Groups of Order 2025 Are Not Simple

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Let G be a group with $|G| = 2025 = 3^4 \cdot 5^2$. Prove that G is not simple, i.e. has a proper (nontrivial and a proper subset) normal subgroup.

Proof. Assume that $n_5(G) > 1$ (otherwise we are done). Sylow's theorem says $n_5(G)$ divides 3^4, hence $n_5(G) \in \{1, 3, 9, 27, 81\}$. Since $n_5(G) - 1 \equiv 0 \bmod 5$ also, we must have $n_5(G) = 81$. There are now two cases to consider.

- **Case 1). All Sylow 5-subgroups intersect trivially.**

  Here G has $24 \cdot 81 = 1\,944$ distint elements of order 5 or 25. Since $2025 - 1944 = 81$, which is the order of *any* Sylow 3-subgroup, one has $n_3(G) = 1$ necessarily. Thus G has a normal Sylow 3-subgroup.

- **Case 2). There exists distinct $P, Q \in \mathrm{Syl}_5(G)$ that do not intersect trivially.**

  Consider $P \cap Q$. Then $P \cap Q \leq P$, so by Lagrange's theorem, $|P \cap Q| = 5$ (it cannot be 5^2, for this would imply $P \cap Q = P$ i.e. $P = Q$). We will show that $P \cap Q$ is normal in G (so G has a normal subgroup of order 5). Equivalently, we show $N_G(P \cap Q) = G$.

  To that end, let $H := N_G(P \cap Q)$; we will study the order of H.

  First, observe that $|P : P \cap Q| = 5$. Since 5 is the smallest prime dividing $|P| = 5^2$, it follows that $P \cap Q$ is normal in P. This means that $P \leq H$. By Lagrange's theorem, we have that $|P| = 5^2$ divides $|H|$. Since $|H|$ divides $|G| = 3^4 \cdot 5^2$, we must have $|H| = 3^r \cdot 5^2$ for some $0 \leq r \leq 4$. The goal is to show that $r = 4$. To do so, we will apply Sylow's theorems on H now.

  Sylow's theorem says that $n_5(H)$ divides 3^r and $n_5(H) - 1 \equiv 0 \bmod 5$. If $r < 4$, then these two imply $n_5(H) = 1$. This is irrelevant to our case, because we've shown that H contains at least 2 Sylow 5-subgroups (namely P and Q). Note that if $r = 4$, we have either $n_5(H) = 1$ or $n_5(H) = 81$. Again since $n_5(H) \geq 2$, we must have $n_5(H) = 81$.

  Therefore $r = 4$ and $|H| = 2025 = |G|$, i.e. $N_G(P \cap Q) = G$. This means $P \cap Q$ is a normal subgroup of G of order 5, so G is not simple in this case either.

In any case, we've shown G cannot be simple. ❏