Riesz Representation on Compact Metric Spaces

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1. Introduction and Preliminaries

Let (K, d) be a compact metric space; we denote by $\mathscr{B}(K)$ the σ-algebra generated by the open subsets of K, which is called the **Borel sigma algebra** on K. Suppose that $\mu : \mathscr{B}(K) \to [0, +\infty]$ is a finite (Borel) measure on X. Then there is a positive functional $\mathscr{C}^0(K) \to \mathbb{R}$ given by

$$f \mapsto \int f \, d\mu.$$

It turns out that the converse of this observation also holds, namely that given a positive functional $T : \mathscr{C}^0(K) \to \mathbb{R}$ (here *positive* means that if $f \geq 0$, then $Tf \geq 0$), then there a unique finite Borel measure μ on K such that

$$Tf = \int f \, d\mu$$

for all $f \in \mathscr{C}^0(K)$. This nontrivial result is known as the **Riesz Representation theorem** (Thm 3.1), which is the topic of this paper.

We will work exclusively under a compact metric space (K, d); the purpose of this section is to briefly review some topological properties K possesses.

Because K is metrizable, it is Hausdorff. Compactness further implies that K is second-countable (namely, there is a countable base that generates the topology of K). To see this, fix an $n \in \mathbb{N}$ and let $\mathscr{U}_n = \{B(x, \frac{1}{n}) : x \in K\}$. Then K is covered by finitely many members of $\mathscr{U}_n$, say $\{B(x_i^{(n)}, \frac{1}{n}) : 1 \leq i \leq m_n\}$. Repeating this for each $n \in \mathbb{N}$ gives a collection $\bigcup_{n\in\mathbb{N}} \bigcup_{i=1}^{m_n} \{B(x_i^{(n)}, \frac{1}{n})\}$, which is a countable base for K.

Any compact Hausdorff space is also normal (disjoint closed sets may be separated by open sets). Let $A, B \subseteq K$ be disjoint closed sets. Since K is compact, so also are A and B. Now fix $a \in A$. For each $b \in B$, there exists U_b, V_b disjoint open subsets with $a \in U_b$ and $b \in V_b$ since K is Hausdorff. Then $\{V_b : b \in B\}$ is an open cover of B, hence admits a finite subcover $\{V_{b_1}, \ldots, V_{b_n}\}$. Put $U = \bigcap_{i=1}^n U_{b_i}$ and $V = \bigcup_{i=1}^n V_{b_i}$. Then $U, V \subseteq K$ are disjoint, open, while satisfing $a \in U$ and $B \subseteq V$. A symmetric argument shows that there are disjoint open sets U', V' such that $A \subseteq U'$ and $B \subseteq V'$. In other words, K is normal. The importance of this is that Urysohn's lemma applies for K:

LEMMA 1.1 (Urysohn, cf. Thm 33.1 in [3]). *Let X be a normal space and $A, B \subseteq X$ disjoint closed subsets. Given $a < b$ real numbers, there exists a continuous function $f : X \to [a, b]$ such that $f(A) = \{a\}$ and $f(B) = \{b\}$.*

In fact, we will make use of a more general result. An indispensable tool in manifold theory is the notion of **partitions of unity**, the existence of which implies Urysohn's lemma. Let X be any topological space and $\mathscr{U} = \{U_\alpha : \alpha \in J\}$ an open cover of X. A **partition of unity subordinate to** $\mathscr{U}$ is a collection $\{u_\alpha : \alpha \in J\}$ of continuous functions $u_\alpha : U_\alpha \to [0, 1]$ satisfying the following properties:

(i) $\operatorname{supp} u_\alpha \subseteq U_\alpha$,
(ii) Each $x \in X$ has neighborhood U_x such that U_x intersects finitely many $\operatorname{supp} u_\alpha$, which is to say that the collection $\{\operatorname{supp} u_\alpha : \alpha \in J\}$ is **locally finite.**
(iii) For each $x \in X$, the sum $\sum_{\alpha \in J} u_\alpha$ is finite (by (ii)) and is unity.

