The Reeb Transform

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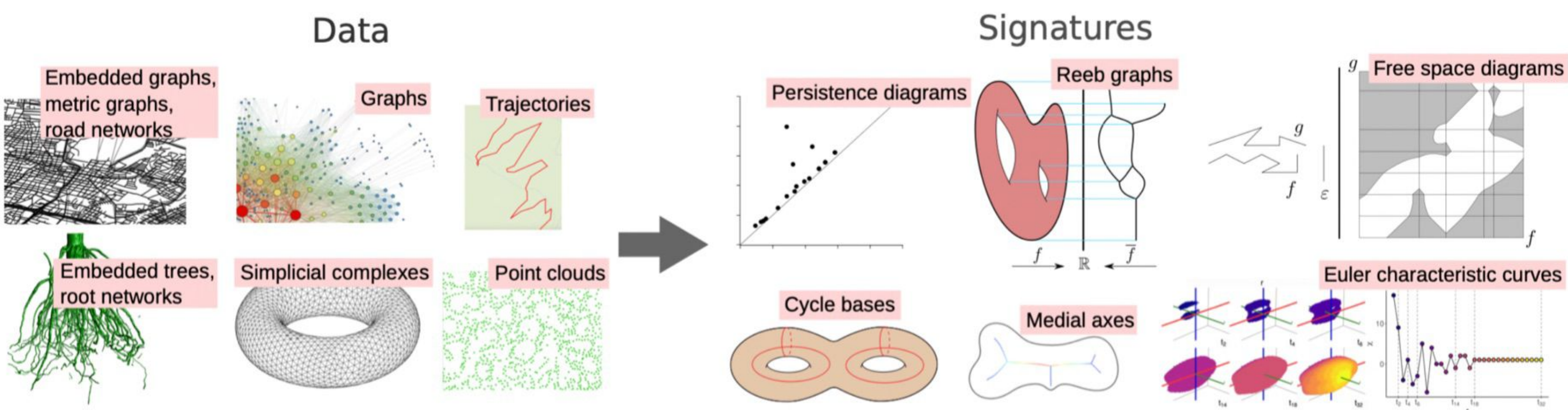
Topological Signatures

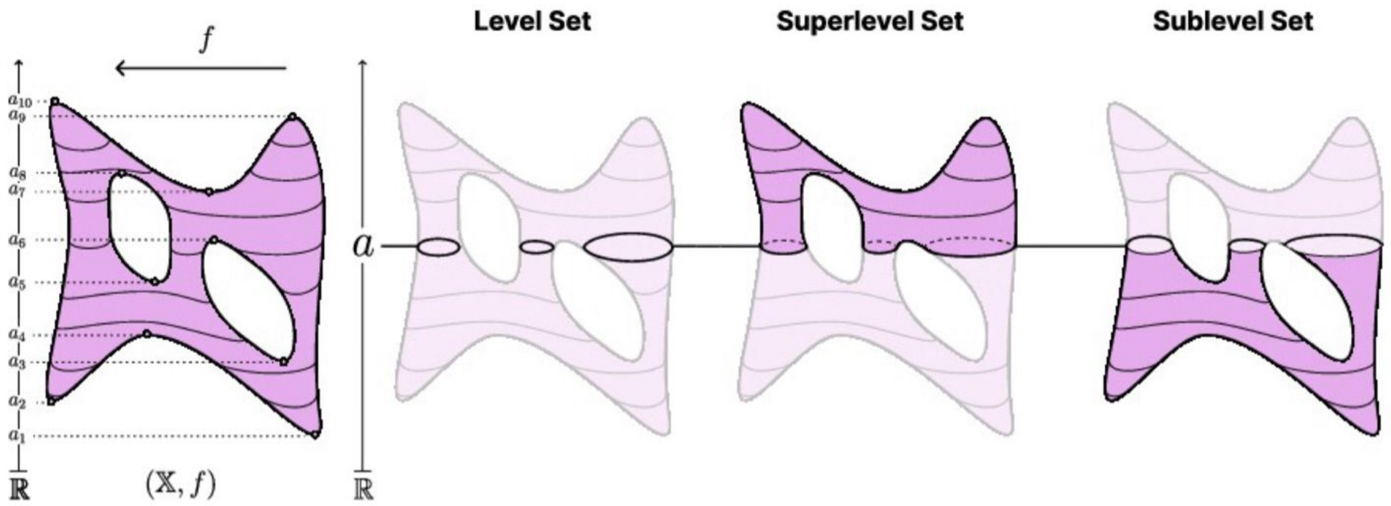
- There are many topological signatures which attempt to capture shape information in a compressed form



Some Definitions

- Scalar Field (or $\mathbb{R}$-space)
  - Space X along with a function $f : X \to \mathbb{R}$

