MATH 6480/STAT 9300/AMCS 6481, Fall 2023, Homework 5

1. **If it must happen once, it happens infinitely often.**
Let $\{A_n\}$ be a sequence of independent events with $\mathbb{P}(A_n) < 1$ for all n. Show that $\mathbb{P}(\bigcup_n A_n) = 1$ implies $\mathbb{P}(A_n \text{ i.o.}) = 1$.

<u>*Solution*</u> :

Given $\mathbb{P}(\bigcup_n A_n) = 1$, we have

$$\mathbb{P}(\bigcap_n A_n^C) = 1 - \mathbb{P}(\bigcup_n A_n) = 0$$

Given $\{A_n\}$ independent, $\{A_n^C\}$ are independent as well. Then

$$\mathbb{P}(\bigcap_n A_n^C) = \Pi_n \mathbb{P}(A_n^C) = \Pi_n (1 - \mathbb{P}(A_n))$$

Since $\mathbb{P}(A_n) < 1$ for all n, we have

$$\mathbb{P}(\bigcap_{n=1}^{m-1} A_n^C) = \Pi_{n=1}^{m-1}(1 - \mathbb{P}(A_n)) > 0$$

for $m = 2, 3, \cdots$. Then for $m = 1, 2, 3, \cdots$, the tail part

$$\mathbb{P}(\bigcap_{n=m}^{\infty} A_n^C) = 0$$

To prove $\mathbb{P}(A_n \text{ i.o.}) = 1$, it's sufficient to prove $\mathbb{P}(A_n^C \text{ eventually}) = 0$

$$\mathbb{P}(A_n^C \text{ eventually}) = \mathbb{P}(\liminf_{n\to\infty} A_n^C) = \lim_{m\to\infty} \mathbb{P}(\bigcap_{n\geq m} A_n^C) = 0$$

Or

$$\mathbb{P}(A_n^C \text{ eventually}) = \mathbb{P}(\liminf_{n\to\infty} A_n^C) = \mathbb{P}(\bigcup_{m=1}^{\infty} \bigcap_{n\geq m} A_n^C) \leq \sum_{m=1}^{\infty} \mathbb{P}(\bigcap_{n\geq m} A_n^C) = 0$$

2. Durrett chapter 1 exercise 7.4 (an investment problem). Fix a real number $a > 0$. At each time $n = 0, 1, 2, \dots$ you can buy bonds for \$1 that are worth \$a at the end of the year (deterministically) or you can buy stock that is worth a random amount $Y_n \geq 0$. Suppose you always invest a fixed proportion p of your wealth in bonds and the rest in stock (there is only one type of stock). Let X_n denote your wealth at the beginning of year n. Then $X_{n+1} = X_n \cdot (pa + (1-p)Y_{n+1})$. Assume that $\{Y_n\}$ are IID and that both Y_n and $1/Y_n$ have finite second moments.

   (a) Show that $n^{-1} \log X_n \to c$ almost surely, where c depends on p.

   (b) Show that c is a concave function of p.

   (c) By investigating $c'(0)$ and $c'(1)$, give conditions on the law of Y_n that guarantee that the optimal choice of p is strictly between zero and one.

   (d) Suppose Y_n is 1 or 4 with probability 1/2 each. Find the optimal value of p.

   Solution : (a) In this problem, a single case that $p = 0$ and $Y_i = 0$ is exempted to avoid integration issues.

$$\begin{aligned} X_{n+1} &= X_n(pa + (1-p)Y_{n+1}) \\ &= X_{n-1}(pa + (1-p)Y_n)(pa + (1-p)Y_{n+1}) \\ &= X_0 \Pi_{i=1}^{n+1}(pa + (1-p)Y_i) \end{aligned}$$

