Deformation Quantization

With an Application to Quantum Field Theory

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What makes a physical system?

Key Ingredients

- Phase spaces: consists of pure states.
- Observables: experimentally measurable properties of pure states (e.g. momentum of particle).
- Dynamics: governs the temporal evolution of system.

Classical Physical System

- Classical Phase Space: A real symplectic $\mathscr{C}^\infty$-manifold (W, ω).
- Classical Observables: The set $\mathscr{C}^\infty(W)$ of smooth functions on W.
- Dynamics: Hamilton's equations $\frac{df_t}{dt} = \{f_t, H\}$, where H is the Hamiltonian (function).

A $\mathscr{C}^\infty$-manifold W is said to be symplectic if there exists a nondegenerate closed smooth 2-form $\omega \in \Omega^2(W)$ on W.

The study of classical mechanics is most generally rooted in Poisson geometry.

Def (Poisson Manifolds and Algebras)

Let $\mathscr{A}$ be an algebra over a commutative ring. Then $\mathscr{A}$ is said to be a Poisson algebra if it admits a Lie bracket $\{\cdot,\cdot\}$ such that $\{x,\cdot\} : \mathscr{A} \to \mathscr{A}$ is a derivation for each $x \in \mathscr{A}$.

A $\mathscr{C}^\infty$-manifold is a Poisson manifold if $\mathscr{C}^\infty(M)$ admits a Poisson algebra structure.

Poisson brackets on $\mathscr{C}^\infty(M)$ are determined by certain bivector fields $\pi \in \Gamma(\bigwedge^2 TM)$ in a bijective way: say (M, π) is a Poisson manifold.

Quantum Physical System

- Quantum Phase Space: the set $\mathbb{P}\mathscr{H}$ of rays in a complex Hilbert space $\mathscr{H}$.
- Quantum Observables: self-adjoint operators $\mathscr{H} \to \mathscr{H}$.
- Quantum Dynamics: governed by $i\hbar\frac{d\hat{A}_t}{dt} = [\hat{A}_t, \hat{H}]$ where $\hat{H}$ is the Hamiltonian (operator).

Typically one takes $\mathscr{H} = L^2(\mathbb{R})$.

Define position resp. momentum operators $\hat{x}$ resp. $\hat{p}$ on e.g. Schwartz space by

$$\hat{x}\psi = (x \mapsto x\psi(x)) \quad \text{and} \quad \hat{p}\psi = -i\hbar\frac{d\psi}{dx}.$$

Canonical commutation relation: $[\hat{x}, \hat{p}] = i\hbar$.

What is Quantization?

Quantization is a “recipe" for translating classical systems to quantum systems.

- Difficulty: an ambiguous definition.
- Difficulty: mathematics of classical & quantum mechanics are disjoint.
- Difficulty: seems to depend on physical situation at hand.

Let $\mathfrak{S}(\mathscr{H})$ be a set of self-adjoint linear operators $\mathscr{H} \to \mathscr{H}$.

- Lie bracket: $\frac{i}{\hbar}[\cdot,\cdot]$, where $[\cdot,\cdot]$ is the commutator.

Def (Dirac's Quantization)

Let $A \subseteq \mathscr{C}^\infty(W)$ resp. $B \subseteq \mathfrak{S}(\mathscr{H})$ be Poisson resp. Lie subalgebras. A quantization is a Lie algebra morphism $\varrho : A \to B$.

In particular ϱ satisfies:

1. Linearity.
2. $\varrho(1) = 1$ (identity operator).
3. $[\varrho(f), \varrho(g)] = -i\hbar\varrho(\{f, g\})$ for all $f, g \in \mathscr{C}^\infty(W)$.

There can be no quantization in the sense of Dirac.

Theorem (Groenewold–Van Hove, weak version) [Hall(2013)]
Let $\mathscr{P}_k$ denote homogenous polynomials of degree $\leq k$, and $\mathfrak{D}(\mathbb{R}^n)$ the differential operators on $\mathbb{R}^n$ with polynomial coefficients. There is no linear map $\varrho : \mathscr{P}_4 \to \mathfrak{D}(\mathbb{R}^n)$ with $\varrho(x) = \hat{x}$ and $\varrho(p) = \hat{p}$ that satisfies Dirac's quantization conditions.

