\right)^{c}\right\} & \text{for}\quad y\le a, \\ 1-0.5\,\exp\left\{-\left(\dfrac{y-a}{b}\right)^{c}\right\} & \text{for}\quad y\ge a. \end{array}\right\} \tag{3.40b}$$

The density function is symmetric about a vertical line in $y = a$ (see Fig. 3/6 for $a = 0$) and the CDF is symmetric about the point $(y = a,\ F_D = 0.5)$. When $c = 1$, we have the **double exponential distribution** or **LAPLACE distribution**.

In its general three–parameter version the double WEIBULL distribution is not easy to analyze. Thus, BALAKRISHNAN/KOCHERLAKOTA (1985), who introduced this model, concentrated on the reduced form with $a = 0$ and $b = 1$. Then, (3.40a,b) turn into

$$f_D(y\,|\,0,1,c) = \frac{c}{2}\,|y|^{c-1}\,\exp\{-\,|y|^{c}\} \tag{3.41a}$$

and

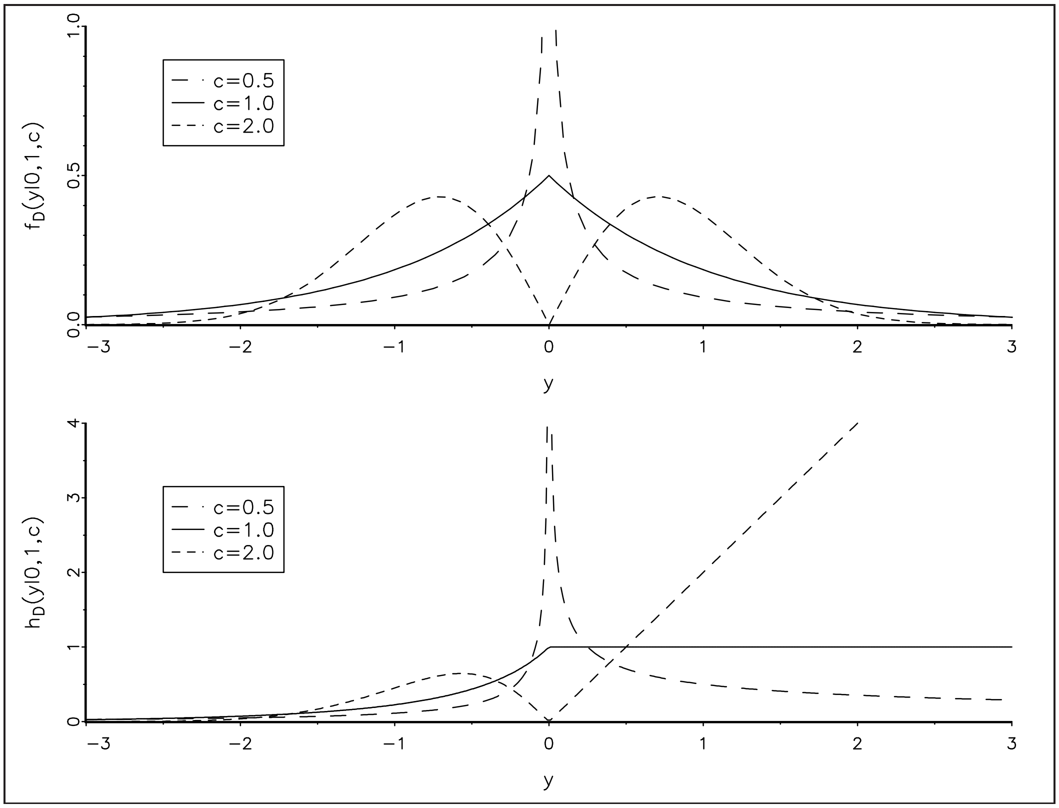
$$F_D(y\,|\,0,1,c) = \left\{\begin{array}{cl} 0.5\,\exp\big[-\big(-y\big)^{c}\big] & \text{for}\quad y\le 0, \\ 1-0.5\,\exp[-y^{c}] & \text{for}\quad y\ge 0. \end{array}\right\} \tag{3.41b}$$

(3.41a) is depicted in the upper part of Fig. 3/6 for several values of c. Notice that the distribution is bimodal for $c > 1$ and unimodal for $c = 1$ and has an improper mode for $0 < c < 1$. The lower part of Fig. 3/6 shows the hazard rate

$$h_D(y\,|\,0,1,c) = \left\{\begin{array}{cl} \dfrac{c\,(-y)^{c-1}\,\exp\{-\big(-y\big)^{c}\}}{2-\exp\{-(-y)^{c}\}} & \text{for}\quad y\le 0, \\ \dfrac{c\,y^{c-1}\,\exp\{-y^{c}\}}{\exp\{-y^{c}\}} & \text{for}\quad y\ge 0, \end{array}\right\} \tag{3.41c}$$

which — except for $c = 1$ — is far from being monotone; it is asymmetric in any case.

Figure 3/6: Density function and hazard rate of the double WEIBULL distribution $(a = 0,\ b = 1)$



The moments of this reduced form of the double WEIBULL distribution are given by

$$\mathrm{E}(Y^r) = \left\{ \begin{array}{cl} 0 & \text{for } r \text{ odd,} \\ \Gamma\left(1 + \dfrac{r}{c}\right) & \text{for } r \text{ even.} \end{array} \right\} \tag{3.41d}$$

Thus, the variance follows as

$$\mathrm{Var}(Y) = \Gamma\left(1 + \frac{2}{c}\right) \tag{3.41e}$$

and the kurtosis as

$$\alpha_4(Y) = \Gamma\left(1 + \frac{4}{c}\right) \Big/ \Gamma^2\left(1 + \frac{2}{c}\right). \tag{3.41f}$$

The absolute moments of Y are easily found to be

$$\mathrm{E}\big[\,|Y|^r\,\big] = \Gamma\left(1 + \frac{r}{c}\right), \quad r = 0, 1, 2, \ldots \tag{3.41g}$$

Parameter estimation for this distribution — mainly based on order statistics — is considered by BALAKRISHNAN/KOCHERLAKOTA (1985), RAO/NARASIMHAM (1989) and RAO/RAO/NARASIMHAM (1991).

3.3.3 Inverse WEIBULL distribution[13]

In Sect. 3.2.2. we have shown that — when $X \sim We(a,b,c)$, i.e., X has a classical WEIBULL distribution — the transformed variable

$$Y = \frac{b^2}{X-a}$$

has the DF

$$f_I(y\,|\,b,c) = \frac{c}{b}\left(\frac{y}{b}\right)^{-c-1}\exp\left\{-\left(\frac{y}{b}\right)^{-c}\right\};\;\; y \geq 0;\; b,c > 0 \tag{3.42a}$$

and the CDF

$$F_I(y\,|\,b,c) = \exp\left\{-\left(\frac{y}{b}\right)^{-c}\right\}. \tag{3.42b}$$

This distribution is known as **inverse WEIBULL distribution**.[14] Other names for this distribution are **complementary WEIBULL distribution** (DRAPELLA, 1993), **reciprocal WEIBULL distribution** (MUDHOLKAR/KOLLIA, 1994) and **reverse WEIBULL distribution** (MURTHY et al., 2004, p. 23).The distribution has been introduced by KELLER et al. (1982) as a suitable model to describe degradation phenomena of mechanical components (pistons, crankshafts) of diesel engines.

The density function generally exhibits a long right tail (compared with that of the commonly used distributions, see the upper part of Fig. 3/7, showing the densities of the classical and the inverse WEIBULL distributions for the same set of parameter values). In contrast to the classical WEIBULL density the inverse WEIBULL density always has a mode y^* in the interior of the support, given by

$$y^* = b\left(\frac{c}{1+c}\right)^{1/c}. \tag{3.42c}$$

and it is always positively skewed.

Inverting (3.42b) leads to the following percentile function

$$y_P = F_I^{-1}(P) = b\,\big(-\ln P\big)^{1/c}. \tag{3.42d}$$

The r–th moment about zero is given by

$$\begin{aligned} \mathrm{E}(Y^r) &= \int\limits_0^\infty y^r\left(\frac{c}{b}\right)\left(\frac{y}{b}\right)^{-c-1}\exp\left[-\left(\frac{y}{b}\right)^{-c}\right]\mathrm{d}y \\ &= b^r\,\Gamma\Big(1-\frac{r}{c}\Big)\;\;\text{for } r < c \text{ only.} \end{aligned} \tag{3.42e}$$

The above integral is not finite for $r \geq c$. As such, when $c \leq 2$, the variance is not finite. This is a consequence of the long, fat right tail.

[13] Suggested reading for this section: CALABRIA/PULCINI (1989, 1990, 1994), DRAPELLA (1993), ERTO (1989), ERTO/RAPONE (1984), JIANG/MURTHY/JI (2001), MUDHOLKAR/KOLLIA (1994).

[14] The distribution is identical to the type–II maximum distribution.

The hazard rate of the inverse WEIBULL distribution

$$h_I(y\,|\,b,c) = \frac{\left(\dfrac{c}{b}\right)\left(\dfrac{y}{b}\right)^{-c-1}\exp\left\{-\left(\dfrac{y}{b}\right)^{-c}\right\}}{1-\exp\left\{-\left(\dfrac{y}{b}\right)^{-c}\right\}} \tag{3.42f}$$

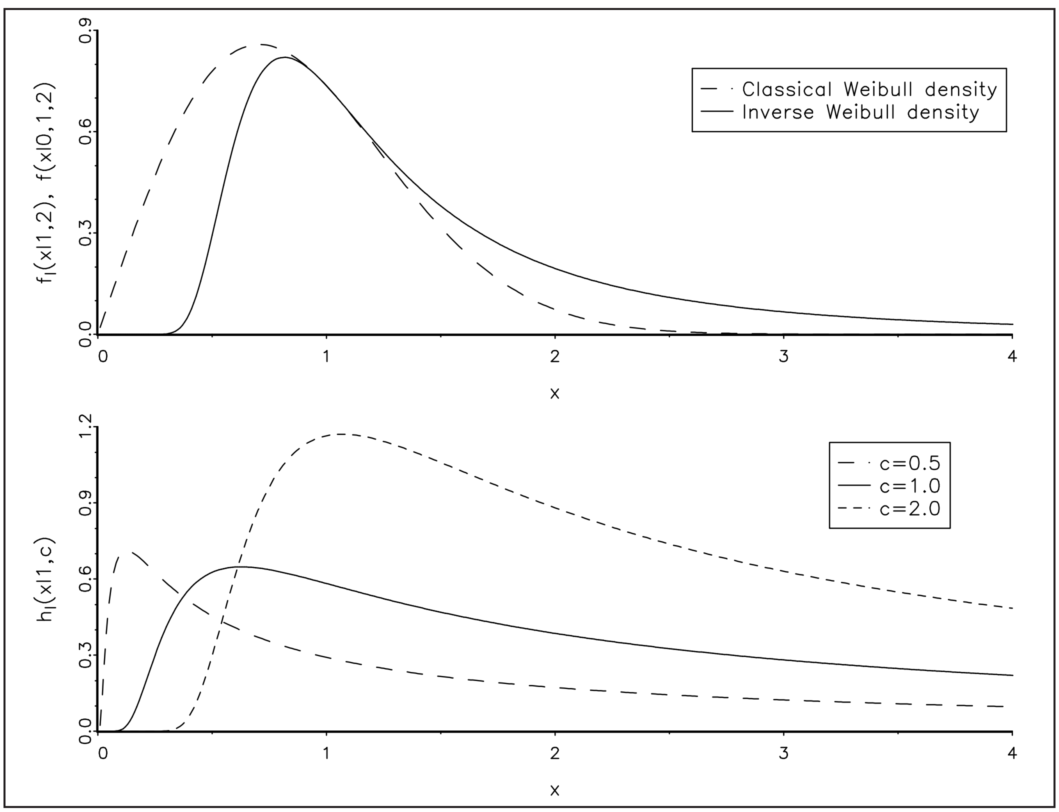
is an upside down bathtub (see the lower part of Fig. 3/7) and has a behavior similar to that of the lognormal and inverse GAUSSIAN distributions:

$$\lim_{y\to 0} h_I(y\,|\,b,c) = \lim_{y\to\infty} h_I(y\,|\,b,c) = 0$$

with a maximum at y_h^*, which is the solution of

$$\frac{\left(\dfrac{b}{y}\right)^c}{1-\exp\left[-\left(\dfrac{b}{y}\right)^c\right]} = \frac{c+1}{c}\,.$$

Figure 3/7: Density function and hazard rate of the inverse WEIBULL distribution



CALABRIA/PULCINI (1989, 1994) have applied the ML–method to estimate the two parameters. They also studied a BAYESIAN approach of predicting the ordered lifetimes in a future sample from an inverse WEIBULL distribution under type–I and type–II censoring.

3.3.4 Log-WEIBULL distribution[15]

In Sect. 3.2.2 we have introduced the following transformation of $X \sim We(a,b,c)$:

$$Y = \ln(X-a);\;\; X \geq a.$$

Y is called a **Log–WEIBULL variable**. Starting from

$$F_X(t\,|\,a,b,c) = \Pr(X \leq t) = 1 - \exp\left\{-\left(\frac{t-a}{b}\right)^c\right\},$$

we have

$$\begin{aligned}\Pr(Y \leq t) &= \Pr\big[\ln(X-a) \leq t\big]\\ &= \Pr\big[X - a \leq e^t\big] = \Pr\big[X \leq a + e^t\big]\\ &= 1 - \exp\big\{-\big(e^t/b\big)^c\big\}\\ &= 1 - \exp\big\{-\exp\big[c\,(t - \ln b)\big]\big\}. \end{aligned} \tag{3.43a}$$

(3.43a) is nothing but $F_L(t\,|\,a^*,b^*)$, the CDF of a type–I–minimum distribution (see (3.17b)) with location parameter

$$a^* := \ln b \tag{3.43b}$$

and scale parameter

$$b^* := 1/c. \tag{3.43c}$$

Provided, a of the original WEIBULL variate is known the log–transformation results in a distribution of the location–scale–family which is easy to deal with. So one approach to estimate the parameters b and c rests upon a preceding log–transformation of the observed WEIBULL data.

DF and HR belonging to (3.43a) are

$$f_L(t\,|\,a^*,b^*) = \frac{1}{b^*}\,\exp\left\{\frac{t-a^*}{b^*} - \exp\left[\frac{t-a^*}{b^*}\right]\right\} \tag{3.43d}$$

and

$$h_L(t\,|\,a^*,b^*) = \frac{1}{b^*}\,\exp\left(\frac{t-a^*}{b^*}\right). \tag{3.43e}$$

(3.43d) has been graphed in Fig. 1/1 for $a^* = 0$ and $b^* = 1$. The hazard rate is an increasing function of t. The percentile function is

$$t_P = a^* + b^*\,\ln\big[-\ln(1-P)\big];\;\; 0 < P < 1. \tag{3.43f}$$

In order to derive the moments of a Log–WEIBULL distribution, we introduce the reduced variable

$$Z = c\,(Y - \ln b) = \frac{Y - a^*}{b^*}$$

[15] Suggested reading for this section: KOTZ/NADARAJAH (2000), LIEBLEIN/ZELEN (1956), WHITE (1969).

with

$$F_L(z\,|\,0,1) = 1 - \exp\big(-e^z\big), \quad z \in \mathbb{R}, \tag{3.44a}$$

$$f_L(z\,|\,0,1) = \exp\big(z - e^z\big). \tag{3.44b}$$

The corresponding raw moment generating function is

$$\begin{aligned} M_Z(\theta) &= \mathrm{E}\Big(-e^{\theta\,Z}\Big) \\ &= \int\limits_{-\infty}^{\infty} e^{\theta\,z}\, \exp\big(z - e^{-z}\big)\, \mathrm{d}z, \\ &= \int\limits_{0}^{\infty} u^{\theta}\, e^{-u}\, \mathrm{d}u, \text{ setting } \; u = e^z, \\ &= \Gamma(1+\theta). \end{aligned} \tag{3.44c}$$

Using (2.65c) we easily find

$$\begin{aligned} \mathrm{E}(Z) &= \left.\frac{\mathrm{d}\Gamma(1+\theta)}{\mathrm{d}\theta}\right|_{\theta=0} = \Gamma'(1) \\ &= \psi(1)\,\Gamma(1) \\ &= -\gamma \approx -0.577216; \end{aligned} \tag{3.44d}$$

$$\begin{aligned} \mathrm{E}\big(Z^2\big) &= \left.\frac{\mathrm{d}^2\Gamma(1+\theta)}{\mathrm{d}\theta^2}\right|_{\theta=0} = \Gamma''(1) \\ &= \Gamma(1)\left[\psi^2(1) + \psi'(1)\right] \\ &= \gamma^2 + \pi^2/6 \approx 1.97811; \end{aligned} \tag{3.44e}$$

$$\begin{aligned} \mathrm{E}\big(Z^3\big) &= \left.\frac{\mathrm{d}^3\Gamma(1+\theta)}{\mathrm{d}\theta^3}\right|_{\theta=0} = \Gamma'''(1) \\ &= \Gamma(1)\left[\psi^3(1) + 3\,\psi(1)\,\psi'(1) + \psi''(1)\right] \\ &= -\gamma^3 - \frac{\gamma\,\pi^2}{2} + \psi''(1) \approx -5.44487; \end{aligned} \tag{3.44f}$$

$$\begin{aligned} \mathrm{E}\big(Z^4\big) &= \left.\frac{\mathrm{d}^4\Gamma(1+\theta)}{\mathrm{d}\theta^4}\right|_{\theta=0} = \Gamma^{(4)}(1) \\ &= \Gamma(1)\Big\{\psi^4(1) + 6\,\psi^2(1)\,\psi'(1) + 3\left[\psi'(1)\right]^2 + 4\,\psi(1)\,\psi''(1) + \psi'''(1)\Big\} \\ &= \gamma^4 + \gamma^2\,\pi^2 + \frac{3\,\pi^4}{20} - 4\,\gamma\,\psi''(1) \approx 23.5615. \end{aligned} \tag{3.44g}$$

(3.44c–d) lead to the following results for a general Log–WEIBULL variable:

$$\mathrm{E}(Y) = a^* + b^*(-\gamma) \approx a^* - 0.577216\, b^*; \tag{3.45a}$$

$$\mathrm{Var}(Y) = (b^*)^2\, \pi^2/6 \approx 1.644934\, (b^*)^2; \tag{3.45b}$$

$$\alpha_3 \approx -1.13955; \tag{3.45c}$$

$$\alpha_4 \approx 5.4. \tag{3.45d}$$

So the Log–WEIBULL density is negatively skewed with a mode $y^* = a^*$ and leptokurtic.

Continuing with the reduced Log–WEIBULL variable, we can state the following property of the first order statistic (sample minimum) in a sample of size n

$$Z_{1:n} = \min_{1 \le i \le n}(Z_i); \quad Z_i \text{ iid};$$

$$\begin{aligned}\Pr\big(Z_{1:n} \le t\big) &= 1 - \big[1 - F_L(t\,|\,0,1)\big]^n \\ &= 1 - \exp[-n\, e^t] \\ &= F_L\big[t + \ln(n)\big],\end{aligned} \tag{3.46a}$$

which implies that $Z_{1:n}$ has the same distribution as $Z - \ln(n)$, and hence

$$\mathrm{E}\big(Z_{1:n}\big) = -\gamma - \ln(n) \tag{3.46b}$$

$$\mathrm{Var}\big(Z_{1:n}\big) = \mathrm{Var}(Z) = \pi^2/6. \tag{3.46c}$$

We will revert to order statistics in greater detail in Sect. 5.

Parameter estimation of the Log–WEIBULL distribution can be viewed as a special case of the estimation for extreme value distributions. DEKKERS et al. (1989) and CHRISTOPEIT (1994) have investigated estimation by the method of moments. More information about statistical inference is provided in the monograph by KOTZ/NADARAJAH (2000). The first, but still noteworthy case study applying the Log–WEIBULL distribution as the lifetime distribution for ball bearing was done by LIEBLEIN/ZELEN (1956).

We finally note that the relationship between WEIBULL and Log–WEIBULL variables is the reverse of that between normal and log–normal variables. That is, if log X is normal then X is called a log-normal variable, while if $\exp(Y)$ has a WEIBULL distribution then we say that Y is a Log–WEIBULL variable. Perhaps a more appropriate name than Log–WEIBULL variable could have been chosen, but on the other hand one might also argue that the name "log-normal" is misapplied.

3.3.5 Truncated WEIBULL distributions[16]

In common language the two words *truncation* and *censoring* are synonyms, but in statistical science they have different meanings. We join COHEN (1991, p. 1) when we say

[16] Suggest reading for this section: AROIAN (1965), CROWDER (1990), GROSS (1971), HWANG (1996), MARTINEZ/QUINTANA (1991), MCEWEN/PARRESOL (1991), MITTAL/DAHIYA (1989), SHALABY (1993), SHALABY/AL–YOUSSEF (1992), SUGIURA/GOMI (1985), WINGO (1988, 1989, 1998).

truncation is a notion **related to populations** whereas **censoring** is **related to samples**. A truncated population is one where — according to the size of the variable — some part of the original population has been removed thus restricting the support to a smaller range. A truncated distribution is nothing but a conditional distribution,[17] the condition being that the variate is observable in a restricted range only. The most familiar types of truncation, occurring when the variate is a lifetime or a duration, are

- **left–truncation** In such a lower truncated distribution the smaller realizations below some left truncation point t_ℓ have been omitted. This will happen when all items of an original distribution are submitted to some kind of screening, e.g., realized by a burn–in lasting t_ℓ units of time.
- **right–truncation** An upper truncated distribution is missing the greater realizations above some right truncation point t_r. This might happen when the items of a population are scheduled to be in use for a maximum time of t_r only.
- **double–truncation** A doubly truncated distribution only comprises the "middle" portion of all items.

Censored samples are those in which sample specimens with measurements that lie in some restricted areas of the sample space may be identified and thus counted, but are not otherwise measured. So the censored sample units are not omitted or forgotten but they are known and are presented by their number, but not by the exact values of their characteristic.

In a truncated population a certain portion of units is eliminated without replacement thus the reduced size of the population, which would be less than 1 or 100%, has to be rescaled to arrive at an integrated DF amounting to 1; see (3.48a). Censored samples are extensively used in life–testing. We will present the commonly used types of censoring in Sect. 8.3 before turning to estimation and testing because the inferential approach depends on whether and how the sample has been censored.

Starting with a three–parameter WEIBULL distribution, the general truncated WEIBULL model is given by the following CDF

$$
\begin{aligned}
F_{DT}(x\,|\,a,b,c,t_\ell,t_r) &= \frac{F(x\,|\,a,b,c)-F(t_\ell\,|\,a,b,c)}{F(t_r\,|\,a,b,c)-F(t_\ell\,|\,a,b,c)};\;\; a\le t_\ell\le x\le t_r<\infty,\\
&= \frac{\exp\left\{-\left(\dfrac{t_\ell-a}{b}\right)^c\right\}-\exp\left\{-\left(\dfrac{x-a}{b}\right)^c\right\}}{\exp\left\{-\left(\dfrac{t_\ell-a}{b}\right)^c\right\}-\exp\left\{-\left(\dfrac{t_r-a}{b}\right)^c\right\}}\\
&= \frac{1-\exp\left\{\left(\dfrac{t_\ell-a}{b}\right)^c-\left(\dfrac{x-a}{b}\right)^c\right\}}{1-\exp\left\{\left(\dfrac{t_\ell-a}{b}\right)^c-\left(\dfrac{t_r-a}{b}\right)^c\right\}}. \qquad (3.47a)
\end{aligned}
$$

This is also referred to as the **doubly truncated WEIBULL distribution**. We notice the following special cases of (3.47a):

[17] See Sect. 2.6.

- $(t_\ell > a,\ t_r = \infty)$ gives the **left truncated WEIBULL distribution** with CDF

$$F_{LT}(x\,|\,a,b,c,t_\ell,\infty) = 1 - \exp\left\{\left(\frac{t_\ell - a}{b}\right)^c - \left(\frac{x-a}{b}\right)^c\right\}. \tag{3.47b}$$

- $(t_\ell = a, a < t_r < \infty)$ gives the **right truncated WEIBULL distribution** with CDF

$$F_{RT}(x\,|\,a,b,c,a,t_r) = \frac{1 - \exp\left\{-\left(\dfrac{x-a}{b}\right)^c\right\}}{1 - \exp\left\{-\left(\dfrac{t_r - a}{b}\right)^c\right\}}. \tag{3.47c}$$

- $(t_\ell = a; t_r = \infty)$ leads to the original, non–truncated WEIBULL distribution.

In the sequel we will give results pertaining to the doubly truncated WEIBULL distribution. Special results for the left and right truncated distributions will be given only when they do not follow from the doubly truncated result in a trivial way.

