曲面积分

第一型曲面积分

对于一个曲面方程$r(u,v)=\begin{pmatrix}x(u,v)\\y(u,v)\\z(u,v)\end{pmatrix}$,$(u,v)\in D$为光滑曲面定义曲面$\sum$的第一基本量

$$||\vec{r_u}\times\vec{r_v}||=\sqrt{||r_u||^2||r_v||^2-(r_u\cdot r_v^2)}=\sqrt{EG-F^2}$$

$$E=||r_u||^2=(\frac{\partial x}{\partial u})^2+(\frac{\partial y}{\partial u})^2+(\frac{\partial z}{\partial u})^2$$

$$G=||r_u||^2=(\frac{\partial x}{\partial v})^2+(\frac{\partial y}{\partial v})^2+(\frac{\partial z}{\partial v})^2$$

$$F=\vec{r_u}\cdot\vec{r_v}=\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}+\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}+\frac{\partial z}{\partial u}\frac{\partial z}{\partial v}$$

$$\iint f(x,y,z)dS=\iint f(x,y,z)\sqrt{EG-F^2}dxdy$$

第二型曲面积分

定义

微小分割平面的法向量$\vec{n}=(\cos\alpha,\cos\beta,\cos\gamma)$当法向量与$z$轴正向夹角为锐角时为正，否则为负.

$$\iint_{\sum}Pdydz+Qdzdx+Rdxdy$$

性质:

- 方向性
- 线性性质
- 曲面可加性

两型面积分的关系

外法向量的方向向量的方向余弦$(-\frac{f_x}{\sqrt{1+f_x^2+f_y^2}},-\frac{f_y}{\sqrt{1+f_x^2+f_y^2}},\frac{1}{\sqrt{1+f_x^2+f_y^2}})$

$$\iint_{\sum} Pdydz + Qdzdx + Rdxdy = \iint_{\sum} (P\cos\alpha + Q\cos\beta + R\cos\gamma)dS$$

其余派生的转换显而易见

高斯公式

$$\iiint_V (\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z})dxdydz = \iint_{\sum} Pdydz + Qdzdx + Rdxdy$$

斯托克斯公式

设光滑曲面$\sum$的边界$\partial\sum$是分段光滑封闭曲线，P, Q, R在$\sum$及其边界$\partial\sum$上连续且具有一阶连续偏导数

$$\iint_{\sum} (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})dydz + (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})dzdx + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy$$

$$= \int_{\partial\sum} Pdx + Qdy + Rdz$$

$$\iint_{\sum} \begin{bmatrix} dydz & dzdx & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & Q \end{bmatrix} = \int_L Pdx + Qdy + Rdz$$