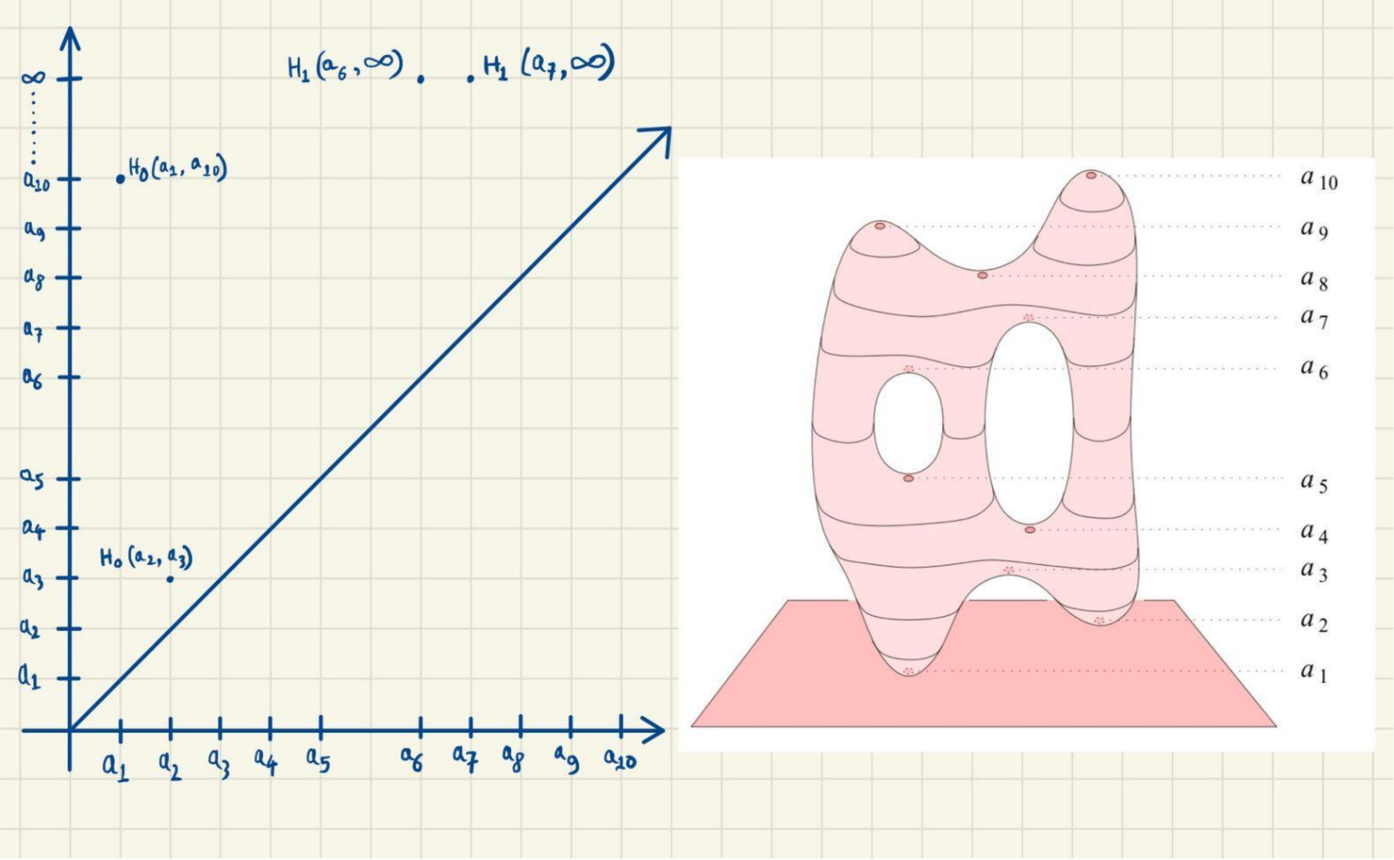
- Level Sets: $f^{-1}(a),\ X_a$



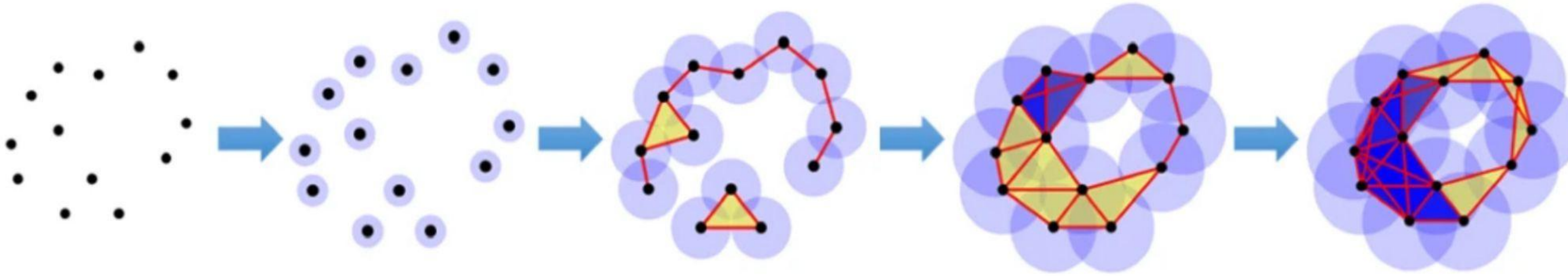
Filtrations

- We can use sublevel (or superlevel) sets to build filtrations
- Assuming (X, f) to be "nice"(i.e. Morse, o-minimal, finite, etc.), have a finite number of critical points, where structures changes. We denote these $a_{0,} a_{1,\ldots,}\ a_n$
  - For Sublevel Sets: $X_{a_0} \subseteq X_{a_1} \subseteq \ldots \subseteq X_{a_n}$
  - For Superlevel Sets: $X_{a_n} \subseteq X_{a_{n-1}} \subseteq \ldots \subseteq X_{a_0}$

Persistent Homology



Persistence Homology (contd.)



