

Lecture 6: More on Coordinate-Independent CNNs: Convolutions and Isometry

Geometry and Topology in Machine Learning Seminar

June 30th, 2025

Lecture 6:
More on
Coordinate-
Independent
CNNs:
Convolutions and
Isometry

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1×1 GM-convolution

GM-Convolution

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Recall: Why “Feature Spaces” on Manifolds?

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- Classical CNNs store a feature map $f : \mathbb{R}^d \rightarrow \mathbb{R}^c$.
- A manifold M has no global coordinate, so we use **sections of a vector bundle** over M .
- Goal: make features **independent of any chosen gauge / coordinates**.

$$\boxed{G\text{-equivariant feature field}} \iff \boxed{\text{section of } \mathcal{A} := GM \times_{\rho} \mathbb{R}^c}$$

Key Ingredients

- 1 FM – frame bundle, principal $GL_n(\mathbb{R})$.
- 2 $GM \subset FM$ – chosen **G-structure** for $G \leq GL_n(\mathbb{R})$.
- 3 $\rho : G \rightarrow GL(\mathbb{R}^c)$ – channel representation.

Definition

$$FM = \bigsqcup_{p \in M} F_p M, \quad F_p M = \{[v_1, \dots, v_d] \mid \{v_i\} \text{ is a basis of } T_p M\}$$

- Action: $[v_1, \dots, v_d] \cdot g = [gv_1, \dots, gv_d]$ with $g \in GL_n(\mathbb{R})$ is **free** and **transitive** on each fibre.
- Projection $\pi : FM \rightarrow M$ makes FM a principal $GL_n(\mathbb{R})$ -bundle.
- With the fundamental representation $\rho : GL_n(\mathbb{R}) \rightarrow GL_n(\mathbb{R})$ being the identity map, $FM \times_{\rho} \mathbb{R}^n \cong TM$.

Examples

- $M = S^2$: each fibre $F_p M \cong GL_2(\mathbb{R})$.
- $M = T^2$: same as above but the bundle is **trivial**, i.e., there exists a principal-bundle isomorphism $FT^2 \cong T^2 \times GL_2(\mathbb{R})$.
- M Möbius band: FM is **non-trivial**; no global frame exists.

Definition

A G -structure on M is a principal G -sub-bundle $GM \subset FM$, where $G \leq GL_n(\mathbb{R})$.

Given a representation $\rho: G \rightarrow GL(\mathbb{R}^c)$, the feature bundle $A = GM \times_{\rho} \mathbb{R}^c$ carries the fields used by gauge-/manifold-equivariant CNNs.

Examples

- S^2 : $G = O(2)$ — orthonormal frame bundle $O(S^2) \subset F(S^2)$.
- T^2 : $G = SO(2)$ — oriented orthonormal frames; bundle is trivial $T^2 \times SO(2)$.
- Möbius band M : locally there is $G = e$ but *no* global reduction.
- Parallelizable M (e.g., T^2, \mathbb{R}^n): $G = \{e\}$ — a single global frame (classical CNN limit).

Definition

Given the principal bundle $GM \xrightarrow{\pi} M$ and a representation $\rho : G \rightarrow GL(\mathbb{R}^c)$, the **associated feature bundle** is

$$\mathcal{A} = GM \times_{\rho} \mathbb{R}^c, \quad (g, v) \sim (gh, \rho(h)^{-1}v), \quad h \in G$$

A **feature field** is a smooth section $s : M \rightarrow \mathcal{A}, x \mapsto [g_x, f(x)]$.

If you switch a local frame g_{α} to $g_{\alpha}h$ by $(g_{\alpha}, v) \sim (g_{\alpha}h, \rho(h)^{-1}v)$, meaning the fibre element v is simultaneously rotated by $\rho(h)^{-1}$.

The ψ 's typically refer to **local trivializations**.

The **associated feature bundle** \mathcal{A} makes the same abstract point in \mathcal{A} represented consistently in every chart. **Sections** of \mathcal{A} can make local frames (e.g., $g_\alpha, g_\alpha h$) into a single globally defined feature field.

$$\boxed{G\text{-equivariant feature field}}^1 \iff \boxed{\text{section of } \mathcal{A} := GM \times_\rho \mathbb{R}^c}$$

¹called G -equivariant since we are considering the G -sub-bundle instead of the whole frame bundle

Recall there are still some questions we want to address:

- ① What are the objects we actually want the CNN to work with and produce? In other words, what are the **feature fields** on M ?
- ② How do we define convolutions in this set-up?
- ③ What kind of symmetry does M have? How do we design the model to respect the symmetries?

