MATH 6480/STAT 9300/AMCS 6481, Fall 2023, Homework Set 2

1. Let $X_1, X_2, \ldots$ be random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let L be the random variable defined by

$$L(\omega) = \inf\{t : \text{ only finitely many values } X_1, X_2, \ldots \text{ are greater than } t\}\,.$$

Prove or disprove:

$$\mathbb{P}\left(L < \frac{1}{7}\right) \leq \liminf_{n\to\infty} \mathbb{P}\left(X_n \leq \frac{1}{7}\right)\,.$$

<u>*Solution*</u> : Take $A_n = \{\omega : X_n(\omega) \leq \frac{1}{7}\}$

$$\mathbb{P}(L < \frac{1}{7}) = \mathbb{E}(\mathbf{1}_{L<\frac{1}{7}}) \leq \mathbb{E}(\mathbf{1}_{\liminf_{n\to\infty} A_n}) \leq \mathbb{E}(\liminf_{n\to\infty} \mathbf{1}_{A_n}) \leq \liminf_{n\to\infty} \mathbb{E}(\mathbf{1}_{A_n}) = \liminf_{n\to\infty} \mathbb{P}(X_n \leq \frac{1}{7})$$

Claim 1: $\{L < \frac{1}{7}\} \subseteq \{\liminf_{n\to\infty} A_n\}$

Take $\omega \in \{L < \frac{1}{7}\}$, there exists $\epsilon > 0$ such that there are finitely many $X_i(\omega)$ greater than $\frac{1}{7} - \epsilon$, equivalently speaking, there exists N such that $\forall n > N, X_n(\omega) \leq \frac{1}{7} - \epsilon$, which implies $\omega \in \{\liminf_{n\to\infty} A_n\}$. Therefore, we have $\{L < \frac{1}{7}\} \subseteq \{\liminf_{n\to\infty} A_n\}$ and thus $\mathbb{P}(\{L < \frac{1}{7}\}) \leq \mathbb{P}(\{\liminf_{n\to\infty} A_n\})$, that is, $\mathbb{E}(\mathbf{1}_{L<\frac{1}{7}}) \leq \mathbb{E}(\mathbf{1}_{\liminf_{n\to\infty} A_n})$.

Claim 2: $\mathbf{1}_{\liminf_{n\to\infty} A_n} \leq \liminf_{n\to\infty} \mathbf{1}_{A_n}$

$\mathbf{1}_{\liminf_{n\to\infty} A_n} = 1$ iff there exists N that for any $n > N$, we have $X_n \leq \frac{1}{7}$, which implies $\liminf_{n\to\infty} \mathbf{1}_{A_n} = 1$. Thus, we have $\mathbb{E}(\mathbf{1}_{\liminf_{n\to\infty} A_n}) \leq \mathbb{E}(\liminf_{n\to\infty} \mathbf{1}_{A_n})$.

Claim 3: $\mathbb{E}(\liminf_{n\to\infty} \mathbf{1}_{A_n}) \leq \liminf_{n\to\infty} \mathbb{E}(\mathbf{1}_{A_n})$. we can easily see it by Fatou's lemma.

2. Let Ω be the set of permutations of $\{1, \ldots, n\}$, let $\mathcal{F}$ be the σ-field of all subsets and let $\mathbb{P}$ be the uniform probability measure. Let $N : \Omega \to \mathbb{Z}^+$ count the number of fixed points.

   (a) Compute $\mathbb{E}N$ and $\mathrm{Var}(N)$ exactly.

   (b) Find a formula for $\mathbb{E}(N)_k$ where $(m)_k$ denotes the "falling factorial" product $m(m-1)\cdots(m-k+1)$.

   (c) Compute $\mathbb{E}(X)_k$ where X is a Poisson variable of mean 1.