   Then $\log X_n = X_0 + \sum_{i=1}^n \log(pa + (1-p)Y_i)$ where $\frac{X_0}{n} \to 0$ a.s.
   Given $\mathbb{E}Y_n^2 < \infty$ and $\mathbb{E}(1/Y_n^2) < \infty$, we have $\mathbb{E}Y_n < \infty$ and $\mathbb{E}\frac{1}{Y_n} < \infty$ by Jensen's inequality. And by our assumption of $Y_n \geq 0$, $a > 0$ and $p \in [0,1]$, the wealth $(pa + (1-p)Y_i) > 0$. Consider the inequality $\log x < x$, then

$$\mathbb{E}|\log(pa + (1-p)Y_i| < \mathbb{E}|\frac{1}{(pa + (1-p)Y_i}\mathbf{1}_{Y_i<1}| + \mathbb{E}|(pa + (1-p)Y_i\mathbf{1}_{Y_i>1}| < \infty$$

   By SLLN,

$$\frac{\log X_n}{n} \to \mathbb{E}(\log(pa + (1-p)Y_i) \quad \text{a.s}$$

   Denote $c(p) = \mathbb{E}(\log(pa + (1-p)Y_i)$.

   (b) By Leibniz integral rule,

$$c'(p) = \int \frac{d}{dp}(\log(pa + (1-p)Y_i)d\mathbb{P} = \mathbb{E}(\frac{a - Y_i}{pa + (1-p)Y_i})$$

   And

$$c''(p) = -\mathbb{E}[(\frac{a - Y_i}{pa + (1-p)Y_i})^2] < 0$$

   indicating that c is a concave function of p.

(c) We have $c'(0) = \mathbb{E}(\frac{a-Y_i}{Y_i}) = -1 + a\mathbb{E}(\frac{1}{Y_i})$ and $c'(1) = \mathbb{E}(\frac{a-Y_i}{a}) = 1 - \frac{1}{a}\mathbb{E}(Y_i)$. To guarantee the optimal choice of p in $(0,1)$, we need the maxima of $c(p)$ occurs in $(0,1)$, then $c'(0)$ and $c'(1)$ must satisfy $c'(0) > 0$ and $c'(1) < 0$, gives conditions on the law Y_n:

$$\forall a > 0, \;\; \mathbb{E}(Y_i) > a, \;\; \mathbb{E}(\frac{1}{Y_i}) > \frac{1}{a}$$

(d) Given the description, $\mathbb{E}(Y_i) = \frac{1}{2} + 2 = \frac{5}{2}$ and $\mathbb{E}(\frac{1}{Y_i}) = \frac{1}{2} + \frac{1}{4}\frac{1}{2} = \frac{5}{8}$
From (c), if $\frac{8}{5} < a < \frac{5}{2}$, then p is optimal in $(0,1)$ where

$$\frac{1}{2}\frac{a-1}{pa+1-p} + \frac{1}{2}\frac{a-4}{pa+4(1-p)} = 0$$

By separating a and p, and cross multiplying, we have

$$(a-1)(a-4)p + 4(a-1) = p(a-1)(4-a) + (4-a)$$

And solving the equation we have

$$p = \frac{8-5a}{2(a-1)(a-4)}$$

The optimal value of p is 0 when $a \leq \frac{8}{5}$, and the optimal value of p is 1 when $a \geq \frac{5}{2}$,

3. Let $\{X_n\}$ and $\{Y_n\}$ be sequences of real random variables. Suppose that $X_n \to X$ in distribution and $Y_n \to Y$ in distribution for some random variables X and Y. Does it follow that $(X_n, Y_n) \to (X, Y)$ in distribution? Now suppose instead that $X_n \to X$ in probability and $Y_n \to Y$ in probability for some random variables X and Y. Does it follow that $(X_n, Y_n) \to (X, Y)$ in probability?

