MATH 6480/STAT 9300/AMCS 6481, Fall 2023, Homework 1

1. Let $\mathcal{F}$ be all subsets of the set $\{1, 2, 3, 4\}$. Give an example of two probability measures $\mu \neq \nu$ on $\mathcal{F}$ that agree on a collection of sets $\mathcal{C}$ for which $\sigma(\mathcal{C}) = \mathcal{F}$.

Solution :

$X = \{1, 2, 3, 4\}, \mathcal{F} = \mathcal{P}(X)$. Take $\mathcal{C} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}\}$, where $\sigma(\mathcal{C}) = \mathcal{F}$.
Define the probability measure μ s.t. $\mu(\{x\}) = \frac{1}{4}$ for any $x \in X$, also define ν as $\nu(\{x\}) = \frac{1}{8}$ for $x \in \{1, 4\}$, $\nu(\{x\}) = \frac{3}{8}$ for $x \in \{2, 3\}$. These two probability measures μ and ν on $\mathcal{F}$ are not equal and agree on $\mathcal{C}$ above for which $\sigma(\mathcal{C}) = \mathcal{F}$

2. Let X and Y be random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $X \leq Y$ a.s. If c is a real number, show that $\mathbb{P}(X \leq \lambda) \geq \mathbb{P}(Y \leq \lambda)$. Now suppose C is another random variable. Is it true that $\mathbb{P}(X \leq C) \geq \mathbb{P}(Y \leq C)$? [Note: the assertion $X \leq Y$ a.s. implies that X and Y are defined on the same probability space.]

Solution :

Given $X \leq Y$ a.s., we have $\mathbb{P}(\omega \in \Omega : X(\omega) \leq Y(\omega)) = 1$, implying $\{\omega \in \Omega : Y(\omega) \leq \lambda\} \subseteq \{\omega \in \Omega : X(\omega) \leq \lambda\}$ a.s., i.e., $\mathbb{P}(\omega \in \Omega : \{\{Y(\omega) \leq \lambda\} \subseteq \{X(\omega) \leq \lambda\}\}) = 1$. We can therefore conclude that $\mathbb{P}(Y \leq \lambda) \leq \mathbb{P}(X \leq \lambda)$ by the monotonicity of probability measure. Let $X_C \equiv X - C$ and $Y_C \equiv Y - C$ be new random variables with the same C, then $X_C \leq Y_C$. From the formal part, we have $\mathbb{P}(Y_C \leq \lambda) \leq \mathbb{P}(X_C \leq \lambda)$ for any constant λ. Take $\lambda = 0$, we thus have $\mathbb{P}(X \leq C) \geq \mathbb{P}(Y \leq C)$.

3. Is it possible to have random variables X, Y and Z for which simultaneously $\mathbb{P}(X > Y) > 1/2$, $\mathbb{P}(Y > Z) > 1/2$ and $\mathbb{P}(Z > X) > 1/2$? Determine with proof the maximum possible value for

$$\min\{\mathbb{P}(X > Y), \mathbb{P}(Y > Z), \mathbb{P}(Z > X)\}.$$

[Hint: Consider the sum $\mathbb{P}(X > Y) + \mathbb{P}(Y > Z) + \mathbb{P}(Z > X)$]. symmetry triangle loop

<u>*Solution*</u> :

Yes, it's possible to have such X, Y and Z.
For any R.V.s X, Y and Z, at most two out of the three events $\{X < Y\}, \{Y < Z\}, \{Z < X\}$ can occur simultaneously, which implies $\mathbb{E}(\mathbb{1}_{X>Y} + \mathbb{1}_{Y>Z} + \mathbb{1}_{Z>X}) = \mathbb{P}(X > Y) + \mathbb{P}(Y > Z) + \mathbb{P}(Z > X) \leq 2$.
Take $a = \min\{\mathbb{P}(X > Y), \mathbb{P}(Y > Z), \mathbb{P}(Z > X)\}, c = \max\{\mathbb{P}(X > Y), \mathbb{P}(Y > Z), \mathbb{P}(Z > X)\}$, and the remaining probability is b, we have $a \leq b \leq c$. Then $3a \leq a+b+c \leq 2$ lead to $a \leq \frac{2}{3}$. We can thus conclude that the maximum possible value of $\min\{\mathbb{P}(X > Y), \mathbb{P}(Y > Z), \mathbb{P}(Z > X)\}$ is upper bounded by $\frac{2}{3}$.
The following is an example of $\mathbb{P}(X > Y) = \mathbb{P}(Y > Z) = \mathbb{P}(Z > X) = \frac{2}{3}$. Let X, Y and Z be random variables with possible values $\{0, 1, 2\}$ with assigned probabilities $\mathbb{P}(X = 0, Y = 1, Z = 2) = \mathbb{P}(X = 1, Y = 2, Z = 0) = \mathbb{P}(X = 2, Y = 0, Z = 1) = \frac{1}{3}$, $\mathbb{P}(X = Y) = 0, \mathbb{P}(Y = Z) = 0, \mathbb{P}(Z = X) = 0$, and all the other arrangements of X, Y and Z have probability 0. Then we have $\mathbb{P}(X < Y) = \frac{2}{3}, \mathbb{P}(Y < Z) = \frac{2}{3}$ and $\mathbb{P}(Z < X) = \frac{2}{3}$

4. Let $q_1, q_2, \ldots$ be an enumeration of the rational numbers in $[0,1]$ and let f be the function mapping $x \in [0,1]$ to $\sum_{n=1}^{\infty} 2^{-n}|x-q_n|^{-1/3}$. Note that this sum is equal to $+\infty$ if $x = q_n$ for some n or if the sum diverges. Prove or disprove: the random variable f on $([0,1], \mathcal{B}, \mathbf{m})$ has finite expectation.

<u>*Solution*</u> :

$$\mathbb{E}(f) = \int_{\Omega} f dx = \int_0^1 \sum_{n=1}^{\infty} 2^{-n}|x-q_n|^{-1/3} dx = \int_0^1 \lim_{N\to\infty} \sum_{n=1}^{N} 2^{-n}|x-q_n|^{-1/3} dx$$

where $2^{-n}|x-q_n|^{-1/3} \geq 0$ for any x and q_n and the sum is monotonic increasing. By MCT and Fubini's theorem,

$$\mathbb{E}(f) = \lim_{N\to\infty} \sum_{n=1}^{N} \int_0^1 2^{-n}|x-q_n|^{-1/3} dx = \lim_{N\to\infty} \sum_{n=1}^{N} 2^{-n} \int_0^1 |x-q_n|^{-1/3} dx$$

$$\int_0^1 |x-q_n|^{-1/3} dx = \int_0^{q_n} (q_n - x)^{-1/3} dx + \int_{q_n}^1 (x-q_n)^{-1/3} dx = \frac{3}{2} q_n^{-2/3} + \frac{3}{2}(1-q_n)^{-2/3}$$

$$= \frac{3}{2}(q_n^{-2/3} + (1-q_n)^{-2/3}) \leq \frac{3}{2} \cdot 2 = 3 \text{ as } q_n \in [0,1]$$

Therefore, $\mathbb{E}(f) = \lim_{N\to\infty} \sum_{n=1}^{N} 2^{-n} \int_0^1 |x-q_n|^{-1/3} dx \leq \lim_{N\to\infty} \sum_{n=1}^{N} 3 \cdot 2^{-n} = 3$ is finite.