Lecture 5: Riemannian Manifolds and the Start of Coordinate-Independent CNNs

Geometry and Topology in Machine Learning Seminar

June 23rd, 2025

Recall: Manifolds

- A topological n-dimensional manifold M is a second countable Hausdorff space that is locally Euclidean, ie. every point $p \in M$ has a neighborhood that locally looks like $\mathbb{R}^n$.
- These neighborhood description comes in the form of charts $(U_\alpha, \varphi_\alpha : U_\alpha \to \mathbb{R}^n)$. M is said to be **smooth** if the transition functions:

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \text{Open Subset of } \mathbb{R}^n \to \text{Open Subset of } \mathbb{R}^n$$

  is smooth in the usual sense.
- A function $f : M \to N$ with coordinate charts $(U_\alpha, \varphi_\alpha), (V_\beta, \psi_\beta)$ is smooth if for all α, β, the following map is smooth in the usual sense

$$\psi_\beta \circ f \circ \varphi_\alpha^{-1} : \text{Open Subset of } \mathbb{R}^{\dim M} \to \text{Open Subset of } \mathbb{R}^{\dim N}$$

Recall: (Co)Tangent Spaces

- For each smooth manifold M and $p \in M$, there is a well-defined construction of a $\mathbb{R}$-vector space of tangent vectors at $p \in M$, called T_pM.
- The linear dual of T_pM is called the cotangent space T_p^*M. If M is embedded as some submanifold, there is also a well-defined notion of a normal vector space N_pM.
- These vector spaces can be "bundled" together into vector bundles - the tangent bundle TM, the cotangent bundle T^*M, and the normal bundle NM.

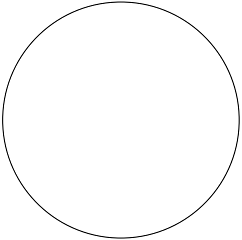
Today, we will first talk about an additional structure we can give to smooth manifolds - a Riemannian metric.

Outline

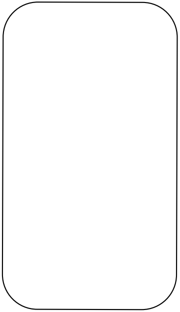
Motivation: Curvature

Question:

Are these two shapes the same?

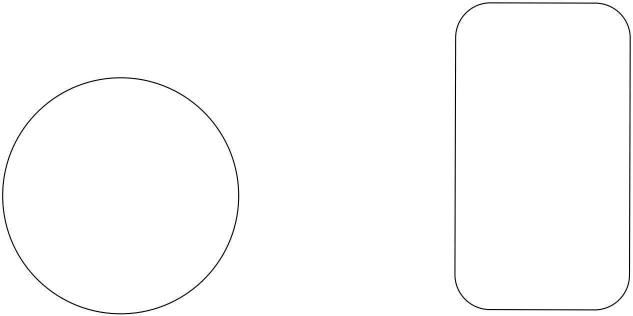


Standard Sphere



Also a Sphere, but Stretched out

Motivation: Curvature



Standard Sphere

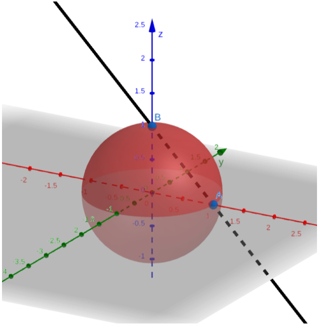
Also a Sphere, but Stretched out

- These two shapes are diffemorphic as manifolds!
- But if your tasks concern rigidity and curvature, morally they should be different!

Motivation: Measuring Distance on the Manifold

Given two points $p, q \in M$, we would like a way to measure how far away they are! This is not something smooth manifolds can usually give.

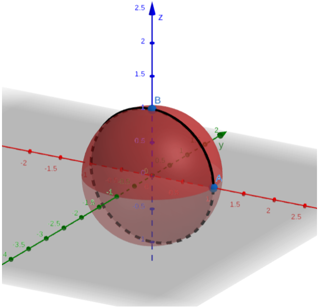
If M is embedded in $\mathbb{R}^n$, the straight line metric may not be the most suitable for M.[1]



[1] Here we suppress a discussion on how Riemannian metric can technically restrict to a Riemannian metric, not in the way outlined here.

Motivation: Measuring Distance on the Manifold

Instead, we would like a way to define a metric/distance based on the manifold M itself, without necessarily in an embedding.



