MATH 6480/STAT 9300/AMCS 6481, Fall 2023, Homework Set 4

1. Let $\{X_n\}$ be IID with $\mathbb{P}(X_n = -1) = p_0$ and $\mathbb{P}(X_n = 2^k - 1) = p_k$, where $p_k := \frac{1}{2^k k(k+1)}$ for $k \geq 1$ and $p_0 := 1 - \sum_{k=1}^{\infty} p_k$. Prove that

$$\frac{S_n}{n/\log_2 n} \to -1$$

in probability.

<u>*Solution*</u> :

Let $S_n = X_1 + \cdots + X_n$, use the weak law for triangular arrays with $b_n = 2^{m(n)}$ where $m(n) = \min\{m : 2^{-m} m^{-\frac{3}{2}} \leq n^{-1}\}$

Check conditions:

$(0) b_n = 2^{m(n)} > 0$ as $n \to \infty$ holds from definition

$(i) \sum_{k=1}^{n} P(|X_{n,k}| > b_n) \to 0$

$$\begin{aligned}
\sum_{k=1}^{n} P(|X_k| > b_n) &= n\mathbb{P}(|X_1| \geq 2^m) \\
&= n \sum_{k=m+1}^{\infty} p_k \\
&= n \sum_{k=m+1}^{\infty} \frac{1}{2^k k(k+1)} \\
&\leq n \sum_{k=m+1}^{\infty} \frac{1}{2^k m^2} \quad \text{as } k > m, \frac{1}{k} \leq \frac{1}{m} \\
&= \frac{n}{2^m m^2} \\
&\leq m^{-\frac{1}{2}} \quad \text{as } n \leq 2^m m^{\frac{3}{2}}
\end{aligned}$$

$\sum_{k=1}^{n} P(|X_k| > b_n) = m^{-\frac{1}{2}} \to 0$ as $n \to \infty (m \to \infty$ as well)

$(ii) b_n^{-2}\sum_{k=1}^n \mathbb{E}\bar{X}_{n,k}^2 \to 0$

$$
\begin{aligned}
\mathbb{E}(\bar{X}_1)^2 &= \mathbb{E}(X_1^2\mathbf{1}_{\{|X_1|\le 2^m\}}) \\
&= (-1)^2 p_0 + \sum_{k=1}^{m}(2^k-1)^2 p_k \\
&\le 1+\sum_{k=1}^{m} 2^{2k}p_k \\
&= 1+\sum_{k=1}^{m} 2^k\frac{1}{k(k+1)} \\
&= 1+\sum_{1\le k\le \frac{m}{2}} 2^k\frac{1}{k(k+1)}+\sum_{\frac{m}{2}\le k\le m} 2^k\frac{1}{k(k+1)} \\
&\le 1+\sum_{k=1}^{\frac{m}{2}} 2^k+\frac{4}{m^2}\sum_{\frac{m}{2}}^{m} 2^k \text{ replace k by the smallest value in each piece} \\
&= 1+2^{\frac{m}{2}+1}-1+\frac{4}{m^2}(2^{m+1}-2^{\frac{m}{2}+1}) \\
&\le 2\cdot 2^{\frac{m}{2}}+\frac{8}{m^2}\cdot 2^m
\end{aligned}
$$

Then

$$
\begin{aligned}
b_n^{-2}\sum_{k=1}^{n}\mathbb{E}\bar{X}_k^2 &= \frac{n\mathbb{E}\bar{X_1}^2}{b_n^2} \\
&\le \frac{2^m m^{\frac{3}{2}}(2\cdot 2^{\frac{m}{2}}+\frac{8}{m^2}\cdot 2^m)}{2^{2m}} \\
&= \frac{m^{3/2}}{2^{m/2-1}}+\frac{8}{m^2}
\end{aligned}
$$