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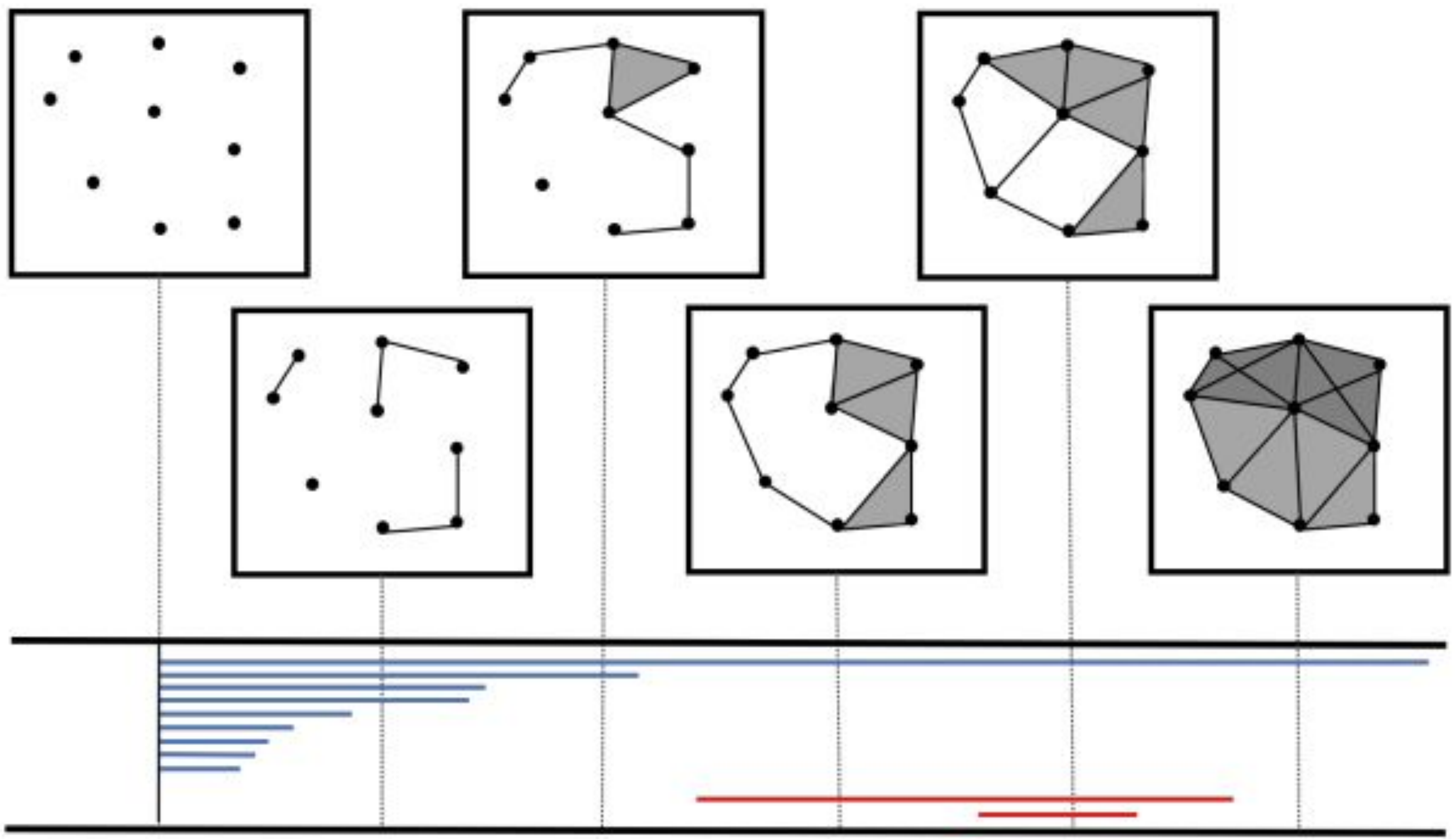
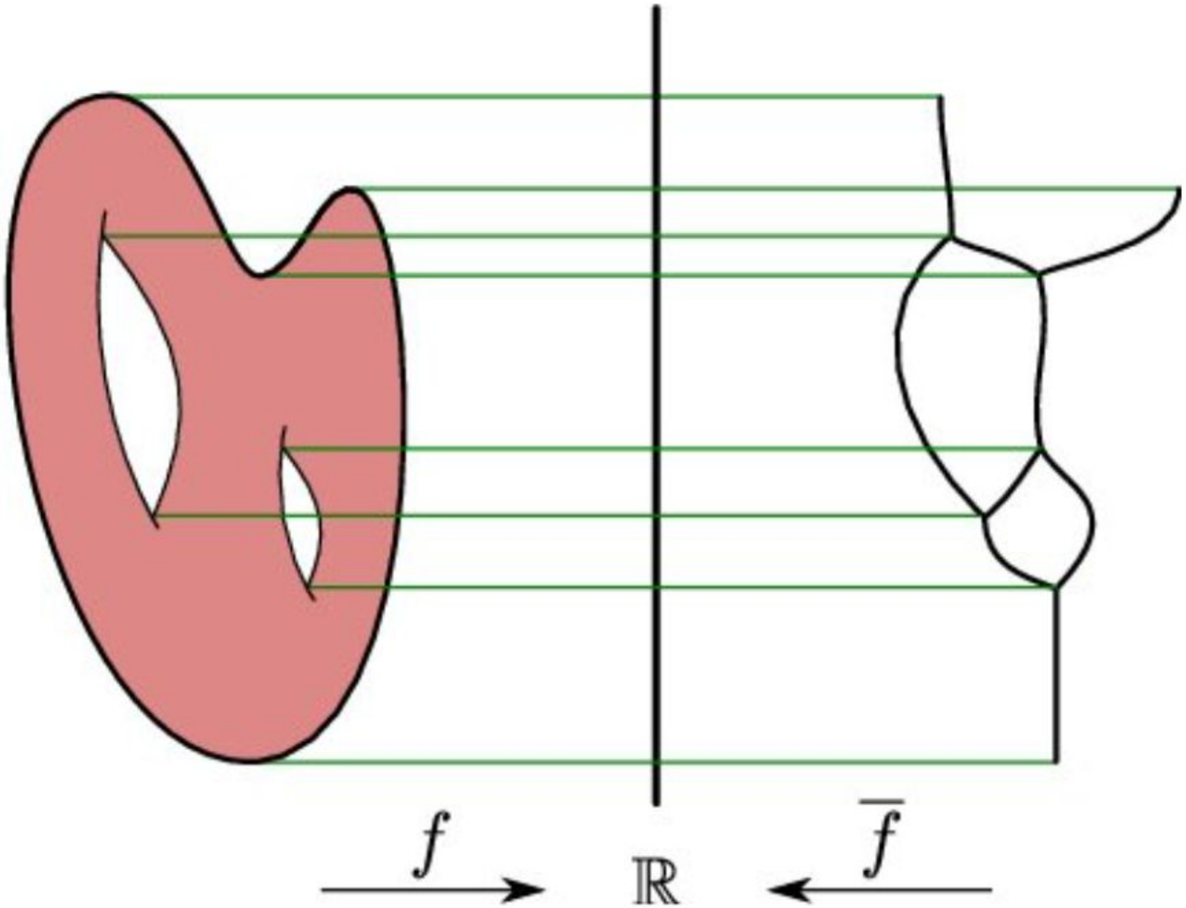
Persistence Modules

- It is Functor from a Poset to Vector Spaces
  - $F : (I, \leq) \to Vect_{\mathbb{K}}$
- For a feature that is born at a parameter $b \in I$, there exists a vector $v \notin im(F(s \to b))$for any $s < b$
- If the feature dies at $d \in I$, then for all $t \geq d$, the element $\varphi_{b,t}(v)$ becomes trivial in the sense that it either:
  - Maps to zero, or
  - Is no longer independent (e.g., merges with other generators)

Barcode



Reeb Graphs



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Construction

- We define a Morse function $f : M \to \mathbb{R}$ which satisfy:
  - Smoothness
  - Non-Degeneracy (hessian matrix at critical points is non-singular)