Last time, we answered the first question. Let us look at the **second question now**.

Neural Layers as Bundle Morphisms

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Recall that a convolution layer is precisely a G -equivariant maps between two feature spaces, i.e. **bundle morphism of sections** $\Gamma(\mathcal{A}_{\text{in}}) \rightarrow \Gamma(\mathcal{A}_{\text{out}})$. And, neural networks are the **stack of such layers**.

Given a sequence of G -associated feature vector bundles

$$\mathcal{A}_0 \xrightarrow{\pi_{\mathcal{A}_0}} M, \dots, \mathcal{A}_N \xrightarrow{\pi_{\mathcal{A}_N}} M \quad \text{with} \quad \mathcal{A}_i := GM \times_{\rho_i} \mathbb{R}^{c_i}.$$

- Feature space of layer i : $\Gamma(\mathcal{A}_i)$ (global sections).
- A network is a sequence of parameterized maps

$$\Gamma(\mathcal{A}_0) \xrightarrow{L_1} \Gamma(\mathcal{A}_1) \xrightarrow{L_2} \dots \xrightarrow{L_N} \Gamma(\mathcal{A}_N).$$

- Each L_i must be G -equivariant and coordinate independent.

- ① **1×1 GM-convolution²** This is a **pointwise convolution map** that linearly sends each feature vector $f_{in}(p) \in \mathcal{A}_{in,p} \cong \mathbb{R}^{c_{in}}$ to an output vector $f_{out}(p) \in \mathcal{A}_{out,p} \cong \mathbb{R}^{c_{out}}$. (Mathematically, these are certain vector bundle morphisms).

- ② **Kernel-field transforms and GM-convolutions:**

- A (smooth) **kernel field** is a "certain" map $\mathcal{K} : TM \rightarrow \text{Hom}(\mathcal{A}_{in}, \mathcal{A}_{out})$. \mathcal{K} defines a **kernel field transform**

$$\mathcal{T}_{\mathcal{K}} : \Gamma(\mathcal{A}_{in}) \rightarrow \Gamma(\mathcal{A}_{out})$$

$$[\mathcal{T}_{\mathcal{K}}(f_{in})](x) := \int_{T_x M} \mathcal{K}_x(v) \text{Exp}^* f_{in}(v) dv.$$

Exp^* is related to the exponential map and **to be defined**.

- GM-convolutions** can be viewed as specific **kernel field transforms** with what are so called **GM-convolutional kernel fields**.

²Gauge-Manifold-Convolution

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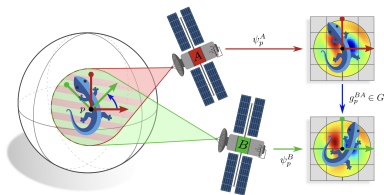
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Why point-wise (1×1) GM-convolution?

Same point, different gauges



Picture from [Weiler et al., 2021].

- Two observers A, B choose local frames ψ_p^A, ψ_p^B at **one** point $p \in M$.
- Coordinates of the **same** input vector differ by $f^B(p) = \rho_{\text{in}}(g_p^{BA})f^A(p)$, where $g_p^{BA} \in G$ (blue arrow in figure).
- A valid layer must output exactly the same **physical** feature, independent of the chosen frame.

Solution: apply a 1×1 GM-convolution that operates only at the point p and multiplies by a weight matrix $K_{1 \times 1} \in \mathbb{R}^{C_{out} \times C_{in}}$ that is G -equivariant, i.e.

$$K_{1 \times 1} \rho_{in}(g) = \rho_{out}(g) K_{1 \times 1} \text{ for all } g \in G. \quad (\dagger)$$

We denote collection of such $K_{1 \times 1}$ as $\text{Hom}_G(\rho_{in}, \rho_{out})$.

Intuition: A 1×1 GM-convolution is a **point-wise channel mixer**: at each $p \in M$ it applies the **same** weight matrix $K_{1 \times 1}$, and the equivariance condition (\dagger) guarantees that "mix then change frame" equals "change frame then mix", making the output gauge-independent.

Let's figure out why this is a solution to the problem in the proceeding slides.

