Lecture 4: Differentiable Manifolds

Geometry and Topology in Machine Learning Seminar

June 20th, 2025

Motivation

$\mathbb{R}^n$ is a very nice space:

- We love calculus: derivatives, integration, optimization, etc.
- Defined by a coordinate system: convenient, intuitive
- Intuitive definitions of distances and angles.
- Linear algebra also works well here.

Problem: A variety of spaces we need to deal with can't inherit the coordinate system from $\mathbb{R}^n$.

- $S^2 = \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| = 1\}$
- Rotation matrices $\mathrm{SO}(3) = \{R \in \mathbb{R}^{3\times 3} : R^T R = I, \det R = 1\}$.
- Projective space $\mathbb{P}^n = \mathbb{R}^{n+1} \setminus \{0\} / \sim$.
- ... and many more.

Key observation: these are each "locally like $\mathbb{R}^n$."

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Motivation: What goes wrong?

Consider $S^2 = \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| = 1\}$.

How about spherical coordinates?

Recall

$$x = \sin\phi\cos\theta, \quad y = \sin\phi\sin\theta, \quad z = \cos\phi$$

Problem: Coordinates break down at the poles!

- At north pole $(0, 0, 1)$: $\phi = 0$, but θ is undefined
- At south pole $(0, 0, -1)$: $\phi = \pi$, but θ is undefined
- Can't compute $\frac{\partial f}{\partial \theta}$ at the poles

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Outline

Topological Manifolds

Definition (Topological Manifold)

A **topological n-manifold** is a topological space M such that

- M is Hausdorff and second countable.
- (**Locally Euclidean**): Every point $p \in M$ has a neighborhood U homeomorphic to an open set $U \subseteq \mathbb{R}^n$.

Definition (Chart, Atlas)

A **chart** (or coordinate patch) is a pair (U, φ) where $U \subset M$ is open and $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$ is a homeomorphism

An **atlas** is a collection of charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ (A some index set) such that $\bigcup_\alpha U_\alpha = M$

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Topological Manifolds: Examples

Examples:

- $S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} : \|\mathbf{x}\| = 1\}$
- The torus $T^2 = S^1 \times S^1$ (product of two circles)
- Real projective space $\mathbb{RP}^n = (S^n)/\{\pm 1\}$, identifying antipodal points

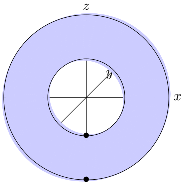

Figure 1: A picture of the torus.

Non-examples:

- The letter X and “figure-8” space will fail to be “locally Euclidean” at the crossing point

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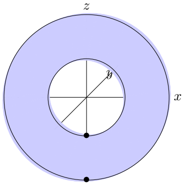

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Smooth Manifolds

Definition (Smooth Manifold)

A **smooth n-manifold** is a topological n-manifold M equipped with a **smooth atlas**: a collection of charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ such that:

1. $\bigcup_\alpha U_\alpha = M$
2. For overlapping charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β), the **transition map**

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^n \to \varphi_\beta(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^n$$

   is smooth (i.e., C^∞)

Main point: We can do calculus here!

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Smooth Submanifolds

Definition (Smooth Submanifold)

Let M be a smooth manifold of dimension n. A subset $N \subset M$ is a **smooth submanifold of dimension** k if for every point $p \in N$, there exists a chart (U, φ) around p such that:

$$\varphi(U \cap N) = \varphi(U) \cap (\mathbb{R}^k \times \{0\}^{n-k})$$

where $\mathbb{R}^k \times \{0\}^{n-k} = \{(x_1, \ldots, x_k, 0, \ldots, 0) \in \mathbb{R}^n\}$.

Intuition: Locally, the submanifold looks like a "slice" of $\mathbb{R}^n$.

Note: Every smooth submanifold $N \subset M$ is itself a smooth manifold.