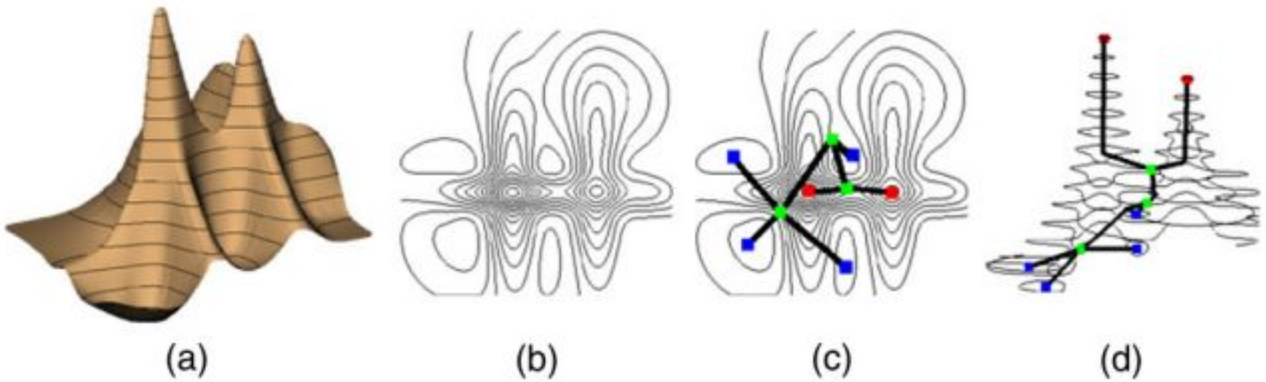
- For each value $c \in \mathbb{R}$, we consider the level set $f^{-1}(c)$

- We define an equivalence relation $x \sim y$ if $f(x) = f(y)$ and x and y lie in the same connected component of $f^{-1}(f(x))$

- We collapse each connected component into a single point. The resulting space is the Reeb Graph $R(f) = M/\sim$

Some Examples

S. Biasotti et al. / Theoretical Computer Science 392 (2008) 5–22



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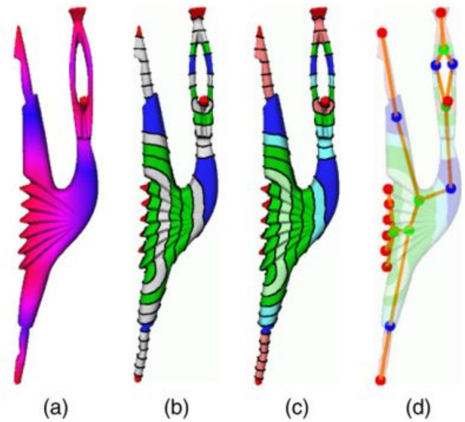

Euler Characteristic Curve

- Euler Characteristic of a simplicial complex K is:
  χ(K)= *#vertices - #edges + #faces - …*

- By Poincare duality:
  χ(K)= *#connected components - #loops + #voids - ...*

- Euler Characteristic Curve of filtered space (X,f) that induces the filtration $X_{a_i} = f^{-1}(-\infty, a_i]$:

$$\chi(X) = \chi(X_{a_0}) + \sum_{i=1}^{n} [\chi(X_{a_i}) - \chi(X_{a_{i-1}})]$$

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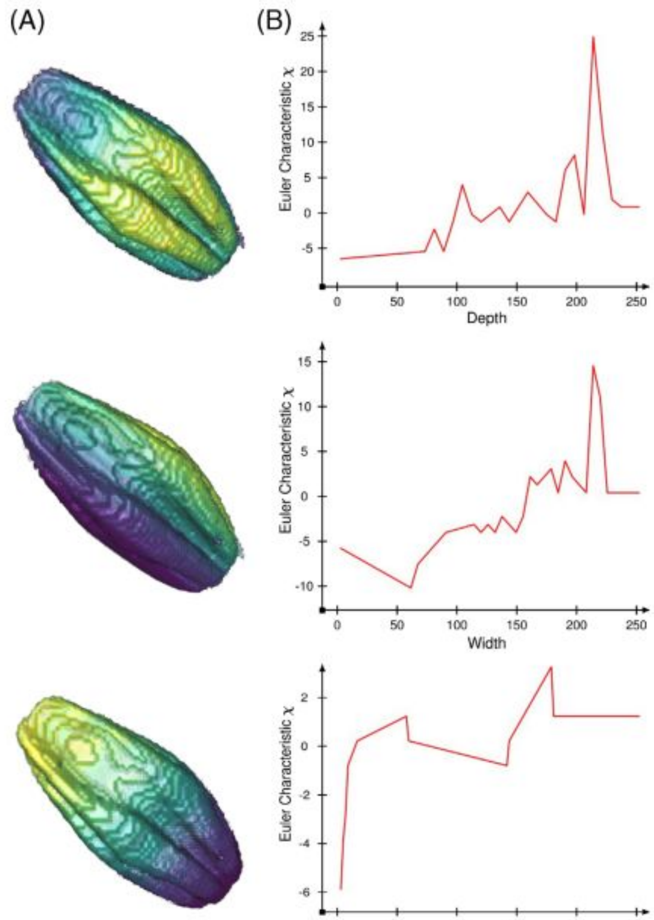
Example

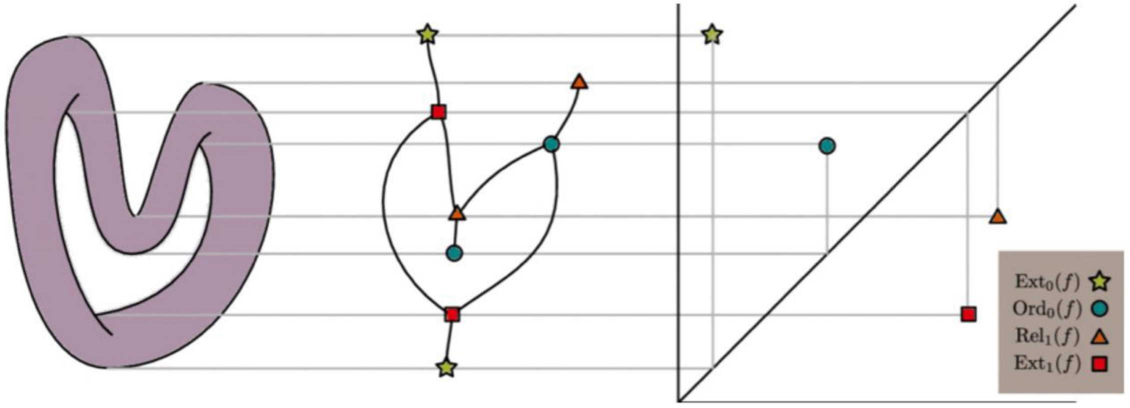


The shape of things to come: Topological data analysis and biology, from molecules to organisms

EJ Amézquita, MY Quigley, T Ophelders, E Munch… - Developmental Dynamics, 2020

Not totally independent!