Step 1: What a bundle morphism looks like in gauges

We said earlier 1×1 GM-convolutions are certain bundle maps.

At the illustrated point $p \in M$, the bundle morphism

$$C|_p : \mathcal{A}_{\text{in},p} \longrightarrow \mathcal{A}_{\text{out},p}$$

sends the input fibre (red frame) of \mathcal{A}_{in} to the corresponding output fibre (green frame) of \mathcal{A}_{out} , with both fibres anchored at the same base point p .

Such a vector bundle M -morphism \mathcal{C} is a smooth bundle map satisfying the following commutative diagram:

$$\begin{array}{ccc} \mathcal{A}^{\text{in}} & \xrightarrow{\mathcal{C}} & \mathcal{A}^{\text{out}} \\ & \searrow \pi_{\mathcal{A}^{\text{in}}} & \swarrow \pi_{\mathcal{A}^{\text{out}}} \\ & M & \end{array}$$

Step 2: Conjugation Rule after Trivializations

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- Fix a gauge³ ψ^A on a chart U^A , at each point $p \in U^A$, the bundle map is the following matrix

$$C_{|p}^A := \psi_{\mathcal{A}_{\text{out},p}}^A \circ C_{|p} \circ (\psi_{\mathcal{A}_{\text{in},p}}^A)^{-1} \in \mathbb{R}^{c_{\text{out}} \times c_{\text{in}}}$$

- Change to a second gauge ψ^B with $g_p^{BA} = \psi^B \psi^{A^{-1}}$ on the overlap. Conjugation rule (Eq. 219, also gauge transformations last lecture):

$$C_{|p}^B = \rho_{\text{out}}(g_p^{BA}) C_{|p}^A \rho_{\text{in}}(g_p^{BA})^{-1}.$$

- So **any smooth bundle map** C transforms like a **conjugation of its local matrix** when gauges change.

³Picking a particular gauge initializes the local trivialization

Step 3: Weight Sharing by the Intertwiner Condition

Goal: use **one and the same** kernel template at every point and in every gauge:

$$C_{K_{1 \times 1}}^X|_p = K_{1 \times 1} \quad \text{for all gauges } X \text{ and all } p \in U^X \quad (\text{Eq. 223}).$$

- Insert the conjugation rule from Step 2:

$$\rho_{\text{out}}(g) K_{1 \times 1} \rho_{\text{in}}(g)^{-1} = K_{1 \times 1} \quad \forall g \in G$$

- In other words, any such $K_{1 \times 1}$ has to satisfy the linear constraint above, which is equivalent to being in the **intertwiner space**

$$\text{Hom}_G(\rho_{\text{in}}, \rho_{\text{out}}) := \{K \in \mathbb{R}^{c_{\text{out}} \times c_{\text{in}}} \mid \rho_{\text{out}}(g)K = K\rho_{\text{in}}(g) \quad \forall g \in G\}$$

Formal Definition: 1×1 GM-Convolution

Definition

A 1×1 **GM-convolution** is a map

$$K_{1 \times 1} \otimes : \Gamma(\mathcal{A}_{in}) \rightarrow \Gamma(\mathcal{A}_{out}), f_{in} \mapsto K_{1 \times 1} \otimes f_{in} := C_{K_{1 \times 1}} \circ f_{in}.$$

where $K_{1 \times 1} \in \text{Hom}_G(\rho_{in}, \rho_{out})$ and $C_{K_{1 \times 1}}|_p := \psi_{\mathcal{A}_{out}, p}^{-1} \circ K_{1 \times 1} \circ \psi_{\mathcal{A}_{in}, p}$ for arbitrary gauges $\psi_{\mathcal{A}_{out}, p}$ and $\psi_{\mathcal{A}_{in}, p}$.

This guarantees gauge-independent (coordinate-free) channel mixing. Here, to give some heuristics of this definition:

- ρ_{in} and ρ_{out} can be thought of as **channel representations** $\rho_{in}, \rho_{out} : G \longrightarrow GL(\mathbb{R}^{c_*})$ that tell us how an element $g \in G$ acts on the input and output feature vectors.
- Each feature bundle is $\mathcal{A}_* = GM \times_{\rho_*} \mathbb{R}^{c_*}$, so keeping a layer G -equivariant means **its weights must respect these two representations**.

