Global and Local Pictures of Principal Bundles

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Let M be a smooth manifold and G a Lie group. By a **local G-system** on M, we understand an open cover $\{U_i : i \in I\}$ of M together with functions $g_{ij} : U_i \cap U_j \to G$ that satisfy the so-called cocycle condition $g_{ij} g_{jk} = g_{ik}$. Observe that then also $g_{ii} = e$ and $g_{ij}^{-1} = g_{ji}$. We will show that any principal G-bundle $\pi : P \to M$ gives rise to a local G-system on M, and conversely any local G-system constructs a principal G-bundle over M with a certain property. These constructions are actually compatible with the definitions of connections presented in Bleecker in the sense that, roughly, a collection of $\mathscr{G}$-valued 1-forms defined locally on M (which has a specified local G-system) will give rise to the usual connection 1-form on the constructed bundle $\pi : P \to M$, and vice versa.

1 Global to Local

Let $\pi : P \to M$ be a principal G-bundle; let $\psi_i : U_i \times G \to \pi^{-1}U_i$ and $\psi_j : U_j \times G \to \pi^{-1}P$ be two of its local trivializations that overlap. For each $x \in U_i \cap U_j$, there is a $g_{ij}(x) \in G$ such that

$$(\psi_i^{-1}\psi_j)(x, e) = (x, g_{ij}(x)).$$

More precisely, there is a function $g_{ij} : U_i \cap U_j \to G$ explicitly given by $g_{ij} = \mathrm{pr}_2 \psi_i^{-1}\psi_j$, where pr_2 is projection onto the second factor. It is then immediate that g_{ij} is smooth. Now in $U_i \cap U_j \cap U_k$, the G-equivariance[1] of the corresponding trivializations imply that

$$\begin{aligned}
(\psi_i^{-1}\psi_k)(x, e) &= (\psi_i^{-1}\psi_j)(\psi_j^{-1}\psi_k)(x, e) \\
&= (\psi_i^{-1}\psi_j)(x, g_{jk}(x)) \\
&= \psi_i^{-1}\big(\psi_j(x, e) g_{jk}(x)\big) \\
&= \big(x, g_{ij}(x) g_{jk}(x)\big)
\end{aligned}$$

On the other hand, $(\psi_i^{-1}\psi_k(x, e)) = (x, g_{ik}(x))$, so we've shown $g_{ij}g_{jk} = g_{ik}$. Thus we have a local G-system on M from the principal G-bundle $\pi : P \to M$. We remark that the functions $g_{ij} : U_i \cap U_j \to G$ obtained from $\pi : P \to M$ in this manner are called the bundle's **transition functions**.

Local Picture of Connections Now let ω be a connection 1-form on $\pi : P \to M$. We show there is a collection $\{\omega_i : i \in I\}$ of $\mathscr{G}$-valued 1-forms, defined on each $U_i \subseteq M$, that are compatible with the local G-system on M in the sense that

$$\omega_j = g_{ij}^{-1} dg_{ij} + g_{ij}^{-1} \omega_i g_{ij} \tag{1}$$

on each overlap $U_i \cap U_j$. From the local trivialization $\psi_i : U_i \times G \to \pi^{-1}U_i$, one obtains a smooth local section $\sigma_i : U_i \to P$ by defining $\sigma_i(x) := \psi_i(x, e)$. Put $\omega_i = \sigma_i^* \omega \in \Omega^1(U_i, \mathscr{G})$ for each $i \in I$. Observe that

$$\sigma_j(x) = \psi_j(x, e) = (\psi_i \psi_i^{-1}\psi_j)(x, e) = \psi_i(x, g_{ij}(x)) = \sigma_i(x) g_{ij}(x),$$

[1] We are also using the fact that ψ_i^{-1} has the form $\psi_i^{-1} = \pi \times s_i$, where $s_i : \pi^{-1}U_i \to G$ is a G-equivariant function, so $s_i\big(\psi_j(x, e) g_{jk}(x)\big) = s_i\big(\psi_j(x, e)\big) \cdot g_{jk}(x) = g_{ij}(x) g_{jk}(x)$.

