# MATH 6480/STAT 9300/AMCS 6481, Fall 2023, Homework Set X (two pages)

1. Let $\{X_n\}$ be IID random variables on some probability space and let $\mathcal{F}_n := \sigma(X_1, \ldots, X_n)$. Which of the following are stopping times? Give a one-sentence reason for each.

   (a) $\inf\{k : X_{k-1} \in A\}$

   (b) $\inf\{k : X_{k+1} \in A\}$

   (c) $\sup\{k : X_1, \ldots, X_k \geq 0\}$

   (d) $\sup\{k : X_1 = X_k \notin \{X_2, \cdots, X_{k-1}\}$

   <u>*solution*</u>

   Recall the definition of stopping time is $\{\tau \leq n\} \subseteq \mathcal{F}_n$ or $\{\tau = n\} \subseteq \mathcal{F}_n$

   (a) It is a stopping time, as the set includes all information of $X_1$ up to $X_k$. The set can be determined for $\tau \leq n$
   (b) It is not a stopping time, as we're not able to infer if $X_{k+1} \in A$ based on only information of previous R.V.s $X_1, \cdots, X_k$.
   (c) It is not a stopping time. Consider the process of finding $k$ satisfying sup $\{k : X_1 \cdots, X_k \geqslant 0\}$ as an iteration with index $i$. Initialize $i = 1$, if $X_1 \geqslant 0$, then continue to check $X_2$, if the condition $X_i \geqslant 0$ success then further $X_{i+1}$ till we meet $m$ s.t. $X_m < 0$, then $\sup\{k : X_1, \cdots, X_k \geq 0\} = m - 1$. Besides $X_1 \ldots X_k$, we also need the information of $X_{k+1}$ to determine $k$. $\{\tau \leq n\} \not\subset \mathcal{F}_n$, so it's not a stopping time.
   (d) It is a stopping time. The set describes the first labeling $k$ s.t. $X_1 = X_k$ i.e. $X_1 \neq X_2, X_1 \neq X_3, \cdots, X_1 \neq X_{k-1}$ but $X_1 = X_k$. There is $\{\tau = n\} \subseteq \mathcal{F}_n$, which implies that it's a stopping time

2. Let $(\Omega, \mathcal{F}, \mathbf{P}, \{\mathcal{F}_n\})$ be as in the previous problems and let $\tau$ denote the stopping time $\inf\{k \geq 2 : X_k > X_{k-1}\}$. Which of the following events or random variables are in $\mathcal{F}_\tau$? Again, give a one-sentence reason.

   (a) $\{X_2 \leq X_1\}$

   (b) $\{X_3 \leq X_2\}$

   (c) $\{S_{\tau-1} > 10\}$ (as usual, $S_n$ are the partial sums)

   (d) $\inf\{k \geq 2 : X_k \leq X_{k-1}\}$

   _solution_ Recall that $A \in \mathcal{F}_\tau$ iff $A \cap \{\tau = n\} \subseteq \mathcal{F}_n$ for any $n$.

   (a) The event is in $\mathcal{F}_\tau$. $\tau \geq 2$ so any $\mathcal{F}_\tau$ contains information of $X_1$ and $X_2$. A rigorous proof: The set $\{X_2 \leq X_1\}$ holds iff $\tau > 2$. When $n = 1$, we have $\{X_3 \leq X_2\} \cap \{\tau = 2\} = \emptyset \subseteq \mathcal{F}_n$; when $n \geq 2$, we have $\{\tau > 2\} \cap \{\tau = n\} = \{\tau = n\} \subseteq \mathcal{F}_n$.
   (b) The event is not in $\mathcal{F}_\tau$. Consider when $n = 2$, $\{X_3 \leq X_2\} \cap \{\tau = 2\} = \{X_3 \leq X_2\} \not\subset \mathcal{F}_2 = \sigma(X_1 X_2)$ as $\mathcal{F}_2$ only contains information of $X_1$ and $X_2$, the information of $X_3$ is not included.
   (c) The event is in $\mathcal{F}_\tau$, since $\{S_{\tau-1} > 10\} \cap \{\tau = n\} = \{X_1 + \cdots + X_{n-1} > 10\} \subseteq \mathcal{F}_n$. $\mathcal{F}_n$ contains all information from $X_1$ up to $X_n$ and thus contains the information of $S_{n-1}$.
   (d) It is not in $\mathcal{F}_\tau$. Consider $X_1 = 1$, $X_2 = 3$ and $X_3 = 2$, the infimum of $\{k \geq 2 : X_k \leq X_{k-1}\}$ is 3 in this case. However, $\tau = 2$ in this example, the information of $X_3$ is not included. As a result, we cannot say $\inf\{k \geq 2 : X_k \leq X_{k-1}\}$ is in $\mathcal{F}_\tau$ as $\mathcal{F}_\tau$ has missed out information which is which is necessary for $\inf\{k \geq 2 : X_k \leq X_{k-1}\}$ .