The DF belonging to (3.47a) is

$$\begin{aligned}
f_{DT}(x\,|\,a,b,c,t_\ell,t_r) &= \frac{f(x\,|\,a,b,c)}{F(t_r\,|\,a,b,c) - F(t_\ell\,|\,a,b,c)};\ \ a \le t_\ell \le x \le t_r < \infty;\\
&= \frac{\dfrac{c}{b}\left(\dfrac{x-a}{b}\right)^{c-1}\exp\left\{-\left(\dfrac{x-a}{b}\right)^c\right\}}{\exp\left\{-\left(\dfrac{t_\ell-a}{b}\right)^c\right\} - \exp\left\{-\left(\dfrac{t_r-a}{b}\right)^c\right\}}\\
&= \frac{\dfrac{c}{b}\left(\dfrac{x-a}{b}\right)^{c-1}\exp\left\{\left(\dfrac{t_\ell-a}{b}\right)^c - \left(\dfrac{x-a}{b}\right)^c\right\}}{1 - \exp\left\{\left(\dfrac{t_\ell-a}{b}\right)^c - \left(\dfrac{t_r-a}{b}\right)^c\right\}}.
\end{aligned} \tag{3.48a}$$

The shape of $f_{DT}(x\,|\,a,b,c,t_\ell,t_r)$ is determined by the shape of $f(x\,|\,a,b,c)$ over the interval $t_\ell \le x \le t_r$ (see Fig. 3/8) with the following results:

- $(c < 1)$ or $(c > 1$ and $t_\ell > x^*)$ $\Longrightarrow$ $f_{DT}(\cdot)$ is decreasing,
- $c > 1$ and $t_r < x^*$ $\Longrightarrow$ $f_{DT}(\cdot)$ is increasing,
- $c > 1$ and $t_\ell < x^* < t_r$ $\Longrightarrow$ $f_{DT}(\cdot)$ is unimodal,

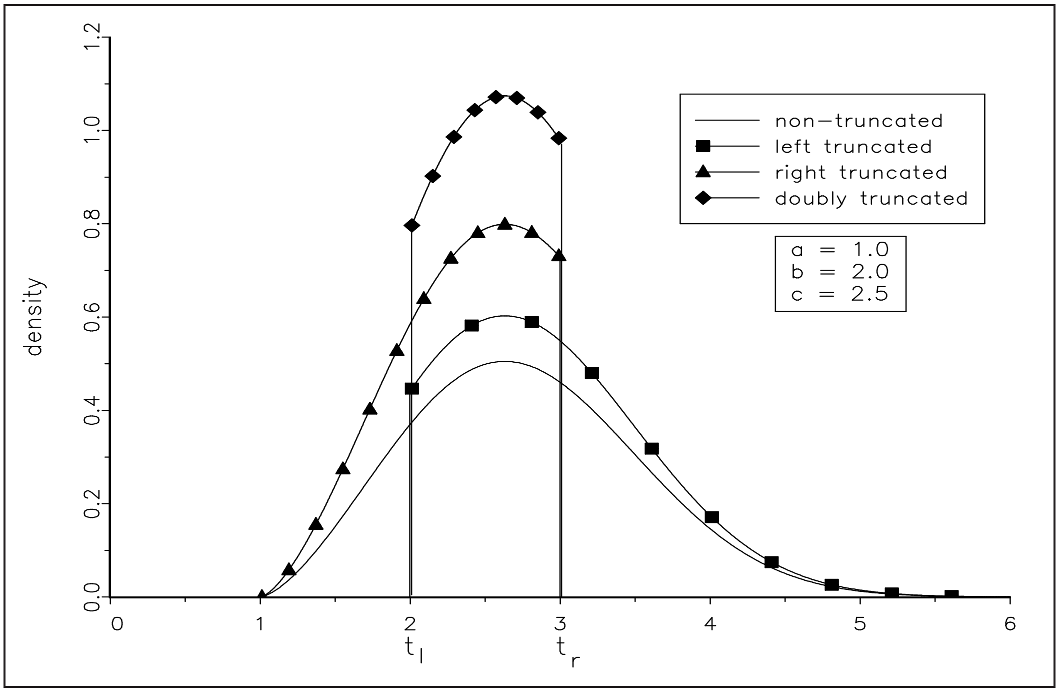
where $x^* = a + b\left[(c-1)/c\right]^{1/c}$ is the mode of the non–truncated distribution.

The hazard rate belonging to (3.47a) is

$$\begin{aligned}
h_{DT}(x\,|\,a,b,c,t_\ell,t_r) &= \frac{f_{DT}(x\,|\,a,b,c)}{1 - F_{DT}(x\,|\,a,b,c,t_\ell,t_r)};\ \ a \le t_\ell \le x \le t_r < \infty;\\
&= \frac{f(x\,|\,a,b,c)}{F(t_r\,|\,a,b,c) - F(x\,|\,a,b,c)}\\
&= \frac{f(x\,|\,a,b,c)}{1 - F(x\,|\,a,b,c)}\,\frac{1 - F(x\,|\,a,b,c)}{F(t_r\,|\,a,b,c) - F(x\,|\,a,b,c)}.
\end{aligned} \tag{3.48b}$$

The first factor on the right–hand side of (3.48b) is the hazard rate (2.36b) of the non–truncated distribution. The second factor is a monotonically increasing function, approaching ∞ as $x \to t_r$. So $h_{DT}(\cdot)$ can be either decreasing, increasing or bathtub–shaped. The bathtub shape will come up in the doubly and right truncated cases when $0 < c < 1$.

Figure 3/8: Densities of truncated and non-truncated WEIBULL distributions



The percentile function belonging to (3.47a) is

$$F_{DT}^{-1}(P) = a + b\left\{\ln\left[\frac{1}{1 - F(t_\ell \,|\, a,b,c) - P\left[F(t_r \,|\, a,b,c) - F(t_\ell \,|\, a,b,c)\right]}\right]\right\}^{1/c}. \tag{3.48c}$$

MCEWEN/PARRESOL (1991) give the following formula for the raw moments of the doubly truncated WEIBULL distribution:

$$\begin{aligned} \mathrm{E}\big(X_{DT}^r\big) \;=\; & \frac{\exp\left\{\left(\dfrac{t_\ell - a}{b}\right)^c\right\}}{1 - \exp\left\{\left(-\dfrac{t_r - a}{b}\right)^c\right\}} \sum_{j=0}^{r} \binom{r}{j} b^{r-j}\, a^j \,\times \\ & \left\{\gamma\left[\frac{r-j}{c} + 1 \,\middle|\, \left(-\frac{t_r - a}{b}\right)^c\right] - \gamma\left[\frac{r-j}{c} + 1 \,\middle|\, \left(\frac{t_\ell - a}{b}\right)^c\right]\right\}, \end{aligned} \tag{3.48d}$$

where $\gamma[\cdot\,|\,\cdot]$ is the incomplete gamma function (see the excursus in Sect. 2.9.1). So, the moments can be evaluated only numerically. Looking at the reduced WEIBULL variate U, i.e., $a = 0$ and $b = 1$, (3.48d) becomes a little bit simpler:

$$\mathrm{E}\big(U_{DT}^r\big) = \frac{\exp\{t_\ell^c\}}{1-\exp\{-t_r^c\}}\left\{\gamma\left[\frac{r}{c}+1\,\middle|\,t_r^c\right]-\gamma\left[\frac{r}{c}+1\,\middle|\,t_\ell^c\right]\right\}. \tag{3.48e}$$

In the left truncated case (3.48d,e) turn into[18]

$$\begin{aligned}\mathrm{E}\big(X_{LT}^r\big) &= \exp\left\{\left(\frac{t_\ell-a}{b}\right)^c\right\}\sum_{j=0}^{r}\binom{r}{j}\,b^{r-j}\,a^j\times\\ &\qquad\left\{\Gamma\left(\frac{r-j}{c}+1\right)-\gamma\left[\frac{r-j}{c}+1\,\middle|\,\left(\frac{t_\ell-a}{b}\right)^c\right]\right\}\end{aligned} \tag{3.49a}$$

and

$$\mathrm{E}\big(U_{LT}^r\big) = \exp\{t_\ell^c\}\left\{\Gamma\left(\frac{r}{c}+1\right)-\gamma\left[\frac{r}{c}+1\,\middle|\,t_\ell^c\right]\right\}. \tag{3.49b}$$

When the distribution is right truncated we have

$$\mathrm{E}\big(X_{RT}^r\big) = \frac{1}{1-\exp\left\{-\left(\dfrac{t_r-a}{b}\right)^c\right\}}\sum_{j=0}^{r}\binom{r}{j}\,b^{r-j}\,a^j\,\gamma\left[\frac{r-j}{c}+1\,\middle|\,\left(\frac{t_r-a}{b}\right)^c\right] \tag{3.50a}$$

and

$$\mathrm{E}\big(U_{RT}^r\big) = \frac{1}{1-\exp\{-t_r^c\}}\,\gamma\left[\frac{r}{c}+1\,\middle|\,t_r^c\right]. \tag{3.50b}$$

SUGIURA/GOMI (1985) give a moment–ratio diagram showing the right truncated WEIBULL distribution in comparison with the doubly truncated normal distribution.

Testing hypotheses on the parameters of a truncated WEIBULL distribution is treated by CROWDER (1990) and MARTINEZ/QUINTANA (1991). Parameter estimation is discussed by MITTAL/DAHIYA (1989) and WINGO (1988, 1989, 1998) applying the maximum likelihood approach and by SHALABY (1993) and SHALABY/AL–YOUSSEF (1992) using a BAYES approach. See CHARERNKAVANICH/COHEN (1984) for estimating the parameters as well as the unknown left point of truncation by maximum likelihood. HWANG (1996) gives an interesting application of the truncated WEIBULL model in reliability prediction of missile systems.

3.3.6 Models including two or more distributions

This section presents models which result when several variates or several distributions are combined in one way or another whereby at least one of these variates or distributions has to be WEIBULL. We will first (Sect. 3.3.6.1) study the distribution of the sum of WEIBULL variates. Then (Sect. 3.3.6.2) we turn to the lifetime distribution of a system made up

[18] AROIAN (1965) gives an evaluation based on the χ^2–distribution.

by linking two or more randomly failing components. Sections 3.3.6.3 to 3.3.6.5 deal with composing and with discrete as well as continuous mixing; topics which have been explored intensively in theory and practice.

3.3.6.1 WEIBULL folding

The distribution of the **sum of** two or more **variates** is found by convoluting or folding their distributions. We will only report on the sum of iid WEIBULL variates. Such a sum is of considerable importance in renewal theory (Sect. 4.4) and for WEIBULL processes (Sect. 4.3).

First, we will summarize some general results on a sum

$$Y = \sum_{i=1}^{n} X_i$$

of n iid variates. The DF and CDF of Y are given by the following recursions:

$$f_n(y) = \int_R f(x)\, f_{n-1}(y-x)\,\mathrm{d}x;\quad n = 2,3,\ldots; \tag{3.51a}$$

$$F_n(y) = \int_R f(x)\, F_{n-1}(y-x)\,\mathrm{d}x;\quad n = 2,3,\ldots; \tag{3.51b}$$

with

$$f_1(\cdot) = f(\cdot) \quad \text{and} \quad F_1(\cdot) = F(\cdot).$$

R, the area of integration in (3.51a,b), contains all arguments for which the DFs and CDFs in the integrand are defined. The raw moments of Y are given by

$$\begin{aligned} \mathrm{E}(Y^r) &= \mathrm{E}\big[(X_1 + X_2 + \ldots + X_n)^r\big] \\ &= \sum \frac{r!}{r_1!\, r_2! \ldots r_n!}\, \mathrm{E}\big(X_1^{r_1}\big)\, \mathrm{E}\big(X_2^{r_2}\big) \ldots \mathrm{E}\big(X_n^{r_n}\big). \end{aligned} \tag{3.51c}$$

The summation in (3.51c) runs over all n–tuples $(r_1, r_2, \ldots, r_n)$ with $\sum\limits_{i=1}^{n} r_i = r$ and $0 \le r_i \le r$. The number of summands is $\binom{r+n-1}{r}$. In particular we have

$$\mathrm{E}(Y) = \sum_{i=1}^{n} \mathrm{E}(X_i). \tag{3.51d}$$

The variance of Y is

$$\mathrm{Var}(Y) = \sum_{i=1}^{n} \mathrm{Var}(X_i). \tag{3.51e}$$

Assuming

$$X_i \sim We(a,b,c);\quad i = 1,2,\ldots,n,$$

(3.51a,b) turn into

$$f_n(y) = \int_{n\,a}^{y} \frac{c}{b}\left(\frac{x-a}{b}\right)^{c-1} \exp\left\{-\left(\frac{x-a}{b}\right)^{c}\right\} f_{n-1}(y-x)\,\mathrm{d}x; \quad n = 2,3,\ldots; \tag{3.52a}$$

$$F_n(y) = \int_{n\,a}^{y} \frac{c}{b}\left(\frac{x-a}{b}\right)^{c-1} \exp\left\{-\left(\frac{x-a}{b}\right)^{c}\right\} F_{n-1}(y-x)\,\mathrm{d}x; \quad n = 2,3,\ldots. \tag{3.52b}$$

Whereas the evaluation of (3.51d,e) for WEIBULL variates is quite simple there generally do not exist closed form expressions for the integrals[19] on the right–hand sides of (3.52a,b), not even in the case $n = 2$.

An exception is the case $c = 1$ for all $n \in \mathbb{N}$, i.e. the case of folding identically and independently exponentially distributed variates. The n–fold convolution is a gamma distribution, more precisely an ERLANG distribution as n is a positive integer. For $n = 2$ and $c = 1$ we have

$$f_2(y\,|\,a,b,1) = \frac{y-2\,a}{b^2}\,\exp\left[-\left(\frac{y-2\,a}{b}\right)\right],\ \ y \geq 2\,a,$$

$$F_2(y\,|\,a,b,1) = 1-\exp\left[-\frac{y-2\,a}{b}\right]\left(\frac{y-2\,a+b}{b}\right),\ y \geq 2\,a.$$

For $n \in \mathbb{N},\ c = 1$ and $a = 0$, we have the handsome formulas

$$f_n(y\,|\,0,b,1) = \frac{y^{n-1}}{b^n\,(n-1)!}\,\exp\left(-\frac{y}{b}\right),$$

$$F_n(y\,|\,0,b,1) = 1-\exp\left(-\frac{y}{b}\right)\sum_{i=0}^{n-1}\frac{y^i}{b^i\,i!}\,.$$

Turning to a genuine WEIBULL distribution, i.e., $c \neq 1$, we will only evaluate (3.52a,b) for $n = 2$ and $a = 0$ when $c = 2$ or $c = 3$.

$\boxed{\text{c = 2}}$

$$f_2(y\,|\,0,b,2) = \frac{\exp\left[-\left(\frac{y}{b}\right)^2\right]}{2\,b^3}\left\{2\,b\,y-\sqrt{2\,\pi}\,\exp\left[\frac{1}{2}\left(\frac{y}{b}\right)^2\right](b^2-y^2)\,\mathrm{erf}\left(\frac{y}{b\,\sqrt{2}}\right)\right\}$$

$$F_2(y\,|\,0,b,2) = 1-\exp\left[-\left(\frac{y}{b}\right)^2\right]-\frac{\exp\left[-\frac{1}{2}\left(\frac{y}{b}\right)^2\right]}{b}\sqrt{\frac{\pi}{2}}\,\mathrm{erf}\left(\frac{y}{b\,\sqrt{2}}\right)$$

[19] In Sect. 3.1.4 we have already mentioned that the WEIBULL distribution is not reproductive through summation.

$\boxed{\text{c=3}}$

$$f_2(y\,|\,0,b,3) = \frac{\exp\left[-\left(\frac{y}{b}\right)^3\right]}{16\,b^{9/2}\,y^{5/2}}\left\{6\,(b\,y)^{3/2}\,(y^3-2\,b^3)+\sqrt{3\,\pi}\,\exp\left[\frac{3}{4}\left(\frac{y}{b}\right)^3\right]\times\right.$$
$$\left.\left[4\,b^6-4\,(b\,y)^3+3\,y^6\right]\,\mathrm{erf}\left[\frac{\sqrt{3}}{2}\left(\frac{y}{b}\right)^{3/2}\right]\right\}$$

$$F_2(y\,|\,0,b,3) = 1-\frac{\exp\left[-\left(\frac{y}{b}\right)^3\right]}{2}-\frac{\exp\left[-\frac{1}{4}\left(\frac{y}{b}\right)^3\right]}{4\,(b\,y)^{3/2}}\sqrt{\frac{\pi}{3}}\,(2\,b^3+3\,y^3)\,\mathrm{erf}\left[\frac{\sqrt{3}}{2}\left(\frac{y}{b}\right)^{3/2}\right]$$

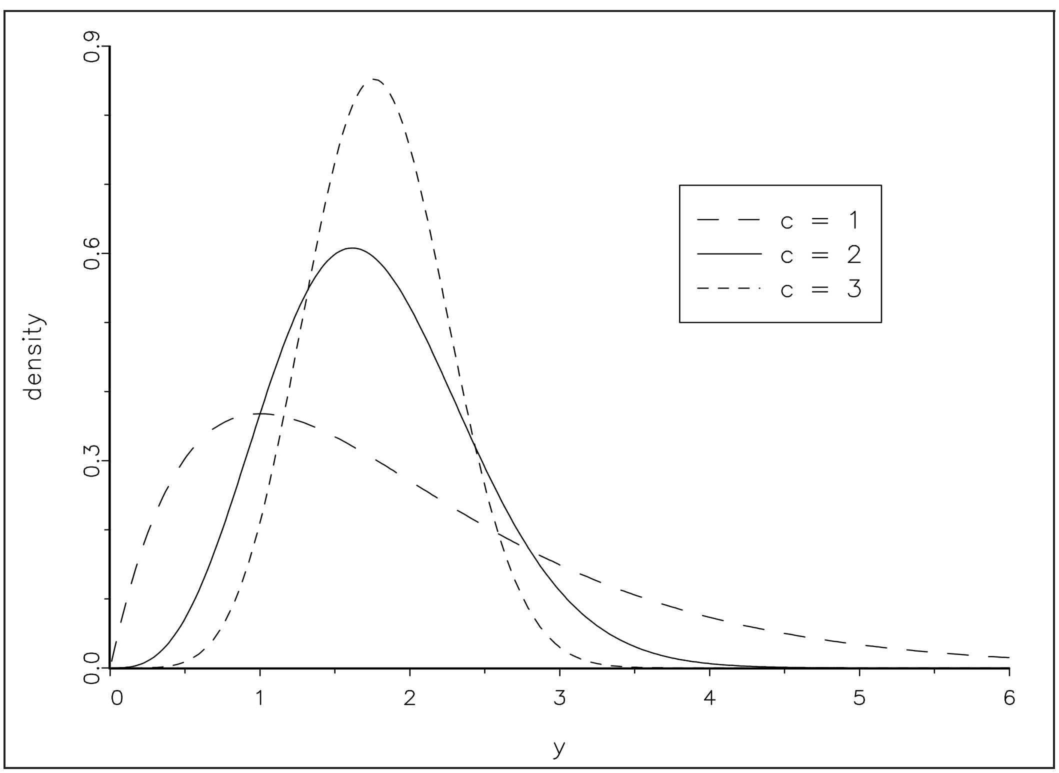
$\mathrm{erf}(\cdot)$ is the **GAUSSIAN error function**:

$$\mathrm{erf}(\tau)=\frac{2}{\sqrt{\pi}}\int\limits_0^\tau e^{-t^2}\,\mathrm{d}t, \tag{3.53a}$$

which is related to $\Phi(z)$, the CDF of the standardized normal variate by

$$\Phi(\tau)=\frac{1}{2}\left[1+\mathrm{erf}(\tau/\sqrt{2})\right] \quad \text{for } \tau\geq 0. \tag{3.53b}$$

Figure 3/9: Densities of a two-fold convolution of reduced WEIBULL variates



3.3.6.2 WEIBULL models for parallel and series systems

Two simple systems in reliability theory are the **series system** and the **parallel system**, each consisting of n components that will fail at random. For component i $(1 \le i \le n)$ let $F_i(t) = \Pr(X_i \le t)$ be the probability of failure up to time t, then $R_i(t) = 1 - F_i(t) = \Pr(X_i > t)$ is the probability of its surviving of time t.

A series system is working as long as all its n components are working, so that the series survival probability of time t is given by

$$\begin{aligned} R_S(t) &= \Pr(X_1 > t,\ X_2 > t,\ \ldots,\ X_n > t) \\ &= \prod_{i=1}^{n} R_i(t). \end{aligned} \tag{3.54a}$$

(3.54a) holds only for independently failing components. $R_S(t)$ is a decreasing function of n with $R_S(t) \le R_i(t)\ \forall\ i$. So, a series system is more reliable the less components are combined into it. The series system's failure distribution follows as

$$F_S(t) = 1 - \prod_{i=1}^{n} R_i(t) = 1 - \prod_{i=1}^{n} \big[1 - F_i(t)\big] \tag{3.54b}$$

with DF

$$\begin{aligned} f_S(t) &= \sum_{i=1}^{n} \left\{ \prod_{j=1, j\ne i}^{n} \big[1 - F_j(t)\big] \right\} f_i(t), \\ &= R_S(t) \sum_{i=1}^{n} \frac{f_i(t)}{R_i(t)}\,. \end{aligned} \tag{3.54c}$$

Thus, the HR of a series system simply is

$$\begin{aligned} h_S(t) &= \frac{f_S(t)}{R_S(t)} = \sum_{i=1}^{n} \frac{f_i(t)}{R_i(t)} \\ &= \sum_{i=1}^{n} h_i(t), \end{aligned} \tag{3.54d}$$

i.e., the sum of each component's hazard rate.

A parallel system is working as long as at least one of its n components is working and fails when all its components have failed. The failure distribution CDF is thus

$$\begin{aligned} F_P(t) &= \Pr(X_1 \le t,\ X_2 \le t,\ \ldots,\ X_n \le t) \\ &= \prod_{i=1}^{n} F_i(t) \end{aligned} \tag{3.55a}$$

for independently failing units. $F_P(t)$ is a decreasing function of n with $F_P(t) \leq F_i(t)\ \forall\, i$. Thus, a parallel system is more reliable the more components it has. The system's reliability function or survival function is

$$R_P(t) = 1 - \prod_{i=1}^{n} F_i(t) = 1 - \prod_{i=1}^{n} \big[1 - R_i(t)\big] \tag{3.55b}$$

and its DF is

$$\begin{aligned} f_P(t) &= \sum_{i=1}^{n} \left\{ \prod_{j=1, j\neq i}^{n} F_j(t) \right\} f_i(t) \\ &= F_P(t) \sum_{i=1}^{n} \frac{f_i(t)}{F_i(t)}\,. \end{aligned} \tag{3.55c}$$

The HR follows as

$$\begin{aligned} h_P(t) &= \frac{f_P(t)}{R_P(t)} = \frac{F_P(t)}{1 - F_P(t)} \sum_{i=1}^{n} \frac{f_i(t)}{F_i(t)} \\ &= \frac{F_P(t)}{R_P(t)} \sum_{i=1}^{n} \frac{f_i(t)}{R_i(t)}\, \frac{R_i(t)}{F_i(t)} \\ &= \frac{F_P(t)}{R_P(t)} \sum_{i=1}^{n} h_i(t)\, \frac{R_i(t)}{F_i(t)}\,, \end{aligned} \tag{3.55d}$$

i.e., HR is a weighted sum of the individual hazard rates, where the weights $R_i(t)\big/F_i(t)$ are decreasing with t. This sum is multiplied by the factor $F_P(t)\big/R_P(t)$ which is increasing with t.

Before evaluating (3.54a–d) and (3.55a-d) using WEIBULL distributions, we should mention that the two preceding systems are borderline cases of the more general "***k*–out–of–*n*–system**", which is working as long as at least k of its n components have not failed. Thus, an series system is a "n–out–of–n" system and a parallel system is a "1–out–of–n" system. The reliability function of a "k–out–of–n" system is given by

$$R_{k,n}(t) = \sum_{m=k}^{n} \left[\sum_{Z(m,n)} R_{i_1}(t) \cdot\ \ldots\ \cdot R_{i_m}(t) \cdot F_{i_{m+1}}(t) \cdot\ \ldots\ \cdot F_{i_n}(t) \right]. \tag{3.56}$$

$Z(m, n)$ is the set of all partitions of n units into two classes such that one class comprises m non–failed units and the other class $n-m$ failed units. An inner sum gives the probability that exactly m of the n units have not failed. The outer sum rums over all $m \geq k$.

We first evaluate the series system, also known as **competing risk model** or **multi–risk model**, with WEIBULL distributed lifetimes of its components.[20] Assuming $X_i \overset{\text{iid}}{\sim}$

[20] MURTHY/XIE/JIANG (2004) study multi–risk models with inverse WEIBULL distributions and hybrid models for which one component has a WEIBULL distributed lifetime and the remaining components are non–WEIBULL.