- PDs: No connectivity, but contains information of all dims
- Reeb Graphs: Has all H_0 and H_1 information, but loses higher dimensions
- Merge Trees: Some H_0 data and some connectivity
- Euler Characteristics: Extremely compact, but non-injective and loss of information



One Observation!

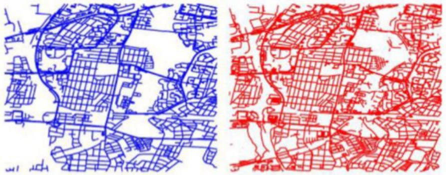
- The filtration function does not necessarily have to be a height function
  - Any Real valued function will do, if it is "Nice"!
- Examples:
  - Curvature
  - Temperature
  - Wind Velocity

Directional Transforms

- If the space is embedded in $\mathbb{R}^n$, we consider all possible "height" functions

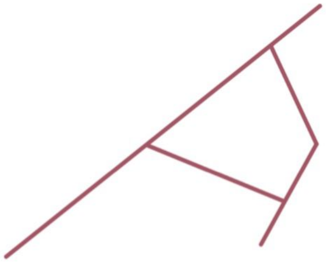
- In $\mathbb{R}^n$, we consider $\mathbb{S}^{n-1}$

- Consider $\mathbb{R}^2$, embedded graphs:



Two reconstructed maps of Berlin

Green fluorescent protein



Letter "A"

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HOW MANY DIRECTIONS DETERMINE A SHAPE AND OTHER SUFFICIENCY RESULTS FOR TWO TOPOLOGICAL TRANSFORMS

JUSTIN CURRY, SAYAN MUKHERJEE, AND KATHARINE TURNER

Euler Characteristic Transform

Definition 3.1. *The* ***Euler Characteristic Transform*** *takes a constructible function ϕ on $\mathbb{R}^d$ and returns a constructible function on $S^{d-1} \times \mathbb{R}$ whose value at a direction v and real parameter $t \in \mathbb{R}$ is the Euler integral of the restriction of ϕ to the half space $x \cdot v \leq t$. In equational form, we have*

$$\mathrm{ECT} : CF(\mathbb{R}^d) \to CF(S^{d-1} \times \mathbb{R}) \qquad \textit{where} \qquad \mathrm{ECT}(\phi)(v,t) := \int_{x \cdot v \leq t} \phi \, d\chi.$$

- A key technical step uses an inversion theorem of Schapira (originally stated for “constructible” functions under Euler integration). In essence:
  - The ECT is injective on compact definable subsets of
  - Intuitively, if two distinct shapes M and M' gave the same ECT, they would have the same Euler Characteristic across every slice and thus be forced to coincide

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Persistence Diagram Space

Definition 4.4. ***Persistence Diagram Space***, *written Dgm, is the set of all possible countable multi-sets of* $\mathbb{R}^{2+} := \{(b,d) \in (\{-\infty\} \cup \mathbb{R}) \times (\mathbb{R} \cup \{\infty\}) \mid b \leq d\}$ *where the number of points of the form* (b, ∞) *and* $(-\infty, d)$ *are finite and* $\sum_{d-b<\infty} d - b < \infty$. *Points of the form* (b, ∞) *or* $(-\infty, d)$ *are called* ***essential classes*** *and points of the form* (b, d) *where neither coordinate is* ∞ *are called* ***inessential classes***. *For persistence diagrams that encode the sublevel set persistent homology* $PH_k(M, h_v)$ *of a constructible set* M *there are no points of the form* $(-\infty, d)$.

- A point (b, ∞) is called an essential feature (it never dies)
- A point $(-\infty, d)$ is a feature that was born before the filtration
- A point (b, d) is an inessential feature (it is fully captured between finite birth and death times)
- Any diagram $B \in Dgm$ has a finite number of points either of form (b, ∞) or $(-\infty, d)$
- $\sum_{(b,d)\in B} (d - b) < \infty$ (for all points with finite coordinates) This ensures that the diagram does not contain infinitely many features

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Persistent Homology Transform

Definition 4.6. *The* ***Persistent Homology Transform*** PHT *of a constructible set* $M \in CS(\mathbb{R}^d)$ *is the map* $\mathrm{PHT}(M) : S^{d-1} \to Dgm^d$ *that sends a direction* $v \in S^{d-1}$ *to the persistent diagrams gotten by filtering* M *in the direction of* v*, recording one diagram for each homological degree* $0 \leq k \leq d-1$*, i.e.*

$$\mathrm{PHT}(M) : v \mapsto (PH_0(M, h_v), PH_1(M, h_v), \ldots, PH_{d-1}(M, h_v)).$$

Letting the set M *vary gives us the map*

$$\mathrm{PHT} : CS(\mathbb{R}^d) \to C(S^{d-1}, Dgm^d)$$

where $C(S^{d-1}, Dgm^d)$ *is the set of continuous functions from* S^{d-1} *to* Dgm^d*, the latter being equipped with some Wasserstein p-distance.*

- The Wasserstein-p distance is used to measure the distance between two d-tuples of diagrams. This is typically done by combining the distances of the corresponding diagrams in each coordinate (for example, by taking an l^p norm of the individual W-p distances)

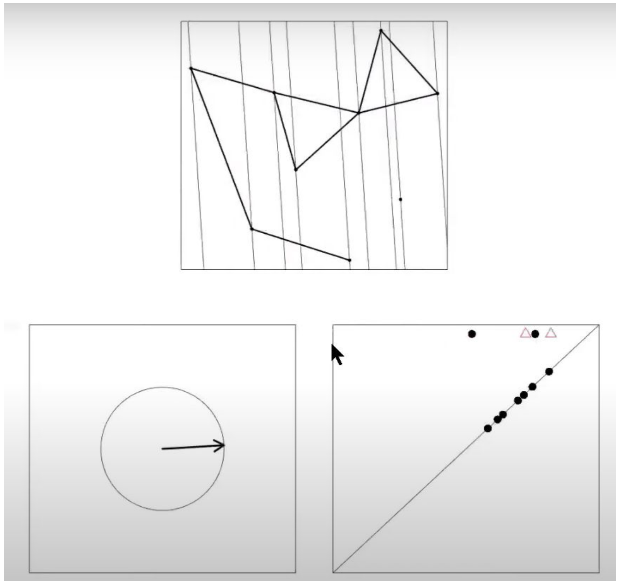
How good are transforms?