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Submanifold Examples

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- $S^n \subset \mathbb{R}^{n+1}$ (spheres in Euclidean space)
- Any smooth curve in $\mathbb{R}^2$ or surface in $\mathbb{R}^3$
- $SO(3) \subset \mathbb{R}^{3\times 3}$ (rotation matrices in matrix space)
- Open subsets: if M is a smooth manifold and $U \subset M$ is open, then U is a submanifold

Non-examples:

- A square in $\mathbb{R}^2$ (corners are not smooth)
- The union of two intersecting planes in $\mathbb{R}^3$
- Any subset with "kinks," "corners," or "self-intersections"

The test: Can you find smooth coordinate charts that make the subset look "flat"?

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Tangent Spaces

Recall: Our goal is to "do calculus" on manifolds.

Problem: An (intrinsic) manifold M doesn't live in $\mathbb{R}^n$ globally, so we can't just take derivatives in the usual sense.

Solution: At each point $p \in M$, define the **tangent space** T_pM as the space of "infinitesimal directions" we can move from p.

Definition (Tangent Space (Informal))

The **tangent space** T_pM at point p is the vector space of all "tangent vectors" at p.

If M has dimension n, then $T_pM \cong \mathbb{R}^n$.

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Two Perspectives on Tangent Vectors

Perspective 1: Tangent vectors to curves
A tangent vector $v \in T_pM$ is the "velocity vector" $\gamma'(0)$ of a smooth curve $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = p$.

Perspective 2: Directional derivatives
A tangent vector $v \in T_pM$ is a linear map $v : C^\infty(M) \to \mathbb{R}$ that acts like "directional derivative in direction v":

$$v(f) = \lim_{t \to 0} \frac{f(\gamma(t)) - f(p)}{t}$$

where γ is any curve with $\gamma(0) = p$ and $\gamma'(0) = v$.

Key insight: Both perspectives give the same vector space structure!

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Example: Tangent Planes in $\mathbb{R}^3$

Recall from Calculus III: For a surface S given by $F(x, y, z) = c$, the tangent plane at point $p = (x_0, y_0, z_0)$ has equation:

$$F_x(p)(x - x_0) + F_y(p)(y - y_0) + F_z(p)(z - z_0) = 0$$

In our language: The tangent space T_pS is the 2-dimensional subspace of $\mathbb{R}^3$ given by:

$$T_pS = \{v \in \mathbb{R}^3 : \nabla F(p) \cdot v = 0\}$$

Example: For the unit sphere $x^2 + y^2 + z^2 = 1$:

- $F(x, y, z) = x^2 + y^2 + z^2 - 1$
- $\nabla F = (2x, 2y, 2z)$
- At $p = (x_0, y_0, z_0)$: $T_pS^2 = \{v \in \mathbb{R}^3 : (x_0, y_0, z_0) \cdot v = 0\}$

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Tangent Spaces: Smooth vs. Non-Smooth

Circle S^1 in $\mathbb{R}^2$:

- At every point $p \in S^1$
- T_pS^1 is 1-dimensional
- Always a line tangent to the circle
- Smoothly varying as p moves

Square in $\mathbb{R}^2$:

- At smooth points: T_p(square) is 1-dimensional
- At corners: tangent space is not well-defined!
- Multiple "tangent directions" possible
- This is why squares aren't smooth manifolds

General principle: Smooth submanifolds have well-defined tangent spaces of (locally) constant dimension everywhere. Non-smooth spaces have "singularities" where the tangent space dimension could jump or become ill-defined.

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Cotangent Spaces

Definition (Cotangent Space)

The **cotangent space** T_p^*M at point p is the dual vector space to the tangent space:

$$T_p^*M = (T_pM)^* = \{\text{linear maps } \omega : T_pM \to \mathbb{R}\}$$

Elements: Elements of T_p^*M are called **cotangent vectors** or **1-forms**.