A partition of unity allows a continuous function $f : X \to \mathbb{R}$ to be decomposed into a sum of continuous functions, whose support lay in "smaller" subsets, via $f = \sum_{\alpha \in J} u_\alpha f$. This will be used in the proof of the Riesz Representation theorem later, and more detail will be discussed. For our purposes, a partition of unity subordinate to any open cover of a compact metric space K always exists (in fact, one needs paracompactness), but the proof of this lies outside the scope of this paper and may be found in essentially any differential geometry textbook (see e.g. [2]).

We will need the following result in proving the uniqueness of the measure given by the Riesz Representation theorem:

COROLLARY 1.2 (cf. Prop 7.1.9 in [1]). *Let $F \subseteq K$ be closed. If U is an open set containing F, then there exists $f \in \mathscr{C}^0(K)$ such that $1_F \leq f \leq 1_U$.*

PROOF. Apply Urysohn's lemma with $a = 0$ and $b = 1$. Full details omitted for space, since the proof itself is not crucial for Riesz Representation. $\square$

2. A Candidate Measure

Let $T : \mathscr{C}^0(K) \to \mathbb{R}$ be a positive linear functional. For any open subset $U \subseteq K$, define $\mu^*(U) = \sup_{f \in \mathscr{C}^0(K)}\{Tf : f \leq 1_U\}$. We extend to a function $\mu^* : \mathscr{P}(K) \to [0, +\infty]$ via

$$E \mapsto \mu^*(E) = \inf_{U \subseteq K \text{ open}} \{\mu^*(U) : E \subseteq U\}. \tag{2.1}$$

This extension is well-defined in the sense that the two definitions coincide when E is open. Indeed, suppose E is open. If U is any open subset containing E, then $1_E \leq 1_U$, so

$$\{Tf : f \leq 1_E\} \subseteq \{Tf : f \leq 1_U\}.$$

Therefore $\mu^*(E) \leq \mu^*(U)$ for all $U \subseteq K$ open (where $\mu^*(E)$ is given by the definition involving open sets). Thus the infimum of the set $\{\mu^*(U) : E \subseteq U\}$ is attained as $\mu^*(E)$.

Let us show that μ^* is an outer measure on K. Then, according to Carathéodory, the restriction $\mu^*|_{\mathscr{F}^*}$ where

$$\mathscr{F}^* = \{A \subseteq K : \mu^*(E) \geq \mu^*(A \cap E) + \mu^*(A^c \cap E)\}$$

is a σ-algebra on K, is a measure. It turns out that this is the measure specified by the Riesz Representation theorem. The goal of this section is to develop μ^* as above and to show that μ^* restricts to a Borel measure on K.

Proposition 2.1 (cf. Theorem 13.4.23 in [4]). *Let $\mu^* : \mathscr{P}(K) \to [0, +\infty]$ be the function given by (2.1). Then μ^* is an outer measure on K.*

Proof. We have $\mu^*(\varnothing) = \sup\{Tf : f \leq 1_\varnothing\} = T(0) = 0$ since T is linear. Given $A \subseteq B$, observe that if U is an open subset of K for which $B \subseteq U$, then $A \subseteq U$ also; in particular, $\mu^*(A) \leq \mu^*(B)$.

It remains to show that μ^* is σ-subadditive. First let $(U_n : n \in \mathbb{N})$ be a sequence of open subsets of K. Choose any $f \in \mathscr{C}^0(K)$ with $\operatorname{supp} f \subseteq \bigcup_{n=1}^\infty U_n$. Since $\operatorname{supp} f$ is a closed subset of a compact space, it itself is compact. In particular there exists $N \in \mathbb{N}$ for which $\operatorname{supp} f \subseteq U_1 \cup \cdots \cup U_N$. Now let $(u_i : 1 \leq i \leq N)$ be a partition of unity subordinate to $\{U_i : 1 \leq i \leq N\}$. Since $\operatorname{supp} f$ is covered by these U_i, it follows that $f = \sum_{i=1}^N f u_i$. The fact that $f u_i \leq 1_{U_i}$ gives $T(1_{U_i} - f u_i) \geq 0$ since T is positive, hence $T(f u_i) \leq T(1_{U_i})$ and

$$Tf = \sum_{i=1}^N T(f u_i) \leq \sum_{i=1}^N T(1_{U_i}) = \sum_{i=1}^N \mu^*(U_i) \leq \sum_{i=1}^\infty \mu^*(U_i).$$