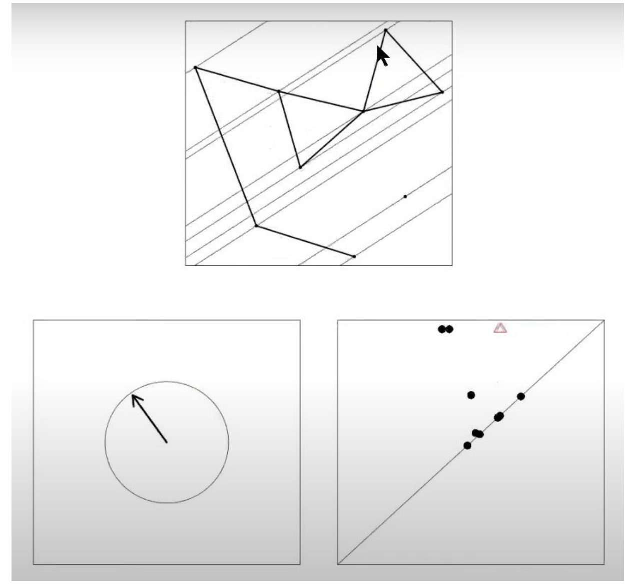
- Natural Question: When does a set of signatures completely determine the input shape?
- Theoretical Result: If we have all directions, both PD and ECT completely determine the shape

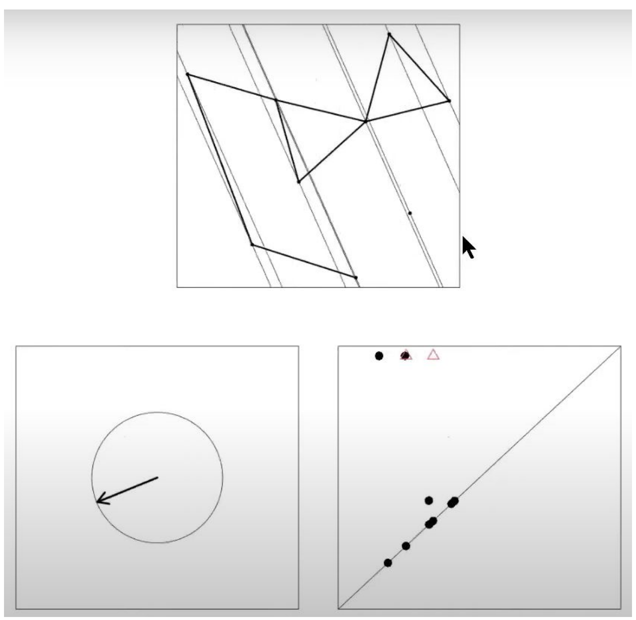
Theorem 3.5. *Let $CS(\mathbb{R}^d)$ be the set of constructible sets, i.e. compact definable sets. The map* $\mathrm{ECT} : CS(\mathbb{R}^d) \to CF(S^{d-1} \times \mathbb{R})$ *is injective. Equivalently, if M and M' are two constructible sets that determined the same association of directions to Euler curves, then they are, in fact, the same set. Said symbolically:*

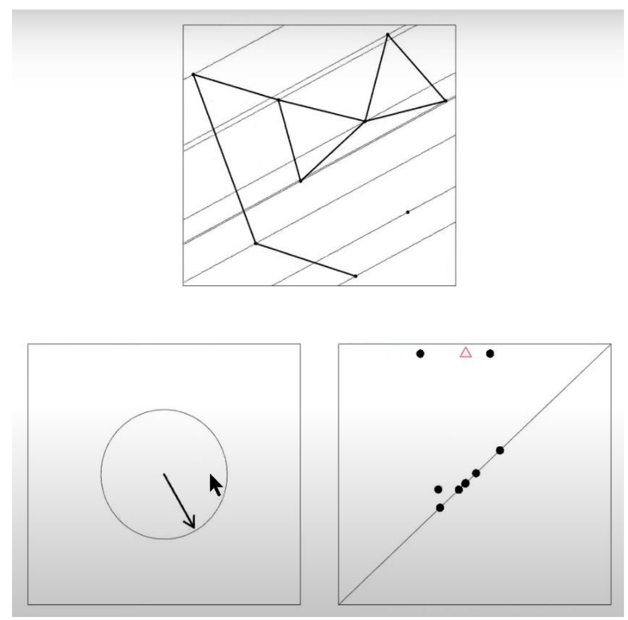
$$\mathrm{ECT}(M) = \mathrm{ECT}(M') : S^{d-1} \to CF(\mathbb{R}) \qquad \Rightarrow \qquad M = M'$$

Theorem 4.16. *Let $CS(\mathbb{R}^d)$ be the set of constructible sets, i.e. compact definable subsets of $\mathbb{R}^d$. The Persistent Homology Transform* $\mathrm{PHT} : CS(\mathbb{R}^d) \to C(S^{d-1}, Dgm^d)$ *and Betti Curve Transform* $\mathrm{BCT} : CS(\mathbb{R}^d) \to C(S^{d-1}, CF(\mathbb{R})^d)$ *are both injective.*









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ELSEVIER
Computational Geometry
Volume 90, October 2020, 101658

Reconstructing embedded graphs from persistence diagrams

Robin Lynne Belton [a], Brittany Terese Fasy [a b], Rostik Mertz [b], Samuel Micka [b], David L. Millman [b], Daniel Salinas [c], Anna Schenfisch [a], Jordan Schupbach [a], Lucia Williams [b]

Do we really need infinite directions?