One solution: satisfy the correspondence principle asymptotically as $\hbar \to 0$.

$$[\varrho(f), \varrho(g)] = -i\hbar\varrho(\{f, g\}) + \mathscr{O}(\hbar^2)$$

This idea motivates the deformation approach to quantization.

An Example: Elementary Star-Products (1/4)

Let $W = T^*\mathbb{R} \cong \mathbb{R}^2$ with coordinates (q, p) and take $\mathscr{H} = L^2(\mathbb{R})$.

Define $\varrho : \mathscr{C}^\infty(W) \supseteq \mathbb{C}[q, p] \to \mathfrak{D}(\mathbb{R}) \subseteq \mathfrak{L}(\mathscr{H}, \mathscr{H})$ by $\varrho(x^n) = \hat{x}^n$ and $\varrho(p^m) = \hat{p}^m$ and...

- Various ways to order mixed monomials $x^n p^m$.
- One way: do standard ordering and put $\varrho(x^n p^m) = \hat{x}^n \hat{p}^m$. Call this map $\varrho = \varrho_S$.

Extend $\mathbb{C}$-linearly.

ϱ_S is a linear isomorphism, but in general $\varrho_S(f)$ is not self-adjoint.

- e.g., $\varrho_S(qp) = \hat{q}\hat{p}$ is not even symmetric.

An Example: Elementary Star-Products (2/4)

Another way to order mixed monomials: completely symmetrize with Weyl–Moyal ordering.

Define $T : \mathbb{C}[q,p] \to \mathbb{C}[q,p]$ by

$$Tf = \exp\left(\frac{\hbar}{2i}\frac{\partial^2}{\partial q \partial p}\right) f = f + \frac{1}{1!}\frac{\hbar}{2i}\frac{\partial^2 f}{\partial q \partial p} + \cdots$$

which is a linear isomorphism.

Put $\varrho_W(f) = \varrho_S(Tf)$, which defines linear isomorphism $\mathbb{C}[q,p] \to \mathfrak{D}(\mathbb{R})$. We have e.g.

$$\varrho_W(qp) = \varrho_S\left(qp + \frac{\hbar}{2i}\right) = \frac{1}{2}(\hat{q}\hat{p} + \hat{p}\hat{q})$$

Does ϱ_W define a quantization of $\mathbb{C}[q, p]$?

Compute $[\varrho_W(f), \varrho_W(g)]$ indirectly by working within $\mathbb{C}[q, p]$ instead.

Proposition [Bordemann(2008)]

Define an $\mathbb{C}$-bilinear map $\star : \mathbb{C}[q, p] \times \mathbb{C}[q, p] \to \mathbb{C}[q, p]$ by

$$(f, g) \mapsto f \star g = \varrho_W^{-1}(\varrho_W(f)\varrho_W(g)).$$

Then $\star$ defines a noncommutative associative product on $\mathbb{C}[q, p]$
Furthermore, the following equality holds:

$$f \star g = \sum_{n=0}^{\infty} \frac{(i\hbar/2)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{\partial^n f}{\partial q^k \partial p^{n-k}} \frac{\partial^n g}{\partial q^{n-k} \partial p^k}$$

Form a star-commutator

$$[f,g]_\star := f\star g - g\star f = (fg-gf) + \frac{i\hbar}{2}(\{g,f\}-\{f,g\}) + \mathscr{O}(\hbar^2)$$

which is equivalent to $[\varrho_W(f),\varrho_W(g)] = -i\hbar\varrho_W(\{f,g\}) + \mathscr{O}(\hbar^2)$. Therefore ϱ_W quantizes $\mathbb{C}[q,p]$.

The Motivation

Interestingly, $\star$ can be viewed as a deformation of $\cdot$ in the direction of $\{\cdot,\cdot\}$. Deformation quantization is an abstraction of this idea.

The main framework lies in Gerstenhaber's deformation theory.