Then $b_n^{-2}\sum_{k=1}^n \mathbb{E}\bar{X}_k^2 \to 0$ as $n\to\infty(m\to\infty$ as well)

$$
\begin{aligned}
a_n &= n\mathbb{E}(X_1 \mathbf{1}_{|X_1| \le 2^m}) \\
&= n(\sum_{k=1}^{m} (2^k - 1)p_k - p_0) \\
&= n(\sum_{k=m+1}^{\infty} \frac{1}{2^k k(k+1)} - \sum_{k=m+1}^{\infty} \frac{1}{k(k+1)}) \\
&= \sum_{k=1}^{n} \mathbb{P}(|X_k| > b_n) - \frac{n}{m+1}
\end{aligned}
$$

From (i), we know that $a_n \to -\frac{n}{m}$ as $n \to \infty$
Since $2^{m-1}(m-1)^{3/2} \le n \le 2^m m^{3/2}$, by taking $\log_2$ of each piece and divided by m we can get

$$
\frac{m - 1 + 3/2 \log_2(m-1)}{m} \le \frac{\log_2 n}{m} \le \frac{m + 3/2 \log_2 m}{m}
$$

where each side approaches to 1 as $n \to \infty$, it follows that $\frac{\log_2 n}{m} \to 1$ as $n \to \infty$. Then

$$
a_n \to -\frac{n}{\log_2 n} \text{ as } n \to \infty
$$

From the definition of b_n and $\frac{\log_2 n}{m} \to 1$ as $n \to \infty$, we can get

$$
2^{m-1} \le \frac{n}{m^{3/2}} \to \frac{n}{(\log_2 n)^{\frac{3}{2}}} \text{ as } n \to \infty
$$

that is

$$
b_n \to \frac{n}{(\log_2 n)^{\frac{3}{2}}} \text{ as } n \to \infty
$$

Finally, by the weak law for triangular arrays

$$
\frac{S_n - a_n}{b_n} \sim \frac{S_n + \frac{n}{\log_2 n}}{\frac{n}{(\log_2 n)^{\frac{3}{2}}}} \to 0 \quad \text{in probability}
$$

Since $(\log_2 n)^{-1/2} \to 0$ as $n \to \infty$

$$
\frac{S_n + n/\log_2 n}{n/\log_2 n} \to 0 \quad \text{in probability}
$$

Therefore,

$$
\frac{S_n}{n/\log_2 n} \to -1 \quad \text{in probability}
$$

2. (a) Write a formal probability model for the following scenario. You start with an amount of money X_0. At each time $n \geq 1$, you wager 2/3 of your money. Each time, your chance of winning is 2/3. If you win your fortune increases by the amount of the wager and if you lose, your fortune decreases by this amount.

   (b) Are you likely to be happy in the long term?

<u>*Solution*</u> :

(a) At time n, suppose the player have won Z_n times and lost $n - Z_n$ times, then the remaining fortune, denoted as X_n, is $X_n = (\frac{5}{3})^{Z_n}(\frac{1}{3})^{n-Z_n}X_0$, where $Z_n \sim \text{Binomial}(n, \frac{2}{3})$. We are looking for the probability $\mathbb{P}(X_n \geq X_0)$ to see if it's a fair game.

Or, we can consider each time separately. At time 1, the total fortune X_1 can be either $\frac{5}{3}X_0$ with probability $\mathbb{P}(X_1 = \frac{5}{3}X_0) = \frac{2}{3}$ or $\frac{1}{3}X_0$ with $\mathbb{P}(X_1 = \frac{1}{3}X_0) = \frac{1}{3}$. At time 2, the total fortune X_2 can be either $\frac{5}{3}X_1$ with $\mathbb{P}(X_2 = \frac{5}{3}X_1) = \frac{2}{3}$ or $\frac{1}{3}X_0$ with $\mathbb{P}(X_2 = \frac{1}{3}X_1) = \frac{1}{3}$. Then at time n, the total fortune $X_n = X_0\Pi_{i=1}^{n}C_i$ where $\mathbb{P}(C_i = \frac{5}{3}) = \frac{2}{3}$ and $\mathbb{P}(C_i = \frac{1}{3}) = \frac{1}{3}$.