Basis of cotangent space: If (U, φ) is a chart around p with coordinates $(x^1, \ldots, x^n)$, then the cotangent space has basis:

$$\{dx^1|_p, dx^2|_p, \ldots, dx^n|_p\}$$

where $dx^i|_p : T_pM \to \mathbb{R}$ is defined by $dx^i|_p \left(\left. \frac{\partial}{\partial x^j} \right|_p \right) = \delta^i_j$.

Intuition: dx^i measures the "rate of change in the x^i direction."

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The Tangent Bundle

Definition (Tangent Bundle)

The **tangent bundle** of a manifold M is:

$$TM = \bigcup_{p \in M} T_pM = \{(p, v) : p \in M, v \in T_pM\}$$

Key fact: TM is itself a smooth manifold of dimension $2n$ (if $\dim M = n$).

Natural projection: $\pi : TM \to M$ given by $\pi(p, v) = p$.

Definition (Vector Bundle)

$\pi : TM \to M$ is an example of a (rank n) **vector bundle** over M: a manifold that "looks locally like" $U \times \mathbb{R}^n$, where each fiber $\pi^{-1}(p) = T_pM$ is a vector space.

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Vector Fields

Definition (Vector Field)

A **vector field** on M is a smooth section of the tangent bundle TM:

$$X : M \to TM \quad \text{such that } \pi \circ X = \mathrm{id}_M$$

In other words, X assigns to each point $p \in M$ a tangent vector $X(p) \in T_pM$.

Local coordinate expression: In local coordinates $(x^1, \ldots, x^n)$:

$$X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}$$

where $X^i : M \to \mathbb{R}$ are smooth functions.

Examples:

- Gradient vector field: $X = \nabla f$ for a function f
- Flow of a differential equation: $\frac{d}{dt}\gamma(t) = X(\gamma(t))$

Vector Fields

Definition (Vector Field)

A **vector field** on M is a smooth section of the tangent bundle TM:

$$X : M \to TM \quad \text{such that } \pi \circ X = \mathrm{id}_M$$

In other words, X assigns to each point $p \in M$ a tangent vector $X(p) \in T_pM$.

Local coordinate expression: In local coordinates $(x^1, \ldots, x^n)$:

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Cotangent Bundle and 1-Forms

The **cotangent bundle** is:

$$T^*M = \bigcup_{p \in M} T_p^*M$$

Definition (1-Form)

A (differential) **1-form** on M is a smooth section of T^*M:

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Action on vector fields: $\omega(X)|_p = \omega_p(X_p) \overset{\text{"locally"}}{=} \sum_{i=1}^{n} \omega_i X^i$ (view $\omega_p : T_pM \to \mathbb{R}$ and $X_p \in T_pM$).

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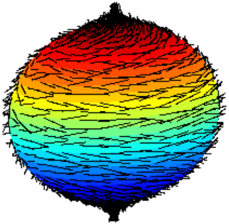
Why Vector Bundles Matter

The tangent bundle is not always trivial!

Example: For the 2-sphere S^2, the tangent bundle TS^2 cannot be written as $S^2 \times \mathbb{R}^2$.

Hairy Ball Theorem: There is no non-vanishing continuous tangent vector field on S^2.



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Intuition: You can't "comb the hair" on a sphere without creating a cowlick!

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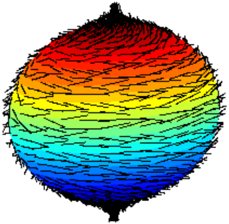

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Contrast: For the torus T^2, the tangent bundle $TT^2 \cong T^2 \times \mathbb{R}^2$ is trivial.

Significance:

- Topological obstructions to global coordinate systems
- Important for understanding the global geometry of manifolds
- Connects to characteristic classes, Euler characteristic, etc.

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Normal Bundles

For submanifolds: If $N \subset M$ is a submanifold, we get additional structure.

Definition (Normal Bundle)

The **normal bundle** to N in M is:

$$\nu N = \bigcup_{p \in N} (T_p M / T_p N)$$

Each fiber $\nu_p N$ consists of directions "perpendicular" to N at p.