Let $(\mathscr{A}, \mu_0)$ be an algebra over a commutative ring R. Recall the linear space $\mathscr{A}[\![\hbar]\!]$ of formal power series

$$a(\hbar) = \sum_{n=0}^{\infty} \hbar^n a_n, \qquad a_n \in \mathscr{A}.$$

If $\mathscr{A} = R$ is a ring, so is $R[\![\hbar]\!]$ under

$$a(\hbar)b(\hbar) = \sum_{n=0}^{\infty} \hbar^n \sum_{r=0}^{n} a_r b_{n-r}.$$

Definition

A (formal associative) deformation of $(\mathscr{A}, \mu_0)$ is a sequence $(\mu_n)_{n=0}^{\infty}$ of R-bilinear maps $\mu_n : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ such that for all $x, y, z \in \mathscr{A}$,

1. $\sum_{s=0}^{r}(\mu_r(\mu_{r-s}(x, y), z) - \mu_r(x, \mu_{r-s}(y, z))) = 0$ for all $r \geq 0$.
2. $\mu_r(1, x) = 0 = \mu_r(x, 1)$ for all $r \geq 1$.

Gives a deformed product $\mu_\hbar : \mathscr{A} \to \mathscr{A}[\![\hbar]\!]$ defined by

$$\mu_\hbar(x, y) = \sum_{n=0}^{\infty} \hbar^n \mu_n(x, y) = xy + \sum_{n=1}^{\infty} \hbar^n \mu_n(x, y)$$

Extends $R[\![\hbar]\!]$-linearly to $\mathscr{A}[\![\hbar]\!]$, giving an $R[\![\hbar]\!]$-structure on $\mathscr{A}[\![\hbar]\!]$.

When does an algebra admit a (nontrivial) formal deformation?

Hochschild Cohomology

Any (associative) algebra $\mathscr{A}$ gives rise to Hochschild cochain complex $(C^\bullet(\mathscr{A}), \beta)$:

- $C^n(\mathscr{A}) = \{\text{n-multilinear maps } \mathscr{A}^n \to \mathscr{A}\},\ n \geq 0.$
- If $c \in C^n(\mathscr{A})$, then $\beta c \in C^{n+1}(\mathscr{A})$ is the map given by

$$\begin{aligned}\beta c(x_1, \ldots, x_{n+1}) = {} & x_1 c(x_2, \ldots, x_{n+1}) \\ & + \sum_{r=1}^{n} (-1)^r c(x_1, \ldots, x_{r-1}, x_r x_{r+1}, \ldots, x_{n+1}) \\ & + (-1)^{n+1} c(x_1, \ldots, x_n) x_{n+1},\end{aligned}$$

- The associated cohomology theory is denoted $HH^\bullet(\mathscr{A})$.

Hochschild cohomology is a measure of “how associative" deformations can be.

Lemma

Let $(\mu_n)_{n=0}^{\infty}$ be a deformation of $(\mathscr{A}, \mu_0)$. Then condition (1) of definition 3.1 is equivalent to:

$$\beta\mu_r(x,y,z) = \sum_{\substack{s+t=r \\ s,t>0}} (\mu_s(\mu_t(x,y),z) - \mu_s(x,\mu_t(y,z)).$$

This implies that if $\mu_1 \in C^2(\mathscr{A})$ is a cocycle, then $\mu = \mu_0 + \hbar\mu_1$ is a deformation up to linear order.

In general, given 2-cochains $\mu_1, \ldots, \mu_{n-1}$, the cochain on the RHS of the preceding lemma

$$\sum_{\substack{s+t=n \\ s,t>0}} (\mu_s(\mu_t(x,y),z) - \mu_s(x,\mu_t(y,z)))$$

is a 3-cocycle, and the deformation $\mu = \mu_0 + \hbar\mu_1 + \cdots + \hbar^{n-1}\mu_{n-1}$ can be extended to order n if and only if the above is a 3-coboundary $\beta\mu_n$. In other words:

Theorem
The obstructions to formal deformations of an algebra $\mathscr{A}$ lie in $HH^3(\mathscr{A})$.

Definition

Let μ and $\tilde{\mu}$ be two deformations of an algebra $\mathscr{A}$. A morphism of formal deformations from μ to $\tilde{\mu}$ is a formal power series $T = \sum_{n=0}^{\infty} \hbar^n T_n$ of 1-cochains $T_n : \mathscr{A} \to \mathscr{A}$ such that $T\mu = \tilde{\mu} T^2$.

