MATH 6480/STAT 9300/AMCS 6481, Fall 2023, Homework Set 3

1. Let μ be the uniform distribution on $[0,1]$. Compute the density of the convolution $\mu * \mu$ and the convolution $\mu * \mu * \mu$.

   Solution :

   For the convolution $\mu * \mu(x) = \int_{-\infty}^{\infty} \mu(t)\mu(x-t)dx$, consider the following three cases:
   i) $0 \leq x \leq 1$, then $\mu * \mu(x) = \int_0^x 1 \cdot 1dt = x$
   ii) $1 \leq x \leq 2$, i.e., then $0 \leq x-1 \leq 1$, $\mu * \mu(x) = \int_{x-1}^1 1 \cdot 1dt = 2-x$
   iii) $x \geq 2, x \leq 0$, then $\mu * \mu(x) = 0$

   For the convolution $\mu * \mu * \mu(x) = \int_{-\infty}^{\infty} \mu * \mu(t)\mu(x-t)dx$, consider the following four cases: i) $0 \leq x \leq 1$, then $\mu * \mu * \mu(x) = \int_0^x t \cdot 1dt = \frac{x^2}{2}$
   ii) $1 \leq x \leq 2$, i.e., $0 \leq x-1 \leq 1$, then $\mu * \mu * \mu(x) = \int_{x-1}^1 t \cdot 1dt + \int_1^x (2-t) \cdot 1dt = -x^2 + 3x - \frac{3}{2}$
   iii) $2 \leq x \leq 3$, i.e., $1 \leq x-1 \leq 2$, then $\mu * \mu * \mu(x) = \int_{x-1}^2 (2-t) \cdot 1dt = \frac{(x-3)^2}{2}$
   iv) $x \leq 0, x \geq 3$, then $\mu * \mu * \mu(x) = 0$

2. In each case compute $\mathbb{E}e^{tX}$, differentiate twice to compute the second moment and variance.

   (a) $X \sim \exp(\lambda)$;

   (b) $X \sim N(0, \sigma^2)$;

   (c) $X \sim \Gamma(k, \lambda)$;

   (d) $X \sim \mathcal{P}(\lambda)$ (Poisson);

   (e) $X \sim U[0,1] * U[0,1]$ (convolution).

   <u>*Solution*</u> :

   (a) For $X \sim \exp(\lambda)$, $f(x) = \lambda e^{-\lambda x}, \ \ \forall x \geq 0$

$$\begin{aligned} m(t) = \mathbb{E}(e^{tX}) &= \int_{-\infty}^{\infty} e^{tx} f(x) dx \\ &= \int_0^{\infty} \lambda e^{\lambda x} \cdot e^{tx} dx = \lambda \int_0^{\infty} e^{-(\lambda - t)x} \\ &= \frac{\lambda}{\lambda - t} \int_0^{\infty} (\lambda - t) e^{-(\lambda - t)x} dx \text{ (integral=total mass of } \exp(\lambda - t)) \\ &= \frac{\lambda}{\lambda - t} \quad \text{for } \ t \leq \lambda \end{aligned}$$

$$\mathbb{E}(X) = m'(0) = \frac{\lambda}{(\lambda - t)^2}\bigg|_{t=0} = \frac{1}{\lambda}, \ \ \mathbb{E}(X^2) = m''(0) = \frac{2\lambda}{(\lambda - t)^3}\bigg|_{t=0} = \frac{2}{\lambda^2}$$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{1}{\lambda^2}$$

   (b) For $X \sim N(0, \sigma^2)$, $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$

$$\begin{aligned} m(t) = \mathbb{E}(e^{tX}) &= \int_{-\infty}^{\infty} e^{tx} f(x) dx = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{tx} e^{\frac{-x^2}{2\sigma^2}} dx \\ &= e^{\frac{t^2\sigma^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x - t\sigma^2)^2}{2\sigma^2}} dx \text{ (integral=total mass of } (t\sigma^2, \sigma^2) \\ &= e^{\frac{t^2\sigma^2}{2}}) \end{aligned}$$

$$\mathbb{E}(X) = m'(0) = t\sigma^2 e^{\frac{t^2\sigma^2}{2}}\bigg|_{t=0} = 0, \ \ \mathbb{E}(X^2) = m''(0) = \sigma^2 e^{\frac{t^2\sigma^2}{2}}\bigg|_{t=0} + \text{a zero term when t=0} = \sigma^2$$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \sigma^2$$