$We(a,b,c)$; $i = 1, 2, \ldots, n$; (3.54a) turns into

$$R_S(t) = \exp\left\{-n\left(\frac{t-a}{b}\right)^c\right\} = \exp\left\{-\left(\frac{t-a}{b\,n^{-1/c}}\right)^c\right\}. \tag{3.57a}$$

(3.57a) is easily recognized as the CCDF of a variate $T \sim We(a, b\,n^{-1/c}, c)$, the minimum of n iid WEIBULL variates (see Sect. 3.1.4). Assuming $X_i \overset{\text{iid}}{\sim} We(a, b_i, c)$; $i = 1, 2, \ldots, n$; i.e., the scale factor is varying with i and the series system's lifetime T is still WEIBULL with

$$R_S(t) = \exp\left\{-(t-a)^c \sum_{i=1}^{n} \frac{1}{b_i^c}\right\}. \tag{3.57b}$$

Now, the scale parameter of T is $\left(\sum\limits_{i=1}^{n} b_i^{-c}\right)^{-1/c}$. T has no WEIBULL distribution when the shape parameters and/or the location parameters of the components' lifetimes are differing. In the general case, i.e., $X_i \sim We(a_i, b_i, c_i)$ and independent, assuming $c_1 < c_2 < \ldots < c_n$ we can state some results on the hazard rate $h_S(t)$ of (3.54d).

- The asymptotic properties are
  - ◇ $h_S(t) \approx h_1(t)$ for $t \to 0$,
  - ◇ $h_S(t) \approx h_n(t)$ for $t \to \infty$.

  For small t the hazard rate of the series system is nearly the same as that for the component with the smallest shape parameter. For large t, it is approximately the hazard rate for the component with the largest shape parameter.

- $h_S(t)$ can have only one of three possible shapes:
  - ◇ $h_S(t)$ is decreasing when $c_1 < c_2 < \ldots < c_n < 1$.
  - ◇ $h_S(t)$ is increasing when $1 < c_1 < c_2 < \ldots < c_n$.
  - ◇ $h_S(t)$ is bathtub–shaped when $c_1 < \ldots < c_{j-1} < 1 < c_j < \ldots < c_n$.

Parameter estimation in this general case is extensively discussed by MURTHY/XIE/JIANG (2004, pp. 186–189) and by DAVISON/LOUZADA–NETO (2000), who call this distribution a **poly-WEIBULL model**.

A parallel system with WEIBULL distributed lifetimes of its components is sometimes called a **multiplicative WEIBULL model**, see (3.55a). Assuming $X_i \overset{\text{iid}}{\sim} We(a,b,c)$; $i = 1, 2, \ldots, n$, (3.55a) turns into

$$F_P(t) = \left[1 - \exp\left\{-\left(\frac{t-a}{b}\right)^c\right\}\right]^n. \tag{3.58a}$$

(3.58a) is a special case of the **exponentiated WEIBULL distribution**, introduced by MUDHOLKAR/SRIVASTAVA (1993) and analyzed in subsequent papers by MUDHOLKAR et al. (1995), MUDHOLKAR/HUTSON (1996) and JIANG/MURTHY (1999). Whereas in (3.58a) the parameter n is a positive integer, it is substituted by $\nu \in \mathbb{R}$ to arrive at the exponentiated WEIBULL distribution.

The DF belonging to (3.58a) is

$$\begin{aligned} f_P(t) &= n\, f_X(t)\,\big[F_X(t)\big]^{n-1} \\ &= n\,\frac{c}{b}\left(\frac{x-a}{b}\right)^{c-1}\exp\left\{-\left(\frac{x-a}{b}\right)^{c}\right\}\left[1-\exp\left\{-\left(\frac{x-a}{b}\right)^{c}\right\}\right]^{n-1} \end{aligned} \tag{3.58b}$$

$f_P(t)$ is (see the upper part of Fig. 3/10)

- monotonically decreasing for $n\,c \le 1$ with $f_P(0) = \infty$ for $n\,c < 1$ and $f_P(0) = 1/b$ for $n\,c = 1$,
- unimodal for $n\,c > 1$ with — according to MUDHOLKAR/HUTSON (1996) — a mode approximated by

$$t^* = a + b\left\{\frac{1}{2}\left[\frac{\sqrt{c\,(c-8\,n+2\,c\,n+9\,c\,n^2)}}{c\,n} - 1 - \frac{1}{n}\right]\right\}^{n}. \tag{3.58c}$$

It is noteworthy that the effect of the product $c\,n$ on the shape of the DF for the exponentiated WEIBULL distribution is the same as that of c in the simple WEIBULL distribution.

The HR belonging to (3.58a) is

$$\begin{aligned} h_P(t) &= \frac{f_P(t)}{1-F_P(t)} \\ &= n\,\frac{c}{b}\left(\frac{x-a}{b}\right)^{c-1}\frac{\exp\left\{-\left(\frac{x-a}{b}\right)^{c}\right\}\left[1-\exp\left\{-\left(\frac{x-a}{b}\right)^{c}\right\}\right]^{n-1}}{1-\left[1-\exp\left\{-\left(\frac{x-a}{b}\right)^{c}\right\}\right]^{n}} \end{aligned} \tag{3.58d}$$

For small t we have

$$h_P(t) \approx \left(\frac{n\,c}{b}\right)\left(\frac{t-a}{b}\right)^{n\,c-1},$$

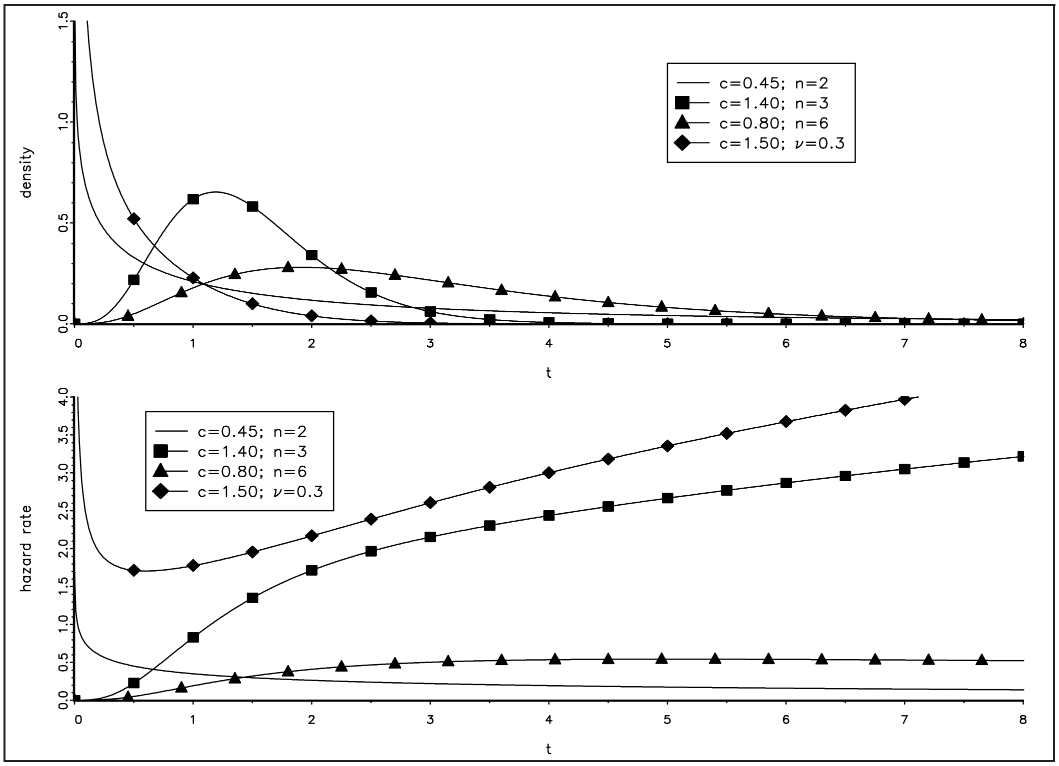
i.e., the system's hazard rate can be approximated by the hazard rate of a WEIBULL distribution with shape parameter $n\,c$ and scale parameter b. For large t we have

$$h_P(t) \approx \left(\frac{c}{b}\right)\left(\frac{t-a}{b}\right)^{c-1},$$

i.e., the approximating hazard rate is that of the underlying WEIBULL distribution. The shape of the HR is as follows (also see the lower part of Fig. 3/10):

- monotonically decreasing for $c \le 1$ and $n\,c \le 1$,
- monotonically increasing for $c \ge 1$ and $n\,c \ge 1$,
- unimodal for $c < 1$ and $n\,c > 1$,
- bathtub–shaped for $c > 1$ and $\nu\,c < 1,\ \nu \in \mathbb{R}^+$.

Figure 3/10: Densities and hazard rates of multiplicative WEIBULL models $(a = 0,\ b = 1)$



The raw moments belonging to (3.58a), but with $a = 0$, are

$$\mu_r' = n\, b^r\, \Gamma\Big(1 + \frac{r}{c}\Big) \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} \frac{1}{(j+1)^{1+r/c}}\,. \tag{3.58e}$$

The percentile function is simply

$$F_P^{-1}(P) = a + b\Big\{-\ln\big[1 - P^{1/n}\big]\Big\}^{1/c},\ 0 < P < 1. \tag{3.58f}$$

General results on a multiplicative WEIBULL model with components having differing parameter values are not easy to find. When the components all have a two–parameter WEIBULL distribution $(a_i = 0\ \forall\ i)$ and assuming $c_1 \le c_2 \le \ldots \le c_n$ and $b_i \ge b_j$ for $i < j$ if $c_i = c_j$, MURTHY/XIE/JIANG (2004, p. 198/9) state the following results:

$$F_P(t) \approx F_0(t) \ \text{ for small } \ t \tag{3.59a}$$

with $F_0(t)$ as the two–parameter WEIBULL CDF having

$$c_0 = \sum_{i=1}^{n} c_i \ \text{ and } \ b_0 = \sum_{i=1}^{n} b_i^{c_i/c_0},$$

$$F_P(t) \approx (1-k) + k\, F_1(t) \ \text{ for large } \ t, \tag{3.59b}$$

where k is the number of components with distribution identical to $F_1(t)$ having c_1 and b_1. The DF can be decreasing or may have k modes. The HR can never have a bathtub shape, but has four possible shapes (decreasing, increasing, k modal followed by increasing or decreasing followed by k modal). The asymptotes of the HR are

$$h_P(t) \approx \left\{ \begin{array}{lll} h_0(t) = \dfrac{c_0}{b_0}\left(\dfrac{t}{c_0}\right)^{c_0-1} & \text{for} & t \to 0 \\ h_1(t) = \dfrac{c_1}{b_1}\left(\dfrac{t}{b_1}\right)^{c_1-1} & \text{for} & t \to \infty. \end{array} \right\} \tag{3.59c}$$

3.3.6.3 Composite WEIBULL distributions

A **composite distribution**, also known as **sectional model** or **piecewise model**, is constructed by knotting together pieces of two or more distributions in such a manner that the resulting distribution suffices some smoothness criteria. The task is somewhat similar to that of constructing a spline function. In reliability theory and life testing the composite WEIBULL distribution, first introduced by KAO (1959) to model the lifetime distribution of electronic tubes, is primarily used to arrive at a non–monotone hazard rate which is not encountered in only one WEIBULL distribution. The bathtub shape, regularly found in life–tables as the behavior of the conditional probability of death at age x, is of interest, see ELANDT–JOHNSON/JOHNSON (1980, Chapter 7.5). We will present the WEIBULL composite distribution having $n \geq 2$ WEIBULL subpopulations. Of course, it is possible to join WEIBULL and other distributions into a so–called hybrid composite model.

An n–fold composite WEIBULL CDF is defined as

$$F_{nc}(x) = F_i(x \,|\, a_i, b_i, c_i) \;\text{ for }\; \tau_{i-1} \leq x \leq \tau_i;\; i = 1, 2, \ldots, n, \tag{3.60a}$$

where $F_i(x \,|\, a_i, b_i, c_i) = 1 - \exp\left\{-\left(\dfrac{x - a_i}{b_i}\right)^{c_i}\right\}$ is the i–th component in CDF form. The quantities τ_i are the knots or points of component partition. Because the end points are $\tau_0 = a_1$ and $\tau_n = \infty$, for a n–fold composite model there are only $n - 1$ partition parameters. The n–fold composite WEIBULL DF is

$$f_{nc}(x) = f_i(x \,|\, a_i, b_i, c_i) \tag{3.60b}$$

$$= \frac{c_i}{b_i}\left(\frac{x - a_i}{b_i}\right)^{c_i-1} \exp\left\{-\left(\frac{x - a_i}{b_i}\right)^{c_i}\right\} \text{ for } \tau_{i-1} \leq x \leq \tau_i;\; i = 1, 2, \ldots, n; \tag{3.60c}$$

with corresponding HR

$$h_{nc}(x) = \frac{f_i(x \,|\, a_i, b_i, c_i)}{1 - F_i(x \,|\, a_i, b_i, c_i)} = h_i(x \,|\, a_i, b_i, c_i) \;\text{ for }\; \tau_{i-1} \leq x \leq \tau_i;\; i = 1, 2, \ldots, n. \tag{3.60d}$$

In the interval $[\tau_{i-1}; \tau_i]$ the portion

$$P_i = F_i(\tau_i \,|\, a_i, b_i, c_i) - F_{i-1}(\tau_{i-1} \,|\, a_{i-1}, b_{i-1}, c_{i-1});\; i = 1, \ldots, n, \tag{3.60e}$$

with

$$F_0(\cdot) = 0 \text{ and } F_n(\cdot) = 1,$$

is governed by the i–th WEIBULL distribution.

We first look at a composite WEIBULL model requiring only that the CDFs join continuously:

$$F_i(\tau_i \,|\, a_i, b_i, c_i) = F_{i+1}(\tau_i \,|\, a_{i-1}, b_{i+1}, c_{i+1});\; i = 1, \ldots, n. \tag{3.61a}$$

Fulfillment of (3.61a) does not guarantee that the DFs and the HRs join without making a jump (see Fig. 3/11). At those endpoints, where two CDFs have to be joined, seven parameters are involved: three parameters for each distribution and the partition parameter τ_i. Because of the restriction (3.61a) six of the seven parameters are independent. The most common way to construct (3.60a) is to fix the six parameters $\tau_i,\ a_i,\ b_i,\ c_i,\ b_{i+1},\ c_{i+1}$ $(i = 1, 2, \ldots, n-1)$ and to shift the right–hand CDF with index $i+1$, i.e., to determine a_{i+1} so as to fulfill (3.61a). Thus, the resulting location parameter a_{i+1} is given by

$$a_{i+1} = \tau_i - b_{i+1}\left(\frac{\tau_i - a_i}{b_i}\right)^{c_i/c_{i+1}};\; i = 1, 2, \ldots, n-1. \tag{3.61b}$$

Fig. 3/11 shows the CDF, DF and HR together with the WEIBULL probability plot for a three–fold composite model with the following parameter set:

$$a_1 = 0;\; b_1 = 1;\; c_1 = 0.5;\; \tau_1 = 0.127217 \implies P_1 = 0.3;$$
$$a_2 = -0.586133;\; b_2 = 2;\; c_2 = 1;\; \tau_2 = 1.819333 \implies P_2 = 0.4;$$
$$a_3 = 1.270987;\; b_3 = 0.5;\; c_3 = 2 \implies P_3 = 0.3.$$

When we want to have a composite WEIBULL model with both the CDFs and the DFs joining continuously, we have to introduce a second continuity condition along with (3.61a):

$$f_i(\tau_i \,|\, a_i, b_i, c_i) = f_{i+1}(\tau_i \,|\, a_{i+1}, b_{i+1}, c_{i+1});\; i = 1, 2, \ldots, n-1. \tag{3.62a}$$

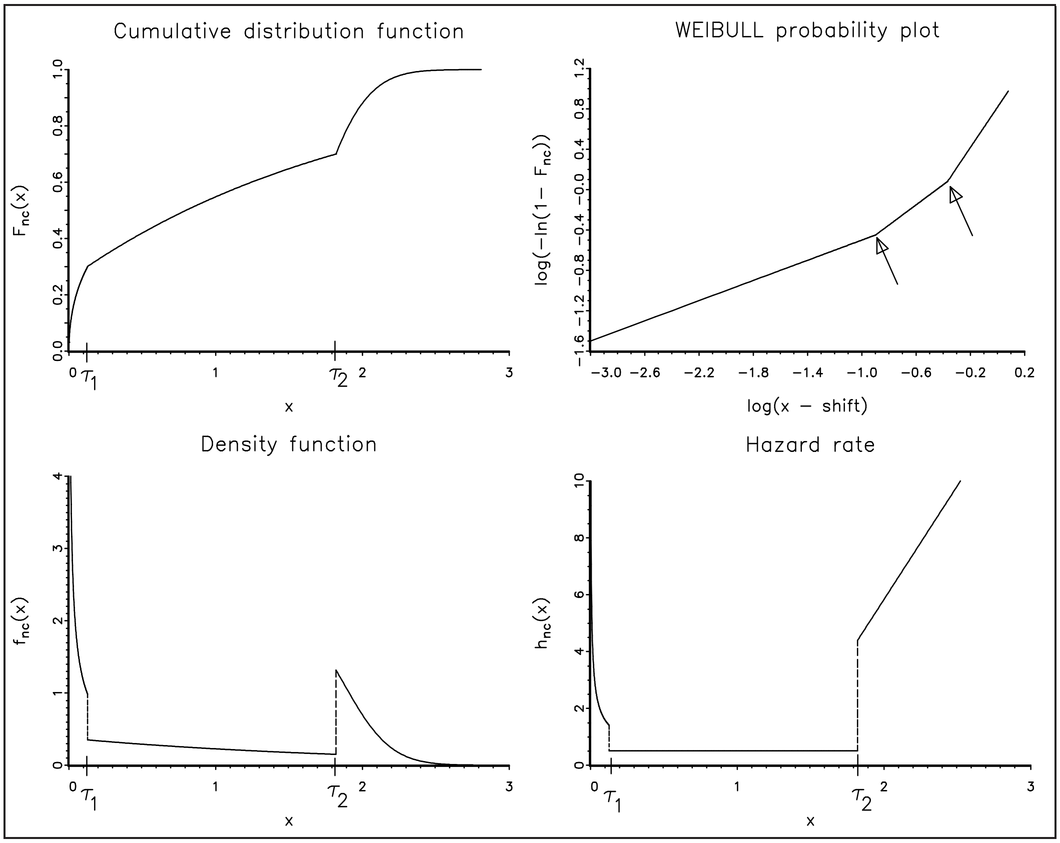
(3.61a) and (3.62a) guarantee a continuous hazard rate, too. In order to satisfy (3.61a) and (3.62a) the seven parameters coming up at knot i have to fulfill the following two equations:

$$a_i = \tau_i - \left(\frac{c_{i+1}}{c_i}\,\frac{b_i^{c_i/c_{i+1}}}{b_{i+1}}\right)^{c_{i+1}/(c_i-c_{i+1})}, \tag{3.62b}$$

$$a_{i+1} = \tau_i - b_{i+1}\left(\frac{\tau_i - a_i}{b_i}\right)^{c_i/c_{i+1}}, \tag{3.62c}$$

so five out of the seven parameters are independent.

Figure 3/11: Composite WEIBULL distribution with continuously joining CDFs



A common procedure to construct a composite WEIBULL distribution supposes that at least the knots τ_i and the shape parameters c_i are fixed. Then, the algorithm is as follows:

1. Given $\tau_1,\ b_1,\ c_1,\ b_2,\ c_2$, the location parameters a_1 and a_2 of the first two distributions are determined using (3.62b,c).
2. For the following knots $(i = 2, \ldots, n-1)$ the parameters τ_i, a_i (from the preceding step), $b_i,\ c_i$ and c_{i+1} are known. b_{i+1}, the scale parameter of distribution $i+1$, is calculated as

$$b_{i+1} = \frac{c_{i+1}}{c_i}\, b_i^{c_i/c_{i+1}} \left(\tau_i - a_i\right)^{-(c_i - c_{i+1})/c_{i+1}}. \tag{3.62d}$$

   Then a_{i+1} follows from (3.62c).

Fig. 3/12 gives an example for a three–fold composite and completely continuous WEIBULL model. The predetermined parameters are

$$\tau_1 = 1;\ b_1 = 1;\ c_1 = 0.5;\ b_2 = 2;\ c_2 = 1.0;\ \tau_2 = 2.0;\ c_2 = 3.$$

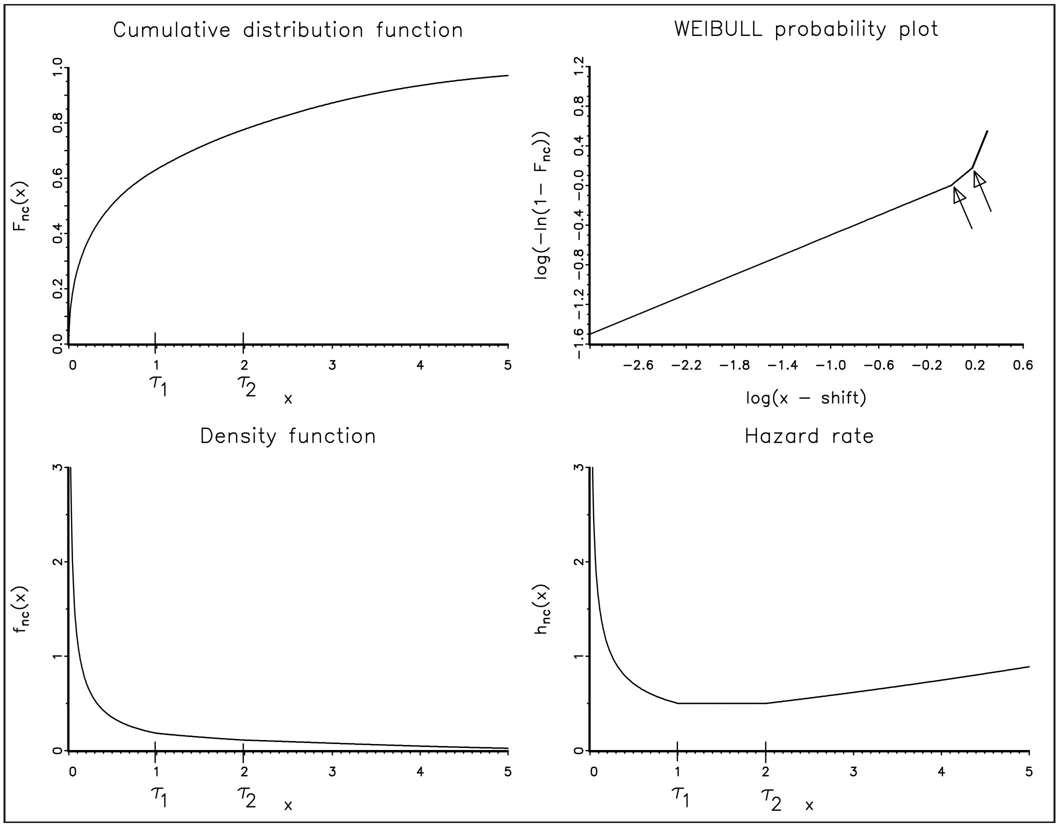
The algorithm yields

$$a_1 = 0;\ a_2 = -1;\ b_3 \approx 7.862224;\ a_3 \approx -7.$$

The portions P_i of the three–sectional distributions are

$$P_1 \approx 0.6321;\ P_2 \approx 0.1448;\ P_3 \approx 0.2231.$$

Figure 3/12: Composite WEIBULL distribution with continuously joining CDFs, DFs and HRs



Moments of composite WEIBULL distribution have to be determined by numerical integration. Percentiles are easily calculated because (3.60a) can be inverted in closed form.

There are mainly two approaches to estimate the parameters of a composite WEIBULL distribution. The graphical approach rests either upon the WEIBULL probability–plot, see KAO (1959) or MURTHY/XIE/JIANG (2004), or on the WEIBULL hazard–plot, see ELANDT–JOHNSON/JOHNSON (1980), and adheres to the kinks (Fig. 3/11 and 3/12) that will also come up in the empirical plots. The numerical approach consists in a maximum likelihood estimation; see AROIAN/ROBINSON (1966) and COLVERT/BOARDMAN (1976).

3.3.6.4 Mixed WEIBULL distributions[21]

The mixing of several WEIBULL distributions is another means to arrive at a non–monotone hazard rate, but this will never have a bathtub–shape as will be shown further down. A

[21] Suggested reading for this section: Since the first paper on mixed WEIBULL models by KAO (1959), the literature on this topic has grown at an increasing pace. MURTHY/XIE/JIANG (2004) give an up–to–date overview of this relative to the classical WEIBULL distribution.

discrete or **finite mixture**[22] is a linear combination of two or more WEIBULL distributions that — contrary to the composite model of the preceding section — have no restricted support.[23] A simple explanation of discretely mixed distributions is the following: The population under consideration is made up of $n \geq 2$ subpopulations contributing the portion $\omega_i,\ i = 1, 2, \ldots, n; 0 < \omega_i < 1; \sum_{i=1}^{n} \omega_i = 1;$ to the entire population. For example, a lot of items delivered by a factory has been produced on n production lines which are not working perfectly identically so that some continuous quality characteristic X of the items has DF $F_i(x)$ when produced on line i.