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- **Limitation of 1×1 GM-Conv.** Only limited to using a “point-like kernel”.
- **Kernel Fields:** Instead of the point-wise (1×1) weight matrix

$$K_{1 \times 1} \in \text{Hom}_G(\rho_{\text{in}}, \rho_{\text{out}}),$$

we use a (unconstrained) **spatial kernel field** defined as a map

$$\begin{array}{ccc} TM & \xrightarrow{\mathcal{K}} & \text{Hom}(\mathcal{A}_{\text{in}}, \mathcal{A}_{\text{out}}) \\ & \searrow \pi_{TM} & \swarrow \pi_{\text{Hom}} \\ & M & \end{array}$$

Remark: \mathcal{K} is not required to be a vector bundle map (ie. $\mathcal{K}_p : T_p M \rightarrow \text{Hom}(\mathbb{R}^{\text{in}}, \mathbb{R}^{\text{out}})$ does not have to be linear).

Step 1: Coordinate-free Spatial Kernel

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To detect local patterns around a point $p \in M$, we rewrite the coordinate-free kernel \mathcal{K}_p in the numerical coordinates provided by a chosen gauge A :

Definition (Coordinate-free Kernel)

A coordinate-free kernel \mathcal{K}_p at a point $p \in M$ can be expressed relative to gauges $\psi_{TM,p}^A$ (for the tangent bundle) and $\psi_{\text{Hom},p}^A$ (for the homomorphism bundle) of a G -atlas via

$$\mathcal{K}_p^A : \mathbb{R}^d \longrightarrow \mathbb{R}^{c_{\text{out}} \times c_{\text{in}}}, \quad \mathcal{K}_p^A = \psi_{\text{Hom},p}^A \circ \mathcal{K}_p \circ (\psi_{TM,p}^A)^{-1}$$

For another gauge B , there is a relation

$$\mathcal{K}_p^B = \rho_{\text{Hom}}(g_p^{BA}) \circ \mathcal{K}_p^A \circ (g_p^{BA})^{-1}.$$

Step 2: From a Template Kernel to a Kernel Field

In order for kernel fields to arise from a convolution, it should come from a **template (spatial) kernel**

$$K : \mathbb{R}^d \longrightarrow \mathbb{R}^{c_{\text{out}} \times c_{\text{in}}}$$

be the single weight tensor learnt by the network
($K \in C^\infty(\mathbb{R}^d, \mathbb{R}^{c_{\text{out}} \times c_{\text{in}}})$).

Definition

From here we define a **kernel field** \mathcal{K}_K locally by

$$\mathcal{K}_{K,p}^X := \frac{K}{\sqrt{|\eta_p^X|}},$$

where X is any gauge, $p \in U^X$, and $\sqrt{|\eta_p^X|}$ is the **reference frame volume** (think of this as normalization).

Step 3: Template sharing & volume normalization

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Question: How do \mathcal{K}_K differ across different gauges?

Note that for gauges A, B and $p \in U^A \cap U^B$, the **change of coordinates** gives a essentially “**Jacobian-like**” relationship with

$$\sqrt{|\eta_p^A|} = |\det(g_p^{BA})| \sqrt{|\eta_p^B|}.$$

The global version of this constraint and the **(aforementioned) transformation law** $\mathcal{K}_p^B = \rho_{\text{Hom}}(g_p^{BA}) \circ \mathcal{K}_p^A \circ (g_p^{BA})^{-1}$ implies that for every $g \in G$ and offset $\nu \in \mathbb{R}^d$, the shared template kernel $K : \mathbb{R}^d \rightarrow \mathbb{R}^{c_{\text{out}} \times c_{\text{in}}}$ must satisfy the following **G-steerability kernel constraint**:

$$\rho_{\text{out}}(g) K(\nu) \rho_{\text{in}}(g)^{-1} = |\det g| \cdot K(g \cdot \nu)$$

Observation: Volume normalization still only handles frame changes at a **fixed point** p . **However**, convolutions involve integrating the kernel over the **entire space**, which includes transformations between different points via actions g . We need the kernel to remain equivariant under **global symmetry transformations**. So instead of normalizing w.r.t. frame volume, we do normalization w.r.t. the group volume. To do so, we require the kernel K to satisfy a specific constraint under actions of G .

