References:

- Paper presented:
  ReLU Neural Networks, Polyhedral Decompositions, and Persistent Homology

- "Follow-up" paper:
  Locally linear attributes of ReLU Neural Networks

- Berzins' edge subdivision algorithm:
  Polyhedral Complex Extraction from ReLU Networks using Edge Subdivision

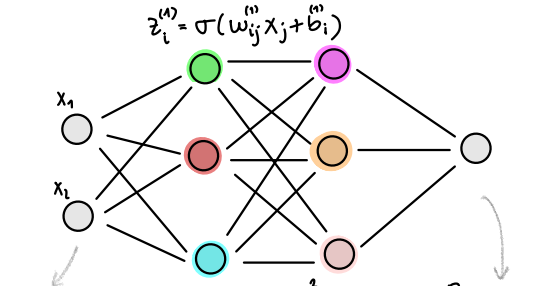
- CW-complex structureof the polytope decomposition:
  Algorithmic Determination of the Combinatorial Structure of the Linear region of ReLU Neural Networks

- How the topology of manifolds is reduced through the layers of a FFRNN:
  Topology of Deep Neural Networks

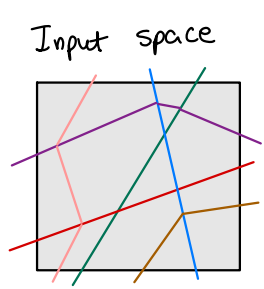
- About grokking and decision boundaries:
  Deep Networks Always Grok and Here is Why

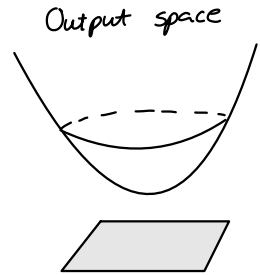
- About relation of FFRNN and tropical rational maps
  TRopical Geometry of Deep Neural Networks

Feed Forward ReLU Neural Network (FFRNN)

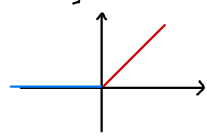


$z^{(1)} = \sigma(W^{(1)} x + b^{(1)})$ $\quad z^{(i)} = \sigma(W^{(i)} z^{(i-1)} + b^{(i)}) \dots$

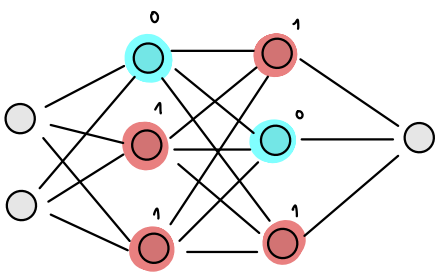




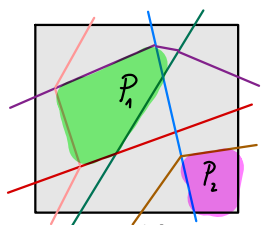
$y = \begin{bmatrix} 1 \\ y_i \\ 1 \end{bmatrix} \in \mathbb{R}^n \Rightarrow \sigma(y) = \begin{bmatrix} 1 \\ \sigma(y_i) \\ 1 \end{bmatrix}$



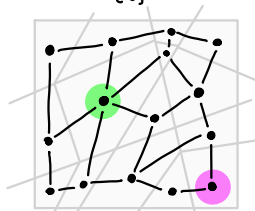
$\sigma(y_i) = \max\{0, y_i\}$



Binary vector: $b_{P_1} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$



$b_{P_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$



Observations:
- Convex polyhedra
- Within each P the NN is affine: $A_p x + b_p$ [reference]
- Unique binary vector
- Share a facet $\Leftrightarrow$ b_p's differ by one bit

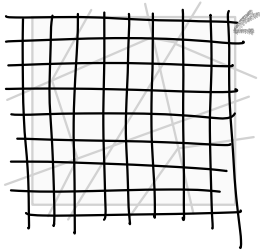
Dual graph

$d(b_{P_1}, b_{P_2}) =$ # of bit flips to make them equal $= 4$

- Construct the polyhedron equation $Ax \le c$ from binary vector
- Nice summary at the end of section 2
- # of polyhedra $\ll 2^{\#\text{ hidden layers}}$ ← a priori estimate [reference]
- Section 3: algorithms
- Section 4: persistent homology

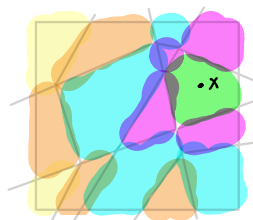
Algorithms

- Brute force grid subdivision



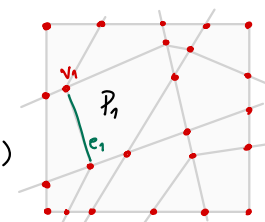
evaluate NN on each square
↳ exponential with dim (input)

- Find adjacent polytopes



- Start with random x
- Find adjacent polytopes by bit flipping

- Fastest (IMO) : Arthur Berzin's "edge subdivision"
  - $\sim 10^4$ vertices in seconds
  - Give you vertices & sign vectors
  - Does not construct complex (but tells you how)



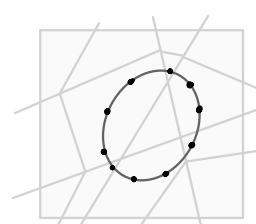
$$b_{P_1} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow S_{P_1} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

Extends to lower dim cells

$$S_{e_1} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \qquad S_{v_1} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

Persistent homology

- Sample some points from manifold inside input
- Look at the polytopes these points correspond to
- Build the dual graph and (Hamming) distance matrix
- Rips complex & persistent homology
- Detect homology of Sampling manifold



Extra

- Topology of DNN : ReLU > tanh & width < depth
- Deep NN always grow : decision boundaries $\longleftrightarrow$ polytope decomposition
- Tropical geometry of NN : FFRNN $\longleftrightarrow$ Tropical rational maps, dec. bdry. $\leftrightarrow$ tropical hypersurface