Furthermore T is called an isomorphism if $T_0 = 1$ and μ is said to be c-equivalent to $\tilde{\mu}$. In this case μ has the form

$$\mu(x, y) = T^{-1}\tilde{\mu}(Tx, Ty).$$

Definition

Let (M, π) be a Poisson manifold. A deformation quantization of M is a formal associative deformation $(\mu_n)_{n=0}^{\infty}$ of $\mathscr{C}^{\infty}(M)$ such that the μ_n are bi-differential operators for $n \geq 1$ satisfying the so-called *classical limit*:

$$\mu_1(f, g) - \mu_1(g, f) = i\pi(df, dg) \quad \text{for all } f, g \in \mathscr{C}^{\infty}(M).$$

The deformed product is called a (differential) star-product on M and is denoted by $\star$, i.e.

$$f \star g = fg + \hbar \mu_1(f, g) + \mathscr{O}(\hbar^2), \qquad f, g \in \mathscr{C}^{\infty}(M).$$

The associated star-commutator deforms the Poisson bracket on $\mathscr{C}^\infty(M)$ essentially by definition:

$$[f,g]_\star = \hbar(\mu_1(f,g) - \mu_1(g,f)) + \mathscr{O}(\hbar^2) = i\hbar\{f,g\} + \mathscr{O}(\hbar^2).$$

One can go further:

Theorem
Let $\tilde{\star} = \sum_{n=0}^{\infty} \hbar^n \tilde{\mu}_n$ be a star-product on M. Then $\tilde{\star}$ is c-equivalent to a star-product $\star = \sum_{n=0}^{\infty} \hbar^n \mu_n$ with μ_1 a Poisson bracket on $\mathscr{C}^\infty(M)$.

The idea is to replace cochains via a c-equivalence operator. This is a key ingredient in the deformation quantization of fields, as we will see later.

Lemma

Let $\tilde{\star} = \sum_{n=0}^{\infty} \tilde{\mu}_n$ be a star-product. Then:

1. The bracket $\{f, g\} = \tilde{\mu}_1(f, g) - \tilde{\mu}_1(g, f)$ is a Lie bracket.
2. If $\tilde{\mu}_1$ is skew-symmetric, then $\tilde{\mu}_1$ satisfies the Leibniz identity.

Proof.

1. Jacobi identity is a direct computation.
2. Follows from $\beta\tilde{\mu}_1(f, g, h) - \beta\tilde{\mu}_1(g, h, f) = 0$ and $\beta\tilde{\mu}_1(h, f, g) - \beta\tilde{\mu}_1(g, h, f) = 0$.

$\square$

Proof (of theorem).

Decompose $\tilde{\mu}_1 = \tilde{\mu}_1^- + \tilde{\mu}_1^+$ where

$$\tilde{\mu}_1^{\pm}(f,g) = \frac{1}{2}\Big(\tilde{\mu}_1(f,g) \pm \tilde{\mu}_1(g,f)\Big)$$

Since $\tilde{\mu}_1^+$ is symmetric, it is zero in $HH^2(\mathscr{C}^\infty(M))$, hence $\tilde{\mu}_1^+ = \beta T_1$ for some 1-cochain T_1. Put $T = \sum_{n=0}^{\infty} \hbar^n T_n$, where $T_n = 1$ for $n \neq 1$, and define $f \star g = T^{-1}(Tf \tilde{\star} Tg)$. The $\hbar$-term of $\tilde{\star}$ transforms under T as

$$\mu_1(f,g) = \tilde{\mu}_1(f,g) + T_1(fg) - fT_1(g) - T_1(f)g$$

Thus $\mu_1 = \tilde{\mu}_1^-$ is a Poisson bracket on $\mathscr{C}^\infty(M)$.

$\square$

Example: The Moyal Product

Let V be a real vector space with $\dim V = m$. For each $1 \leq i, j \leq m$ let $\Lambda^{ij} = -\Lambda^{ji} \in \mathbb{R}$. Then $\Lambda = \frac{1}{2}\Lambda^{ij}\partial_i \wedge \partial_j$ defines a (constant) Poisson structure on V.

For $f, g \in \mathscr{C}^\infty(V)$, define the Moyal (star-)product by

$$(f \star g)(z) = \exp\left(\frac{i\hbar}{2}\Lambda\right) f(x)g(y)\Big|_{x=y=z}$$

If $m = 2$ and $\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, recover "elementary" star-product on $\mathbb{C}[q, p]$ from Weyl-Moyal ordering.