Before turning to WEIBULL mixtures we summarize some general results on mixed distributions:

Mixed DF

$$f_m(x) = \sum_{i=1}^{n} \omega_i\, f_i(x);\ \ 0 < \omega_i < 1;\ \ \sum_{i=1}^{n} \omega_i = 1; \tag{3.63a}$$

Mixed CDF und CCDF

$$F_m(x) \ = \ \sum_{i=1}^{n} \omega_i\, F_i(x); \tag{3.63b}$$

$$R_m x) \ = \ \sum_{i=1}^{n} \omega_i\, R_i(x); \tag{3.63c}$$

Mixed HR

$$\begin{aligned} h_m(x) &= \frac{f_m(x)}{R_m(x)} \\ &= \sum_{i=1}^{n} g_i(x)\, h_i(x), \end{aligned} \tag{3.63d}$$

where $h_i(x) = f_i(x)/R_i(x)$ is the hazard rate of subpopulation i and the weights $g_i(x)$ are given by

$$g_i(x) = \frac{\omega_i\, R_i(x)}{R_m(x)},\quad \sum_{i=1}^{n} g_i(x) = 1 \tag{3.63e}$$

and are thus varying with x;

Mixed raw moments

$$\mathrm{E}\big(X^r\big) = \sum_{i=1}^{n} \omega_i\, \mathrm{E}\big(X_i^r\big),\ \text{where } \mathrm{E}\big(X_i^r\big) = \int x_i^r\, f_i(x)\,\mathrm{d}x. \tag{3.63f}$$

[22] A **continuous** or **infinite mixture**, where the weighting function is a DF for one of the WEIBULL parameters, will be presented in Sect. 3.3.6.5. In this case the WEIBULL parameter is a random variable.

[23] We will not present **hybrid mixtures** where some of the mixed distributions do not belong to the WEIBULL family. Some papers on hybrid mixtures are AL–HUSSAINI/ABD–EL–HAKIM (1989, 1990, 1992) on a mixture of an inverse GAUSSIAN and a two–parameter WEIBULL distribution, CHANG (1998) on a mixture of WEIBULL and GOMPERTZ distributions to model the death rate in human life–tables, LANDES (1993) on a mixture of normal and WEIBULL distributions and MAJESKE/HERRIN (1995) on a twofold mixture of WEIBULL and uniform distributions for predicting automobile warranty claims.

From (3.63f) we have two special results:

Mean of the mixed distribution

$$\mathrm{E}(X) = \sum_{i=1}^{n} \omega_i \, \mathrm{E}\big(X_i\big); \tag{3.63g}$$

Variance of the mixed distribution

$$\begin{aligned}
\mathrm{Var}(X) &= \mathrm{E}\big\{\big[X - \mathrm{E}(X)\big]^2\big\} = \mathrm{E}\big(X^2\big) - \big[\mathrm{E}(X)\big]^2 \\
&= \sum_{i=1}^{n} \omega_i \, \mathrm{E}\big(X_i^2\big) - \left[\sum_{i=1}^{n} \omega_i \, \mathrm{E}\big(X_i\big)\right]^2 \\
&= \sum_{i=1}^{n} \omega_i \left[\mathrm{Var}(X_i) + \mathrm{E}(X_i)\right]^2 - \left[\sum_{i=1}^{n} \omega_i \, \mathrm{E}(X_i)\right]^2 \\
&= \sum_{i=1}^{n} \omega_i \mathrm{Var}(X_i) + \sum_{i=1}^{n} \omega_i \left[\mathrm{E}(X_i) - \mathrm{E}(X)\right]^2 .
\end{aligned} \tag{3.63h}$$

Thus, the variance of the mixed distribution is the sum of two non–negative components. The first component on the right–hand side of (3.63h) is the mean of the n variances. It is termed the **internal variance** or **within–variance**. The second component is the **external variance** or **between–variance**, showing how the means of the individual distributions vary around their common mean $\mathrm{E}(X)$.

Inserting the formulas describing the functions and moments of n WEIBULL distributions into (3.63a–h) gives the special results for a mixed WEIBULL model. Besides parameter estimation (see farther down) the behavior of the density and the hazard rate of a mixed WEIBULL distribution have attracted the attention of statisticians.

We start by first developing some approximations to the CDF of a mixed WEIBULL model where the n subpopulations all have a two–parameter WEIBULL distribution[24] and assuming in the sequel, without loss of generality, that $c_i \le c_j,\ i < j$, and $b_i > b_j$, when $c_i = c_j$. Denoting

$$y_i = \left(\frac{x}{b_i}\right)^{c_i}$$

we have

$$\lim_{x \to 0}\left(\frac{y_i}{y_1}\right) = \left\{\begin{matrix} 0 & \text{if} \quad c_i > c_1 \\ \left(\dfrac{b_1}{b_i}\right)^{c_1} & \text{if} \quad c_i = c_1 \end{matrix}\right\} \tag{3.64a}$$

and

$$\lim_{x \to \infty}\left(\frac{y_i}{y_1}\right) = \left\{\begin{matrix} \infty & \text{if} \quad c_i > c_1 \\ \left(\dfrac{b_1}{b_i}\right)^{c_1} > 1 & \text{if} \quad c_i = c_1 \end{matrix}\right\}. \tag{3.64b}$$

[24] The results for a three–parameter WEIBULL distribution are similar because the three–parameter distribution reduces to a two–parameter distribution under a shifting of the time scale.

From (3.64a,b) JIANG/MURTHY (1995) derive the following results from which the behavior of the hazard and density functions may be deduced:

1. For small x, i.e., very close to zero,
$$F_m(x) = \sum_{i=1}^{n} \omega_i \, F_i(x \,|\, 0, b_i, c_i)$$
can be approximated by
$$F_m(x) \approx g \, F_1(x \,|\, 0, b_1, c_1), \tag{3.65a}$$
where
$$g = \sum_{j=1}^{m} \omega_j \left(\frac{b_1}{b_j}\right)^{c_1} \tag{3.65b}$$
and m is the number of the subpopulations with the common shape parameter c_1. When $m = 1$, then $g = \omega_1$.
2. For large x, $F_m(x)$ can be approximated by
$$F_m(x) \approx 1 - \omega_1 \Big[1 - F_1(x \,|\, 0, b_1, c_1)\Big]. \tag{3.65c}$$

From (3.65a,b) the **density of the mixed WEIBULL model** can be approximated by
$$f_m(x) \approx g \, f_1(x \,|\, 0, b_1, c_1) \;\text{ for small }\; x \tag{3.66a}$$
and
$$f_m(x) \approx \omega_1 \, f_1(x \,|\, 0, b_1, c_1) \;\text{ for large }\; x, \tag{3.66b}$$
implying that $f_m(x)$ is increasing (decreasing) for small x if $c_1 > 1$ $(c_1 < 1)$. The shape of $f_m(x)$ depends on the model parameters, and the possible shapes are

- decreasing followed by $k - 1$ modes $(k = 1, 2, \ldots, n - 1)$,
- k–modal $(k = 1, 2, \ldots, n)$.

Of special interest is the two–fold mixture. The possible shapes in this case are

- decreasing,
- unimodal,
- decreasing followed by unimodal,
- bimodal.

Although the two–fold mixture model has five parameters ($b_1,\ b_2,\ c_1,\ c_2$ and ω_1), as $\omega_2 = 1 - \omega_1$, the DF–shape is only a function of the two shape parameters, the ratio of the two scale parameters and the mixing parameter. JIANG/MURTHY (1998) give a parametric characterization of the density function in this four–dimensional parameter space.

From (3.66a,c) the **hazard rate of the mixed WEIBULL model** can be approximated by
$$h_m(x) \approx g \, h_1(x \,|\, 0, b_1, c_1) \text{ for small } x \tag{3.67a}$$
and
$$h_m(x) \approx h_1(x \,|\, 0, b_1, c_1) \text{ for large } x. \tag{3.67b}$$

Thus, $h_m(x)$ is increasing (decreasing) for small x if $c_1 > 1$ $(c_1 < 1)$. The shape of $h_m(x)$ depends on the model parameters leading to the following possible shapes; see GUPTA/GUPTA (1996):

- decreasing,
- increasing,
- decreasing followed by k modes $(k = 1, 2, \ldots, n-1)$,
- k modal followed by increasing $(k = 1, 2, \ldots, n)$.

A special case of a decreasing mixed HR has already been given by PROSCHAN (1963).[25] He proved that a mixture of exponential distributions, being WEIBULL distribution with $c_i = 1;\ i = 1, 2, \ldots, n$, which have differing scale parameters b_i, will lead to a mixed model with decreasing hazard rate.

We now turn to the hazard rate of a two–fold WEIBULL mixture, which has been studied in some detail. This hazard rate follows from (3.63d,e) as

$$h_m(x) = \frac{\omega_1\, R_1(x\,|\,0,b_1,c_1)}{R_m(x)}\, h_1(x\,|\,0,b_1,c_1) + \frac{(1-\omega_1)\, R_2(x\,|\,0,b_2,c_2)}{R_m(x)}\, h_2(x\,|\,0,b_2,c_2). \tag{3.68a}$$

Assuming $c_1 \le c_2$ JIANG/MURTHY (1998) show

$$h_m(x) \to h_1(x\,|\,0,b_1,c_1) \ \text{ for large } \ x, \tag{3.68b}$$

$$h_m(x) \to g\, h_1(x\,|\,0,b_1,c_1) \ \text{ for small } \ x, \tag{3.68c}$$

with

$$g = \omega_1 \ \text{ for } c_1 < c_2 \ \text{ and } \ g = \omega_1 + (1-\omega_1)\left(\frac{b_1}{b_2}\right)^{c_1} \ \text{ for } \ c_1 = c_2.$$

(3.68b,c) imply that for small and large x the shape of $h_m(x)$ is similar to that of $h_1(x\,|\,\boldsymbol{.})$. When $c_1 < 1$ $(c_1 > 1)$, then $h_1(x|\boldsymbol{.})$ is decreasing (increasing) for all x. As a result, $h_m(x)$ is decreasing (increasing) for small and large x, so $h_m(x)$ cannot have a bathtub shape.[26] The possible shapes of (3.68a) are

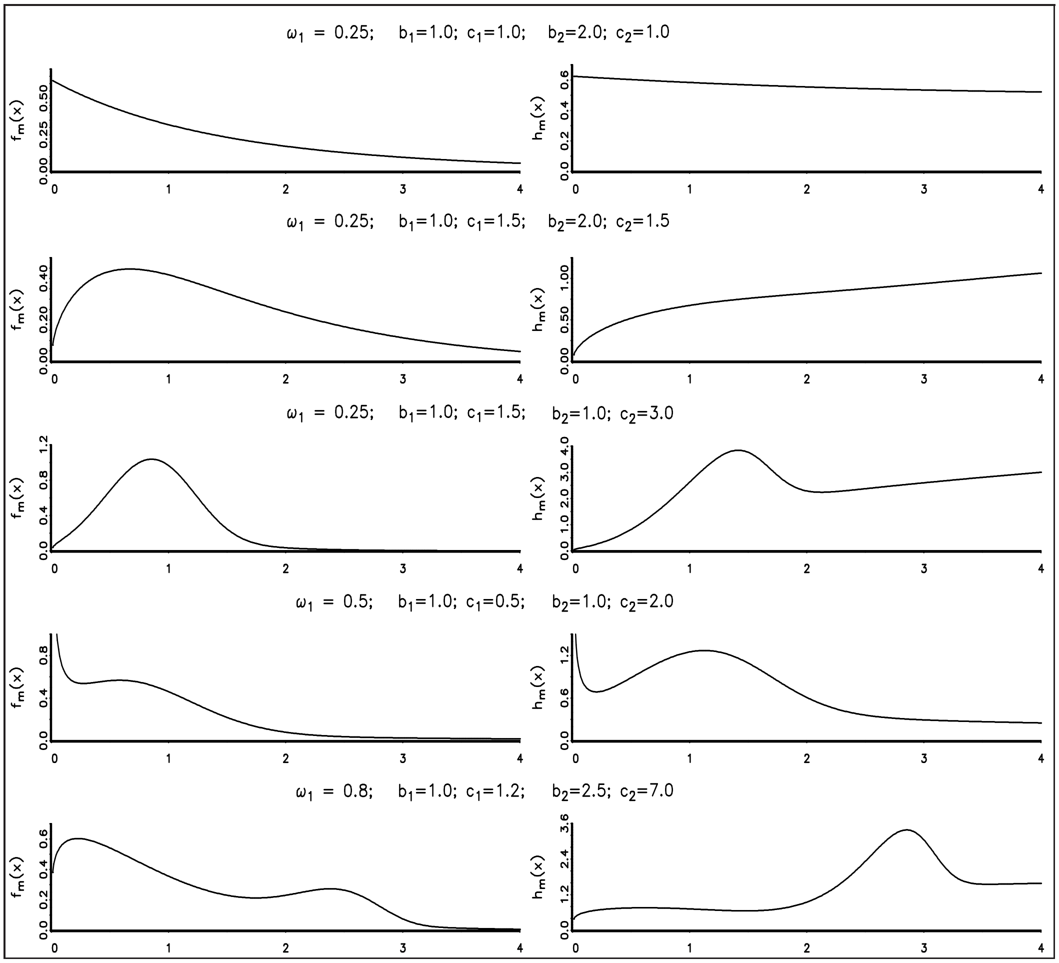
- decreasing,
- increasing,
- uni–modal followed by increasing,
- decreasing followed by uni–modal,
- bi–modal followed by increasing.

The conditions on the parameters $b_1,\ b_2,\ c_1,\ c_2$ and ω, leading to these shapes, are given in JIANG/MURTHY (1998). Fig. 3/13 shows a hazard rate (right–hand side) along with its corresponding density function (left–hand side) for the five possible hazard shapes.

[25] A paper with newer results on mixtures of exponential distributions is JEWELL (1982).

[26] One can even show that n–fold mixtures of WEIBULL distributions $(n > 2)$ can never have a bathtub shape; thus falsifying the assertion of KROHN (1969) that a three–fold mixture of WEIBULL distributions having $c_1 < 1,\ c_2 = 1$ and $c_3 > 1$ leads to a bathtub shape.

Figure 3/13: Hazard rate and density function of some two-fold WEIBULL mixtures



The literature on **parameter estimation** for the mixture model is vast. So, we will cite only some key papers. **Graphical methods** based the WEIBULL probability plot seem to be dominant. The first paper based on this approach is KAO (1959) on a two–fold mixture. His approach has been applied by several authors introducing minor modifications. A newer paper circumventing some of the drawbacks of older approaches is JIANG/MURTHY (1995). The graphical method may lead to satisfactory results for a two–fold mixture in well–separated cases, i.e., $b_1 \gg b_2$ or $b_1 \ll b_2$. Some papers on **analytical methods** of parameter estimation are

- RIDER (1961) and FALLS (1970) using the method of moments,
- ASHOUR (1987a) and JIANG/KECECIOGLU (1992b) applying the maximum likelihood approach,
- WOODWARD/GUNST (1987) taking a minimum–distance estimator,
- CHENG/FU (1982) choosing a weighted least–squares estimator,
- ASHOUR (1987b) and AHMAD et al. (1997) considering a BAYES approach.

3.3.6.5 Compound WEIBULL distributions[27]

The weights ω_i, used in mixing n WEIBULL densities $f_i(x \,|\, a_i, b_i, c_i)$, may be interpreted as the probabilities of n sets of values for the parameter vector (a, b, c). Generally, each parameter of a parametric distribution may be regarded as a continuous variate, so that the probabilities ω_i have to be substituted against a density function. The resulting continuous mixture is termed **compound distribution**. In the sequel we will discuss only the case where one of the parameters, denoted by Θ, is random. Let θ be a realization of Θ. The density of X, given θ, is conditional and denoted by $f_X(x \,|\, \theta)$. In the context of compounding it is termed **parental distribution**. The density of Θ, denoted by $f_\Theta(\theta)$, is termed **compounding distribution** or **prior distribution**.[28]

The **joint density** of the variates X and Θ is given by

$$f(x, \theta) = f_X(x \,|\, \theta)\, f_\Theta(\theta), \tag{3.69a}$$

from which the **marginal density** of X follows as

$$f(x) = \int f(x, \theta)\, \mathrm{d}\theta = \int f_X(x, \theta) f_\Theta(\theta)\, \mathrm{d}\theta. \tag{3.69b}$$

This is the DF of the compound distribution. In statistics and probability theory a special notation is used to express compounding. For example, let the parental distribution be normal with random mean Θ, θ being a realization of Θ, and fixed variance $\mathrm{Var}(X \,|\, \theta) = \sigma_*^2$, i.e., $X \,|\, \theta \sim \mathit{No}(\theta, \sigma_*^2)$. Supposing a prior distribution, which is also normal, $\Theta \sim \mathit{No}(\xi, \sigma_{**}^2)$, compounding in this case is denoted by

$$\mathit{No}(\Theta, \sigma_*^2) \bigwedge_{\Theta} \mathit{No}(\xi, \sigma_{**}^2).$$

$\bigwedge$ is the **compounding operator**. We shortly mention that in this example compounding is reproductive,

$$\mathit{No}(\Theta, \sigma_*^2) \bigwedge_{\Theta} \mathit{No}(\xi, \sigma_{**}^2) = \mathit{No}(\xi, \sigma_*^2 + \sigma_{**}^2).$$

Compound WEIBULL distributions have been intensively discussed with respect to a **random scale factor**. This scale factor is the reciprocal of the combined scale–shape parameter introduced by W. WEIBULL; see (2.26a,b):

$$\beta := 1/b^c = b^{-c}. \tag{3.70a}$$

Thus, the parental WEIBULL DF reads as

$$f_X(x \,|\, \beta) = c\, \beta\, (x - a)^{c-1} \exp\{-\beta\, (x - a)^c\}. \tag{3.70b}$$

The results pertaining to this conditional WEIBULL distribution which have been reported by DUBEY (1968) und HARRIS/SINGPURWALLA (1968) are special cases of a more general **theorem of ELANDT–JOHNSON** (1976):

[27] Suggested reading for this section: DUBEY (1968), ELANDT–JOHNSON (1976), HARRIS/SINGPUWALLA (1968, 1969).

[28] The term *prior distribution* is used in BAYES inference, see Sect. 14, which is related to compounding.

Let

$$F_X(x,\boldsymbol{\alpha}\,|\,\lambda) = \left\{ \begin{array}{ccc} 0 & \text{for} & x \le x_0 \\ 1-\exp\{-\lambda\,u(x,\boldsymbol{\alpha})\} & \text{for} & x > x_0 \end{array} \right\} \tag{3.71a}$$

be the CDF of the parental distribution, which is of **exponential type** with λ as a "special" parameter and $\boldsymbol{\alpha}$ as a vector of "ordinary" parameters. $u(x,\boldsymbol{\alpha})$ is an increasing function of x with

$$\begin{array}{llcl} u(x,\boldsymbol{\alpha}) & \longrightarrow 0 & \text{as} & x \to x_0, \\ u(x,\boldsymbol{\alpha}) & \longrightarrow \infty & \text{as} & x \to \infty. \end{array}$$

(The parental WEIBULL distribution (3.70b) is of this type with $\lambda = \beta$ and $u(x,\boldsymbol{\alpha}) = (x-a)^c$.) Further, let the compounding distribution have a raw moment generating function

$$\mathrm{E}_\Lambda(e^{t\,\lambda}) =: M_\Lambda(t).$$

Then, the CDF of the compound distribution is given by

$$F(x) = \left\{ \begin{array}{ccc} 0 & \text{for} & x \le x_0 \\ 1-M_\Lambda\big[-u(x,\boldsymbol{\alpha})\big] & \text{for} & x > x_0, \end{array} \right\} \tag{3.71b}$$

provided that $M_\Lambda\big[-u(x,\boldsymbol{\alpha})\big]$ exists for $x > x_0$. ■

We will apply this theorem to a parental WEIBULL distribution of type (3.70b) where the stochastic scale factor, denoted by B, has either a uniform (= rectangular) prior or a gamma prior. These prior distributions are the most popular ones in BAYES inference.

First case: $We(a, \sqrt[c]{1/B}, c) \underset{B}{\bigwedge} Re(\xi,\delta)$

The uniform or rectangular distribution over $[\xi, \xi+\delta]$ has DF

$$f_B(y) = \left\{ \begin{array}{ccl} 1/\delta & \text{for} & \xi \le y \le \xi+\delta \\ 0 & \text{for} & \xi \text{ elsewhere}, \end{array} \right\} \tag{3.72a}$$

denoted by

$$B \sim Re(\xi,\delta).$$

The corresponding raw moment generating function is

$$M_B(t) = \frac{\exp(\xi\,t)}{\delta\,t}\big[\exp(\delta\,t)-1\big]. \tag{3.72b}$$

Putting

$$t = -u(x,a,c) = -(x-a)^c$$

into (3.71b) yields the following CDF of the compound WEIBULL distribution (or **WEIBULL–uniform distribution**):

$$F(x\,|\,a,\delta,\xi,c) = 1 - \frac{\exp\big\{-\xi\,(x-a)^c\big\} - \exp\big\{-(\xi+\delta)\,(x-a)^c\big\}}{\delta\,(x-a)^c} \tag{3.72c}$$

with corresponding DF

$$\left.\begin{array}{l} f(x\,|\,a,\delta,\xi,c) = \dfrac{c}{\delta}\times \\ \dfrac{\exp\big\{-(\xi+\delta)(x-a)^c\big\}\big[\exp\big\{\delta\,(x-a)^c\big\}\big(1+\xi\,(x-a)^c\big)-(\xi+\delta)(x-a)^c-1\big]}{(x-a)^{c+1}}. \end{array}\right\} \tag{3.72d}$$

With some integration and manipulation we find

$$\mathrm{E}(X) = a + \frac{c}{\delta\,(c-1)}\,\Gamma\left(1+\frac{1}{c}\right)\left[(\xi+\delta)^{1-1/c} - \xi^{1-1/c}\right], \tag{3.72e}$$

$$\left.\begin{array}{rcl} \mathrm{Var}(X) & = & \dfrac{c}{\delta\,(c-1)}\,\Gamma\left(1+\dfrac{2}{c}\right)\left[(\xi+\delta)^{1-1/c} - \xi^{1-1/c}\right] - \\ & & \dfrac{c^2}{\delta^2\,(c-1)^2}\,\Gamma^2\left(1+\dfrac{1}{c}\right)\left[(\xi+\delta)^{1-1/c} - \xi^{1-1/c}\right]^2. \end{array}\right\} \tag{3.72f}$$

When the prior density (3.72a) degenerates ($\delta \to 0$), the formulas (3.72c–f) show the familiar results of the common three–parameter distribution for $b = \xi^{-1/c}$.

Second case: $We(a, \sqrt[c]{1/B}, c) \underset{B}{\bigwedge} Ga(\xi,\delta,\gamma)$

The three–parameter gamma prior $Ga(\xi,\delta,\gamma)$ over $[\xi,\infty)$ has DF

$$f_B(y) = \frac{\delta^\gamma\,(y-\xi)^{\gamma-1}}{\Gamma(\gamma)}\,\exp\big\{-\delta\,(y-\xi)\big\} \tag{3.73a}$$

with raw moment generating function

$$M_B(t) = \frac{\delta^\gamma\,\exp\{\xi\,t\}}{(\delta-t)^\gamma}. \tag{3.73b}$$

Taking

$$t = -u(x,a,c) = -(x-a)^c$$

in (3.73b) turns (3.71b) into the following CDF of this compound WEIBULL distribution (or **WEIBULL–gamma distribution**):

$$F(x\,|\,a,\delta,\xi,\gamma,c) = 1 - \frac{\delta^\gamma\,\exp\big\{-\xi\,(x-a)^c\big\}}{\big[\delta+(x-a)^c\big]^\gamma} \tag{3.73c}$$

with DF

$$f(x\,|\,a,\delta,\xi,\gamma,c) = \frac{c\,\delta^\gamma (x-a)^{c-1}\,\big\{\gamma+\xi\,\big[\delta+(x-a)^c\big]\big\}\,\exp\big\{-\xi\,(x-a)^c\big\}}{\big[\delta+(x-a)^c\big]^{\gamma+1}}. \tag{3.73d}$$

In the special case $\xi = 0$, there exist closed form expressions for the mean and variance:

$$\mathrm{E}(X\,|\,\xi=0) = a + \frac{\delta^{1/c}\,\Gamma\left(\gamma-\dfrac{1}{c}\right)\,\Gamma\left(\dfrac{1}{c}+1\right)}{\Gamma(\gamma)}, \tag{3.73e}$$

$$\operatorname{Var}(X\,|\,\xi=0)=\frac{\delta^{2/c}}{\Gamma(\gamma)}\left\{\Gamma\left(\gamma-\frac{2}{c}\right)\Gamma\left(\frac{2}{c}+1\right)-\frac{1}{\Gamma(\gamma)}\,\Gamma^2\left(\frac{1}{c}+1\right)\right\}. \tag{3.73f}$$

We further mention that in the special case $\xi = 0$ and $\delta = 1$, (3.73c,d) turn into the BURR type–XII distribution; see (3.3a,b). Finally, we refer to HARRIS/SINGPURWALLA (1969) when parameters have to be estimated for each of the two compound WEIBULL distributions.