Tubular Neighborhood Theorem: Every submanifold $N \subset M$ has a neighborhood that looks like a neighborhood in normal bundle νN.

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Outline

Smooth Maps

After having defined manifolds themselves, we want want an appropriate notion of a smooth map between them.

Definition

A function $f : M \to N$ between manifolds with charts $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ is smooth if for all α, β, the map

$$\psi_\beta \circ f \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \subset \mathbb{R}^n \to \psi_\beta(V_\beta) \subset \mathbb{R}^m$$

is smooth in the usual sense.

In other words, smoothness is a local property and can be verified in the usual sense locally.

1. The height function $h : S^n \to \mathbb{R}$ is smooth.
2. The anti-podal map $f : S^n \to S^n$ is smooth.

The Jacobian Map

Let $f : M \to N$ be a smooth function, there is a generalized notion of the Jacobian matrix that can be defined as follows.

Definition

Let $q = f(p) \in N$, take a local coordinate φ_α of p and a local coordinate ψ_β of q, the Jacobian of f at p is

$$Df(p) = \text{Jacobian matrix of the map } \psi_\beta \circ f \circ \varphi_\alpha^{-1} \text{ at } \varphi_\alpha(p).$$

Note that this definition is quite coordinate dependent right now. A coordinate independent formulation is to view this as a map between tangent spaces.

Jacobian Between Tangent Spaces

Let $f : M \to N$ be a smooth function, and $q = f(p) \in N$.

Definition

There is an induced linear map $f_* : T_pM \to T_qN$ given as follows,

1. View $v \in T_pM$ as the directional derivative of some curve $\gamma(t) : (-\epsilon, \epsilon) \to M$ at $t = 0$.
2. $f_*(v)$ is the directional derivative of the curve $f \circ \gamma(t) : (-\epsilon, \epsilon) \to N$ at $t = 0$.

In local coordinates, this is exactly the Jacobian!

We say f is an immersion (resp. submersion) at $p \in M$ if $f_* : T_pM \to T_qN$ is injective (resp. surjective).

Regular Value Theorem

Definition

Let $f : M \to N$ be a smooth map, a point $q \in N$ is said to be a **regular value** if for all $p \in f^{-1}(q)$, $Df|_p$ is surjective (ie. a submersion between $p \to q$).

In practice, detecting regular values is one common way to show some geometric object is a manifold.

Theorem (The Regular Value Theorem)

Let $q \in N$ be a regular value of $f : M \to N$ and N is connected. If $f^{-1}(q)$ is not empty, then $f^{-1}(q)$ is a submanifold of codimension equal to $\dim N$.

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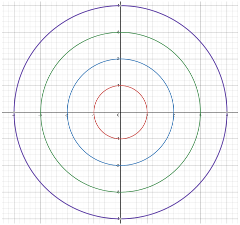
Examples of Regular Values

Consider the distance function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^2 + y^2$. In this case we have that

$$Df(x, y) = (2x, 2y) : T_{(x,y)}\mathbb{R}^2 \cong \mathbb{R}^2 \to T_{(x,y)}\mathbb{R}^2 \cong \mathbb{R}^2.$$

We see that r is a regular value for all $r > 0$, which makes sense since $f^{-1}(r)$'s are concentric circles:



At $f^{-1}(0)$, the preimage is a point. This is still a manifold, but notice that the dimension jumped.

Examples with Matrix Groups

1. Let $M_{n,n}(\mathbb{R})$ be the vector space of $n \times n$ real matrices, and $M_{n,n}^{\mathrm{sym}}(\mathbb{R})$ be the vector space of symmetric $n \times n$ real matrices.
2. Consider the function $f : M_{n,n}(\mathbb{R}) \to M_{n,n}^{\mathrm{sym}}(\mathbb{R})$ that sends a matrix M to $M^T M$. The preimage $f^{-1}(I)$ is exactly $O(n)$, and one can check I is a regular value.
3. This shows that $O(n)$ is an $n(n-1)/2$-dimensional manifold.
4. $O(n)$ has two connected components, with $SO(n)$ being one of them. This shows $SO(n)$ is also a $n(n-1)/2$-dimensional manifold.