which implies that if $Y \in T_x M$ is a tangent vector, then (with $\gamma : \mathbb{R} \to M$ any curve at x with $\gamma'(0) = Y$):

$$\begin{aligned}
\sigma_{j*}(Y) &= \frac{d}{dt}\bigg|_{t=0} (\sigma_j \circ \gamma)(t) \\
&= \frac{d}{dt}\bigg|_{t=0} \sigma_i(\gamma(t)) \cdot g_{ij}(\gamma(t)) \\
&= \sigma_i(x) g_{ij*}(Y) + \sigma_{i*}(Y) g_{ij}(x) \\
&= \big(\sigma_j(x) g_{ij}(x)^{-1}\big) g_{ij*}(Y) + \sigma_{i*}(Y) g_{ij}(x) \\
&= \frac{d}{dt}\bigg|_{t=0} \sigma_j(x) \cdot \exp\big(t g_{ij}(x)^{-1} g_{ij*}(Y)\big) + \sigma_{i*}(Y) g_{ij}(x) \\
&= \big[g_{ij}(x)^{-1} g_{ij*}(Y)\big]^*_{\sigma_j(x)} + \sigma_{i*}(Y) g_{ij}(x),
\end{aligned}$$

where A^* means the fundamental vector field corresponding to A. I should also say I am using the fact that if $R : P \times G \to P$ is the action map, then $R_{*(p,g)}(X, A) = Xg + pA$, whose proof is a straightforward calculation. Therefore,

$$\omega_j(Y) = \omega(\sigma_{j*}(Y)) = g_{ij}(x)^{-1} dg_{ij}(Y) + g_{ij}(x)^{-1} \omega\big(\sigma_{i*}(Y)\big) g_{ij}(x),$$

which was what we wanted to show (we are using also that $R_g^* \omega = g^{-1} \omega g$).

Summary A principal G-bundle induces a local G-system on its base space. If a connection 1-form is specified on the bundle, we obtain as above a collection of 1-forms on the base that are compatible with the local G-system.

2 Local to Global

Now suppose that M is a smooth manifold with a local G-system $\{g_{ij}, U_i : i, j \in I\}$. Let us first construct a principal G-bundle over M; this bundle shall have the property that its transition functions are precisely the g_{ij}.

Consider the set $\tilde{P} = \bigsqcup_{i \in I} (U_i \times G)$; points of $\tilde{P}$ will be written (i, x, g), which is the usual convention for disjoint unions. We define a relation on $\tilde{P}$ as follows: declare (i, x, g) and (j, x', g') equivalent if and only if $y = x$ and $g' = g_{ij}(x) g$. The fact that this relation is actually an equivalence relation follows readily from the cocycle condition on the local G-system. We then form the quotient P from $\tilde{P}$ by this equivalence relation: points of P will be written $[i, x, g]$, understanding these are equivalence classes of points in $\tilde{P}$. There is a natural projection $\pi : P \to M$ given by $\pi([i, x, g]) = x$.

Let us show that P is a smooth manifold. For each $i \in I$, define $\psi_i : U_i \times G \to \pi^{-1} U_i$ by

$$\psi_i(x, g) = [i, x, g].$$

Then ψ_i has a well-defined inverse given by $\psi_i^{-1}([i, x, g]) = (x, g)$, since if $[j, x', g'] = [i, x, g] \in \pi^{-1} U_i$, then $j = i$, $x' = x$, and $g' = g_{ii}(x) g = g$. So every ψ_i is a bijection. We need to show that each $\psi_i^{-1} \psi_j$ is smooth as a map from $(U_i \cap U_j) \times G$ to itself. Indeed,

$$(\psi_i^{-1} \psi_j)(x, g) = \psi_i^{-1}([j, x, g]) = \psi_i^{-1}([i, x, g_{ji}(x) g]) = (x, g_{ji}(x) g),$$

and g_{ji} was assumed to be smooth. Thus, P inherits a unique smooth manifold structure making each ψ_i a diffeomorphism. Observe that π locally corresponds (via ψ_i) to the map $(x, g) \mapsto x$, so π is smooth.