3.3.7 WEIBULL distributions with additional parameters

Soon after the WEIBULL distribution had been presented to a broader readership by WEIBULL in 1951, statisticians began to re–parameterize this distribution. By introducing additional parameters, those authors aimed at more flexibility to fit the model to given datasets. But, the more parameters that are involved and have to be estimated, the greater will be the risk to not properly identify these parameters from the data.

In the preceding sections, we have already encountered several WEIBULL distributions with more than the traditional location, scale and shape parameters, namely:

- the doubly truncated WEIBULL distribution (3.47a) with five parameters,
- the left and the right truncated WEIBULL distributions (3.47b,c) with four parameters each,
- the WEIBULL competing risk model (3.57a,b) with four parameters,
- the multiplicative WEIBULL model (3.58a,b) and its generalization, the exponentiated WEIBULL model, each having four parameters.

In this section we will present further WEIBULL distributions, enlarged with respect to the parameter set.

3.3.7.1 Four-parameter distributions

MARSHALL/OLKIN extension

In their paper from 1997, MARSHALL/OLKIN proposed a new and rather general method to introduce an additional parameter into a given distribution having $F(x)$ and $R(x)$ as CDF and CCDF, respectively. The CCDF of the new distribution is given by

$$G(x)=\frac{\alpha\,R(x)}{1-(1-\alpha)\,R(x)}=\frac{\alpha\,R(x)}{F(x)+\alpha\,R(x)}\,,\quad \alpha>0. \tag{3.74}$$

Substituting $F(x)$ and $R(x)$ by the CDF and CCDF of the three–parameter WEIBULL distribution, we arrive at the following CCDF of the **extended WEIBULL distribution**:

$$R(x\,|\,a,b,c,\alpha)=\frac{\alpha\,\exp\left\{-\left(\dfrac{x-a}{b}\right)^c\right\}}{1-(1-\alpha)\,\exp\left\{-\left(\dfrac{x-a}{b}\right)^c\right\}}\,. \tag{3.75a}$$

The corresponding CDF is

$$F(x\,|\,a,b,c,\alpha) = \frac{1-\exp\Big\{-\Big(\dfrac{x-a}{b}\Big)^c\Big\}}{1-(1-\alpha)\,\exp\Big\{-\Big(\dfrac{x-a}{b}\Big)^c\Big\}} \tag{3.75b}$$

with DF

$$f(x\,|\,a,b,c,\alpha) = \frac{\dfrac{\alpha\,c}{b}\Big(\dfrac{x-a}{b}\Big)^{c-1}\,\exp\Big\{-\Big(\dfrac{x-a}{b}\Big)^c\Big\}}{\Big[1-(1-\alpha)\,\exp\Big\{-\Big(\dfrac{x-a}{b}\Big)^c\Big\}\Big]^2} \tag{3.75c}$$

and HR

$$h(x\,|\,a,b,c,\alpha) = \frac{\dfrac{c}{b}\Big(\dfrac{x-a}{b}\Big)^{c-1}}{1-(1-\alpha)\,\exp\Big\{-\Big(\dfrac{x-a}{b}\Big)^c\Big\}}\,. \tag{3.75d}$$

The hazard rate given by (3.75d) has the following behavior for growing x.

- for $\alpha \ge 1$ and $c \ge 1$ it is increasing,
- for $\alpha \le 1$ and $c \le 1$ it is decreasing,
- for $c > 1$ it is initially increasing and eventually increasing, but there may be one interval where it is decreasing,
- for $c < 1$ it is initially decreasing and eventually decreasing, but there may be one interval where it is increasing,

When $|1-\alpha| \le 1$, moments of this distribution can be given in closed form. Particularly we have

$$\mathrm{E}\big[(X-a)^r\big] = \frac{r}{b\,c}\sum_{j=0}^{\infty}\frac{(1-\alpha)^j}{(1+j)^{r/c}}\,\Gamma\Big(\frac{r}{c}\Big)\,. \tag{3.75e}$$

MARSHALL/OLKIN proved that the distribution family generated by (3.74) possesses what they call **geometric–extreme stability**: If $X_i,\ i \le N$, is a sequence of iid variates with CCDF (3.74) and N is geometrically distributed, i.e. $\Pr(N=n) = P\,(1-P)^n;\ n = 1,2,\ldots$; then the minimum (maximum) of X_i also has a distribution in the same family.

LAI et al. extension

LAI et al. (2003) proposed the following extension of the WEIBULL model,[29] assuming $a = 0$ and using the combined scale–shape parameter[30] $b_1 = 1/b^c$:

$$F(x\,|\,0,b_1,c,\alpha) = 1-\exp\{-b_1\,x^c\,e^{\alpha\,x}\};\ \ b_1,\ c>0;\ \alpha \ge 0. \tag{3.76a}$$

[29] In a recent article NADARAJAH/KOTZ (2005) note that this extension has been made some years earlier by GURVICH et al. (1997); see also LAI et al. (2005).

[30] For parameter estimation under type–II censoring, see NG (2005).

The corresponding DF, HR and CHR are

$$f(x\,|\,0,b_1,c,\alpha) = b_1\,(c+\alpha\,x)\,x^{c-1}\,\exp\big\{\alpha\,x - b_1\,x^c\,e^{\alpha\,x}\big\}, \tag{3.76b}$$

$$h(x\,|\,0,b_1,c,\alpha) = b_1\,(c+\alpha\,x)\,x^{c-1}\,\exp\{\alpha\,x\}, \tag{3.76c}$$

$$H(x\,|\,0,b_1,c,\alpha) = b_1\,x^c\,\exp\{\alpha\,x\}. \tag{3.76d}$$

We have the following relationships with other distributions:

- $\alpha = 0$ gives the general WEIBULL distribution.
- $c = 0$ results in the type–I extreme value distribution for the minimum, also known as a log–gamma distribution.
- (3.76a) is a limiting case of the beta–integrated model with CHR

  $$H(x) = b_1\,x^c\,(1-g\,x)^k;\;\; 0 < x < g^{-1};$$

  and $g > 0,\ k < 0$. Set $g = 1/\nu$ and $k = \alpha\,\nu$. With $\nu \to \infty$ we have $(1-x/\nu)^{-\alpha\,\nu} \to \exp\{\alpha\,x\}$, and $H(x)$ is the same as (3.76d).

Moments of this extended WEIBULL model have to be evaluated by numeric integration. Percentiles x_P of order P follow by solving

$$x_P = \left[-\frac{\ln(1-P)}{b_1}\,\exp\{-\alpha\,x_P\}\right]^{1/c}. \tag{3.76e}$$

The behavior of the hazard rate (3.76c) is of special interest. Its shape solely depends on c, and the two other parameters α and b_1 have no effect. With respect to c two cases have to be distinguished (see Fig. 3/14):

- $c \geq 1$

  HR is increasing. $\big[h(x\,|\,\cdot) \to \infty$ as $x \to \infty$, but $h(0\,|\,\cdot) = 0$ for $c > 1$ and $h(0\,|\,\cdot) = b_1\,c$ for $c = 1.\big]$

- $0 < c < 1$

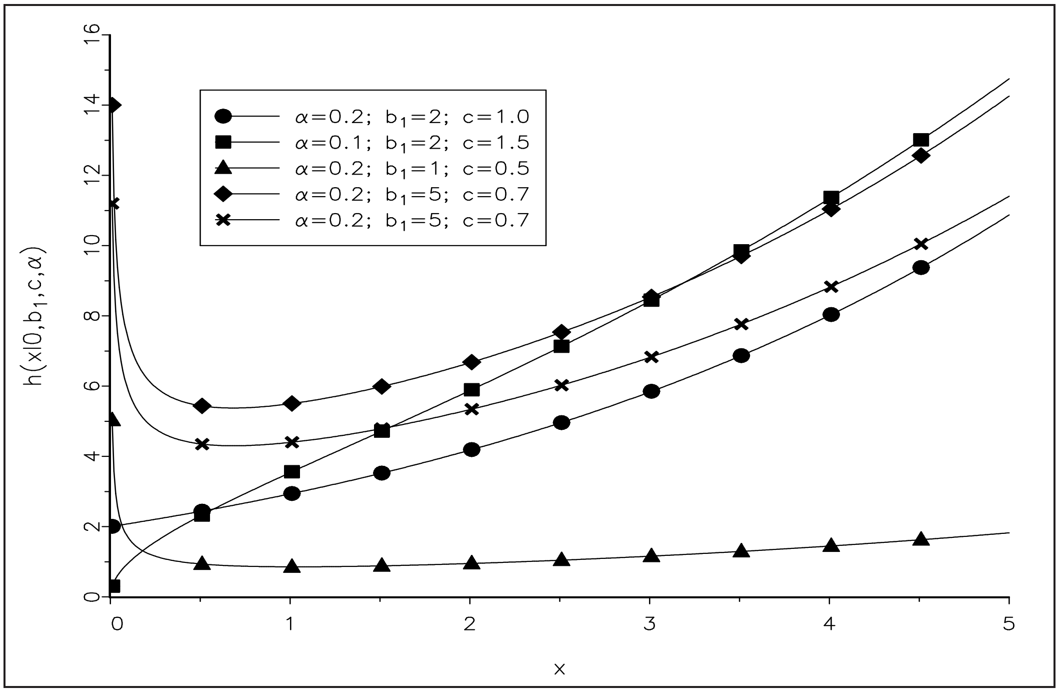
  $h(x\,|\,\cdot)$ is initially decreasing and then increasing, i.e. it has a bathtub shape with $\lim\limits_{x\to 0} h(x\,|\,\cdot) = \lim\limits_{x\to\infty} h(x\,|\,\cdot) = \infty$. The minimum of $h(x\,|\,\cdot)$ is reached at

  $$x_{h,\min} = \frac{\sqrt{c}-c}{\alpha}, \tag{3.76f}$$

  which is increasing when α decreases.

We finally mention that LAI et al. (2003) show how to estimate the model parameters by the ML–method.

Figure 3/14: Hazard rate of the LAI et al. extension of the WEIBULL model



Extensions by modifying the percentile function

The percentile function of the general three–parameter WEIBULL distribution is given by (see (2.57b)):

$$x_P = a + b\left[-\ln(1-P)\right]^{1/c}. \tag{3.77}$$

Modifying this function and introducing a fourth parameter ν and re–inverting to $P = F(x\,|\,\ldots)$ is another way to arrive at extended WEIBULL models. We will discuss two approaches along this line.

MUDHOLKAR et al. (1996) transformed (3.77) into

$$x_P = a + b\left[\frac{1-(1-P)^{\lambda}}{\lambda}\right]^{1/c}, \quad \lambda \in \mathbb{R}. \tag{3.78a}$$

The limit of the ratio $[1-(1-P)^{\lambda}]/\lambda$ in (3.78a) for $\lambda \to 0$ is $-\ln(1-P)$, which is similar to the definition of the **BOX–COX–transformation**, used to stabilize the variance of a time series. So, for $\lambda = 0$ we have the general WEIBULL distribution. Solving for $P = F(x\,|\,\cdot)$ gives

$$F(x\,|\,a,b,c,\lambda) = 1 - \left[1-\lambda\left(\frac{x-a}{b}\right)^{c}\right]^{1/\lambda}. \tag{3.78b}$$

The support of (3.78b) depends not only on a in the sense that $x \geq a$, but also on some further parameters:

- $\lambda \leq 0$ gives the support (a, ∞).
- $\lambda > 0$ gives the support $(a, b/\lambda^{1/c})$.

The DF and HR belonging to (3.78b) are

$$f(x \,|\, a, b, c, \lambda) = \frac{c}{b} \left(\frac{x-a}{b}\right)^{c-1} \left[1 - \lambda \left(\frac{x-a}{b}\right)^{c}\right]^{-1+1/\lambda}, \tag{3.78c}$$

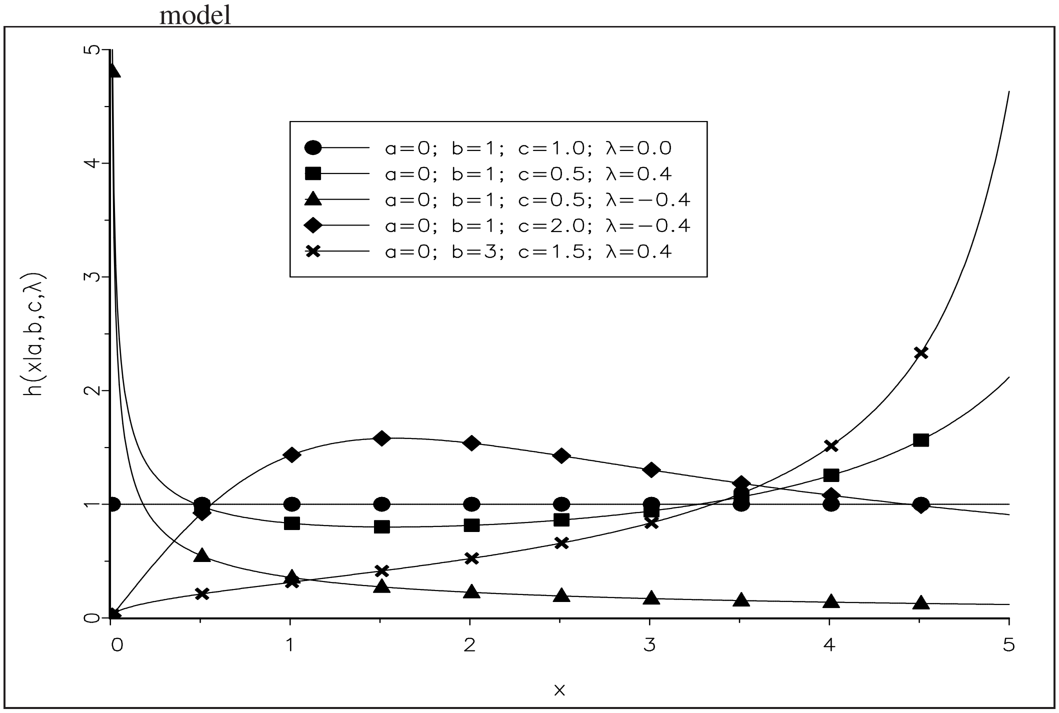
$$h(x \,|\, a, b, c, \lambda) = \frac{c}{b} \left(\frac{x-a}{b}\right)^{c-1} \left[1 - \lambda \left(\frac{x-a}{b}\right)^{c}\right]^{-1}. \tag{3.78d}$$

The behavior of (3.78d) is as follows:

- $c = 1$ and $\lambda = 0 \;\;\Rightarrow\;\; h(x \,|\, \cdot) = 1/b =$ constant,
- $c < 1$ and $\lambda > 0 \;\;\Rightarrow\;\; h(x \,|\, \cdot)$ is bathtub–shaped,
- $c \leq 1$ and $\lambda \leq 0 \;\;\Rightarrow\;\; h(x \,|\, \cdot)$ is decreasing,
- $c > 1$ and $\lambda < 0 \;\;\Rightarrow\;\; h(x \,|\, \cdot)$ is inverse bathtub–shaped,
- $c \geq 1$ and $\lambda \geq 0 \;\;\Rightarrow\;\; h(x \,|\, \cdot)$ is increasing

(see Fig. 3/15).

Figure 3/15: Hazard rate of the MUDHOLKAR et al. (1996) extension of the WEIBULL model



The r–th moment of $(X - a)$ according to MUDHOLKAR/KOLLIA (1994) is

$$\mathrm{E}\big[(X-a)^r\big] = \begin{cases} b^r \dfrac{\Gamma\Big(\dfrac{1}{\lambda}\Big)\,\Gamma\Big(\dfrac{r}{c}+1\Big)}{\Gamma\Big(\dfrac{1}{\lambda}+\dfrac{r}{c}+1\Big)\,\lambda^{r/c+1}} & \text{for}\quad \lambda>0, \\[2ex] b^r \dfrac{\Gamma\Big(-\dfrac{r}{c}-\dfrac{1}{\lambda}\Big)\,\Gamma\Big(\dfrac{r}{c}+1\Big)}{\Gamma\Big(1-\dfrac{1}{\lambda}\Big)\,\big(-\lambda\big)^{r/c+1}} & \text{for}\quad \lambda<0. \end{cases} \tag{3.78e}$$

Hence, if $\lambda > 0$ and $c > 0$, moments of all orders exist. For $\lambda < 0$ the r–th moment exists if $\lambda/c \le -r^{-1}$, and in case of $1/c < 0$ — see the next paragraph — if $1/c \ge -r^{-1}$. We should mention that MUDHOLKAR et al. (1996) show how to estimate the parameters by ML.

MUDHOLKAR/KOLLIA (1994) have studied a more generalized percentile function than (3.78a), circumventing its discontinuity at $\lambda = 0$. Instead, they proposed the following percentile function, assuming $a = 0$ and $b = 1$:

$$x_P = c\left[\frac{1-(1-P)^\lambda}{\lambda}\right]^{1/c} - c,\quad \lambda\in\mathbb{R}. \tag{3.79a}$$

Here, c may be negative, too, so that the inverse WEIBULL distribution (Sect. 3.3.3) is included in this model. Solving for $P = F(x\,|\,\cdot)$ leads to

$$F_G(x\,|\,0,1,c,\lambda) = 1-\left[1-\lambda\left(1+\frac{x}{c}\right)^c\right]^{1/\lambda} \tag{3.79b}$$

with DF and HR

$$f_G(x\,|\,0,1,c,\lambda) = \left[1-\lambda\left(1+\frac{x}{c}\right)^c\right]^{\frac{1}{\lambda}-1}\left(1+\frac{x}{c}\right)^{c-1}, \tag{3.79c}$$

$$h_G(x\,|0,1,c,\lambda) = \left(1+\frac{x}{c}\right)^{c-1}\left[1-\lambda\left(1+\frac{x}{c}\right)^c\right]^{-1}. \tag{3.79d}$$

The support of these last three functions depends on the parameters c and λ as follows:

- $c < 0$
  - ∗ If $\lambda < 0$, we have $x \in (-\infty, -c)$.
  - ∗ If $\lambda > 0$, we have $x \in \big(-\infty, c/\lambda^{1/c} - c\big)$.
- $c > 0$
  - ∗ If $\lambda < 0$, we have $x \in (-c, \infty)$.
  - ∗ If $\lambda > 0$, we have $x \in \big(-c, c/\lambda^{1/c} - c\big)$.

The raw moments of X, distributed according to (3.79b), can be expressed by the moments given in (3.78e). (3.79a) includes the following special cases:

- $\lambda = 0,\ c > 0 \;\Rightarrow$ reduced WEIBULL distribution,
- $\lambda = 0,\ c < 0 \;\Rightarrow$ inverse reduced WEIBULL distribution,

- $\lambda = 1,\ c = 1\ \Rightarrow$ uniform distribution over $[0, 1]$,
- $\lambda = 0,\ c = 1\ \Rightarrow$ exponential distribution,
- $\lambda = -1,\ c = \infty\ \Rightarrow$ logistic distribution,

The limiting cases $(c \to \infty,\ \lambda \to 0)$ are interpreted according to L' HOPITAL's rule.

XIE et al. extension

XIE et al. (2002) proposed a distribution that we have enlarged to a four–parameter model by introducing a location parameter a. The CDF is

$$F(x) = 1 - \exp\left\{\lambda\,\alpha\,\left(1 - \exp\left[\left(\frac{x-a}{\alpha}\right)^{\beta}\right]\right)\right\},\quad x \geq a, \tag{3.80a}$$

with $\lambda, \alpha, \beta > 0$ and $a \in \mathbb{R}$. The corresponding DF and HR are

$$f(x) = \lambda\,\beta\left(\frac{x-a}{\alpha}\right)^{\beta-1}\exp\left\{\left(\frac{x-a}{\alpha}\right)^{\beta}\right\}\exp\left\{\lambda\,\alpha\left(1 - \exp\left[\left(\frac{x-a}{\alpha}\right)^{\beta}\right]\right)\right\}, \tag{3.80b}$$

$$h(x) = \lambda\,\beta\left(\frac{x-a}{\alpha}\right)^{\beta-1}\exp\left\{\left(\frac{x-a}{\alpha}\right)^{\beta}\right\}. \tag{3.80c}$$

The behavior of HR is determined by the shape parameter β:

- $\beta \geq 1\ \Rightarrow\ h(x)$ is increasing from $h(a) = 0$, if $\beta > 1$, or from $h(a) = \lambda$, if $\beta = 1$.
- $0 < \beta < 1\ \Rightarrow\ h(x)$ is decreasing from $x < x^*$ and decreasing for $x > x^*$, where
$$x^* = a + \alpha\,(1/\beta - 1)^{1/\beta},$$
i.e., $h(x)$ has a bathtub shape.

We find the following relations to other distributions:

- For $\alpha = 1$ we have the original version of this distribution introduced by CHEN (2000).
- When α is large, then $1 - \exp\left\{\left(\frac{x-a}{\alpha}\right)^{\beta}\right\} \approx -\exp\left(\frac{x-a}{\alpha}\right)^{\beta}$, so that (3.80a) can be approximated by a WEIBULL distribution.
- When $\lambda\,\alpha = 1$, this model is related to the exponential power model of SMITH/BAIN (1975).

KIES extension

KIES (1958) introduced the following four–parameter version of a WEIBULL distribution that has a finite interval as support:

$$F(x) = 1 - \exp\left\{-\lambda\left(\frac{x-a}{b-x}\right)^{\beta}\right\},\quad 0 \leq a \leq x \leq b < \infty, \tag{3.81a}$$

with $\lambda, \beta > 0$. The corresponding DF and HR are

$$f(x) = \frac{\lambda\,\beta\,(b-a)\,\exp\left\{-\lambda\left(\frac{x-a}{b-x}\right)^{\beta}\right\}(x-a)^{\beta-1}}{(b-x)^{\beta+1}}, \tag{3.81b}$$

$$h(x) = \frac{\lambda\,\beta\,(b-a)\,(x-a)^{\beta-1}}{(b-x)^{\beta+1}}. \tag{3.81c}$$

$h(x)$ is increasing for $\beta \geq 1$, and it has a bathtub shape for $0 < \beta < 1$. The approach of KIES represents another way to doubly truncate the WEIBULL distribution (see Sect. 3.3.5). We will revert to (3.81a) in Sect. 3.3.7.2, where this model is extended by a fifth parameter.

FAUCHON et al. extension

In Sect. 3.2.3 we have presented several extensions of the gamma distribution. One of them is the four–parameter gamma distribution, originally mentioned by HARTER (1967). Its density is given by

$$f(x\,|\,a,b,c,d) = \frac{c}{b\,\Gamma(d)}\left(\frac{x-a}{b}\right)^{c\,d-1}\exp\left\{-\left(\frac{x-a}{b}\right)^{c}\right\};\ x \geq a. \tag{3.82a}$$

(3.82a) contains a lot of other distributions as special cases for special values of the parameters (see Tab. 3/1). For $d = 1$ we have the three–parameter WEIBULL distribution, $c = 1$ leads to gamma distributions, and $a = 0,\ b = 2,\ c = 1,\ d = \nu/2$ gives a χ^2–distribution with ν degrees of freedom. As the χ^2–distribution is related to a normal distribution in the following way

$$X = \frac{1}{\sigma^2}\sum_{j=1}^{\nu} Y_j^2 \ \text{and}\ Y_j \overset{\text{iid}}{\sim} No(0,\sigma^2) \implies X \sim \chi^2(\nu),$$

and as both, the χ^2– and the WEIBULL distributions, are linked via the general gamma distribution, we can find a bridge connecting the WEIBULL and the normal distributions. This relationship, which was explored by FAUCHON et al. (1976), gives an interesting interpretation of the second form parameter d in (3.82a).

First, we will show how to combine iid normal variates to get a three–parameter WEIBULL distribution:

1. $Y_1,\ Y_2 \overset{\text{iid}}{\sim} No(0,\sigma^2) \implies (Y_1^2 + Y_2^2) \sim \sigma^2\,\chi^2(2)$,
2. $\sqrt{Y_1^2 + Y_2^2} \sim \sigma\,\chi(2) = We\big(0, \sigma\sqrt{2}, 2\big)$ (see Tab. 3/1).