Great Circle Metric

Tangent Vectors

Observation: We can detect both curvature and distance using tangent vectors in Calculus III.

1. How fast the tangent vector changes measures the curvature of the curve.
2. The arc length of a smooth curve $(x(t), y(t), z(t)) : [0, 1] \to \mathbb{R}^3$ is given by

$$\int_0^1 \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}dt.$$

3. Note that $x'(t)^2 + y'(t)^2 + z'(t)^2$ is really the dot product of $(x'(t), y'(t), z'(t))$ with itself. In other words, we are implicitly assuming that $T_p\mathbb{R}^3$ has a positive-definite bilinear form.

Riemannian Metric

We would like to add this structure to the setting of general smooth manifolds too.

Definition

Let M be a smooth manifold, a Riemannian metric g on M is a smooth choice of a positive-definite bilinear form $g_p : T_pM \times T_pM \to \mathbb{R}$ for each $p \in M$. (M, g) is called a Riemannian manifold.

Note that g_p is allowed to vary as p changes!

Theorem

Every smooth manifold has a[2] Riemannian metric.

[2] in fact, usually many.

Norm and Distance

The additional structure of g allows one to define:

1. **The norm of a tangent vector**: For $v \in T_pM$,

$$|v| = g_p(v, v).$$

2. **Length of a Curve**: Let $\gamma : [0,1] \to M$ be a smooth curve, then

$$\mathrm{Length}(\gamma) := \int_0^1 \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))} dt$$

where we note that $\gamma'(t) \in T_{\gamma(t)}M$ is a tangent vector.

3. The length function defines a metric on M, where for $p, q \in M$,

$$d(p,q) := \inf\{\mathrm{Length}(\gamma) \mid \gamma : [0,1] \to M, \gamma(0) = p, \gamma(1) = q\}.$$

In other words, the distance is given by the shortest converging "path" between p and q.

Riemannian Gradient

The metric g can be used to construct a canonical notion of gradient.

Prop:

Let (M, g) be a Riemannian manifold and $f : M \to \mathbb{R}$ be smooth, then there is a unique vector field $\operatorname{grad} f$ on M such that $g(\operatorname{grad} f, Y) = df(Y)$ for any vector field Y.

Proof: The proof quite literally follows from the **Riesz representation theorem** in linear algebra, as pointwise this is a positive symmetric bilinear form.

Affine Connection

An (affine) connection on a general **smooth** manifold is informally a mathematical tool that connects one tangent space to another.

More formally, it is a bilinear map $\nabla_\bullet(\bullet) : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ such that for vector fields X, Y and smooth function $f : M \to \mathbb{R}$,

1. $\nabla_{fX}(Y) = f\nabla_X(Y)$.
2. $\nabla_X(fY) = X(f)Y + f\nabla_X Y$, where $X(f)$ is the directional derivative of f in X

Warning: An arbitrary smooth manifold can have many many different possible affine connections.

Theorem (Fundamental Theorem of Riemannian Geometry)

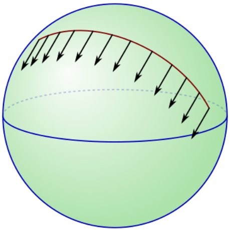
Every Riemannian manifold (M, g) admits a unique connection[3] ∇ that respects the metric structure.

[3] Called a Levi-Civita Connection

Parallel Transport

For a smooth manifold M, it is very hard to compare two tangent vectors $v \in T_pM$ and $w \in T_qM$ with $p \neq q$.

With a fixed choice of affine connection ∇, we can transport tangent vectors via smooth curves such that the vector is "parallel" with respect to ∇



Picture from Wikipedia

Parallel Transport

More formally, fix a connection ∇ on M, and let $\gamma : [0, 1] \to M$ be a smooth curve with $\gamma(0) = p, \gamma(1) = q$. Fix $v \in T_pM$, a vector field X along γ is a parallel transport of v if

1. $X_p = v$.
2. $\nabla_{\gamma'(t)} X = 0$ for $0 \leq t \leq 1$.

For all cases we will care about, a parallel transport always exists.

Isometry

Let $f : (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds, we say f is an isometry if:

1. f is a diffeomorphism.
2. For all $v, w \in T_pM$, $h(f_*v, f_*w) = g(v, w)$ (ie. the maps between tangent spaces are all isometries).

Two Riemannian manifolds are essentially the same if they are isometric!

