MATH 6480/STAT 9300/AMCS 6481, Fall 2023, Homework Set 6

1. Prove or find a counterexample: if ϕ is a characteristic function then there is another characteristic function ψ with $\psi^2 = \phi$.

   <u>*solution*</u>

   A counterexample is $\phi(t) = \cos t$, which has corresponding probability measure μ for the random variable X such that,

   $$\mathbb{P}(x = 1) = \frac{1}{2}, \mathbb{P}(X = -1) = \frac{1}{2}$$

   Checking the characteristic function, we get

   $$\begin{aligned} \phi(t) = \mathbb{E}e^{itx} &= \frac{1}{2}e^{it} + \frac{1}{2}e^{-it} \\ &= \frac{1}{2}(\cos t + i \sin t) + \frac{1}{2}(\cos t - i \sin t) \\ &= \cos t \end{aligned}$$

   Suppose there is a a random variable Y with law ν and characteristic function ψ satisfying $\psi^2 = \phi$. There are random variables Y_1 and Y_2 follows law ν such that

   $$\phi(t) = \mathbb{E}^{itx} = \mathbb{E}(e^{it(y_1+y_2)}) = \mathbb{E}(e^{ity_1})\mathbb{E}(e^{ity_2}) = \psi^2(t)$$

   Consider the support of μ, $\text{supp}(\mu) = \{-1, 1\}$, we need the support of ν must be a set S such that $S + S$ has cardinality 2 to achieve $\psi^2 = \phi$. However, the support S would be suspicious in this case. Consider all possible cases of the size of S:
   (i) If $|S| = 1$, then $|S + S| = 1$, does not meet the requirement of $S + S = 2$.
   (ii) If $|S| \geq 2$, then $|S + S| \geq 3$, still does not meet the requirement of $S + S = 2$.

   Therefore, for $\phi = \cos t$, there exists no characteristic function ψ with $\psi^2 = \phi$.

2. Let N be a Poisson random variable with mean λ, whose characteristic function is given in Example 1 of the notes for Unit 6. Let $X_1, X_2, ...$ be i.i.d. with characteristic function ϕ and independent of N. Compute the characteristic function of the random sum

$$S = \sum_{i=1}^{N} X_i.$$

<u>*solution*</u>

Given $\mathbb{P}(N = n) = \frac{e^{-\lambda}\lambda^n}{n!}$, and i.i.d. R.V.s $X_1, X_2, \cdots$ with $\phi(t) = \mathbb{E}[e^{itx_i}]$

$$\begin{aligned}
\phi(\sum i = 1^N X_i) &= \mathbb{E}(e^{it\sum i=1^N X_i}) \\
&= \sum_{n=0}^{\infty} \mathbb{E}(e^{it\sum i=1^N X_i} | N = n)\mathbb{P}(N = n) \\
&= \sum_{n=0}^{\infty} \mathbb{E}(e^{it\sum i=1^n X_i} | N = n)\mathbb{P}(N = n) \\
&= \sum_{n=0}^{\infty} \phi^n(t)\frac{e^{-\lambda}\lambda^n}{n!} \\
&= e^{-\lambda}\sum_{n=1}^{\infty} \frac{(\phi(t)\lambda)^n e^{-\lambda\phi(t)}}{n!} e^{\lambda\phi(t)} \\
&= e^{\lambda(\phi(t)-1)}
\end{aligned}$$

3. A probability distribution μ on the real numbers has characteristic function $\phi(t) = 1/(2-e^{it})$. (i) determine whether μ is a lattice distribution and if so, what the lattice is. (ii) determine μ. **Hint:** You could use lattice inversion, based on the Hilbert space basis approach discussed before the inversion formula, but it might be easier to do it another way. Write down the generating function $f(u) = \sum_n \mu(n)u^n$ for the probability distribution, substitute $u = e^{ix}$, see the relation between the generating function and characteristic function, and work backwards to guess the generating function. Really either of these two methods should work (I haven't checked this), but the first may require (undergraduate) residue methods to do the integral.