<u>*Solution*</u> :

(a) No. Take the continuous function $f(x, y) = x + y$. If $(X_n, Y_n) \to (X, Y)$, by continuous mapping theorem, then there should be $X_n + Y_n \to X + Y$. A counterexample presented as below

Take $X_n \sim uniform(0, 1)$, and take $X_n + Y_n = 0$. Then $Y_n \to uniform(-1, 0)$ in distribution. However, $X_n + Y_n$ is not convergent to $X + Y$ in distribution, since $X + Y$, where $X \sim uniform(0, 1)$ and $Y \sim uniform(-1, 0)$, is not always 0 if X_n and Y_n are independent

(b) Yes. Start with $d([X_n, Y_n), (X, Y)] \leq d(X_n, X) + d(Y_n, Y)$. If $d([X_n, Y_n), (X, Y)] > \epsilon$, there is at least one of $d(X_n, X)$ and $d(Y_n, Y)$ required to be larger than $\frac{\epsilon}{2}$.

Consider sets $A = \{\omega : d[(X_n, Y_n), (X, Y)] > \epsilon\}$ and $B = \{\omega : d(X_n, X) > \frac{\epsilon}{2} \cup d(Y_n, Y) > \frac{\epsilon}{2}\}$, we see $A \subseteq B$ from the last statement. Then

$$\begin{aligned}\mathbb{P}(d[(X_n, Y_n), (X, Y)] > \epsilon) &\leq \mathbb{P}(d(X_n, X) > \frac{\epsilon}{2} \cup d(Y_n, Y) > \frac{\epsilon}{2}) \\ &\leq \mathbb{P}(d(X_n, X) > \frac{\epsilon}{2}) + \mathbb{P}(d(Y_n, Y) > \frac{\epsilon}{2})\end{aligned}$$

Given $X_n \to X$ in probability and $Y_n \to Y$ in probability, we have

$$\mathbb{P}(d(X_n, X) > \frac{\epsilon}{2}) + \mathbb{P}(d(Y_n, Y) > \frac{\epsilon}{2}) \to 0$$

and thus

$$\mathbb{P}(d[(X_n, Y_n), (X, Y)] > \epsilon) \leq \mathbb{P}(d(X_n, X) > \frac{\epsilon}{2} \cup d(Y_n, Y) > \frac{\epsilon}{2}) \to 0$$

We thus conclude that $(X_n, Y_n) \to (X, Y)$ in probability

4. Say which of the following familes of random variables is tight (do not give a proof, just yes or no and a brief reason):

   (a) $X_n = n$ with probability $1/n$ and zero otherwise;

   (b) The set of all random variables having mean zero and variance one.

   (c) The family of all bounded random variables on the space $(\mathbb{R}, \mathcal{B})$;

Solution : (a) The set of $X_n = n$ with probability $1/n$ and zero otherwise is tight.

$$\limsup_{M\to\infty} \mathbb{P}(|X_n| \geq M) = \limsup_{M\to\infty} \mathbb{P}(n \geq M) = \limsup_{M\to\infty} \frac{1}{M}\frac{1}{M+1}\frac{1}{M+2}\cdots = 0$$

(b) The set $\{X_n\}$ with $\mathbb{E}X_n = 0$ and $\mathrm{Var}X_n = 1$ is tight.

$$\mathbb{P}(|X_n| \geq M) \leq \frac{\mathbb{E}(|X_n|^2)}{M^2} = \frac{\mathrm{Var}X_n + (\mathbb{E}X_n)^2}{M^2} = \frac{1}{M^2}$$

Then $\limsup_{M\to\infty} \mathbb{P}(|X_n| \geq M) = 0$, the given set is tight.

(c) The set $\{X_n\}$ with bounded R.V.s is not tight. A counterexample is presented as below
The condition that X_n is bounded can be precisely written as $\forall \epsilon > 0,\ \exists M > 0$ s.t $\mathbb{P}(|X_n| > M) < \epsilon$. Notice that R.V.s are not uniformly bounded. Consider $X_1 = 1, X_2 = 2, \cdots, X_i = i, \cdots$ is a countable set of bounded R.V.s, it can be observed that $\forall M > 0,$ and for $i \geq M+1, \mathbb{P}(|X_i| > M) = 1$, which indicates that the set $\{X_n\}$ with bounded R.V.s is not tight.