Question:

For smooth manifold M, every $p \in M$ has a neighborhood diffeomorphic to $\mathbb{R}^n$. If (M, g) is a **Riemannian manifold**, is it the case that every $p \in M$ has a neighborhood **isometric to** $\mathbb{R}^n$?

Gauss's Lemma and the Exponential Map

The answer is no! Due to the presence of curvatures! There is a partially correct answer, to this question though.

Gauss's Lemma

Every point has a neighborhood that is radially isometric to $\mathbb{R}^n$.

In particular, the radial isometry is implemented by what is called the exponential map. For $p \in (M, g)$, one can construct a map

$$\exp_p : T_pM \to (M, g),$$

i think we should stick with the interval [0,1], tho the original definitions usually use $[-\epsilon, \epsilon]$ or something. we only need [0,1] to record where the geodesic reaches at time $t = 1$. a forward interval is sufficient and implicitly implies the "time" idea. also [0,1] seems to be a regular choice in literatures where $\exp_p(v) = \gamma_v(1)$, and $\gamma_v : [0, 1] \to M$ is the unique distance minimizing curve[4] with $\gamma_v(0) = p$ and $\gamma_v'(0) = v$.

[4] ie. a geodesic

Outline

CNNs on Manifolds?

There are many data that are naturally valued on manifolds. Let us try to build CNNs on manifolds now!

But there are some questions we should answer ....

1. What are the objects we actually want to the CNN to work with and produce? In other words, what are the feature fields on M?
2. How do we define convolutions in this set-up?
3. What kind of symmetry does M have? How do we design the model to respect the symmetries?
4. ...

Feature Fields on Manifolds

Question:

What are the feature fields on M?

Recall for homogeneous spaces in Lecture 2/3, a feature, for us, is a section of an associated vector bundle

$$s : G/H \to E$$

to keep track of some geometric quantities.

For manifolds, we still want some kind of function

$$s : M \to E$$

that associates each point on M some geometric quantities.

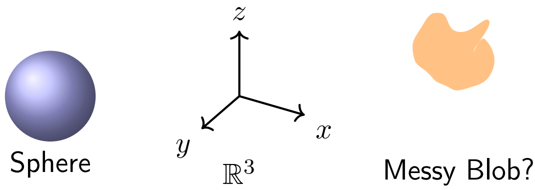
Coordinatization

For manifolds, we still want some kind of function $s : M \to E$.

Question:

But how is a computer implementation going to, in practice, represent this map numerically?

In practice, these types of numerical implementations would need to choose some coordinatization of M, but many manifolds do not have a canonical choice of coordinates.



Coordinate Independence

It would be quite undesirable if we make a CNN that performs well for one choice of coordinates and produces drastically different results for another choice.

- Thus, our design decision should aim to create a CNN architecture that is independent of the choice of coordinates.
- To achieve coordinate independence, we need to know how features are transformed between different choices of coordinates!
- As we will see in the upcoming slides, the study of how to regulate these choices is essentially the subject of gauge theory.

Two Ways to Design Coordinate Independence

In this lecture, we will focus on designing coordinate independent feature fields. There are two equivalent ways [Weiler et al., 2021] to achieve this coordinate independence that we will discuss.

1. Construct the global feature field dependent on an arbitrary choice of coordinates and show it is independent of the choice. usually start with picking a local trivialization, then build feature field and the resulting section does NOT depend on the choice of trivialization, useful for implementations
2. Define the global feature field using a coordinate free object to begin with.treats the feature field directly as a section of an associated bundle, so a trivialization exists in principle but is never fixed or used in the construction, useful for mathematical analysis

There are advantages to both perspectives. It may be easier to deduce theoretical properties from the second perspective. But in practice, to concretely write down a global feature field by hand, one would most likely go through the first perspective.

Perspective 2

For our purposes, we will introduce the framework through the second perspective.

Let $p \in M$ and T_pM be its tangent space. We know from the last lecture that

$$T_pM \cong \mathbb{R}^n, n = \dim M.$$

However, there is no canonical way to write down this isomorphism. It requires a choice of basis $v_1, ..., v_n$ of T_pM.

Observation:

Rather than keeping track of any one specific choice of basis, why don't we look at all of them at the same time? (ie. a "moduli" of frames)

The Frame Bundle

Thus, we define the **frame bundle** FM as

$$FM = \bigsqcup_{p \in M} F_p M,$$

$$F_p M = \{[v_1, ..., v_d] \mid \{v_1, ..., v_d\} \text{ forms a basis of } T_p M\}$$

Now observe that $\mathrm{GL}_n(\mathbb{R})$ acts both freely and transitively on $F_p M$ by matrix multiplication!