(c) For $X \sim \Gamma(k, \lambda),\ f(x) = \frac{\lambda^k}{\Gamma(k)} x^{k-1} e^{-\lambda x},\ \ \forall x > 0$

$$\begin{aligned}
m(t) = \mathbb{E}(e^{tX}) &= \int_{-\infty}^{\infty} e^{tx} f(x) dx \\
&= \frac{\lambda^k}{\Gamma(k)} \int_0^{\infty} e^{tx} x^{k-1} e^{\lambda x} dx \\
&= \frac{\lambda^k}{\Gamma(k)} \int_0^{\infty} e^{-(\lambda-t)x} x^{k-1} dx \text{ (content in integral=kernal of } \Gamma(k, \lambda - t)) \\
&= \frac{\lambda^k}{\Gamma(k)} \cdot \frac{\Gamma(k)}{(\lambda-t)^k} = (\frac{\lambda}{\lambda-t})^k \quad \text{for } t < \lambda
\end{aligned}$$

$$\mathbb{E}(X) = m'(0) = \frac{k\lambda^k}{(\lambda-t)^{k+1}}\Big|_{t=0} = \frac{k}{\lambda},\ \ \mathbb{E}(X^2) = m''(0) = k(k+1)\frac{\lambda^k}{(\lambda-t)^{k+2}}\Big|_{t=0} \frac{k(k+1)}{\lambda^2}$$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{k}{\lambda^2}$$

(d) For $X \sim \mathcal{P}(\lambda),\ f(x) = \frac{e^{-\lambda}\lambda^x}{x!},\ \ \forall x \in 0, 1, 2, \cdots$

$$\begin{aligned}
m(t) = \mathbb{E}(e^{tX}) &= \int_{-\infty}^{\infty} e^{tx} f(x) dx \\
&= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{e^{tx}\lambda^x}{x!} \\
&= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^t\lambda)^x}{x!} \text{ (kernal of } \mathcal{P}(\lambda e^t), e^{-e^t\lambda} \sum_{x=0}^{\infty} \frac{(e^t\lambda)^x}{x!} = 1) \\
&= e^{-\lambda} \cdot e^{e^t\lambda} = e^{\lambda(e^t-1)}
\end{aligned}$$

$$\mathbb{E}(X) = m'(0) = \lambda e^t e^{\lambda(e^t-1)}\Big|_{t=0} = \lambda,\ \ \mathbb{E}(X^2) = m''(0) = \lambda e^t e^{\lambda(e^t-1)} + \lambda^2 e^{2t} e^{\lambda(e^t-1)}\Big|_{t=0} = \lambda^2 + \lambda$$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \lambda$$

(e) For $X \sim U[0,1] * U[0,1]$,

$$\begin{aligned}
m(t) = \mathbb{E}(e^{tX}) &= \int_{-\infty}^{\infty} e^{tx} f(x) dx \\
&= \int_0^1 e^{tx} x dx + \int_1^2 e^{tx}(2-x) dx \\
&= \frac{e^{tx}(tx-1)}{t^2}\Big|_{x=0}^{1} + \frac{e^{tx}(1-t(x-2))}{t^2}\Big|_{x=1}^{2} \\
&= \frac{e^{2t} - 2e^t + 1}{t^2} = \frac{(e^t-1)^2}{t^2}
\end{aligned}$$

$$\mathbb{E}(X) = \lim_{t=0} m'(t) = \lim_{t=0} \frac{2te^{2t} - 2e^{2t} - 2 - 2te^t + 4e^t}{t^3} = 1 \text{ (by L'Hospital)}$$

$$\mathbb{E}(X^2) = \lim_{t=0} m''(t) = \lim_{t=0} \frac{(4t^2 - 8t + 6)e^{2t} - (2t^2 - 8t + 12)e^t + 6}{t^3} = \frac{7}{6} \text{ (by L'Hospital)}$$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{1}{6}$$

3. Let $X_1, X_2, \cdots$ be uncorrelated with $\mathbb{E}[X_i] = \mu_i$ and $\text{var}[X_i]/i \to 0$ as $i \to \infty$. Let $S_n = X_1 + \cdots + X_n$ and $\nu_n = \mathbb{E}[S_n]/n$. Prove as $n \to \infty$, $S_n/n - \nu_n \to 0$ in L^2 and in probability.