3. Let $\{X_n\}$ be IID with $X_1^+ = \max\{0, X_1\}$ and $\mathbf{E}X_1^+ < \infty$, define

$$Y_n := \max_{1 \le k \le n} X_k - cn\,.$$

Informally, we get credit for the highest of $n$ values, but we have to pay $c$ per value we have looked at. $(i)$ Let $T_a := \inf\{n : X_n > a\}$, let $p(a) := \mathbf{P}(X_n > a)$ and compute $\mathbf{E}Y_{T_a}$. $(ii)$ Let $\alpha = \alpha(c)$, possibly negative, be the unique solution to $\mathbf{E}(X_1 - \alpha)^+ = c$. Show that $\mathbf{E}Y_{T_\alpha} = \alpha$ and use the inequality

$$Y_n \le \alpha + \sum_{k=1}^{n} \big((X_k - \alpha)^+ - c\big)$$

for $n \ge 1$ to conclude that if $\tau > 1$ is a stopping time with $\mathbf{E}\tau < \infty$ then $\mathbf{E}Y_\tau \le \alpha$. In other words, the stopping time $\tau = T_\alpha$ achieves the best expecation of $Y_\tau$ over all stopping times, and this expectation is $\alpha$. **Note:** The above analysis assumes you have to view at least one value, $X_1$. If not, then when $\alpha < 0$, the optimal strategy is not to play at all!

*Solution*

(i) From the definition of $Y_n$ and $T_a, \forall k < T_a, X_k \leqslant a$ and $X_{T_a} > a$. Then $\mathbb{E}Y_{T_a} = \mathbb{E}\left(\max_{1\le k\le T_a} X_k - cT_a\right) = \mathbb{E}\left(X_{T_a} - cT_a\right) = \mathbb{E}X_{T_a} - c\mathbb{E}\left(T_a\right)$

Consider $T_a$ as the number of trials to get the first success, indicating that it follows geometric distribution, then $\mathbb{E}(T_a) = \frac{1}{p(a)}$, where $p(a) := \mathbb{P}(X_n > a)$

$$\begin{aligned}
\mathbb{E}\left(T_a\right) &= \sum_{i=1}^{\infty} \mathbb{E}\left(T_a \mid T_a = i\right)\mathbb{P}\left(T_a = i\right) \\
&= \sum_{i=1}^{\infty} i(1-p)^{i-1}p = p\sum_{i=1}^{\infty} i(1-p)^{i-1} \\
&= p \cdot \frac{d}{dp}\left(-\frac{1}{p}\right) = \frac{1}{p},
\end{aligned}$$

Therefore,

$$\mathbb{E}Y_{T_a} = \mathbb{E}X_{T_a} - c\mathbb{E}\left(T_a\right) = \mathbb{E}X_{T_a} - \frac{c}{p(a)}$$

since $X_{T_a} > a$ and $X_i$s are iid, we can further write $\mathbb{E}X_{T_a} = \mathbb{E}(X_1|X_1 > a)$

(ii) Expanding the formula $\mathbb{E}(X_1-\alpha)^+$, we have

$$\begin{aligned}
\mathbb{E}\left(X_1-\alpha\right)^+ &= \mathbb{E}\left(\left(X_1-\alpha\right)^+ \mid X_1>\alpha\right)\mathbb{P}\left(X_1>\alpha\right)+\mathbb{E}\left(\left(X_1-\alpha\right)^+ \mid X_1\leq\alpha\right)\mathbb{P}\left(X_1\leq\alpha\right)\\
&= \mathbb{E}\left(X_1-\alpha \mid X_1>\alpha\right)\mathbb{P}\left(X_1>\alpha\right)+\mathbb{E}(0)\\
&= \left[\mathbb{E}\left(X_1 \mid X_1>\alpha\right)-\alpha\right]p(\alpha)
\end{aligned}$$