- Answer: No
  - In $\mathbb{R}^2$, we need $O\left(n^2\right)$, where n is the size of the graph

- How many directions are needed in d-dimension?
  - In $\mathbb{R}^d$, we need exponential no. of directions, i.e., $O\left(n^d\right)$

Theorem 9 (Vertex Reconstruction in Higher Dimensions). *Let G be a straight-line embedded graph in $\mathbb{R}^d$ for $d > 1$. We can can compute the coordinates of all n vertices of G using $d+1$ directional augmented persistence diagrams in $\Theta(dn^{d+1} + dT_G)$ time, where $\Theta(T_G)$ is the time complexity of computing a persistence diagram.*

Reeb Transform [Chambers, Mukherjee, Turner]

Definition 8 (Reeb Transform). *Let $A \in Set(d)$ be a compact, definable, smooth manifold of intrinsic dimension $(d-1)$ embedded in $\mathbb{R}^d$. For every unit vector $v \in S^{d-1}$ define the height function*

$$h_v : \mathbb{R}^d \longrightarrow \mathbb{R}, \qquad h_v(x) = v \cdot x.$$

The Reeb graph *associated with v is*

$$\mathcal{R}_{h_v} := (h_v \restriction_A)/\sim,$$

The Reeb Transform *of A is the collection of all these graphs, indexed by direction:*

$$RT(A) = \{ \mathcal{R}_{h_v} \mid v \in S^{d-1} \}.$$

Equivalently, one may view $RT(A)$ as the map

$$RT(A) \colon S^{d-1} \longrightarrow \mathcal{G}, \qquad v \longmapsto \mathcal{R}_{h_v},$$

where $\mathcal{G}$ denotes the set of finite, connected 1-complexes (Reeb graphs).

Equivalence of reeb transform

Definition 9 (Equivalence Relation $\sim^{\mathcal{RT}}$). *We define an equivalence relation $\sim^{\mathcal{RT}}$ on $Set(d)$ as follows: for $A, B \in Set(d)$,*

$$A \sim^{\mathcal{RT}} B \iff \mathcal{RT}(A) \cong \mathcal{RT}(B),$$

where $\cong$ denotes an isomorphism of Reeb graphs (with preserved structure) in every unit direction S^{d-1}.

Thus, two spaces are considered equivalent under $\sim^{\mathcal{RT}}$ if and only if their Reeb transforms coincide (up to isomorphism).

- $\sim^{\mathcal{RT}}$ partitions the tame sets by comparing their entire directional data.
- Equivalence is therefore determined by connectivity of all level sets, along all unit vector $v \in S^{d-1}$

Injectivity in 2D

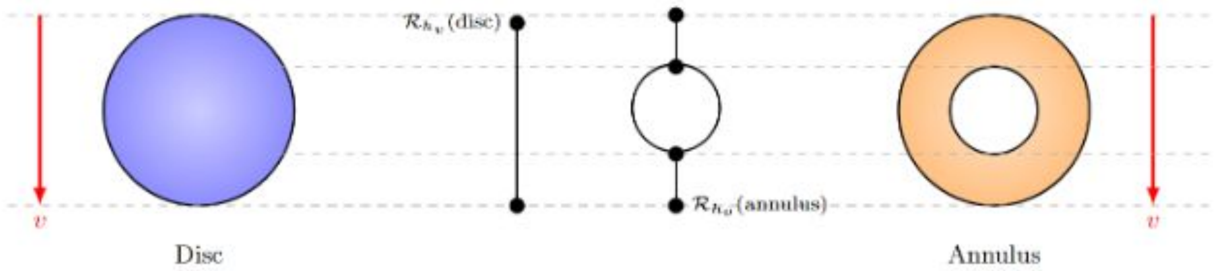


Figure 2: Example of two different surfaces in $\mathbb{R}^2$ (a disc and an annulus) and a direction where their Reeb graphs differ. Such a direction must exist for any pair of distinct surfaces in $\mathbb{R}^2$

- For a planar tame set $A \subset \mathbb{R}^2$ each line slice $L \cap A$ is a finite set of points, so its Euler characteristic equals the number of connected components.
- The Reeb transform records exactly that count for every direction $v \in S^1$ and every height t; hence $\mathcal{RT}(A)$ fully determines the function $(v, t) \mapsto \chi(L \cap A)$.
- Scholium ("Curry–Ghrist"): knowing $\chi(L \cap A)$ for all lines uniquely reconstructs any definable subset of $\mathbb{R}^2$.
- Therefore, if $\mathcal{RT}(A) = \mathcal{RT}(B)$ then the slice-Euler data coincide, implying $A = B$; the Reeb transform is injective on the class $\mathcal{S}(2)$.

Injectivity in 3D

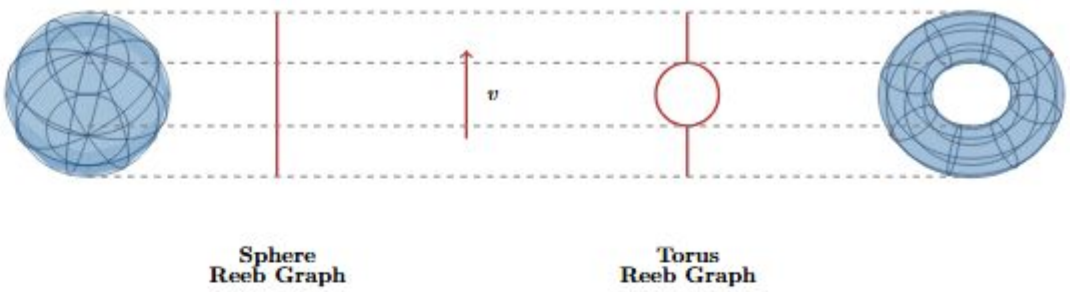
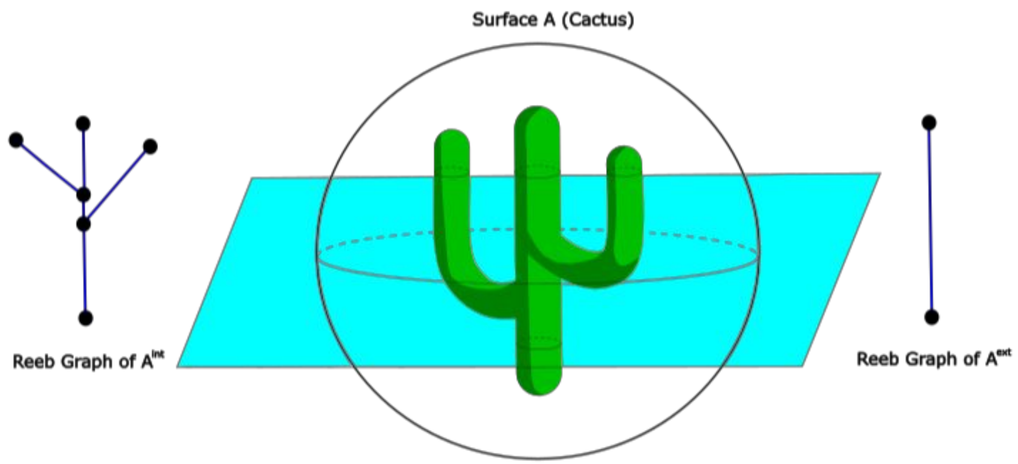