More generally,

$$Y_1,\ Y_2 \overset{\text{iid}}{\sim} No(0,\sigma^2) \implies a + \sqrt[c]{Y_1^2 + Y_2^2} \sim We\Big(a, \big(2\,\sigma^2\big)^{1/c}, c\Big).$$

So, when we add $\nu = 2$ squared normal variates which are centered and of equal variance, the second form parameter d in (3.82a) always is unity. If, instead of $\nu = 2$, we add $\nu = 2\,k$ such squared normal variates, the second form parameter will be $d = k$:

$$Y_j \overset{\text{iid}}{\sim} No(0,\sigma^2),\ j = 1,2,\ldots,2\,k \implies a + \sqrt[c]{\sum_{j=1}^{2\,k} Y_j^2} \sim We\Big(a, \big(2\,\sigma^2\big)^{1/c}, c, k\Big).$$

We mention some moments belonging to (3.82a):

$$\operatorname{E}(X^r) = \frac{b^r}{\Gamma(d)} \sum_{i=0}^{r} \binom{r}{i} \left(\frac{a}{b}\right)^{r-i} \Gamma\left(d + \frac{i}{c}\right), \tag{3.82b}$$

$$\operatorname{E}(X) = a + b\,\frac{\Gamma\left(d + \dfrac{1}{c}\right)}{\Gamma(d)}, \tag{3.82c}$$

$$\operatorname{Var}(X) = \left(\frac{b}{\Gamma(d)}\right)^2 \left[\Gamma(d)\,\Gamma\left(d + \frac{2}{c}\right)\,\Gamma^2\left(d + \frac{1}{c}\right)\right]. \tag{3.82d}$$

The hazard rate behaves as follows: When

- $(1 - d\,c)/[c\,(c-1)] > 1$, then it is
  - $*$ bathtub–shaped for $c > 1$,
  - $*$ upside–down bathtub–shaped for $0 < c < 1$.
- Otherwise it is
  - $*$ constant for $c = 1$,
  - $*$ increasing for $c > 1$,
  - $*$ decreasing for $c < 1$.

A five–parameter extension of (3.82a) given by FAUCHON et al. (1976) will be presented in the following section.

3.3.7.2 Five-parameter distributions

PHANI et al. extension

PHANI (1987) has introduced a fifth parameter into (3.81a) in order to attain more flexibility in fitting the distribution to given datasets. Instead of only one exponent β in (3.81a), this distribution has different exponents β_1 und β_2 for the nominator and denominator in the exp–term:

$$F(x) = 1 - \exp\left\{-\lambda \frac{(x-a)^{\beta_1}}{(b-x)^{\beta_2}}\right\}, \quad 0 \le a \le x \le b < \infty, \tag{3.83a}$$

$$f(x) = \frac{\lambda\,(x-a)^{\beta_1 - 1}\left[(b\,\beta_1 - a\,\beta_2) + (\beta_2 - \beta_1)\,x\right]}{(b-x)^{\beta_2+1}} \exp\left\{-\lambda\,\frac{(x-a)^{\beta_1}}{(b-x)^{\beta_2}}\right\}, \tag{3.83b}$$

$$h(x) = \frac{\lambda\,(x-a)^{\beta_1 - 1}\left[(b\,\beta_1 - a\,\beta_2) + (\beta_2 - \beta_1)\,x\right]}{(b-x)^{\beta_2+1}}. \tag{3.83c}$$

The hazard rate has a bathtub shape when $0 < \beta_1 < 1$ and $0 < \beta_2 < 1$. All other β_1–β_2–combinations lead to an increasing hazard rate.

FAUCHON et al. extension

The FAUCHON et al. (1976) extension in the preceding section rests upon the generalized gamma distribution and its relation to the χ^2–distribution, the latter being the sum of squared and centered normal variates. If, instead, the normal variates have differing means unequal to zero but are still homoscedastic, we know that:

$$Y_j \sim No(\mu_j, \sigma^2);\ j = 1, 2, \ldots, 2k \text{ and independent } \Longrightarrow Y = \sum_{j=1}^{2k} Y_j^2 \sim \sigma^2\, \chi^2(2k, \lambda),$$

i.e., we have a non–central χ^2–distribution:

$$f(y) = \eta^k\, y^{k-1} \exp\{-(\lambda + \eta\, y)\} \sum_{j=0}^{\infty} \frac{(\eta\,\lambda\, y)^j}{j!\,\Gamma(j+k)}$$

with non–centrality parameter

$$\lambda = \frac{1}{2\,\sigma^2} \sum_{j=1}^{2k} \mu_j^2$$

and scale factor

$$\eta = 2\,\sigma^2.$$

So, if we start with $Y_j = No(\mu_j, \sigma^2)$, we get

$$X = a + \sqrt[c]{\sum_{j=1}^{2\,k} Y_j^2} \sim We\big(a, \eta^{-1/c}, c, k, \lambda\big)$$

with a CDF given by

$$F(x) = e^{-\lambda}\, \frac{c}{2\,k} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!\,\Gamma(j+k)} \int\limits_a^{(x-a)^c} \left(\frac{t-a}{b}\right)^{c\,(k+j)-1} \exp\left\{-\left(\frac{t-a}{b}\right)^c\right\} \mathrm{d}t \tag{3.84a}$$

and corresponding DF

$$f(x) = e^{-\lambda}\, \frac{c}{b} \left(\frac{x-a}{b}\right)^{c\,k-1} \exp\left\{-\left(\frac{x-a}{b}\right)^c\right\} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!\,\Gamma(j+k)} \left(\frac{x-a}{b}\right)^{c\,k}. \tag{3.84b}$$

The new, fifth, parameter in this WEIBULL distribution is the non–centrality parameter λ. The raw moments of this distribution can be expressed by the **KUMMER–function**, but have to be evaluated numerically:

$$\mathrm{E}\big(X^r\big) = b^r\, e^{-\lambda}\, \frac{\Gamma\Big(k + \dfrac{r}{c}\Big)}{\Gamma(k)}\, K\Big(k + \frac{r}{c}, k, \lambda\Big), \tag{3.84c}$$

where $K(\cdot)$ is the KUMMER–function (E.E. KUMMER, 1810 – 1893):

$$K(\alpha, \beta, z) := \sum_{j=0}^{\infty} \frac{(\alpha)_j\, z^j}{(\beta)_j\, j!}$$

with

$$\begin{aligned}(\alpha)_j &:= \alpha\,(\alpha+1)\ldots(\alpha+j-1),\ \ (\alpha)_0 := 1;\\ (\beta)_j &:= \beta\,(\beta+1)\ldots(\beta+j-1),\ \ (\beta)_0 := 1.\end{aligned}$$

3.3.8 WEIBULL distributions with varying parameters

Mixed and compound WEIBULL distributions are models with varying parameters, but that variation is at random and cannot be attributed to some other distinct variable. In this section we will present some approaches where the WEIBULL parameters are functionally dependent on one or more other variables. In Sect. 3.3.8.1 the explanatory variable is time itself; in Sect. 3.3.8.2 the WEIBULL parameters will be made dependent on measurable variables other than time, called **covariates**.

3.3.8.1 Time-dependent parameters

The scale parameter b and/or the shape parameter c have been made time–varying in some generalizations of the WEIBULL distribution where the latter is the model for a stochastic duration. We first present an approach of ZUO et al. (1999) with both, b and c, dependent on time.

Let $Y(t)$ be the **degradation** of a device at time t. $Y(t)$ is assumed to be a random variable with CDF

$$F(y \,|\, t) = \Pr\big[Y(t) \le y\big],$$

which at each point of time is a two–parameter WEIBULL distribution. Of course, $Y(t)$ is a non–decreasing function of t. The device will not fail as long as the degradation has not surpassed some given critical value D. Thus, the reliability function (CCDF) of a device is given by

$$R(t) = \Pr(T \ge t) = \Pr\big[Y(t) \le D\big].$$

ZUO et al. suppose the following WEIBULL CHR with respect to t:

$$H(t) = \left(\frac{D}{b(t)}\right)^{c(t)}, \tag{3.85a}$$

where $b(t)$ and $c(t)$ are specified as

$$b(t) = \alpha\, t^{\beta} \exp(\gamma\, t), \tag{3.85b}$$

$$c(t) = \alpha \left(1 + \frac{1}{t}\right)^{\beta} \exp(\gamma/t). \tag{3.85c}$$

The parameters $\alpha,\ \beta$ and γ have to be such that $H(t)$ goes to infinity as $t \to \infty$. Starting with (3.85a) and using Tab. 2/1 we arrive at

$$R(t) = \exp\left\{-\left[\frac{D}{b(t)}\right]^{c(t)}\right\}, \tag{3.85d}$$

$$\begin{aligned} f(t) &= -\frac{\mathrm{d}\exp\left\{-\left[\dfrac{D}{b(t)}\right]^{c(t)}\right\}}{\mathrm{d}t} \\ &= \exp\left\{-\left[\frac{D}{b(t)}\right]^{c(t)}\right\} \frac{\mathrm{d}\left[D/b(t)\right]^{c(t)}}{\mathrm{d}t}. \end{aligned} \tag{3.85e}$$

SRIVASTAVA (1974) studies a WEIBULL model with a scale parameter which periodically changes between b_1 and b_2, attributed to two different conditions of usage of the device. Introducing $\tau_i(t);\ i = 1, 2;$ as the cumulative times, starting at $t = 0$, spent in phase i, the hazard rate $h(t)$ is

$$h(t) = \frac{c}{b_1}\left[\frac{\tau_1(t)}{b_1}\right]^{c-1} + \frac{c}{b_2}\left[\frac{\tau_2(t)}{b_2}\right]^{c-1}. \tag{3.86a}$$

This approach is easily recognized to be a special case of a two–component series system or a competing risk model with continuously joining hazard rates; see (3.54d). The difference between (3.86a) and (3.54d) is that we have two subpopulations with independent variables $\tau_1(t)$ and $\tau_2(t)$ instead of only one variable t. Some other functions describing the time to failure of this model are

$$\begin{aligned} H(t) &= \left[\frac{\tau_1(t)}{b_1}\right]^{c-1} + \left[\frac{\tau_2(t)}{b_2}\right]^{c-1}, && (3.86b)\\ R(t) &= \exp\left\{-\left[\frac{\tau_1(t)}{b_1}\right]^{c-1} - \left[\frac{\tau_2(t)}{b_2}\right]^{c-1}\right\}, && (3.86c)\\ f(t) &= \left\{\frac{c}{b_1}\left[\frac{\tau_1(t)}{b_1}\right]^{c-1} + \frac{c}{b_2}\left[\frac{\tau_2(t)}{b_2}\right]^{c-1}\right\}\exp\left\{-\left[\frac{\tau_1(t)}{b_1}\right]^{c} - \left[\frac{\tau_2(t)}{b_2}\right]^{c}\right\}. && (3.86d) \end{aligned}$$

ZACKS (1984) presents a WEIBULL model where the shape parameter is given by

$$c \left\{\begin{array}{ll} = & \text{for} \quad 0 \le t \le \tau \\ > & \text{for} \quad t > \tau \end{array}\right\} \tag{3.87a}$$

with the following hazard rate

$$h(t) = \left\{\begin{array}{ll} \dfrac{1}{b} & \text{for} \quad 0 \le t \le \tau \\ \dfrac{1}{b} + \dfrac{c}{b}\left(\dfrac{t-\tau}{b}\right)^{c-1} & \text{for} \quad t > \tau. \end{array}\right\} \tag{3.87b}$$

ZACKS calls this an **exponential–WEIBULL distribution**. In the first phase of lifetime up to the change point τ the device has a constant hazard rate that afterwards is super–positioned by a WEIBULL hazard rate. This model is nothing but a special two–fold composite or sectional model (see Sect. 3.3.6.3). Some other functions describing the time to failure of this model are

$$\begin{aligned} H(t) &= \left\{\begin{array}{ll} \dfrac{t}{b} & \text{for} \quad 0 \le t \le \tau \\ \dfrac{\tau}{b} + \left(\dfrac{t-\tau}{b}\right)^{c} & \text{for} \quad t > \tau, \end{array}\right\} && (3.87c)\\ F(t) &= 1 - \exp\left\{\frac{t}{b} - \left[\left(\frac{t-\tau}{b}\right)_+\right]^{c}\right\},\ t \ge 0, && (3.87d)\\ f(t) &= \frac{1}{b}\,e^{-t/b}\left\{1 + c\left[\left(\frac{t-\tau}{b}\right)_+\right]^{c-1}\right\}\exp\left\{-\left[\left(\frac{t-\tau}{b}\right)_+\right]^{c}\right\},\ t \ge 0, && (3.87e) \end{aligned}$$

where $(y)_+ := \max(0, y)$. ZACKS gives the following formula for raw moments

$$\mathrm{E}\big(X^r\big) = b^r\,\mu'_r(c, \tau/b) \tag{3.87f}$$

with

$$\mu'_r(c, \tau) = r!\left[1 - e^{-1/b}\sum_{j=0}^{r}\frac{1}{b^j\,j!}\right] + e^r\sum_{j=0}^{r}\binom{r}{j}\tau^{r-j}\,M_j(c) \tag{3.87g}$$

and

$$M_j(c) = \left\{\begin{array}{cl} 1 & \text{for} \quad j = 0 \\ j\displaystyle\int_0^\infty x^{j-1}\,\exp\big\{-\big(x + x^c\big)\big\}\,\mathrm{d}x & \text{for} \quad j \geq 1. \end{array}\right\}$$

3.3.8.2 Models with covariates[31]

When the WEIBULL distribution, or any other distribution, is used to model lifetime or some other duration variable, it is often opportune to introduce one or more other variables that explain or influence the spell length. Two such approaches will be discussed

- The scale parameter b is influenced by the supplementary variable(s). This gives the **accelerated life model**.
- The hazard rate $h(t)$ is directly dependent on the covariate(s). The prototype of this class is the **proportional hazard model**.

Farther down we will show that the WEIBULL accelerated life model is identical to the WEIBULL proportional hazard model.

In reliability applications a supplementary variable s represents the **stress** on the item such that its lifetime is some function of s. The stress may be electrical, mechanical, thermal etc. In life testing of technical items the application of some kind of stress is a means to shorten the time to failure (see Sect.16), thus explaining the term **accelerated life testing**. The general form of the WEIBULL accelerated life model with one stress variable s is

$$F(t\,|\,s) = 1 - \exp\left\{-\left[\frac{t}{b\,\beta(s)}\right]^c\right\},\ \ t \geq 0. \tag{3.88}$$

The function $\beta(s)$ is the so–called **acceleration factor** which —in the parametrization above —must be a decreasing function of s, leading to a higher probability of failure up to time t the higher the stress exercised on an item. Three types of $\beta(s)$ are extensively used:

- The **ARRHENIUS–equation**[32]

  $$\beta(s) = \exp(\alpha_0 + \alpha_1/s) \tag{3.89a}$$

  gives the **ARRHENIUS–WEIBULL distribution**.

[31] Suggested reading for this section: LAWLESS (1982), NELSON (1990).

[32] S.A. ARRHENIUS (1859 – 1927) was a Swedish physico–chemist who — in 1903 — received the NOBEL prize in chemistry for his theory of electrolytical dissociation. (3.89a) in its original version describes the dependency of temperature on the speed of reaction of some kind of material.

- The **EYRING–equation** (H. EYRING, 1901 – 1981) is an alternative to (3.89a):

$$\beta(s) = \alpha_0 \, s^{\alpha_1} \, \exp(\alpha_2 \, s) \tag{3.89b}$$

  and leads to the **EYRING–WEIBULL distribution**.
- The equation

$$\beta(s) = \alpha_0 / s^{\gamma_1} \tag{3.89c}$$

  gives the **power WEIBULL distribution**.

An acceleration factor which depends on a vector $\boldsymbol{s}$ of m stress variables s_i is given by

$$\beta(\boldsymbol{s}) = \exp\left\{\sum_{i=1}^{m} \gamma_i \, s_i\right\} = \exp\left(\boldsymbol{\gamma}' \, \boldsymbol{s}\right). \tag{3.89d}$$

The hazard rate corresponding to (3.88) is

$$h(t \,|\, s) = \frac{c}{b \, \beta(s)} \left(\frac{t}{b \, \beta(\boldsymbol{s})}\right)^{c-1}. \tag{3.90a}$$

Thus, the ratio of the hazard rates for two items with covariates values s_1 and s_2 is

$$\frac{h(t \,|\, s_1)}{h(t \,|\, s_2)} = \left[\frac{\beta(s_2)}{\beta(s_1)}\right]^c, \tag{3.90b}$$

which is independent of t. Looking at the CDF of (3.88) for two different stress levels s_1 and s_2, we can state

$$F(t \,|\, s_1) = F\left(\frac{\beta(s_2)}{\beta(s_1)} \, t \,\middle|\, s_2\right). \tag{3.90c}$$

So, if an item with stress s_1 has a lifetime T_1, then an item with stress s_2 has a lifetime T_2 given by

$$T_2 = \frac{\beta(s_2)}{\beta(s_1)} \, T_1. \tag{3.90d}$$

Because $\beta(s)$ is supposed to be decreasing with s, we will have $T_2 < T_1$ for $s_2 > s_1$.

The moments belonging to (3.88) are the same as those of a common two–parameter WEIBULL distribution with the scale parameter b replaced by $b \, \beta(\boldsymbol{s})$.

A proportional hazard model is characterized by

$$h(t \,|\, s) = g(s) \, h_0(t), \tag{3.91a}$$

where $h_0(t)$ is called the **baseline hazard rate**. After its introduction by COX (1972) the proportional hazard model has been extensively applied in the social and economic sciences to model the sojourn time of an individual in a given state, e.g., in unemployment or in a given job. The argument of the scalar–valued function $g(\cdot)$ may be a vector too. The most popular proportional hazard model takes the WEIBULL hazard rate as its baseline hazard rate, turning (3.91a) into

$$h(t \,|\, s) = g(s)\,\frac{c}{b}\left(\frac{t}{b}\right)^{c-1} \tag{3.91b}$$

with CDF

$$F(t \,|\, s) = 1 - \exp\left\{-g(s)\left(\frac{t}{b}\right)^{c}\right\}, \;\; t \geq 0. \tag{3.91c}$$

When

$$\frac{1}{g(s)} = \big[\beta(s)\big]^c,$$

the WEIBULL proportional hazard model and the WEIBULL accelerated life model coincide.

It is also instructive to view (3.88) through the distribution of $Y = \ln T$, when the acceleration factor — in most general form — is $\beta(\boldsymbol{s})$, i.e., it depends on a vector of stress variables. The distribution of Y is of extreme value form (see (3.43a–d)) and has DF

$$f(y \,|\, \boldsymbol{s}) = \frac{1}{b^*}\,\exp\left\{\frac{y - a^*(\boldsymbol{s})}{b^*} - \exp\left[\frac{y - a^*(\boldsymbol{s})}{b^*}\right]\right\}, \;\; y \in \mathbb{R}, \tag{3.92a}$$

with

$$b^* = \frac{1}{c} \;\text{ and }\; a^*(\boldsymbol{s}) = \ln\big[b\,\beta(\boldsymbol{s})\big].$$

Written another way, (3.92a) is the DF of

$$Y = a^*(\boldsymbol{s}) + b^*\,Z, \tag{3.92b}$$

where Z has a reduced extreme value distribution with DF $\exp\{z - e^z\}$. (3.92b) is a **location–scale regression model** with error Z. The constancy of c in (3.88) corresponds to the constancy of b^* in (3.92b), so $\ln T$ has a constant variance. A variety of functional forms for $\beta(\boldsymbol{s})$ or $a^*(\boldsymbol{s})$ is often employed together with either (3.88) or (3.92a). The most popular one is perhaps the log–linear form for which

$$a^*(\boldsymbol{s}) = \gamma_0 + \boldsymbol{\gamma}'\,\boldsymbol{s} \tag{3.92c}$$

with $\gamma_0 = \ln b$ and $\beta(\boldsymbol{s}) = \boldsymbol{\gamma}'\,\boldsymbol{s} = \sum\limits_{i=1}^{m} \gamma_i\,s_i$. This linear regression can be estimated by a multitude of methods, see LAWLESS (1982, pp. 299ff.).

We finally mention a generalization of the accelerated life model where the stress changes with time; i.e., stress is growing with time. This approach is applied in life testing to shorten the testing period. NELSON (1990) shows how to estimate the parameters for a **step–stress model** where the stress is raised at discrete time intervals.

3.3.9 Multidimensional WEIBULL models[33]

In Sect. 3.3.6.2 we have studied the lifetime distribution of two special types of a multi–component system, i.e., the parallel and the series systems, under the rather restrictive assumption of independently failing components. When the failure times of the components are dependent, we need a multivariate distribution that is not simply the product of its marginal distributions. There are mainly two approaches to introduce multivariate extensions of the one–dimensional WEIBULL distribution:

- One simple approach is to transform the bivariate and multivariate exponential distribution through power transformation. As the extension of the univariate exponential distribution to a bi- or multivariate exponential distribution is not unique, there exist several extensions all having marginal exponentials; see KOTZ/BALAKRISHNAN/JOHNSON (2000, Chap. 47). A similar approach involves the transformation of some multivariate extreme value distribution.
- A very different approach is to specify the dependence between two or more univariate WEIBULL variates so that the emerging bi- or multivariate WEIBULL distribution has WEIBULL marginals.

We start (Sect. 3.3.9.1) by presenting — in detail — several bivariate WEIBULL models and finish by giving an overview (Sect. 3.3.9.2) on some multivariate WEIBULL models.

3.3.9.1 Bivariate WEIBULL distributions

We will first present some bivariate WEIBULL distributions, **BWD** for short, that are obtained from a bivariate exponential distribution, **BED** for short, by power transformation. A genuine BED has both marginal distributions as exponential. As KOTZ/BALAKRISHNAN/JOHNSON (2000) show, many of such BEDs exist. The BED models of FREUND (1961) and MARSHALL/OLKIN (1967) have received the most attention in describing the statistical dependence of component's life in two–component systems. These two systems rest upon a clearly defined physical model that is both simple and realistic. So, it is appropriate to study possible WEIBULL extensions to these two BEDs.

BWDs based on FREUND's BED

FREUND (1961) proposed the following failure mechanism of a two–component system: Initially the two components have constant failure rates, λ_1 and λ_2, with independent DFs when both are in operation:

$$f_i(x_i \,|\, \lambda_i) = \lambda_i \, \exp\{-\lambda_i \, x_i\}; \;\; x_i \ge 0, \; \lambda_i > 0; \; i = 1, 2. \tag{3.93a}$$

But the lifetimes X_1 and X_2 are dependent because a failure of either component does not result in a replacement, but changes the parameter of the life distribution of the other component to λ_i^*, mostly to $\lambda_i^* > \lambda_i$ as the non–failed component has a higher workload.

[33] Suggested reading for this section: CROWDER (1989), HANAGAL (1996), HOUGAARD (1986), KOTZ/BALAKRISHNAN/JOHNSON (2000), LEE (1979), LU (1989, 1990, 1992a,b), LU/BHATTACHARYYA (1990), MARSHALL/OLKIN (1967), PATRA/DEY (1999), ROY/MUKHERJEE (1988), SARKAR (1987), SPURRIER/WEIER (1981) and TARAMUTO/WADA (2001).