<u>*solution*</u>

(i) Use the proposition that if $\phi_X(t) = 1$ and $t \neq 0$, then X is a lattice distribution. Check $\frac{1}{2-e^{it}} = 1$, then $e^{it} = 1$, that is, $\cos t + i \sin t = 1$. Then $t = 2\pi k$, and thus the span of the lattice is the greatest $a = \frac{2\pi}{t} = 1$.
For any $t \in (0, 2\pi)$, $|\phi(t)| = \frac{1}{|2-e^{it}|} < \frac{1}{|2-1|} = 1$, we can take the starting point as 0.
Therefore, μ is a lattice distribution and the lattice is $\{j : j \in \mathbb{Z}\}$ for some integer b

(ii) Suppose X is a discrete random variable taking values in the non-negative integers $\{0, 1, \cdots\}$. Substituting u with e^{ix} in the generating function $f(u) = \sum_n \mu(n)u^n$ induces

$$f(e^{ix}) = \sum_{n\geq 0} \mu(n)e^{inx}$$

Also can be represented as

$$\phi(t) = \mathbb{E}(e^{itx}) = \sum_{n\geq 0} e^{itx}\mathbb{P}(x = n)$$

And the c.f. of μ is

$$\phi(t) = \frac{1}{2-e^{it}} = \frac{1}{2}\frac{1}{1-\frac{e^{it}}{2}} = \frac{1}{2}\sum_{k=0}^{\infty}(\frac{e^{it}}{2})^k = \sum_{k=0}^{\infty} e^{ikt}(\frac{1}{2})^{k+1}$$

The taylor expansion is valid because $\phi(t)$ is entire. Comparing the two functions of μ, we can give a guess that $\mu(n) = \frac{1}{2^{n+1}}$, which is indeed a probability measure. The corresponding c.f. is

$$\mathbb{E}(e^{itX}) = \sum_{k=0}^{\infty} e^{itk}\mu(n) = \sum_{k=0}^{\infty} e^{ikt}(\frac{1}{2})^{k+1} = \phi(t)$$

As the characteristic function on real numbers completely defines its probability distribution (bijection). We can therefore conclude that the probability measure is $\mu(n) = (\frac{1}{2})^{n+1}$ for $n = 0, 1, \cdots$, and the the lattice is $\{j : j \in \mathbb{Z}\}$

4. Let $\{X_n\}$ be IID standard normal with partial sums $\{S_n\}$.

(a) Compute the covariance matrix for the centered Gaussian process $\{S_n\}$.

(b) For $n = 3$, write the vector (S_1, S_2, S_3) as a linear image of a standard normal vector (Y_1, Y_2, Y_3).

(c) Use the change of variables between $\{S_n\}$ and $\{Y_n\}$ to compute the probability of $S_1, S_2, S_3 \geq 0$.

<u>*solution*</u>

(a) $S_n = X_1 + \cdots + X_n$, where $X_1 \ldots X_n$ are IID standard normal R.V.s. that $\mathbb{E}(X_i) = 0$ and $\mathrm{Var}(X_i) = 1$

$$\mathbb{E}(S_i) = \mathbb{E}(X_1 + \cdots + X_i) = 0, \quad \mathrm{Var}(S_i) = \mathrm{Var}(X_1 + \cdots + X_i) = \sum_{i=1}^{n} \mathrm{Var}(X_i) = i$$

Then the covariance of Gaussian process X_i and X_j is

$$\begin{aligned}
\operatorname{cov}(S_i, S_j) &= \mathbb{E}\left[S_i - \mathbb{E}S_i\right]\left[S_j - \mathbb{E}S_j\right] \\
&= \mathbb{E}\left[S_i S_j - S_i \mathbb{E}S_j - S_j \mathbb{E}S_i - \mathbb{E}S_i \mathbb{E}S_j\right] \\
&= \mathbb{E}\left[S_i S_j\right] \\
&= \mathbb{E}\left[(X_1 + \cdots + X_i)(X_1 + \cdots + X_j)\right] \\
&= \sum_{m=1}^{\min\{i,j\}} \mathbb{E}\left(X_i^2\right) + \sum_{m \neq n} \mathbb{E}\left(X_m X_n\right) \\
&= \min\{i, j\}
\end{aligned}$$

where $\mathbb{E}\left(X_1^2\right) = \operatorname{var}(X_1) - \mathbb{E}^2(X_1) = 1$ and $\mathbb{E}(X_m X_n) = \mathbb{E}(X_m)\mathbb{E}(X_n) = 0$ when $m \neq n$

so the covariance matrix looks like

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & \cdots & n \end{bmatrix}$$