Figure 3: Example of two different surfaces in $\mathbb{R}^3$ (a sphere and a torus) and a direction where their Reeb graphs differ. Such a direction must exist for any pair of distinct surfaces in $\mathbb{R}^3$

- For a surface $A \subset \mathbb{R}^3$, each directional graph $\mathcal{RT}(A, v)$ lets us recover the Euler characteristic of every plane slice normal to v: $\chi(A \cap \Pi_{v,t}) = 2C_t - C_{t-\varepsilon} - C_{t+\varepsilon}$.
- Collecting these graphs for every $v \in S^2$ yields the complete table $(v, t) \mapsto \chi(A \cap \Pi_{v,t})$.
- Scholium (Curry–Ghrist): knowing slice-Euler values for all planes uniquely determines a compact surface.

Injectivity in 3-D (contd.)

Theorem 5. *Let $A \in Surface(3)$ be a dividing surface. Let A^{int} denote the A union the interior of A and A^{ext} denote A union the exterior of A. A as a subset of $\mathbb{R}^3$ is uniquely determined by the combination of $\mathcal{RT}(A^{int})$ and $\mathcal{RT}(A^{ext})$.*



- A closed surface in $\mathbb{R}^3$ splits space into an "inside" and an "outside."
  If we extend the surface by filling in its interior (A^{int}) or its exterior (A^{ext}), we can compute a Reeb Transform for each filled region.
- Taken together, these two transforms contain all the slice–connectivity information of the surface itself, so knowing $\mathcal{RT}(A^{\text{int}})$ and $\mathcal{RT}(A^{\text{ext}})$ lets us reconstruct A uniquely.

Properties

Proposition 6 (Characterisation via Height Functions). *Let* $h_{\hat{n}}(x) = \hat{n} \cdot x$ *and let* $y \in Y$.

(a) y *satisfies* (C_−) $\iff$ y *is a strict local* minimum *of* $h_{\hat{n}}$ *on* Y.

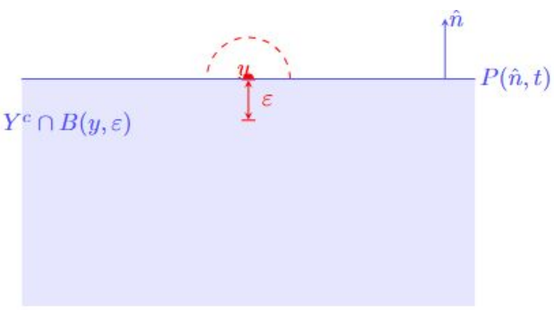
(b) y *satisfies* (C_+) $\iff$ y *is a strict local* maximum *of* $h_{\hat{n}}$ *on* Y.

- Pick a direction $\hat{n}$; the dot-product height $h_{\hat{n}}(x) = \hat{n} \cdot x$ measures how far a point lies along that direction.
- A point y on the set Y is **concave** in direction $\hat{n}$ exactly when $h_{\hat{n}}$ attains a strict local minimum at y; all nearby points of Y sit at the same or higher "height."
- Dually, y is **convex** in direction $\hat{n}$ precisely when $h_{\hat{n}}$ reaches a strict local maximum there—nearby points lie at equal or lower height.

Properties [contd.]

Proposition 7. *Let $A \in Set(d)$ for $d \geq 3$, and let $X \subseteq A$ be an open, simply connected subset. Then*

$$\mathcal{RT}(A \setminus X) = \mathcal{RT}(A) \iff \textit{the closure of } X \textit{ contains no concave points of } X.$$



- **Concave points drive change.** If the open region X contains even one concave point (a strict directional minimum/maximum), deleting X removes that extremum, so the Reeb transform *must* change.
- **No concave points ⇒ no change.** Conversely, if every point of X sits "flat" in every direction—that is, the closure of X holds no concave extremes—then slicing connectivity is unaffected, and $\mathcal{RT}(A \setminus X)$ remains identical to $\mathcal{RT}(A)$.

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Stability of Reeb Transform

Proposition 2 (Directional stability of Reeb graphs). *Let $A \in Set(d)$ be non-empty, and let $D > 0$ such that $A \subset B(0, D)$ (the ball of radius D). Fix a unit direction $v \in S^{d-1}$ and perturb it to a second direction $w \in S^{d-1}$. Then*

$$d_{\mathrm{FD}}(\mathcal{R}_v(A), \mathcal{R}_w(A)) \leq D\,\|v - w\|_2 = D\sqrt{2 - 2\langle v, w\rangle}\,,$$

where d_{FD} is the functional–distortion distance [7]. In particular, if the angle $\theta = \arccos\langle v, w\rangle$ is small, the graph moves by at most $D\,\sin(\frac{\theta}{2})$.

- Rotating the slicing direction from v to w by a small angle θ shifts every point's height by at most $D\sin(\theta/2)$, so the Reeb graph changes no more than that in functional–distortion distance.
- The inequality

  $$d_{\mathrm{FD}}\big(\mathcal{R}_v(A), \mathcal{R}_w(A)\big) \leq D\sqrt{2 - 2\langle v, w\rangle}$$

  shows the deformation is Lipschitz with constant D.
- Because the bound applies for every pair of nearby directions, the whole Reeb Transform varies continuously over the sphere; neighbouring directions yield nearly identical graphs.

Future Work

- Reeb Spaces
- Developing algorithms for faster calculation of Reeb Transform
- Adding decorations within the Reeb Transform

Thank You!