Theorem

$\pi : FM \to M$ is a principal $\mathrm{GL}_n(\mathbb{R})$-bundle.

Remark: The identity map $\rho : \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}(\mathbb{R}^n)$, as a representation, also shows that the associated vector bundle of FM w.r.t ρ is exactly the tangent bundle TM.

G-Structures

Sometimes, there are additional structural information on M that we want our CNN to respect. They are mathematically defined as so called G-structures.

Definition

Let $G \leq \mathrm{GL}_n(\mathbb{R})$ be a subgroup, a G-structure is a principal G-sub-bundle $GM \to M$ of $\pi : FM \to M$.

Here are some examples:

1. If M has a Riemannian metric, we can define $OM = \bigsqcup_{p \in M} O_pM$ with

   $$O_pM = \{[v_1, ..., v_n] \mid v_1, ..., v_n \text{ orthonormal basis of } T_pM\},$$

   $OM \to M$ is a principal $O(n)$-bundle.

G-Structure Examples

2. If M is orientable, we can define $\mathrm{GL}^+ M = \bigsqcup_{p \in M} \mathrm{GL}_p^+ M$ where

$$\mathrm{GL}_p^+ M = \{[v_1, \ldots, v_n] \mid v_1, \ldots, v_n \text{ positively oriented basis of } T_pM\}$$

   $\mathrm{GL}^+ M \to M$ is a principal $\mathrm{GL}_n^+(\mathbb{R})$-bundle[5].

3. Similarly, if M is both orientable and has a Riemannian metric structure, we can define a $\mathrm{SO}\, M \to M$, which is a principal $\mathrm{SO}(n)$-bundle.

4. Let e be the identity elment, $\{e\}$-structures correspond exactly to sections of FM!

[5] The subgroup with positive determinant

Associated Feature Vector Bundle and Features

Let $GM \to M$ be a G-structure and $\rho : G \to \mathrm{GL}(\mathbb{R}^c)$ be a G-representation, our model of the associated feature vector bundle is

$$\mathcal{A} := GM \times \mathbb{R}^c / \sim .$$

Definition

A coordinate free feature field is a (smooth) section of the vector bundle $\mathcal{A} \to M$.

Often in CNNs, we would like to consider a stack of features rather than just one. We achieve this by taking multiple independent sections and direct sum them (equivalently this is taking the section of the vector bundle direct sum $\bigoplus_i \mathcal{A}_i$).

Perspective 1: What is Going on Locally?

Now we will re-examine the definition of the G-structures and feature vector fields locally. Let $p \in M$ and T_pM be its tangent space.

$$T_pM \cong \mathbb{R}^n, n = \dim M.$$

However, there is no canonical way to write down this isomorphism. It requires a choice of coordinates.

Instead of trying to find a canonical way to write down this isomorphism, let us instead try to quantify this arbitrary choice.

Definition

Let $p \in U^A \subset M$ be an open neighborhood where the $TM|_U$ is trivial. A gauge is a smooth choice of linear isomorphisms

$$\psi_p^A : T_pM \xrightarrow{\cong} \mathbb{R}^n, \quad \forall p \in U_A.$$

Reference Frames of Tangent Spaces

Definition

Let $p \in U^A \subset M$ be an open neighborhood where the $TM|_U$ is trivial. A gauge is a smooth choice of linear isomorphisms

$$\psi_p^A : T_pM \xrightarrow{\cong} \mathbb{R}^n, \quad \forall p \in U_A.$$

Observation: A smooth gauge is really a map

$$\psi^A : U^A \to \mathrm{GL}_n(\mathbb{R}).$$

Let $e_1, e_2, ..., e_n$ be the standard normal basis of $\mathbb{R}^n$, for all $p \in U_A$, a gauge ψ_p^A gives a reference frame of T_pM as

$\{e_1^A := (\psi_p^A)^{-1}(e_1), ..., e_n^A := (\psi_p^A)^{-1}(e_n)\}$ forms a basis of T_pM

Gauge Transformation

Similar to coordinate charts, we can consider an atlas of smooth gauges $\{(\psi^A, U^A)\}_{A \in I}$ where U^A forms an open cover of M.