There is no other dependence. The time to first failure is exponentially distributed with parameter

$$\lambda = \lambda_1 + \lambda_2.$$

The probability that component i is the first to fail is $\lambda_i \big/ \lambda$, whenever the first failure occurs. The distribution of the time from first failure to failure of the other component is thus a mixture of two exponential distributions with parameters λ_1^* or λ_2^* in proportions $\lambda_2 \big/ \lambda$ and $\lambda_1 \big/ \lambda$, respectively. Finally, the joint DF of X_1 and X_2 is

$$f_{1,2}^{FR}(x_1, x_2) = \begin{Bmatrix} f_1(x_1 \,|\, \lambda_1)\, R_2(x_1 \,|\, \lambda_2)\, f_2(x_2 - x_1 \,|\, \lambda_2^*) & \text{for} \quad 0 \le x_1 < x_2 \\ f_2(x_2 \,|\, \lambda_2)\, R_1(x_2 \,|\, \lambda_2)\, f_1(x_1 - x_2 \,|\, \lambda_1^*) & \text{for} \quad 0 \le x_2 < x_1 \end{Bmatrix} \tag{3.93b}$$

$$= \begin{Bmatrix} \lambda_1\, \lambda_2^* \exp\big\{ - \lambda_2^*\, x_2 - (\lambda_1 + \lambda_2 - \lambda_2^*)\, x_1 \big\} & \text{for} \quad 0 \le x_1 < x_2 \\ \lambda_2\, \lambda_1^* \exp\big\{ - \lambda_1^*\, x_1 - (\lambda_1 + \lambda_2 - \lambda_1^*)\, x_2 \big\} & \text{for} \quad 0 \le x_2 < x_1, \end{Bmatrix} \tag{3.93c}$$

where

$$R_i(x_i \,|\, \lambda_i) = \exp\big\{ - \lambda_i\, x_i \big\}.$$

$f_{1,2}^{FR}(x_1, x_2)$ is not continuous at $x = x_1 = x_2$, unless $\lambda_1\, \lambda_2^* = \lambda_2\, \lambda_1^*$. The joint survival function CCDF is

$$R_{1,2}^{Fr}(x_1, x_2) = \begin{Bmatrix} \dfrac{1}{\lambda_1 + \lambda_2 - \lambda_2^*} \Big[\lambda_1 \exp\big\{ - (\lambda_1 + \lambda_2 - \lambda_2^*)\, x_1 - \lambda_2^*\, x_2 \big\} + \\ \qquad (\lambda_2 - \lambda_2^*) \exp\big\{ - (\lambda_1 + \lambda_2)\, x_2 \big\} \Big] \quad \text{for} \quad 0 \le x_1 < x_2 \\ \dfrac{1}{\lambda_1 + \lambda_2 - \lambda_2^*} \Big[\lambda_2 \exp\big\{ - (\lambda_1 + \lambda_2 - \lambda_1^*) x_2 - \lambda_1^* x_1 \big\} + \\ \qquad (\lambda_1 - \lambda_1^*) \exp\big\{ - (\lambda_1 + \lambda_2)\, x_1 \big\} \Big] \quad \text{for} \quad 0 \le x_2 < x_1. \end{Bmatrix} \tag{3.93d}$$

Provided $\lambda_1 + \lambda_2 \ne \lambda_i^*$, the marginal DF of X_i is

$$f_i(x_i) = \frac{1}{\lambda_1 + \lambda_2 - \lambda_i^*} \begin{Bmatrix} (\lambda_i - \lambda_i^*)\, (\lambda_1 + \lambda_2) \exp\big\{ - (\lambda_1 + \lambda_2)\, x_i \big\} + \\ \lambda_i^*\, \lambda_{3-i} \exp\big\{ - \lambda_i^*\, x_i \big\} \end{Bmatrix}; \quad x_i \ge 0. \tag{3.93e}$$

(3.93e) is not exponential but rather a mixture of two exponential distributions for $\lambda_i > \lambda_i^*$, otherwise (3.93e) is a weighted average. So (3.93c,d) do not represent a genuine BED.

We now substitute the exponential distributions in (3.93a) by WEIBULL distributions in the following parametrization:

$$f_i(y_i \,|\, \lambda_i, c_i) = c_i\, \lambda_i \left(\lambda_i\, y_i \right)^{c_i - 1} \exp\big\{ -(\lambda_i\, y_i)^{c_i} \big\}; \;\; y_i \ge 0; \; \lambda_i, c_i > 0; \; i = 1, 2. \tag{3.94a}$$

When one component fails, the remaining lifetime of the other component is still WEIBULL, but with possibly changed parameters λ_i^* and c_i^*. Time is reset on the surviving component. The shape parameter c_i^* might be equal to c_i, but the scale parameter λ_i^* is not equal to λ_i generally, which makes the hazard rate of the non–failed component change.

As time is reset on the surviving component, it has a shifted WEIBULL distribution[34] and (3.93b) turns into the following BWD:

$$f_{1,2}(y_1,y_2)=\begin{cases} f_1(y_1\,|\,\lambda_1,c_1)\,\exp\big\{-\big(\lambda_2\,y_1\big)^{c_2}\big\}\times \\ c_2^*\,\lambda_2^*\big[\lambda_2^*\,(y_2-y_1)\big]^{c_2^*-1}\exp\big\{-\big[\lambda_2^*\,(y_2-y_1)\big]^{c_2^*}\big\}\ \text{ for } 0\le y_1<y_2 \\ f_2(y_2\,|\,\lambda_2,c_2)\,\exp\big\{-\big(\lambda_1\,y_2\big)^{c_1}\big\}\times \\ c_1^*\,\lambda_1^*\big[\lambda_1^*\,(y_1-y_2)\big]^{c_1^*-1}\exp\big\{-\big[\lambda_1^*\,(y_1-y_2)\big]^{c_1^*}\big\}\ \text{ for } 0\le y_2<y_1. \end{cases} \tag{3.94b}$$

(3.94b) yields (3.93c) for the special case of all shape parameters equal to one.

LU (1989) shows that the above WEIBULL extension to FREUND's BED is not equal to a bivariate WEIBULL model obtained by using a direct power transformation of the marginals of FREUND's BED resulting in

$$f_{1,2}^*(y_1,y_2)=\begin{cases} f_1(y_1\,|\,\lambda_1,c_1)\,\exp\big\{-\big[(\lambda_2-\lambda_2^*)\,y_1\big]^{c_1}\big\}\times \\ \quad f_2(y_2\,|\,\lambda_2^*,c_2)\ \text{ for } 0\le y_1^{c_1}<y_2^{c_2} \\ f_2(y_2\,|\,\lambda_2,c_2)\,\exp\big\{-\big[(\lambda_1-\lambda_1^*)\,y_2\big]^{c_2}\big\}\times \\ \quad f_1(y_1\,|\,\lambda_1^*,c_1)\ \text{ for } 0\le y_2^{c_2}<y_1^{c_1}. \end{cases} \tag{3.95}$$

SPURRIER/WEIER (1981) modified the FREUND idea of a hazard rate change to derive another bivariate WEIBULL model. Let U be the time to the first failure and W the time from the first to the second failure. So the time of system failure is $V=U+W$. The time U until the first failure is distributed as the minimum of two iid WEIBULL variates with shape parameter c and scale factor λ, so

$$f_U(u)=2\,c\,\lambda\,(\lambda\,u)^{c-1}\,\exp\big\{-2\,(\lambda\,u)^c\big\},\quad u\ge 0 \tag{3.96a}$$

(see Sect. 3.1.4). Upon the first failure the remaining component changes its scale factor to $\theta\,\lambda, \theta>0$. Then we have the following conditional density of $W|u$:

$$f_{W|u}(w)=\frac{c\,\lambda\,\theta\big[\lambda\,(\theta\,w+u)\big]^{c-1}\,\exp\big\{-\big[\lambda\,(\theta\,w+u)\big]^c\big\}}{\exp\{-(\lambda\,u)^c\}},\quad w\ge 0. \tag{3.96b}$$

The joint DF of U and W is

$$\begin{aligned} f(u,w)=f_U(u)\,f_{W\,|\,u}(w) &= 2\,c^2\,\theta\,\lambda^{2\,c}\,u^{c-1}\,(\theta\,w+u)\ \times \\ &\quad \exp\big\{-\big[\lambda\,(\theta\,w+u)\big]^c-(\lambda\,u)^c\big\};\ \ u,w\ge 0. \end{aligned} \tag{3.96c}$$

The DF of the time to system failure $V=U+W$ does not in general have a closed form and has to be determined by numerical integration of (3.96c) over the set $\{(u,w):\ u+w=v\}$.

[34] The shift or location parameter is equal to the time of the first failure of a component.

Three important special cases of (3.96c) are

Case I: $c = 1;\ \theta = 1 \implies$ The lifetimes of the two components are iid exponential variates.

Case II: $c = 1; \implies$ The model reduces to a reparametrization of the FREUND model.

For cases I and II we have :

$$f(v) = \begin{cases} 2\,\theta\,\lambda\,(2-\theta)^{-1}\,\exp\big\{-\theta\,\lambda\,v\big\}\,\big[1-\exp\big\{-(2-\theta\,\lambda\,v)\big\}\big]\,; & v \geq 0 \text{ and } \theta \neq 2, \\ 4\,\lambda^2\,v\,\exp\big\{-2\,\lambda\,v\big\}; & v \geq 0 \text{ and } \theta = 2. \end{cases}$$

Case III: $\theta = 1; \implies$ The lifetimes of the two components are iid WEIBULL variates.

In this case we get

$$f(v) = 2\,c\,\lambda^c\,v^{c-1}\big[\exp\{-(\lambda\,v)^c\}\exp\{-2\,(\lambda\,v)^c\}\big],\ \ v \geq 0.$$

We finally mention that LU (1989) has generalized the model of SPURRIER/WEIER by introducing different shape parameters for the two components and by also allowing a change of the shape parameter upon the first failure.

BWDs based on MARSHALL/OLKIN's BED

The physical model behind the MARSHALL/OLKIN (1967) BED differs from that of FREUND's BED insofar as there is a third failure mechanism that hits both components simultaneously. To be more precise, their model is as follows: The components of a two–component system fail after receiving a shock which is always fatal. The occurrence of shocks is governed by three independent POISSON processes $N_i(t;\lambda_i);\ i = 1,2,3$. By $N(t;\lambda) = \{N(t), t \geq 0;\ \lambda\}$, we mean a POISSON process (see Sect. 4.2) with parameter λ, giving the mean number of events per unit of time. Events in the processes $N_1(t;\lambda_1)$ and $N_2(t;\lambda_2)$ are shocks to components 1 and 2, respectively, and events in the process $N_3(t;\lambda_3)$ are shocks to both components. The joint survival function of X_1 and X_2, the lifetimes of the two components, is:

$$\begin{aligned} R^{MO}_{1,2}(x_1,x_2) &= \Pr(X_1 > x_1, X_2 > x_2) \\ &= \Pr\big\{N_1(x_1;\lambda_1) = 0,\ N_2(x_2;\lambda_2) = 0,\ N_3(\max[x_1,x_2];\lambda_3) = 0\big\} \\ &= \exp\big\{-\lambda_1\,x_1 - \lambda_2\,x_2 - \lambda_3\,\max[x_1,x_2]\big\}. \qquad (3.97a) \\ &= \left\{\begin{array}{ll} \exp\big\{-\lambda_1\,x_1 - (\lambda_2+\lambda_3)\,x_2\big\} & \text{for } 0 \leq x_1 \leq x_2 \\ \exp\big\{-(\lambda_1+\lambda_3)\,x_1 - \lambda_2\,x_2\big\} & \text{for } 0 \leq x_2 \leq x_1. \end{array}\right\} \qquad (3.97b) \end{aligned}$$

The marginal distributions are genuine one–dimensional exponential distributions:

$$R_i(x_i) = \exp\big\{-(\lambda_i+\lambda_3)\,x_i\big\};\ \ i = 1,2. \qquad (3.97c)$$

The probability that a failure on component i occurs first is

$$\Pr(X_i < X_j) = \frac{\lambda_i}{\lambda_1 + \lambda_2 + \lambda_3}\,;\;\; i,j = 1,2;\; i \neq j;$$

and we have a positive probability that both components fail simultaneously:[35]

$$\Pr(X_1 = X_2) = \frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3}\,.$$

The latter probability is responsible for the **singularity of the distribution** along the line $x_1 = x_2$. MARSHALL/OLKIN (1967) show that the joint survival function (3.97a,b) can be written as a mixture of an absolutely continuous survival function

$$R_c(x_1,x_2) = \left\{ \begin{array}{l} \dfrac{\lambda_1+\lambda_2+\lambda_3}{\lambda_1+\lambda_2}\,\exp\big\{-\lambda_1\,x_1 - \lambda_2\,x_2 - \lambda_3\,\max[x_1,x_2]\big\} - \\ \\ \dfrac{\lambda_3}{\lambda_1+\lambda_2}\,\exp\big\{-(\lambda_1+\lambda_2+\lambda_3)\,\max[x_1,x_2]\big\} \end{array} \right\} \tag{3.97d}$$

and a singular survival function

$$R_s(x_1,x_2) = \exp\big\{-(\lambda_1+\lambda_2+\lambda_3)\,\max[x_1,x_2]\big\} \tag{3.97e}$$

in the form

$$R_{1,2}^{MO}(x_1,x_2) = \frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2+\lambda_3}\,R_c(x_1,x_2) + \frac{\lambda_3}{\lambda_1+\lambda_2+\lambda_3}\,R_s(x_1,x_2). \tag{3.97f}$$

The joint DF of the MARSHALL/OLKIN model is

$$f_{1,2}^{MO}(x_1,x_2) = \left\{ \begin{array}{ll} \lambda_1\,(\lambda_2+\lambda_3)\,R_{1,2}^{MO}(x_1,x_2) & \text{for} \quad 0 \le x_1 < x_2 \\ \lambda_2\,(\lambda_1+\lambda_3)\,R_{1,2}^{MO}(x_1,x_2) & \text{for} \quad 0 \le x_2 < x_1 \\ \lambda_3\,R_{1,2}^{MO}(x,x) & \text{for} \quad 0 \le x_1 = x_2 = x. \end{array} \right\} \tag{3.97g}$$

A first WEIBULL extension to this model has been suggested by MARSHALL/OLKIN themselves, applying the power–law transformations $X_i = Y_i^{c_i};\;\; i = 1,2;$ thus changing (3.97a,b) into

$$R_{1,2}(y_1,y_2) = \exp\big\{-\lambda_1\,y_1^{c_1} - \lambda_2\,y_2^{c_2} - \lambda_3\,\max\big[y_1^{c_1},y_2^{c_2}\big]\big\} \tag{3.98a}$$

$$= \left\{ \begin{array}{ll} \exp\big\{-\lambda_1\,y_1^{c_1} - (\lambda_2+\lambda_3)\,y_2^{c_2}\big\} & \text{for} \quad 0 \le y_1^{c_1} \le y_2^{c_2} \\ \exp\big\{-(\lambda_1+\lambda_3)\,y_1^{c_1} - \lambda_2\,y_2^{c_2}\big\} & \text{for} \quad 0 \le y_2^{c_2} \le y_1^{c_1}. \end{array} \right\} \tag{3.98b}$$

[35] The correlation coefficient between X_1 and X_2 is $\varrho(X_1,X_2) = \Pr(X_1 = X_2) > 0$.

The areas in the y_1–y_2–plane, where the two branches of (3.98b) are valid, may be expressed by $y_2 \geq y_1^{c_1/c_2}$ and $y_2 \leq y_1^{c_1/c_2}$.

Set y_1 or y_2 to 0 in (3.98a,b) to get the marginal survival functions:

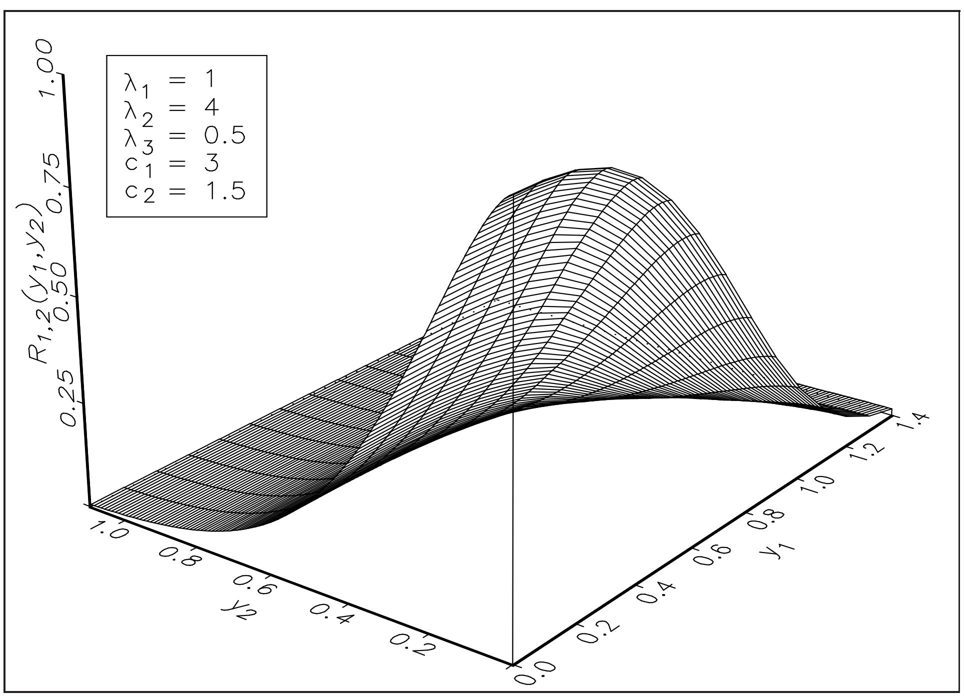
$$R_i(y_i) = \exp\big\{-(\lambda_i + \lambda_3)\, y_i^{c_i}\big\}, \quad i = 1, 2. \tag{3.98c}$$

These are genuine univariate WEIBULL distributions. Whereas the joint survival function is absolutely continuous (see Fig. 3/16), the joint DF is not. It is given by

$$f_{1,2}(y_1,y_2)=\begin{cases}\lambda_1(\lambda_2+\lambda_3)c_1y_1^{c_1-1}c_2y_2^{c_2-1}\exp\big\{-\lambda_1y_1^{c_1}-(\lambda_2+\lambda_3)y_2^{c_2}\big\}; & 0\leq y_1^{c_1}<y_2^{c_2}\\ \lambda_2(\lambda_1+\lambda_3)c_1y_1^{c_1-1}c_2y_2^{c_2-1}\exp\big\{-(\lambda_1+\lambda_3)y_1^{c_1}-\lambda_2y_2^{c_2}\big\}; & 0\leq y_2^{c_2}<y_1^{c_1}\\ \lambda_3c_1y_1^{c_1-1}\exp\big\{-(\lambda_1+\lambda_2+\lambda_3)y_1^{c_1}\big\}; & 0\leq y_1^{c_1}=y_2^{c_2}.\end{cases} \tag{3.98d}$$

The third branch of (3.98d) represents the discrete part of the joint DF and it lies on the curve $y_2 = y_1^{c_1/c_2}$ in the y_1–y_2–plane. This locus is a straight line through the origin for $c_1 = c_2$. The discontinuity is clearly seen in Fig. 3/17, where we have marked some points of discontinuity on the surface grid. The discontinuity is best seen in the contour plot.

Figure 3/16: Joint survival function of a BWD of MARSHALL/OLKIN type (surface plot and contour plot)



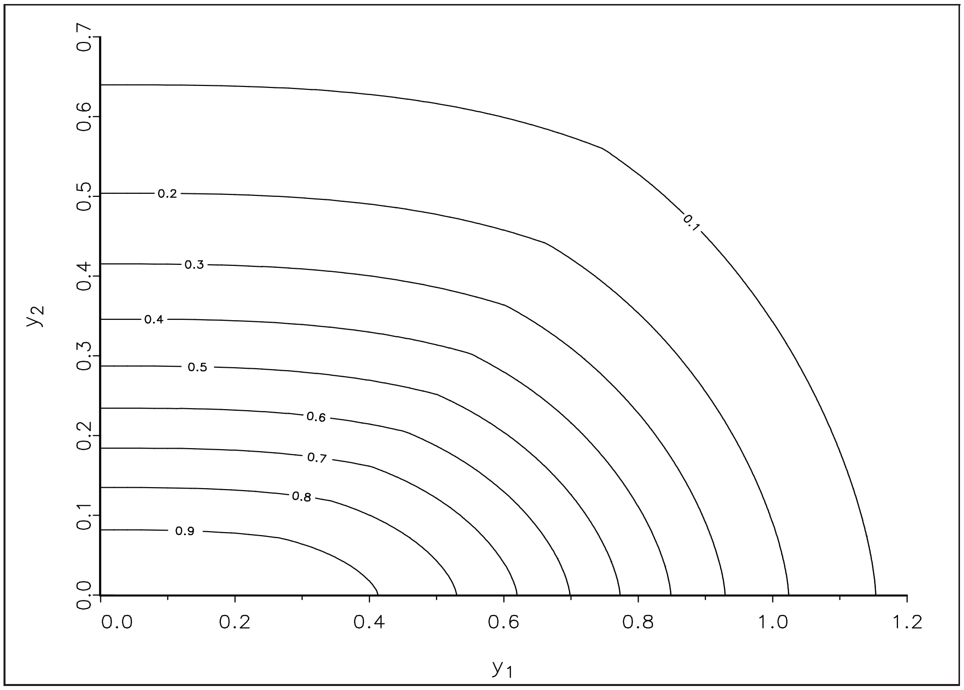
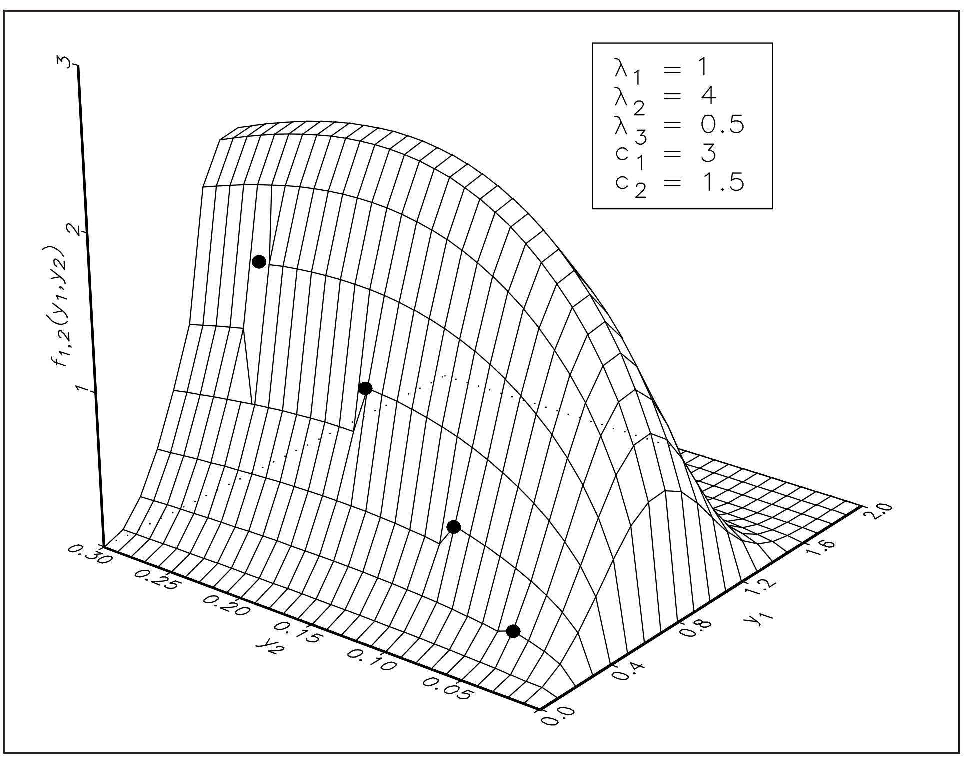
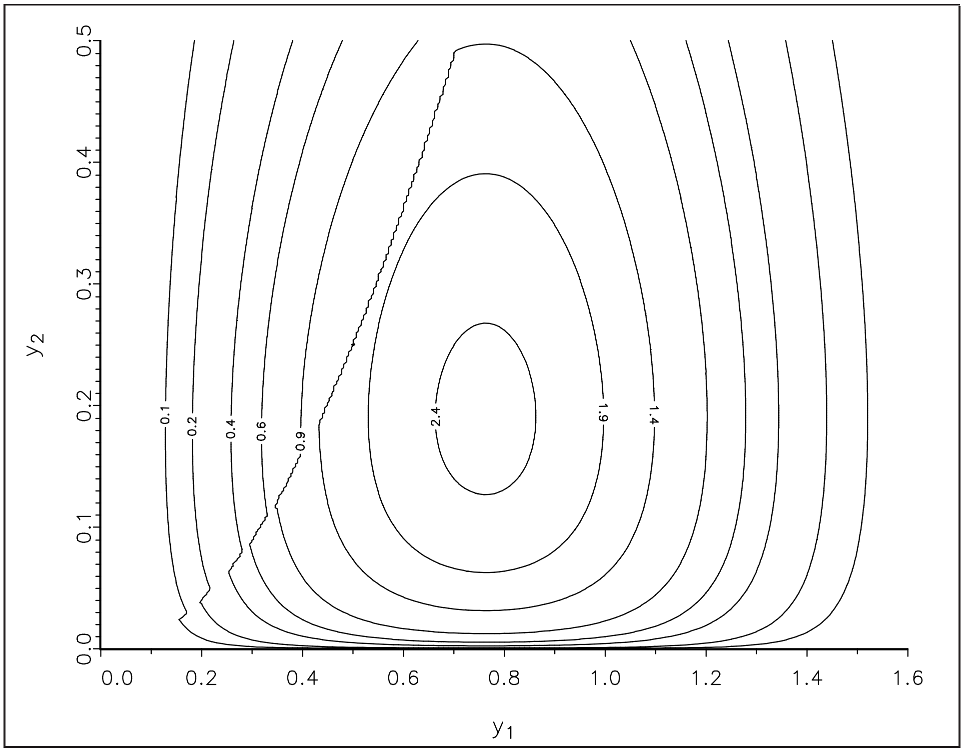


Figure 3/17: Joint density function of a BWD of MARSHALL/OLKIN type (surface plot and contour plot)





LU (1989) has proposed a second WEIBULL extension to MARSHALL/OLKIN's BED by generalizing their fatal shock model, which is given by three homogeneous POISSON processes, to non–homogeneous POISSON processes $N_i(t;\lambda_i,c_i)$ with power-law intensities[36] $\lambda_i\,c_i\,(\lambda_i\,t)^{c_i-1};\ i=1,2,3$. As in the original MARSHALL/OLKIN model's the processes N_1 and N_2 cause failure of the component 1 and 2, respectively, while events in the process N_3 are shocks under which both components will fail simultaneously. The joint survival function for the lifetimes Y_1, Y_2 of the two components are

$$\begin{aligned} R_{1,2}(y_1,y_2) &= \Pr(Y_1>y_1, Y_2>y_2) \\ &= \Pr\big\{N_1(y_1;\lambda_1;c_1)=0,\ N_2(y_2;\lambda_2,c_2)=0,\ N_3\big(\max[y_1,y_2];\lambda_3,c_3\big)\big\} \\ &= \exp\big\{-\lambda_1\,y_1^{c_1}-\lambda_2\,y_2^{c_2}-\lambda_3\,\big(\max[y_1,y_2]\big)^{c_3}\big\}. \qquad (3.99a) \end{aligned}$$

$$= \left\{\begin{array}{lll} \exp\big\{-\lambda_1\,y_1^{c_1}-\lambda_2\,y_2^{c_2}-\lambda_3\,y_2^{c_3} & \text{for} & 0\le y_1\le y_2 \\ \exp\big\{-\lambda_1\,y_1^{c_1}-\lambda_2\,y_2^{c_2}-\lambda_3\,y_1^{c_3} & \text{for} & 0\le y_2\le y_1. \end{array}\right\} \qquad (3.99b)$$

(3.99a/b) differ from (3.98a,b) in two respects:

- The LU extension has one more parameter c_3.
- The admissible regions for the two branches are different from those of the first extension, and the borderline is given by $y_1=y_2$.