Lie Groups

These matrix groups are special examples of manifolds known as Lie groups!

Definition

A Lie group is a group G that is also a smooth manifold such that the following two maps are smooth:

$$\text{Group Multiplication: } \bullet : G \times G \to G, (g,h) \mapsto gh,$$

$$\text{Inversion: } (-)^{-1} : G \to G, g \mapsto g^{-1}.$$

Other Examples of Lie groups:

- $\mathbb{R}^n$, S^1, the torus, ...

The Lie Algebra of a Lie Group

We can think of vector fields X (viewed as directional derivatives) as a map $C^\infty(M) \to C^\infty(M)$. For a Lie group G, the Lie algebra of G is

$$\mathfrak{g} := \{\text{vector fields } X \mid \pi_g X = X\pi_g, \forall g \in G\},$$

where $\pi_g : C^\infty(G) \to C^\infty(G)$ is given by $\pi_g(f)(h) := f(g^{-1}h)$.

$\mathfrak{g}$ is a vector space equipped with the Lie bracket operation

$$[X, Y] : C^\infty(G) \to C^\infty(G) := XY - YX.$$

Fact: $\mathfrak{g}$ can be identified with T_eG where e is the identity.

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Parallelizability

Fact: The tangent bundle of a Lie group is trivial.

Corollary:

S^2 is not a Lie group because its tangent bundle is not trivial.

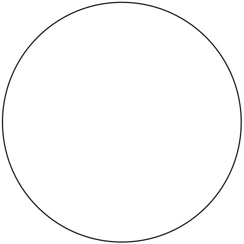
More General Fact: Only S^0, S^1, S^3 in the sphere family S^n's are Lie groups.

Outline

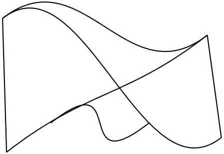
Manifolds with Boundaries

Recall previous we defined manifolds with the condition that every point should locally look like $\mathbb{R}^n$. A manifold with boundary relaxes the condition by allowing a point to either

- Locally look like $\mathbb{R}^n$.
- Or locally look like the upper half space $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$.



Closed Disk

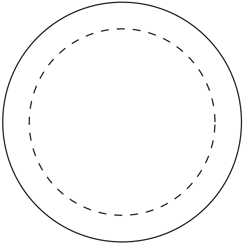


Mobius Strip

Collar Neighborhood Theorem

Theorem

Let M be a smooth manifold with boundary ∂M, then ∂M has a neighborhood that is diffeomorphic to $\partial M \times [0, 1)$. This is called a collar neighborhood.



Collar Neighborhood of a Closed Disk.

Orientation of a Manifold

Recall a smooth manifold M is equipped with a collection of charts $\{(U_\alpha, \varphi_\alpha)\}$, such that the **transition map**

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^n \to \varphi_\beta(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^n$$

is smooth (i.e., C^∞).

We say that M is orientable if the Jacobian of its transition functions all have positive determinant, ie.

$$\det(J(\varphi_\beta \circ \varphi_\alpha^{-1})) > 0, \forall \alpha, \beta.$$

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Examples of Orientation

Fact:

A co-dimension 1 submanifold M of $\mathbb{R}^n$ is orientable if and only if it has a non-vanishing normal vector field.

For example,

1. $S^{n-1} \subset \mathbb{R}^n$ is orientable with its normal vector field being $(x_1, ..., x_n)$ at $(x_1, ..., x_n)$.
2. The Mobius strip in $\mathbb{R}^3$ is not orientable! Take a pencil pendicular on the Mobius strip and travel one circle, the pencil would face the other way.