Definition

Let $\{(\psi^A, U^A)\}_{A \in I}$ be an atlas of smooth gauges. For $A, B \in I$, a gauge transformation is the map

$$g^{BA} : U^A \cap U^B \to \mathrm{GL}_n(\mathbb{R}), g_p^{BA} := \psi_p^B \circ (\psi_p^A)^{-1}.$$

Note here the inverse means the matrix inverse.

Clearly we have that

$$\psi_p^B = g_p^{BA} \circ \psi_p^A$$

Thus, g_p^{BA} can be thought of as the transition functions.

G-Structures and Feature Space

Definition

Let $\{(\psi^A, U^A)\}_{A\in I}$ be an atlas of smooth gauges. If for all A, B,

$$g^{BA} : U^A \cap U^B \to G \leq \mathrm{GL}_n(\mathbb{R}),$$

this is called a G-atlas, and we have a G-structure!

Let $\{(\psi^A, U^A)\}_{A\in I}$ be a G-structure and $\rho : G \to \mathrm{GL}(\mathbb{R}^c)$ be a G-representation, a feature field for us is a collection of maps

$$f^A : U^A \to \mathbb{R}^c \text{ such that } f^B(p) = \rho(g_p^{BA}) \cdot f^A(p).$$

Outline

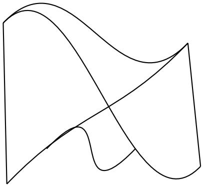
No $\{e\}M$-structures on Mobius Band

Let us try to build the feature fields on a Mobius Band M!

Prop:

You cannot build a $\{e\}M$ structure on the **Mobius Band**!

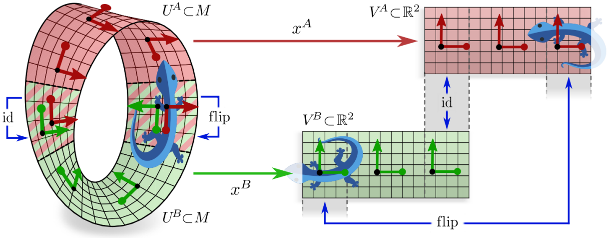
Proof: Suppose you can, then this means you have a globally defined frame of tangent vectors. Now travel one circle along the Mobius strip, the frame has to end up with the opposite orientation.



Mobius Band

What About A $\mathbb{Z}/2\mathbb{Z}$M Structure?

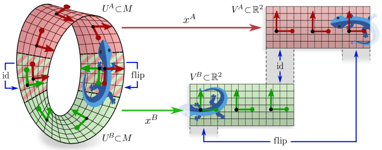
Yes! Consider the following two charts and gauges:



Very Nice Picture from [Weiler et al., 2021].

Here we see one gauge transformation is the identity and the other is the reflection, which gives the group $\mathbb{Z}/2\mathbb{Z} = \{e, r\}$.

Interpreting Feature Fields for Mobius Bands

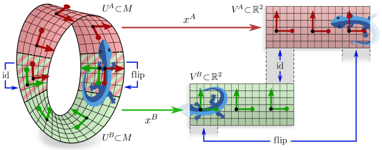


Very Nice Picture from [Weiler et al., 2021].

Our feature is given by $\frac{\mathbb{Z}}{2\mathbb{Z}}$-representations $\rho : \mathbb{Z}/2\mathbb{Z} \to \mathrm{GL}(\mathbb{R}^c)$.

1. If $c = 1$, $\rho(e) = 1$ and $\rho(r) = 1$. This is the trivial representation and the maps $f^A, f^B : U^A, U^B \to \mathbb{R}$ agree on all intersections.
2. If $c = 1$, $\rho(e) = 1$ and $\rho(r) = -1$. The maps f^A, f^B differ by a sign on the intersection labeled flip.

Interpreting Feature Fields for Mobius Bands



Very Nice Picture from [Weiler et al., 2021].

Algebra Fact: The two representations we described are the only two irreducible representations of $\mathbb{Z}/2\mathbb{Z}$.

This means for a general representation $\rho : \mathbb{Z}/2\mathbb{Z} \to \mathbb{R}^c$, we can always "decompose" the features in terms of the two representations before.

Next Time

So far, we have defined the feature spaces for our coordinate independent CNNs, but we still need to discuss how these feature spaces connect with each other with convolutions. More next lecture!

References I

Weiler, M., Forré, P., Verlinde, E., and Welling, M. (2021). Coordinate independent convolutional networks – isometry and gauge equivariant convolutions on riemannian manifolds.