[36] This intensity function looks like a WEIBULL hazard rate. Its relation to a WEIBULL process will be discussed in Sect. 4.3.

(3.99a,b) have the marginals

$$R_i(y_i) = \exp\big\{-\lambda_3\, y_i^{c_3} - \lambda_i\, y_i^{c_i}\big\};\;\; i = 1, 2; \tag{3.99c}$$

which, in general, are not WEIBULL. Of course, this extension has no absolutely joint DF as well.

We mention the following special cases of (3.99a,b):

- When $c_1 = c_2 = c_3 = c$, (3.99a,b) coincide with the original MARSHALL/OLKIN extension, where the marginals and the minimum $T = \min[Y_1, Y_2]$; i.e., the lifetime of the series system, are all WEIBULL variates with

$$\begin{aligned} R(y_1) &= \exp\big\{-(\lambda_1+\lambda_3)\, y_1^c\big\},\\ R(y_2) &= \exp\big\{-(\lambda_2+\lambda_3)\, y_2^c\big\},\\ R(t) &= \exp\big\{-(\lambda_1+\lambda_2+\lambda_3)\, t^c\big\}. \end{aligned}$$

- The univariate minimum–type distribution, introduced by FRIEDMAN/GERTSBAKH (1980) has the survival function

$$R(x) = \exp\big\{-\lambda^*\, x - \lambda^{**}\, x^c\big\}.$$

  It belongs to a series system with two statistically dependent components having exponential and WEIBULL as lifetime distributions. This model is appropriate where system failure occurs with constant and/or increasing intensity. With $c_1 = c_2 = 1$ and $c_3 = c$, (3.99a) turns into

$$R(y_1, y_2) = \exp\big\{-(\lambda_1\, y_1 - \lambda_2\, y_2 - \lambda_3\,\big(\max[y_1, y_2]\big)^c\big\},$$

  which can be regarded as a bivariate extension of the minimum–type distribution because it has marginals of the univariate minimum–type.

- The univariate linear hazard rate distribution with $h(x) = a + b\,x$ has the survival function

$$R(x) = \exp\big\{-a\,x - 0.5\,b\,x^2\big\}.$$

  This is a special case $(c = 2)$ of the univariate minimum–type distribution. Taking $c_1 = c_2 = 2$ and $c_3 = 1$ in (3.99a), we get the following bivariate extension of the linear hazard rate distribution:

$$R(y_1, y_2) = \exp\big\{-\lambda_1\, y_1^2 - \lambda_2\, y_2^2 - \lambda_3\, \max[y_1, y_2]\big\},$$

  which has marginals of the form of the linear hazard rate distribution. With $c_1 = c_2 = 1$ and $c_3 = 2$, we would get the same bivariate model.

BWD's based on LU/BHATTACHARYYA (1990)

LU/BHATTACHARYYA (1990) have proposed several new constructions of bivariate WEIBULL models. A first family of models rests upon the general form

$$R_{1,2}(y_1, y_2) = \exp\left\{-\left(\frac{y_1}{b_1}\right)^{c_1} - \left(\frac{y_2}{b_2}\right)^{c_2} - \delta\,\psi(y_1, y_2)\right\}. \tag{3.100}$$

Different forms of the function $\psi(y_1, y_2)$ together with the factor δ yield special members of this family:

- $\delta = 0$ and any $\psi(y_1, y_2)$ lead to a BWD with independent variables Y_1 and Y_2.
- $\delta = \lambda_3$ and $\psi(y_1, y_2) = \max\left(y_1^{c_1}, y_2^{c_2}\right)$ together with $\lambda_1 = b_1^{-c_1},\ \lambda_2 = b_2^{-c_2}$ gives the MARSHALL/OLKIN extension (3.98a).
- LU/BHATTACHARYYA proposed the two following functions:

$$\begin{aligned}\psi(y_1, y_2) &= \left\{\left(\frac{y_1}{b_1}\right)^{c_1/m} + \left(\frac{y_2}{b_2}\right)^{c_2/m}\right\}^m,\\ \psi(y_1, y_2) &= \left[1 - \exp\left\{-\left(\frac{y_1}{b_1}\right)^{c_1}\right\}\right]\left[1 - \exp\left\{-\left(\frac{y_2}{b_2}\right)^{c_2}\right\}\right].\end{aligned}$$

- A BWD to be found in LEE (1979):

$$R_{1,2}(y_1, y_2) = \exp\left\{-\lambda_1\, d_1^c\, y_1^c - \lambda_2\, d_2^c\, y_2^c - \lambda_3\, \max\left[d_1^c\, y_1^c, d_2^c\, y_2^c\right]\right\}$$

  is of type (3.100) for $c_1 = c_2 = c$.

All BWD models presented up to here have a singular component. A BWD being absolutely continuous is given by LEE (1979):

$$R_{1,2}(y_1, y_2) = \exp\left\{-(\lambda_1\, y_1^c + \lambda_1\, y_2^c)^\gamma\right\};\quad y_1, y_2 \geq 0; \tag{3.101a}$$

with $\lambda_1, \lambda_2 > 0$, $c > 0$ and $0 < \gamma \leq 1$. The corresponding DF is

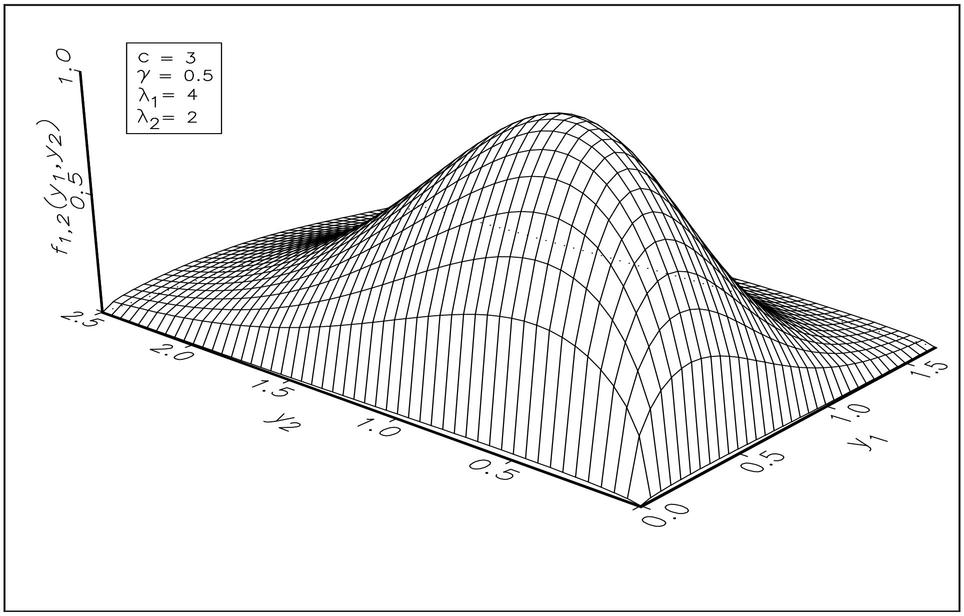
$$\begin{aligned}f_{1,2}(y_1, y_2) &= \gamma^2\, c^2\, (\lambda_1\, \lambda_2)^\gamma\, (y_1\, y_2)^{c-1}\left[\left(y_1\, y_2\right)^c\right]^{\gamma-1} \times\\ &\quad \exp\left\{-\left(\lambda_1\, y_1^c + \lambda_2\, y_2^c\right)^\gamma\right\}.\end{aligned} \tag{3.101b}$$

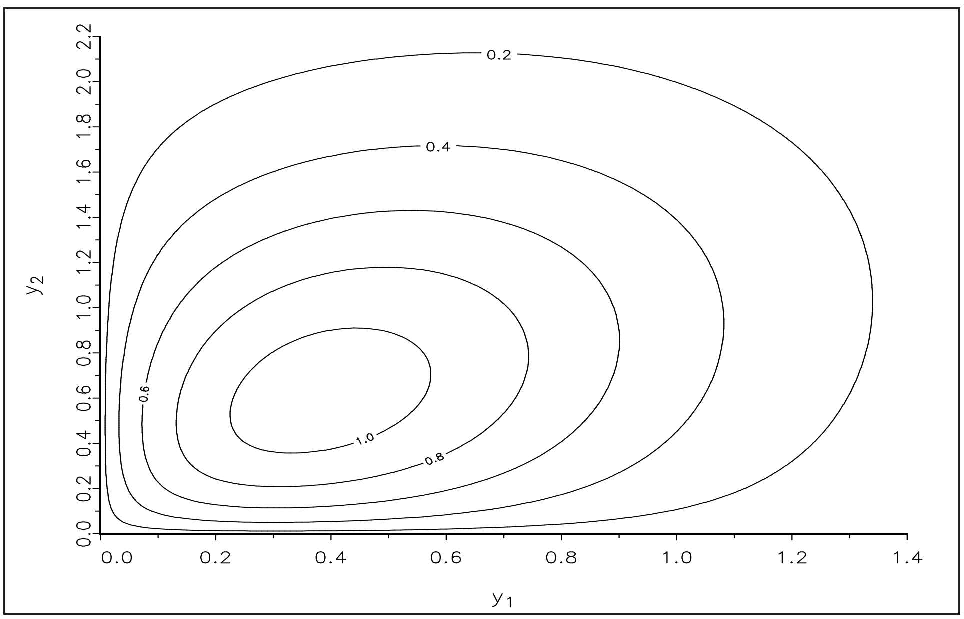
The surface plot and the contour plot of (3.101b) in Fig. 3/18 show the continuity of this BWD. The marginals of (3.101a) are all WEIBULL:

$$\left.\begin{aligned}R_1(y_1) &= \exp\{-\lambda_1^\gamma\, y_1^{c\,\gamma}\},\\ R_2(y_2) &= \exp\{-\lambda_2^\gamma\, y_2^{c\,\gamma}\}.\end{aligned}\right\} \tag{3.101c}$$

The minimum of Y_1 and Y_2 is also WEIBULL with shape parameter $c\,\gamma$.

Figure 3/18: Joint density function of the absolutely continuous BWD given by LEE (surface plot and contour plot)





3.3.9.2 Multivariate WEIBULL distributions

MARSHALL/OLKIN (1967) have proposed the following multivariate exponential distribution (MVE) for a vector $\boldsymbol{x}$ of p variates giving the joint survival function

$$\left.\begin{aligned} R(\boldsymbol{x}) &= \Pr(X_1 > x_1, X_2 > x_2, \ldots, X_p > x_p) \\ &= \exp\Bigg\{-\sum_{i=1}^{p} \lambda_i\, x_i - \sum_{i<j} \lambda_{ij}\, \max[x_i, x_j] - \\ &\qquad \sum_{i<j<k} \lambda_{ijk}\, \max[x_i, x_j, x_k] - \ldots - \lambda_{12\ldots p}\, \max[x_2, x_2, \ldots, x_p]\Bigg\}. \end{aligned}\right\} \tag{3.102a}$$

To obtain a more compact notation for (3.102a), let S denote the set of vectors $(s_1, s_2, \ldots, s_p)$ where each $s_j = 0$ or 1 but $(s_1, s_2, \ldots, s_p) \neq (0, 0, \ldots, 0)$. For any vector $\boldsymbol{s} \in S$, $\max(s_i\, x_i)$ is the maximum of the x_i's for which $s_i = 1$. Thus,

$$R(\boldsymbol{x}) = \exp\Bigg\{-\sum_{\boldsymbol{s}\in S} \lambda_{\boldsymbol{s}}\, \max(s_i\, x_i)\Bigg\}. \tag{3.102b}$$

The zero–one–variable s_j in $\boldsymbol{s}$ indicates which component(s) of the p–component system is (are) simultaneously hit by a POISSON process with shock rate $\lambda_{\boldsymbol{s}}$. For example, for $p = 3$ we get

$$\begin{aligned} R(x_1, x_2, x_3) &= \exp\big\{-\lambda_{100}\, x_1 - \lambda_{010}\, x_2 - \lambda_{001}\, x_3 \\ &\qquad - \lambda_{110}\, \max[x_1, x_2] - \lambda_{101}\, \max[x_1, x_2] - \lambda_{011}\, \max[x_2, x_3] \\ &\qquad - \lambda_{111}\, \max[x_1, x_2, x_3]\big\}. \end{aligned}$$

The k–dimensional marginals $(k = 1, 2, \ldots, p-1)$ are all exponential. But the MVE of dimension $k \geq 2$ is not absolutely continuous because a singular part is present. At least one of the hyperplanes $x_i = x_j\ (i \neq j);\ x_i = x_j = x_k\ (i, j, k$ distinct), etc., has a positive probability.

Introducing the power transformations $X_i = Y_i^{c_i};\ i = 1, \ldots, p$; gives a first multivariate WEIBULL distribution — MWD for short — with joint survival function

$$R(\boldsymbol{y}) = \exp\Bigg\{-\sum_{\boldsymbol{s}\in S} \lambda_{\boldsymbol{s}}\, \max\left(s_i\, y_i^{c_i}\right),\ \ \boldsymbol{y} \geq \boldsymbol{o}. \tag{3.103a}$$

This MWD is not absolutely continuous but has WEIBULL marginals of all dimensions $k = 1, 2, \ldots, p$. HANAGAL (1996) discusses the following special case of (3.103a):

$$R(\boldsymbol{y}) = \exp\Bigg\{-\sum_{i=1}^{p} \lambda_i\, y_i^c - \lambda_0\, \big(\max[y_1, y_2, \ldots, y_p]\big)^c\Bigg\}. \tag{3.103b}$$

For $\lambda_0 = 0$ the joint distribution is the product of p independent one–dimensional WEIBULL distributions. The one–dimensional marginals of (3.103b) are

$$R(y_i) = \exp\big\{-(\lambda_i + \lambda_0)\, y_i^c\big\}. \tag{3.103c}$$

The distribution of $T = \min(Y_1, \ldots, Y_n)$, i.e., of the system's lifetime, is

$$R(t) = \exp\left\{-\left(\sum_{i=0}^{p} \lambda_i\right) t^c\right\}. \tag{3.103d}$$

A MWD which is absolutely continuous and which has WEIBULL marginals of all dimensions has been proposed by HOUGAARD (1986) and later on by ROY/MUKHERJEE (1988). The joint survival function is

$$R(\boldsymbol{y}) = \exp\left\{\left[-\sum_{i=1}^{p} \lambda_i\, y_i^c\right]^{\nu}\right\};\ \ \lambda_i > 0;\ c, \nu > 0;\ y_i \geq 0. \tag{3.104}$$

$T = \min\left(\dfrac{\lambda_1}{a_1}\, Y_1, \ldots, \dfrac{\lambda_p}{a_p}\, Y_p\right)$ has a one–dimensional WEIBULL distribution with shape parameter $c\,\nu$ when $a_i \geq 0$ and such that $\left(\sum\limits_{i=2}^{p} a_i^c\right)^{1/c} = 1.$

CROWDER (1989) has extended (3.104) in two aspects:

- The exponent c is made specific for variate Y_i.
- He introduced another parameter $\kappa > 0$.

Thus, the MWD of CROWDER has the joint survival distribution

$$R(\boldsymbol{y}) = \exp\left\{\kappa^{\nu} - \left[\kappa + \sum_{i=1}^{p} \lambda_i\, y_i^{c_i}\right]^{\nu}\right\}. \tag{3.105}$$

$\kappa = 0$ gives (3.104). The marginal distributions for Y_i are each WEIBULL when $\kappa = 0$ or $\nu = 1$. For general κ and ν the marginal distributions are such that $\big(\kappa + \lambda_i\, Y_i^{c_i}\big)^{\nu} - \kappa^{\nu}$ is exponential with unit mean.

PATRA/DEY (1999) have constructed a class of MWD in which each component has a mixture of WEIBULL distributions. Specifically, by taking

$$Y_i = \sum_{j=1}^{m} a_{ij}\, Y_{ij}\ \text{ with }\ Y_{ij} \sim We(0, \lambda_{ij}, c_{ij});\ \ i = 1, 2, \ldots, p, \tag{3.106a}$$

where $(a_{i1}, \ldots, a_{im})$ is a vector of mixing probabilities ($a_{ij} \geq 0\, \forall\, i, j$ and $\sum_{j=1}^{m} a_{ij} = 1$) and by further taking an exponentially distributed variate Z with DF

$$f(z) = \lambda_0\, \exp\big\{-\lambda_0\, z\big\}, \tag{3.106b}$$

where the Y_i and Z are all independent, they consider the multivariate distribution of

$$X_i = \min(Y_i, Z);\ \ i = 1, \ldots, p.$$

The joint survival function of $(X_1, \ldots, X_p)$ is

$$\begin{aligned} R(\boldsymbol{x}) &= \prod_{i=1}^{p} \Pr(X_i > x_i)\big[\Pr(Z > x_0)\big]^{1/p} \\ &= \prod_{i=1}^{p} \sum_{j=1}^{m} a_{ij}\, \exp\bigg\{-\bigg(\lambda_{ij}\, x_i^{c_{ij}} + \frac{\lambda\, x_0}{r}\bigg)\bigg\}, \end{aligned} \tag{3.106c}$$

where $x_0 = \max(x_1, \ldots, x_p) > 0$. The special case $m = p = 2$ is studied by PATRA/DEY in some detail.

3.3.10 Miscellaneous

In this last section of Chapter 3 we will list some relatives to the WEIBULL distribution which do not fit into one of the classes or models mentioned above.

The model proposed by VODA (1989) — called **pseudo–WEIBULL distribution** — has the DF

$$g(y\,|\,b, c) = \frac{c}{b\,\Gamma\big(1+\frac{1}{c}\big)} \left(\frac{y}{c}\right)^c \exp\bigg\{-\left(\frac{y}{c}\right)^c\bigg\};\;\; b, c, y > 0. \tag{3.107a}$$

(3.107a) is easily recognized to be

$$g(y\,|\,b, c) = \frac{y}{\mu^*}\, f(y\,|\,0, b, c), \tag{3.107b}$$

where μ^* is the mean and $f(y\,|\,0, b, c)$ is the DF (2.8) of a conventional WEIBULL distribution. The CDF belonging to (3.107a) is given by

$$\begin{aligned} G(y\,|\,b, c) &= \int_0^y g(y\,|\,b, c)\,\mathrm{d}u \\ &= \frac{\Gamma_{\sqrt{b^c y}}\big(1+\frac{1}{c}\big)}{\Gamma\big(1+\frac{1}{c}\big)} \end{aligned} \tag{3.107c}$$

and — because of its dependence on the incomplete gamma function $\Gamma.(\cdot)$ (see the excursus on the gamma function in Sect. 2.9.1) — has to be evaluated by numerical integration. The moments are given by

$$\mathrm{E}\big(Y^r\big) = \frac{b^{r/c}\,\Gamma\big(1+\frac{r+1}{c}\big)}{\Gamma\big(1+\frac{1}{c}\big)}\,. \tag{3.107d}$$

We have

$$\mathrm{E}(Y) = \frac{b^{1/c}\,\Gamma(1+2/c)}{\Gamma(1+1/c)}, \tag{3.107e}$$

$$\mathrm{Var}(Y) = b^{2/c}\left\{\frac{\Gamma(1+3/c)}{\Gamma(1+1/c)} - \frac{\Gamma^2(1+2/c)}{\Gamma^2(1+1/c)}\right\}. \tag{3.107f}$$

(3.107a) gives a gamma density — see (3.24) — with b and $d = 2$:

$$f(x\,|\,b, 2) = \frac{x}{b^2}\,\exp(-x/b).$$

But the pseudo–WEIBULL distribution is a generalization neither of the gamma distribution nor of the WEIBULL distribution.

FREIMER et al. (1989) developed what they call an **extended WEIBULL distribution** which rests upon a generalization of the percentile function $x_P = a + b\big[-\ln(1-P)\big]^{1/c}$ of the conventional three–parameter WEIBULL distribution:[37]

$$u_P = \left\{ \begin{array}{lll} d\left\{\big[-\ln(1-P)\big]^{1/d} - 1\right\} & \text{for} & d \neq 0 \\ \ln\big[-\ln(1-P)\big] & \text{for} & d = 0. \end{array} \right\} \tag{3.108a}$$

Inverting, i.e., solving for $P = F(u)$, gives the CDF

$$F(u\,|\,d) = \left\{ \begin{array}{lll} 1 - \exp\left\{-\left(1 + \dfrac{u}{d}\right)^d\right\} & \text{for} & d \neq 0 \\ 1 - \exp\big\{-e^u\big\} & \text{for} & d = 0. \end{array} \right\} \tag{3.108b}$$

The support of $F(u\,|\,d)$ depends on d as follows:

- $d < 0$ gives $-\infty < u < -d$,
- $d > 0$ gives $-d < u < \infty$,
- $d = 0$ gives $-\infty < u < \infty$.

SRIVASTAVA (1989) calls his special version a **generalized WEIBULL distribution**. It has some resemblance to a WEIBULL model with time–depending parameters (see Sect. 3.3.8.1). The general CDF

$$F(x) = 1 - e^{-\Psi(x)} \tag{3.109}$$

with non–decreasing function $\Psi(x)$ where $\Psi(0) = 0$ and $\Psi(\infty) = \infty$. The special case

$$\Psi(x) = \left(\frac{x-a}{b}\right)^c; \;\; a \in \mathbb{R};\; b, c > 0$$

yields the conventional three–parameter WEIBULL distribution.

In Fig. 2/8 we have presented the WEIBULL-probability–paper. On this paper the following transformation of the conventional three–parameter WEIBULL CDF

$$\ln\Big\{-\ln\big[1 - F(y\,|\,a,b,c)\big]\Big\} = -c\,\ln b + c\,\ln(x-a) \tag{3.110}$$

gives a straight line. The SLYMEN/LACHENBRUCH (1984) model rests upon a generalization of the WEIBULL transformation (3.110):

$$\ln\Big\{-\ln\big[1 - F(x)\big]\Big\} = \alpha + \beta\,w(x), \tag{3.111a}$$

where $w(x)$ is a monotonically increasing function depending on one or more parameters satisfying

$$\lim_{x\to 0} w(x) = -\infty \;\text{ and }\; \lim_{x\to\infty} w(x) = \infty.$$

[37] See also the extensions given in (3.78a) – (3.79b).

They discussed the special function

$$w(x) = \frac{x^{\nu} - x^{-\nu}}{2\,\nu}\,;\;\; x, \nu \geq 0. \tag{3.111b}$$

Re–transformation of (3.111a) using (3.111b) gives

$$F(x\,|\,\alpha,\beta,\nu) = 1 - \exp\left\{-\exp\left[\alpha + \frac{\beta\left(x^{\nu} - x^{-\nu}\right)}{2\,\nu}\right]\right\},\;\; x \geq 0. \tag{3.111c}$$

The corresponding HR is

$$h(x\,|\,\alpha,\beta,\nu) = \frac{\beta}{2}\left(x^{\nu} + x^{-\nu-1}\right)\,\exp\left\{\alpha + \frac{\beta\left(x^{\nu} - x^{-\nu}\right)}{2\,\nu}\right\}, \tag{3.111d}$$

which is monotonically increasing when

$$\beta > \frac{2\,\left[(\nu+1)\,x^{-\nu-2} - (\nu-1)\,x^{\nu+2}\right]}{(x^{\nu-1} + x^{-\nu-1})^2};$$

otherwise it may start with a decreasing part but finally (for x great) it will increase.