The above holds for any $f \in \mathscr{C}^0(K)$ with $f \leq 1_{\bigcup_{i=1}^\infty U_i}$. Taking the supremum over Tf for such f gives $\mu^*(\bigcup_{i=1}^\infty U_i) \leq \sum_{i=1}^\infty \mu^*(U_i)$, which shows σ-subadditivity over the open subsets. In general, let $(E_n : n \in \mathbb{N})$ be any sequence of subsets of K. We may assume WLOG that $\mu^*(E_n) < +\infty$, otherwise there is nothing to prove. By the definition in (2.1), there is a sequence $(U_n : n \in \mathbb{N})$ of open subsets with $E_n \subseteq U_n$ and $\mu^*(U_n) \leq \mu^*(E_n)+2^{-n}\varepsilon$. Then σ-subadditivity for the open subsets gives

$$\mu^*\left(\bigcup_{n=1}^\infty E_n\right) \leq \mu^*\left(\bigcup_{n=1}^\infty U_n\right) \leq \sum_{n=1}^\infty \mu^*(U_n) \leq \sum_{n=1}^\infty \mu^*(E_n) + \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, we've shown μ^* is σ-additive. Altogether, μ^* is an outer measure on K.

$\square$

Before we show that the σ-algebra $\mathscr{F}^*$, given by Carathéodory, contains the Borel subsets, we will need the following helpful fact in the next section.

Lemma 2.2 (cf. Lemma 7.2.10 in [1]). *Let $f \in \mathscr{C}^0(K)$. For any $A \subseteq K$, if $1_A \leq f$, then $\mu^*(A) \leq Tf$.*

Proof. Let $\varepsilon > 0$ and assume further that $\varepsilon < 1$. Put $U = f^{-1}((1-\varepsilon, \infty))$, which is open in K by continuity of f. Consider $g \in \mathscr{C}^0(K)$ with $Tg \in \{Th : h \leq 1_U\}$. If $x \in U$, then $g(x) \leq 1_U(x) = 1 < \frac{1}{1-\varepsilon} f(x)$. Now observe that f is non-negative since $0 \leq 1_A \leq f$; thus, if $x \notin U$, then $g(x) \leq 0 \leq \frac{1}{1-\varepsilon} f(x)$. We've shown that $g \leq \frac{1}{1-\varepsilon} f$, so that

$$\mu^*(U) = \sup\{Th : h \leq 1_U\} \leq T\left(\frac{1}{1-\varepsilon} f\right) = \frac{1}{1-\varepsilon} Tf.$$

If $x \in A$, then $f(x) \geq 1 > 1-\varepsilon$, so $x \in U$. Therefore,

$$\mu^*(A) \leq \mu^*(U) \leq \frac{1}{1-\varepsilon} Tf.$$

But $0 < \varepsilon < 1$ was arbitrary; we conclude that $\mu^*(A) \leq Tf$.

$\square$

We conclude this section by showing that $\mathscr{F}^*$ contains the Borel subsets, following [1]. Let $U \subseteq K$ be open and $A \subseteq K$ any subset (we may once again assume WLOG $\mu^*(A) < +\infty$, otherwise there is nothing to show). We need to show that $\mu^*(A) \geq \mu^*(U \cap A) + \mu^*(U^c \cap A)$.