There is a natural right G-action on P given by $[i, x, g]h = [i, x, gh]$. It is well-defined:

$$[j, x, g_{ij}(x) g]h = [j, x, g_{ij}(x) gh] = [i, x, gh] = [i, x, g]h.$$

The action $P \times G \to P$ locally corresponds (again via ψ_i) to the map $((x, g), h) \mapsto (x, gh)$, so this action is also smooth. For each $g \in G$, the map $R_g : P \to P$ given by $R_g([i, x, h]) = [i, x, hg]$ is a diffeomorphism with inverse $R_{g^{-1}}$. The action is also free and transitive, so actually $\pi : P \to M$ is a principal G-bundle.

Global Picture of Connections In the above notation, let $\{\omega_i : i \in I\}$ be a collection of $\mathscr{G}$-valued 1-forms such that

$$\omega_j = g_{ij}^{-1} dg_{ij} + g_{ij}^{-1} \omega_i g_{ij}.$$

Construct the principal G-bundle $\pi : P \to M$ from M and $\{U_i, g_{ij} : i, j \in I\}$ as above. We will create a connection 1-form ω on $\pi : P \to M$ such that $\omega|_{U_i} = \omega_i$ for each $i \in I$.

Since $\psi_i : U_i \times G \to \pi^{-1}U_i$ is a diffeomorphism, we have $T_xU_i \oplus \mathscr{G} \cong T_pP$ (here we write p for $[i, x, e]$, and we remark that p can be arbitrary since every tangent space of G is isomorphic to $\mathscr{G}$). Because $\sigma_i : U_i \to P$ is injective, we have the decomposition

$$T_pP \cong \sigma_{i*}(T_xU_i) \oplus \mathscr{G}.$$

Thus the formula $\omega_p^i(\sigma_{i*}X + A^*) = \omega_i(X) + A$ defines a map $\omega_p^i : T_pP \to \mathscr{G}$. We can extend to $T_{pg}P$ by the formula

$$\omega_{pg}^i(X_{pg}) = g^{-1}\omega^i(X_{pg}g^{-1})g,$$

which defines a 1-form ω^i on all of $\pi^{-1}U_i$. This is necessary so that ω^i is a connection 1-form on the restricted bundle $\pi^{-1}U_i \to U_i$, since then $R_g^*\omega^i = g^{-1}\omega^i g$. Repeating this construction for all $i \in I$, we need now to check that $\omega^i = \omega^j$ on the overlaps. Since $\omega^i(A^*) = A = \omega^j(A^*)$, it suffices to check that they agree on the horizontal vectors. Indeed,

$$\begin{aligned}
\omega^i\big(\sigma_{j*}X\big) &= \omega^i\bigg(\big[g_{ij}(x)^{-1}g_{ij*}(X)\big]^*_{\sigma_j(x)} + \sigma_{i*}(X)g_{ij}(x)\bigg) \\
&= g_{ij}(x)^{-1}g_{ij*}(X) + g_{ij}(x)^{-1}\omega^i(\sigma_{i*}X)g_{ij}(x) \\
&= \omega^j(\sigma_{j*}X),
\end{aligned}$$

since $\omega^i(\sigma_{i*}X) = \omega_i(X)$. Thus each ω^i piece together to form a connection 1-form ω on P.

Summary Given a smooth manifold with a local G-system, one constructs a principal G-bundle over it in such a way that the local G-system constitute its transition functions. Then, a family of 1-forms compatible with the local G-system defines local connection 1-forms on P, which piece together into the usual connection 1-form.