(b) The expected value of convergence rate of the total X_n is

$$\begin{aligned}
\mathbb{E}\log[(\frac{5}{3})^{Z_n}(\frac{1}{3})^{n-Z_n}] &= \mathbb{E}[Z_n\log(\frac{5}{3}) + (n - Z_n)\log(\frac{1}{3})] \\
&= \mathbb{E}[n(\frac{Z_n}{n})\log(\frac{5}{3}) + n(1 - \frac{Z_n}{n})\log(\frac{1}{3})] \\
&= n(\frac{2}{3}\log\frac{5}{3} + \frac{1}{3}\log\frac{1}{3}) \approx -0.01114n
\end{aligned}$$

The third step achieved because $\mathbb{E}(\frac{Z_n}{n}) = \frac{2}{3}$.

For the single change of each time, $\mathbb{E}[\log C_i] = \frac{2}{3}\log\frac{5}{3} + \frac{1}{3}\log\frac{1}{3}$ and $\text{Var}[\log C_i] = \mathbb{E}[(\log C_i)^2] - \mathbb{E}^2[\log C_i] =$ some constant $< \infty$. Applying L^2 weal law, we can see that $\frac{\sum \log C_i}{n} \to \mathbb{E}[\log C_i]$, and thus $\sum \log C_i = n\mathbb{E}[\log C_i]$. The result is consistent with the expected value of convergence rate, suggesting that the change rate $(\frac{5}{3})^{Z_n}(\frac{1}{3})^{n-Z_n} = e^{-0.01114n} \to 0$ as $n \to \infty$, it's not likely to be happy in the long term.

3. Durrett Chapter 1 Exercise 5.3 (Monte Carlo integration):

(a) Let f be a Borel-measurable function on $[0,1]$ which is integrable, i.e., $\int_0^1 |f|\,dx < \infty$. Let $U_1, U_2, \ldots$ be independent and uniformly distributed on $[0,1]$ and let

$$I_n := \frac{1}{n}\sum_{k=1}^{n} f(U_k)\,.$$

Show that $I_n \to I := \int_0^1 f\,dx$ in probability.

(b) Suppose further that $\int_0^1 f(x)^2\,dx < \infty$. Use Chebyshev's inequality to estimate $\mathbb{P}(|I_n - I| > an^{-1/2})$.

<u>*Solution*</u> :

(a) Since all U_i are iid and f is measurable, all $f(U_i)$ are also iid.
Let $\mu_n = \mathbb{E}[f(U_n)] = \int_0^1 f(x)dx$ as $U_n \sim \text{Uniform}(0,1)$, and $\mathbb{E}[f(U_n)] = \int_0^1 f(x)dx = \int_0^1 |f(x)|dx < \infty$. Let $S_n = \sum_{k=1}^n f(U_k)$, by WLLN, $\frac{S_n}{n} \to \mu_n$ in probability, that is,

$$I_n = \frac{1}{n}\sum_{k=1}^{n} f(U_k) \to \int_0^1 f dx = I$$

(b) Using Chebyshev's inequality, we have

$$\begin{aligned}
\mathbb{P}(|I_n - I| > an^{-1/2}) \le \frac{\mathbb{E}[(I_n - I)^2]}{(an^{\frac{1}{2}})^2} &= \frac{n\mathbb{E}[(I_n - I)^2]}{a^2} \\
&= \frac{n}{a^2}\text{Var}(I_n) \\
&= \frac{1}{a^2}\text{Var}(f(U_i)) \\
&= \frac{1}{a^2}(\mathbb{E}[f(U_i)^2] - \mathbb{E}^2[f(U_i)]) \\
&= \frac{1}{a^2}[\int_0^1 f^2 dx - (\int_0^1 f dx)^2] < \infty
\end{aligned}$$