<u>*Solution*</u> : (a) Denote $\mathbf{1}_i$ as the indicator of the i-th point in Ω that $\mathbf{1}_i = 1$ if the i-th point is fixed.

$$\mathbb{E}(N) = \mathbb{E}(\sum_{i=1}^{n} \mathbf{1}_i) = \sum_{i=1}^{n} \mathbb{E}(\mathbf{1}_i) = \sum_{t=i}^{n} \mathbb{P}(\text{the i-th element is fixed}) = n \cdot \frac{(n-1)!}{n!} = 1$$

$$\mathbb{E}(N^2) = \mathbb{E}[(\sum_{i=1}^{n} \mathbf{1}_i)^2] = \sum_{i=1}^{n} \mathbb{E}(\mathbf{1}_i^2) + 2\sum_{i<j} \mathbb{E}(\mathbf{1}_i \mathbf{1}_j) = 1 + 2 \cdot \binom{n}{2} \frac{(n-2)!}{n!} = 1 + 2 \cdot \frac{n!}{(n-2)!2!} \cdot \frac{(n-2)!}{n!} = 2$$

where $\mathbb{E}(\mathbf{1}_i \mathbf{1}_j) = \mathbb{P}(\text{both the i-th\&j-th element are fixed})$

Thus, the variance is $\mathrm{Var}(N) = \mathbb{E}(N^2) - \mathbb{E}^2(N) = 1$

(b)

$$\begin{aligned}
\mathbb{E}[(N)_k] = \mathbb{E}[N(N-1)\cdots(N-k+1)] &= k!\mathbb{E}[\binom{n}{k}] \\
&= k!\mathbb{E}(\sum_{A \subseteq \Omega} \mathbf{1}_{A \text{ is fixed}}) \\
&= k! \sum_{A \subseteq \Omega} \mathbb{P}(A \text{ is fixed}) \\
&= k! \binom{n}{k} \cdot \frac{(n-k)!}{n!} = 1
\end{aligned}$$

Consider $\mathbb{E}[\binom{n}{k}]$ as the expected number of k elements among all N fixed numbers. Take A as a subset of Ω contains k elements of fixed numbers, then the probability of all elements in A are fixed is $\frac{(n-k)!}{n!}$ and there are number of $\binom{n}{k}$ such of k-tuples.
We can consider (b) as a generalization of (a), in other words, (a) is a special case of (b) by taking k=1.

(c) As $X \sim Poission(1)$, i.e., $\lambda = \mathbb{E}(X) = 1$, we have $\mathbb{P}(X = i) = \frac{e^{-1}}{i!}$

$$\begin{aligned}
\mathbb{E}[(X)_k] &= \mathbb{E}[X(X-1)\cdots(X-k+1)] \\
&= \sum_{i=-\infty}^{\infty} \mathbb{E}[X(X-1)\cdots(X-k+1)|X=i]\mathbb{P}(X=i) \\
&= \sum_{i=k}^{\infty} i(i-1)\cdots(i-k+1)\frac{e^{-1}}{i!} \text{ as } i(i-1)\cdots(i-k+1) = 0 \text{ for } i < k \\
&= \sum_{i=k}^{\infty} \frac{e^{-1}}{(i-k)!} \\
&= \sum_{j=0}^{\infty} \frac{e^{-1}}{j!} \text{ by taking } j = i - k \\
&= 1 \text{ (total mass of poission distribution)}
\end{aligned}$$

Generally speaking,

$$\begin{aligned}
\mathbb{E}[(X)_k] &= \mathbb{E}[X(X-1)\cdots(X-k+1)] \\
&= \sum_{i=-\infty}^{\infty} \mathbb{E}[X(X-1)\cdots(X-k+1)|X=i]\mathbb{P}(X=i) \\
&= \sum_{i=k}^{\infty} i(i-1)\cdots(i-k+1)\frac{e^{-\lambda}\lambda^i}{i!} = \sum_{i=k}^{\infty} \frac{e^{-\lambda}\lambda^i}{(i-k)!} \\
&= \lambda^k \sum_{i=k}^{\infty} \frac{e^{-\lambda}\lambda^{i-k}}{(i-k)!} = \lambda^k \sum_{j=0}^{\infty} \frac{e^{-\lambda}\lambda^j}{j!} = \lambda^k
\end{aligned}$$