Let $\varepsilon > 0$. By the definition (2.1), there is an open subset $V \subseteq K$ such that $A \subseteq V$ and

$$\mu^*(V) < \mu^*(A) + \varepsilon/3. \tag{2.2}$$

The set $U \cap V$ is open; the definition of $\mu^*(U \cap V)$ implies there exists $g \in \mathscr{C}^0(K)$ such that $\operatorname{supp} g \subseteq U \cap V$ and $Tg > \mu^*(U \cap V) - \varepsilon/3$. Finally, by the same reasoning, choose $h \in \mathscr{C}^0(K)$ with $\operatorname{supp} h \subseteq V \cap (\operatorname{supp} g)^c$ and $Th > \mu^*(V \cap (\operatorname{supp} g)^c) - \varepsilon/3$. Then $\operatorname{supp}(g+h) \subseteq \operatorname{supp} g \cup \operatorname{supp} h \subseteq V$, which means

$$\begin{aligned}
\mu^*(V) &= \sup_{f \in \mathscr{C}^0(K)} \{Tf : \operatorname{supp} f \subseteq V\} \\
&\geq T(g+h) = Tg + Th \\
&> \mu^*(U \cap V) + \mu^*(V \cap (\operatorname{supp} g)^c) - 2\varepsilon/3 \\
&\geq \mu^*(U \cap V) + \mu^*(U^c \cap V) - 2\varepsilon/3,
\end{aligned}$$

where the last step follows since $V \cap (\operatorname{supp} g)^c \subseteq (V \cap U^c) \cup (V \cap V^c) = V \cap U^c$. By (2.2),

$$\mu^*(A) > \mu^*(U \cap V) + \mu^*(U^c \cap V) - \varepsilon \geq \mu^*(U \cap A) + \mu^*(U^c \cap A) - \varepsilon.$$

Since $\varepsilon > 0$, we've shown that $U \in \mathscr{F}^*$ and therefore, since $\mathscr{F}^*$ is a σ-algebra, that $\mathscr{B}(K) \subseteq \mathscr{F}^*$. So we have a Borel measure on K, obtained by restricting μ^*, which we shall denote by μ.

3. The Riesz Representation Theorem

We are now ready to prove the main result. The idea behind the proof comes from proposition 7.2.11 in [1] by decomposing f, but we will modify it and use instead partitions of unity as described in section 1.

Theorem 3.1 (Riesz Representation). *Let $T : \mathscr{C}^0(K) \to \mathbb{R}$ be a positive linear functional, where K is a compact metric space. Then there is a unique Borel measure μ on K such that*

$$Tf = \int f \, d\mu \tag{3.1}$$

for all $f \in \mathscr{C}^0(K)$.

Proof. We first show that the measure μ described in section 2 is one Borel measure satisfying (3.1). Let $f \in \mathscr{C}^0(K)$. Since f can be rewritten as a difference of non-negative functions $f = f_+ - f_-$, we may assume that f itself is non-negative. The general case follows by applying this argument to each $f_\pm$, by linearity of T.

Since any closed subset of a compact space is compact, $\operatorname{supp} f$ is compact. By continuity of f, its image $f(\operatorname{supp} f) \subseteq \mathbb{R}$ is compact. Accordingly choose $a < b$ such that $f(\operatorname{supp} f) \subseteq [a, b]$. Let us fix some notation. Given $\varepsilon > 0$, choose a partition

$$a = y_0 < y_1 < \cdots < y_n = b$$

such that $y_i - y_{i-1} < \varepsilon$. Put $E_i = f^{-1}((y_{i-1}, y_i])$. Then E_i is Borel, hence μ-measurable. By definition of μ, there is an open subset $U_i \subseteq K$ for which $E_i \subseteq U_i$ and $\mu(U_i) < \mu(E_i) + \varepsilon$. The U_i can be chosen such that $f < y_i + \varepsilon$ on U_i, by continuity. The collection $\{U_i : 1 \leq i \leq n\}$ is an open cover of $\operatorname{supp} f$, so there is a partition of unity $\{u_i : 1 \leq i \leq n\}$ subordinate to this cover.