Solving the equation $\left[\mathbb{E}\left(X_1 \mid X_1>\alpha\right)-\alpha\right]p(\alpha)=\mathbb{E}(X_1-\alpha)^+=c$, we can get

$$\mathbb{E}\left(X_1 \mid X_1>\alpha\right)=\alpha+\frac{c}{p(\alpha)}$$

Recall that $\mathbb{E}X_{T_a}$ is a conditional expected value with the construction $X_{T_a}>a$, and $X_i$s are iid R.V.s, then

$$\mathbb{E}\left(X_{T\alpha}\right)=\mathbb{E}\left(X_1 \mid X_1>\alpha\right)=\alpha+\frac{c}{p(\alpha)}$$

Substituting $\mathbb{E}(X_{T_\tau})$ in the last equation in part (a) gives

$$\mathbb{E}\left(Y_{T_\alpha}\right)=\mathbb{E}\left(X_{T_\alpha}\right)-\frac{c}{p(\alpha)}=\alpha+\frac{c}{p(\alpha)}-\frac{c}{p(\alpha)}=\alpha$$

As for the inequality, given $\mathbb{E}\tau<\infty$ taking expected value of both sides then we have

$$\mathbb{E}\left(Y_n\right)\leq\mathbb{E}(\alpha+\sum_{k=1}^{n}\left(\left(X_k-\alpha\right)^+-c\right))=\alpha+\sum_{k=1}^{n}\left(\mathbb{E}\left(X_k-\alpha\right)^+-c\right)=\alpha$$

for any $n\geq1$, therefore if $\tau>1$ is a stopping time with $\mathbb{E}\tau<\infty$ then $\mathbb{E}Y_\tau\leq\alpha$.

4. Let $\{S_n\}$ be a simple random walk in one dimension, with $S_0 = 0$, and let

$$\tau = \tau_{[0,5]^c} = \inf\{n : S_n \notin [0,5]\}$$

be the first time the random walk exits the set $\{0,1,2,3,4,5\}$. Evaluate $\mathbf{E}S_{\tau-1}$.

<u>*Solution*</u>

We start with $\mathbb{E}S_{\tau-1} = \mathbb{E}S_\tau - \mathbb{E}X_\tau$, then consider $\mathbb{E}S_\tau$ and $\mathbb{E}X_\tau$ separately.
(1) Notice that $\mathbb{P}(\tau = \infty) = 0$ almost surely, i.e., the hitting time $\tau$ is almost surely finite, and thus $\mathbb{E}(\tau) < \infty$. By Wald's identity, $S_\tau = (\mathbb{E}X_1)(\mathbb{E}\tau) = 0$
(2) Let $p := \mathbb{P}(S_\tau = 6) = 1 - \mathbb{P}(S_\tau = -1)$, then there is $\mathbb{E}S_\tau = 6p - (1-p) = 0$. Solving it we have $p = \frac{1}{7}$, then $\mathbb{P}(S_\tau = 6) = \frac{1}{7}$ and $\mathbb{P}(S_\tau = -1) = \frac{6}{7}$. Consider the expected value of $X_\tau$, which can only be 1 or $-1$ in a 1d simple random walk. $X_\tau = 1$ has a corresponding escaping value $S_\tau = 6$, i.e. $\{X_\tau = 1\} = \{S_\tau = 6\}$, while $X_\tau = -1$ is related to $S_\tau = -1$. So $\mathbb{P}(X_\tau = -1) = \mathbb{P}(S_\tau = -1) = \frac{6}{7}$, and $\mathbb{P}(X_\tau = 1) = \mathbb{P}(S_\tau = 6) = \frac{1}{7}$. Therefore, $\mathbb{E}X_\tau = 1 \times \frac{1}{7} - 1 \times \frac{6}{7} = -\frac{5}{7}$. Overall,

$$\mathbb{E}S_{\tau-1} = \mathbb{E}S - \mathbb{E}X_\tau = 0 - (-\frac{5}{7}) = \frac{5}{7}$$