Main focus: G. Dito's star-product approach to QFT [Dito(1990)].

- An active area of research: deformation quantization of infinite-dimensional manifolds is not well-understood.
- Relevant formulation is perturbative field theory.
- This means one should consider a space of functionals in place of $\mathscr{C}^\infty$-functions.

The initial data (φ, π) of solutions to the equation $(\Box + m^2)\Phi = 0$, i.e. $\varphi(x) = \Phi(x, 0)$ and $\pi(x) = \partial_t \Phi(x, 0)$, are elements of $E_\infty := \mathscr{S}(\mathbb{R}^3, \mathbb{R}) \oplus \mathscr{S}(\mathbb{R}^3, \mathbb{R})$.

Let $\mathscr{M} \subseteq E_\infty$ be the phase space of such initial data.

Let $(\varphi, \pi) \in \mathscr{M}$. Decompose into Fourier modes (a^*, a) by

$$\varphi(x) = \int \left(\frac{a^*(k)e^{-i\langle k,x\rangle} + a(k)e^{i\langle k,x\rangle}}{2(2\pi)^{3/2}\omega_k} \right) dk$$
$$\pi(x) = \int \left(\frac{a^*(k)e^{-i\langle k,x\rangle} + a(k)e^{i\langle k,x\rangle}}{2(2\pi)^{3/2}} \right) dk$$

where $\omega_k = (\|k\|^2 + m^2)^{1/2}, m > 0$. The free scalar Hamiltonian is given by

$$H = \int \left(\omega_k a^*(k)a(k) \right) dk$$

Some notation

Put $a_1(k) = a(k)$, $a_2(k) = a^*(k)$ and

$$\langle \delta_{i_1 \cdots i_n} \Psi_1, \delta_{j_1 \cdots j_n} \Psi_2 \rangle :=$$
$$\int \left(\frac{\delta^n \Psi_1}{\delta a_{i_1}(k_1) \cdots \delta a_{i_n}(k_n)} \frac{\delta^n \Psi_2}{\delta (a_{j_1})^*(k_1) \cdots \delta (a_{j_n})^*(k_n)} \right) dk,$$

and $\Lambda^{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

There is a Poisson structure Λ on scalar-valued functionals given by

$$\{\Psi_1, \Psi_2\} = \Lambda^{ij} \langle \delta_i \Psi_1, \delta_j \Psi_2 \rangle.$$

Can attempt to define Moyal product on a certain space of functionals...

$$\Psi_1 \star_M \Psi_2 := \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i\hbar}{2} \right)^n \Lambda^{i_1 j_1} \cdots \Lambda^{i_n j_n} \langle \delta_{i_1 \cdots i_n} \Psi_1, \delta_{j_1 \cdots j_n} \Psi_2 \rangle.$$

...but Moyal–Weyl ordering is not suitable for QFT.

This is reflected in the fact that $H \star_M H$ diverges, i.e. $\mathrm{Exp}_\star(tH/i\hbar)$ does not exist.

The solution presented by G. Dito is to "renormalize" using cohomology, i.e. cohomological renormalization.

For the series

$$T = \exp\left(\frac{\hbar}{2}\int \frac{\delta^2}{\delta a(k)\delta a^*(k)}\, dk\right)$$

define a normal ordered star-product by $\Psi_1 \star_N \Psi_2 = T^{-1}\left(T\Psi_1 \star_M T\Psi_2\right)$. It has an explicit form given by

$$(\Psi_1 \star_N \Psi_2)(a^*, a) = \int_{\mathscr{S}'\oplus\mathscr{S}'} \left(\Psi_1(a^*, a+\xi)\Psi_2(a^*+\xi^*, a)\right)\, d\mu(\xi^*, \xi),$$

where μ is the unique (Gaussian) measure on $\mathscr{S}'\oplus\mathscr{S}'$ given by the characteristic function $\exp(-\frac{1}{\hbar}\int a^*(k)a(k)\, dk)$.

It turns out that this normal ordered star-product ensures $\mathrm{Exp}_\star(tH/i\hbar)$ exists.

This means that one can derive the energy levels using the star-exponential.

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