Step 4: Formal Definition of a G -steerable kernelDefinition (vector space of G -steerable kernels)

The vector space of **smooth G -steerable kernels** that map between field types ρ_{in} and ρ_{out} is defined by

$$\begin{aligned} \mathcal{H}_{\rho_{\text{in}}, \rho_{\text{out}}}^G &:= \left\{ K : \mathbb{R}^d \rightarrow \mathbb{R}^{c_{\text{out}} \times c_{\text{in}}} \text{ smooth} \mid \rho_{\text{out}}(g)K(\nu)\rho_{\text{in}}(g)^{-1} \right. \\ &\quad \left. = |\det g|K(g \cdot \nu) \forall g \in G, \nu \in \mathbb{R}^d \right\} \end{aligned}$$

Here $\det(g)$ refers to taking the determinant of the linear map given by the representation.

- The LHS rotates the **channels**, while the RHS rotates the **spatial offset** and rescales by $|\det g|^{-1}$ to keep the integral measure invariant.
- Check: At $\nu = 0$, we have $\rho_{\text{out}}(g)K(0) = K(0)\rho_{\text{in}}(g)$, i.e. $K(0)$ is exactly the intertwiner $K_{1 \times 1}$.

Step 5: Kernel Field Transform

Recall given a kernel field \mathcal{K} , we define its corresponding kernel field transform to be:

$$\mathcal{T}_{\mathcal{K}} : \Gamma(\mathcal{A}_{in}) \rightarrow \Gamma(\mathcal{A}_{out})$$

$$[\mathcal{T}_{\mathcal{K}}(f_{in})](x) := \int_{T_x M} \mathcal{K}_x(v) \text{Exp}^* f_{in}(v) dv,$$

where Exp^* is **to be defined**.

Definition

Let $f \in \Gamma(\mathcal{A})$, its **transporter pullback** is a map $\text{Exp}^* f : TM \rightarrow \mathcal{A}$ such that

$$v \mapsto P_{A, \pi_{TM}(v) \leftarrow \exp(v)} \circ f \circ \exp(v).$$

where $P_{A, \pi_{TM}(v) \leftarrow \exp(v)}$ is some **parallel transport** with respect to a " **G -compatible connection**".⁴

⁴More details in the reference [Weiler et al., 2021].

Step 6: GM-Convolution

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Definition

A **GM-convolution** $K \star$ is a **kernel field transformation** coming from a G -steerable template kernel K .

Is a GM-convolution (or more generally a kernel field well-defined?). It turns out that this is if we add a **compactly-supported condition**.

Theorem

Suppose \mathcal{K} is a kernel field such that $\mathcal{K}_p : T_p M \rightarrow \mathbb{R}^{c_{out} \times c_{in}}$ is compactly supported in a closed ball of radius R , then $\mathcal{T}_{\mathcal{K}}$ exists and is smooth.

The Convolution Operator: Summary

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Putting it together, let (M, G) be a **Riemannian manifold** with an associated **G -structure**. Let $\mathcal{A}_{in}, \mathcal{A}_{out}$ be two **G -associated vector bundles** over M .

Let $(W_{in}, \rho_{in}), (W_{out}, \rho_{out})$ be the two associated representations and K be a **G -steerable template convolution kernel**, the convolution operator is

$$L : \Gamma(\mathcal{A}_{in}) \rightarrow \Gamma(\mathcal{A}_{out}), f_{in} \mapsto f_{out} := L(f_{in}),$$

where L is a **GM-convolution**.

Theorem (Coordinate Independence)

The output f_{out} is a well-defined global section of \mathcal{A}_{out} .

Remark: Why the spatial kernel field can vary over TM

- **Point-wise case.** The matrix $K_{1 \times 1} \in \text{Hom}_G(\rho_{\text{in}}, \rho_{\text{out}})$ is **constant**: it has **no dependence** on position or direction, so it only mixes channels at the **single point** p .
- **In contrast**, the kernel field

$$\mathcal{K} : TM \rightarrow \text{Hom}(\mathbb{R}^{c_{\text{in}}}, \mathbb{R}^{c_{\text{out}}}), \quad \mathcal{K}(g \cdot u) = \rho_{\text{out}}(g)\mathcal{K}(u)\rho_{\text{in}}(g)^{-1},$$

depends on the offset $u \in T_p M$. For each u we

- 1 parallel-transport $f(x')$ from $x' = \exp_p(u)$ back to p ,
- 2 multiply by $\mathcal{K}(u)$,
- 3 integrate over $\|u\| \leq r$.