(b)

$$(s_1, s_2, s_3) = (x_1, x_1 + x_2, x_1 + x_2 + x_3)$$

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(c) Give gaussian density $\mathbb{P}(X=x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$

$$
\begin{aligned}
&\mathbb{P}\left(S_1,S_2,S_3\geqslant 0\right)=\mathbb{P}\left(x_1\geqslant 0,x_2\geqslant -x_1,x_3\geqslant -x_1-x_2\right)\\
&=\int_0^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{x_2^2}{2}}\int_{-x_1}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\int_{-x_1-x_2}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{x_3}{2}}dx_3dx_2dx_1\\
&=\int_0^\infty\int_{-x_1}^\infty\int_{-x_1-x_2}^\infty\left(\frac{1}{\sqrt{2\pi}}\right)^3 e^{-\frac{x_1^2+x_2^2+x_3^3}{2}}dx_3dx_2dx_1
\end{aligned}
$$

By symmetry of joint gaussian distribution, the integral is invariant under rotation. Consider the integral as a part of a unit ball, the probability were looking for can be represented as the proportion of the area constructed by integral ranges and the sphere area. that is, $\frac{A}{4\pi r^2}=\frac{\Omega}{4\pi}$, where Ω can be represented by the solid angle generated from the intersection of $x_1\geqslant 0$, $x_1+x_2\geqslant 0$ and $x_1+x_2+x_3\geqslant 0$

The unit direction vector of the intersection of sphere and construction half-lines $\{x_1\geqslant 0,x_1+x_2\geqslant 0\}$, $\{x_1\geqslant 0,x_1+x_2+x_2\geqslant 0\}$ and $\{x_1+x_2\geqslant 0,x_1+x_2+x_3\geqslant 0\}$ are $\vec{a}=(0,0,1)^T$, $\quad\vec{b}=\left(0,\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)^T$ $\vec{c}=\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0\right)^T$ respectively.

Then the solid angle Ω subtended by the triangular surface is:

$$
\begin{aligned}
\tan\left(\frac{1}{2}\Omega\right)&=\frac{|\vec{a}\vec{b}\vec{c}|}{|\vec{a}||\vec{b}||\vec{c}|+(\vec{a}\cdot\vec{b})|\vec{c}|+(\vec{a}\cdot\vec{c}||\vec{b}|+(\vec{b}\cdot\vec{c}),\vec{a}|}\\
&=\frac{\frac{1}{2}\ |}{1-\frac{1}{\sqrt{2}}+0-\frac{1}{2}}\\
&=\frac{1}{1-\sqrt{2}}
\end{aligned}
$$

And the corresponding sine and cosine are

$$
\begin{aligned}
\cos(\Omega)&=\frac{1-\tan^2\left(\frac{1}{2}\Omega\right)}{1+\tan^2\left(\frac{1}{2}\Omega\right)}-\frac{1-\frac{1}{(1-\sqrt{2})^2}}{1+\frac{1}{(1+\sqrt{2})^2}}\\
&=\frac{\frac{2-2\sqrt{2}}{(1-\sqrt{2})^2}}{\frac{4-2\sqrt{2}}{(1-\sqrt{2})^2}}=\frac{1-\sqrt{2}}{2-\sqrt{2}}=\frac{(1-\sqrt{2})(2+\sqrt{2})}{(2-\sqrt{2})(2+\sqrt{2})}\\
&=\frac{-\sqrt{2}}{2}
\end{aligned}
$$

$$
\begin{aligned}
\sin(\Omega) &= \frac{2\tan\left(\frac{\Omega}{2}\right)}{1+\tan^2\left(\frac{\Omega}{2}\right)} = \frac{\frac{2}{1-\sqrt{2}}}{1+\frac{1}{(1-\sqrt{2})^2}} = \frac{\frac{2}{1-\sqrt{2}}}{\frac{4-2\sqrt{2}}{(1-\sqrt{2})^2}} \\
&= \frac{1-\sqrt{2}}{2-\sqrt{2}} = -\frac{\sqrt{2}}{2}
\end{aligned}
$$

As $\cos(\Omega) < 0$ and $\sin(\Omega) < 0$, we have $\sigma \in (\pi, \frac{3}{2}\pi$, then $\pi = \frac{5}{4}\pi$, and therefore $\mathbb{P}(S_1, S_2, S_3 \geq 0) = \frac{5}{16}$