3. Fix $n \geq 5$ and let $\{U_k : 1 \leq k \leq N\}$ be IID Uniform $[0,1]$ random variables. Let G_k be the event of a local maximum at site k, in other words the event that $U_k \geq \max\{U_{k+1}, U_{k-1}\}$; here we interpret the indices mod N. Let $Z_N := \sum_{k=1}^{N} \mathbf{1}_{G_k}$ count the number of local maxima. Compute $\mathbb{E}Z_N$ and $\mathrm{Var}(Z_N)$.

<u>*Solution*</u> :

$$\mathbb{E}(Z_N) = \mathbb{E}(\sum_{k=1}^{N} \mathbf{1}_{G_k}) = \sum_{k=1}^{N} \mathbb{E}(\mathbf{1}_{G_k}) = \sum_{k=1}^{N} \frac{1}{3} = \frac{N}{3}$$

$$\begin{aligned}
\mathbb{E}(Z_N^2) &= \mathbb{E}[(\sum_{k=1}^{N} \mathbf{1}_{G_k})^2] = \sum_{k=1}^{N} \mathbb{E}(\mathbf{1}_{G_k}^2) + \sum_{i \neq j} \mathbb{E}(\mathbf{1}_{G_i} \mathbf{1}_{G_j}) \\
&= \frac{N}{3} + \sum_{|i-j|=1} \mathbb{P}(G_i \cap G_j) + \sum_{|i-j|=2} \mathbb{P}(G_i \cap G_j) + \sum_{|i-j|\geq 3} \mathbb{P}(G_i \cap G_j) \\
&= \frac{N}{3} + 2N \cdot 0 + 2N \cdot \frac{2}{15} + (N^2 - 5N) \cdot \frac{1}{9} = \frac{2N}{45} + \frac{N^2}{9}
\end{aligned}$$

where $\mathbb{P}(G_i \cap G_j) = 0$ when $|i-j| = 1$ as U_k is a continuous random variable the probability of both adjacency sites local maxima is 0, and apply the probability to all $2N$ pairs of (G_i, G_j) with distance 1. For each i, there are 2 such j that $|i-j| = 1$, then there are $2N$ pairs in total.
Then we move to the case $|i-j| = 2$. For any 5 consecutive sites $(U_k, U_{k+1}, U_{k+2}, U_{k+3}, U_{k+4})$, there are two possible scenarios that makes U_{k+1} and U_{k+3} local maxima: i) U_{k+1} and U_{k+3} are maxima among all five sites. There are 2!3! possible permutations in total. ii) the second large number is at the boundary next to the largest number, for example, (*, 3, *, 5, 4). There are 4 possible such permutations. Therefore, $\mathbb{P}(G_i \cap G_j) = \frac{2!3!+4}{5!} = \frac{2}{15}$ when $|i-j| = 2$. And similar to the case $|i-j| = 2$, there are $2N$ pairs of $|i-j| = 2$ in total.
As for the case $|i-j| \geq 3$, G_is are independent that $\mathbb{P}(G_i \cap G_j) = (\frac{1}{3})^2 = \frac{1}{9}$. And by partition, there are $2 \cdot \binom{N}{2} - 2N - 2N = N^2 - 5N$ such pairs in total. Another approach to count the total pair number of $|i-j| \geq 3$ is considering $\sum_{i \neq j} = \sum_{k=3}^{N-3} \sum_{i-j=k}$, which is also $N^2 - 5N$

$$\mathrm{Var}(Z_N) = \mathbb{E}(Z_N^2) - \mathbb{E}(Z_N)^2 = \frac{2N}{45}$$