Intuition: The steerability condition ensures this direction-dependent kernel still commutes with every gauge change, making the filter both **spatially aware** and **G -equivariant**.

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So far, we have been considering GM structures, with $G \leq \text{GL}_n(\mathbb{R})$. This is done without much compatibility with a possible metric on M .

Thus, we would like to consider the following data.

$$\text{Isom}_{GM} := \{\phi \in \text{Isom}(M) \mid [\phi_*(e_i)]_{i=1}^d \in GM, \forall [e_1, \dots, e_d] \in GM\}.$$

In other words, this is exactly the **subgroup of isometries** respecting the GM structure.

Note: When G contains $O(d)$, $\text{Isom}_{GM} = \text{Isom}_M$.

Action of the G-Isometry Group

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There is a clear **action** of Isom_{GM} on GM by

$$\phi \cdot [e_1, \dots, e_d] \mapsto [\phi_*(e_1), \dots, \phi_*(e_d)]$$

with the pushforward induced on tangent spaces.

This leads to an induced action of Isom_{GM} on \mathcal{A}_{in} and \mathcal{A}_{out} given by

$$\phi_{*, \mathcal{A}_\bullet} : \mathcal{A}_\bullet \rightarrow \mathcal{A}_\bullet, [e, v] \mapsto [\phi_{*, GM}(e), v],$$

for $\bullet \in \{\text{in}, \text{out}\}$. This clearly gives an action on $\Gamma(\mathcal{A}_\bullet)$ as well.

We say a map

$$L : \Gamma(\mathcal{A}_{in}) \rightarrow \Gamma(\mathcal{A}_{out})$$

is **G-isometry equivariant** if for each $\phi \in \text{Isom}_{GM}(M)$,

$$L(\phi \cdot f_{in}) = \phi \cdot L(f_{in}).$$

Theorem

A GM-convolution given earlier $K \star : \Gamma(\mathcal{A}_{in}) \rightarrow \Gamma(\mathcal{A}_{out})$ with a G -steerable kernel K is **G-isometry equivariant**.

Corollary: Full Isometry Equivariant

For any G that contains $O(d)$, a GM-convolution operator is **fully isometry equivariant**.

Proof: Recall if G contains $O(d)$, then

$$\text{Isom}_{GM}(M) = \text{Isom}(M)$$

Corollary: $\text{Isom}_+(M)$ -Equivariant

When $G = \text{SO}(d)$, then the GM-convolutions are equivariant w.r.t to all **orientation preserving isometries**.

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Gauge Equivariant CNNs on the icosahedron

[Cohen et al., 2019] provide a handcrafted approach to processing spherical signals.

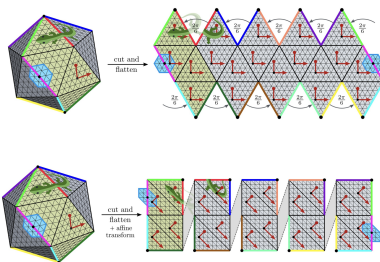
Utilizing the regular grid of the **icosahedral discretization** as the approximation of the sphere that **sampling from the icosahedron** (illustration next slide), the convolution operation can be equivariant to both local gauge transformations and global icosahedral symmetries.

Given a manifold M (the icosahedron) and a feature field $f : M \rightarrow \mathbb{R}^n$, the **gauge equivariant convolution** is defined as:

$$(K \star f)(p) = \int_{\mathbb{R}^d} K(v) \rho_{\text{in}}(g_{p \leftarrow q_v}) f(q_v) dv,$$

where K is a learnable kernel, $q_v = \exp_p w_p(v)$ is obtained via the exponential map, and ρ_{in} represents the transformation of features under gauge changes.

- each hexagonal grid is a locally flat approximation of S^2
- each hexagonal grid admits \mathbb{Z}_6 -structure as the G -structure since the Levi-Civita connection has holonomy group \mathbb{Z}_6
- flatten icosahedron by cutting it at the north and south pole (upper). So we also need to apply an affine transformation (lower) to align boundaries.



Pictures from [Cohen et al., 2019]



Cohen, T. S., Weiler, M., Kicanaoglu, B., and Welling, M. (2019).

Gauge equivariant convolutional networks and the icosahedral CNN.

CoRR, abs/1902.04615.

Proc. ICML 2019.



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