This is all the notation needed and we may now resume the proof. Since the U_i cover $\operatorname{supp} f$, we have $1_{\operatorname{supp} f} \leq \sum_{i=1}^n u_i$. By lemma 2.2,

$$\mu(\operatorname{supp} f) \leq T\left(\sum_{i=1}^n u_i\right) = \sum_{i=1}^n Tu_i \leq \sum_{i=1}^n \mu(U_i),$$

where the last inequality follows from the fact that $u_i \leq 1_{U_i}$. Altogether,

$$\begin{aligned}
Tf = T\left(\sum_{i=1}^n u_i f\right) &= \sum_{i=1}^n T(u_i f) \\
&< \sum_{i=1}^n (y_i + \varepsilon) Tu_i = \sum_{i=1}^n (y_i + \varepsilon + 1) Tu_i - \sum_{i=1}^n Tu_i \\
&\leq \sum_{i=1}^n (y_i + \varepsilon + 1)(\mu(E_i) + \varepsilon) - \mu(\operatorname{supp} f) \\
&\leq \sum_{i=1}^n (y_i - \varepsilon)\mu(E_i) + \left(2\varepsilon \sum_{i=1}^n \mu(E_i) + \sum_{i=1}^n \mu(E_i) + \varepsilon(y_i + \varepsilon + 1)\right) - \mu(\operatorname{supp} f) \\
&\leq \sum_{i=1}^n (y_i - \varepsilon)\mu(E_i) + (2\mu(\operatorname{supp} f) + y_i + \varepsilon + 1)\varepsilon,
\end{aligned}$$

where we have used the fact that the E_i are disjoint, and in particular $\sum_{i=1}^n \mu(E_i) = \mu(\bigcup_{i=1}^n E_i) = \mu(\operatorname{supp} f)$. Observe that $(y_i - \varepsilon)1_{E_i} \leq f$ on E_i, so by definition of the integral,

$$(y_i - \varepsilon)\mu(E_i) \leq \int_{E_i} f \, d\mu,$$

for all $1 \le i \le n$. Therefore,

$$Tf \le \int f \, d\mu + C(\varepsilon),$$

where $C(\varepsilon) = (2\mu(\operatorname{supp} f) + y_i + \varepsilon + 1)\varepsilon \to 0$ as $\varepsilon \to 0$. We've shown that $Tf \le \int f \, d\mu$, and in fact this holds for general (not necessarily nonnegative) $f \in \mathscr{C}^0(K)$ by the prior discussion. Thus, by replacing f with $-f$, one obtains

$$-Tf = T(-f) \le \int -f \, d\mu = -\int f \, d\mu,$$

which means that

$$Tf = \int f \, d\mu.$$

This proves that the measure μ given in section 2 is a Borel measure that satisfies the Riesz Representation theorem. It remains to show that μ is unique.

Indeed, suppose ν is another Borel measure on K with $Tf = \int f \, d\nu$ for all $f \in \mathscr{C}^0(K)$. We will first show that

$$\nu(U) = \sup \left\{ \int f \, d\nu : 0 \le f \le 1_U \right\},$$

which is Lemma 7.2.7 in [1]. It is clear that $\nu(U) \ge \sup\{\int f \, d\nu : 0 \le f \le 1_U\}$, so we show the other inequality. Since K is a 2nd-countable compact Hausdorff space, any open set U is F_σ. Thus U is a union of a sequence of compact subsets $\{K_n : n \in \mathbb{N}\}$ in K, so that $\nu(U) = \lim_{n\to\infty} \nu(K_n)$. Let $0 < \varepsilon < \nu(U)$; accordingly choose a compact subset F such that $\varepsilon < \nu(F) < \nu(U)$. By corollary 1.2, there exists $f \in \mathscr{C}^0(K)$ for which $1_F \le f$. In particular, $\varepsilon < \nu(F) < \int f \, d\nu$, so that

$$\varepsilon < \sup \left\{ \int f \, d\nu : 0 \le f \le 1_U \right\}.$$

Taking $\varepsilon \to \nu(F)$ gives the desired equality. A similar argument gives an analogous result for μ in the sense that

$$\mu(U) = \sup \left\{ \int f \, d\mu : 0 \le f \le 1_U \right\},$$

for any open $U \subseteq K$. This means that $\nu(U) = \mu(U)$ for any open subset, so that $\nu = \mu$. Therefore, μ is unique.

$\square$

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