<u>*Solution*</u> :

For this problem, we don't need to consider the condition that $\text{Var}[X_i] = \infty$

To prove as $n \to \infty$, $S_n/n - \nu_n \to 0$, necessarily prove $\mathbb{E}[(S_n/n - \nu_n)^2] \to 0$ as $n \to \infty$.

$$\mathbb{E}[(\frac{S_n}{n} - \nu_n)^2] = \frac{1}{n^2}\mathbb{E}[(\sum_{i=1}^n X_i - \mathbb{E}S_n)^2] = \frac{1}{n^2}\sum_{i=1}^n \mathbb{E}[(X_i - \mathbb{E}S_n)^2] = \frac{1}{n^2}\sum_{i=1}^n \text{Var}(X_i)$$

The second step achieved as $X_1, X_2, \cdots$ are uncorrelated.

Proof by mean of convergent sequence:

As $\text{Var}(X_i)/i \to 0$, there exists N_1 s.t $\forall \epsilon > 0, \forall i \geq N_1, \text{Var}(X_i)/i < \epsilon$. Choosing N_2 sufficiently large, we have $\sum_{i=1}^{N_1} \text{Var}(X_i)/i + N_1 \cdot 0 \leq N_2\epsilon$. Then for $n \leq \max\{N_1, N_2\}$,

$$\begin{aligned}
\frac{1}{n^2}\sum_{i=1}^n \text{Var}(X_i) &= \frac{1}{n}\sum_{i=1}^n \frac{\text{Var}(X_i)}{n} \leq \frac{1}{n}\sum_{i=1}^n \frac{\text{Var}(X_i)}{i} \\
&= \frac{1}{n}(\sum_{i=1}^{N_1} \frac{\text{Var}(X_i)}{i} + \sum_{i=N_1+1}^{n} \frac{\text{Var}(X_i)}{i}) \\
&\leq \frac{1}{n}(N_2\epsilon + (n - N_1)\epsilon) \\
&= \frac{(N_2 - N_1)\epsilon}{n} + \epsilon = \epsilon \text{ as } n \to 0
\end{aligned}$$

$\mathbb{E}[(\frac{S_n}{n} - \nu_n)^2] = \frac{1}{n^2}\sum_{i=1}^n \text{Var}(X_i) \leq \epsilon$ implies $\mathbb{E}[(S_n/n - \nu_n)^2] \to 0$ as $n \to \infty$ as ϵ can be arbitrarily small. Therefore, as $n \to \infty$, $S_n/n - \nu_n \to 0$ in L^2, and also in probability by Chebyshev's inequality.

Proof by convergent sequence bounded(similar but consider single $\frac{\text{Var}(X_i)}{i}$ instead of the summation):

As $\text{Var}(X_i)/i \to 0$, there exists N such that $\forall i \geq N_1, \text{Var}(X_i)/i < \epsilon$ for any $\epsilon > 0$. Then $|\text{Var}(X_i)| < i\epsilon$ for each X_i, and thus $|\text{Var}(X_i)| < n\epsilon$ for all $X_1, \cdots, X_n$ (equ 1). For all $i \leq N$, we have $\text{Var}(X_i) \leq \max\{|\text{Var}(X_i)|, \forall i \in \{1, \cdots, N\}\}$ (equ 2), denoted the maxima as M. Combining 1&2, we have $|\text{Var}(X_i)| < \epsilon n + M$. Then

$$\frac{1}{n^2}\sum_{i=1}^n \text{Var}(X_i) < \frac{1}{n^2} \cdot n(\epsilon n + M) = \frac{\epsilon n + M}{n} = \epsilon$$

as $n \to 0$, implying $\mathbb{E}[(S_n/n - \nu_n)^2] \to 0$ as $n \to \infty$ as ϵ can be arbitrarily small.